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Thermodynamics, Hydrodynamics and Critical Phenomena in Strongly Coupled Gauge Theories

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by

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Abstract

The gauge theory / string theory correspondence has led to great progress in the study of strongly-coupled gauge theories. In this work, we start with a detailed treatment of some simple examples of this correspondence in order to establish some of the concepts and techniques are used on a more complicated system. We then consider a (3+1)-dimensional theory of gravity with a translationally invariant horizon, that is assumed to be dual to a (2+1)-dimensional non-conformal gauge theory at finite temperature. We study the thermodynamics of this model and find that there exists an exotic type of second-order phase transition wherein the symmetry-broken phase occurs above the critical temperature. We also study the hydrodynamics of this model and find that the speed of sound in the various phases of the model suggests that the symmetry broken phases are thermodynamically stable, yet their higher free energy with respect to the symmetric phase suggests that they are not thermodynamically preferred. We calculate the bulk-to-shear viscosity ratio and find that, in the symmetry-broken phase, it diverges at the phase transition. Finally, we study the critical behaviour of this model close to the phase transition and compute the static and dynamic critical exponents, which turn out to be of mean-field type. We conclude that, although the symmetry-broken phases are thermodynamically stable, they are perturbatively unstable. Thus, this model is a counter-example to the Correlated Stability Conjecture, which relates thermodynamic and classical (in)stabilities of black branes with translationally invariant horizons.

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Chapter 1

Introduction

The standard model of fundamental particles and interactions is formulated in terms of quantum field theories [13]. This framework is most useful in regimes where the couplings measuring the strength of the interactions is weak, whereby perturbation theory can be used to calculated observable quantities. This approach has had a great deal of success when applied to high-energy particle collisions in accelerator experiments. However, in the first decade of this century, heavy ion collisions at the Relativistic Heavy Ion Collider gave results suggesting that the collisions produce a strongly coupled quark-gluon plasma [19], whose theoretical description should be that of strongly coupled quantum chromodynamics.

Perturbation theory fails when the couplings in the field theory become strong. One way of dealing with strongly coupled field theories is to use the Anti de Sitter / Conformal Field Theory (AdS/CFT) correspondence proposed by Maldacena [32]. The prototypical example of this correspondence is the duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in 3+1 dimensions, which is a conformal gauge theory. This correpondence has been extended to conjecture a correspondence between asymptotically AdS spacetimes and non-conformal gauge theories. In support of this, particular massive deformations of strongly coupled $\mathcal{N} = 4$ SU($N \to \infty$) SYM are conjectured to be dual to AdS_5 spacetime coupled to a particular set of scalar fields [35]. This proposed duality has survived all of the holographic tests made so far [12, 9]. This observation may justify one to devise a phenomenological model of gravity and conjecture that it *defines* a dual field theory. This approach was used by Erlich et al. [18] to create a phenomenological model that captures some of the key features of QCD. Herzog and collaborators [28, 27] created phenomenological models of superfluidity and superconductivity. The non-conformal extension of the AdS/CFT correspondence a central theme in this thesis[1]. We will not go into the fine details of the correspondence here. Instead we will give the some of the concepts, results and parts of the AdS/CFT dictionary that relates quantities in the gravity theory to quantities in the field theory. We refer the reader to reference [2] and references therein for a thorough treatment.

1.1 General relativity, AdS spacetime and black holes

In this section we review some of the principles of general relativity (GR). For more detail the reader is referred to references [36, 41].

The central object in GR is the metric tensor $g_{\mu\nu}(x)$, whose components are generally functions of the spacetime coordinates $\left(x^0, x^1, x^2, ..., x^{d-1}\right)$, and they are encoded in the infinitesimal line element[2]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1.1}$$

[1]Although we will be considering non-conformal field theories, we will continue to use the terminology ”AdS/CFT correspondence”.

[2]We use Einstein’s summation convention, and signature (-+++...)

The metric components are governed by Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} + \frac{\Lambda}{2}g_{\mu\nu}, \tag{1.2}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $T_{\mu\nu}$ is the stress-energy tensor associated with the matter fields, and Λ is the cosmological constant. Einstein's equation can be derived by minimizing the Einstein-Hilbert action,[3]

$$\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = 0, \quad \text{where} \quad S_{EH} = \frac{1}{16\pi G_D}\int d^Dx\,\sqrt{-g}\,(R + \Lambda + 16\pi G_D\mathcal{L}_m)\,, \tag{1.3}$$

and G_D is the D-dimensional Newton's constant, and $\mathcal{L}_m$ is the lagrangian describing the matter fields.

A metric of particular importance to us here is that of AdS_4. This is the solution to Einstein's equations in $D = 4$ dimensions when we assume radial symmetry, and set

$$T_{\mu\nu} = 0, \qquad \Lambda = 6. \tag{1.4}$$

We will not solve this explicitly here since it will be done in detail in chapter 3. The result is the AdS_4 metric in Poincaré coordinates,

$$ds^2 = -r^2\left(1 - \frac{r_0^3}{r^3}\right)dt^2 + r^2\left(dx_1^2 + dx_2^2\right) + \frac{dr^2}{r^2\left(1 - \frac{r_0^3}{r^3}\right)}, \tag{1.5}$$

where $x_0 \equiv t$ is time, $x_3 \equiv r$ is the radial coordinate and r_0 is an integration constant.

The final concept we need to cover in this section is that of a black hole. There are many entire textbooks devoted to this subject, but here we will be very brief. A black hole is defined by a hypersurface in spacetime called the horizon, which

[3]Here we are being cavalier about boundary contributions and other counter-terms, but these will be treated carefully in later chapters.

is a boundary that separates the spacetime into two causally disconnected regions. Observers inside the horizon cannot communicate with those outside the horizon. To locate the horizon in spacetime we need to introduce the concept of a Killing vector. A Killing vector is a vector, denoted ξ^μ, that satisfies the Killing equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0, \tag{1.6}$$

where ∇_μ is the covariant derivative, defined by

$$\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\lambda{}_{\mu\nu} \xi_\lambda, \tag{1.7}$$

where $\Gamma^\lambda{}_{\mu\nu}$ are the Christoffel symbols. A static spacetime always admits a time-like Killing vector of the form

$$\xi^\mu = [1, 0, 0, 0...] \tag{1.8}$$

The horizon is defined by the hypersurface where this vector becomes null. That is

$$\text{horizon} \quad \Leftrightarrow \quad g_{\mu\nu}\xi^\mu \xi^\nu = g_{tt} = 0, \tag{1.9}$$

with the added condition that the other metric components do not vanish, or they vanish more slowly than g_{tt} as we approach the horizon. For our AdS_4 spacetime described by the metric in (1.5) we have

$$g_{tt} = -r^2 \left(1 - \frac{r_0^3}{r^3}\right), \tag{1.10}$$

so the horizon is given by the surface

$$r = r_0. \tag{1.11}$$

Notice that the geometry of the horizon is found by fixing $r = r_0$ =constant in (1.5), giving

$$ds^2\Big|_{horizon} = r_0^2\left(dx_1^2 + dx_2^2\right), \tag{1.12}$$

which is the geometry of the plane $\mathbb{R}^2$. Thus we have a planar black hole.

1.2 Conformal Field Theories

This thesis will focus on a phenomenological model of the AdS/CFT correspondence. The Hamiltonian of the field theory we will study is not known, but it is not conformally invariant. Nevertheless, we will briefly review some of the main features of CFTs here, as they will be important when we consider the conformal limit of our model. For a detailed study of CFTs, the reader is referred to [20].

A CFT is a quantum field theory that is invariant under conformal transformations, which are spacetime coordinate transformations that are angle-preserving. More precisely, the infinitesimal conformal transformation is

$$x^\mu \to \tilde{x}^\mu = x^\mu + v^\mu(x), \tag{1.13}$$

where

$$v^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + \left(b^\mu x^2 - 2b^\lambda x_\lambda x^\mu\right). \tag{1.14}$$

The parameters of the transformation have the following interpretation: a^μ is a translation, $\omega^\mu{}_\nu x^\nu$ is a Lorentz transformation (i.e. rotations and boosts), λx^μ is a rescaling, and $\left(b^\mu x^2 - 2b^\lambda x_\lambda x^\mu\right)$ is a special conformal transformation. Under a conformal transformation, the metric tranforms as

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu}(x) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} = \Omega(x) g_{\mu\nu}(x). \tag{1.15}$$

For an infinitesimal transformation, we assume the form $\Omega(x) = 1 - \omega(x)$, and we can show that v^μ satisfies the conformal Killing equation,

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{p+1}\left(\partial_\lambda v^\lambda\right) g_{\mu\nu}, \tag{1.16}$$

where p is the spatial dimension of the spacetime in which the CFT lives.

The stress-energy tensor of a CFT describes the reaction of the system under a perturbation of the metric, and it is given by[4]

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{1.17}$$

Let us prove a key feature of CFTs - that the stress-energy tensor is traceless. The conserved current j^μ associated with a symmetry of the form (1.13) is given by

$$j^\mu = T^{\mu\nu} v_\nu. \tag{1.18}$$

Since j^μ is conserved, we have

$$\partial_\mu j^\mu = v_\nu \partial_\mu T^{\mu\nu} + \frac{1}{2} T^{\mu\nu}\left(\partial_\mu v_\nu + \partial_\nu v_\mu\right) = 0. \tag{1.19}$$

Note that $T^{\mu\nu}$ is symmetric, so we take only the symmetric part of $\partial_\mu v_\nu$ in the second

[4]Strictly speaking this is the Belinfante stress-energy tensor, which is related to the canonical stress-energy tensor by $T^{\mu\nu} = T^{\mu\nu}_{canonical} + \partial_\alpha V^{\alpha\mu\nu}$ where $V^{\alpha\mu\nu} = -V^{\mu\alpha\nu}$. Both stress-energy tensors lead to the same conservation equations and Ward identities [20].

term. It is always the case that $\partial_\mu T^{\mu\nu} = 0$. Using (1.16), equation (1.19) becomes

$$\frac{1}{p+1}\left(\partial^\lambda v_\lambda\right) g_{\mu\nu} T^{\mu\nu} = 0. \tag{1.20}$$

Thus, the stress energy tensor is traceless

$$T^\mu{}_\mu = 0. \tag{1.21}$$

An important quantity in a CFT is the central charge, c. The central charge is essentially a way to measure the number of degrees of freedom in the CFT. This is often defined in terms of the two-point function of the stress-energy tensor,

$$\left\langle T_{\mu\nu}(x) T_{\alpha\beta}(0)\right\rangle = \frac{c}{(x^2)^d}\frac{1}{\omega_{d-1}^2}\left(I_{\mu\alpha}I_{\nu\beta} + I_{\mu\beta}I_{\nu\alpha} - \frac{2}{d}\delta_{\mu\nu}\delta_{\alpha\beta}\right), \tag{1.22}$$

where

$$I_{\mu\nu} = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}, \qquad \omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}, \tag{1.23}$$

and $\delta_{\mu\nu}$ is the Kronecker delta function. In the next section we will see that the AdS/CFT correspondence provides a much more convenient way to calcuate the central charge.

From Noether's theorem, we can find the conserved charges associated with conformal tranformations, giving rise to the following generators of the conformal group,

$$\begin{aligned} a^\mu &\to P^\mu \\ \omega_{\mu\nu} &\to M_{\mu\nu} \\ \lambda &\to D \\ b^\mu &\to K_\mu, \end{aligned} \tag{1.24}$$

which obey the conformal algebra (see [20]). Of particular importance here is the generator D of scale transformations, which is called the dilatation operator. Given an operator $\mathcal{O}$ in the CFT, the dilatation operator acts as

$$D\mathcal{O} = \Delta\mathcal{O}, \tag{1.25}$$

where Δ is the scaling dimension of the operator $\mathcal{O}$, and the operator $\mathcal{O}$ transforms as

$$\mathcal{O}(x) \to \tilde{\mathcal{O}}(x) = \lambda^{\Delta}\mathcal{O}(\lambda x) \tag{1.26}$$

under a scale transformation parameterized by λ.

Operators in a CFT can be separated in to three classes: relevant, irrelevant, and marginal. This characterization of operators can be made in terms of their scaling dimensions;

$$\begin{aligned} \Delta < p+1 \quad &\Rightarrow \quad \text{relevant} \\ \Delta > p+1 \quad &\Rightarrow \quad \text{irrelevant} \\ \Delta = p+1 \quad &\Rightarrow \quad \text{marginal.} \end{aligned} \tag{1.27}$$

Loosely speaking, a relevant operator is one that can appear in the Hamiltonian of the CFT, while an irrelevant operator does not (however, we may still consider correlation functions containing irrelevant operators). In other words, Hamiltonians with relevant operators are renormalizable, while those with irrelevant ones are not.

Let us now look at how CFTs are related to non-conformal field theories. Quite generally, we can construct a field theory by identifying the field content and writing down an action of the form,

$$S = S_0 + \int d^{p+1}x \sum_i g_i\mathcal{O}_i, \tag{1.28}$$

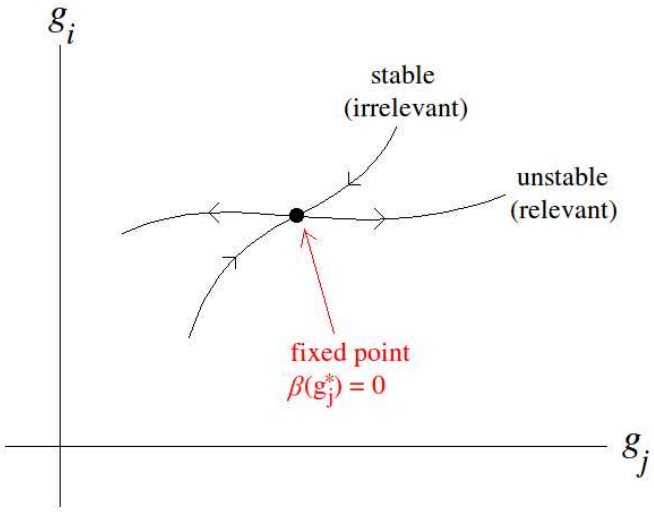

Figure 1.1: Fixed points and RG flow in coupling space.

where S_0 is the free-field part, O_i are operators characterizing all possible interactions between the fields, and g_i are couplings governing the strength of the interactions. Under the renomalization scheme, the couplings acquire a dependence on the energy scale μ, and this dependence is governed by the so-called renormalization group (RG) flow equations,

$$\mu \frac{dg_i}{d\mu} = \beta_i \left(g_j \right). \tag{1.29}$$

A set of fixed points in coupling space is found by solving the system

$$\beta_i \left(g_j^* \right) = 0. \tag{1.30}$$

At the fixed points, the theories become scale invariant and can be described a CFT. These fixed points are very special because, if we have a CFT corresponding to a fixed point, the couplings of that theory do not run as we lower the energy scale. One can linearize the system (1.29) to determine the stability of these fixed points in coupling space, where trajectories leaving the fixed point are unstable and those entering the fixed point are stable. This allows us to classify the operators associated with the couplings in these directions. Operators whose couplings flow along unstable (stable) trajectories are called irrelevant (relevant) operators. The point here is that if we perturb a CFT by an operator, we move our theory away from the fixed point and induce the RG flow.

1.3 AdS/CFT correspondence

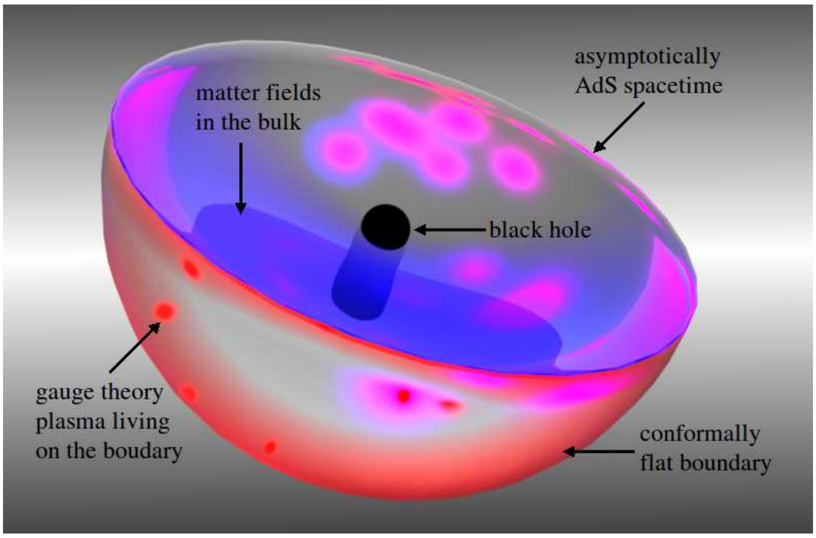

Figure 1.2: Artists interpretation of the AdS/CFT correspondence. In the bulk of an asymptotically AdS spacetime, we may have some matter fields, represented by the blue colour, and a black hole. The dual gauge theory plasma, represented in red, lives on the boundary of the spacetime.

Figure 1.2 shows a cartoon depicting how we can interpret the AdS/CFT correspondence. The idea is that we have a $(d + 1)$-dimensional, asymptotically AdS spacetime containing a black hole, and perhaps some matter fields. This defines our gravity theory; that is, the AdS side of the correspondence. A key feature of AdS spacetimes is that they have a boundary such that light rays emitted in the bulk of the spacetime can reach the boundary in finite time.[5] The dual field theory lives on this boundary. The existence of a black hole in the bulk spacetime corresponds to thermal states in the field theory, where the radius of the black hole's horizon is related to the temperature of the ”plasma” living on the boundary. In the absence of matter fields on the gravity side, we assume that the dual gauge theory is a CFT. If we have matter fields propagating in the bulk spacetime from the gravity perspective, this corresponds to the turning on of massive fields in the field theory, thus deforming away from the fixed point and breaking the conformal invariance.

There are several examples of the AdS/CFT correspondence that arise from various limits of string theory. In these cases, the identity of both the gravity theory and the field theory theory are known [32, 35, 2]. The model that we will focus on here does not have such a string theory embedding. As such, we will adopt the philosophy that our gravity theory on the AdS side *defines* a field theory on the boundary. Although this means that we do not know exactly what field theory we are dealing with, the AdS/CFT correspondence allows us to glean many of its physical features.

The most deeply studied example of the AdS/CFT correspondence is that of $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [32, 2, 38, 5]. Having a duality between a field theory and a ”classical” gravity theory is only valid if the 't Hooft coupling $g_{YM}^2 N$ is infinite. We assume a similar argument for phenomenological models, so really we are considering very strongly coupled field theories. To study finite coupling corrections, we typically would have to find a string theory embedding of our model and look at subleading terms in a $1/N$ expansion.

[5]This is much different from Minkowski spacetime, where a light ray reaches the boundary in infinite time

1.3.1 Thermodynamics

A formal statement of the AdS/CFT correspondence that will be useful to us is that the partition function of the CFT is related to the gravitational (AdS) action by

$$Z_{CFT} = e^{-\frac{F}{T}} = e^{-S_g}, \tag{1.31}$$

where S_g is the Euclidean action of the gravity theory, which must be renormalized to remove divergences at the boundary. This divergence-cancelling prescription is called holographic renormalization, and the techniques are described in [3, 4, 40]; although, we will work out several examples in detail in later chapters. There is a dictionary that enables us to relate certain quantities in the gravity theory to quantities in the gauge theory. Here we will list those items that we will use later; however, we will review them as they arise later on in the context of our specific model.

Beginning with thermodynamics, the Hawking temperature, and entropy density of the black hole in the gravity theory are equal to the temperature and entropy in the field theory correspondingly,

$$T_{BH} = T_{field} \equiv T, \quad \text{and} \quad s_{BH} = s_{field} \equiv s. \tag{1.32}$$

The stress-energy tensor of AdS gravity is given by [3]

$$T^{\mu\nu} = \frac{2}{\sqrt{\gamma}}\frac{\delta S_{grav}}{\delta\gamma_{\mu\nu}}, \tag{1.33}$$

where $\gamma_{\mu\nu}$ is the induced metric on the AdS boundary, and S_{grav} is the renormalized graviational action. We will see later that $T^{\mu\nu}$ has divergences that must be properly cancelled. When all is said and done, the stress-energy tensor is related to the energy density and pressure of the gauge theory plasma by

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{E} & 0 & 0 & ... \\ 0 & \mathcal{P} & 0 & ... \\ 0 & 0 & \mathcal{P} & 0 \\ \vdots & 0 & 0 & \ddots \end{pmatrix} \tag{1.34}$$

The diagonal structure of $T^{\mu\nu}$ is a result of the fact that we consider only spherically symmetric spacetimes, but non-zero off-diagonal may be present in more general cases. The free energy density $\mathcal{F}$ can be computed from (1.31), and we may check that $\mathcal{P} = -\mathcal{F}$, as expected for a homogeneous system.

The AdS/CFT correspondence allows for a simple method to calcuate the central charge in CFTs at finite temperature with dual gravity descriptions. In a d-dimensional CFT at finite temperature, T is the only scale in the system. By dimensional analysis, we expect that $s \propto T^{d-1}$. In a thermal system, the entropy density is a measure of the number of degrees of freedom, and so should be related to the central charge. This relation is [31]

$$s = c\frac{\Gamma\left(\frac{d}{2}\right)^3}{4\pi^{\frac{d}{2}}\Gamma(d)}\left(\frac{4\pi}{d}\right)^d\left(\frac{d-1}{d+1}\right)T^{d-1}. \tag{1.35}$$

So computing the entropy density in the gravity theory also determines the central charge in the CFT.

1.4 Hydrodynamics

On the field theory side, at finite temperature, the hydrodynamics are that of a standard viscous relativistic fluid. The local stress-energy tensor is given by [11]

$$\begin{aligned} T^{\mu\nu} &= \epsilon u^\mu u^\nu + P(\epsilon)\Delta^{\mu\nu} - \eta(\epsilon)\sigma^{\mu\nu} - \zeta(\epsilon)\Delta^{\mu\nu}\nabla_\alpha u^\alpha \\ \Delta^{\mu\nu} &= g^{\mu\nu} + u^\mu u^\nu, \qquad \sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha\right) - \frac{1}{d-1}\Delta^{\mu\nu}\Delta^{\alpha\beta}\nabla_\alpha u_\beta, \end{aligned} \tag{1.36}$$

where ϵ is the local energy density, P is the pressure, u^μ is the local d-velocity of the plasma, and η and ζ are the shear and bulk viscosities respectively. A plasma with such a stress-energy tensor allows for the propagation of hydrodynamic sound waves with the following dispersion relation

$$\hat{\omega} = c_s\hat{q} - i\Gamma\hat{q}^2 + \mathcal{O}\left(\hat{q}^3\right), \qquad \hat{q} \to 0. \tag{1.37}$$

where c_s is the speed of the sound waves, and Γ is their attenuation,

$$c_s^2 = \left(\frac{\partial P}{\partial \epsilon}\right)_T, \qquad \Gamma = 2\pi\frac{\eta}{s}\left(\frac{d-2}{d-1} + \frac{\zeta}{2\eta}\right), \tag{1.38}$$

and

$$\hat{\omega} = \frac{\omega}{2\pi T}, \qquad \hat{q} = \frac{|\vec{q}|}{2\pi T}, \tag{1.39}$$

where ω and $\vec{q}$ are the frequency and momentum of the waves, respectively. Also, s is the entropy density, and $d = p + 1$ is the spacetime dimension. The AdS/CFT correspondence identifies the dispersion relation (1.37) of sound waves in the field theory (plasma) with that of quasinormal modes of the dual gravity theory in the limit where $q \to 0$. We will see how this works in chapters 4 and 6.

In a conformal plasma, the trace of the stress-energy tensor vanishes,

$$T^{\mu}{}_{\mu} = 0. \tag{1.40}$$

Let us show that a result of this is that for a CFT plasma,

$$c_s^2 = \frac{1}{d-1}, \quad \text{and} \quad \frac{\zeta}{\eta} = 0 \quad \text{(conformal).} \tag{1.41}$$

First, we assume that u^μ is a timelike vector field normalized to $u^\mu u_\mu = -1$. Now, taking the trace of (1.36), we get

$$T^{\mu}{}_{\mu} = -\epsilon + P\Delta^{\mu}{}_{\mu} - \eta\sigma^{\mu}{}_{\mu} - \zeta\Delta^{\mu}{}_{\mu}\left(\nabla_\alpha u^\alpha\right) = 0. \tag{1.42}$$

We have

$$\Delta^{\mu}{}_{\mu} = g^{\mu}{}_{\mu} + u^\mu u_\mu = d - 1, \tag{1.43}$$

and using $g^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$ it is straightforward to show that

$$\begin{aligned} \sigma^{\mu}{}_{\mu} &= u^\alpha u^\beta \left(\nabla_\beta u_\alpha - \nabla_\alpha u_\beta\right). \\ &= 0, \end{aligned} \tag{1.44}$$

where the second equality follows from the fact that $u^\alpha u^\beta$ is a symmetric tensor and $\left(\nabla_\beta u_\alpha - \nabla_\alpha u_\beta\right)$ is an antisymmetric tensor, so their product must vanish. We now have

$$T^{\mu}{}_{\mu} = \left[-\epsilon + (d-1)P\right] - \zeta(d-1)\left(\nabla_\alpha u^\alpha\right) = 0. \tag{1.45}$$

Since the first term is u-independent and the second term is u-dependent, both terms must vanish. The first term gives

$$P = \frac{\epsilon}{d-1}, \quad \text{so} \quad c_s^2 = \left(\frac{\partial P}{\partial \epsilon}\right)_T = \frac{1}{d-1}. \tag{1.46}$$

The second term gives

$$\zeta = 0. \tag{1.47}$$

The shear viscosity η of the plasma is related to the low-energy graviton absorption cross-section σ of the black hole by [37],

$$\eta = \frac{1}{16\pi G}\sigma. \tag{1.48}$$

It turns out that the ratio η/s is universal for all strongly coupled gauge-theories dual to a two-derivative gravitational theory[6]. The universal value is [7]

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{1.49}$$

We will not prove this here, but we will give a brief argument of why η/s is constant, following [33]. In [16] the authors give a theorem that says that the low-energy absorption of any black hole is equal to the horizon area, which in turn is proportional to the black hole entropy,

$$\sigma = A \propto s. \tag{1.50}$$

Thus

$$\frac{\eta}{s} = \text{constant} \tag{1.51}$$

Having a universally constant value for $\frac{\eta}{s}$ now allows us to compute the bulk-to-

[6]Camanho et al. [14] propose higher derivative corrections. Erdmenger et al. [17] found that in a two-derivative gravity theory coupled to an SU(2) gauge field, spontaneous breaking of rotational invariance induces additional shear modes whose corresponding viscosities are not universal.

shear viscosity ratio ζ/η, provided that we can compute the dispersion relation in the hydrodynamic limit, i.e. $\hat{q} \to 0$.

1.5 Exotic Model

The goal of this thesis is to demonstrate the techniques used to study thermodynamics, hydrodynamics and critical phenomena in an AdS/CFT model. The latter requires that the gravity theory be asymptotically AdS, and that the system undergoes a phase transition at some critical temperature. Gubser [25] remarks that a theory with an action of the form

$$\begin{aligned} S &= \int d^5x \sqrt{-g} \left(R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} (\partial\chi)^2 - V(\phi,\chi) \right) \\ V(\phi,\chi) &= -6 + \frac{1}{2} m_\phi^2 \phi^2 + \frac{1}{2} m_\chi^2 \chi^2 + g\phi^2\chi^2, \end{aligned} \tag{1.52}$$

exhibits a second order phase transition. Here we will consider a gravitational action with similar form[7],

$$\begin{aligned} S &= \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(\mathcal{L}_{CFT} + \mathcal{L}_r + \mathcal{L}_i \right), \\ \mathcal{L}_{CFT} &= R + 6, \quad \mathcal{L}_r = -\frac{1}{2} (\nabla\phi)^2 + \phi^2, \quad \mathcal{L}_i = -\frac{1}{2} (\nabla\chi)^2 - 2\chi^2 - g\phi^2\chi^2. \end{aligned} \tag{1.53}$$

We can find asymptotically AdS_4 solutions as long as we require that the scalars $\phi, \chi \to 0$ at the AdS boundary. Let us interpret what the form of this action means to the dual field theory. In the absence of $\mathcal{L}_r$ and $\mathcal{L}_i$, the remaining part $\mathcal{L}_{CFT}$, of which pure AdS-Schwarzschild black holes are a solution, is dual to a UV fixed point (as in figure 1.1) decribed by a CFT with Hamiltonian $\mathcal{H}_{CFT}$. According to the AdS/CFT correspondence, scalar fields in the gravity theory are associated with

[7] We chose to work in four dimensions because we initially wanted to study this system in an external magnetic field.

operators in the field theory. In our model, ϕ and χ are associated with operators $\mathcal{O}_r$ and $\mathcal{O}_i$. The inclusion of the terms $\mathcal{L}_r$ and $\mathcal{L}_i$ then deform our field theory away from the fixed point and induces the renormalization group (RG) flow,

$$\mathcal{H}_{CFT} \to \tilde{\mathcal{H}} = \mathcal{H}_{CFT} + \lambda_r \mathcal{O}_r + \lambda_i \mathcal{O}_i. \tag{1.54}$$

The masses of scalar fields in the gravity theory (see the ϕ^2 and χ^2 terms in (1.53)) are related to the scaling dimensions of the operators in the field theory by [42]

$$m^2 = \Delta \left(\Delta - p - 1\right). \tag{1.55}$$

Thus, we infer scaling dimensions[8] and classification of the field theory operators,

$$\begin{aligned} m_\phi^2 &= -2 \quad \Rightarrow \quad \Delta_{\mathcal{O}_r} = 2 < p+1 = 3 \quad \text{(relevant)} \\ m_\chi^2 &= 4 \quad \Rightarrow \quad \Delta_{\mathcal{O}_i} = 4 > p+1 = 3 \quad \text{(irrelevant).} \end{aligned} \tag{1.56}$$

The AdS/CFT correspondence dictates that the asymptotic behaviour of the scalar fields near the AdS boundary takes the form

$$\begin{aligned} \phi &= \lambda_r r^{d-\Delta_{\mathcal{O}_r}} + \langle \mathcal{O}_r \rangle \, r^{-\Delta_{\mathcal{O}_r}} + \ldots \\ \chi &= \lambda_i r^{d-\Delta_{\mathcal{O}_i}} + \langle \mathcal{O}_i \rangle \, r^{-\Delta_{\mathcal{O}_i}} + \ldots \end{aligned} \tag{1.57}$$

where $d = 3$ in our case. Having determined that $\mathcal{O}_i$ is an irrelevant operator, we are forced to set $\lambda_i = 0$ in (1.54) in order to maintain a well-defined field theory. Thus we expect that for our model with $r \to \infty$

[8]Note that $\Delta_{\mathcal{O}_r} = 1$ is also possible, but we will not consider this case here. See reference [10].

$$\begin{aligned}\phi &= \lambda_r r + \frac{\langle O_r \rangle}{r^2} + \ldots \\ \chi &= \frac{\langle O_i \rangle}{r^4} + \ldots\end{aligned} \tag{1.58}$$

The action (1.53) has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry associated with the parity transformation

$$\phi \to -\phi \qquad \chi \to -\chi. \tag{1.59}$$

The former symmetry is broken by hand via a relevant deformation of the type (1.54). The latter, as we will see, is broken spontaneously. In other words, we will find a critical temperature where $\langle O_i \rangle = 0$ in one phase, and $\langle O_i \rangle \neq 0$ in the other.

In the field theory, to have a properly renormalized correlation function, say $\langle O_i \rangle$, we may have to mix O_r with O_i in a linear combination. For example

$$O_i \to Z_0 O_i + Z_1 O_r, \tag{1.60}$$

where Z_0 and Z_1 are coefficients that must be carefully chosen in order to cancel any UV divergences. This mixing of the field theoretic operators under the RG flow is accounted for in the gravity theory by the interaction term $g\phi^2\chi^2$. Also, as pointed out in [25] we observe a phase transition only if $g < 0$. Unless explicitly stated otherwise, we will take $g = -100$.

This concludes what we can learn about the field theory corresponding to the gravity theory (1.53) without doing any real computations. But before exploring the thermodynamics, hydrodynamics, and critical phenomena of this model, we will warm up on some simpler examples; namely, we will consider pure AdS_5, which is dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, and is the most well understood example of the AdS/CFT correspondence. As well, we will consider pure AdS_4, which is the conformal limit $\lambda_r \to 0$ of our Exotic Model.

Chapter 2

AdS$_5$

In this chapter we will study anti-de Sitter gravity in five dimensions (AdS$_5$). Starting from the gravitational action, we derive the equations of motion and find their solution. We consider black hole solutions with 3-sphere horizons. From the solution we compute the temperature and entropy density, and we perform the holographic renormalization of the gravitational action to determine the free energy density. We compute the stress-energy tensor and, in turn, the energy density and pressure. These exercises are done in painful detail, so many steps may be skipped by the experienced reader.

2.1 Action

Consider a (4+1)-dimensional Einstein gravity with a cosmological constant. The gravitational action is [3]

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^5x\,\sqrt{-g}\,(R_5+\Lambda) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^4x\,\sqrt{-\gamma}\Theta, \tag{2.1}$$

where g is the metric determinant, R_5 is the five-dimensional scalar curvature and Λ is the cosmological constant. For AdS gravity in $d+1$ dimensions, we take $\Lambda = \frac{d(d-1)}{l^2}$ where l is a characteristic AdS length much akin to the radius for a d-sphere. Here, $d = 4$ and since l is the only relevant length scale in the problem, we are free to choose units such that $l = 1$. Also, γ is the determinant of the boundary metric and Θ is the trace of the extrinsic curvature, which is given by

$$\Theta^{\mu\nu} = -\frac{1}{2}\left(\nabla^{\mu} n^{\nu} + \nabla^{\nu} n^{\mu}\right) \tag{2.2}$$

where n^{μ} is the outward pointing unit normal 5-vector to the boundary $\partial\mathcal{M}$, and ∇_{μ} is the covariant derivative operator.

2.2 Equations of motion

In order to find the equations of motion (i.e. the equations that govern the metric $g_{\mu\nu}$) we must vary the action with respect to the metric; that is, we must find $\frac{\delta S}{\delta g^{\mu\nu}}$. Let us look for spherically symmetric solutions. As such, we choose an ansatz for the metric of the form

$$ds^2 = -c_1^2(r)dt^2 + c_2^2(r)dS_3^2 + c_3^2(r)dr^2, \tag{2.3}$$

where $dS_3^2 = d\psi^2 + \sin^2(\psi)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)$ is the metric for a 3-sphere. The determinant of the metric (2.3) is

$$g = -c_1^2 c_2^6 c_3^2 \sin^4\psi \sin^2\theta. \tag{2.4}$$

To simplify the calculation of the scalar curvature R_5, let us consider the metric

$$ds^2 = -c_1^2(r)dt^2 + c_2^2(r)d\mathbf{x}^2 + c_3^2(r)dr^2, \tag{2.5}$$

where $d\mathbf{x}^2 = dx^2 + dy^2 + dz^2$. The Christoffel symbols are defined to be [36]

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\mu}\left(\partial_\gamma g_{\mu\beta} + \partial_\beta g_{\mu\gamma} - \partial_\mu g_{\beta\gamma}\right). \tag{2.6}$$

Since the metric (2.3) is diagonal, only one term in the summation over μ on the right-hand side survives, i.e. the term for which $\mu = \alpha$. Also, since all components of the metric depend only of the coordinate r, then all components of $\Gamma^{\alpha}{}_{\beta\gamma}$ are zero except those with either the form $\Gamma^{r}{}_{\beta\beta}$ or $\Gamma^{\beta}{}_{\beta r}$.[1] This is clearly seen if we write (2.6) as

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}\delta_{\alpha\mu}g^{\alpha\mu}\left(\delta^{\beta\mu}\delta^{\gamma r}\partial_\gamma g_{\mu\beta} + \delta^{\gamma\mu}\delta^{\beta r}\partial_\beta g_{\mu\gamma} - \delta^{\mu r}\delta^{\beta\gamma}\partial_\mu g_{\beta\gamma}\right). \tag{2.7}$$

The Kronecker-delta symbols reflect the facts that the metric is diagonal and depends only on the coordinate r. The result is

$$\Gamma^{r}{}_{\beta\beta} = \frac{1}{2}g^{rr}\left(2\delta^{\beta r}\partial_\beta g_{\beta r} - \partial_r g_{\beta\beta}\right) \text{ and } \Gamma^{\beta}{}_{\beta r} = \frac{1}{2}g^{\beta\beta}\partial_r g_{\beta\beta}, \tag{2.8}$$

and all other components are zero.

The scalar curvature is defined to be

$$R_5 = g^{\mu\nu}\left(\partial_\lambda\Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu\Gamma^{\lambda}{}_{\mu\lambda} + \Gamma^{\lambda}{}_{\mu\nu}\Gamma^{\sigma}{}_{\lambda\sigma} - \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma}\right). \tag{2.9}$$

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[1] Here we are not contracting over the index β but instead just fixing them to be the same.

Using (2.8) we get the scalar curvature for the metric (2.59)

$$\tilde{R}_5 = 2\left(-3\frac{c_1'c_2'}{c_1c_2c_3^2} + \frac{c_1'c_3'}{c_1c_3^3} - \frac{c_1''}{c_1c_3^2} - 3\frac{(c_2')^2}{c_2^2c_3^2} + 3\frac{c_2'c_3'}{c_2c_3^3} - 3\frac{c_2''}{c_2c_3^2}\right). \tag{2.10}$$

The only difference between the metrics (2.3) and (2.59) is that we traded the 3-sphere for Euclidean 3-space. In order to get the scalar curvature R_5 for the metric (2.3) we now must add to $\tilde{R}_5$ the contribution to the curvature due to the 3-sphere with radius c_2, which is known to be $R_{S_d} = d(d-1)/c_2^2$ (here $d = 3$). So the scalar curvature that should be inserted into the action is

$$R_5 = 2\left(-3\frac{c_1'c_2'}{c_1c_2c_3^2} + \frac{c_1'c_3'}{c_1c_3^3} - \frac{c_1''}{c_1c_3^2} - 3\frac{(c_2')^2}{c_2^2c_3^2} + 3\frac{c_2'c_3'}{c_2c_3^3} - 3\frac{c_2''}{c_2c_3^2} + \frac{3}{c_2^2}\right). \tag{2.11}$$

Now the first term in the action (2.1), called the bulk term, becomes

$$\begin{aligned} S_{bulk} = \frac{1}{8\pi G}\int_{\mathcal{M}} d^5x \sin^2\psi \sin\theta \\ \times \begin{pmatrix} 3c_1c_2c_3 + \frac{c_2^3c_1'c_3'}{c_3^2} - 3\frac{c_2^2c_1'c_2'}{c_3} - \frac{c_2^3c_1''}{c_3} \\ -3\frac{c_1c_2(c_2')^2}{c_3} + 3\frac{c_1c_2^2c_2'c_3'}{c_3^2} - 3\frac{c_1c_2^2c_2''}{c_3} - 6c_1c_2^3c_3 \end{pmatrix}. \end{aligned} \tag{2.12}$$

Now we will turn to the second term in the action, called the boundary term, $S_{boundary}$. In our coordinate system, we will consider the boundary $\partial\mathcal{M}$ to be defined by the hypersurface $r = \rho$ where ρ is a constant that we will eventually take to infinity in order to cover the entire space. The outward normal vector to the boundary will then be in the direction of $v^\mu = (0, 0, 0, 0, 1)$. To construct a unit normal, first notice that $v^\mu v_\mu = g_{\mu\nu}v^\mu v^\nu = g_{rr} = c_3^2$. So the unit normal to the boundary is $n^\mu = (0, 0, 0, 0, \frac{1}{c_3})$. From equation (2.2), we see that the trace of the extrinsic curvature is the negative of the divergence of n^μ. That is,

$$\begin{aligned}\Theta &= -\nabla_\mu n^\mu \\ &= -\left(\partial_\mu n^\mu + \Gamma^\mu{}_{\mu\nu} n^\nu\right) \\ &= -\left(\partial_r n^r + \Gamma^\mu{}_{\mu r} n^r\right) \\ &= -\left[(c_3^{-1})' + \Gamma^\mu{}_{\mu r} c_3^{-1}\right] \\ &= -\left[-\frac{c_3'}{c_3^2} + \Gamma^\mu{}_{\mu r}\frac{1}{c_3}\right]\end{aligned} \tag{2.13}$$

From equation (2.6) we can find

$$\begin{aligned}\Gamma^\mu{}_{\mu r} &= \frac{1}{2} g^{\mu\lambda}\left(\partial_r g_{\lambda\mu} + \partial_\mu g_{\lambda r} - \partial_\lambda g_{\mu r}\right) \\ &= \frac{1}{2}\left(\frac{(c_1^2)'}{c_1^2} + 3\frac{(c_2^2)'}{c_2^2} + \frac{(c_3^2)'}{c_3^2}\right) \\ &= \frac{c_1'}{c_1} + 3\frac{c_2'}{c_2} + \frac{c_3'}{c_3}\end{aligned} \tag{2.14}$$

so that

$$\Theta = -\left(\frac{c_1'}{c_1 c_3} + 3\frac{c_2'}{c_2 c_3}\right) \tag{2.15}$$

Notice this is also the extrinsic curvature for the metric (2.5) because the angular factors played no role in the calculation. That is, the angular factors cancelled since, in (2.14), the metric components with lower indices always come with a partial derivative with respect to r, so the angular factors are not affected in the numerator, and the components with upper indices carry the same angular factors in the denominator, and thus the angular factors cancel completely.

The boundary $\partial\mathcal{M}$ is defined by $r = \rho$, so we have $dr = 0$. The induced metric on the boundary is

$$ds^2 = -c_1^2(\rho)dt^2 + c_2^2(\rho)dS_3^2, \tag{2.16}$$

and the determinant is

$$\gamma = -c_1^2 c_2^6 \sin^4\psi \sin^2\theta\Big|_{r=\rho}. \tag{2.17}$$

The boundary term in the action is

$$S_{boundary} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^4x \sin^2\psi \sin\theta \left(\frac{c_2^3 c_1'}{c_3} + 3\frac{c_1 c_2^2 c_2'}{c_3}\right)\Bigg|_{r=\rho} \tag{2.18}$$

To find the equations of motion, we demand that $\delta S/\delta g_{\mu\nu} = 0$, or, considering only the non-zero components of the metric, we demand that $\delta S/\delta c_1 = 0$, and similarly for c_2 and c_3. We will impose that the variation $\delta c_1 = 0$ at the boundary, but we must allow $\delta c_1' \neq 0$ at the boundary, and similarly for c_2 and c_3, in order to get non-trivial results. Now,

$$\begin{aligned}
\delta S_{bulk} &= \frac{\partial S_{bulk}}{\partial c_1}\delta c_1 + \frac{\partial S_{bulk}}{\partial c_1'}\delta c_1' + \frac{\partial S_{bulk}}{\partial c_1''}\delta c_1'' + \text{...similiar terms for } c_2 \text{ and } c_3 \\
&= \frac{1}{8\pi G}\int_{\mathcal{M}} d^5x \sin^2\psi \sin\theta \\
&\times \left[\begin{array}{l} \left(3c_2c_3 - 3\frac{c_2(c_2')^2}{c_3} + 3\frac{c_2^2 c_2' c_3'}{c_3} - 3\frac{c_2^2 c_2''}{c_3} - 6c_2^3 c_3\right)\delta c_1 \\ + \left(\frac{c_2^3 c_3'}{c_3^2} - 3\frac{c_2^2 c_2'}{c_3}\right)\delta c_1' - \frac{c_2^3}{c_3}\delta c_1'' \end{array}\right].
\end{aligned} \tag{2.19}$$

Integrating the second and third terms by parts with respect to r we get

$$\begin{aligned}\frac{\delta S_{bulk}}{\delta c_1} &= \frac{1}{8\pi G}\int_{\mathcal{M}} d^5x \sin^2\psi \sin\theta \\ &\times\left[\begin{array}{l}\left(3c_2c_3 - 3\frac{c_2(c_2')^2}{c_3} + 3\frac{c_2^2c_2'c_3'}{c_3} - 3\frac{c_2^2c_2''}{c_3} - 6c_2^3c_3\right)\delta c_1 \\ -\left(\frac{c_2^3c_3'}{c_3^2} - 3\frac{c_2^2c_2'}{c_3}\right)'\delta c_1 + \left(\frac{c_2^3}{c_3}\right)'\delta c_1'\end{array}\right] \\ &- \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^4x \sin^2\psi \sin\theta\left[\left(\frac{c_2^3c_3'}{c_3^2} - 3\frac{c_2^2c_2'}{c_3}\right)\delta c_1 - \left(\frac{c_2^3}{c_3}\right)\delta c_1'\right].\end{aligned} \tag{2.20}$$

Integrating the final term in the integral over $\mathcal{M}$ we get

$$\begin{aligned}\frac{\delta S_{bulk}}{\delta c_1} &= \frac{1}{8\pi G}\int_{\mathcal{M}} d^5x \sin^2\psi \sin\theta \\ &\times\left[\begin{array}{l}\left(3c_2c_3 - 3\frac{c_2(c_2')^2}{c_3} + 3\frac{c_2^2c_2'c_3'}{c_3} - 3\frac{c_2^2c_2''}{c_3} - 6c_2^3c_3\right) \\ -\left(\frac{c_2^3c_3'}{c_3^2} - 3\frac{c_2^2c_2'}{c_3}\right)' - \left(\frac{c_2^3}{c_3}\right)''\end{array}\right]\delta c_1 \\ &- \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^4x \sin^2\psi \sin\theta\left(\frac{c_2^3}{c_3}\right)\delta c_1',\end{aligned} \tag{2.21}$$

where we dropped the δc_1 term in the integral over $\partial\mathcal{M}$ since the variation vanishes there.

Now let us vary the boundary action $S_{boundary}$ with respect to c_1.

$$\begin{aligned}\frac{\delta S_{boundary}}{\delta c_1} &= \frac{\partial S_{boundary}}{\partial c_1}\delta c_1 + \frac{\partial S_{boundary}}{\partial c_1'}\delta c_1' \\ &= \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^4x \sin^2\psi \sin\theta\left(\frac{c_2^3}{c_3}\right)\delta c_1',\end{aligned} \tag{2.22}$$

where we dropped the δc_1 term as it vanishes on the boundary.

Now, since $S = S_{bulk} + S_{boundary}$, when we vary the total action, the boundary integrals cancel exactly[2] and we are left with

$$\frac{\delta S}{\delta c_1} = \frac{1}{8\pi G}\int_{\mathcal{M}} d^5x \sin^2\psi \sin\theta \times \left[\begin{array}{l} \left(3c_2c_3 - 3\frac{c_2(c_2')^2}{c_3} + 3\frac{c_2^2c_2'c_3'}{c_3} - 3\frac{c_2^2c_2''}{c_3} - 6c_2^3c_3\right) \\ -\left(\frac{c_2^3c_3'}{c_3^2} - 3\frac{c_2^2c_2'}{c_3}\right)' - \left(\frac{c_2^3}{c_3}\right)'' \end{array} \right]\delta c_1 \tag{2.23}$$

Since the variation δc_1 is arbitrary and the functions c_1, c_2 and c_3 are assumed to be continuous, demanding that $\delta S = 0$ means that the quantity in square brackets must vanish. This gives one of the equations of motion

$$c_2''c_2c_3 + (c_2')^2c_3 - c_2'c_3'c_2 - 2c_2^2c_3^3 - c_3^3 = 0. \tag{2.24}$$

Similarly, varying the action with respect to c_2 and c_3 gives the other equations of motion

$$c_1''c_2^2c_3 + 2c_2''c_1c_2c_3 + 2c_1'c_2'c_2c_3 - c_1'c_3'c_2^2 + (c_2')^2c_1c_3 - 2c_2'c_3'c_1c_2 - c_1c_3^3 - 6c_1c_2^2c_3 = 0 \tag{2.25}$$

$$c_1'c_2'c_2 + (c_2')^2c_1 - c_1c_3^2 - 2c_1c_2^2c_3^2 = 0. \tag{2.26}$$

We can put these equations into a nicer form by using (2.24) and (2.25) to eliminate c_2'' from the former, then to eliminate c_1'' from the latter. We find that the equations of motion can be put in the form

$$c_1'' + 2\frac{c_1'c_2'}{c_2} - \frac{c_1'c_3'}{c_3} - \frac{c_1(c_2')^2}{c_2^2} + \frac{c_1c_3^2}{c_2^2} - 2c_1c_3^2 = 0. \tag{2.27}$$

$$c_2'' + \frac{(c_2')^2}{c_2} - \frac{c_2'c_3'}{c_3} - \frac{c_3^2}{c_2} - 2c_2c_3^2 = 0 \tag{2.28}$$

[2] This cancellation is precisely the reason for inserting the boundary term in the action.

$$c_1' c_2' c_2 + (c_2')^2 c_1 - c_1 c_3^2 - 2 c_1 c_2^2 c_3^2 = 0. \tag{2.29}$$

2.3 3-sphere black hole solutions

Now we wish to solve the equations (2.27)-(2.28). Invariance under diffeomorphisms permits us to choose $c_2(r) = r$, which is a natural choice as it assigns to the 3-sphere part of the metric a radius r. Also, let us look for solutions of the form $c_1(r) = r f(r)$ and $c_3(r) = 1/r f(r)$. With these choices[3], equation (2.28) and (2.29) are the same; they are

$$r^3 f(r) \frac{df}{dr} + 2r^2 \left[(f(r))^2 - 1 \right] - 1 = 0. \tag{2.30}$$

Equation (2.27) becomes

$$r^4 f(r) \frac{d^2 f}{dr^2} + r^4 \left(\frac{df}{dr} \right)^2 + 6r^3 f(r) \frac{df}{dr} + 2r^2 \left[(f(r))^2 + 1 \right] - 1 = 0. \tag{2.31}$$

The solution to (2.30) is[4]

$$f(r) = \pm \sqrt{1 + \frac{1}{r^2} + \frac{C}{r^4}}. \tag{2.32}$$

The solution to (2.31) is

$$f(r) = \pm \sqrt{1 + \frac{1}{r^2} + \frac{C_1}{r^3} + \frac{C_2}{r^4}} \tag{2.33}$$

These solutions are mutually compatible only if $C_1 = 0$ and $C_2 = C$. The signature of the metric $(-, +, +, +)$ is preserved if we take the positive solution. Also, we are

[3]The motivation for these choices is that we expect that $f(r) \to 1$ as $r \to \infty$, in which case we have the Poincare coordinate representation of a pure AdS spacetime

[4]Maple was used to solve (2.30) and (2.31)

looking for black hole solutions that are asymptotically AdS. In order for a horizon to exist, we must have $g_{tt} \equiv c_1(r_H) = 0$ while $c_2(r_H) \neq 0$ and $c_3(r_H) \neq 0$, where $r = r_H$ determines the hypersurface of the horizon. For this to be possible, we must take C to be negative, say $C = -r_0^4$. Thus, we take

$$f(r) = \sqrt{1 + \frac{1}{r^2} - \frac{r_0^4}{r^4}}, \tag{2.34}$$

so our metric is

$$ds^2 = -r^2\left(1 + \frac{1}{r^2} - \frac{r_0^4}{r^4}\right)dt^2 + r^2 dS_3^2 + \frac{dr^2}{r^2\left(1 + \frac{1}{r^2} - \frac{r_0^4}{r^4}\right)}. \tag{2.35}$$

By solving $c_1(r) = r f(r) = 0$ for the real, positive root, we find that the horizon is at

$$r = r_H = \frac{1}{\sqrt{2}}\sqrt{-1 + \sqrt{1 + 4r_0^4}} \tag{2.36}$$

Because of our choice of ansatz (2.3), the horizon defined by $r = r_H$ and $t = constant$ (i.e. $dr = dt = 0$) takes the form of a 3-sphere. Thus, we have found the 3-sphere black hole solutions.

2.4 Asymptotic geometry

Asymptotically, i.e. as $r \to \infty$, we have $f(r) \to 1 - 1/r^2$, and the metric becomes

$$ds^2 = -(1 + r^2)dt^2 + r^2 dS_3^2 + \frac{dr^2}{1 + r^2}. \tag{2.37}$$

Let $r^2 = \sinh^2\rho$, so $dr = \cosh\rho d\rho$. Also, $\cosh^2\rho = 1 + r^2$. Under this change of variables, the metric (2.37) becomes

$$ds^2 = -\cosh^2\rho dt^2 + \sinh^2\rho dS_3^2 + d\rho^2, \tag{2.38}$$

which is precisely the AdS_5 metric in global coordinates [2].

2.5 Minimized action

Now we will calculate on-shell value of the action, and we will insist on getting a finite result. This process is called holographic renormalization [3, 40]. Since we have the exact solution (2.35), we may calculate the bulk action by substituting our solution for $c_1(r)$, $c_2(r)$ and $c_3(r)$ into (2.12), which simplifies to

$$\begin{aligned} S_{bulk} &= -\frac{1}{2\pi G}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\pi} \sin^2\psi d\psi \int_{r_H}^{\rho} r^3 dr \int_0^{\tau} dt \\ &= -\frac{\pi}{4G}\left(\rho^4 - r_H^4\right)\tau, \end{aligned} \tag{2.39}$$

where τ is some finite time that we will determine later. What is important to note for now is that the bulk action diverges like ρ^4 as $\rho \to \infty$. Substituting our solution for $c_1(r)$, $c_2(r)$ and $c_3(r)$ at $r = \rho$ into the boundary action, (2.18) simplifies to

$$S_{boundary} = \frac{\pi}{4G}\left(4\rho^4 + 3\rho^2 - 2r_0^4\right)\tau. \tag{2.40}$$

Adding the results together, we get the total action $S = S_{bulk} + S_{boundary}$ to be

$$S = \frac{\pi}{4G}\left(3\rho^4 + 3\rho^2 + r_H^4 - 2r_0^4\right)\tau. \tag{2.41}$$

The action diverges as $\rho \to \infty$. To remove this divergence, we must add a counter-term S_{ct} to the action that depends only on the boundary metric and that is invariant under diffeomorphisms on the boundary so that the equations of motion are unchanged. Also, the counter-term must not depend on any derivatives of the metric so that it does not affect the cancellation of the boundary terms that we achieved earlier. Since $\sqrt{\gamma}$ and the four-dimensional scalar curvature on the boundary R_4 are both invariants on the boundary, a candidate for the counter-term is[5]

$$\frac{1}{8\pi G} S_{ct} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^4x \sqrt{-\gamma} \left(\alpha_1 + \alpha_2 R_4\right), \tag{2.42}$$

where α_1 and α_2 are constants that must be chosen to cancel the divergent part of the action. A straightforward calculation gives

$$\frac{1}{8\pi G} S_{ct} = \frac{\pi}{4G} \rho^2 \sqrt{\rho^4 + \rho^2 - r_0^4} \left(\alpha_1 + \alpha_2 \frac{6}{\rho^2}\right) \tau. \tag{2.43}$$

Since we are interested in the limit as $\rho \to \infty$, we may approximate $\sqrt{\rho^4 + \rho^2 - r_0^4} \approx \rho^2$, and $\rho^2 \sqrt{\rho^4 + \rho^2 - r_0^4} \approx \rho^4 + \rho^2/2 - r_0^4/2$. Then (2.43) becomes

$$\frac{1}{8\pi G} S_{ct} = \frac{\pi}{4G} \left[\alpha_1 \left(\rho^4 - \frac{r_0^4}{2}\right) + \left(\frac{\alpha_1}{2} + 6\alpha_2\right) \rho^2\right] \tau. \tag{2.44}$$

[5] In principle, we could add higher powers of R_4, however, these terms would not diverge, so they are not included.

Comparing equations (2.54) and (2.44) we see that choosing $\alpha_1 = -3$ and $\alpha_2 = -1/4$ will remove the divergent part of the action, which is now given by

$$\begin{aligned} S &= S_{bulk} + S_{boundary} + \frac{1}{8\pi G} S_{ct} \\ &= \frac{\pi}{4G}\left(r_H^4 - \frac{r_0^4}{2}\right)\tau \\ &= \frac{\pi}{8G}\left(r_H^4 - r_H^2\right)\tau \end{aligned} \tag{2.45}$$

When we add scalar hair to the action in chapter 5 we will, in general, not have the luxury of knowing the exact solution. As such, we will demonstrate a way to calculate the result (2.58) assuming that we know only the asymptotic form of the solution about the horizon and about the boundary.

Looking at the bulk action (2.12), the only non-trivial integration is over the r coordinate. Let us define the r-dependent part of the integrand to be

$$\tilde{\mathcal{L}}_{bulk} = 3c_1c_2c_3 + \frac{c_2^3c_1'c_3'}{c_3^2} - 3\frac{c_2^2c_1'c_2'}{c_3} - \frac{c_2^3c_1''}{c_3} - 3\frac{c_1c_2(c_2')^2}{c_3} + 3\frac{c_1c_2^2c_2'c_3'}{c_3^2} - 3\frac{c_1c_2^2c_2''}{c_3} - 6c_1c_2^3c_3. \tag{2.46}$$

If c_1, c_2, and c_3 are solutions to the equations of motion (2.27)-(2.29), then we are free to add terms containing the left-hand sides of those equations to $\tilde{\mathcal{L}}_{bulk}$. In particular, denoting the left-hand side of (2.28) by A, we find

$$\begin{aligned} \tilde{\mathcal{L}}_{bulk} &= \tilde{\mathcal{L}}_{bulk} + 3\frac{c_1c_2^2}{c_3}A \\ &= -3\frac{c_2^2c_1'c_2'}{c_3} + \frac{c_2^3c_1'c_3'}{c_3^2} - \frac{c_2^3c_1''}{c_3} \\ &= \frac{d}{dr}\left(-\frac{c_2^3c_1'}{c_3}\right), \end{aligned} \tag{2.47}$$

where the first equality holds since $A = 0$. Now the action (2.12) can be written as

$$\begin{aligned} S_{bulk} &= \frac{\pi\tau}{4G}\int_{r_H}^{\rho}\frac{d}{dr}\left(-\frac{c_2^3 c_1'}{c_3}\right)dr \\ &= -\frac{\pi\tau}{4G}\left(\frac{c_2^3 c_1'}{c_3}\right)\Bigg|_{r_H}^{\rho}, \end{aligned} \tag{2.48}$$

where we take the limit $\rho \to \infty$. This shows that in order to evaluate the bulk action, we only need the asymptotic solutions. Using equation (2.36), we find that $r_0^4 = r_H^4 + r_H^2$, so the metric (2.35) can be written as

$$ds^2 = -r^2\left(1 + \frac{1}{r^2} - \frac{r_H^4 + r_H^2}{r^4}\right)dt^2 + r^2 dS_3^2 + \frac{dr^2}{r^2\left(1 + \frac{1}{r^2} - \frac{r_H^4+r_H^2}{r^4}\right)}. \tag{2.49}$$

For $r \to \infty$, we find

$$\begin{aligned} c_1 &\sim r + \frac{1}{2r} - \frac{4r_H^4 + 4r_H^2 + 1}{8r^3} + \mathcal{O}\left(\frac{1}{r^5}\right), \\ c_2 &= r, \\ c_3 &\sim \frac{1}{r} - \frac{1}{2r^3} + \frac{4r_H^4 + 4r_H^2 + 3}{8r^5} + \mathcal{O}\left(\frac{1}{r^3}\right). \end{aligned} \tag{2.50}$$

For $r \gtrsim r_H$, we find

$$\begin{aligned} c_1 &\sim \left(\frac{4r_H^2 + 2}{r_H}\right)^{\frac{1}{2}}(r - r_H)^{\frac{1}{2}} + \mathcal{O}\left((r - r_H)^{\frac{3}{2}}\right), \\ c_2 &= r, \\ c_3 &\sim \left(\frac{4r_H^2 + 2}{r_H}\right)^{-\frac{1}{2}}(r - r_H)^{-\frac{1}{2}} + \mathcal{O}\left((r - r_H)^{\frac{1}{2}}\right). \end{aligned} \tag{2.51}$$

Putting these expansions into (2.48) and keeping non-vanishing orders we find

$$S_{bulk} = -\frac{\pi}{4G}\left(\rho^4 - r_H^4\right)\tau \tag{2.52}$$

Putting the expansion into (2.18) and keeping non-vanishing orders we find

$$S_{boundary} = \frac{\pi}{4G}\left(4\rho^4 + 3\rho^2 - 2r_0^4\right)\tau. \tag{2.53}$$

Adding the results together, we get the total action $S = S_{bulk} + S_{boundary}$ to be

$$S = \frac{\pi}{4G}\left(3\rho^4 + 3\rho^2 + r_H^4 - 2r_0^4\right)\tau. \tag{2.54}$$

As before, the action diverges as $\rho \to \infty$, so we add a counter-term of the form

$$\begin{aligned}\frac{1}{8\pi G}S_{ct} &= \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^4x\,\sqrt{-\gamma}\,(\alpha_1 + \alpha_2 R_4)\,, \\ &= \frac{\pi\tau}{4G}\left(\alpha_1 c_1 c_2^3 + 6\alpha_2 c_1 c_2\right).\end{aligned} \tag{2.55}$$

Inserting our expansions for c_1 and c_2 about the boundary and keeping non-vanishing order, we find

$$\frac{1}{8\pi G}S_{ct} = \frac{\pi\tau}{4G}\left[\alpha_1\rho^4 + \left(\frac{1}{2}\alpha_1 + 6\alpha_2\right)\rho^2 - \frac{\alpha_1}{8}\left(4r_H^4 + 4r_H^2 + 1\right) + 3\alpha_2\right], \tag{2.56}$$

Comparing this to (2.54) we see that in order to cancel the divergences, we must satisfy

$$\begin{aligned}\alpha_1 &= -3 \\ \frac{1}{2}\alpha_1 + 6\alpha_2 &= -3\end{aligned} \tag{2.57}$$

So if we choose $\alpha_1 = -3$ and $\alpha_2 = -1/4$ in our counterterm, then we cancel the divergences in the action exactly. Altogether, we arrive at a finite value for the minimized action

$$\begin{aligned} S &= S_{bulk} + S_{boundary} + \frac{1}{8\pi G} S_{ct} \\ &= \frac{\pi}{8G}\left(r_H^4 - r_H^2\right)\tau, \end{aligned} \tag{2.58}$$

which is identical to the answer we obtained using the exact metric.

2.6 Planar black hole solutions

Now let us look for black hole solutions whose horizons are three dimensional hyperplanes. Most of the details will be omitted because the calculations are similar to, yet simpler than, those performed in the previous sections. Let us return to the metric ansatz

$$ds^2 = -c_1^2(r)dt^2 + c_2^2(r)d\mathbf{x}^2 + c_3^2(r)dr^2, \tag{2.59}$$

where $d\mathbf{x}^2 = dx^2 + dy^2 + dz^2$. The metric determinant is

$$g = -c_1^2 c_2^6 c_3^2. \tag{2.60}$$

We found in section 2.2 that the scalar curvature for this metric is

$$R_5 = 2\left(-3\frac{c_1' c_2'}{c_1 c_2 c_3^2} + \frac{c_1' c_3'}{c_1 c_3^3} - \frac{c_1''}{c_1 c_3^2} - 3\frac{(c_2')^2}{c_2^2 c_3^2} + 3\frac{c_2' c_3'}{c_2 c_3^3} - 3\frac{c_2''}{c_2 c_3^2}\right), \tag{2.61}$$

and the extrinsic curvature of the boundary $r = \rho$ is given by

$$\Theta = -\left(\frac{c_1'}{c_1 c_3} + 3\frac{c_2'}{c_2 c_3}\right), \tag{2.62}$$

and the determinant of the boundary metric is

$$\gamma = -c_1^2 c_2^6. \tag{2.63}$$

Putting everything into the action (2.1) and setting the variation of the action with respect to the metric components to zero, we find the equations of motion

$$c_1'' + 2\frac{c_1'c_2'}{c_2} - \frac{c_1'c_3'}{c_3} - \frac{c_1(c_2')^2}{c_2^2} - 2c_1c_3^2 = 0 \tag{2.64}$$

$$c_2'' + \frac{(c_2')^2}{c_2} - \frac{c_2'c_3'}{c_3} - 2c_2c_3^2 = 0 \tag{2.65}$$

$$c_1'c_2'c_2 + (c_2')^2 c_1 - 2c_1c_2^2c_3^2 = 0 \tag{2.66}$$

Letting $c_1(r) = rf(r)$, $c_2(r) = r$, $c_3(r) = 1/rf(r)$, and solving for $f(r)$ we find

$$f(r) = \sqrt{1 - \frac{r_0^4}{r^4}}. \tag{2.67}$$

The metric is

$$ds^2 = -r^2\left(1 - \frac{r_0^4}{r^4}\right)dt^2 + r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2\left(1 - \frac{r_0^4}{r^4}\right)}, \tag{2.68}$$

which has a horizon at $r_H = r_0$. This metric is also asymptotically anti de Sitter. Setting $r = r_0$ and $t = const$ we see that the the horizon is a three dimensional hyperplane. Putting our solution (2.68) into the action gives

$$S = \frac{V\tau}{8\pi G}\left(3\rho^4 - r_0^4\right), \tag{2.69}$$

where $V = \int d^3x$, and τ is some, as of yet undetermined, upper limit on the integration over time. The action diverges as we take the boundary to infinity (i.e. $\rho \to \infty$).

We may add a counterterm of the form

$$\begin{aligned} S_{ct} &= \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^4x\alpha\sqrt{-\gamma} \\ &= \frac{V\tau}{8\pi G}\alpha\rho^4\sqrt{1-\frac{r_0^4}{\rho^4}} \\ &\approx \frac{V\tau}{8\pi G}\alpha\left(\rho^4-\frac{r_0^4}{2}\right) \end{aligned} \tag{2.70}$$

without affecting the equations of motion. Here we have assumed that $r_0^4/\rho^4 << 1$. Comparing (2.69) and (2.70) we see that choosing $\alpha = -3$ will cancel the divergence, and we get a finite action density

$$\frac{S}{V} = \frac{\tau r_0^4}{16\pi G}. \tag{2.71}$$

2.7 Thermodynamics of black holes in AdS$_5$

2.7.1 3-sphere black hole

Here we will derive the thermodynamics of the black hole described by the metric

$$ds^2 = -r^2 f^2(r)dt^2 + \frac{dr^2}{r^2 f^2(r)} + r^2 dS_3^2, \tag{2.72}$$

where

$$f^2(r) = 1 + \frac{1}{r^2} - \frac{r_0^4}{r^4}, \tag{2.73}$$

which has a horizon is at

$$r_H = \frac{1}{\sqrt{2}}\sqrt{-1+\sqrt{1+4r_0^4}}. \tag{2.74}$$

In particular, we will examine the cases where $r_0 >> 1$ and $r_0 << 1$. Note that

$$r_H \approx r_0, \text{ if } r_0 >> 1, \tag{2.75}$$

and

$$r_H \approx r_0^2, \text{ if } r_0 << 1, \tag{2.76}$$

Before moving on to calculating the thermodynamics of our black hole, we will need to establish a relationship between imaginary time and temperature in quantum field theories (QFT) at finite temperature

Imaginary time and temperature

Following [22] the path integral of a QFT is given by

$$Z \equiv \langle \phi_2, t_2 | \phi_1 t_1 \rangle = \int d[\phi] e^{iS[\phi]}, \tag{2.77}$$

where ϕ denotes the fields in the theory and $S[\phi]$ is the action. The integral is over all field configurations where ϕ has the value ϕ_1 at t_1 and the value ϕ_2 at t_2. The Schrodinger and Heisenberg pictures are related by

$$\langle \phi_2, t_2 | \phi_1 t_1 \rangle = \left\langle \phi_2 | e^{iH(t_2 - t_1)} | \phi_1 \right\rangle \tag{2.78}$$

where H is the Hamiltonian of the theory. If we set $i(t_1 - t_2) = 1/T$ and $\phi_1 = \phi_2$, then sum over all ϕ_1, then we get

$$Z = \text{Tr} e^{-\frac{H}{T}} = \int d[\phi] e^{iS(\phi)}, \tag{2.79}$$

where the integral is now over all field configurations that are periodic in imaginary time $t_E = it$ with period $1/T$. The left hand side is nothing other than the canonical partition function of a statistical system. The interpretation here is that in a QFT at finite temperature, imaginary time t_E is periodic with period $1/T$. When we integrate over imaginary time (e.g. as in the action), we should integrate over one period $t_E \in [0, 1/T]$.

(*a*) *Temperature*

First let us calculate the Hawking temperature of our black hole. We will do this by removing a conical singularity in the induced metric on the horizon[6]. Expanding the function $r^2 f^2(r)$ about the horizon, $r \approx r_H$, we get

$$\begin{aligned} r^2 f^2(r) &\approx r_H^2 f^2(r_H) + \frac{d}{dr}\left(r^2 f^2(r)\right)|_{r=r_H}(r - r_H) \\ &= 2\kappa\,(r - r_H)\,, \end{aligned} \tag{2.80}$$

where $2\kappa = \frac{d}{dr}(r^2 f^2(r))|_{r=r_H}$. The factor 2 has been inserted for later convenence. The first term vanishes since $f(r_H) = 0$ by definition of the horizon. The metric near the horizon takes the form

$$ds^2 \approx -2\kappa(r - r_H)dt^2 + \frac{dr^2}{2\kappa(r - r_H)} + r_H^2 dS_3^2. \tag{2.81}$$

If we perform the coordinate transformation

$$\begin{aligned} &y^2 = r - r_H \implies dr^2 = 4y^2 dy^2 \\ &t = it_E \implies dt^2 = -dt_E^2, \end{aligned} \tag{2.82}$$

---

[6]In general, we should not have spacetime singularities on a black hole horizon.

then the near-horizon metric becomes

$$\begin{aligned} ds^2 &\approx 2\kappa y^2 dt_E^2 + \frac{2}{\kappa} dy^2 + r_H dS_3^2 \\ &= \frac{2}{\kappa}\left[dy^2 + y^2 d\left(\kappa t_E\right)^2\right] + r_H dS_3^2. \end{aligned} \tag{2.83}$$

We may identify the first term as being the metric of a cone in polar coordinates (i.e. y being the radial coordinate and κt_E being the angular coordinate). Note that there is a conical singularity unless we impose that κt_E is periodic as $\kappa t_E = \kappa t_E + 2\pi$, in which case we get the plane $\mathbb{R}^2$ in polar coordinates. Since spacetime is regular (i.e. not singular) at a black hole horizon, we must have

$$t_E = t_E + 2\pi/\kappa. \tag{2.84}$$

According to the AdS/CFT correspondence, the AdS black hole and the dual field theory have a common temperature, and the bulk spacetime and bounary share the same time coordinate. Thus we identify the period of the imaginary time t_E with the inverse temperature. So,

$$T = \frac{\kappa}{2\pi}. \tag{2.85}$$

In the language of GR, we call κ the *surface gravity* [36]. In our language, it is just given by the leading coefficient in the series expansion of the full metric (2.80) about the horizon. We find that

$$\begin{aligned} \kappa &= \frac{1}{2}\frac{d}{dr}\left(r^2 f^2(r)\right)|_{r=r_H} \\ &= \frac{r_H^4 + r_0^4}{r_H^3} \\ &= \frac{2r_H^4 + r_H^2}{r_H^3}, \end{aligned} \tag{2.86}$$

Figure 2.1: Plot of AdS_5 black hole temperature versus r_0.

so the temperature of the black hole is

$$T = \frac{2r_H^4 + r_H^2}{2\pi r_H^3}. \tag{2.87}$$

Figure 2.1 shows a plot of the temperature T versus r_0. There is a minimum, which occurs at $r_0 = (3/4)^{1/4} \approx 1$ and $T_{min} = \sqrt{2}/\pi$. It is apparent that for a given temperature $T >> T_{min}$ there are two possible black holes; i.e., one for $r_0 >> 1$ and one for $r_0 << 1$ (both regimes correspond to large temperatures). Later, we will find out that only the $r_0 >> 1$ black hole is physical. In each of these two regimes, we find that the temperature is given by

$$T \approx \frac{r_0}{\pi}, \text{ if } r_0 >> 1$$
$$T \approx \frac{1}{2\pi r_0^2}, \text{ if } r_0 << 1. \tag{2.88}$$

Inverting the formulas (2.87) and (2.88) we have

$$r_0 = \frac{1}{\sqrt{2}}\left(2T^4\pi^4 \pm 2T^3\pi^3\sqrt{T^2\pi^2 - 2} - 2T^2\pi^2 - 1\right)^{\frac{1}{4}} \quad \text{(exact)}$$
$$r_0 \approx \pi T, \text{ if } r_0 >> 1 \tag{2.89}$$
$$r_0 \approx \sqrt{\frac{1}{2\pi T}}, \text{ if } r_0 << 1,$$

where for the equality we must choose the real, positive roots. The ”plus” root corresponds to the right branch of the temperature curve in figure 2.1 (i.e. for $r_0 > (3/4)^{1/4}$), and the ”minus” root corresponds to the left branch (i.e. $r_0 < (3/4)^{1/4}$).

(*b*) *Entropy*

The Hawking formula for the entropy, s, of a black hole is

$$s = \frac{A}{4G}, \tag{2.90}$$

where A is the proper area of the horizon. To find the area of the horizon, we set $r = r_H$ and $t = constant$ in (2.129) to get the induced metric on the horizon

$$ds^2 = r_H^2 dS_3^2, \tag{2.91}$$

then we integrate the square root of the determinant $\sqrt{\sigma} = r_H^3 \sin^2\psi \sin\theta$, over all angles (i.e. integrate the area measure) to get

$$\begin{aligned} A &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} r_H^3 \sin^2\psi \sin\theta d\psi d\theta d\phi \\ &= 2\pi^2 r_H^3. \end{aligned} \tag{2.92}$$

So

$$\begin{aligned} s &= \frac{\pi^2 r_H^3}{2G} \qquad \text{(exact)} \\ &= \begin{cases} \frac{\pi^2 r_0^3}{2G}, & \text{if} \quad r_0 >> 1 \\ \frac{\pi^2 r_0^6}{2G}, & \text{if} \quad r_0 << 1, \end{cases} \end{aligned} \tag{2.93}$$

or using (2.89) to get this in terms of temperature, we get

$$\begin{aligned} s &= \frac{\pi}{4G}\left(-1 + T\pi\sqrt{2\left(T^2\pi^2 \pm T\pi\sqrt{T^2\pi^2 - 2} - 1\right)}\right) \quad \text{(exact)} \\ s &= \frac{\pi^5}{2G}T^3, \text{ if } r_0 >> 1 \\ s &= \frac{1}{16\pi G T^3}, \text{ if } r_0 << 1. \end{aligned} \tag{2.94}$$

Here we begin to see that the $r_0 << 1$ black hole is non-physical because its entropy *decreases* with temperature, which goes against our intuition about thermal systems. However, we will not rule out this black hole yet.

(*c***)** *Mass / Energy*

There is a well-known analogy that can be drawn between the laws of black hole mechanics and the laws of thermodynamics. For the first law, if U is the internal energy, we have

$$dU = Tds \iff dM = \frac{\kappa}{2\pi} d\left(\frac{A}{4G}\right), \tag{2.95}$$

from which the formulas $T = \kappa/2\pi$ and $s = A/4G$ that we have already used are apparent. This analogy also tells us that the black hole mass M plays the role of U, and from now on we might as well regard them as equivalent. Now, using the first law we have

$$
\begin{aligned}
dM &= T ds = T\frac{ds}{dT}dT \\
&= \begin{cases} \frac{\sqrt{2}T\pi^2}{4G}\left(\frac{2\sqrt{T^2\pi^2-2}T^2\pi^2 \pm 2T^3\pi^3 \mp 3T\pi - \sqrt{T^2\pi^2-2}}{\sqrt{\left(T^2\pi^2 \pm T\pi\sqrt{T^2\pi^2-2}-1\right)\left(T^2\pi^2-2\right)}}\right)dT & \text{(exact)} \\ \frac{3\pi^5}{2G}T^3 dT, & \text{if} \quad r_0 >> 1 \\ -\frac{3}{16\pi G T^3}dT, & \text{if} \quad r_0 << 1, \end{cases}
\end{aligned} \tag{2.96}
$$

Integrating gives

$$
\begin{aligned}
M &= \frac{3}{8G}\pi^5 T^4, \text{ if } r_0 >> 1 \\
M &= \frac{3}{32\pi G T^2} + const, \text{ if } r_0 << 1,
\end{aligned} \tag{2.97}
$$

where we have neglected the integration constant for the $r_0 >> 1$ case because it is irrelevant for large temperatures. We refrain from calculating the exact mass because the integration is intractable.

(d**)** *Heat capacity*

From classical thermodynamics, the heat capacity of a system is given by

$$
c_V = T\left(\frac{ds}{dT}\right)_V . \tag{2.98}
$$

From equation (2.96) we can just pick off the result

$$\begin{aligned} c_V &= \frac{\sqrt{2}T\pi^2}{4G}\left(\frac{2\sqrt{T^2\pi^2-2}T^2\pi^2 \pm 2T^3\pi^3 \mp 3T\pi - \sqrt{T^2\pi^2-2}}{\sqrt{\left(T^2\pi^2 \pm T\pi\sqrt{T^2\pi^2-2}-1\right)(T^2\pi^2-2)}}\right) \quad \text{(exact)} \\ c_V &= \frac{3\pi^5}{2G}T^3, \text{ if } r_0 >> 1 \\ c_V &= -\frac{3}{16\pi G T^3}, \text{ if } r_0 << 1. \end{aligned} \tag{2.99}$$

Now the instability of the $r_0 << 1$ black hole is clear; its heat capacity is negative. This is not surprising from a GR perspective. Consider, for example, the Schwarzschild black hole in flat spacetime. The mass is well-known to be $M = 1/8\pi T$, giving a negative heat capacity $c_V = \partial M/\partial T = -1/8\pi T^2$. For the $r_0 << 1$ case (i.e. the horizon radius is tiny compared to the characteristic AdS curvature, which we set to unity), the black hole is so small that it does not feel the AdS curvature and is effectively living in flat space. So the situation is similar to the standard Schwarzschild case. But our AdS spacetimes should be dual to a well-defined field theory. Physically, the heat capacity of a system is the amount of heat required to raise the temperature of the system by a certain amount. If the heat capacity is negative, then this means that the system can increase its temperature by *losing* heat. Thus, a $r_0 << 1$ black hole in a heat sink will spontaneously heat up forever, which is an unphysical scenario. So the $r_0 < 1$ branch of solutions should be ignored as they do not correspond to a well-defined dual CFT.

(*e***)** *Free Energy*

The free energy of a system, as defined in classical thermodynamics, is given by

$$F = U - sT. \tag{2.100}$$

Regarding $U = M$ and plugging in the expressions we have found for M and s we get

$$\begin{aligned} F &= -\frac{\pi^5}{8G}T^4, \text{ if } r_0 >> 1 \\ F &= \frac{1}{32\pi G T^2}, \text{ if } r_0 << 1. \end{aligned} \tag{2.101}$$

Finally we see without any doubt that the $r_0 << 1$ black hole is unstable, as it is the $r_0 >> 1$ black hole that corresponds to the minimum of the free energy.

Equation (2.84) tells us how we should have treated the integration over time in the calculation of the action of equation (2.58); we should have transormed to imaginary time, i.e. let $t = it_E$, and taken the integral over one period $(0, 1/T)$. Under this change of time coordinate integration measure becomes

$$\int d^4x \int_0^\tau dt\,\sqrt{-g} \rightarrow \int d^4x \int_0^{\frac{1}{T}} (idt_E)\,(i\,\sqrt{g_E})\,, \tag{2.102}$$

where $g_E = r^6 \sin^4\psi \sin^2\theta$. Thus, after the integration this amounts to the replacement

$$\tau \rightarrow -\frac{1}{T}, \tag{2.103}$$

and the Euclidean action (2.58) becomes

$$S_E = -\frac{1}{T}\frac{\pi}{4G}\left(r_H^4 - \frac{r_0^4}{2}\right) \tag{2.104}$$

For a theory of a field ϕ (here the field in question is the metric $g_{\mu\nu}$) described by an action S, the path integral is given by

$$Z = \int d[\phi] \exp(iS[\phi]), \tag{2.105}$$

where this is understood to be a functional integral. This integral is dominated by the field ϕ that minimizes S; that is, the dominant contribution is that for ϕ that is a solution to the equations of motion. Contributions from fields away from the solution cancel because e^{iS} oscillates rapidly when S is large. So we can write

$$\begin{aligned} Z &\approx \exp(iS) \\ &= \exp(-S_E) \end{aligned} \tag{2.106}$$

where S is understood to be evaluated for a solution to the equations of motion. Comparing this to the partition function for a statistical thermal system

$$Z = \exp(-\beta F), \tag{2.107}$$

where $\beta = 1/T$, we see that the free energy is given in terms of the Euclidean action by

$$\frac{F}{T} = S_E. \tag{2.108}$$

Writing (2.104) in terms of T, for $r_0 >> 1$,

$$S_E = -\frac{\pi^5}{8G} T^3, \tag{2.109}$$

and comparing this with (2.101), we see that, indeed, (2.108) is satisfied.

Equation (2.108) gives us an exact answer for the free energy and energy,

$$F = TS_E = -\frac{\pi}{4G}\left(r_H^4 - \frac{r_0^4}{2}\right), \tag{2.110}$$

and we can get U from

$$U = F + sT \tag{2.111}$$

(*f*) *Stress-energy tensor*

The stress-energy tensor for AdS gravity is given by [3]

$$T^{\mu\nu} = \frac{1}{\sqrt{-\gamma}}\frac{\delta S}{\delta \gamma_{\mu\nu}}. \tag{2.112}$$

For the action

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^5x\,\sqrt{-g}\,(R_5 + \Lambda) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^4x\,\sqrt{-\gamma}\Theta + \frac{1}{8\pi G}S_{ct}, \tag{2.113}$$

we find that

$$T^{\mu\nu} = \frac{1}{8\pi G}\left(\Theta^{\mu\nu} - \Theta\gamma^{\mu\nu} + \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma_{\mu\nu}}\right). \tag{2.114}$$

For a counterterm of the form

$$S_{ct} = \int_{\partial\mathcal{M}} d^4x\,\sqrt{-\gamma}\,(\alpha_1 + \alpha_2 R_4)\,, \tag{2.115}$$

we can use the well-known results that

$$\delta\sqrt{-\gamma} = \frac{1}{2}\sqrt{-\gamma}\gamma^{\mu\nu}\delta\gamma_{\mu\nu}, \qquad \text{and} \qquad \delta\left(\sqrt{-\gamma}R_4\right) = -\sqrt{-\gamma}G^{\mu\nu}\delta\gamma_{\mu\nu}, \tag{2.116}$$

where $G^{\mu\nu}$ is the Einstein tensor for the boundary metric $\gamma_{\mu\nu}$, we find that

$$\frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta \gamma_{\mu\nu}} = \alpha_1 \gamma^{\mu\nu} - 2\alpha_2 G^{\mu\nu}. \tag{2.117}$$

Finally, we have

$$T^{\mu\nu} = \frac{1}{8\pi G}\left(\Theta^{\mu\nu} - \Theta\gamma^{\mu\nu} + \alpha_1 \gamma^{\mu\nu} - 2\alpha_2 G^{\mu\nu}\right). \tag{2.118}$$

Using the metric (2.35) and $\alpha_1 = -3$, $\alpha_2 = -1/4$, we can get the stress-energy tensor explicitly. Taking the boundary to infinity, i.e. $\rho \to \infty$, and $r_0 >> 1$, we find that to leading order

$$T_{\mu\nu} = \frac{1}{8\pi G}\begin{pmatrix} \frac{3r_0^4}{2\rho^2} & 0 & 0 & 0 \\ 0 & \frac{r_0^4}{2\rho^2} & 0 & 0 \\ 0 & 0 & \frac{r_0^4}{2\rho^2} & 0 \\ 0 & 0 & 0 & \frac{r_0^4}{2\rho^2} \end{pmatrix} \tag{2.119}$$

In conformal field theories, the one-point function of the stress-energy tensor is given by

$$\left\langle \hat{T}_{\mu\nu} \right\rangle = \frac{2}{\sqrt{-\eta}}\frac{\delta S}{\delta \eta^{\mu\nu}}, \tag{2.120}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. Our boundary metric is related to the Minkowski metric by $\gamma^{\mu\nu} = c_\mu^{-2}\eta^{\mu\nu}$, where no summation is implied here. We can write

$$\left\langle \hat{T}_{\mu\nu} \right\rangle = \frac{2c_1 c_2^3}{\sqrt{-\gamma}}\frac{\delta S}{\delta \gamma^{\mu\nu}}\frac{\delta \gamma^{\mu\nu}}{\delta \eta^{\mu\nu}}. \tag{2.121}$$

Since $\frac{\delta\gamma^{\mu\nu}}{\delta\eta^{\mu\nu}} = c_\mu^{-2}$, we end up with

$$\left\langle \hat{T}_{\mu\nu} \right\rangle = c_1 c_2^3 c_\mu^{-2} T_{\mu\nu} \tag{2.122}$$

As $\rho \to \infty$, $c_1 c_2^3 c_\mu^{-2} \to \rho^2$, so we get a finite stress-energy tensor for the field theory

$$\left\langle \hat{T}_{\mu\nu} \right\rangle = \frac{1}{8\pi G} \begin{pmatrix} \frac{3r_0^4}{2} & 0 & 0 & 0 \\ 0 & \frac{r_0^4}{2} & 0 & 0 \\ 0 & 0 & \frac{r_0^4}{2} & 0 \\ 0 & 0 & 0 & \frac{r_0^4}{2} \end{pmatrix}, \tag{2.123}$$

from which we identify the mass/energy density $\mathcal{E} = M/V$ and pressure or free energy density $\mathcal{F} = F/V$

$$\mathcal{E} = \frac{3r_0^4}{16\pi G}, \qquad \text{and} \qquad P = -\mathcal{F} = \frac{r_0^4}{16\pi G}, \tag{2.124}$$

giving exact agreement with our previous results (2.97) and (2.101).

Speed of sound

In a thermal system, the speed of sound is given by

$$c_s^2 = \left(\frac{\partial P}{\partial E} \right) \tag{2.125}$$

For our AdS$_5$ black hole we get

$$c_s^2 = \frac{\left(\frac{r_0^4}{16\pi G} \right)}{\left(\frac{3r_0^4}{16\pi G} \right)} = \frac{1}{3}, \tag{2.126}$$

in agreement with the first equation in (1.41) for a CFT in $d = 3 + 1$ dimensions.

Also note that

$$c_s^2 = \frac{\left(\frac{\partial P}{\partial T} \right)}{\left(\frac{\partial \mathcal{E}}{\partial T} \right)} = \frac{s}{c_V}. \tag{2.127}$$

In (2.94) and (2.99) we found that

$$s = \frac{\pi^5}{2G}T^3, \quad \text{and} \quad c_V = \frac{3\pi^5}{2G}T^3. \tag{2.128}$$

So (2.127) is satisfied and our results are consistent.

2.7.2 Planar black hole

Following the same procedure as for the 3-sphere black hole, but with the metric

$$ds^2 = -r^2 f^2(r)dt^2 + \frac{dr^2}{r^2 f^2(r)} + r^2 d\mathbf{x}^2, \tag{2.129}$$

where

$$f^2(r) = 1 - \frac{r_0^4}{r^4}, \tag{2.130}$$

which has a horizon is at

$$r_H = r_0, \tag{2.131}$$

we find the following results:

Given r_0, there is a single (stable) black hole with temperature

$$T = \frac{r_0}{\pi}. \tag{2.132}$$

The entropy density s/V, mass/energy density M/V, heat capacity per volume c_V/V, free energy density F/V, and speed of sound squared c_s^2 are given by

$$s \equiv \frac{S}{V} = \frac{\pi^3 T^3}{4G}, \qquad \mathcal{E} \equiv \frac{M}{V} = \frac{3\pi^3}{16G} T^4, \qquad c_V \equiv \frac{C_V}{V} = \frac{3\pi^3}{4G} T^3$$
$$\mathcal{F} \equiv \frac{F}{V} = -\frac{\pi^3}{16G} T^4, \qquad c_s^2 = \frac{1}{3}. \tag{2.133}$$

The stress-energy tensor is given by (2.123). All of these expressions are exact.

The thermodynamics of the planar black hole are identical to those of the 3-sphere black hole in the limit $r_0 >> 1$. First, we've already established that the temperatures are the same. Furthermore, for the planar black hole we simply set $V = \int d^3x$, but for the 3-sphere black hole we were able to explicity carry out the corresponding integral

$$\int_{S_3} dS_3 = 2\pi^2. \tag{2.134}$$

If we define $V = 2\pi^2$ (i.e. the volume of the unit 3-sphere), then it is easy to check that the thermodynamics of the planar black hole are identical to those of the 3-sphere black hole with $r_0 >> 1$.

2.7.3 The first law

As an independent check that our thermodynamics make sense, let us verify that our thermodynamics are consistent with the first law

$$dF = -s dT. \tag{2.135}$$

For the planar black hole, this is easy to check using the formulas in (2.133)

$$\frac{d\mathcal{F}}{dT} = -\frac{\pi^3}{4G} T^3 = -s. \tag{2.136}$$

Having established that $S_E = F/T$, we can now find the exact free energy and mass for the 3-sphere black hole. From equation (2.104) we find

$$F = TS_E = -\frac{\pi}{8G}\left(r_H^4 - r_H^2\right). \tag{2.137}$$

Also, recall that

$$T = \frac{2r_H^4 + r_H^2}{2\pi r_H^3}, \qquad s = \frac{\pi^2 r_H^3}{2G}. \tag{2.138}$$

Now it can be easily shown that

$$\frac{dF}{dr_H} = -\frac{\pi}{4G}\left(2r_H^3 - r_H\right) \tag{2.139}$$

and that

$$-s\frac{dT}{dr_H} = -\frac{\pi}{4G}\left(2r_H^3 - r_H\right). \tag{2.140}$$

So $\frac{dF}{dr_H} = -s\frac{dT}{dr_H}$, and the first law holds.

Chapter 3

AdS_4

In this chapter we repeat the analysis of the previous chapter, but in four spacetime dimensions. The results of this chapter will be important because they will be the expected results for our Exotic Model when we take the conformal limit. Since the calculations are similar to those in the last chapter, we will spare most of the details.

3.1 Solution for the metric

Setting $d = 3$ in (2.1), and using the same notations as before, the action for AdS_4 is

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\,(R_4 + 6) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\,\sqrt{-\gamma}\Theta. \tag{3.1}$$

We are interested in black hole solutions with planar horizons, thus we choose an ansatz for the metric to be

$$ds^2 = -c_1^2(r)dt^2 + c_2^2(r)d\mathbf{x}^2 + c_3^2(r)dr^2, \tag{3.2}$$

where $d\mathbf{x}^2 = dx_1^2 + dx_2^2$. Using this ansatz, the action is given by $S = S_{bulk} + S_{boundary}$ where

$$S_{bulk} = \frac{1}{8\pi G}\int_{\mathcal{M}} d^4x \left(\begin{array}{c} -2\frac{c_1' c_2' c_2}{c_3} + \frac{c_1' c_3' c_2^2}{c_3^2} - \frac{c_1'' c_2^2}{c_3} - \frac{c_1 (c_2)^2}{c_3} \\ +2\frac{c_2' c_3' c_1 c_2}{c_3^2} - 2\frac{c_2'' c_1 c_2}{c_3} + 3c_1 c_2^2 c_3 \end{array} \right), \tag{3.3}$$

$$S_{boundary} = -\int_{\partial\mathcal{M}} d^3x \left(\frac{c_1' c_2^2}{c_3} + 2\frac{c_2' c_1 c_2}{c_3} \right). \tag{3.4}$$

If we vary the action with respect to c_1, c_2 and c_3 respectively, we get the equations of motion

$$c_1'' + \frac{c_1' c_2'}{c_2} - \frac{c_1' c_3'}{c_3} - \frac{1}{2}\frac{c_1 (c_2')^2}{c_2^2} - \frac{3}{2} c_1 c_3^2 = 0 \tag{3.5}$$

$$c_2'' + \frac{1}{2}\frac{(c_2')^2}{c_2} - \frac{c_2' c_3'}{c_3} - \frac{3}{2} c_2 c_3^2 = 0 \tag{3.6}$$

$$2c_1' c_2' c_2 + (c_2')^2 c_1 - 3c_1 c_2^2 c_3^2 = 0 \tag{3.7}$$

Seeking solutions of the form $c_1(r) = rf(r)$, $c_2(r) = r$, $c_3(r) = 1/rf(r)$, the equations of motion become

$$2r^2 f f'' + 2r^2 (f')^2 + 10 r f f' + 3f^2 = 0 \tag{3.8}$$

$$2r f f' + 3f^2 - 3 = 0 \tag{3.9}$$

Finally we get the solution

$$ds^2 = -r^2\left(1 - \frac{r_0^3}{r^3}\right) dt^2 + r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2\left(1 - \frac{r_0^3}{r^3}\right)}, \tag{3.10}$$

which has a horizon at $r_H = r_0$.

3.2 Thermodynamics

First we will calculate the temperature. Expanding the metric (3.10) about the horizon $r = r_0$ we get

$$ds^2 = -2\kappa(r - r_0)dt^2 + r_0^2 d\mathbf{x}^2 + \frac{dr^2}{2\kappa(r - r_0)}, \tag{3.11}$$

where

$$2\kappa = \frac{d}{dr}\left(c_1^2\right)\Big|_{r=r_0} = 3r_0. \tag{3.12}$$

Applying the transformations given in section 2.7.1 (a) we get the temperature

$$T = \frac{\kappa}{2\pi} = \frac{3r_0}{4\pi}. \tag{3.13}$$

To calculate the entropy, we use the Hawking formula $s = A/4G$, where A is the area of the horizon. Since our horizon is an infinite plane, the entropy is infinite; however, the entropy density is finite. The metric on the horizon is

$$ds_H^2 = r_0^2(dx_1^2 + dx_2^2). \tag{3.14}$$

The area element on the horizon is given by $r_0^2 dxdy$, so that the entropy is given by

$$s = \frac{r_0^2 \iint dx_1 dx_2}{4G}. \tag{3.15}$$

If we define $V = \iint dx_1 dx_2$, then the entropy density is

$$\mathfrak{s} = \frac{s}{V} = \frac{r_0^2}{4G} = \frac{4\pi^2}{9G}T^2, \tag{3.16}$$

where we used (3.13) in the last equality. Recall from (1.35) that black hole entropy and central charge of the dual field theory are related by

$$s = c\frac{\Gamma\left(\frac{d}{2}\right)^3}{4\pi^{\frac{d}{2}}\Gamma(d)}\left(\frac{4\pi}{d}\right)^d\left(\frac{d-1}{d+1}\right)T^{d-1}. \tag{3.17}$$

We find that the central charge is

$$c = \frac{24}{\pi G} = \frac{192}{\kappa^2}, \tag{3.18}$$

where $\kappa^2 = 8\pi G$ is commonly used in place of the gravitational constant G.

To find the free energy we must compute the finite Euclidean action. Putting the solution (3.10) into the action (3.1) and taking the boundary $\partial\mathcal{M}$ to be the surface $r = \rho$, we get

$$\frac{S}{V} = \frac{\tau}{8\pi G}\left(2\rho^3 - \frac{r_0^3}{2}\right), \tag{3.19}$$

where $\tau = \int_0^\tau dt$. The action diverges as $\rho \to \infty$. To remove this divergence we may add a counterterm to the action as follows

$$\frac{S}{V} = \frac{\tau}{8\pi G}\left(2\rho^3 - \frac{r_0^3}{2}\right) + \frac{\alpha}{8V\pi G}\int_{\partial\mathcal{M}} d^3x\,\sqrt{-\gamma}. \tag{3.20}$$

The addition of the second term on the right-hand side does not change the equations of motion because it does not contain any derivatives of the metric components and we assume that the variations vanish on the boundary. Putting our solution into the counterterm and expanding it for large ρ, the action density becomes

$$\frac{S}{V} = \frac{\tau}{8\pi G}\left[\left(2\rho^3 - \frac{r_0^3}{2}\right) + \alpha\left(\rho^3 - \frac{1}{2}r_0^3\right)\right]. \tag{3.21}$$

If we choose $\alpha = -2$ we get a finite action density,

$$\frac{S}{V} = \frac{\tau}{16\pi G} r_0^3. \tag{3.22}$$

Changing to imaginary time we make the replacement $\tau = -1/T$. The Euclidean action density is

$$\frac{S_E}{V} = -\frac{1}{16\pi G}\frac{r_0^3}{T}. \tag{3.23}$$

Then the free energy is $F = TS_E$, or

$$\mathcal{F} = \frac{F}{V} = -\frac{4\pi^2}{27G} T^3. \tag{3.24}$$

The best way to find the mass density and pressure is to compute the stress-energy tensor. Using the formulas in section 2.7.1 (*f*), but with $\sqrt{-\gamma} = c_1 c_2^2$ we find

$$\left\langle \hat{T}_{\mu\nu} \right\rangle = \begin{pmatrix} \frac{8\pi^2}{27G} T^3 & 0 & 0 \\ 0 & \frac{4\pi^2}{27G} T^3 & 0 \\ 0 & 0 & \frac{4\pi^2}{27G} T^3 \end{pmatrix}, \tag{3.25}$$

giving the mass/energy density and pressure respectively as

$$\mathcal{E} = \frac{8\pi^2}{27G} T^3, \qquad \text{and} \qquad P = \frac{4\pi^2}{27G} T^3. \tag{3.26}$$

It is straight forward to check our thermodynamics by verifying the first law

$$\mathcal{F} = \mathcal{E} - sT. \tag{3.27}$$

We can now calculate the speed of sound c_s using the thermodynamic relation

$$c_s^2 = -\frac{\partial F}{\partial E} = -\frac{\left(\frac{\partial F}{\partial T}\right)}{\left(\frac{\partial E}{\partial T}\right)} = \frac{1}{2}, \tag{3.28}$$

as expected for a CFT in $d = 2 + 1$ dimensions.

Chapter 4

AdS$_4$ fluctuations

In this chapter we will study small fluctuations of the background metric $g^{BG}_{\mu\nu}$ given by (3.10). The results of this chapter will serve as a check of the results of our Exotic Model when we look at fluctuations in the conformal limit.

4.1 Fluctuation equations

Consider fluctuations of the background metric of the form

$$g_{\mu\nu} = g^{BG}_{\mu\nu} + a h_{\mu\nu}, \tag{4.1}$$

where $g^{BG}_{\mu\nu}$ is given by (3.10), and $h_{\mu\nu}$ are regarded to be small fluctuations about the background. We respect the symmetry of the metric, so $h_{\mu\nu} = h_{\nu\mu}$, and we can orient the coordinate system such that the x_2 axis is directed along the momentum vector of the fluctuations so that $h_{\mu\nu} = h_{\mu\nu}(t, x_2, r)$. If we vary the bulk action

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} \,(R + 6)\,, \tag{4.2}$$

where R is the scalar curvature for the full metric $g_{\mu\nu}$, with respect to the ten independent metric fluctuation components $h_{\mu\nu}$, and expand the result in powers of a, then one finds that the coefficient of the linear term vanishes by virtue of the background equations of motion. The coefficient of the quadratic term then gives the linearized equations of motion governing the fluctuations. After performing the variations we may use diffeomorphism invariance to choose $h_{tr} = h_{x_i r} = h_{rr} = 0$. Also let $h_{ii} = h_{x_1x_1} + h_{x_2x_2}$. The result is the following system of partial differential equations,

$$\frac{\partial^2 h_{ii}}{\partial r^2} + \frac{1}{r\left(r^3 - r_0^3\right)}\left(\frac{\partial^2 h_{x_1x_1}}{\partial x_2^2} + \frac{3}{2}r_0^3\frac{\partial h_{ii}}{\partial r} - \frac{2r^3 + r_0^3}{r}h_{ii}\right) = 0, \tag{4.3}$$

$$\frac{\partial^2 h_{tx_1}}{\partial r^2} + \frac{1}{r\left(r^3 - r_0^3\right)}\left(\frac{\partial^2 h_{tx_1}}{\partial x_2^2} - \frac{\partial^2 h_{x_1x_2}}{\partial t\partial x_2}\right) - \frac{2}{r^2}h_{tx_1} = 0, \tag{4.4}$$

$$\frac{\partial^2 h_{tx_2}}{\partial r^2} + \frac{1}{r\left(r^3 - r_0^3\right)}\frac{\partial^2 h_{x_1x_1}}{\partial t\partial x_2} - \frac{2}{r^2}h_{tx_2} = 0, \tag{4.5}$$

$$\frac{\partial^2 h_{ii}}{\partial t\partial r} - \frac{\partial^2 h_{tx_2}}{\partial x_2\partial r} + \frac{1}{r\left(r^3 - r_0^3\right)}\left(\left(2r^3 + r_0^3\right)\frac{\partial h_{tx_2}}{\partial x_2} - \frac{1}{2}\left(4r^3 - r_0^3\right)\frac{\partial h_{ii}}{\partial t}\right) = 0, \tag{4.6}$$

$$\begin{aligned}&\frac{\partial^2 h_{x_2x_2}}{\partial t^2} + \frac{\partial^2 h_{tt}}{\partial x_2^2} - 2\frac{\partial^2 h_{tx_2}}{\partial t\partial x_2} + r\left(r^3 - r_0^3\right)\frac{\partial^2 h_{tt}}{\partial r^2} - \left(\frac{r^3 - r_0^3}{r}\right)^2\frac{\partial^2 h_{x_2x_2}}{\partial r^2} - \frac{3}{2}r_0^3\frac{\partial h_{tt}}{\partial r}\\ &- 3\frac{r_0^3}{r^3}\left(r^3 - r_0^3\right)\frac{\partial h_{x_2x_2}}{\partial r} - \frac{1}{2r}\left(\frac{4r^6 - 14r^3r_0^3 + r_0^6}{r^3 - r_0^3}\right)h_{tt} + \frac{2}{r^4}\left(r^3 - r_0^3\right)\left(r^3 + 2r_0^3\right)h_{x_2x_2} = 0,\end{aligned} \tag{4.7}$$

$$\begin{aligned}&\frac{\partial^2 h_{x_1x_2}}{\partial t^2} - \frac{\partial^2 h_{tx_1}}{\partial t\partial x_2} - \left(\frac{r^3 - r_0^3}{r}\right)^2\frac{\partial^2 h_{x_1x_2}}{\partial r^2} - 3\frac{r_0^3}{r^3}\left(r^3 - r_0^3\right)\frac{\partial h_{x_1x_2}}{\partial r}\\ &\qquad + \frac{2}{r^4}\left(r^3 - r_0^3\right)\left(r^3 + 2r_0^3\right)h_{x_1x_2} = 0,\end{aligned} \tag{4.8}$$

$$\frac{\partial^2 h_{x_1x_2}}{\partial x_2\partial r} - \frac{r^2}{r^3 - r_0^3}\left(r\frac{\partial^2 h_{tx_1}}{\partial t\partial r} - 2\frac{\partial h_{tx_1}}{\partial t}\right) - \frac{2}{r}\frac{\partial h_{x_1x_2}}{\partial x_2} = 0, \tag{4.9}$$

$$\frac{\partial^2 h_{x_1x_1}}{\partial t^2} + r\left(r^3 - r_0^3\right)\frac{\partial^2 h_{tt}}{\partial r^2} - \left(\frac{r^3 - r_0^3}{r}\right)^2 \frac{\partial^2 h_{x_1x_1}}{\partial r^2} - 3\frac{r_0^3}{r^3}\left(r^3 - r_0^3\right)\frac{\partial h_{x_1x_1}}{\partial r} + \frac{2}{r^4}\left(r^3 - r_0^3\right)\left(r^3 + 2r_0^3\right)h_{x_1x_1} - \frac{3}{2}r_0^3\frac{\partial h_{tt}}{\partial r} - \frac{1}{2r}\left(\frac{4r^6 - 14r^3r_0^3 + r_0^6}{r^3 - r_0^3}\right)h_{tt} = 0, \tag{4.10}$$

$$\frac{\partial^2 h_{tx_2}}{\partial t \partial r} - \frac{\partial^2 h_{tt}}{\partial x_2 \partial r} + \left(\frac{r^3 - r_0^3}{r^3}\right)\left(\frac{\partial^2 h_{x_1x_1}}{\partial x_2 \partial r} - \frac{2}{r}\frac{\partial h_{x_1x_1}}{\partial x_2}\right) - \frac{2}{r}\frac{\partial h_{tx_2}}{\partial t} + \frac{1}{2r}\left(\frac{4r^3 - r_0^3}{r^3 - r_0^3}\right)\frac{\partial h_{tt}}{\partial x_2} = 0, \tag{4.11}$$

$$\frac{\partial^2 h_{tt}}{\partial x_2^2} + \frac{\partial^2 h_{ii}}{\partial t^2} - 2\frac{\partial^2 h_{tx_2}}{\partial t \partial x_2} - \left(r^3 - r_0^3\right)\left(\frac{1}{r^3}\frac{\partial^2 h_{x_1x_1}}{\partial x_2^2} - 2\frac{\partial h_{tt}}{\partial r} + \frac{1}{2r^3}\left(4r^3 + r_0^3\right)\left(\frac{\partial h_{ii}}{\partial r} + \frac{2}{r}h_{ii}\right)\right) - \frac{2}{r}\left(2r^3 + r_0^3\right)h_{tt} = 0. \tag{4.12}$$

We will be interested in fluctuations that carry a plane-wave profile in the x_2 direction. Letting[1]

$$\begin{aligned} h_{tt}(t, x_2, r) &= e^{-i\omega t + iqx_2} r^2 \left(1 - \frac{r_0^3}{r^3}\right) H_{tt}(r), \\ h_{tx_i}(t, x_2, r) &= e^{-i\omega t + iqx_2} r^2 H_{tx_i}(r), \\ h_{x_ix_i}(t, x_2, r) &= e^{-i\omega t + iqx_2} r^2 H_{x_ix_i}(r), \qquad i = 1, 2, \end{aligned} \tag{4.13}$$

where ω is the frequency and q is momentum, we get a system of ordinary differential equations for the components H_{tt}, $H_{x_1x_1}$, $H_{x_2x_2}$, and H_{tx_2},

$$H_{ii}'' + \frac{1}{2r}\left(\frac{8r^3 - 5r_0^3}{r^3 - r_0^3}\right)H_{ii}' - \frac{q^2}{r\left(r^3 - r_0^3\right)}H_{x_1x_1} = 0, \tag{4.14}$$

$$H_{tx_2}'' + \frac{4}{r}H_{tx_2}' + \frac{\omega q}{r\left(r^3 - r_0^3\right)}H_{x_1x_1} = 0, \tag{4.15}$$

[1] We can choose this form of the fluctuations because spacetime boundary is conformally flat. Thus, for fixed r we expect a plane-wave solution to the fluctuation (wave) equations.

$$H''_{tt}+H''_{x_2x_2}+\frac{1}{r\left(r^3-r_0^3\right)}\left(\left(4r^3-\frac{r_0^3}{2}\right)H'_{tt}-\left(4r^3-r_0^3\right)H'_{x_2x_2}-q^2H_{tt}\right)$$
$$-\omega\left(\frac{r}{r^3-r_0^3}\right)^2\left(\omega H_{x_2x_2}+2qH_{tx_2}\right)=0, \tag{4.16}$$

$$H''_{x_1x_1}-H''_{tt}-\frac{1}{r\left(r^3-r_0^3\right)}\left(\left(4r^3-\frac{r_0^3}{2}\right)H'_{tt}-\left(4r^3-r_0^3\right)H'_{x_1x_1}-q^2H_{tt}\right)$$
$$+\left(\frac{\omega r}{r^3-r_0^3}\right)^2H_{x_1x_1}=0. \tag{4.17}$$

There are three first-order constraints,

$$H'_{ii}+\frac{q}{\omega}H'_{tx_2}-\frac{3r_0^3}{r\left(r^3-r_0^3\right)}\left(\frac{1}{2}H_{ii}+\frac{q}{\omega}H_{tx_2}\right)=0, \tag{4.18}$$

$$H'_{tt}-\frac{r^3-\frac{r_0^3}{4}}{r^3-r_0^4}H'_{ii}-\frac{\omega r^3}{\left(r^3-r_0^3\right)^2}\left(\frac{\omega}{2}H_{x_2x_2}+qH_{tx_2}\right)-\frac{q^2}{2}\left(\frac{1}{r^3-r_0^3}\right)H_{tt}$$
$$+\frac{q^2\left(r^3-r_0^3\right)-\omega^2r^3}{\left(r^2-r_0^2\right)^2}H_{x_1x_1}=0, \tag{4.19}$$

$$H'_{tt}-H'_{x_1x_1}+\frac{1}{r^3-r_0^3}\left(\frac{\omega r^3}{q}H'_{tx_2}+\frac{3r_0^3}{2r}H_{tt}\right)=0. \tag{4.20}$$

We also get a second system of ODEs, decoupled from the first, for the components $H_{x_1x_2}$ and H_{tx_1},

$$H''_{tx_1}+\frac{4}{r}H'_{tx_1}-\frac{q}{r\left(r^3-r_0^3\right)}\left(\omega H_{x_1x_2}+qH_{tx_1}\right)=0, \tag{4.21}$$

$$H''_{x_1x_2}+\frac{4r^3-r_0^3}{r\left(r^3-r_0^3\right)}H'_{x_1x_2}+\frac{\omega r^2}{\left(r^3-r_0^3\right)^2}\left(\omega H_{x_1x_2}+qH_{tx_1}\right)=0, \tag{4.22}$$

and the first order constraint

$$H'_{x_1x_2} + \frac{\omega r^3}{q\left(r^3 - r_0^3\right)} H'_{tx_1} = 0. \tag{4.23}$$

The main goal of the rest of this chapter is to calculate the dispersion relation $\omega = \omega(q)$, from which we can infer the hydrodynamics of the dual CFT.

4.2 Gauge-invariant fluctuations

4.2.1 Metric fluctuations

Let us focus on the first system (4.14)-(4.20). The redundancy of the system (that there are four equations and three constraints, but only 4 unknown functions) implies that our gauge choice $h_{tr} = h_{x_ir} = h_{rr} = 0$ does not completely remove all of the gauge freedom. In other words, our metric $g_{\mu\nu}$ is not invariant under a general coordinate transformation. In principle, we can think that the three constraints eliminate three of our unknown functions so that there is only a single function describing the fluctuations. Of course, since the constraints are first-order in the derivatives, we cannot pursue this direct route. Following [38] our task, then, is to find an infinitesimal transformation of the form

$$x^{\mu'} = x^\mu + \xi^\mu(t, x_2, r) \tag{4.24}$$

that preserves our gauge choice and leaves the metric invariant. Now, $g'_{\mu\nu}(x')$ is the transformed metric evaluated at the transformed point, and $g_{\mu\nu}(x')$ is the original metric evaluated at the transformed point. Their difference is, by definition, the Lie derivative of the original metric along the curve whose tangent is ξ^μ, that is [36]

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x') - \pounds_\xi g_{\mu\nu}(x), \tag{4.25}$$

where the Lie derivative is given by

$$\begin{aligned} \pounds_\xi g_{\mu\nu} &= \xi^\alpha \nabla_\alpha g_{\mu\nu} + g_{\alpha\nu}\nabla_\mu \xi^\alpha + g_{\mu\alpha}\nabla_\nu \xi^\alpha \\ &= \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu, \end{aligned} \tag{4.26}$$

where we used the fact that $\nabla_\alpha g_{\mu\nu} = 0$ always, and we contracted the second and third terms in the first line. So we now have the transformed metric

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x') - \nabla_\nu \xi_\mu(t, x_2, r) - \nabla_\mu \xi_\nu(t, x_2, r). \tag{4.27}$$

In order to obtain such a transformation that preserves our gauge choice $h_{tr} = h_{x_i r} = h_{rr} = 0$, we must satisfy the system of partial differential equations

$$\begin{aligned} &\nabla_i \xi_r + \nabla_r \xi_i = 0 \\ &\nabla_r \xi_r = 0. \end{aligned} \tag{4.28}$$

Let us return to the metric

$$ds^2 = -c_1^2(r)dt^2 + c_2^2(r)dx_1^2 + c_2^2(r)dx_2^2 + c_3^2(r)dr^2 \tag{4.29}$$

so that we may derive a general result that will be useful in later chapters. Explicitly in this metric, the system (4.28) is

$$\begin{aligned}
&\partial_r \xi_0 - 2\frac{c_1'}{c_1}\xi_0 + \partial_t \xi_3 = 0 \\
&\partial_r \xi_1 - 2\frac{c_2'}{c_2}\xi_1 = 0 \\
&\partial_r \xi_2 - 2\frac{c_2'}{c_2}\xi_2 + \partial_{x_2} \xi_3 = 0 \\
&\partial_r \xi_3 - 2\frac{c_3'}{c_3}\xi_3 = 0.
\end{aligned} \tag{4.30}$$

It is straightforward to find the general solution

$$\begin{aligned}
\xi_0 &= c_1^2\left(K_{x_2} - (\partial_t K_r)\int \frac{c_3}{c_1^2}dr\right) \\
\xi_1 &= c_2^2 K_{x_1} \\
\xi_2 &= -c_2^2\left(K_t + (\partial_{x_2} K_r)\int \frac{c_3}{c_2^2}dr\right) \\
\xi_3 &= c_3 K_r,
\end{aligned} \tag{4.31}$$

where $K_\mu = K_\mu(t, x_2)$ are arbitrary functions which we will take to be plane waves; that is, $K_\mu(t, x_2) = e^{-i\omega t + iqx_2}$. We are guaranteed that the above solution for the coordinate transformation (4.24) preserves our gauge choice. From this solution we may construct a set of linearly independent solutions as follows

$$\xi^{\mu}_{(1)} = e^{-i\omega t+iqx_2}\begin{pmatrix} 0 \\ 0 \\ c_2^2 \\ 0 \end{pmatrix}, \quad \xi^{\mu}_{(2)} = e^{-i\omega t+iqx_2}\begin{pmatrix} -c_1^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\xi^{\mu}_{(3)} = e^{-i\omega t+iqx_2}\begin{pmatrix} i\omega c_1^2 \int \frac{c_3}{c_1^2} dr \\ 0 \\ -i\omega c_2^2 \int \frac{c_3}{c_2^2} dr \\ c_3 \end{pmatrix}, \quad \xi^{\mu}_{(4)} = e^{-i\omega t+iqx_2}\begin{pmatrix} 0 \\ c_2^2 \\ 0 \\ 0 \end{pmatrix} \tag{4.32}$$

Now, we wish to find a linear combination of the metric fluctuations $h_{\mu\nu}$ that is invariant under our gauge transformation (4.24). That is, we need to find α, β and δ such that

$$\alpha h'_{tt} + \beta h'_{tx_2} + h'_{x_2x_2} + \delta h'_{x_1x_1} = \alpha h_{tt} + \beta h_{tx_2} + h_{x_2x_2} + \delta h_{x_1x_1}. \tag{4.33}$$

Transforming the fluctuations according to

$$h'_{\mu\nu} = h_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu \tag{4.34}$$

equation (4.33) becomes

$$2\alpha\left(\frac{c_1c_1'}{c_3^2}\xi_3 - \partial_t\xi_0\right) - \beta\left(\partial_{x_2}\xi_0 + \partial_t\xi_2\right) - 2\left(\partial_{x_2}\xi_2 + \frac{c_2c_2'}{c_3^2}\xi_3\right) - 2\delta\frac{c_2c_2'}{c_3^2}\xi_3 = 0. \tag{4.35}$$

Putting the independent solutions $\xi^\mu_{(1)}, \xi^\mu_{(2)}$ and $\xi^\mu_{(3)}$ for ξ_μ into the above equation we get the system

$$\begin{aligned} &\beta\omega - 2q = 0 \\ &\beta q - 2\alpha\omega = 0 \\ &\alpha c_1 c_1' - (1+\delta)\, c_2 c_2' = 0, \end{aligned} \tag{4.36}$$

which has the solution

$$\alpha = \frac{q^2}{\omega^2}, \qquad \beta = 2\frac{q}{\omega}, \qquad \delta = \frac{q^2}{\omega^2}\frac{c_1 c_1'}{c_2 c_2'} - 1. \tag{4.37}$$

Thus, we have found a gauge invariant fluctuation

$$\zeta_H = \frac{q^2}{\omega^2} h_{tt} + 2\frac{q}{\omega} h_{tx_2} + h_{x_2x_2} + \left(\frac{q^2}{\omega^2}\frac{c_1c_1'}{c_2c_2'} - 1\right) h_{x_1x_1}. \tag{4.38}$$

Letting

$$\begin{aligned} h_{tt}(t, x_2, r) &= e^{-i\omega t + iqx_2} r^2 \left(1 - \frac{r_0^3}{r^3}\right) H_{tt}(r), \\ h_{tx_i}(t, x_2, r) &= e^{-i\omega t + iqx_2} r^2 H_{tx_i}(r), \\ h_{x_ix_i}(t, x_2, r) &= e^{-i\omega t + iqx_2} r^2 H_{x_ix_i}(r), \qquad i = 1, 2, \end{aligned} \tag{4.39}$$

we get the gauge invariant fluctuation

$$Z_H = 2\frac{q^2}{\omega^2}\frac{c_1^2}{c_2^2} H_{tt} + 4\frac{q}{\omega} H_{tx_2} + 2H_{x_2x_2} + 2\left(\frac{q^2}{\omega^2}\frac{c_1c_1'}{c_2c_2'} - 1\right) H_{x_1x_1}, \tag{4.40}$$

where $Z_H = 2e^{i\omega t - iqx_2}\zeta_H/c_2^2$. Note from the right-hand side that $Z_H = Z_H(r)$ is a function of r only. With the gauge freedom completely fixed, the function Z_H alone describes the metric fluctuations.

4.2.2 Scalar fluctuations

For our Exotic model we will add scalar fields to the action, and we will need to consider fluctuations of these scalar fields. Consider a field of the form $\phi = \phi(r)$. Consider fluctuating this field as $\phi \to \phi + \alpha$ where α is considered to be a small fluctuation of the background field ϕ. Under an infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu$, the fluctuation transforms as

$$\begin{aligned} \alpha \to \alpha' &= \alpha - \xi^\mu \nabla_\mu \phi \\ &= \alpha - \xi^r \frac{d\phi}{dr}. \end{aligned} \tag{4.41}$$

The fluctuation α will only transform under $\xi^\mu_{(3)}$. Also note that $h_{x_1 x_1}$ only transforms under $\xi^\mu_{(1)}$ as well. All other metric fluctuations transform under either $\xi^\mu_{(1)}$ and/or $\xi^\mu_{(1)}$. So our only hope of finding a diffeomorphism-invariant fluctuation involving the scalar α is to write down an invariant linear combination of the form

$$\alpha' + \beta h'_{x_1 x_1} = \alpha + \beta h_{x_1 x_1}. \tag{4.42}$$

Using the transformation $\xi^\mu_{(3)}$ in the left hand side, this becomes

$$c_3^2 \frac{d\phi}{dr} + 2\beta c_2 \frac{dc_2}{dr} = 0. \tag{4.43}$$

So choosing

$$\beta = -\frac{c_3^2 \phi'}{2c_2' c_2} \tag{4.44}$$

leads to the invariant combination

$$Z_\alpha = \alpha - \frac{c_2 c_3^2 \phi'}{2c_2'} H_{x_1 x_1} \tag{4.45}$$

where the prime in the last two equations denotes differentiation with respect to r.

4.3 Gauge-invariant fluctuation equation

Now we must find that equation which governs the single gauge-invariant fluctuation (4.40). The intermediate steps in what follows are very cumbersome, so they will be formally outlined, but the explicit formulas will not be given until the final result is obtained.

First, we can solve equations (4.14)-(4.17) for the second derivatives,

$$H''_{x_1x_1} = f_1\left(H'_{x_1x_1}, H'_{x_2x_2}, H_{x_1x_1}, H_{x_2x_2}, H_{tt}, H_{tx_2}, r\right), \tag{4.46}$$

$$H''_{x_2x_2} = f_2\left(H'_{x_1x_1}, H'_{x_2x_2}, H_{x_1x_1}, H_{x_2x_2}, H_{tt}, H_{tx_2}, r\right), \tag{4.47}$$

$$H''_{tx_2} = f_3\left(H'_{tx_2}, H_{x_1x_1}, r\right), \tag{4.48}$$

$$H''_{tt} = f_4\left(H'_{tt}, H'_{x_1x_1}, H'_{x_2x_2}, H_{x_1x_1}, H_{x_2x_2}, H_{tt}, H_{tx_2}, r\right). \tag{4.49}$$

If we put the solution (3.10) for the background c_1, c_2 and c_3 into (4.40), then solve the result for H_{tt} we get

$$H_{tt} = \frac{\omega^2}{q^2}\frac{r^3}{\left(r^3 - r_0^3\right)}\left[Z_H - 4\frac{q}{\omega}H_{tx_2} + 2H_{x_1x_1}\left(1 - \frac{q^2}{\omega^2}\left(1 + \frac{r_0^3}{2r^3}\right)\right) - 2H_{x_2x_2}\right]. \tag{4.50}$$

Putting (4.50) into (4.18)-(4.20) we can solve for first derivatives as follows,

$$H'_{x_1x_1} = f_5\left(Z'_H, Z_H, H_{x_1x_1}, r\right), \tag{4.51}$$

$$H'_{x_2x_2} = f_6\left(Z'_H, Z_H, H_{x_1x_1}, r\right), \tag{4.52}$$

$$H'_{tx_2} = f_7\left(Z'_H, Z_H, H_{x_1x_1}, H_{x_2x_2}, H_{tx_2}, r\right). \tag{4.53}$$

Consistency ensures that

$$\frac{d f_5}{dr} - f_1 = 0, \tag{4.54}$$

which yields an equation of the form

$$F\left(Z''_H, Z'_H, Z_H, H'_{x_1x_1}, H_{x_1x_1}, H'_{x_2x_2}\right) = 0. \tag{4.55}$$

Substituting equations (4.51) and (4.52) into (4.55) to eliminate $H'_{x_1x_1}$ and $H'_{x_2x_2}$ we get

$$Z''_H + AZ'_H + BZ_H = 0, \tag{4.56}$$

where the prime denotes differentiation with respect to r, and

$$A = \frac{16q^2r^6 - 16\omega^2r^6 - 14q^2r^3r_0^3 + 4\omega^2r^3r_0^3 + 7q^2r_0^6}{4q^2r^7 - 4\omega^2r^7 - 5q^2r^4r_0^3 + 4\omega^2r^4r_0^3 + q^2rr_0^6}, \tag{4.57}$$

$$B = -\frac{4q^4r^7 - 5q^4r_0^3r^4 + q^4r_0^6r - 8q^2\omega^2r^7 + 5q^2\omega^2r_0^3r^4 - 9q^2r_0^6r^3 + 9q^2r_0^9 + 4\omega^4r^7}{r^2\left(r^3 - r_0^3\right)\left(4q^2r^6 - 4\omega^2r^6 - 5q^2r^3r_0^3 + 4\omega^2r_0^3r^3 + q^2r_0^6\right)}. \tag{4.58}$$

All dependence on the functions $H_{\mu\nu}$ has vanished, so (4.56) is a diffeomorphism invariant.

We will have to solve the equation (4.56) numerically. It is convenient to transform to a new radial coordinate

$$y = \sqrt{1 - \frac{r_0^3}{r^3}} \quad \Leftrightarrow \quad r = r_0\left(1 - y^2\right)^{-\frac{1}{3}}. \tag{4.59}$$

In this variable, the horizon $r = r_0$ becomes $y = 0$, and the boundary $r \to \infty$

becomes $y = 1$. Derivatives become

$$\frac{dZ_H}{dr} = \frac{3}{2r_0}\frac{\left(1-y^2\right)^{\frac{4}{3}}}{y}\frac{dZ_H}{dy}, \tag{4.60}$$

$$\frac{d^2Z_H}{dr^2} = \frac{3}{4r_0^2}\left[3\frac{\left(1-y^2\right)^{\frac{8}{3}}}{y^2}\frac{d^2Z_H}{dy^2} - \frac{\left(3+5y^2\right)\left(1-y^2\right)^{\frac{5}{3}}}{y^3}\frac{dZ_H}{dy}\right]. \tag{4.61}$$

Equation (4.56) becomes

$$Z_H'' + \mathcal{A}Z_H' + \mathcal{B}Z_H = 0, \tag{4.62}$$

where the prime now denotes differentiation with respect to y, and

$$\mathcal{A} = \frac{1}{y}\left(\frac{3q^2\left(1-y^2\right)-4\omega^2}{3q^2\left(1+y^2\right)-4\omega^2}\right), \tag{4.63}$$

$$\mathcal{B} = -\frac{4}{9}\left(\frac{q^4y^2\left(3+y^2\right) - q^2\omega^2\left(3+5y^2\right) - 9r_0^2q^2y^2\left(1-y^2\right)^{\frac{4}{3}} + 4\omega^4}{r_0^2y^2\left(1-y^2\right)^{\frac{4}{3}}\left(3q^2+q^2y^2-4\omega^2\right)}\right). \tag{4.64}$$

4.3.1 Near-horizon behaviour

It is also convenient to extract the leading behaviour near the horizon. To do this, we let $Z_H = y^n$ and expand equation (4.62) about the horizon $y = 0$. To leading order, equation (4.62) gives

$$\left(n^2 + \frac{4\omega^2}{8r_0^2}\right)\frac{1}{y^2} + \mathcal{O}(1) = 0. \tag{4.65}$$

This is satisfied to leading order if $n = \pm\frac{2\omega}{3r_0}i$, or

$$Z_H \sim C_1 y^{-\frac{2\omega}{3r_0}i} + C_2 y^{+\frac{2\omega}{3r_0}i}, \quad y \to 0, \tag{4.66}$$

where C_1 and C_2 are constants. Thus we have two independent solutions near the horizon, and we should determine their physical validity. The solution $y^{-\frac{2\omega}{3r_0}i}$ represents a wave propagating *into* the horizon, and $y^{+\frac{2\omega}{3r_0}i}$ represents a wave propagating *out of* the horizon. Imposing that the wave is completely absorbed at the horizon, the latter solution is precluded, so we get

$$Z_H \sim y^{-\frac{2\omega}{3r_0}i}, \quad y \to 0. \tag{4.67}$$

Having extracted the leading behaviour near the horizon, we may write the solution away from the horizon as

$$Z_H = y^{-i\hat{\omega}} z_h, \qquad \text{where} \quad \hat{\omega} = \frac{2\omega}{3r_0} = \frac{\omega}{2\pi T} \tag{4.68}$$

and $z_h(y)$ is well-behaved at the horizon, admitting a regular Taylor series expansion. Letting[2] $\hat{q} = \frac{2q}{3r_0} = \frac{q}{2\pi T}$ along with (4.68), equation (4.62) becomes

$$z_H'' + \hat{\mathcal{A}} z_H' + \hat{\mathcal{B}} z_H = 0, \tag{4.69}$$

where

$$\hat{\mathcal{A}} = -\frac{1}{y}\left(\frac{4\hat{\omega}^2 - 3\hat{q}^2\left(1-y^2\right) + 2i\hat{\omega}\left(\hat{q}^2y^2 + 3\hat{q}^2 - 4\hat{\omega}^2\right)}{\hat{q}^2y^2 + 3\hat{q}^2 - 4\hat{\omega}^2}\right), \tag{4.70}$$

and

$$\begin{aligned}\hat{\mathcal{B}} =& \frac{4\left(1-y^2\right)^{\frac{4}{3}}\left(\hat{\omega}^4 + \hat{q}^2y^2\left(1+i\hat{\omega}\right)\right) + \hat{\omega}^2\hat{q}^2\left(1-y^2\right)^{\frac{1}{3}}\left(y^2\left(2+y^2\right)-3\right)}{y^2\left(1-y^2\right)^{\frac{4}{3}}\left(\hat{q}^2y^2 - 4\hat{\omega}^2 + 3\hat{q}^2\right)} \\ &+ \frac{-4\hat{\omega}^4 + \hat{\omega}^2\hat{q}^2\left(3+5y^2\right) - \hat{q}^4y^2\left(3+y^2\right)}{y^2\left(1-y^2\right)^{\frac{4}{3}}\left(\hat{q}^2y^2 - 4\hat{\omega}^2 + 3\hat{q}^2\right)}.\end{aligned} \tag{4.71}$$

[2]The frequency $\hat{\omega}$ and momentum $\hat{q}$ are dimensionless.

4.4 Boundary conditions

At the horizon, z_H is a regular function, and so may be expanded as a Taylor series about $y = 0$. Since equation (4.69) is second-order, there are two integration constants; that is,

$$z_H = \zeta_0 + \zeta_1 y^2 + \mathcal{O}\left(y^4\right), \tag{4.72}$$

where ζ_0 and ζ_1 are coefficients that should be fixed by the boundary conditions. Coefficients of higher-order terms can be found in terms of ζ_0 and ζ_1. Since (4.69) is invariant under rescalings of the form $z_H \to \lambda z_H$, where λ is any constant, we are free to choose the integration constant $\zeta_0 = 1$ (i.e. $\lambda = 1/\zeta_0$) so that

$$z_H\Big|_{y\to 0_+} = 1. \tag{4.73}$$

At the boundary the metric component $c_1 \to 0$. The fluctuation z_H must also vanish at the boundary because, if it didn't, then the fluctuation would dominate the background at the boundary. If this were the case, we could not interpret z_H as a small perturbation on the background. Thus, to keep our theory intact we must impose the boundary condition

$$z_H\Big|_{y\to 1_-} = 0. \tag{4.74}$$

4.5 Dispersion relation

Our goal is to calculate the dispersion relation $\hat{\omega} = \hat{\omega}(\hat{q})$. We will first consider the dispersion relation in the hydrodynamic limit where we will be able to extract the speed of sound and the attenuation of sound, the latter of which is related to the bulk viscosity. We will then numerically compute the full dispersion relation and find that there is a discrete spectrum of frequencies at fixed momentum.

4.5.1 Hydrodynamic limit

Let us first compute the dispersion relation to leading order when the momentum $\hat{q}$ is small. The limit $\hat{q} \to 0$ is called the hydrodynamic limit. Begin by expanding $\hat{\omega}$ and z_H with respect to $\hat{q}$ about $\hat{q} = 0$ as follows[3],

$$\hat{\omega} = c_s \hat{q} - ib\hat{q}^2, \tag{4.75}$$

$$z_h(y) = z_0(y) + i\hat{q} z_1(y). \tag{4.76}$$

c_s and b are the speed and attentuation of sound respectively. Putting the expansions (4.75) and (4.76) into (4.69) and expanding the result about $\hat{q} = 0$ to leading order, we get

$$z_0'' + \frac{1}{y}\left(\frac{4c_s^2 - 3 + 3y^2}{4c_s^2 - 3 - y^2}\right) z_0' - \frac{4}{4c_s^2 - 3 - y^2} z_0 = 0. \tag{4.77}$$

The general solution is

$$z_0 = C_1\left(4c_s^2 - 3 + y^2\right) + C_2\left[\left(4c_s^2 - 3 + y^2\right)\ln(y) + 8c_s^2 - 6\right], \tag{4.78}$$

where C_1 and C_2 are integration constants. The boundary conditions (4.73) and (4.74) translate into

$$\begin{aligned} z_0\Big|_{y\to 0_+} &= 1, & z_1\Big|_{y\to 0_+} &= 0, \\ z_0\Big|_{y\to 1_-} &= 0, & z_1\Big|_{y\to 1_-} &= 0. \end{aligned} \tag{4.79}$$

[3]We expect that $\omega \to 0$ as $q \to 0$. The imaginary unit appears in the subleading terms so that b and z_1 are real.

The boundary condition for $y \to 0_+$ is satisfied if

$$C_1 = \frac{1}{4c_s^2 - 3}, \quad C_2 = 0, \tag{4.80}$$

giving

$$z_0 = \frac{4c_s^2 - 3 + y^2}{4c_s^2 - 3}. \tag{4.81}$$

Expanding (4.81) about the boundary $y = 1$ we get

$$z_0 = \frac{4c_s^2 - 2}{4c_s^2 - 3} - \frac{2}{4c_s^2 - 3}(1 - y) + \mathcal{O}\left[(1 - y)^2\right]. \tag{4.82}$$

The boundary condition for $y \to 1_-$ is satisfied if

$$c_s^2 = \frac{1}{2}, \tag{4.83}$$

which is in agreement with (3.28). Now (4.81) becomes

$$z_0 = 1 - y^2. \tag{4.84}$$

Putting (4.75), (4.76), (4.83) and (4.84) into (4.69) and expanding the result about $\hat{q} = 0$ to leading order, we get

$$z_1'' - \frac{3y^2 - 1}{y\,(y^2 + 1)} z_1' + \frac{4}{y^2 + 1} z_1 + \frac{4\sqrt{2}\,(-4b + 1)}{y^2 + 1} = 0. \tag{4.85}$$

The general solution is

$$z_1 = A_1\left(y^2 - 1\right) + A_2\left[(y - 1)\ln(y) - 2\right] - \sqrt{2}\,(1 - 4b), \tag{4.86}$$

where A_1 and A_2 are integration constants. The boundary condition for $y \to 0_+$ is satisfied if

$$A_1 = -\sqrt{2}\,(1-4b)\,, \qquad A_2 = 0. \tag{4.87}$$

So

$$z_1 = -\sqrt{2}\,(1-4b)\,y^2. \tag{4.88}$$

Expanding (4.88) about $y = 1$ we get

$$z_1 = -\sqrt{2}\,(1-4b) + 2\sqrt{2}\,(1-4b)\,(1-y) + O\left[(1-y)^2\right]. \tag{4.89}$$

The boundary condition for $y \to 1_-$ is satisfied if

$$b = \frac{1}{4}. \tag{4.90}$$

Recall that the fluctuations are of the form

$$flucs \sim F(r)e^{-i\omega t + iqx_2}. \tag{4.91}$$

With $\omega = c_s q - \frac{ib}{2\pi T}q^2$, this becomes

$$flucs \sim F(r)e^{-bq^2 t}e^{i(x_2 - c_s t)q}. \tag{4.92}$$

So if

$$\begin{array}{llll} b > 0, & \text{fluctuations decay} & \Rightarrow & \text{stable} \\ b < 0, & \text{fluctuations grow} & \Rightarrow & \text{unstable.} \end{array} \tag{4.93}$$

We have $b = 1/4 > 0$ so our AdS_4 black hole solution and dual CFT plasma are perturbatively stable.

4.5.2 Hydrodynamics

Recall from (1.37) that b is related to the viscosities of the dual plasma by

$$b = 2\pi\frac{\eta}{s}\left(\frac{1}{2} + \frac{\zeta}{2\eta}\right) = \frac{1}{4}, \tag{4.94}$$

and that the ratio η/s is universal,

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{4.95}$$

Since we know from (3.16) that $s = 4\pi^2T^2/9G$, we can get the shear viscosity

$$\eta = \frac{\pi T^2}{9G}. \tag{4.96}$$

Now we can extract the bulk viscosity from (4.94),

$$\zeta = 0, \tag{4.97}$$

which is the expected result for a CFT (see (1.41)).

4.5.3 Arbitrary momentum

Now we will numerically calculate the dispersion relation for arbitrary momentum $\hat{q}$. In this section we will set up the equations and expansions that are required for our numerical method, then give the results. The details of the numerical method are given in Appendix B. Many of the equations are too cumbersome to give explicitly, in which case we will just give the equations' form. We begin by letting

$$\hat{\omega} = \omega_r + i\omega_i, \tag{4.98}$$

and

$$z_H = z_r + iz_i, \tag{4.99}$$

where $\omega_{r,i} = \omega_{r,i}(\hat{q})$ and $z_{r,i} = z_{r,i}(y)$ are all real functions. Putting (4.98) and (4.99) into (4.69), then separating the real from imaginary parts, we get two equations of the form

$$z_r'' + A_r z_r' + B_r z_i' + C_r z_r + D_r z_r = 0, \tag{4.100}$$

$$z_i'' + A_i z_r' + B_i z_i' + C_i z_r + D_i z_r = 0, \tag{4.101}$$

where $A_{r,i} = A_{r,i}\,(y, \hat{q}, \omega_r, \omega_i)$, and likewise for $B_{r,i}$, $C_{r,i}$, and $D_{r,i}$. It is straightforward to find these coefficients, but the explicit expressions are too lengthy to write here. Now we look for asymptotic solutions near the horizon $y = 0$. The solutions for z_r and z_i have regular Taylor series expansions near the horizon, so we let

$$z_r = 1 + b_1 y + b_2 y^2 + b_3 y^3 + b_4 y^4 ... \tag{4.102}$$

$$z_i = h_1 y + h_2 y^2 + h_3 y^3 + h_4 y^4 ... \tag{4.103}$$

Putting the expansions (4.102) and (4.103) into the equations (4.100) and (4.101) and expanding both equations about $y = 0$, we demand that the coefficient of every power of y must vanish. This leads to a system of equations for the coefficients b_j, h_j, $(j = 1, 2, 3, ...)$. Solving this system we get

$$b_1 = h_1 = 0 \tag{4.104}$$

$$b_2 = \frac{N_1}{12 D_1}, \tag{4.105}$$

$$h_2 = \frac{\omega_r N_2}{12 D_2}, \tag{4.106}$$

where

$$\begin{aligned} N_1 =& -64\omega_r^6 + 48\hat{q}^2\omega_r^2 + 192\hat{q}^2\omega_i^4 + 27\hat{q}^6 + 96\hat{q}^2\omega_r^4 - 72\hat{q}^4\omega_r^2 + 64\omega_i^6 \\ & - 48\hat{q}^2\omega_r^4\omega_i - 48\hat{q}^2\omega_i^2 + 192\hat{q}^2\omega_r^2\omega_i - 96\hat{q}^2\omega_r^2\omega_i^2 - 36\hat{q}^4\omega_r^2\omega_i \\ & + 96\hat{q}^2\omega_r^2\omega_i^3 + 64\omega_i^7 - 36\hat{q}64 + 192\omega_r^2\omega_i^5 + 64\omega_r^2\omega_i^4 + 144\hat{q}^2\omega_i^5 \\ & + 144\hat{q}^4\omega_i^2 + 108\hat{q}^4\omega_i^3 + 27\hat{q}^6\omega_i - 64\omega_r^4\omega_i^2 + 64\omega_r^6\omega_i + 192\omega_r^4\omega_i^3 \end{aligned} \tag{4.107}$$

$$\begin{aligned} D_1 =& \left(32\omega_r^2\omega_i^2 - 24\hat{q}^2\omega_r^2 + 9\hat{q}^4 + 24\hat{q}^2\omega_i^2 + 16\omega_r^4 + 16\omega_i^4\right) \\ & \times \left(\omega_r^2 + \omega_i^2 + 2\omega_i + 1\right) \end{aligned} \tag{4.108}$$

$$\begin{aligned} N_2 =& 96\hat{q}^2\omega_r^2\omega_i^2 - 96\hat{q}^2\omega_i + 36\hat{q}^4\omega_i^2 - 96\hat{q}^2\omega_i^2 - 72\hat{q}^4 + 96\hat{q}^2\omega_r^2 - 12\omega_r^4\omega_i \\ & - 128\omega_i^5 - 256\omega_r^2\omega_i^3 - 96\hat{q}^2\omega_i^3 + 288\hat{q}^2\omega_r^2\omega_i - 72\hat{q}^4\omega_i - 48\omega_i^4\hat{q}^2 \\ & + 27\hat{q}^6 - 64\omega_r^6 - 192\omega_r^4\omega_i^2 + 144\hat{q}^2\omega_r^4 - 108\hat{q}^4\omega_r^2 - 192\omega_r^2\omega_i^4 - 64\omega_i^6 \end{aligned} \tag{4.109}$$

$$\begin{aligned} D_2 =& 32\omega_r^2\omega_i^2 + 18\hat{q}^4\omega_i + 32\omega_r^4\omega_i + 64\omega_r^2\omega_i^3 + 24\hat{q}^2\omega_i^2 + 9\hat{q}^4 + 48\hat{q}^2\omega_i^3 + \\ & 32\omega_i^5 + 16\omega_i^4 + 16\omega_r^6 + 16\omega_r^4 + 48\omega_r^4\omega_i^2 - 24\hat{q}^2\omega_r^4 + 9\hat{q}^4\omega_r^2 + 48\omega_r^2\omega_i^4 \\ & + 9\hat{q}^4\omega_i^2 + 24\hat{q}^2\omega_i^4 - 24\hat{q}^2\omega_r^2 + 16\omega_i^6 - 48\hat{q}^2\omega_r^2\omega_i. \end{aligned} \tag{4.110}$$

We may work to arbitrary order and find as many coefficients b_j, h_j $(j = 1, 2, 3...)$ as we like. Next we repeat this process, but expanding about the boundary $y = 1$. To do this, we first apply the transformation

$$y \to 1 - x, \quad \frac{d}{dy} \to -\frac{d}{dx}, \quad \frac{d^2}{dy^2} \to \frac{d^2}{dx^2} \tag{4.111}$$

horizon: $y = 0 \Leftrightarrow x = 1$

boundary: $y = 1 \Leftrightarrow x = 0$

to the equations (4.100) and (4.101). We get equations of the form

$$z_r'' + \mathcal{A}_r z_r' + \mathcal{B}_r z_i' + C_r z_r + \mathcal{D}_r z_r = 0, \tag{4.112}$$

$$z_i'' + \mathcal{A}_i z_r' + \mathcal{B}_i z_i' + C_i z_r + \mathcal{D}_i z_r = 0, \tag{4.113}$$

where the prime now denotes differentiation with respect to x, and $\mathcal{A}_{r,i} = \mathcal{A}_{r,i}\,(x, \hat{q}, \omega_r, \omega_i)$, and likewise for $\mathcal{B}_{r,i}$, $C_{r,i}$, and $\mathcal{D}_{r,i}$. We need to make an "educated guess" at the asymptotic expansion of z_r and z_i. First note that from (4.59) we have

$$r = r_0 \left(2x - x^2\right)^{\frac{-1}{3}} . \tag{4.114}$$

In the x variable, the background metric components are

$$c_1 = r\sqrt{1 - \frac{r_0^3}{r^3}} = r_0 \frac{1-x}{(2x - x^2)^{\frac{1}{3}}}, \tag{4.115}$$

$$c_2 = r = r_0 \left(2x - x^2\right)^{\frac{-1}{3}} , \tag{4.116}$$

$$c_3 = \frac{1}{r\sqrt{1 - \frac{r_0^3}{r^3}}} = \frac{1}{r_0} \frac{\left(2x - x^2\right)^{\frac{1}{3}}}{1-x}. \tag{4.117}$$

Expanding about the boundary $x = 0$ we get

$$c_1 = 2^{-\frac{1}{3}} r_0 x^{-\frac{1}{3}} - \frac{5}{12} r_0 2^{\frac{2}{3}} x^{\frac{2}{3}} + O\left(x^{\frac{5}{3}}\right), \tag{4.118}$$

$$c_2 = 2^{-\frac{1}{3}} r_0 x^{-\frac{1}{3}} + \frac{1}{12} r_0 2^{\frac{2}{3}} x^{\frac{2}{3}} + O\left(x^{\frac{5}{3}}\right), \tag{4.119}$$

$$c_3 = \frac{2^{\frac{1}{3}}}{r_0} x^{\frac{1}{3}} + \frac{5}{6} \times \frac{2^{\frac{1}{3}}}{r_0} x^{\frac{4}{3}} + O\left(x^{\frac{7}{3}}\right). \tag{4.120}$$

As mentioned in our discussion of the boundary conditions, we must ensure that our fluctuation does not change the background at the boundary. Looking at the expansions for c_1, c_2 and c_3, we may guess[4] that the expansion for z_r and z_i have the form

$$z_r = f_0 x^{\frac{2}{3}} + f_1 x + f_2 x^{\frac{4}{3}} + f_3 x^{\frac{5}{3}} + ... \tag{4.121}$$

$$z_i = g_0 x^{\frac{2}{3}} + g_1 x + g_2 x^{\frac{4}{3}} + g_3 x^{\frac{5}{3}} + ... \tag{4.122}$$

Putting (4.121) and (4.122) into (4.112) and (4.113), expanding the equations about $x = 0$ and setting coefficients of powers of x equal to zero, we get a system of equations for the coefficients $f_j, g_j, (j = 0, 1, 2, ...)$. Solving the system we get

$$f_0 = g_0 = f_2 = g_2 = 0, \tag{4.123}$$

$$f_3 = 2^{\frac{2}{3}} \times \frac{9}{40} \left[f_1 \left(\hat{q}^2 - \omega_r^2 + \omega_i^2 \right) + 2 g_1 \omega_r \omega_i \right] \tag{4.124}$$

$$g_3 = 2^{\frac{2}{3}} \times \frac{9}{40} \left[g_1 \left(\hat{q}^2 - \omega_r^2 + \omega_i^2 \right) - 2 f_1 \omega_r \omega_i \right] \tag{4.125}$$

Our numerical method solves the equations (4.100), (4.101), (4.112) and (4.113) on the intervals $x, y \in \left[\epsilon, \frac{1}{2}\right]$, where ϵ is some small initial value. Here we take $\epsilon = 0.0001$. Notice that the y integration covers the part of the domain from near the horizon, out to the midpoint of the domain, whereas the x integration covers from the boundary, in to the midpoint of the domain. The two integrations meet in the middle, and we use the smoothness of the solution at this point as a measure of the error. As initial conditions we use the expansions (4.102), (4.103), (4.121) and (4.122) and their derivatives evaluated at ϵ. We numerically solve

$$z_r'' + A_r z_r' + B_r z_i' + C_r z_r + D_r z_r = 0, \tag{4.126}$$

[4] We assume that the asymptotic expansion of the fluctuations have the same form as that of the background, but we remove the leading order to ensure that the fluctuations are subdominant.

$$z_i'' + A_i z_r' + B_i z_i' + C_i z_r + D_i z_r = 0, \tag{4.127}$$

$$z_r(\epsilon) = 1 + b_1\epsilon + b_2\epsilon^2 + b_3\epsilon^3 + b_4\epsilon^4, \tag{4.128}$$

$$z_i(\epsilon) = h_1\epsilon + h_2\epsilon^2 + h_3\epsilon^3 + h_4\epsilon^4, \tag{4.129}$$

where the prime denotes differentiation with respect to y, and

$$z_r'' + \mathcal{A}_r z_r' + \mathcal{B}_r z_i' + C_r z_r + \mathcal{D}_r z_r = 0, \tag{4.130}$$

$$z_i'' + \mathcal{A}_i z_r' + \mathcal{B}_i z_i' + C_i z_r + \mathcal{D}_i z_r = 0, \tag{4.131}$$

$$z_r(\epsilon) = f_1\epsilon + f_2\epsilon^{\frac{4}{3}} + f_3\epsilon^{\frac{5}{3}} \tag{4.132}$$

$$z_i(\epsilon) = g_1\epsilon + g_2\epsilon^{\frac{4}{3}} + g_3\epsilon^{\frac{5}{3}}, \tag{4.133}$$

where the prime denotes differentiation with respect to x. There are five parameters that must be fixed. They are $\hat{q}$, ω_r, ω_i, f_1, and g_1. We choose a value for $\hat{q}$, then use a shooting method to fix the remaining four parameters. The shooting method is explained in detail in Appendix B. In particular, we can get $\omega_{r,i} = \omega_{r,i}(\hat{q})$. We should start with small values of $\hat{q}$ and use our results from section 4.5.1 to give us a good initial guess. For small $\hat{q}$, referring to equation (4.75), we have

$$\omega_r \approx c_s\hat{q}, \quad \text{and} \quad \omega_i \approx b\hat{q}^2, \tag{4.134}$$

where $c_s = 1/\sqrt{2}$ and $b = -1/4$. We can choose $\hat{q}$ to be a small number, say, $\hat{q} = 0.001$. Now, from equations (4.84) and (4.88), we find that for small $\hat{q}$

$$z_r \approx z_0 = 2x - x^2, \quad \text{and} \quad z_i \approx z_1 = 0. \tag{4.135}$$

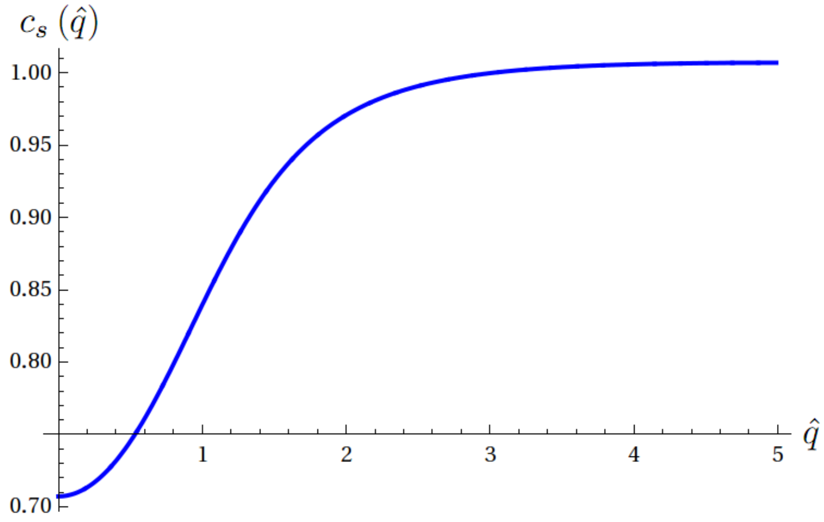

Figure 4.1: Speed of sound versus momentum of fluctuations.

Thus we may make a good guess that for $\hat{q} = 0.01$, we have

$$\begin{aligned} \omega_r &\approx 0.0071 \\ \omega_i &\approx -0.000025 \\ f_1 &\approx 2 \\ g_1 &\approx 0 \end{aligned} \tag{4.136}$$

With these choices, we find a dispersion relation that is typical of a massless mode; that is, $\omega(0) = 0$. Figure 4.1 shows the speed of sound as a function of momentum. We can see that in the hydrodynamic limit $\hat{q} \to 0$, the speed of the fluctuations approaches the conformal value of of the speed of sound $c_s = 1/\sqrt{2}$. In the large momentum limit $\hat{q} \to 0$, the speed of the fluctuations approaches the speed of light $c_s = 1$, as expected. We may also look for solutions corresponding to massive modes by making different initial guesses of the values of the parameters. Finding this spectrum is largely a matter of numerical trial and error, however, each excitation is usually found with $w_r \sim 1, 2, ...$ The result is shown in figure 4.2 where we

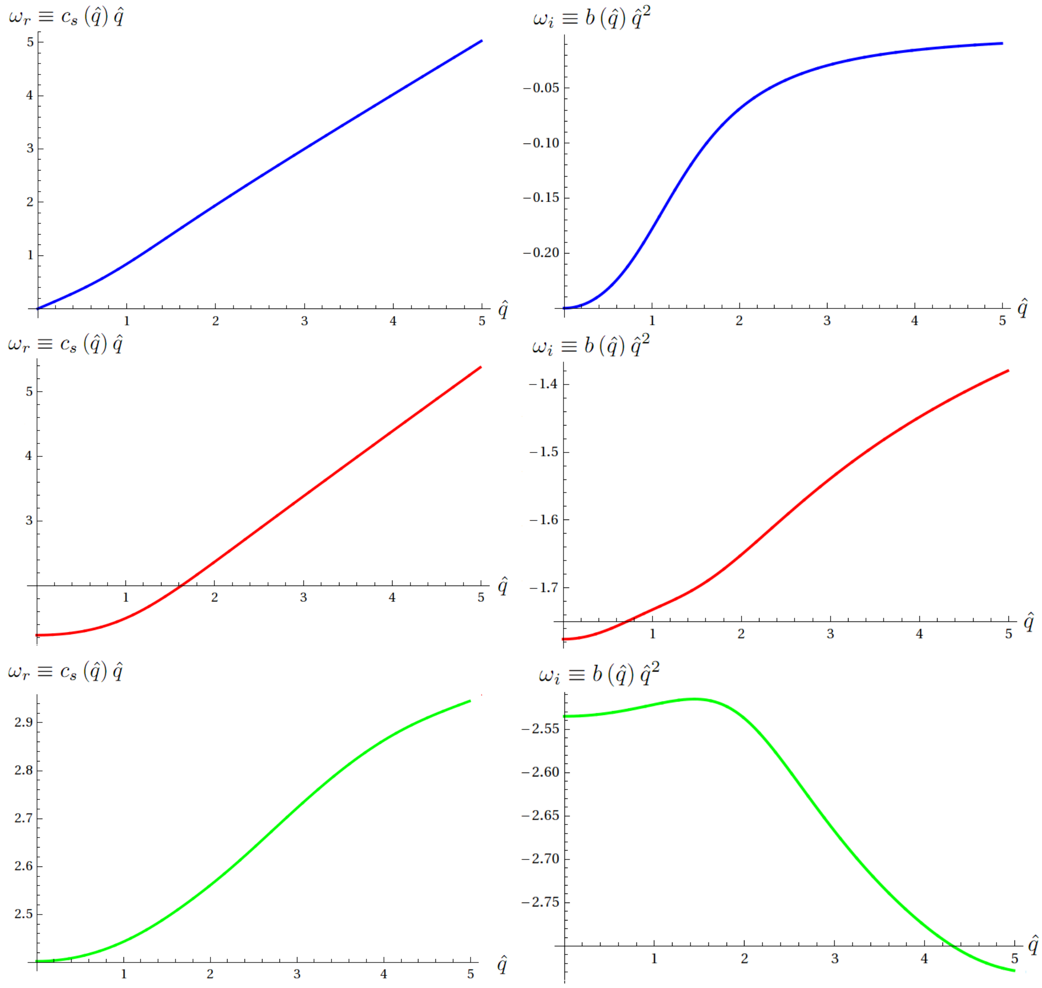

Figure 4.2: Left column: ω_r vs $\hat{q}$. Right column: ω_i vs $\hat{q}$. Each row corresponds to successive excitations.

see that at $\hat{q} = 0$, we have a massless mode (top) and two massive modes (middle and bottom). There are two regimes, $\hat{q} \to 0$ and $\hat{q} \to \infty$, of particular physical importance. For $\hat{q} \to 0$, which is relevant to the study of hydrodynamics, the fluctuations represent sound modes that decay after many oscillation, and thus can be interpreted as genuine excitations. In this regime, Reω grows linearly with a slope equal to the speed of sound. For $\hat{q} \to \infty$, the fluctuations are also long-lived, but behave like massless particles. In this regime, Reω is linear with slope equal to the speed of light (i.e. $c = 1$). This is true for all branches in figure 4.2, although in the third branch (green) we have not shown results for larg enough $\hat{q}$ to make this obvious. This makes sense since the rest mass becomes un-important as the momentum becomes sufficiently large, and the massive ”particle” behaves similarly to a massless one. We may conclude that since $0 < c_s < 1$ always, all modes are thermodynamically stable and causality is never violated. From the first ω_i vs $\hat{q}$ plot, we see that for small (large) values of $\hat{q}$ we get the expected result $\omega_i \to -1/4$ ($\omega_i \to 0$) Also, since $\omega_i < 0$ always, all branches are perturbatively stable as well.

In this chapter we studied small fluctuations of the background spacetime. From the dispersion relation of the fluctuations we were able to extract the speed of sound and bulk viscosity of the dual field theory, which we found are in agreement with the expected values for a CFT. From the sign of the attenuation we determined that AdS_4 black holes, and thus the dual field theory, are perturbatively stable. In the next chapter we will begin our study of the Exotic Model. Although it will be technically more difficult, the techniques used are almost identical to study AdS_4.

Chapter 5

AdS4 black holes with scalar hair

In this chapter we begin our study of the Exotic Model, which is AdS_4 gravity minimally coupled to two massive scalar fields. This model is explained within the context of the AdS/CFT correspondence in section 1.5. We will derive the equations of motion and solve them numerically to compute the exact thermodynamics. We will consider the near-conformal limit and analytically calculate the leading order corrections to the results of the previous chapter.

5.1 Action

Here we will consider the Exotic Model, whose action is given by

$$S = \frac{1}{2\kappa^2}\left(\int_{\mathcal{M}} d^4x\,\sqrt{-g}\,(\mathcal{L}_{\mathrm{AdS}_4} + \mathcal{L}_r + \mathcal{L}_i) - \int_{\partial\mathcal{M}} d^3x\,\sqrt{-\gamma}\Theta + S_{ct}\right), \tag{5.1}$$

where

$$\mathcal{L}_{\mathrm{AdS}_4} = R_4 + 6, \quad \mathcal{L}_r = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \phi^2, \quad \mathcal{L}_i = -\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - 2\chi^2 - g\phi^2\chi^2, \tag{5.2}$$

and $\Theta = g_{\mu\nu}\Theta^{\mu\nu}$, where

$$\Theta^{\mu\nu} = -\frac{1}{2}\left(\nabla^{\mu}n^{\nu} + \nabla^{\nu}n^{\mu}\right). \tag{5.3}$$

The counterterm S_{ct} will be chosen in such a way that the on-shell action is finite.

5.2 Equations of motion

We seek solutions for the metric of the form

$$ds^2 = -c_1(r)^2dt^2 + c_2(r)^2\left(dx_1^2 + dx_2^2\right) + c_3(r)^2dr^2. \tag{5.4}$$

Putting this metric into the bulk part of the action gives

$$S_{bulk} = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x \left(\begin{array}{l} -4\dfrac{c_1'c_2'c_2}{c_3} + 2\dfrac{c_1'c_3'c_2^2}{c_3^2} - 2\dfrac{c_1''c_2^2}{c_3} - 2\dfrac{c_1(c_2')^2}{c_3} + 4\dfrac{c_2'c_3'c_1c_2}{c_3^2} \\ -4\dfrac{c_2''c_1c_2}{c_3} + 6c_1c_2^2c_3 + c_1c_2^2c_3\phi^2 - 2c_1c_2^2c_3\chi^2 \\ -gc_1c_2^2c_3\phi^2\chi^2 - \dfrac{c_1c_2^2(\phi')^2}{2c_3} - \dfrac{c_2c_2^2(\chi')^2}{2c_3} \end{array}\right). \tag{5.5}$$

The equations of motion are derived by setting $\delta S_{bulk} = 0$. It can be checked that the boundary terms arising from the variation of the bulk action are exactly cancelled by the boundary integral in (5.1). This is apparent since the additional terms $\mathcal{L}_r$ and $\mathcal{L}_i$ contain only up to first-order derivatives, so the cancellation is identical to that in chapter 3. As such, we will not concern ourselves with the boundary terms. Varying S_{bulk} with respect to c_1, c_2, c_3, ϕ, and χ repectively gives the equations of motion

$$\begin{aligned} 8c_2''c_2c_3 - 12c_2^2c_3^3 - 2c_2^2c_3^3\phi^2 + 4c_2^2c_3^3\chi^2 + 2gc_2^2c_3^3\phi^2\chi^2 \\ + 4(c_2')^2c_3 + c_2^2c_3(\phi')^2 - 8c_2'c_3'c_2 + c_2^2c_3(\chi')^2 = 0 \end{aligned} \tag{5.6}$$

$$-4c_2'c_3'c_1 + 4c_1'c_2'c_3 + 2gc_1c_2c_3^3\phi^2\chi^2 + 4c_2''c_1c_3 - 4c_1'c_3'c_2 + c_1c_2c_3(\phi')^2 - 12c_1c_2c_3^3 + 4c_1''c_2c_3 + c_1c_2c_3(\chi')^2 - 2c_1c_2c_3^3\phi^2 + 4c_1c_2c_3^3\chi^2 = 0 \tag{5.7}$$

$$8c_1'c_2'c_2 + 4(c_2')^2c_1 - 12c_1c_2^2c_3^2 - 2c_1c_2^2c_3^2\phi^2 + 4c_1c_2^2c_3^2\chi^2 + 2gc_1c_2^2c_3^2\phi^2\chi^2 - c_1c_2^2(\phi')^2 - c_1c_2^2(\chi')^2 = 0 \tag{5.8}$$

$$2c_1c_2c_3^3\phi - 2gc_1c_2c_3^3\phi\chi^2 + 2c_1c_2'c_3\phi' - c_1c_2c_3'\phi' + c_1'c_2c_3\phi' + c_1c_2c_3\phi'' = 0 \tag{5.9}$$

$$4c_1c_2c_3^3\chi + 2gc_1c_2c_3^3\phi^2\chi - 2c_1c_2'c_3\chi' + c_1c_2c_3'\chi' - c_1'c_2c_3\chi' - c_1c_2c_3\chi'' = 0 \tag{5.10}$$

Using the first two equations, we can eliminate c_2'' from the first and c_1'' from the second. Then we can use the third equation to eliminate $(\phi')^2+(\chi')^2$ from both. The equations of motion can be put into the form

$$c_1'' + c_1'\left(\ln\frac{c_2^2}{c_3}\right)' - \frac{c_1c_3^2}{2}\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2 + 6\right) = 0 \tag{5.11}$$

$$c_2'' + c_2'\left(\ln\frac{c_1c_2}{c_3}\right)' - \frac{c_2c_3^2}{2}\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2 + 6\right) = 0 \tag{5.12}$$

$$\phi'' + \phi'\left(\ln\frac{c_1c_2^2}{c_3}\right)' + 2c_3^2\phi\left(1 - g\chi^2\right) = 0 \tag{5.13}$$

$$\chi'' + \chi'\left(\ln\frac{c_1c_2^2}{c_3}\right)' - 2c_3^2\chi\left(2 + g\phi^2\right) = 0, \tag{5.14}$$

with the first-order constraint

$$(\phi')^2 + (\chi')^2 - 4\,(\ln c_2)'\left(\ln c_1^2c_2\right)' + 2c_3^2\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2 + 6\right) = 0. \tag{5.15}$$

5.3 Asymptotic solutions

We will not attempt to solve (5.11)-(5.15) exactly. We will eventually need to resort to using our numerical method. The domain of interest is $r \in [r_H, \infty)$. Having a boundary at infinity is not convenient in a numerical method. We transorm to a new radial coordinate in which the domain becomes the unit interval. We introduce a new radial coordinate y as

$$y = \frac{c_1}{c_2}, \quad \text{such that} \quad y \in [0, 1] \tag{5.16}$$

The horizon is defined by $c_1 = 0$, so in our new coordinate, the horizon is at $y = 0$. We assume that our spacetime is asymptotically AdS_4, in which case we should have $c_1 = c_2$ at the boundary (see equation (3.10) in the limit $r \to \infty$). Thus the boundary is at $y = 1$. We know that if our spacetime is asyptotically AdS_4, then we must have

$$c_2(r) \sim r, \qquad r \to \infty, \tag{5.17}$$

so from (4.59), in terms of y this is

$$c_2(y) \sim \frac{1}{(1-y^2)^{\frac{1}{3}}}, \qquad y \to 1. \tag{5.18}$$

Let us then explicitly pull out this divergent part and introduce the regular function $a(y)$ as

$$c_2(y) = \frac{a(y)}{(1-y^2)^{\frac{1}{3}}}. \tag{5.19}$$

Comparing with pure AdS_4 we have[1]

$$a(y = 1) = \alpha. \tag{5.20}$$

[1]Note that $\alpha = r_0$ using the notations of the previous chapter.

Also, to preserve our asymptotic AdS_4 spacetime, we must have

$$\phi(y \to 1) = \chi(y \to 1) = 0. \tag{5.21}$$

We will now extract the next leading behaviour of the solutions for c_2, ϕ, and χ. From the chain rule, we can write

$$\frac{d}{dr} = \frac{dy}{dr}\frac{d}{dy}, \qquad \frac{d^2}{dr^2} = \left(\frac{dy}{dr}\right)^2 \frac{d^2}{dy^2} + \frac{d^2y}{dr^2}\frac{d}{dy}. \tag{5.22}$$

Using (5.16) and (5.22) we can write the equation of motion (5.11)-(5.12) in terms of y

$$\left(\frac{dy}{dr}\right)^2 (yc_2)'' + (yc_2)' \left[\left(\ln \frac{c_2^2}{c_3}\right)' \left(\frac{dy}{dr}\right)^2 + \frac{d^2y}{dr^2}\right] - \frac{yc_2c_3^2}{2}\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2 + 6\right) = 0 \tag{5.23}$$

$$\left(\frac{dy}{dr}\right)^2 c_2'' + c_2' \left[\left(\ln \frac{yc_2^2}{c_3}\right)' \left(\frac{dy}{dr}\right)^2 + \frac{d^2y}{dr^2}\right] - \frac{c_2c_3^2}{2}\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2 + 6\right) = 0 \tag{5.24}$$

$$\left(\frac{dy}{dr}\right)^2 \phi'' + \phi' \left[\left(\ln \frac{yc_2^3}{c_3}\right)' \left(\frac{dy}{dr}\right)^2 + \frac{d^2y}{dr^2}\right] + 2c_3^2\phi\left(1 - g\chi^2\right) = 0 \tag{5.25}$$

$$\left(\frac{dy}{dr}\right)^2 \chi'' + \chi' \left[\left(\ln \frac{yc_2^3}{c_3}\right)' \left(\frac{dy}{dr}\right)^2 + \frac{d^2y}{dr^2}\right] - 2c_3^2\chi\left(2 + g\phi^2\right) = 0 \tag{5.26}$$

$$\left(\frac{dy}{dr}\right)^2 \left[(\phi')^2 + (\chi')^2 - 4\,(\ln c_2)' \left(\ln y^2 c_2^3\right)'\right] + 2c_3^2\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2 + 6\right) = 0, \tag{5.27}$$

where the prime now denotes differentiation with respect to y. Using (5.24) and (5.27) we can eliminate dy/dr and d^2y/dr^2. Since we will need it later, let us write the formula explicitly

$$\left(\frac{dy}{dr}\right)^2 = \frac{-2c_3^2\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2 + 6\right)}{(\phi')^2 + (\chi')^2 - 4\,(\ln c_2)' \left(\ln y^2 c_2^3\right)'} \tag{5.28}$$

The remaining three equations can be put in the form

$$c_2'' - \frac{c_2'}{y} - 4\frac{\left(c_2'\right)^2}{c_2} + y\frac{\left(c_2'\right)^2 c_3'}{c_2 c_3} + \frac{c_2}{4}\left[(\phi')^2 + (\chi')^2\right] = 0 \qquad (5.29)$$

$$\phi'' + \frac{\phi'}{y} - \frac{1}{2P}\frac{\partial P}{\partial \phi}\left[(\phi')^2 + (\chi')^2 - \frac{8}{y}\frac{c_2'}{c_2} - 12\frac{\left(c_2'\right)^2}{c_2^2}\right] = 0 \qquad (5.30)$$

$$\chi'' + \frac{\chi'}{y} + \frac{1}{2P}\frac{\partial P}{\partial \chi}\left[(\phi')^2 + (\chi')^2 - \frac{8}{y}\frac{c_2'}{c_2} - 12\frac{\left(c_2'\right)^2}{c_2^2}\right] = 0, \qquad (5.31)$$

where $P = \phi^2 - 2\chi^2 - g\phi^2\chi^2 + 6$. Transforming to the y coordinate, the metric becomes

$$ds^2 = -y^2 c_2^2 dt^2 + c_2^2\left(dx_1^2 + dx_2^2\right) - \frac{(\phi')^2 + (\chi')^2 - 4(\ln c_2)'\left(\ln y^2 c_2^3\right)'}{2c_3^2\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2 + 6\right)}dy^2, \qquad (5.32)$$

where we used (5.16) in the first component, and we used the chain rule $dy = \frac{dy}{dr}dr$ and (5.28) in the last component. In order to ensure regularity of solutions at the horizon we impose the condition

$$y \to 0_+ : \qquad [a(y), \phi(y), \chi(y)] \to [\alpha a_0, p_0, c_0], \qquad (5.33)$$

where a_0, p_0 and c_0 are constants. Regularity at the horizon allows us to expand the solution as a Taylor series about $y = 0$,

$$a = \alpha\left(a_0 + a_1 y^2 + O(y^4)\right) \qquad (5.34)$$

$$\phi = p_0 + O(y^2) \qquad (5.35)$$

$$\chi = c_0 + O(y^2) \qquad (5.36)$$

Higher order coefficients can be found in terms of $\{\alpha a_0, p_0, c_0\}$ by substituting these ansatze into (5.29)-(5.31), expanding about $y = 0$, then insisting that coefficients of powers of y vanish gives; however, they get really ugly really fast so we won't write them down.

Now we need to expand the solutions about the boundary. To do this, we introduce a new variable

$$x = 1 - y. \tag{5.37}$$

In the x variable the boundary is given by $x = 0$. Transforming the equations (5.29)-(5.31) to the x variable we get

$$c_2'' + \frac{c_2'}{1-x} - 4\frac{\left(c_2'\right)^2}{c_2} + \frac{c_2}{4}\left[(\phi')^2 + (\chi')^2\right] = 0 \tag{5.38}$$

$$\phi'' - \frac{\phi'}{1-x} - \frac{1}{2P}\frac{\partial P}{\partial \phi}\left[(\phi')^2 + (\chi')^2 + \left(\frac{8}{1-x}\right)\frac{c_2'}{c_2} - 12\frac{\left(c_2'\right)^2}{c_2^2}\right] = 0 \tag{5.39}$$

$$\chi'' - \frac{\chi'}{1-x} + \frac{1}{2P}\frac{\partial P}{\partial \chi}\left[(\phi')^2 + (\chi')^2 + \left(\frac{8}{1-x}\right)\frac{c_2'}{c_2} - 12\frac{\left(c_2'\right)^2}{c_2^2}\right] = 0, \tag{5.40}$$

where the prime denotes differentiation with respect to x. We know that near the boundary we have $\phi, \chi << 0$ and

$$c_2 \sim x^{-\frac{1}{3}}, \tag{5.41}$$

which follows from (5.19) with $y = 1 - x$ and the limit $x \to 0_+$. Under these assumptions, the asymptotic form of equations (5.39) and (5.40) is

$$\phi'' - \phi' + \frac{2}{9}\frac{\phi}{x^2} = 0, \tag{5.42}$$

and

$$\chi'' - \chi' - \frac{4}{9}\frac{\phi}{x^2} = 0, \tag{5.43}$$

which have the general solutions

$$\begin{aligned} \phi &= \sqrt{x}\exp\left(\frac{x}{2}\right)\left[AI_{\frac{1}{6}}\left(\frac{x}{2}\right) + BK_{\frac{1}{6}}\left(\frac{x}{2}\right)\right] \\ &\sim \mathcal{O}\left(x^{\frac{1}{3}}\right) + \mathcal{O}\left(x^{\frac{2}{3}}\right), \qquad x \to 0_+, \end{aligned} \tag{5.44}$$

and

$$\begin{aligned} \phi &= \sqrt{x}\exp\left(\frac{x}{2}\right)\left[CI_{\frac{5}{6}}\left(\frac{x}{2}\right) + DK_{\frac{5}{6}}\left(\frac{x}{2}\right)\right] \\ &\sim \mathcal{O}\left(x^{-\frac{1}{3}}\right) + \mathcal{O}\left(x^{\frac{2}{3}}\right) + \mathcal{O}\left(x^{\frac{4}{3}}\right), \qquad x \to 0_+, \end{aligned} \tag{5.45}$$

where $I_n(z)$ and $K_n(z)$ are the modified Bessel functions of the first and second kinds respectively, and A and B are integration constants. Comparing the latter to (1.57) we see that the coefficient of the $\mathcal{O}\left(x^{-\frac{1}{3}}\right)$ term corresponds to λ_i, which we must set to zero to avoid destroying our dual AdS_4 fixed point. Also, the $\mathcal{O}\left(x^{\frac{2}{3}}\right)$ term will vanish automatically.

Now putting $\phi \sim x^{\frac{1}{3}}$, $\phi \sim x^{\frac{4}{3}}$ and $c_2 \sim x^{-\frac{1}{3}}a(x)$ into (5.38), we get the asymptotic form

$$a'' + \frac{2}{x}a' + \frac{1}{36}x^{-\frac{4}{3}}a = 0, \tag{5.46}$$

which has the general solution

$$\begin{aligned} a &= C\left[x^{-\frac{2}{3}}\cos\left(\frac{x^{\frac{1}{3}}}{2}\right) - \frac{2}{x}\sin\left(\frac{x^{\frac{1}{3}}}{2}\right)\right] \\ &\sim \mathcal{O}(1) + \mathcal{O}\left(x^{\frac{2}{3}}\right), \qquad x \to 0_+ \end{aligned} \tag{5.47}$$

where C is an integration constant, and we fixed the other integration constant by enforcing that a be a real function of x. Now we have enough information to make an educated guess at the form of the asymptotic solution near the boundary. Let us take

$$a = \alpha\left(1 + C_1 x^{\frac{2}{3}} + C_2 x + ...\right) \tag{5.48}$$

$$\phi = p_1 x^{\frac{1}{3}} + p_2 x^{\frac{2}{3}} + p_3 x + ... \tag{5.49}$$

$$\chi = \chi_2 x^{\frac{2}{3}} + \chi_3 x + ... \tag{5.50}$$

Substituting these ansatze into (5.38)-(5.40), expanding about $x = 0$, then insisting that coefficients of powers of x vanish gives

$$a = \alpha\left(1 - \frac{1}{40} p_1^2 x^{\frac{2}{3}} - \frac{1}{18} p_1 p_2 x + O\left(x^{\frac{4}{3}}\right)\right) \tag{5.51}$$

$$\phi = p_1 x^{\frac{1}{3}} + p_2 x^{\frac{2}{3}} + \frac{3}{20} p_1^3 x + O\left(x^{\frac{4}{3}}\right) \tag{5.52}$$

$$\chi = \chi_4\left(x^{\frac{4}{3}} + \left(\frac{1}{7}g - \frac{3}{70}\right) p_1^2 x^2 + O\left(x^{\frac{7}{3}}\right)\right) \tag{5.53}$$

The integration constants at the boundary $\{p_1, p_2, \chi_4\}$ have the following interpretation in the dual field theory: p_1 is the coupling of the relevant operator O_r that deforms the CFT dual to pure AdS_4, p_2 is $\langle O_r \rangle$, and χ_4 is $\langle O_i \rangle$. Note that $\Delta_{O_r} = 2$ so p_1 is a dimensionless coupling with we take to be proportional to $\frac{m}{T}$ where m is the mass of the deformation.

5.4 Thermodynamics

First let us find the temperature. The calculation is done for a general metric in Appendix A. Transforming to the y coordinate, the metric becomes

$$ds^2 = -y^2 c_2^2 dt^2 + c_2^2\left(dx_1^2 + dx_2^2\right) - \frac{(\phi')^2 + (\chi')^2 - 4\left(\ln c_2\right)'\left(\ln y^2 c_2^3\right)'}{2c_3^2\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2 + 6\right)} dy^2, \tag{5.54}$$

where we used (5.16) in the first component, and we used the chain rule $dy = \frac{dy}{dr}dr$ and (5.28) in the last component. Now we expand the metric about the horizon $y = 0$. We have

$$c_2^2 \sim \alpha^2 a_0^2, \tag{5.55}$$

and

$$\frac{(\phi')^2 + (\chi')^2 - 4(\ln c_2)'\left(\ln y^2 c_2^3\right)'}{2c_3^2(\phi^2 - 2\chi^2 - g\phi^2\chi^2 + 6)} \sim 16\left(\frac{3a_1 + a_0}{6a_0\left(6 - 2c_0^2 + p_0^2 - gp_0^2c_0^2\right)}\right) \equiv 16A^2, \tag{5.56}$$

where A^2 is the quantity in large parentheses. Changing to imaginary time via $t \to it_E$, we can put the metric in the form

$$ds^2 = K\left(dy^2 + y^2 d\left(\frac{\alpha^2 a_0^2}{16A^2}t_E^2\right)\right) + \alpha^2 a_0^2 d\mathbf{x}^2, \tag{5.57}$$

where K is an overall constant. We remove the conical singularity by requiring that

$$t_E = t_E + 2\pi\left(\frac{4A}{\alpha a_0^{\frac{3}{2}}}\right). \tag{5.58}$$

Identifying the period of t_E with $1/T$ and restoring A we find the temperature

$$\left(\frac{8\pi T}{\alpha}\right)^2 = \frac{6a_0^3\left(6 - 2c_0^2 + p_0^2 - gp_0^2c_0^2\right)}{3a_1 + a_0}. \tag{5.59}$$

Next we will calculate the entropy. Recall that $c_2 = a(y)(1 - y^2)^{-\frac{1}{3}}$, and that the horizon is given by $y = 0$, and the induced metric is

$$ds^2 = c_2^2(dx_1^2 + dx_2^2), \tag{5.60}$$

whose determinant is $\alpha^2 a_0^2$. Using the Hawking formula for the entropy we get

$$s = \frac{\iint \alpha^2 a_0^2 dx_1 dx_2}{4G}, \tag{5.61}$$

or

$$\mathfrak{s} \equiv \frac{s}{V} = 2\pi \left(\frac{\alpha a_0}{\kappa}\right)^2 = \frac{c}{384} 4\pi\alpha^2 a_0^2, \tag{5.62}$$

where $V \equiv \iint dx_1 dx_2$ and we used (3.18) in the last equality.

We can calculate the free energy by calculating a finite value for the action. We can show that the equations of motion suggest

$$\Gamma \equiv 4c_2'' c_1 c_3 - 4c_2' c_3' c_1 - c_1 c_2 c_3^3 \left(6 - g\phi^2\chi^2\right) + \frac{1}{2} c_1 c_2 c_3 \left((\phi')^2 + (\chi')^2\right) = 0. \tag{5.63}$$

Then, defining the quantity in parentheses in (5.5) to be $\tilde{\mathcal{L}}$ we find that

$$\tilde{\mathcal{L}} + \frac{c_2}{c_3^2}\Gamma = -2\frac{d}{dr}\left(\frac{c_1' c_2^2}{c_3}\right), \tag{5.64}$$

and since $\Gamma = 0$ we can write the bulk action as

$$\frac{S_{bulk}}{V} = -\frac{\tau}{2\kappa^2}\left(\frac{2c_1' c_2^2}{c_3}\right)\Bigg|_{horizon}^{boundary}, \tag{5.65}$$

where $\tau = \int_0^\tau dt$. It is straight forward to calculate the boundary action

$$S_{boundary} = -\frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} d^3x \sqrt{-\gamma}\Theta \tag{5.66}$$

or

$$\frac{S_{boundary}}{V} = -\frac{\tau}{2\kappa^2}\left(2\frac{c_2^2 c_1'}{c_3} + 4\frac{c_1 c_2 c_2'}{c_3}\right)\Bigg|_{boundary}. \tag{5.67}$$

Adding the parts together we get

$$\frac{S}{V} = \frac{\tau}{2\kappa^2}\left[2\left(\frac{c_1' c_2^2}{c_3}\right)\Bigg|_{horizon} - 4\left(\frac{c_1 c_2 c_2'}{c_3}\right)\Bigg|_{boundary}\right] \tag{5.68}$$

Equation (5.68) is written in terms of the r variable. In order to make use of our asymptotic solutions, we must write the first term in terms of the y variable, and the second term in terms of the x variable. Although it is tedious, the calculation is straightforward. Take $c_1 = yc_2$ and use equation (5.28), where under $y \to 1 - x$ we have $dy/dr \to -dx/dr$. Then use the appropriate expansion in each term. The result is

$$\frac{S}{V} = \frac{\tau}{2\kappa^2}\left(-\frac{1}{2}\alpha^3 a_0^3 \sqrt{\frac{6a_0\left(6 + p_0^2 - 2c_0^2 - g p_0^2 c_0^2\right)}{3a_1 + a_0}} - \frac{2\alpha^3}{x} - \frac{1}{10}\frac{\alpha^3 p_1^2}{x^{\frac{1}{3}}} + 3\alpha^3 - \frac{1}{3}\alpha^3 p_1 p_2 \right). \tag{5.69}$$

The action diverges as $x \to 0$. We may add a counterterm of the form

$$\frac{1}{2\kappa^2} S_{ct} = \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} d^3x \sqrt{-\gamma}\left(\beta_1 + \beta_2 \phi^2\right). \tag{5.70}$$

Since this expression is diffeomorphism invariant on the boundary and contains no derivatives of the metric or scalar fields, we are sure that adding it to the action will not alter the equations of motion. Straightforward calculation gives

$$\frac{1}{2\kappa^2}\frac{S_{ct}}{V} = \frac{\tau}{2\kappa^2}\left(\frac{1}{2}\frac{\beta_1\alpha^3}{x} + \frac{\frac{-3}{80}\beta_1\alpha^3 p_1^2 + \frac{1}{2}\beta_2\alpha^3 p_1^2}{x^{\frac{1}{3}}} - \beta_1\left(\frac{1}{12}\alpha^3 p_1 p_2 + \frac{1}{4}\alpha^3\right) + \beta_2\alpha^3 p_1 p_2 \right). \tag{5.71}$$

Choosing $\beta_1 = 4$ and $\beta_2 = 1/2$ we get a finite action

$$\frac{S}{V} = \frac{\tau}{2\kappa^2}\left(2\alpha^3 - \frac{1}{6}\alpha^3 p_1 p_2 - \frac{1}{2}\alpha^3 a_0^3 \sqrt{\frac{6a_0\left(6 + p_0^2 - 2c_0^2 - g p_0^2 c_0^2\right)}{3a_1 + a_0}} \right). \tag{5.72}$$

Taking $\tau = i/T$ and $F = TS_E$ we find the free energy density

$$\mathcal{F} \equiv \frac{F}{V} = \frac{1}{2\kappa^2}\left(2\alpha^3 - \frac{1}{6}\alpha^3 p_1 p_2 - \frac{1}{2}\alpha^3 a_0^3 \sqrt{\frac{6a_0\left(6 + p_0^2 - 2c_0^2 - gp_0^2c_0^2\right)}{3a_1 + a_0}}\right). \tag{5.73}$$

To find the mass/energy density we will calculate the stress-energy tensor using[2]

$$T_{\mu\nu} = \frac{1}{\kappa^2}\left(\Theta_{\mu\nu} - \Theta\gamma_{\mu\nu} + \frac{1}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{\mu\nu}}\right), \tag{5.74}$$

where

$$\frac{\delta S_{ct}}{\delta\gamma^{\mu\nu}} = \frac{1}{2}\sqrt{-\gamma}\gamma_{\mu\nu}\left(4 + \frac{\phi^2}{2}\right), \tag{5.75}$$

which follows from varying equation (5.70). Then using $\left\langle\hat{T}_{\mu\mu}\right\rangle = c_1 c_2^2 c_\mu^{-2} T_{\mu\mu}$, the result is

$$\left\langle\hat{T}_{\mu\nu}\right\rangle = \frac{1}{2\kappa^2}\begin{pmatrix} 2\alpha^3 - \frac{1}{6}\alpha^3 p_1 p_2 & 0 & 0 \\ 0 & \alpha^3 + \frac{1}{6}\alpha^3 p_1 p_2 & 0 \\ 0 & 0 & \alpha^3 + \frac{1}{6}\alpha^3 p_1 p_2 \end{pmatrix}. \tag{5.76}$$

This gives us the mass/energy density and pressure

$$\mathcal{E} \equiv \frac{M}{V} = \frac{1}{2\kappa^2}\left(2\alpha^3 - \frac{1}{6}\alpha^3 p_1 p_2\right), \quad \text{and} \quad P = \frac{1}{2\kappa^2}\left(\alpha^3 + \frac{1}{6}\alpha^3 p_1 p_2\right). \tag{5.77}$$

Now one may ask, why do we not immediately see that $P = -\mathcal{F}$. In fact, this equality holds, which we will show now. The Einstein equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\tilde{T}_{\mu\nu}. \tag{5.78}$$

[2] We do not have the factor of 2 in the last term because the entire action is normalized up to $1/2\kappa^2$.

where

$$\tilde{T}_{\mu\nu} = -2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}. \tag{5.79}$$

For our metric we have $R = 0$. The Taking our lagrangian density to be

$$\mathcal{L} = R + 6 - \frac{(\partial\phi)^2}{2} - \frac{(\partial\chi)^2}{2} + \phi^2 - 2\chi^2 - g\phi^2\chi^2, \tag{5.80}$$

we get

$$\tilde{T}_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) + (\partial_\mu\chi)(\partial_\nu\chi) + g_{\mu\nu}\mathcal{L}, \tag{5.81}$$

or

$$\tilde{T}^{\mu}{}_{\nu} = (\partial^\mu\phi)(\partial_\nu\phi) + (\partial^\mu\chi)(\partial_\nu\chi) + \delta^{\mu}{}_{\nu}\mathcal{L}. \tag{5.82}$$

Since our metric is diagonal and the scalars ϕ and χ depend only on R, then Einstein's equations guarantee that

$$R^{t}{}_{t} - R^{x_1}{}_{x_1} = 0. \tag{5.83}$$

Explicitly in terms of the r variable this is

$$\frac{c_1'c_2'}{c_1c_2c_3^2} + \frac{c_1'c_3'}{c_1c_3^3} - \frac{c_1''}{c_1c_3^2} + \frac{\left(c_2'\right)^2}{c_2^2c_3^2} - \frac{c_2'c_3'}{c_2c_3^3} + \frac{c_2''}{c_2c_3^2} = 0, \tag{5.84}$$

which can be written as

$$\frac{1}{c_1c_2^2c_3}\left(\left(\frac{c_1}{c_2}\right)'\frac{c_2^3}{c_3}\right)' = 0. \tag{5.85}$$

This implies that

$$\left(\frac{c_1}{c_2}\right)'\frac{c_2^3}{c_3} = const \tag{5.86}$$

is a constant of motion. Converting to the y coordinate and expanding about the horizon we get

$$\lim_{y\to 0_+}\left(\frac{c_1}{c_2}\right)'\frac{c_2^3}{c_3} = \frac{1}{4}\alpha^3 a_0^3\sqrt{\frac{6a_0\left(6+p_0^2-2c_0^2-gp_0^2c_0^2\right)}{3a_1+a_0}}. \tag{5.87}$$

Then converting to the x variable and expanding about the boundary we get

$$\lim_{x\to 0_+}\left(\frac{c_1}{c_2}\right)'\frac{c_2^3}{c_3} = \frac{3}{2}\alpha^3. \tag{5.88}$$

Thus we have

$$3\alpha^3 = \frac{1}{2}\alpha^3 a_0^3\sqrt{\frac{6a_0\left(6+p_0^2-2c_0^2-gp_0^2c_0^2\right)}{3a_1+a_0}}. \tag{5.89}$$

Then we can write the pressure as

$$\begin{aligned} P &= \frac{1}{2\kappa^2}\left(-2\alpha^3+\frac{1}{6}\alpha^3 p_1 p_2 + 3\alpha^3\right) \\ &= \frac{1}{2\kappa^2}\left(-2\alpha^3+\frac{1}{6}\alpha^3 p_1 p_2 + \frac{1}{2}\alpha^3 a_0^3\sqrt{\frac{6a_0\left(6+p_0^2-2c_0^2-gp_0^2c_0^2\right)}{3a_1+a_0}}\right) \\ &= -\mathcal{F}. \end{aligned} \tag{5.90}$$

It is easy to check that the relation

$$\mathcal{F} = \mathcal{E} - Ts \tag{5.91}$$

holds.

5.5 Conformal limit

Let us first consider solutions where $\chi = 0$, which corresponds to the case where $\langle \mathcal{O}_i \rangle = 0$. Our equations of interest are

$$c_2'' - \frac{c_2'}{y} - 4\frac{\left(c_2'\right)^2}{c_2} + \frac{c_2}{4}\left(\phi'\right)^2 = 0 \tag{5.92}$$

$$\phi'' + \frac{\phi'}{y} - \frac{\phi\left(y\left(\phi'\right)^2 c_2^2 - 12y\left(c_2'\right)^2 - 8c_2'c_2\right)}{yc_2^2\left(6+\phi^2\right)} = 0 \tag{5.93}$$

To study the conformal limit, we should expand the solutions as a series of some small parameter δ_1 where

$$\delta_1 \propto \frac{m}{T} << 1, \tag{5.94}$$

where m is the mass that deforms the CFT, and is related to the scalar field ϕ. So the conformal limit corresponds to either[3] $m \to 0$ or $T \to \infty$. To leading order, we may write the solution as

$$c_2(y) = e^{A(y)}, \quad \text{and} \quad \phi(y) = \delta_1\phi_1(y), \tag{5.95}$$

where

$$A(y) = \ln\left(\alpha a_0\right) - \frac{1}{3}\ln\left(1-y^2\right) + \delta_1^2 A_1(y). \tag{5.96}$$

Putting this into (5.93) and expanding to leading order in δ_1 we get

$$\phi_1'' + \frac{\phi_1'}{y} + \frac{8}{9}\frac{\phi_1}{\left(1-y^2\right)^2} = 0. \tag{5.97}$$

[3]Thus we may use ”conformal limit” and ”high-temperature limit” interchangably

The general solution is found in terms of hypergeometric functions

$$\phi_1 = C_1 \left(1-y^2\right)^{\frac{1}{3}} {}_2F_1\left(\frac{1}{3},\frac{1}{3},\frac{2}{3};1-y^2\right) + C_2 \left(1-y^2\right)^{\frac{2}{3}} {}_2F_1\left(\frac{2}{3},\frac{2}{3},\frac{4}{3};1-y^2\right) \quad (5.98)$$

This solution is singular at the horizon $y = 0$. We must choose the integration constants so that we cancel the divergence at the horizon. Expanding about the horizon $y = 0$ we get

$$\phi_1 = -2C_1 \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)^2}\left(\gamma + \psi\left(\frac{1}{3}\right) + \ln y\right) - 2C_2 \frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}\right)^2}\left(\gamma + \psi\left(\frac{2}{3}\right) + \ln y\right), \quad (5.99)$$

where γ is the Euler-Mascheroni constant, and $\psi(x)$ is the digamma function. We can remove the logarithmic divergence if we take

$$C_2 = -\frac{\Gamma\left(\frac{2}{3}\right)^3}{\Gamma\left(\frac{1}{3}\right)^2 \Gamma\left(\frac{4}{3}\right)} C_1. \quad (5.100)$$

Now we have

$$\begin{aligned}\phi_1 = {} & \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)^2}\left(1-y^2\right)^{\frac{1}{3}} {}_2F_1\left(\frac{1}{3},\frac{1}{3},\frac{2}{3};1-y^2\right) \\ & - \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)}\left(1-y^2\right)^{\frac{2}{3}} {}_2F_1\left(\frac{2}{3},\frac{2}{3},\frac{4}{3};1-y^2\right),\end{aligned} \quad (5.101)$$

where we redefined our integration constant $C_1 = \tilde{C}_1 \Gamma\left(\frac{1}{3}\right) / \Gamma\left(\frac{2}{3}\right)^2$ and we absorbed $\tilde{C}_1$ into the expansion parameter δ_1 in equation (5.95). The solution (5.101) is still singular at the horizon $y = 0$. Using the transformation [1]

$$
{}_2F_1(a,b,c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b,a+b-c+1;1-z) + (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}{}_2F_1(c-a,c-b,c-a-b+1;1-z) \tag{5.102}
$$

we can analytically continue the solution to get

$$
\phi_1 = \left(1-y^2\right)^{\frac{1}{3}} {}_2F_1\left(\frac{1}{3},\frac{1}{3},1;y^2\right), \tag{5.103}
$$

which is well-behaved at the horizon. Changing (5.101) to the variable $x = 1 - y$ we get

$$
\phi_1 = \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)^2}\left(2x-x^2\right)^{\frac{1}{3}}{}_2F_1\left(\frac{1}{3},\frac{1}{3},\frac{2}{3};2x-x^2\right) - \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)}\left(2x-x^2\right)^{\frac{2}{3}}{}_2F_1\left(\frac{2}{3},\frac{2}{3},\frac{4}{3};2x-x^2\right), \tag{5.104}
$$

which is well-behaved at the boundary $x = 0$. Expanding (5.104) near the boundary we get

$$
\phi = \delta_1\phi_1 = \frac{2^{\frac{4}{3}}\pi}{\sqrt{3}\Gamma\left(\frac{2}{3}\right)^3}\delta_1 x^{\frac{1}{3}} - \frac{9\Gamma\left(\frac{2}{3}\right)^3}{2^{\frac{4}{3}}\pi^2}\delta_1 x^{\frac{2}{3}} + \mathcal{O}\left(x^{\frac{4}{3}}\right). \tag{5.105}
$$

Comparing to the asymptotic solution (5.52)

$$
\phi = p_1 x^{\frac{1}{3}} + p_2 x^{\frac{2}{3}} + \mathcal{O}(x) \tag{5.106}
$$

we see that

$$p_1 = \frac{2^{\frac{4}{3}}\pi}{\sqrt{3}\Gamma\left(\frac{2}{3}\right)^3}\delta_1, \quad \text{and} \quad p_2 = -\frac{9\Gamma\left(\frac{2}{3}\right)^3}{2^{\frac{4}{3}}\pi^2}\delta_1, \tag{5.107}$$

and

$$p_2 = -\frac{9\sqrt{3}\Gamma\left(\frac{2}{3}\right)^6}{2^{\frac{8}{3}}\pi^3}p_1. \tag{5.108}$$

Let us now turn to equation (5.92). Inserting (5.95) and expanding to leading order in δ_1 we get

$$A_1'' - \frac{\left(3y^2+1\right)}{y\left(1-y^2\right)}A_1' + \frac{1}{4}\left(\phi_1'\right)^2 = 0. \tag{5.109}$$

The solution is

$$A_1 = -\frac{1}{4}\int_0^y \frac{z}{\left(1-z^2\right)^2}dz\left(\gamma_1 + \int_0^z \left(\frac{\partial\phi_1}{\partial u}\right)^2 \frac{\left(1-u^2\right)^2}{u}du\right) + \gamma_2, \tag{5.110}$$

where γ_1 and γ_2 are integration constants that are fixed by the boundary conditions. First, note that from (5.95) and (5.96)

$$c_2(y) = \frac{\alpha a_0 e^{\delta_1^2 A_1(y)}}{\left(1-y^2\right)^{\frac{1}{3}}}, \quad \text{so} \quad a(y) = \alpha a_0 e^{\delta_1^2 A_1(y)}. \tag{5.111}$$

Comparing to the asymptotics in (5.34)

$$a(y) = \alpha\left(a_0 + a_1 y^2 + O(y^4)\right), \qquad y \to 0, \tag{5.112}$$

we see that we must insist that $A_1(y=0) = 0$, which fixes the integration constant

$$\gamma_2 = 0. \tag{5.113}$$

The solution (5.110) diverges at the boundary $y = 1$, which destroys the asymtotically AdS_4 nature of our spacetime. We must fix γ_1 such that it removes the divergence. The outer integral diverges only when $z = 1$. So we must fix

$$\gamma_1 = -\int_0^1 \left(\frac{\partial\phi_1}{\partial u}\right)^2 \frac{\left(1-u^2\right)^2}{u} du \approx -0.07689 \tag{5.114}$$

Now, let us find the leading corrections to the conformal thermodynamics that we found in chapter 3. Using the formula (A.7)

$$T = \frac{1}{2\pi}\left(\frac{c_2}{c_3}\frac{dy}{dr}\right)\bigg|_{horizon} \tag{5.115}$$

with $\chi = 0$ in (5.28),

$$\frac{dy}{dr} = c_3\sqrt{\frac{2\left(6+\phi^2\right)}{4A'(y)\left(\frac{2}{y}+3A'(y)\right)-(\phi')^2}} \tag{5.116}$$

and our solutions for $\phi(y) = \delta_1\phi_1(y)$ and $A(y) = \ln(\alpha a_0) - \frac{1}{3}\ln\left(1-y^2\right) + \delta_1^2 A_1(y)$, then expanding in powers of δ_1 we get the correction to the temperature,

$$T = \frac{\alpha a_0}{2\pi}\left[\frac{3}{2} + \frac{1}{32}\left(4+9\gamma_1\right)\delta_1^2 + O\left(\delta_1^4\right)\right]. \tag{5.117}$$

The entropy density is given by equation (5.2)

$$s = \frac{2\pi}{\kappa^2}\left(\alpha a_0\right)^2. \tag{5.118}$$

Using (5.117) to eliminate αa_0 from (5.118), then expanding in δ_1 we get the corrections to the entropy density.

$$s = \frac{2\pi^3 T^2}{\kappa^2}\left[\frac{16}{9} - \frac{2}{27}\left(4+9\gamma_1\right)\delta_1^2 + O\left(\delta_1^4\right)\right] \tag{5.119}$$

To get the corrections to the energy density and free energy density, we solve (5.117) for αa_0 and expand in δ_1 to get

$$\alpha a_0 = \pi T \left[\frac{4}{3} - \frac{1}{36} (4 + 9\gamma_1) \delta_1^2 + O\left(\delta_1^4\right) \right] \tag{5.120}$$

We can use the constant of motion (5.89) to simplify the expression for the temperature (5.59) to get

$$T = \frac{3\alpha}{4\pi a_0^2}. \tag{5.121}$$

Putting (5.121) into (5.120), we can solve for a_0 and expand in δ_1 to get

$$a_0 = 1 - \left(\frac{1}{36} + \frac{\gamma_1}{16} \right) \delta_1^2 + O\left(\delta_1^4\right) \tag{5.122}$$

We can write the energy density and free energy density from (5.77) as

$$\mathcal{E} = \frac{(\alpha a_0)^3}{2\kappa^2} \left(2 - \frac{1}{6} p_1 p_2 \right) \frac{1}{a_0^3}, \quad \text{and} \quad \frac{F}{V} = -\frac{(\alpha a_0)^3}{2\kappa^2} \left(2 - \frac{1}{6} p_1 p_2 \right) \frac{1}{a_0^3} \tag{5.123}$$

Inserting (5.107), (5.120) in the numerator and (5.122) in the denominator, then expanding in δ_1 we get

$$\mathcal{E} = \frac{64\pi^3 T^3}{27\kappa^2} - \frac{8\pi^2 T^3}{81\kappa^2} \left(9\pi\gamma_1 + 4\pi - 6\sqrt{3} \right) \delta_1^2 + O\left(\delta_1^4\right) \tag{5.124}$$

and

$$\mathcal{F} = -\frac{32\pi^3 T^3}{27\kappa^2} + \frac{4\pi^2 T^3}{81\kappa^2} \left(9\pi\gamma_1 + 4\pi + 12\sqrt{3} \right) \delta_1^2 + O\left(\delta_1^4\right). \tag{5.125}$$

The first law of thermodynamics must hold in this limit. Writing $\delta_1 = m/T$, the first law

$$\frac{\partial \mathcal{E}}{\partial T} - T \frac{\partial s}{\partial T} = 0 \quad \text{gives} \quad -\frac{8}{81} \left(\frac{\pi m}{\kappa} \right)^2 \left(9\pi\gamma_1 + 4\pi - 6\sqrt{3} \right) = 0. \tag{5.126}$$

This gives us an exact expression for γ_1

$$\gamma_1 = -\frac{2}{9\pi}\left(2\pi - 3\sqrt{3}\right) \approx -0.07689 \tag{5.127}$$

Finally, we can use the thermodynamic relation for the speed of sound

$$c_s^2 = -\frac{\partial \mathcal{F}}{\partial \mathcal{E}} = -\frac{\left(\frac{\partial \mathcal{F}}{\partial T}\right)}{\left(\frac{\partial \mathcal{E}}{\partial T}\right)} \tag{5.128}$$

which gives

$$c_s^2 = \frac{1}{2} - \frac{\sqrt{3}}{8\pi}\delta_1^2 + \mathcal{O}\left(\delta_1^4\right). \tag{5.129}$$

5.6 Numerical results

In this section we will summarize the input for our numerical procedure and give its results. A detailed description of our numerical method is given in the Appendix B, and the Mathematica code used to solve this particular system is given in Appendix C.

The equations that we need to solve are the following

$$c_2'' - \frac{c_2'}{y} - 4\frac{\left(c_2'\right)^2}{c_2} + \frac{c_2}{4}\left[(\phi')^2 + (\chi')^2\right] = 0 \tag{5.130}$$

$$\phi'' + \frac{\phi'}{y} - \frac{1}{2P}\frac{\partial P}{\partial \phi}\left[(\phi')^2 + (\chi')^2 - \frac{8}{y}\frac{c_2'}{c_2} - 12\frac{\left(c_2'\right)^2}{c_2^2}\right] = 0 \tag{5.131}$$

$$\chi'' + \frac{\chi'}{y} + \frac{1}{2P}\frac{\partial P}{\partial \chi}\left[(\phi')^2 + (\chi')^2 - \frac{8}{y}\frac{c_2'}{c_2} - 12\frac{\left(c_2'\right)^2}{c_2^2}\right] = 0, \tag{5.132}$$

where the prime denotes differentiation with respect to y. Near the horizon, say

$y = \epsilon = 10^{-6}$, we have the asymptotic solutions

$$a = \alpha\left(a_0 + a_1\epsilon^2 + O(\epsilon^4)\right) \tag{5.133}$$

$$\phi = p_0 + O(\epsilon^2) \tag{5.134}$$

$$\chi = c_0 + O(y\epsilon^2). \tag{5.135}$$

Since we will use these expansions as our initial values, we should find a few more higher-order terms. We also have, under $y = 1 - x$

$$c_2'' + \frac{c_2'}{1-x} - 4\frac{\left(c_2'\right)^2}{c_2} + \frac{c_2}{4}\left[(\phi')^2 + (\chi')^2\right] = 0 \tag{5.136}$$

$$\phi'' - \frac{\phi'}{1-x} - \frac{1}{2P}\frac{\partial P}{\partial \phi}\left[(\phi')^2 + (\chi')^2 + \left(\frac{8}{1-x}\right)\frac{c_2'}{c_2} - 12\frac{\left(c_2'\right)^2}{c_2^2}\right] = 0 \tag{5.137}$$

$$\chi'' - \frac{\chi'}{1-x} + \frac{1}{2P}\frac{\partial P}{\partial \chi}\left[(\phi')^2 + (\chi')^2 + \left(\frac{8}{1-x}\right)\frac{c_2'}{c_2} - 12\frac{\left(c_2'\right)^2}{c_2^2}\right] = 0, \tag{5.138}$$

where the prime denotes differentiation with respect to x. Near the boundary, say $x = \epsilon = 10^{-6}$, we have the asymptotic solutions

$$a = \alpha\left(1 - \frac{1}{40}p_1^2\epsilon^{\frac{2}{3}} - \frac{1}{18}p_1 p_2 \epsilon + O\left(\epsilon^{\frac{4}{3}}\right)\right) \tag{5.139}$$

$$\phi = p_1\epsilon^{\frac{1}{3}} + p_2\epsilon^{\frac{2}{3}} + \frac{3}{20}p_1^3\epsilon + O\left(\epsilon^{\frac{4}{3}}\right) \tag{5.140}$$

$$\chi = \chi_4\left(\epsilon^{\frac{4}{3}} + \left(\frac{1}{7}g - \frac{3}{70}\right)p_1^2\epsilon^2 + O\left(\epsilon^{\frac{7}{3}}\right)\right), \tag{5.141}$$

where, again, we should find a few higher-order terms. Since we will always plot dimensionless quantities, α will always cancel out of our calculations, so in practice

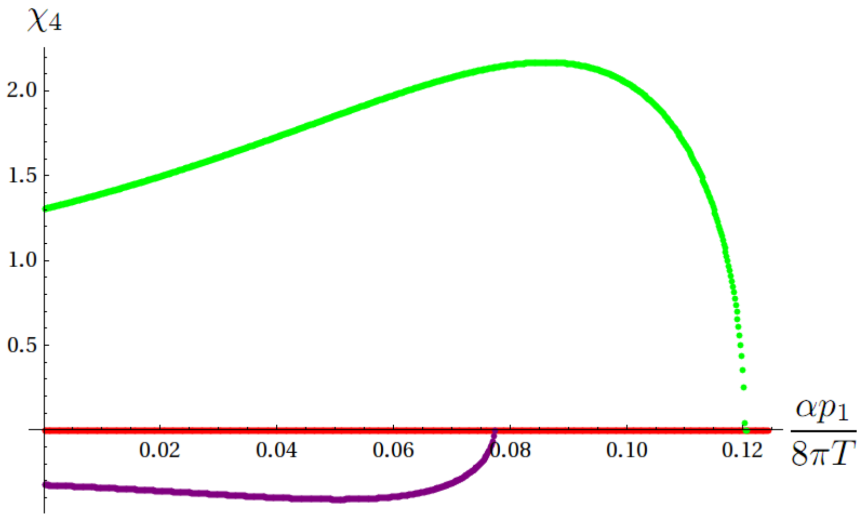

Figure 5.1: Order parameter χ_4 as a function of $\alpha p_1/8\pi T$. The red curve corresponds to the symmetric phase where $\chi_4 = 0$. The purple curve corresponds to the first symmetry-broken phase which exists for $\alpha p_1/8\pi T < 0.0772787$. The green curve corresponds to the second symmetry-broken phase.

we just set it to $\alpha = 1$. We have to fix the parameters p_1, p_2, χ_4, a_0, a_1, p_0, and c_0. We vary p_1 and find all other parameters as functions of it. Starting with a small value of p_1, we may use the pure AdS_4 geometry to guess good starting values for the other parameters. In particular, for $p_1 \approx 0$, we have $p_2 \approx \chi_4 \approx a_1 \approx p_0 \approx c_0 \approx 0$, and $a_0 \approx 1$. This configuration of starting values for the parameters will lead to us finding solutions where $\chi = 0$ for all p_1. However, we may numerically find branches of solutions where $\chi \neq 0$ for certain ranges of p_1. One may wonder why we do eliminate one of the parameters using the constant of motion (5.89). The reason is that the numerics completely destabilize.

Figure 5.1 shows a plot of χ_4 versus $\alpha p_1/8\pi T$. Notice that we can find multiple branches of solutions: the red curve corresponds to the symmetric phase where $\chi_4 = 0$ (or where $\langle O_i \rangle = 0$ in the CFT). The purple curve corresponds to the first symmetry-broken phase with $\chi_4 \neq 0$ (or where $\langle O_i \rangle \neq 0$ in the CFT) which exists for $\alpha p_1/8\pi T < 0.0772787$. The green curve corresponds to the second symmetry-broken phase. Although there is evidence to believe that there are many more (perhaps infinitely many) branches, the numerics on each successive branch are increasingly unstable. We will focus solely on the first two branches (i.e. the red and purple ones). This spontaneous symmetry breaking[4] is associated with a second order phase transition with χ_4 serving as the order parameter. The critical behaviour of our system at the phase transition will be studied in detail in chapter 7. An interesting feature of these phase transitions is that the phases with broken-symmetry exist for temperatures above the critical one. This is in contrast to most typical phase transitions where the symmetry-breaking occurs for low temperatures. For this reason, we consider the phase transitions we see here to be of an exotic type.

Figure 5.2 shows the numerical results for the entropy density. In the top figure, the red curve corresponds to the free energy calculated from (5.2), and the black dashed curve corresponds to (5.119). The excellent agreement indicates that our high-temperature analysis from the previous section is consistent with the full non-conformal thermodynamics. In the bottom figure, the free energy is plotted for the symmetric and symmetry-broken phases. The symmetric phase has the highest entropy, and each successive symmetry-broken phase has lower entropy.

Figure 5.3 shows the numerical results for the free energy. In the top figure, the red curve corresponds to the free energy calculated from (5.73), and the black dashed curve corresponds to (5.125). The excellent agreement indicates that our high-temperature analysis from the previous section is consistent with the full non-conformal thermodynamics. In the bottom figure, the free energy is plotted for the symmetric and symmetry-broken phases. Since the symmetric phase has the lowest free energy for all temperatures, the symmetry-broken phases are metastable at best, and a system in a symmetry broken phase would eventually decay into the

[4]The broken symmetry is the $\mathbb{Z}_2$ associated with the symmetry $\chi \to -\chi$.

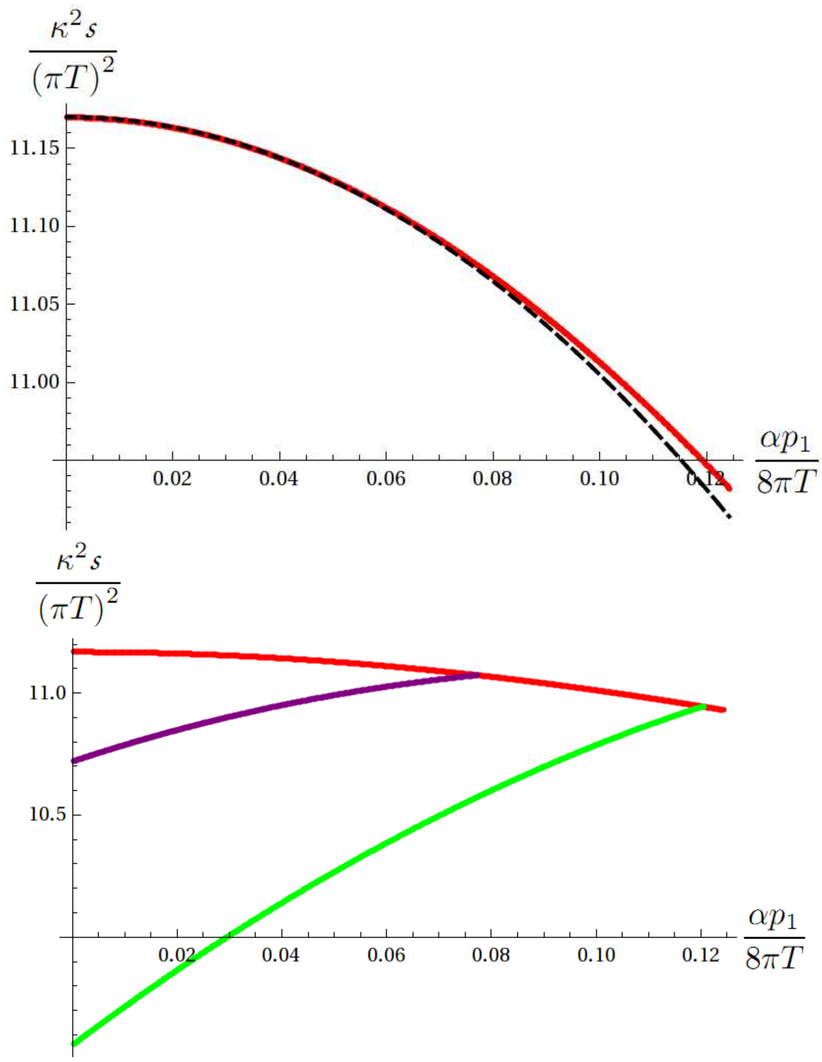

Figure 5.2: Entropy density as a function of $\alpha p_1/8\pi T$. Top: The red curve corresponds to the numerical results for equation (5.2) in the symmetric phase. The black dashed curve is the result from (5.119). Bottom: Numerical results in the symmetric and symmetry-broken phases. The colour scheme is the same as in figure 5.1.

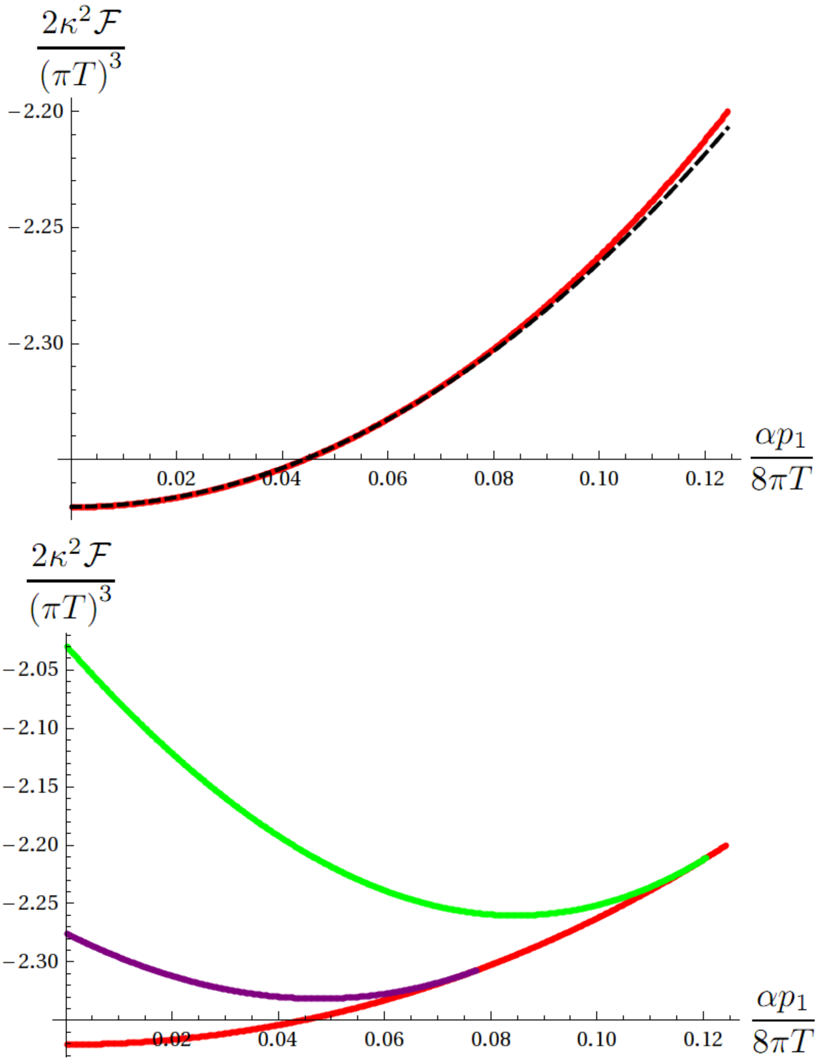

Figure 5.3: Free energy density as a function of $\alpha p_1/8\pi T$. Top: The red curve corresponds to the numerical results for equation (5.73) in the symmetric phase. The black dashed curve is the result from (5.125). Bottom: Numerical results in the symmetric and symmetry-broken phases. The colour scheme is the same as in figure 5.1.

symmetric one. We will study the perturbative stability of these symmetry-broken phases in chapter 7.

Figure 5.4 shows the results for the energy density (top) and the speed of sound (bottom). We find that the speed of sound is positive and finite in the symmetry-broken phases. Thus, these phases are thermodynamically stable [8].

All of the results presented so far have the coupling fixed at $g = -100$. Figure 5.5 shows the natural logarithm of the difference between the free energy density in the first symmetry-broken phase and the symmetric phase for $\alpha p_1/8\pi T \to 0$, versus $\ln(-g)$. As $g \to -\infty$ (i.e. large mixing between O_r and O_i), the symmetry-broken phases approach the symmetric phase at high temperatures ($\alpha p_1/8\pi T \to 0$). We found that no phase transition occurs when $g > 0$.

To summarize this chapter, we have computed the full non-conformal thermodynamics of our Exotic Model. We found the leading corrections in $\alpha p_1/T$ to the conformal thermodynamics in the high-temperature regime. We verified that the thermodynamics computed for our model are consistent with the first law of thermodynamics. We numerically computed the thermodynamic potentials, and we verified that our results are consistent with the high-temperature corrections. We found exotic phase transitions in our model, and we identified the order parameter characterizing them. Before analyzing the critical behaviour near the phase transition, we will study the hydrodynamics of our model in the next chapter.

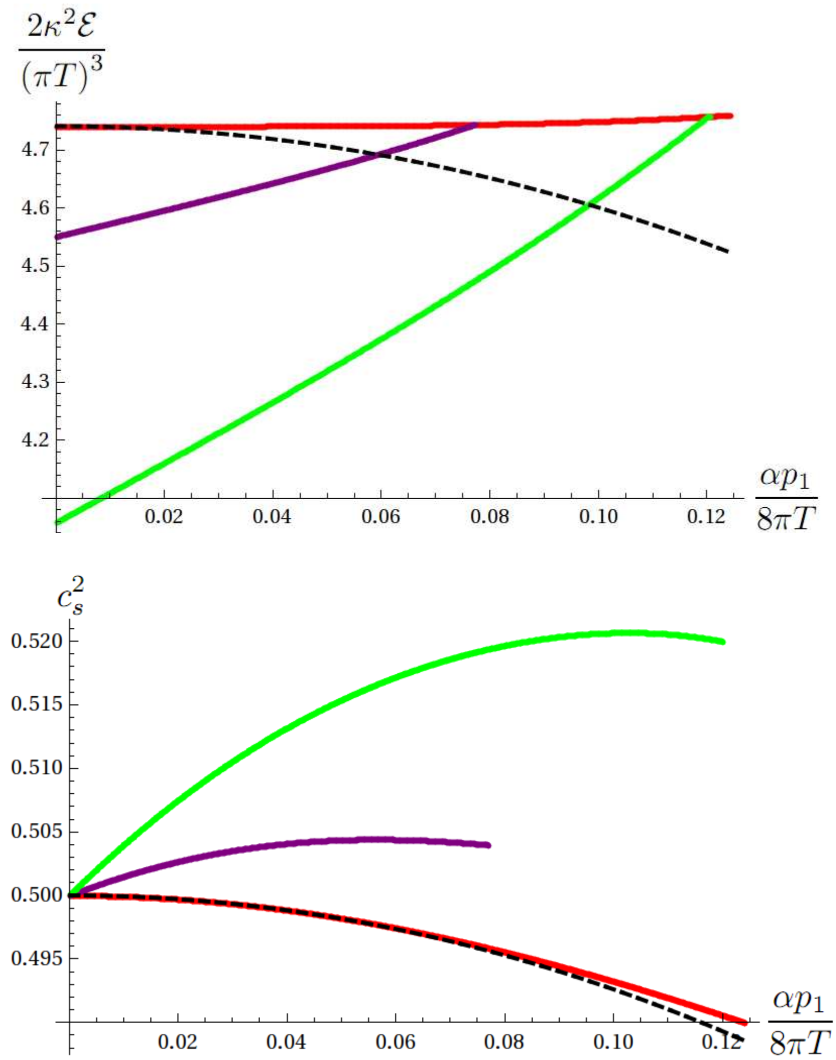

Figure 5.4: Top: Energy density. Bottom: Speed of sound. The colour scheme is the same as in figure 5.3.

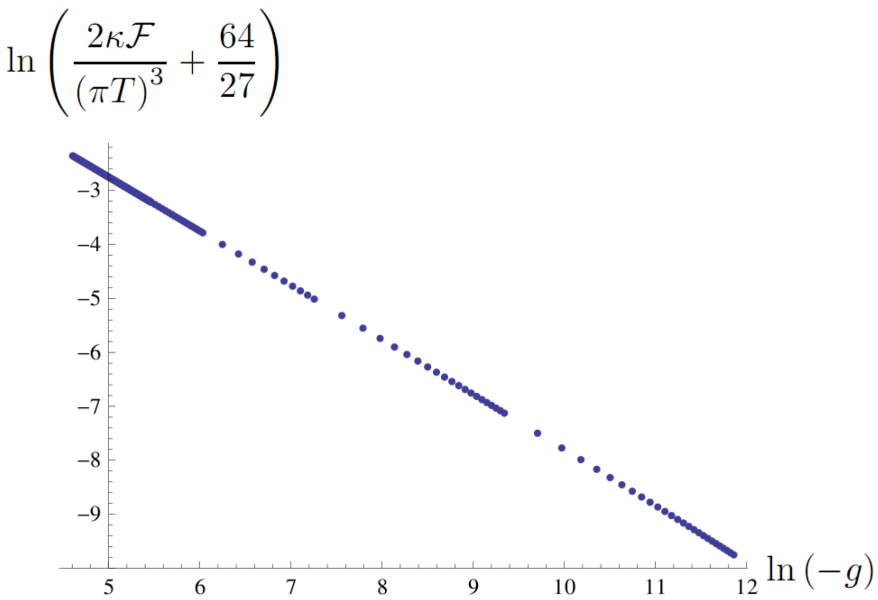

Figure 5.5: Natural logarithm of the difference between the free energy density in the first symmetry-broken phase and the symmetric phase for $\alpha p_1/8\pi T \to 0$, versus $\ln(-g)$.

Chapter 6

AdS4 black holes with scalar hair - fluctuations

In this chapter we will compute the hydrodynamics (i.e. speed of sound and bulk-to-shear viscosity ratio) by considering the dispersion relation of small fluctuations about the background geometry. We let

$$\begin{aligned} g_{\mu\nu} &\to g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \\ \phi &\to \phi' = \phi + A \\ \chi &\to \chi' = \chi + \Psi, \end{aligned} \tag{6.1}$$

where $h_{\mu\nu}$, A and Ψ are understood to be small fluctuations of the background geometry. Explicitly, the line element becomes

$$\begin{aligned} ds^2 =& \left(-c_1^2 + h_{tt}\right) dt^2 + \left(c_2^2 + h_{x_1x_1}\right) dx_1^2 + \left(c_2^2 + h_{x_2x_2}\right) dx_2^2 + \left(c_3^2 + h_{rr}\right) dr^2 \\ &+ 2h_{tx_1} dt dx_1 + 2h_{tx_2} dt dx_2 + 2h_{tr} dt dr + 2h_{x_1x_2} dx_1 dx_2 + 2h_{x_1r} dx_1 dr + 2h_{x_2r} dx_2 dr. \end{aligned} \tag{6.2}$$

Many of the calculations in this chapter are very tedious, so we will often give only the form of very long equations.

6.1 Equations of motion

The bulk action that gives rise to the equations of motion for the fluctuations $h_{\mu\nu}$ is

$$S_{bulk} = \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}'} d^4x\mathcal{L}_{bulk}, \tag{6.3}$$

where

$$\mathcal{L}_{bulk} = \sqrt{-g'}\left[\tilde{R} + 6 - \frac{1}{2}(\partial\phi')^2 - \frac{1}{2}(\partial\chi')^2 + (\phi')^2 - 2(\chi')^2 - g(\phi')^2(\chi')^2\right], \tag{6.4}$$

and $\tilde{R}$ is the scalar curvature calculated from the metric (6.2). Varying this action[1] with respect to the metric (6.2) while keeping only the terms in $\mathcal{L}_{bulk}$ that are at most quadratic[2] in $h_{\mu\nu}$, α, ψ and their derivatives, we get the equations of motion by demanding that $\delta S_{bulk} = 0$. All of the equations of motion are linear in $h_{\mu\nu}$ and its derivatives. Schematically, they have the following form,

$$E_{tt}(\partial_r^2 h_{x_1x_1}, \partial_r^2 h_{x_2x_2}, \partial_{x_2}^2 h_{x_1x_1}, \partial_r h_{x_1x_1}, \partial_r h_{x_2x_2}, \partial_r A, \partial_r\Psi, h_{x_1x_1}, h_{x_2x_2}, A, \Psi) = 0, \tag{6.5}$$

$$E_{tx_1}(\partial_{x_2}^2 h_{tx_1}, \partial_t\partial_{x_2} h_{x_1x_2}, \partial_r^2 h_{tx_1}, \partial_r h_{tx_1}, h_{tx_1}) = 0, \tag{6.6}$$

$$E_{tx_2}(\partial_r^2 h_{tx_2}, \partial_t\partial_{x_2} h_{x_1x_1}, \partial_r h_{tx_2}, h_{tx_2}) = 0, \tag{6.7}$$

$$E_{tr}(\partial_t\partial_r h_{x_2x_2}, \partial_t\partial_r h_{x_1x_1}, \partial_{x_2}\partial_r h_{tx_2}, \partial_t h_{x_1x_1}, \partial_t h_{x_2x_2}, \partial_{x_2} h_{tx_2}, \partial_t A, \partial_t\Psi) = 0, \tag{6.8}$$

---

[1] In principle we should be careful with the boundary terms by adding a Gibbons-Hawking term to the action.

[2] In this case, the linear terms vanish by virtue of the backround equations of motion (5.11)-(5.15). So the first non-vanishing terms are quadratic.

$$E_{x_1x_1}(\partial_t^2 h_{x_2x_2}, \partial_{x_2}^2 h_{tt}, \partial_r^2 h_{x_2x_2}, \partial_t\partial_{x_2} h_{tx_2}, \partial_r^2 h_{tt}, \partial_r h_{x_2x_2}, \partial_r h_{tt}, \partial_r A, h_{tt}, h_{x_2x_2}, A, \Psi) = 0, \tag{6.9}$$

$$E_{x_1x_2}(\partial_t^2 h_{x_1x_2}, \partial_t\partial_{x_2} h_{tx_1}, \partial_r^2 h_{x_1x_2}, \partial_r h_{x_1x_2}, h_{x_1x_2}) = 0, \tag{6.10}$$

$$E_{x_1r}(\partial_t\partial_r h_{tx_1}, \partial_{x_2}\partial_r h_{x_1x_2}, \partial_{x_2} h_{x_1x_2}, \partial_t h_{tx_1}) = 0, \tag{6.11}$$

$$E_{x_2x_2}(\partial_r^2 h_{tt}, \partial_r^2 h_{x_1x_1}, \partial_t^2 h_{x_1x_1}, \partial_r h_{tt}, \partial_r h_{x_1x_1}, \partial_r A, \partial_r \Psi, h_{x_1x_1}, h_{tt}, A, \Psi) = 0, \tag{6.12}$$

$$E_{x_2r}(\partial_{x_2}\partial_r h_{tt}, \partial_t\partial_r h_{tx_2}, \partial_{x_2}\partial_r h_{x_1x_1}, \partial_{x_2} A, \partial_{x_2}\Psi, \partial_{x_2} h_{x_1x_1}, \partial_{x_2} h_{tt}, \partial_t h_{tx_2}) = 0, \tag{6.13}$$

$$\begin{aligned} E_{rr}(&\partial_t^2 h_{x_2x_2}, \partial_{x_2}^2 h_{x_1x_1}, \partial_t\partial_{x_2} h_{tx_2}, \partial_t^2 h_{x_1x_1}, \partial_{x_2}^2 h_{tt}, \partial_r A, \partial_r \Psi, \partial_r h_{x_1x_1}, \partial_r h_{x_2x_2}, \\ &\partial_r h_{tt}, h_{x_1x_1}, h_{x_2x_2}, h_{tt}, A, \Psi) = 0, \end{aligned} \tag{6.14}$$

$$E_A(\partial_r^2 A, \partial_{x_2}^2 A, \partial_r A, \partial_r h_{x_2x_2}, \partial_r h_{x_1x_1}, \partial_r h_{tt}, h_{tt}, h_{x_1x_1}, h_{x_2x_2}, A) = 0, \tag{6.15}$$

$$E_\Psi(\partial_r^2 \Psi, \partial_{x_2}^2 \Psi, \partial_r \Psi, \partial_r h_{x_2x_2}, \partial_r h_{x_1x_1}, \partial_r h_{tt}, h_{tt}, h_{x_1x_1}, h_{x_2x_2}, \Psi) = 0, \tag{6.16}$$

where $E_{\mu\nu}$ is the equation of motion that arises from the variation of the action (6.3) with respect to the fluctuation $h_{\mu\nu}$, and E_A and E_Ψ arise from the variation of the action with respect to the fields A and Ψ. The equations $E_{\mu\nu}$, E_A, and E_Ψ are linear in their arguments whose coefficients[3] are functions of the background fields (c_1, c_2, c_3, ϕ, χ) and their derivatives. Also, after performing the variation of the action, we have used diffeomorphism invariance to fix four of the fluctuation components:

$$h_{tr} = h_{x_1r} = h_{x_2r} = h_{rr} = 0. \tag{6.17}$$

We are assuming that the fluctuations do not change the background; rather, they are propagating *in* the background. Every slice of spacetime with fixed r can be seen to be flat 2+1 dimensional Minkowski space, where $t \to t/c_1$ and $\mathbf{x} \to \mathbf{x}/c_2$. As such, the solutions can be separated into a radial part and a plane-wave part, i.e.

[3]The coefficients are not given explicitly because some of them are very long.

the $(t, \mathbf{x})$ part of the solutions will be eigenfunctions of the flat space wave equation. We may orient the coordinate axes such that the plane-waves propagate along the x_2 direction. Thus we may write the remaining fluctuations as

$$\begin{aligned} h_{tt} &= c_1(r)^2 H_{tt}(r) e^{-i\omega t + iqx_2} \\ h_{tx_2} &= c_2(r)^2 H_{tx_2}(r) e^{-i\omega t + iqx_2} \\ h_{x_1x_1} &= c_2(r)^2 H_{x_1x_1}(r) e^{-i\omega t + iqx_2} \\ h_{x_2x_2} &= c_2(r)^2 H_{x_2x_2}(r) e^{-i\omega t + iqx_2} \\ A &= \alpha(r) e^{-i\omega t + iqx_2} \\ \Psi &= \psi(r) e^{-i\omega t + iqx_2}, \end{aligned} \tag{6.18}$$

where ω and q are the frequency and momentum of the plane wave. The functions c_1^2 and c_2^2 are explicitly extracted for convenience. The equations of motion now take the form

$$E_{tt}(H''_{x_1x_1}, H''_{x_2x_2}, H'_{x_1x_1}, H'_{x_2x_2}, \alpha', \psi', H_{x_1x_1}, H_{x_2x_2}, \alpha, \psi) = 0, \tag{6.19}$$

$$E_{tx_2}(H''_{tx_2}, H'_{tx_2}, H_{tx_2}, H_{x_1x_1}) = 0, \tag{6.20}$$

$$E_{x_1x_1}(H''_{x_2x_2}, H''_{tt}, H'_{x_2x_2}, H'_{tt}, \alpha', H_{tt}, H_{x_2x_2}, H_{tx_2}, \alpha, \psi) = 0, \tag{6.21}$$

$$E_{x_2x_2}(H''_{tt}, H''_{x_1x_1}, H'_{tt}, H'_{x_1x_1}, \alpha', \psi', H_{x_1x_1}, H_{tt}, \alpha, \psi) = 0, \tag{6.22}$$

$$E_A(\alpha'', \alpha', H'_{x_2x_2}, H'_{x_1x_1}, H'_{tt}, H_{tt}, H_{x_1x_1}, H_{x_2x_2}, \alpha) = 0, \tag{6.23}$$

$$E_\Psi(\psi'', \psi', H'_{x_2x_2}, H'_{x_1x_1}, H'_{tt}, H_{tt}, H_{x_1x_1}, H_{x_2x_2}, \psi) = 0, \tag{6.24}$$

along with the first-order constraints

$$E_{tr}(H'_{x_2x_2}, H'_{x_1x_1}, H'_{tx_2}, \alpha, \psi, H_{x_1x_1}, H_{x_2x_2}, H_{tx_2}) = 0, \tag{6.25}$$

$$E_{x_2r}(H'_{tt}, H'_{tx_2}, H'_{x_1x_1}, \alpha, \psi, H_{x_1x_1}, H_{tt}, H_{tx_2}) = 0, \tag{6.26}$$

$$E_{rr}(\alpha', \psi', H'_{x_1x_1}, H'_{x_2x_2}, H'_{tt}, H_{x_1x_1}, H_{x_2x_2}, H_{tt}, H_{tx_2}, \alpha, \psi) = 0, \tag{6.27}$$

In addition, we have the following decoupled system,

$$E_{tx_1}(H''_{tx_1}, H'_{tx_1}, H_{tx_1}, H_{x_1x_2}) = 0, \tag{6.28}$$

$$E_{x_1x_2}(H''_{x_1x_2}, H'_{x_1x_2}, H_{x_1x_2}, H_{tx_1}) = 0, \tag{6.29}$$

$$E_{x_1r}(H'_{tx_1}, H'_{x_1x_2}, H_{x_1x_2}, H_{tx_1}) = 0. \tag{6.30}$$

We focus on the system (6.19)-(6.27) which consists of 6 second-order equations and 3 first-order constraints. The redundancy of these equations suggests that there exists additional diffeomorphic freedom that should be fixed. Following section 4.40 we find three diffeomorphism-invariant combinations of the fluctuations,

$$\begin{aligned} Z_H &= 2\frac{q^2}{\omega^2}\frac{c_1^2}{c_2^2}H_{tt} + 4\frac{q}{\omega}H_{tx_2} + 2H_{x_2x_2} + 2\left(\frac{q^2}{\omega^2}\frac{c_1c_1'}{c_2c_2'} - 1\right)H_{x_1x_1}, \\ Z_\alpha &= \alpha - \frac{c_2c_3^2\phi'}{2c_2'}H_{x_1x_1} \\ Z_\psi &= \psi - \frac{c_2c_3^2\chi'}{2c_2'}H_{x_1x_1} \end{aligned} \tag{6.31}$$

We can solve (6.19)-(6.24) for the second derivatives, and we can solve (6.25)-(6.27) for the first derivatives. Schematically,

$$\begin{aligned} H''_{tt} &= ... \\ H''_{tx_2} &= ... \\ H''_{x_1x_1} &= ... \\ H''_{x_2x_2} &= ... \\ \alpha'' &= ... \\ \psi'' &= ... \end{aligned} \quad , \quad \text{and} \quad \begin{aligned} H'_{tx_2} &= ... \\ H'_{x_1x_1} &= ... \\ H'_{x_2x_2} &= ... \end{aligned} \tag{6.32}$$

We can use (6.31) to eliminate H_{tt}, α, and ψ from the right-hand sides of (6.32), then re-solve for the first derivatives $H'_{tx_2} = ...$, $H'_{x_1x_1} = ...$, and $H'_{x_2x_2} = ...$ Now we demand consistency among the second-order equations and the first-order constraints,

$$\begin{aligned} \frac{d^2 H_{x_1x_1}}{dr^2} &= \frac{d}{dr}\left(\frac{dH_{x_1x_1}}{dr}\right) \\ \frac{d^2\alpha}{dr^2} &= \frac{d}{dr}\left(\frac{d\alpha}{dr}\right) \\ \frac{d^2\psi}{dr^2} &= \frac{d}{dr}\left(\frac{d\psi}{dr}\right), \end{aligned} \tag{6.33}$$

where the left-hand sides are the second derivatives from (6.32), and the right-hand sides are derivatives of the first derivatives in (6.32). It is straightforward, albeit tedious, to show that the equations resulting from (6.33) do not contain any $H_{\mu\nu}$ components or α or ψ; this happens by virtue of the background equations (5.11)-

(5.15). The final diffeomorphism-invariant equations in the r variable are[4]

$$
\begin{aligned}
&A_H Z_H'' + B_H Z_H' + C_H Z_H + E_H Z_\alpha + G_H Z_\psi = 0,\\
&A_\alpha Z_\alpha'' + B_\alpha Z_\alpha' + C_\alpha Z_\alpha + E_\alpha Z_\psi + F_\alpha Z_H' + G_\alpha Z_H = 0,\\
&A_\psi Z_\psi'' + B_\psi Z_\psi' + C_\psi Z_\psi + E_\psi Z_\alpha + F_\psi Z_H' + G_\psi Z_H = 0,
\end{aligned} \tag{6.34}
$$

$$
A_H = 8q^2\omega^2 \left(c_2'\right)^3 c_1^2 c_2^4 c_3 \left(q^2 c_1' c_1 c_2 - 2\omega^2 c_2' c_2^2 + q^2 c_2' c_1^2\right), \tag{6.35}
$$

$$
\begin{aligned}
B_H =& 8q^2\omega^2 \left(c_2'\right)^2 c_1 c_2^3 \Big[q^2 c_1^2 c_2^2 c_3^3 \left(c_1' c_2 - c_2' c_1\right)\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2\right)\\
&-q^2 \left(c_2'\right)^2 c_3' c_1^3 c_2 - 3q^2 \left(c_1'\right)^2 c_2' c_1 c_2^2 c_3 - q^2 c_1' c_2' c_3' c_1^2 c_2^2 + 6q^2 \left(c_2'\right)^3 c_1^3 c_3\\
&+6q^2 c_1' c_1^2 c_2^3 c_3^3 - 6q^2 c_2' c_1^3 c_2^2 c_3^3 + 3q^2 c_1' c_2' c_1^2 c_2 c_3 - 2\omega^2 c_1' \left(c_1'\right)^2 c_2^3 c_3\\
&+2\omega^2 \left(c_2'\right)^2 c_3' c_1 c_2^3 - 4\omega^2 \left(c_2'\right)^3 c_1 c_2^2 c_3\Big]
\end{aligned} \tag{6.36}
$$

$$
\begin{aligned}
C_H =& 8q^2\omega^2 \left(c_2'\right)^2 c_2^2 c_3^2 \Big[q^2 c_1^2 c_2^2 c_3^2 \left(c_2' c_1 - c_1' c_2\right)\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2\right)\\
&-2\omega^4 \left(c_2'\right)^2 c_2^4 c_3^2 - q^4 \left(c_2'\right)^2 c_1^4 c_3^2 - q^4 c_1' c_2' c_1^3 c_2 c_3^2 + 3q^2\omega^2 \left(c_2'\right)^2 c_1^2 c_2^2 c_3^2\\
&+12q^2 c_1' c_2' c_1^3 c_2^3 c_3^2 - 6q^2 \left(c_1'\right)^2 c_1^2 c_2^4 c_3^2 - 6q^2 \left(c_2'\right)^2 c_1^4 c_2^2 c_3^2 - 4q^2 c_1' \left(c_2'\right)^2 c_1^3 c_2\\
&-4q^2 \left(c_1'\right)^2 \left(c_2'\right)^2 c_1^2 c_2^2 + 4q^2 \left(c_1'\right)^3 c_2' c_1 c_2^3 + 4q^2 \left(c_2'\right)^4 c_1^4 + q^2\omega^2 c_1' c_2' c_1 c_2^3 c_3^2\Big]
\end{aligned} \tag{6.37}
$$

$$
\begin{aligned}
E_H =& 16q^4 \left(c_2'\right)^2 c_1^2 c_2^2 c_3 \left(c_1' c_2 - c_2' c_1\right) \Big[c_1 c_2 c_3^2 \phi' \left(\omega^2 c_2^2 - q^2 c_1^2\right)\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2\right)\\
&-2c_1 c_3^2 \phi \left(q^2 c_2' c_1^2 + q^2 c_1' c_1 c_2 - 2\omega^2 c_2' c_2^2\right)\left(1 - g\chi^2\right)\\
&-2c_2 \phi' \left(3q^2 c_1^3 c_3^2 - 3\omega^2 c_1 c_2^2 c_3^2 + 2\omega c_1' c_2' c_2 - 2\omega^2 \left(c_2'\right)^2 c_1\right)\Big]
\end{aligned} \tag{6.38}
$$

[4] Beware typos in the expressions for the coefficients.

$$G_H = 16q^4 \left(c_2'\right)^2 c_1^2 c_2^2 c_3 \left(c_2' c_1 - c_1' c_2\right) \Big[c_1 c_2 c_3^2 \chi' \left(q^2 c_1^2 - \omega^2 c_2^2\right)\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2\right)$$
$$-2c_1 c_3^2 \chi \left(q^2 c_2' c_1^2 - 2\omega^2 c_2' c_2^2 + q^2 c_1' c_1 c_2\right)\left(2 + g\phi^2\right)$$
$$+2c_2 \chi' \left(3q^2 c_1^3 c_3^2 - 3\omega^2 c_1 c_2^2 c_3^2 + 2\omega^2 c_1' c_2' c_2 - 2\omega^2 \left(c_2'\right)^2 c_2\right)\Big] \tag{6.39}$$

$$A_\alpha = 4q^2 \left(c_2'\right)^2 c_1^3 c_2^2 c_3 \left(2\omega^2 c_2' c_2^2 - q^2 c_2' c_1^2 - q^2 c_1' c_1 c_2\right) \tag{6.40}$$

$$B_\alpha = 4q^2 \left(c_2'\right)^3 c_1^2 c_2 \left(q^2 c_1' c_1 c_2 + q^2 c_2' c_1^2 - 2\omega^2 c_2' c_2^2\right)\left(c_3' c_1 c_2 - c_1' c_2 c_3 - 2c_2' c_1 c_3\right) \tag{6.41}$$

$$C_\alpha = 2q^2 \left(c_2'\right)^2 c_1 c_3^3 \Big[c_1^2 c_2^4 \left(\phi'\right)^2 \left(w^2 c_2^2 - q^2 c_1^2\right)\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2\right)$$
$$+2c_1^2 c_2^2 \Big(4\omega^2 \left(c_2'\right)^2 c_2^2 + 4\omega^2 c_2' \phi' c_2^3 \phi - 2q^2 \left(c_2'\right)^2 c_1^2 - q^2 c_1' \phi' c_1 c_2^2 \phi$$
$$-3q^2 c_2' \phi' c_1 c_2 \phi - 2q^2 c_1' c_2' c_1 c_2\Big)\left(1 - g\chi^2\right)$$
$$+2\left(\omega^2 c_2^2 - q^2 c_1^2\right)\left(3\left(\phi'\right)^2 c_1^2 c_2^4 + 2\omega^2 \left(c_2'\right)^2 c_2^2 - q^2 c_1' c_2' c_1 c_2 - q^2 \left(c_2'\right)^2 c_1^2\right)\Big] \tag{6.42}$$

$$E_\alpha = 2q^2 \left(c_2'\right)^2 c_1^3 c_2^2 c_3^3 \Big[c_2^2 \phi' \chi' \left(\omega^2 c_2^2 - q^2 c_1^2\right)\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2\right)$$
$$+2\phi' c_2 \chi \left(q^2 c_1' c_1 c_2 + q^2 c_2' c_1^2 - 2\omega^2 c_2' c_2^2\right)\left(2 + g\phi^2\right)$$
$$+\left(\omega^2 c_2^2 - q^2 c_1^2\right)\left(4c_2' \chi' c_2 \phi \left(1 - g\chi^2\right) + 6\phi' \chi' c_2^2\right)$$
$$+8g c_2' \phi \chi \left(q^2 c_1' c_1 c_2 + q^2 c_2' c_1^2 - 2\omega^2 c_2' c_2^2\right)\Big] \tag{6.43}$$

$$F_\alpha = q^2 \omega^2 \left(c_2'\right)^2 c_1^3 c_2^5 c_3^3 \left[\phi' c_2 \left(6 + \phi^2 - 2\chi^2 - g\phi^2\chi^2\right) + 4c_2' \phi \left(1 - g\chi^2\right)\right] \tag{6.44}$$

$$G_\alpha = q^2 \omega^2 \left(c_2'\right)^2 c_1^2 c_2^4 c_3^3 \left(c_1' c_2 - c_2' c_1\right)\left[4c_2' \phi \left(1 - g\chi^2\right) - \phi' c_2 \left(6 + \phi^2 - 2\chi^2 - g\phi^2\chi^2\right)\right] \tag{6.45}$$

$$A_\psi = 4q^2 (c_2')^2 c_1^3 c_2^2 c_3 \left(q^2 c_1' c_1 c_1 + q^2 c_2' c_1^2 - 2\omega^2 c_2' c_2^2\right) \tag{6.46}$$

$$B_\psi = 4q^2 (c_2')^3 c_1^2 c_2 \left(q^2 c_1' c_1 c_2 + q^2 c_2' c_1^2 - 2\omega^2 c_2' c_2^2\right) (2c_2' c_1 c_3 - c_3' c_1 c_2 + c_1' c_2 c_3) \tag{6.47}$$

$$\begin{aligned} C_\psi = {} & 2q^2 (c_2')^2 c_1 c_3^3 \Big[(\chi')^2 c_1^2 c_2^4 \left(q^2 c_1^2 - \omega^2 c_2^2\right)\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2\right) \\ & -2c_1^2 c_2^2 \Big(q^2 c_1' \chi' c_1 c_2^2 \chi + 3q^2 c_2' \chi' c_1^2 c_2 \chi + 2q^2 c_1' c_2' c_1 c_2 + 2q^2 (c_2')^2 c_1^2 \\ & -4\omega^2 c_2' \chi' c_2^3 \chi - 4\omega^2 (c_2')^2 c_2^2\Big) \\ & -2\left(q^2 c_1^2 - \omega^2 c_2^2\right)\left(q^2 (c_2')^2 c_1^2 + q^2 c_1' c_2' c_1 c_2 - 2\omega^2 (c_2')^2 c_2^2 - 3(\chi')^2 c_1^2 c_2^4\right)\Big] \end{aligned} \tag{6.48}$$

$$\begin{aligned} E_\psi = {} & 2q^2 (c_2')^2 c_1^3 c_2^2 c_3^3 \Big[\phi' \chi' c_2^2 \left(q^2 c_1^2 - \omega^2 c_2^2\right)\left(\phi^2 - 2\chi^2 - g\phi^2\chi^2\right) \\ & +2\chi' c_2 \phi \left(q^2 c_2' c_1^2 + q^2 c_1' c_1 c_2 - 2\omega^2 c_2' c_2^2\right)\left(1 - g\chi^2\right) \\ & \left(q^2 c_1^2 - \omega^2 c_2^2\right)\left(6\phi' \chi' c_2^2 - 4c_2' \phi' c_2 \chi \left(2 + g\phi^2\right)\right) \\ & -8g c_2' \phi \chi \left(q^2 c_2' c_1^2 + q^2 c_1' c_1 c_2 - 2\omega^2 c_2' c_2^2\right)\Big] \end{aligned} \tag{6.49}$$

$$F_\psi = q^2 \omega^2 (c_2')^2 c_1^3 c_2^5 c_3^3 \left[4c_2' \chi \left(2 + g\phi^2\right) - \chi' c_2 \left(6 + \phi^2 - 2\chi^2 - g\phi^2\chi^2\right)\right] \tag{6.50}$$

$$G_\psi = q^2 \omega^2 (c_2')^2 c_1^2 c_2^4 c_3^3 (c_1' c_2 - c_2' c_1) \left[\chi' c_2 \left(6 + \phi^2 - 2\chi^2 - g\phi^2\chi^2\right) - 4c_2' \chi \left(2 + g\phi^2\right)\right] \tag{6.51}$$

Using the change of variables (5.16), (5.22) and (5.28)

$$\begin{aligned} & y = \frac{c_1}{c_2} \qquad \left(\frac{dy}{dr}\right)^2 = \frac{-2c_3^2 \left(\phi^2 - 2\chi^2 - g\phi^2\chi^2 + 6\right)}{(\phi')^2 + (\chi')^2 - 4(\ln c_2)' \left(\ln y^2 c_2^3\right)'} \\ & \frac{d}{dr} = \frac{dy}{dr}\frac{d}{dy}, \qquad \frac{d^2}{dr^2} = \left(\frac{dy}{dr}\right)^2 \frac{d^2}{dy^2} + \frac{d^2 y}{dr^2}\frac{d}{dy}. \end{aligned} \tag{6.52}$$

we can cast the equations into the following form

$$\begin{aligned}
&\mathcal{A}_H Z_H'' + \mathcal{B}_H Z_H' + \mathcal{C}_H Z_H + \mathcal{E}_H Z_\alpha + \mathcal{G}_H Z_\psi = 0, \\
&\mathcal{A}_\alpha Z_\alpha'' + \mathcal{B}_\alpha Z_\alpha' + \mathcal{C}_\alpha Z_\alpha + \mathcal{E}_\alpha Z_\psi + \mathcal{F}_\alpha Z_H' + \mathcal{G}_\alpha Z_H = 0, \\
&\mathcal{A}_\psi Z_\psi'' + \mathcal{B}_\psi Z_\psi' + \mathcal{C}_\psi Z_\psi + \mathcal{E}_\psi Z_\alpha + \mathcal{F}_\psi Z_H' + \mathcal{G}_\psi Z_H = 0,
\end{aligned} \tag{6.53}$$

where the prime now denotes differentiation with respect to y, and the new coefficients are expressed in terms of y. The system (6.53) governs the small gauge-invariant fluctuations of the background geometry.

6.2 Near-horizon behaviour and boundary conditions

It will be convenient to extract the leading behaviour of the fluctuations near the horizon. We make the substitutions

$$Z_H = y^n, \quad Z_\alpha = M_\alpha y^n, \quad Z_\psi = M_\psi y^n, \tag{6.54}$$

where M_α and M_ψ are propotionality constants, and we used the invariance under rescalings of the Z's to set $M_H = 1$. Putting this into the first line of (6.53) and expanding to leading order in y we get an equation of the form

$$K y^4 + \mathcal{O}\left(y^5\right) = 0, \tag{6.55}$$

where the coefficient K depends on n, as well as the background expansion coefficients $\{a_0, a_1, p_0, c_0\}$, the coupling g, and the frequency ω. Setting $K = 0$ and solving for n we get

$$n = \pm i\frac{4\omega}{\alpha}\sqrt{\frac{3a_1 + a_0}{6a_0^3\left(6 + p_0^2 - 2c_0^2 - gp_0^2c_0^2\right)}} \tag{6.56}$$

Looking at equation (5.59) we find that we can write n in a very neat way

$$n = \pm i\frac{\omega}{2\pi T} \equiv \pm i\hat{\omega}. \tag{6.57}$$

The quantity $\hat{\omega}$ is the dimensionless frequency of the fluctuations. We also define the dimensionless momentum to be

$$\hat{q} = \frac{q}{2\pi T}. \tag{6.58}$$

We have found that the leading near-horizon behaviour of the fluctuations Z_i (where i here can be H, α or ψ) is

$$Z_i \sim y^{\pm i\hat{\omega}} \tag{6.59}$$

The plus/minus sign gives components of the wave that are moving out of/into the horizon. We must chose the minus sign to satisfy the boundary condition at the horizon so that we have no outgoing waves. Finally, the leading behaviour is

$$Z_i \sim y^{-i\hat{\omega}} \tag{6.60}$$

near the horizon. It is convenient to explicitly extract this leading behaviour from the full solution. We define the new fluctuation variables $\tilde{Z}_H$, $\tilde{Z}_\alpha$ and $\tilde{Z}_\psi$ to be

$$Z_H(y) = y^{-i\hat{\omega}}\tilde{Z}_H(y), \quad Z_\alpha(y) = y^{-i\hat{\omega}}\tilde{Z}_\alpha(y), \quad Z_\psi(y) = y^{-i\hat{\omega}}\tilde{Z}_\psi(y) \tag{6.61}$$

Following the reasoning in section 4.4 we must impose the following boundary conditions,

$$\tilde{Z}_H(y = 0^+) = 1, \quad \tilde{Z}_H(y = 1^-) = 0. \tag{6.62}$$

6.3 High-temperature hydrodynamic limit

Recall that the goal of this chapter is to calculate the bulk-to-shear viscosity of our dual gauge theory. This will require us to numerically solve the fluctuation equations in the hydrodynamic limit $\hat{q} \to 0$. In order to employ our numerical shooting method, we will need reasonably accurate estimates of the fitting parameters when p_1 is small. Recall from (5.94) and (5.107) that the small p_1 limit corresponds to the high T limit. We can make some progress in these limits to extract initial estimates of the parameters we will need to fit using our shooting method. This will also provide a non-trivial check of our high-temperature corrections to the speed of sound in (5.129), and we will find the high temperature correction to the bulk-to-shear viscocity ratio.

As in section 5.5 we expand the fluctuation variables in powers of δ_1 (high-temperature limit) and also in powers of $\hat{q}$ (hydrodynamic limit). Also, we will consider the symmetric phase where $\chi = 0$, which leads to $Z_\psi = 0$. We write,

$$\begin{aligned} \tilde{Z}_H &= \left(Z_{00} + \delta_1^2 Z_{10}\right) + i\hat{q}\left(Z_{01} + \delta_1^2 Z_{11}\right) \\ \tilde{Z}_\alpha &= \delta_1 \left(Z_{\alpha 0} + i\hat{q} Z_{\alpha 1}\right) \end{aligned} \tag{6.63}$$

The conditions (6.62) become

$$\begin{aligned}\tilde{Z}_{00}\left(y=0^{+}\right)&=1\\ \tilde{Z}_{10}\left(y=0^{+}\right)&=0\\ \tilde{Z}_{01}\left(y=0^{+}\right)&=0\\ \tilde{Z}_{11}\left(y=0^{+}\right)&=0,\end{aligned} \tag{6.64}$$

and $\tilde{Z}_{\alpha 0}$ and $\tilde{Z}_{\alpha 1}$ must be regular on the horizon. The dispersion relation can be expanded as

$$\hat{\omega}=\frac{\hat{q}}{\sqrt{2}}\left(1+\beta_1\delta_1^2\right)-i\frac{\hat{q}^2}{4}\left(1+\beta_2\delta_1^2\right), \tag{6.65}$$

where β_1 and β_2 are new constants that we must fix. Note that we extracted the leading (i.e. $\delta_1=0$) terms in the dispersion relation, which were found in section 4.5.1. Putting (6.63) and (6.65) into (6.61), then those into (6.53), we can expand the latter in powers of δ_1 and $\hat{q}$. Setting the first two leading powers to zero gives back the same system we encountered in section 4.5.1, so we already know that

$$\tilde{Z}_{00}=1-y^2,\quad \text{and}\quad \tilde{Z}_{01}=0 \tag{6.66}$$

which satisfies the boundary conditions. Setting the next four leading terms to zero gives us the following system:

$$y^3\tilde{Z}''_{\alpha 0}+y^2\tilde{Z}'_{\alpha 0}+\frac{8}{9}\frac{y^3}{\left(y^2-1\right)^2}Z_{\alpha 0}+\frac{3}{2}\phi'_1-\frac{2}{3}\frac{y}{y^2-1}\phi_1=0 \tag{6.67}$$

$$\begin{aligned}y^2\left(y^2+1\right)\tilde{Z}''_{10}-y\left(3y^2-1\right)\tilde{Z}'_{10}+4y^2Z_{10}+\frac{8}{3}y^2\left(9y\phi'_1+2\frac{1+y^2}{1-y^2}\phi_1\right)Z_{\alpha 0}\\ +\frac{9}{4}\left(y^2+1\right)\left(y^2-1\right)^2\left(\phi'_1\right)^2+18y\left(y^2-1\right)^2A'_1+16y^2\beta_1=0\end{aligned} \tag{6.68}$$

$$y^3\tilde{Z}_{a1}'' + y^2\tilde{Z}_{a1}' + \frac{8}{9}\frac{y^3}{(y^2-1)^2}\tilde{Z}_{a1} - \sqrt{2}y^2\tilde{Z}_{a0}' - \frac{3}{2}\sqrt{2}\phi_1' + \frac{2\sqrt{2}}{3}\frac{y}{y^2-1}\phi_1 = 0 \quad (6.69)$$

$$\begin{aligned} & y^2\left(y^2+1\right)^2\tilde{Z}_{11}'' - y\left(y^2+1\right)\left(3y^2-1\right)\tilde{Z}_{11}' + 4y^2\left(y^2+1\right)\tilde{Z}_{11} - \sqrt{2}y\left(y^2-1\right)^2\tilde{Z}_{10}' \\ & + 2\sqrt{2}y^2\left(y^2-1\right)\tilde{Z}_{10} - \frac{8}{3}y^2\left(y^2+1\right)\left(9y\phi_1' - 2\frac{y^2+1}{y^2-1}\phi_1\right)\tilde{Z}_{\alpha 1} \\ & - \frac{8\sqrt{2}}{3}y^2\left(9y^3\phi_1' - \frac{\left(y^2+1\right)^2}{y^2-1}\phi_1\right)\tilde{Z}_{\alpha 0} - \frac{9\sqrt{2}}{8}\left(y^4-1\right)^2\phi_1' \\ & - 9\sqrt{2}y\left(y^2+3\right)\left(y^2-1\right)^2 A_1' - 4\sqrt{2}y^2\left[\beta_2\left(y^2+1\right) + 2\beta_1\left(y^2+3\right)\right] = 0 \end{aligned} \quad (6.70)$$

We will solve this system numerically, so it will suffice to find series solutions of this system expanded about the horizon ($y = 0$) and about the boundary ($y = 1$). Let us start with equation (6.67). Recall from section 5.5 that

$$\phi_1(y) = \left(1-y^2\right)^{\frac{1}{3}}\, {}_2F_1\left(\frac{1}{3}, \frac{1}{3}, 1; y^2\right), \quad (6.71)$$

Expanding about the horizon and the boundary gives

$$\begin{aligned} \phi_1 &= 1 - \frac{2}{9}y^2 - \frac{8}{81}y^4 + O\left(y^6\right), \quad y \approx 0 \\ &= \frac{2^{\frac{1}{3}}\pi}{\sqrt{3}\Gamma\left(\frac{2}{3}\right)^3}\left(2x^{\frac{1}{3}} + \frac{1}{3}x^{\frac{4}{3}} + O\left(x^{\frac{7}{3}}\right)\right), \quad x \approx 0, \end{aligned} \quad (6.72)$$

where $x = 1 - y$ is the radial coordinate such that the AdS boundary is at $x = 0$. At the horizon $\tilde{Z}_{\alpha 0}$ should be a regular function. We can solve equation (6.67) perturbatively about the horizon to find that

$$\tilde{Z}_{\alpha 0} = z_0 + \left(\frac{1}{54} - \frac{2}{9}z_0\right)y^2 + O\left(y^4\right), \quad y \approx 0, \quad (6.73)$$

where z_0 is an integration constant that must be determined.

Let us now try to find the perturbative expansion of $\tilde{Z}_{\alpha 0}$ near the boundary. If we make the change of variables $y = 1 - x$ in equation (6.67), then regard x to be very small, then we may approximate the equation near the boundary ($x = 0$) to be

$$\tilde{Z}''_{\alpha 0} - \tilde{Z}'_{\alpha 0} + \frac{2}{9x^2}\tilde{Z}_{\alpha 0} = 0. \tag{6.74}$$

The solution is

$$\tilde{Z}_{\alpha 0} = \sqrt{x}e^{\frac{x}{2}}\left(C_1 I_{\frac{1}{6}}\left(\frac{x}{2}\right) + C_2 K_{\frac{1}{6}}\left(\frac{x}{2}\right)\right) \tag{6.75}$$

where I and K are modified Bessel functions of the first and second kind. Since this solution is only valid for small x, we should expand it as

$$\tilde{Z}_{\alpha 0} \sim x^{\frac{1}{3}} + x^{\frac{2}{3}} + O\left(x^{\frac{4}{3}}\right), \tag{6.76}$$

where we left out the coefficients. This gives us a good guess at what the asymptotic expansion of $\tilde{Z}_{\alpha 0}$ looks like near the boundary; it is a series of powers like $x^{\frac{n}{3}}$ where n is an integer. There is a caveat however, that the fluctuation $\tilde{Z}_{\alpha 0}$ cannot dominate its corresponding background field ϕ. Recall from equation (5.52) that ϕ has a leading $x^{\frac{1}{3}}$ behaviour near the boundary. Thus we must, by hand, remove the leading $x^{\frac{1}{3}}$ term from $\tilde{Z}_{\alpha 0}$ to ensure that it will be sub-dominant. We may solve perturbatively near the boundary to find

$$\tilde{Z}_{\alpha 0} = \zeta_1 x^{\frac{2}{3}} + \frac{\pi}{2^{\frac{2}{3}}\sqrt{3}\Gamma\left(\frac{2}{3}\right)^3}x^{\frac{4}{3}} + O\left(x^{\frac{7}{3}}\right), \quad x \approx 0 \tag{6.77}$$

where ζ_1 is another integration constant to be determined. With equations (6.67) along with its x-variable version, (6.73) and (6.77) we are set up to use our shooting method to determine the constants z_0 and ζ_1. The result is

$$z_0 \approx -0.0833333 \approx -\frac{1}{12} \tag{6.78}$$

$$\zeta_1 \approx -0.224636 \approx -\frac{9\Gamma\left(\frac{2}{3}\right)^3}{8\left(2^{\frac{1}{3}}\right)\pi^2}. \tag{6.79}$$

The exact values on the right-hand side were determined by inspection. These quantities have no physical significance, but they will be used as initial values when we solve the fluctuation equations for arbitrary temperature.

We will not solve the remaining equations (6.3)-(6.70) in detail because the method is similar to the one we just used to solve (6.67). We will just provide the pieces required for the numerics and the final results.

Although we know the exact solution for A_1, it is useful to solve equation (5.109) perturbatively. One finds

$$\begin{aligned} A_1 &= \left(-\frac{1}{36} + \frac{2\pi - 3\sqrt{3}}{72\pi}\right) + \left(\frac{2\pi - 3\sqrt{3}}{36\pi}\right)y^2 + \mathcal{O}\left(y^4\right), \quad && y \approx 0 \\ &= \frac{\pi^2}{15\left(2^{\frac{1}{3}}\right)\Gamma\left(\frac{2}{3}\right)^6}x^{\frac{2}{3}} + \frac{1}{2\pi\sqrt{3}}x + \mathcal{O}\left(x^{\frac{4}{3}}\right), && x \approx 0 \end{aligned} \tag{6.80}$$

Then the asymptotics of $\tilde{Z}_{10}$ are

$$\begin{aligned} \tilde{Z}_{10} &= -\left(\frac{2}{9} + 4\beta_1 + \frac{2\pi - 3\sqrt{3}}{4\pi}\right)y^2 + \frac{2}{81}y^4 + \mathcal{O}\left(y^6\right), \quad && y \approx 0 \\ &= s_1 x - \left(4\beta_1 + \frac{s_1}{2}\right)x^2 + \mathcal{O}\left(x^{\frac{8}{3}}\right), && x \approx 0. \end{aligned} \tag{6.81}$$

We numerically calculate that

$$s_1 \approx -0.102658 \quad \text{and} \quad \beta_1 \approx -0.0689161 \approx -\frac{\sqrt{3}}{8\pi}. \tag{6.82}$$

The asymptotics of $\tilde{Z}_{\alpha 1}$ are

$$\begin{aligned} \tilde{Z}_{\alpha 1} &= u_0 - \frac{2}{9} u_0 y^2 + O\left(y^4\right), & y \approx 0 \\ &= v_0 x^{\frac{2}{3}} - \frac{\pi}{2^{\frac{1}{6}} \sqrt{3} \Gamma\left(\frac{2}{3}\right)^3} x^{\frac{4}{3}} + O\left(x^{\frac{5}{3}}\right), & x \approx 0 \end{aligned} \tag{6.83}$$

and we find numerically that

$$u_0 \approx 0.0457818 \quad \text{and} \quad v_0 \approx 0.188418 \tag{6.84}$$

The asymptotics of $\tilde{Z}_{11}$ are

$$\begin{aligned} \tilde{Z}_{11} &= -\left(\frac{18\sqrt{2}\beta_2 - \frac{27}{\pi}\sqrt{\frac{3}{2}} + \frac{27}{2\sqrt{2}}\left(-\frac{9}{2}\gamma_1\right) + 24 u_0}{9\left(-2 + \sqrt{2}\right)}\right) y^2 + O\left(y^4\right), \quad y \approx 0 \\ &= g_1 x + \left(\sqrt{2}\beta_2 - \frac{g_1}{2} - \frac{1}{\pi}\sqrt{\frac{3}{2}} - \frac{4\pi\left(2^{\frac{1}{3}}\right) v_0}{3\sqrt{3}\Gamma\left(\frac{2}{3}\right)^3}\right) x^2 + O\left(x^{\frac{7}{3}}\right), \quad x \approx 0. \end{aligned} \tag{6.85}$$

Note that $\gamma_1 = -\frac{2}{9\pi}\left(2\pi - 3\sqrt{3}\right)$, and u_0 and v_0 are given above. The remaining unknowns are determined numerically to be

$$g_1 \approx -0.0134491 \quad \text{and} \quad \beta_2 \approx 0.250000 \approx \frac{1}{4} \tag{6.86}$$

Using our results for β_1 and β_2 in (6.65), we have found that the dispersion relation to leading order in δ_1 is

$$\hat{\omega} = \frac{\hat{q}}{\sqrt{2}}\left(1 - \frac{\sqrt{3}}{8\pi}\delta_1^2\right) - i\frac{\hat{q}^2}{4}\left(1 + \frac{1}{4}\delta_1^2\right), \tag{6.87}$$

Comparing this to the generic form of the hydrodynamic dispersion relation [6]

$$\hat{\omega} = c_s \hat{q} - \frac{i}{2}\left(\frac{1}{2} + \frac{\zeta}{2\eta}\right)\hat{q}^2 \tag{6.88}$$

we see that the correction to the speed of sound is in perfect agreement with (5.129), and that the correction to the bulk-to-shear viscosity is found by identifying the $i\hat{q}^2$ coefficients

$$\frac{1}{4}\left(1 + \frac{1}{4}\delta_1^2\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{\zeta}{2\eta}\right) \tag{6.89}$$

or

$$\frac{\zeta}{\eta} \sim 0 + \frac{\delta_1^2}{4} \tag{6.90}$$

where the zero indicates that the bulk viscosity vanishes in the conformal limit.

6.4 Hydrodynamic limit

In this section we will numerically calculate the bulk-to-shear viscosity ratio for arbitrary temperature. In order to do this, we must solve the system (6.53) in the hydrodynamic limit $\hat{q} \to 0$. To get the equations of motion in the hyrodynamic limit, we set

$$\begin{aligned} \tilde{Z}_H &= z_{H0} + i\hat{q} z_{H1} \\ \tilde{Z}_\alpha &= z_{\alpha 0} + i\hat{q} z_{\alpha 1} \\ \tilde{Z}_\psi &= z_{\psi 0} + i\hat{q} z_{\psi 1}. \end{aligned} \tag{6.91}$$

and

$$\hat{\omega} = \frac{\hat{\beta}_1}{\sqrt{2}}\hat{q} - i\frac{\hat{\beta}_2}{4}\hat{q}^2. \tag{6.92}$$

Note that we have extracted the conformal values in (6.92) so that in the conformal limit we have $\hat{\beta}_1 = 1$ and $\hat{\beta}_2 = 1$. Putting all this into (6.61), then into (6.53), then expanding to leading order in $\hat{q}$, we arrive at a system of the form

$$\begin{aligned} z''_{H0} + a_{H0} z'_{H0} + b_{H0} z_{H0} + c_{H0} z_{\alpha 0} + d_{H0} z_{\psi 0} &= 0 \\ z''_{\alpha 0} + a_{\alpha 0} z'_{\alpha 0} + b_{\alpha 0} z_{\alpha 0} + c_{\alpha 0} z'_{H0} + d_{\alpha 0} z_{H0} + e_{\alpha 0} z_{\psi 0} &= 0 \\ z''_{\psi 0} + a_{\psi 0} z'_{\psi 0} + b_{\psi 0} z_{\psi 0} + c_{\psi 0} z'_{H0} + d_{\psi 0} z_{H0} + e_{\psi 0} z_{\alpha 0} &= 0, \end{aligned} \tag{6.93}$$

where the prime denotes differentiation with respect to y. The coefficients are complicated functionals of the background fields[5] and their derivatives, and they contain the parameter $\hat{\beta}_1$. Expanding to the first subleading order in $\hat{q}$, we get the system

$$\begin{aligned} & z''_{H1} + a_{H1} z'_{H1} + b_{H1} z_{H1} + c_{H1} z'_{H0} + d_{H1} z_{H0} + e_{H1} z_{\alpha 1} + f_{H1} z_{\alpha 0} + g_{H1} z_{\psi 1} + h_{H1} z_{\psi 0} = 0 \\ & z''_{\alpha 1} + a_{\alpha 1} z'_{\alpha 1} + b_{\alpha 1} z_{\alpha 1} + c_{\alpha 1} z'_{H0} + d_{\alpha 1} z_{H0} + e_{\alpha 1} z'_{H1} + f_{\alpha 1} z_{H1} \\ & \qquad + g_{\alpha 1} z'_{\alpha 0} + h_{\alpha 1} z_{\alpha 0} + i_{\alpha 1} z_{\psi 1} + j_{\alpha 1} z_{\psi 0} = 0 \\ & z''_{\psi 1} + a_{\psi 1} z'_{\psi 1} + b_{\psi 1} z_{\psi 1} + c_{\psi 1} z'_{H0} + d_{\psi 1} z_{H0} + e_{\psi 1} z'_{H1} + f_{\psi 1} z_{H1} \\ & \qquad + g_{\psi 1} z'_{\psi 0} + h_{\psi 1} z_{\psi 0} + i_{\psi 1} z_{\alpha 1} + j_{\psi 1} z_{\alpha 0} = 0. \end{aligned} \tag{6.94}$$

Again, the coefficients are functionals of the background fields, and they contain $\hat{\beta}_1$ and $\hat{\beta}_2$. The boundary conditions at the horizon are

$$z_{H0}\left(y = 0^+\right) = 1, \quad z_{H1}\left(y = 0^+\right) = 0, \quad z_{\alpha i}\left(y = 0^+\right) = z_{\psi i}\left(y = 0^+\right) = \text{finite} \tag{6.95}$$

In order to set up the numerics, we need to solve the systems (6.93) and (6.94) perturbatively near the horizon. Since the fluctuations are regular at the horizon, we

[5] The explicit forms of the coefficients will be omitted as they are too cumbersome to write down, and would likely be fraught with typos anyway.

may substitute a Taylor series expansion for each fluctuation in (6.93) and (6.94) and demand that the left hand side vanishes order by order. We find the asymptotic behaviour near the horizon is

$$\begin{aligned} z_{H0} &= 1 + \mathcal{O}\left(y^2\right) \\ z_{\alpha 0} &= b_0 + \mathcal{O}\left(y^2\right) \\ z_{\psi 0} &= e_0 + \mathcal{O}\left(y^2\right), \end{aligned} \tag{6.96}$$

and

$$\begin{aligned} z_{H1} &= 0 + \mathcal{O}\left(y^2\right) \\ z_{\alpha 1} &= d_0 + \mathcal{O}\left(y^2\right) \\ z_{\psi 1} &= f_0 + \mathcal{O}\left(y^2\right), \end{aligned} \tag{6.97}$$

where $\{b_0, d_0, e_0, f_0\}$ are integration constants that remain to be fixed. In practice we must explicitly find several more terms than what are given here; however, the coefficients are complicated expressions of the integration coefficients, and they are too cumbersome to write down.

It is straightforward to write change the systems (6.93) and (6.94) in terms of the variable $x = 1 - y$. Following section 6.3, at the boundary $x = 0$ we find the following asymptotics,

$$\begin{aligned} z_{H0} &= z_0 x + \mathcal{O}\left(x^2\right) \\ z_{\alpha 0} &= u_0 x^{\frac{2}{3}} + \mathcal{O}\left(x^{\frac{4}{3}}\right) \\ z_{\psi 0} &= r_0 x^{\frac{4}{3}} + \mathcal{O}\left(x^2\right), \end{aligned} \tag{6.98}$$

and

$$z_{H1} = v_0 x + \mathcal{O}\left(x^2\right)$$
$$z_{\alpha 1} = h_0 x^{\frac{2}{3}} + \mathcal{O}\left(x^{\frac{4}{3}}\right) \tag{6.99}$$
$$z_{\psi 1} = q_0 + \mathcal{O}\left(x^2\right),$$

We are almost set up to solve (6.93) and (6.94) using our shooting method. All we need now are some good initial guesses for our integration constants $\{b_0, d_0, e_0, f_0, h_0, q_0, r_0, u_0, v_0, z_0\}$ as well as $\hat{\beta}_1$ and $\hat{\beta}_2$. In the conformal limit, $p_1 \to 0$, we know from (6.92) that we should start with $\hat{\beta}_1 = 1$ and $\hat{\beta}_2 = 1$. In section 6.3 we considered the small p_1 (near conformal) regime. Identifying our asymptotic expansions in this section with those in section 6.3, we can pick out very good initial guesses for the integration constants if p_1 is small. Namely, if we look at the following equations: the second line of (6.63), (6.73),(6.77), then compare them to the analogous equations: the second lines of equations (6.91), (6.96), and (6.97). So in a sense we are identifying $z_{\alpha 0}$ with $\delta_1 Z_{\alpha 0}$, and $z_{\alpha 1}$ with $\delta_1 Z_{\alpha 1}$. Then the results (6.78) and (6.79) tell us that we should guess that for small p_1 we have

$$b_0 \approx -0.0833333\delta_1, \quad \text{and} \quad u_0 \approx -0.224636\delta_1 \tag{6.100}$$

where (see equation (5.107))

$$\delta_1 = \frac{\sqrt{3}\Gamma\left(\frac{2}{3}\right)^3}{2^{\frac{4}{3}}\pi} p_1 \tag{6.101}$$

Similarly we guess that

$$d_0 \approx 0.0457818\delta_1, \quad \text{and} \quad h_0 \approx 0.188418\delta_1 \tag{6.102}$$

Looking at the first lines of (6.63) and (6.91), we see that we are identifying z_{H0} with $Z_{00} + \delta_1^2 Z_{10}$. Recall that $Z_{00} = 1 - y^2 = 2x - x^2$. The coefficient 2 in front of the x definitely dominates the contribution from $\delta_1^2 Z_{10}$, so a good initial guess is

$$z_0 \approx 2. \tag{6.103}$$

Identifying z_{H1} with $\delta_1^2 Z_{11}$, we guess that

$$v_0 = -0.0134491\delta_1^2. \tag{6.104}$$

In the symmetric phase we know that the scalar field χ, which is associated with the fluctuation Z_ψ, vanishes. As such, we guess that

$$e_0 \approx f_0 \approx r_0 \approx q_0 \approx 0. \tag{6.105}$$

We have twelve ordinary differential equations from (6.93) and (6.94), and we have twelve parameters

$$\{b_0, d_0, e_0, f_0, h_0, q_0, r_0, u_0, v_0, z_0, \hat{\beta}_1, \hat{\beta}_2\} \tag{6.106}$$

that must be fixed and initial guesses for all of them. Thus we are set up to numerically compute these parameters as functions of the deformation parameter p_1.

6.4.1 Numerical results

Figure 6.1 (left) shows the speed of sound squared calculated from the dispersion relation (6.92), i.e.,

$$c_s^2 = \frac{\hat{\beta}_1^{\,2}}{2} \tag{6.107}$$

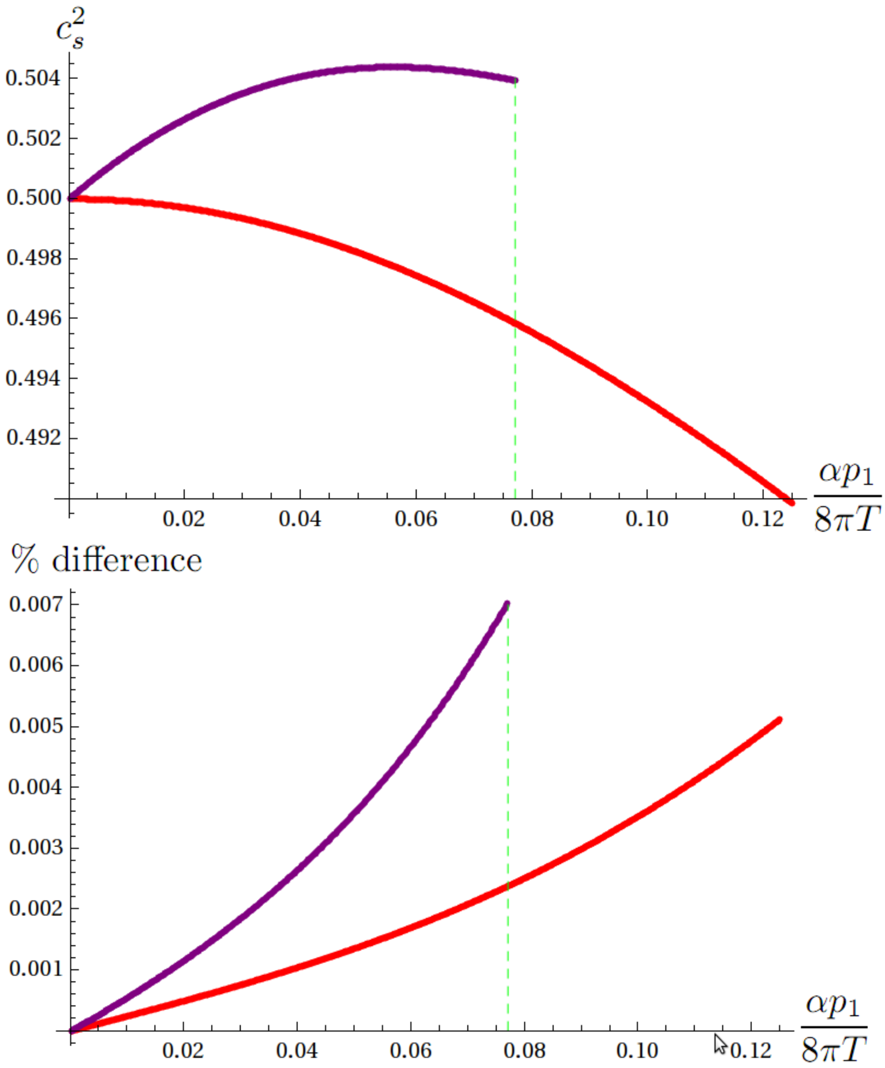

Figure 6.1: Top: Speed of sound squared calculated from the dispersion relation (6.92) versus the dimensionless control parameter $\frac{\alpha p_1}{8\pi T}$. The red curve corresponds to the symmetric phase where $\chi = 0$. The purple curve corresponds to the first symmetry-broken phase with $\chi \neq 0$. Bottom: Percent difference between the speed of sound squared calculated in this section and that shown in figure 5.4.

versus the dimensionless control parameter $\frac{\alpha p_1}{8\pi T}$, where T is given by equation (5.59). Figure 6.1 (bottom) shows the percentage difference between the results in figure 5.4 (bottom) and figure 6.1 (left). The difference is everywhere less than 0.007 %. This is a non-trivial check that our analysis of the background fluctuations in this chapter is correct.

The bulk-to-shear viscosity ratio can be related to $\hat{\beta}_2$ by comparing (6.92) and (6.88). We find

$$\frac{\zeta}{\eta} = \hat{\beta}_2 - 1. \tag{6.108}$$

Figure 6.2 shows the results for the bulk-to-shear viscosity. In the top figure, the red curve corresponds to the symmetric phase, and the black dashed curve is the correction to the conformal value $\zeta/\eta = 0$ given by equation (6.90). There is an excellent agreement at small values of p_1, which is another check that our analyses are consistent and correct. In the bottom figure, the red curve corresponds to the symmetric phase, and the purple curve corresponds to the symmetry-broken phase. The green dashed line corresponds to the critical value of $\frac{\alpha p_1}{8\pi T} \approx 0.0771$. The bulk viscosity in the symmetry-broken phase appears to diverge at the transition. By pushing the calculation closer and closer to the transition, we can confirm that this is indeed the case.

As a final remark in this chapter, we will verify the bulk viscosity bound proposed in [6], which conjectures that

$$\frac{\zeta}{\eta} \geq 2\left(\frac{1}{p} - c_s^2\right), \tag{6.109}$$

where p is the spatial dimension of the field theory. Figure 6.3 shows the bulk-to-shear viscosity ratio as a function of $1/2 - c_s^2$. The red curve corresponds to the symmetric phase, and the purple curve corresponds to the first symmetry-broken

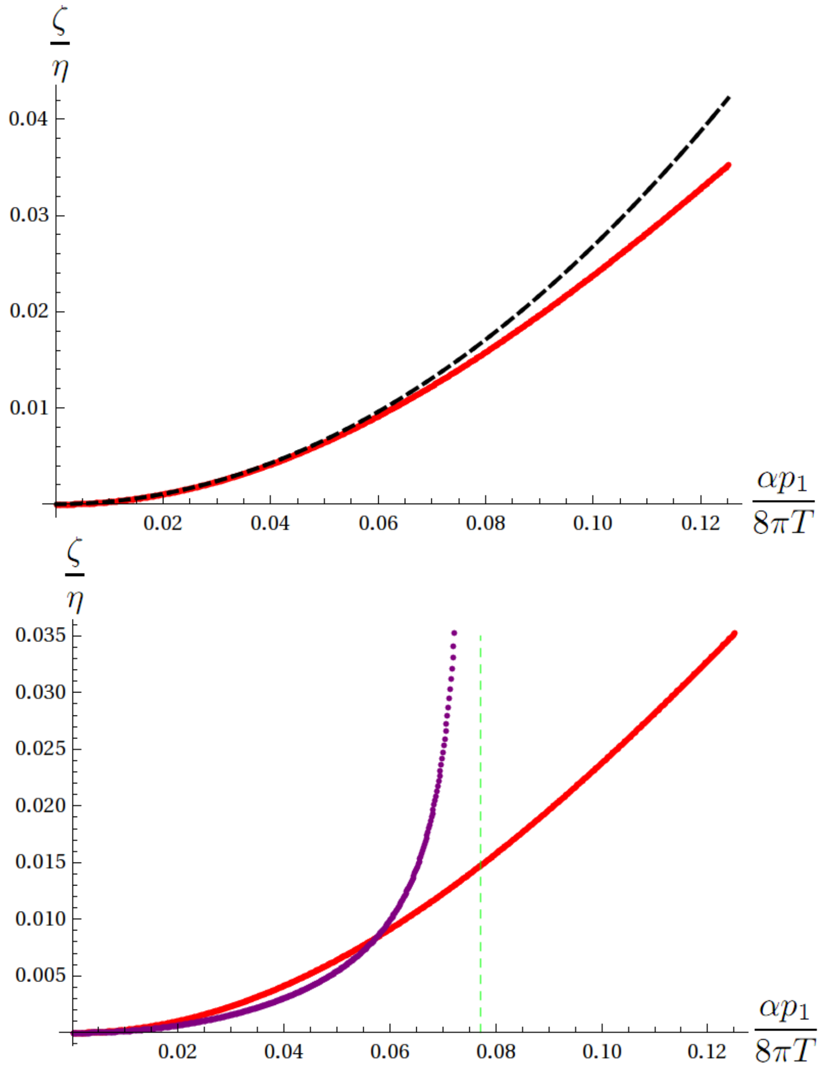

Figure 6.2: Top: Speed of sound squared calculated from the dispersion relation (6.92) versus the dimensionless control parameter $\frac{\alpha p_1}{8\pi T}$. The red curve corresponds to the symmetric phase where $\chi = 0$. The purple curve corresponds to the first symmetry-broken phase with $\chi \neq 0$. Bottom: Percent difference between the speed of sound squared calculated in this section and that shown in figure 5.4.

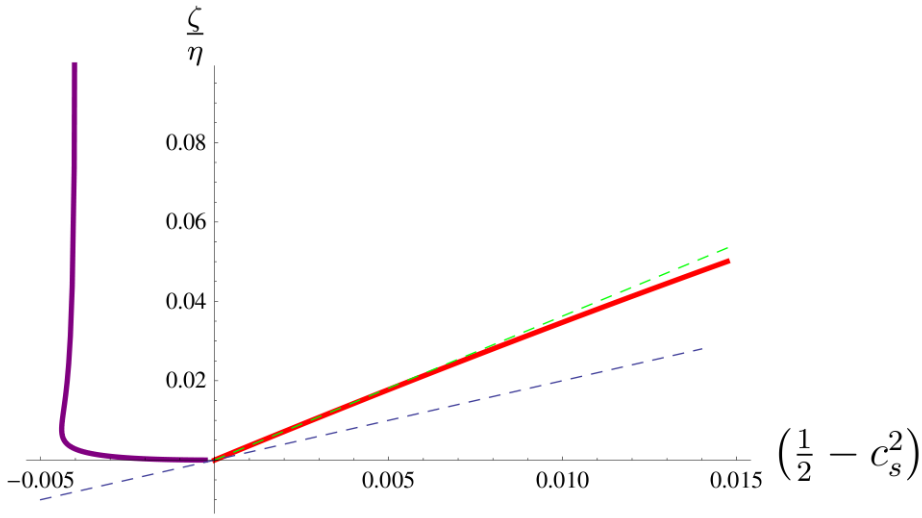

Figure 6.3: Bulk-to-shear viscosity ratio as a function of $1/2 - c_s^2$. The red curve corresponds to the symmetric phase, and the purple curve corresponds to the first symmetry-broken phase. The dashed blue line indicates the lower limit of the bound $\zeta/\eta \geq 2(1/2 - c_s^2)$. The dashed green line corresponds to the high-temperature approximation.

phase. The dashed blue line indicates the lower limit of the bound $\zeta/\eta \geq 2(1/2-c_s^2)$. We find that the bound is satisfied in both the symmetric and symmetry-broken phases. In fact, the bound is satisfied trivially in the symmetry broken phase since $c_s^2 > 1/2$ (see figure 6.1). We also see once again the divergent behaviour of ζ/η in the symmetry-broken phase.

In this chapter we calculated the dispersion relation of small fluctuations of the background fields and extracted the hydrodynamics of the dual field theory. We found that the bulk viscosity in the symmetry-broken phase diverges at the phase transition. We verified the correctness of our results by comparing them against those of the near-conformal limit. Next we will do an in-depth study of the critical behaviour of the symmetry-broken phase near the phase transition.

Chapter 7

Critical behaviour in hairy AdS$_4$

The observation of a second-order phase transition beckons a study of the critical behaviour of this system. In this chapter we will define a set of critical exponents by relating our parameters to those of models of ferromagnets. We will explicitly calculate all of the critical exponents, not all of which will be found to be of mean-field type. We find that some of the scaling relations that arise from the static scaling hypothesis are violated. We will also find that the symmetry broken phases are perturbatively unstable.

7.1 Criticality in ferromagnets and hairy AdS$_4$

To make our analysis as transparent as possible, it is convenient to cast the critical behaviour in terms of the language of ferromagnets. We define the reduced temperature t to be

$$t = \frac{T}{T_c} - 1 \tag{7.1}$$

So that the transition occurs at $t = 0$. In ferromagnetism, we also denote the external magnetic field by $\mathcal{H}$. The Gibbs free energy density is given by

$$\begin{aligned} \mathcal{W}(t, \mathcal{H}) &= \epsilon - sT - \mathcal{M}\mathcal{H} \\ &= \Omega_o (t, \mathcal{H}) - \Omega_d (t, \mathcal{H}), \end{aligned} \tag{7.2}$$

where ϵ is the energy density, s is the entropy density, and Ω_o and Ω_d are the Helmholtz free energies in the ordered phase and disordered phase respectively. As we traverse the critical temperature, there is a spontaneous magnetization in the system given by

$$\mathcal{M} = -\left(\frac{\partial \mathcal{W}}{\partial \mathcal{H}}\right)_t . \tag{7.3}$$

The two-point correlation function of the magnetization is defined as

$$G(\mathbf{r}) = \langle \mathcal{M}(\mathbf{r}) \mathcal{M}(0) \rangle \tag{7.4}$$

The critial exponents[1] $\{\alpha, \beta, \gamma, \delta, \nu, \eta\}$ are defined [43] as follows:

$$c_{\mathcal{H}} \sim |t|^{-\alpha} \tag{7.5}$$

$$\mathcal{M} \sim |t|^{\beta} \tag{7.6}$$

$$\chi_T \sim |t|^{-\gamma} \tag{7.7}$$

$$\mathcal{M}(t = 0) \sim |\mathcal{H}|^{\frac{1}{\delta}} \tag{7.8}$$

$$G(\mathbf{r}) \sim \begin{cases} e^{-\frac{|\mathbf{r}|}{\xi}} \,, & t \neq 0 \\ |\mathbf{r}|^{-p+2-\eta} \,, & t = 0 \end{cases} \,, \quad \text{where} \quad \xi \sim |t|^{-\nu} \,, \tag{7.9}$$

[1]Do not confuse the critical exponent α with the parameter α in (5.51). Likewise with the exponent η and the shear viscosity η. Their use should be clear from the context in which they are used.

where $c_{\mathcal{H}}$ is the specific heat, χ_T is the isothermal susceptibility, and ξ is the correlation length. Under the scaling hypothesis [43], we have the following relations

$$\begin{aligned} &\alpha + 2\beta + \gamma = 2 \\ &\gamma = \beta(\delta - 1) = \nu(2 - \eta) \\ &2 - \alpha = \nu p, \end{aligned} \tag{7.10}$$

where p is the spatial dimension of the system.

In order to employ this language we need to relate the quantities that we have been working with in our Exotic AdS_4 model to those given so far in this section. In ferromagnetism, the order parameter is $\mathcal{M}$; that is, it is zero below the transition, and non-zero above the transition. The quantity in our model that fits this description is χ_4 (see figure 5.1). So we identify that

$$\mathcal{M} \Leftrightarrow \chi_4. \tag{7.11}$$

The external control parameter in our model is αp_1, so we identify it with the external magnetic field,

$$\mathcal{H} \Leftrightarrow \alpha p_1, \tag{7.12}$$

where α here is as in (5.51). Having established the language in which to define the critical exponents, we will now calculate them.

7.2 Static critical exponents

First we will calculate α, which is defined by the scaling of specific heat near the transition,

$$c_{\mathcal{H}} \sim |t|^{-\alpha} \tag{7.13}$$

Specific heat is defined by

$$c_{\mathcal{H}} = \frac{\partial \mathcal{E}}{\partial T}, \tag{7.14}$$

and entropy density is given by

$$s = -\frac{\partial \mathcal{F}}{\partial T}. \tag{7.15}$$

Recall also that the speed of sound squared is given by

$$c_s^2 = -\frac{\partial \mathcal{F}}{\partial \mathcal{E}}. \tag{7.16}$$

Then

$$c_s^2 = -\frac{\left(\frac{\partial \mathcal{F}}{\partial T}\right)}{\left(\frac{\partial \mathcal{E}}{\partial T}\right)} = \frac{s}{c_{\mathcal{H}}} \tag{7.17}$$

Thus we have the relationship

$$c_{\mathcal{H}} \sim \frac{1}{c_s^2}. \tag{7.18}$$

In figure 6.1 (top) we see that the speed of sound c_s on the symmetry-broken branch is finite at the transition; that is,

$$c_{\mathcal{H}} \sim |t|^{-\alpha} \quad \Rightarrow \quad \alpha = 0. \tag{7.19}$$

The critical exponent β is defined by

$$\chi_4 \sim |t|^{\beta}. \tag{7.20}$$

A plot of χ_4^2 versus T_c/T near the transition is given in figure 7.1. The purple dots are the numerical results. The green dashed line is the best linear fit. The excellent linear fit to the data suggests that

$$\chi_4 \sim |t|^{\frac{1}{2}} \quad \Rightarrow \quad \beta = \frac{1}{2}. \tag{7.21}$$

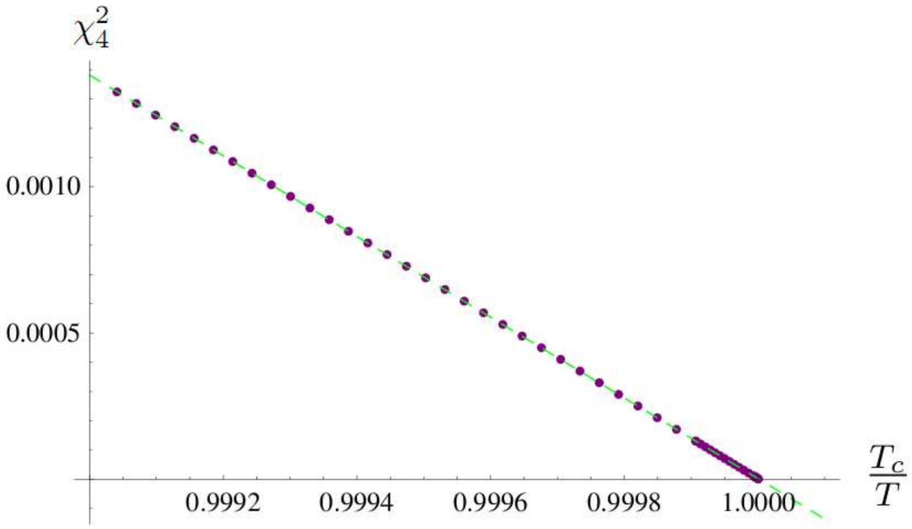

Figure 7.1: Square of the order parameter χ_4 as a function of inverse temperature near the transition. The purple dots are the numerical results. The green dashed line is the best linear fit.

One may now be tempted to use the scaling relations (7.10) to compute the remaining exponents. The result is $\{\alpha, \beta, \gamma, \delta, \nu, \eta\} = \left\{0, \frac{1}{2}, 1, 1, 3, 1, 1\right\}$; however, we will see later that some of these are incorrect.

The critical exponent δ is defined by

$$\chi_4 \sim |\alpha p_1|^{\frac{1}{\delta}} \,. \tag{7.22}$$

Figure 7.2 shows a plot of χ_4^2 versus αp_1. The excellent linear fit suggests that

$$\chi_4 \sim |\alpha p_1|^{\frac{1}{2}} \quad \Rightarrow \quad \delta = 2, \tag{7.23}$$

We may have anticipated this from the value of β. Let us define the dimensionless parameter Λ as

$$\Lambda = \frac{\alpha p_1}{T} \tag{7.24}$$

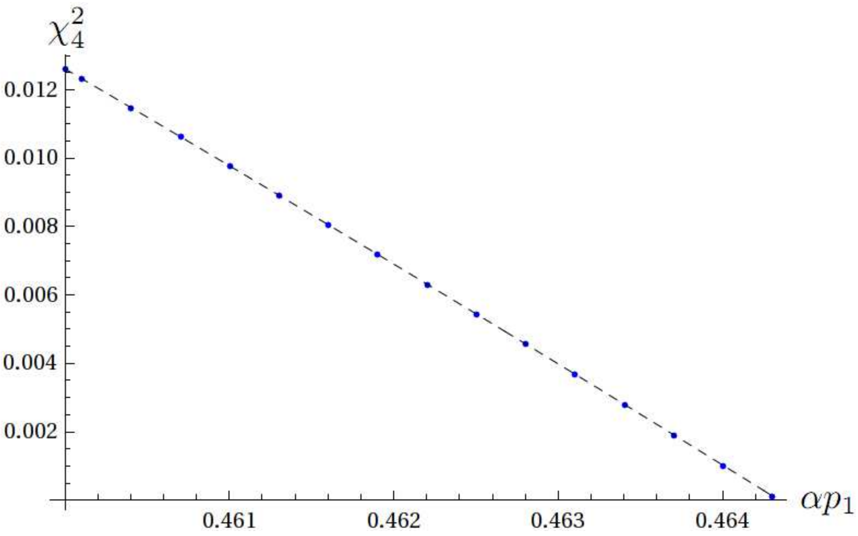

Figure 7.2: Square of the order parameter χ_4 as a function of the external field αp_1. The blue dots are the numerical results. The black dashed line is the best linear fit.

At criticality[2] we have (see figure 6.2)

$$\Lambda_c = \frac{\alpha p_1}{T_c} = \frac{\alpha_c p_c}{T} \approx 8\pi \times 0.0771 \tag{7.25}$$

We can define the reduced temperature t and external field h as

$$t = \frac{T - T_c}{T_c}, \quad \text{and} \quad h = \frac{\alpha_c p_c - \alpha p_1}{\alpha p_1}. \tag{7.26}$$

Using (7.25) to eliminate T and T_c from the first equation in (7.26) we find that

$$t \sim \frac{\alpha_c p_c - \alpha p_1}{\alpha p_1} = h \tag{7.27}$$

[2]Note that we have two dials αp_1 and T which we can tune to establish criticality; that is, we can fix αp_1 and tune T or vice versa. This is similar to how the critical temperature in a ferromagnet is a function of the external field.

This, along with the relations

$$\mathcal{M} \sim |t|^{\beta}, \quad \text{and} \quad \mathcal{M} \sim |h|^{\frac{1}{\delta}} \tag{7.28}$$

implies that

$$\beta = \frac{1}{\delta}. \tag{7.29}$$

From $t \sim h$ we can deduce the critical exponent γ, which is defined by

$$\chi_T = \left(\frac{\partial \mathcal{M}}{\partial \mathcal{H}}\right)\bigg|_T \sim |t|^{-\gamma}. \tag{7.30}$$

Since $t \sim h$ we have

$$\frac{\partial \mathcal{M}}{\partial \mathcal{H}} \sim \frac{\partial \mathcal{M}}{\partial t}, \tag{7.31}$$

and since $\mathcal{M} \sim |t|^{\beta}$, we have

$$\frac{\partial \mathcal{M}}{\partial t} \sim |t|^{\beta - 1}. \tag{7.32}$$

Thus

$$\gamma = 1 - \beta = \frac{1}{2} \tag{7.33}$$

7.3 Dynamic susceptibility

7.3.1 Static critical exponents revisited

In this section we will introduce the concept of dynamic susceptibility. This will allow us to explicity calculate the remaining critical exponents $\{\nu, \eta\}$ without resorting to the scaling relations, and it will provide a non-trivial check of the exponent

γ. To begin we will need to backtrack to the system (6.53), re-stated below,

$$\begin{aligned} &\mathcal{A}_H Z_H'' + \mathcal{B}_H Z_H' + C_H Z_H + \mathcal{E}_H Z_\alpha + \mathcal{G}_H Z_\psi = 0, \\ &\mathcal{A}_\alpha Z_\alpha'' + \mathcal{B}_\alpha Z_\alpha' + C_\alpha Z_\alpha + \mathcal{E}_\alpha Z_\psi + \mathcal{F}_\alpha Z_H' + \mathcal{G}_\alpha Z_H = 0, \\ &\mathcal{A}_\psi Z_\psi'' + \mathcal{B}_\psi Z_\psi' + C_\psi Z_\psi + \mathcal{E}_\psi Z_\alpha + \mathcal{F}_\psi Z_H' + \mathcal{G}_\psi Z_H = 0, \end{aligned} \tag{7.34}$$

where the prime denotes differentiation with respect to the radial coordinate y. Now we extract the leading behaviour in the following way (c.f. equation (6.61))

$$Z_H = \frac{1}{\hat{\omega}^2} y^{-i\hat{\omega}} z_H, \quad Z_\alpha = \frac{1}{\hat{q}^2} y^{-i\hat{\omega}} z_\alpha, \quad Z_\psi = \frac{1}{\hat{q}^2} y^{-i\hat{\omega}} z_\psi. \tag{7.35}$$

Making the substitutions,

$$\hat{\omega} = -i\Omega, \quad \hat{q} = \sqrt{Q} \tag{7.36}$$

we get a new system of ODEs where all the variables are real,

$$\begin{aligned} &a_H z_H'' + b_H z_H' + c_H z_H + e_H z_\alpha + g_H z_\psi = 0, \\ &a_\alpha z_\alpha'' + b_\alpha z_\alpha' + c_\alpha z_\alpha + e_\alpha z_\psi + f_\alpha z_H' + g_\alpha z_H = 0, \\ &a_\psi z_\psi'' + b_\psi z_\psi' + c_\psi z_\psi + e_\psi z_\alpha + f_\psi z_H' + g_\psi z_H = 0, \end{aligned} \tag{7.37}$$

where the coefficients are functionals of the background fields and Ω and Q. Most of the coefficients are too cumbersome to write down explicitly, but here are a few that you can check if you are following along,

$$a_H = a_\alpha = a_\psi = 1 \tag{7.38}$$

$$\mathfrak{b}_H = -\frac{1}{2yc_2'\left(2c_2'\left(\Omega^2+y^2Q\right)+yc_2Q\right)}$$
$$\times\left[yc_2Q\left(yc_2\left((\phi')^2+(\chi')^2\right)-2c_2'\right)+4\Omega c_2'\left(c_2'\left(2\Omega^2-\Omega+2y^2Q\right)+yc_2Q\right)\right], \tag{7.39}$$

$$\mathfrak{b}_\alpha = \mathfrak{b}_\psi = \frac{1-2\Omega}{y}. \tag{7.40}$$

The asymptotics near the horizon $y \approx 0$ are

$$\begin{aligned} z_H &= z_0^h + \mathcal{O}\left(y^2\right) \\ z_\alpha &= \alpha_0^h + \mathcal{O}\left(y^2\right) \\ z_\psi &= \psi_0^h + \mathcal{O}\left(y^2\right). \end{aligned} \tag{7.41}$$

The asymptotics near the boundary $x = 1 - y \approx 0$ are

$$\begin{aligned} z_H &= z_3 x + \mathcal{O}\left(x^{4/3}\right) \\ z_\alpha &= x^{\frac{1}{3}} + \alpha_2 x^{\frac{2}{3}} + \mathcal{O}(x) \\ z_\psi &= \psi_4 x^{\frac{4}{3}} + \mathcal{O}\left(x^2\right). \end{aligned} \tag{7.42}$$

Notice that we have used the scale invariance of (7.37) to fix the first boundary coefficient of z_α to one. The dynamic susceptibility is defined to be the leading boundary coefficient of z_ψ, and we regard it to be a function of Ω and Q. That is,

$$\psi_4 = \psi_4(\Omega, Q) = \text{dynamic susceptibility}. \tag{7.43}$$

Before moving on, let us justify this definition with regard to the theory of dynamical critical phenomena [29, 9]. Consider spacetime-dependent variations of the external magnetic field $\mathcal{H}$,

$$\mathcal{H} \to \mathcal{H} + \delta\mathcal{H}(t, \mathbf{x}). \tag{7.44}$$

The Fourier modes $\mathcal{H}_{\omega,\mathbf{k}}$ of the variation are defined by

$$\delta\mathcal{H}(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} \mathcal{H}_{\omega,\mathbf{k}}. \tag{7.45}$$

This variation induces a corresponding variation in the magnetization; that is,

$$\mathcal{M} \to \mathcal{M} + \delta\mathcal{M}(t, \mathbf{x}), \quad \delta\mathcal{M}(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} \mathcal{M}_{\omega,\mathbf{k}}. \tag{7.46}$$

The dynamic susceptibility is defined to be

$$\chi_{\omega,\mathbf{k}} = \left(\frac{\mathcal{M}_{\omega,\mathbf{k}}}{\mathcal{H}_{\omega,\mathbf{k}}} \right) \Bigg|_T . \tag{7.47}$$

In the static limit $\omega, \mathbf{k} \to 0$ we recover the isothermal susceptibility

$$\lim_{\omega,\mathbf{k} \to 0} \chi_{\omega,\mathbf{k}} = \chi_T = \left(\frac{\partial\mathcal{M}}{\partial\mathcal{H}} \right) \Bigg|_T \tag{7.48}$$

Recall that in the static limit we identified $\mathcal{M} \leftrightarrow \chi_4$ and $\mathcal{H} \leftrightarrow p_1$, where these parameters are defined by the asymptotics of the background scalars near the boundary $x \approx 0^+$,

$$\phi \sim p_1 x^{\frac{1}{3}}, \quad \text{and} \quad \chi \sim \chi_4 x^{\frac{4}{3}}. \tag{7.49}$$

The variation of the background scalars are

$$\phi \to \phi + \delta\phi, \quad \chi \to \chi + \delta\chi. \tag{7.50}$$

with the associated variations in their boundary coefficients

$$p_1 \to p_1 + \delta p_1, \quad \chi_4 \to \chi_4 + \delta\chi_4. \tag{7.51}$$

The fluctuation associated with ϕ is $\delta\phi = z_\alpha$ and that with χ is $\delta\chi = z_\psi$, whose behaviour near the boundary is given by (7.42). Thus we identify

$$\delta p_1 = 1 = \mathcal{H}_{\omega,\mathbf{k}}, \quad \text{and} \quad \delta\chi_4 = \psi_4 = \mathcal{M}_{\omega,\mathbf{k}}. \tag{7.52}$$

So we see that the dynamic susceptibility as defined in (7.47) is given by ψ_4.

We will calculate the dynamic susceptibility in the static limit to calculate the critical exponent γ, which is defined by

$$\chi_{0,0} = \chi_T \propto |t|^{-\gamma} \tag{7.53}$$

To do this, we expand the system (7.37) first about $\Omega = 0$, then expand the result about $Q = 0$, and we keep only the leading order terms. Some of the coefficents are still to cumbersome to write down, but here are a few of the shorter ones you can check to make sure you're on the right track. With $\Omega \to 0$ and $Q \to 0$, we have

$$a_H = a_\alpha = a_\psi = 1 \tag{7.54}$$

$$b_H = \frac{1}{2}\frac{c_2}{yc_2'}\left(\frac{2c_2' - yc_2(\phi')^2 - yc_2(\chi')^2}{2yc_2' + c_2}\right) \tag{7.55}$$

$$b_\alpha = b_\psi = \frac{1}{y}. \tag{7.56}$$

$$c_H = \frac{1}{2}\frac{1}{yc_2'}\left(\frac{4\left(c_2'\right)^2 + c_2^2(\phi')^2 + c_2^2(\chi')^2}{2yc_2' + c_2}\right). \tag{7.57}$$

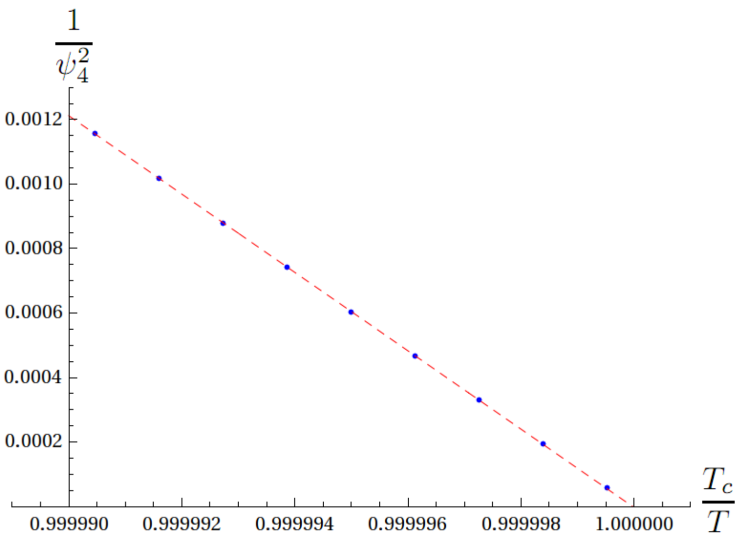

Figure 7.3: Inverse square of dynamic susceptibility ψ_4^{-2} versus inverse temperature T_c/T in the limit $\Omega \to 0$ and $Q \to 0$. The blue dots are the numerical results, and the red curve is the best linear fit.

Likewise, we expand the asymptotic expansions (7.41) and (7.42) about $\Omega \to 0$ and $Q \to 0$ and keep the leading terms. Although the coefficients are still rather large in the horizon expansions, the results for the boundary expansions are

$$\begin{aligned} z_H &= z_3 x - \frac{3}{5} p_1 x^{\frac{5}{3}} + \mathcal{O}\left(x^2\right) \\ z_\alpha &= x^{\frac{1}{3}} + \alpha_2 x^{\frac{2}{3}} + \frac{2}{5} p_1^2 x + \mathcal{O}\left(x^{\frac{4}{3}}\right) \\ z_\psi &= \psi_4 x^{\frac{4}{3}} + \frac{p_1}{70}\left(20\chi_4 (g-1) + p_1 \chi_4 (10g-3)\right) + \mathcal{O}\left(x^{\frac{8}{3}}\right) \end{aligned} \tag{7.58}$$

We can use our shooting method to fix the constants $\{\alpha_2, z_3, \psi_4, \alpha_0^h, z_0^h, \psi_0^h\}$ as functions of p_1. Figure 7.3 shows a plot of ψ_4^{-2} versus T_c/T. The blue dots are the numerical results, and the red curve is the best linear fit, which intersects the horizontal axis precisely at the critical temperature. The excellent linear fit suggests that

$$\frac{1}{\psi_4\left(\Omega = 0, Q = 0\right)^2} = \frac{1}{\chi_T^2} \sim |t| \tag{7.59}$$

or

$$\chi_T \sim |t|^{-\frac{1}{2}} \sim |t|^{-\gamma} \Rightarrow \gamma = \frac{1}{2}, \tag{7.60}$$

in agreement with our previous result (7.33).

Next we will compute the critical exponents ν and η. Let $\tilde{G}(\mathbf{k})$ denote the Fourier transform of the two-point correlation funtion $G(\mathbf{r})$ defined in (7.4). The dynamic susceptibility is related this by

$$\tilde{G}(\mathbf{k}) = T\chi_{\mathbf{k}}, \quad \text{where} \quad \chi_{\mathbf{k}} = \chi_{0,\mathbf{k}}. \tag{7.61}$$

Close to the phase transition we have

$$G(\mathbf{r}) \propto e^{-\frac{|\mathbf{r}|}{\xi}} \Rightarrow \tilde{G}(\mathbf{k}) \propto \int d^3 r e^{-\frac{|\mathbf{r}|}{\xi}} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{7.62}$$

We can do the integral in polar coordinates. The result is

$$\begin{aligned} G(\mathbf{k}) &= \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r}{\xi} + i\hat{q}r\cos\theta} r dr d\theta \\ &= \frac{2\pi\xi^2}{(1+\hat{q}^2\xi^2)^{\frac{3}{2}}}, \end{aligned} \tag{7.63}$$

where $\hat{q} \equiv |\mathbf{k}|$. So the dynamic susceptibility

$$\psi_4(0,\hat{q}) \sim \frac{1}{(\hat{q}^2 + \xi^{-2})^{\frac{3}{2}}} \tag{7.64}$$

has a pole when

$$\hat{q}^2 \equiv Q = -\frac{1}{\xi^2}. \tag{7.65}$$

To locate these poles, we expand (7.37) about $\Omega = 0$, and we keep only the leading order terms. Some of the coefficents are still to long to write down, but here are a few of the shorter ones. With $\Omega \to 0$, we have

$$a_H = a_\alpha = a_\psi = 1 \tag{7.66}$$

$$b_H = \frac{c_2}{2yc_2'}\left(\frac{2c_2' - yc_2\left[(\phi')^2 + (\chi')^2\right]}{c_2 + 2yc_2'}\right) \tag{7.67}$$

$$b_\alpha = b_\psi = \frac{1}{y}, \tag{7.68}$$

and all other coefficients carry Q-dependence. We also expand the asymptotic expansions (7.41) and (7.42) about $\Omega \to 0$ and keep the leading terms. Although the coefficients are still rather large in the horizon expansions, the results for the boundary expansions are

$$\begin{aligned}
z_H &= z_3 x - \left[\frac{3}{5}p_1 + \frac{3}{80}\frac{2^{\frac{2}{3}}\alpha^2 a_0^3 z_3\left(6 + p_0^2 - 2c_0^2 - gp_0^2c_0^2\right)}{3a_1 + a_1}Q\right]x^{\frac{5}{3}} + \mathcal{O}\left(x^2\right) \\
z_\alpha &= x^{\frac{1}{3}} + \alpha_2 x^{\frac{2}{3}} + \left[\frac{2}{5}p_1^2 - \frac{3}{16}\frac{2^{\frac{2}{3}}\alpha^2 a_0^3\left(6 + p_0^2 - 2c_0^2 - gp_0^2c_0^2\right)}{3a_1 + a_1}Q\right]x + \mathcal{O}\left(x^{\frac{4}{3}}\right) \\
z_\psi &= \psi_4 x^{\frac{4}{3}} + \left[\frac{p_1}{70}\left(20\chi_4\left(g-1\right) + p_1\chi_4\left(10g - 3\right)\right)\right. \\
&\qquad \left. - \frac{3}{122}\frac{2^{\frac{2}{3}}\alpha^2 a_0^3\psi_4\left(6 + p_0^2 - 2c_0^2 - gp_0^2c_0^2\right)}{3a_1 + a_1}Q\right] + \mathcal{O}\left(x^{\frac{8}{3}}\right)
\end{aligned} \tag{7.69}$$

We can use our shooting method to fix the constants $\{\alpha_2, z_3, \psi_4, \alpha_0^h, z_0^h, \psi_0^h\}$ as functions of Q with p_1 held fixed.

Figure 7.4 (top) shows a typical plot of $1/\psi_4$ versus Q with p_1 held fixed. In this case $p_1 = 0.450$. We see that there is one pole[3] for $Q > 0$, and multiple poles for $Q < 0$. The spike around $Q \approx -2$ is just a point where the function ψ_4 vanishes. Figure 7.4 (bottom) shows that the positive-Q pole moves to the origin as

[3]By probing very far into the $Q > 0$ regime we checked that there is only one pole.

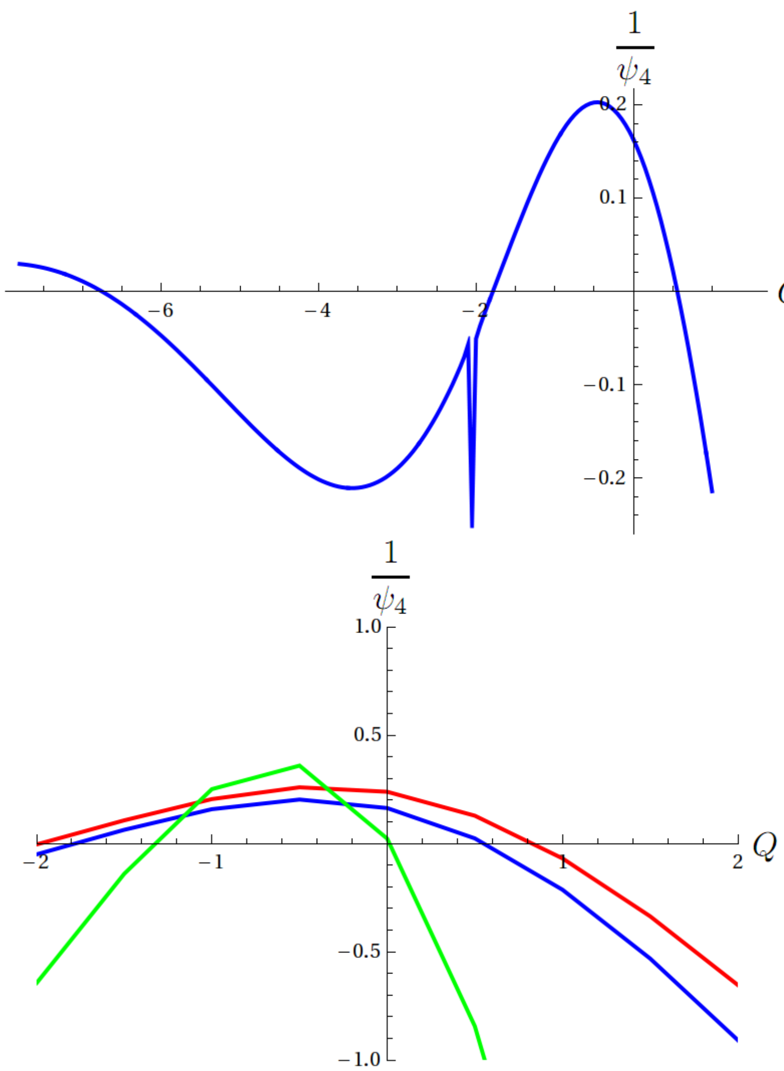

Figure 7.4: Inverse of the dynamic susceptibility $1/\psi_4$ versus the momentum squared Q with p_1 fixed. Top: $p_1 = 0.450$. Bottom: Red curve has $p_1 = 0.440$. Blue curve has $p_1 = 0.450$. Green curve has $p_1 \approx p_c \approx 0.464344$

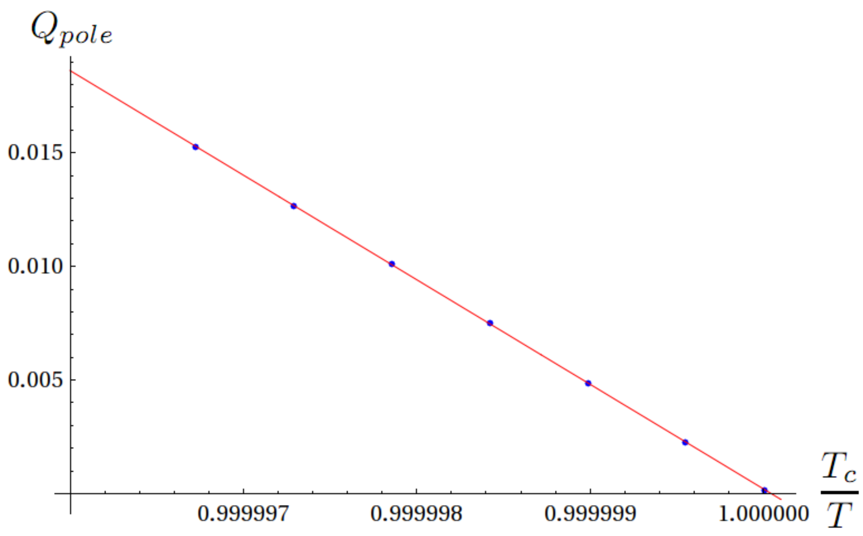

Figure 7.5: Positive-Q pole as a function of p_1. The blue dots are the numerical results. The red curve is the best linear fit.

p_1 moves toward its critical value $p_c \approx 0.464344$. Figure 7.5 shows the position of the positive-Q pole on the Q-axis (as in figure 7.4) as a function of p_1. The blue dots are the numerical results, and the red curve is the best linear fit. The line intersects the horizontal axis precisely at the critical value of p_1. The excellent straight line fit together with (7.65) suggests that

$$Q_{pole} \sim |t| \sim \xi^{-2}. \tag{7.70}$$

Recalling that the critial exponent ν is defined by

$$\xi \sim |t|^{-\nu}, \qquad t \neq 0, \tag{7.71}$$

we conclude that

$$\nu = \frac{1}{2}. \tag{7.72}$$

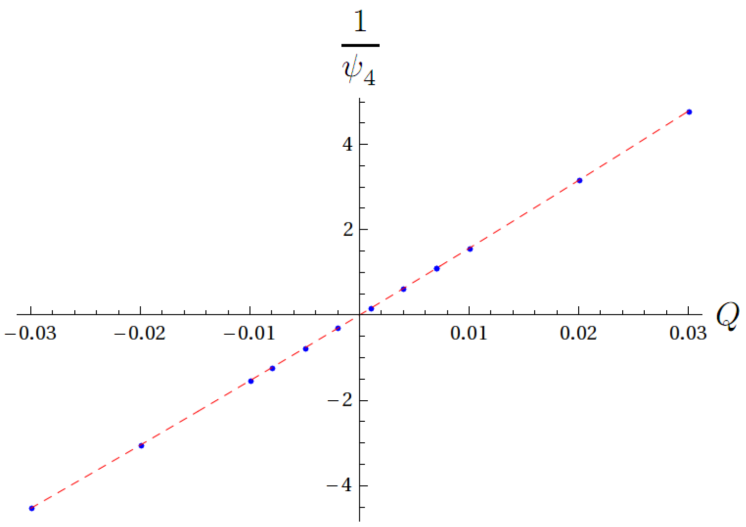

Figure 7.6: Inverse of the dynamic susceptibility $1/\psi_4$ versus the momentum squared Q at criticality $p_1 = p_c \approx 0.464344$ in the limit where $\Omega \to 0$. The blue dots are the numerical results, and the red dashed curve is the best linear fit.

At the criticality we have

$$G(\mathbf{r}) \sim |\mathbf{r}|^{-\eta}, \tag{7.73}$$

where we took the number of spatial dimensions $p = 2$ in (7.9). Taking the Fourier transforms we get

$$\begin{aligned} \tilde{G}(\mathbf{k}) &\sim \int d^3r\, |\mathbf{r}|^{-\eta}\, e^{i\mathbf{k}\cdot\mathbf{r}} \\ &= \int_0^{2\pi}\int_0^{\infty} r^{-\eta} e^{i\hat{q}r\cos\theta} r dr d\theta \\ &= -2\pi\eta 2^{-\eta}\frac{\Gamma\left(-\frac{\eta}{2}\right)}{\Gamma\left(\frac{\eta}{2}\right)}|\hat{q}|^{\eta-2} \end{aligned} \tag{7.74}$$

or since $\hat{q}^2 = Q$,

$$\psi_4(0, Q) \sim \frac{1}{Q^{\frac{2-\eta}{2}}}. \tag{7.75}$$

Figure 7.6 shows a plot of $1/\psi_4$ versus Q with $p_1 = p_c \approx 0.464344$ held fixed. The blue dots are the numerical results, and the red dashed curve is the best linear fit. The linear fit crosses the horizontal axis at $Q \approx 4.27852 \times 10^{-6}$, suggesting that ψ_4 has a pole at the origin $Q = 0$. The excellent linear fit suggests that

$$\psi_4 (0, Q)|_{critical} \sim \frac{1}{Q}. \tag{7.76}$$

Comparing to (7.3.1), we conclude that

$$\eta = 0. \tag{7.77}$$

To summarize the results of this chapter so far, we have computed the static critical exponents

$$\{\alpha, \beta, \gamma, \delta, \eta, \nu\} = \left\{0, \frac{1}{2}, \frac{1}{2}, 2, 0, \frac{1}{2}\right\}. \tag{7.78}$$

Notice that γ and δ are not of mean-field type. Mean-field critical exponents are expected for a classical gravity dual to field theory [5, 9]. This is a result of the fact that classical gravity corresponds to large gauge group number N in the dual field theory. In our Exotic model, however, the dual field theory is not a true gauge theory. In the typical examples of large N field theories the central charge behaves like [2]

$$c \sim N^2, \tag{7.79}$$

but for AdS_4 we have

$$c \sim N^{\frac{3}{2}}. \tag{7.80}$$

This is perhaps the reason why we do not find mean-field critical exponents.

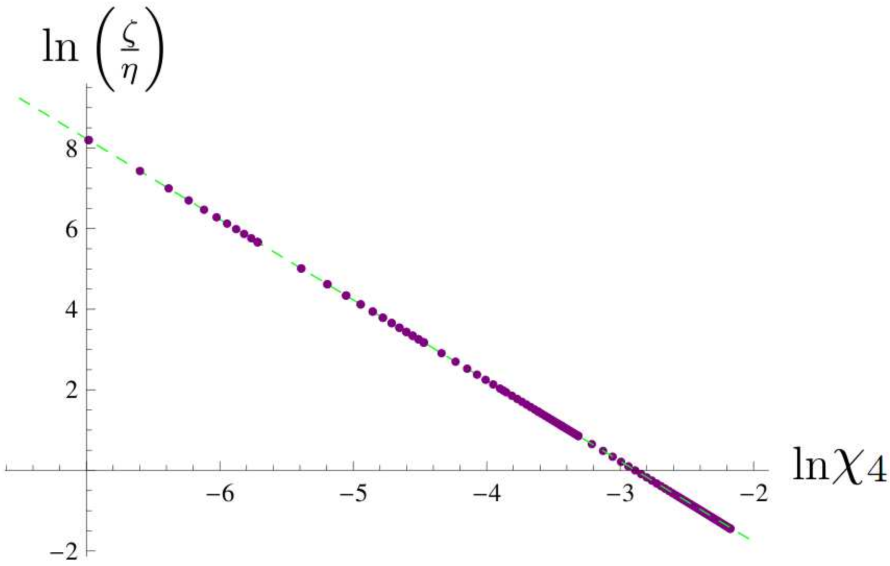

Figure 7.7: Bulk-to-shear viscosity in the first symmetry-broken phase versus the order parameter near criticality on a double logarithmic scale. The purple dots are the numerical results. The dashed green line is the best linear fit. The slope is -1.9999.

7.3.2 Bulk-to-shear viscosity ratio

Figure 7.7 shows the bulk-to-shear viscosity versus the order parameter near criticality on a double logarithmic scale. The purple dots are the numerical results. The dashed green line is the best linear fit. The slope of the line is -1.9999. The power-law behaviour strongly suggests that the bulk viscosity in the first symmetry-broken phase diverges at the phase transition. The excellent linear fit suggests that

$$\frac{\zeta}{\eta} \sim \frac{1}{\chi_4^2}. \tag{7.81}$$

In the previous section we found that

$$\chi_4 \sim |t|^{\frac{1}{2}}. \tag{7.82}$$

Thus we conclude that the scaling behaviour of the bulk-to-shear viscosity ratio near criticality is

$$\frac{\zeta}{\eta} \sim |t|^{-1} \tag{7.83}$$

Several proposals have been made regarding the scaling behaviour of ζ/η near criticality. In [30] the authors propose that

$$\zeta \sim |t|^{-\alpha}. \tag{7.84}$$

We found that $\alpha = 0$, so this model is inconsistent with our results. In [39] it is argued that

$$\zeta \sim |t|^{\alpha+4\beta-1} \tag{7.85}$$

We found that $\beta = 1/2$, so $\alpha + 4\beta - 1 = 1$, so this model is also not consistent with our results. Finally, in [34], it is argued that

$$\zeta \sim |t|^{-z\nu+\alpha}. \tag{7.86}$$

We found that $\nu = 1$, so this model is consistent with our results provided that the dynamical critical exponent is $z = 1$. In the next section we will calculate z explicitly, and show that in fact $z = 2$. Therefore, none of these scaling proposals are consistent with our model.

7.3.3 Dynamical critical exponent

Following [9], the full dynamic susceptibility $\psi_4(\Omega, Q)$ will have a pole at

$$\Omega \sim \xi^{-z}, \tag{7.87}$$

where z is the dynamical critical exponent of the system. In the hydrodynamic limit, this defines a relaxation time τ where

$$\tau \equiv \Omega^{-1} \sim \xi^{z} \sim |t|^{-\nu z}\,. \tag{7.88}$$

To track the pole in $\psi_4\,(\Omega, Q)$ we must numerically solve (7.37) to fix the parameters $\left\{\alpha_2, z_3, \psi_4, \alpha_0^h, z_0^h, \psi_0^h\right\}$ as functions of Ω with both p_1 and Q held fixed. Figure 7.8 (top) shows a typical plot of ψ_4^{-1} versus Ω, in this case with $p_1 = 0.455$ and $Q = 10^{-5}$ held fixed. Generically, there is one pole that occurs at a negative value of Ω. Figure 7.8 (bottom) shows the position of the pole in $\psi_4\,(\Omega, Q)$ as a function of $\alpha p_1 \sim t$. The excellent linear fit suggests that

$$\Omega_{pole} \sim |t|\,. \tag{7.89}$$

Comparing to (7.88) we see that

$$\nu z = 1. \tag{7.90}$$

And since we already found that $\nu = \frac{1}{2}$, we conclude that

$$z = 2. \tag{7.91}$$

Gubser and Mitra [26] conjectured that perturbative instabilities in translationally invariant black holes appear as Gregory-Laflamme instabilities [23, 24], which are defined by

$$\mathrm{Im}\,(\hat{\omega})|_{\hat{q}<\hat{q}_c} > 0, \tag{7.92}$$

where $\hat{q}_c$ is the momentum at the threshold of the instability (i.e. the value of $\hat{q}$ where $\mathrm{Im}\,(\hat{\omega})$ changes sign), and we assume that $\mathrm{Im}\,(\hat{q}) = 0$. Figure 7.9 shows a plot of Q vs T_c/T in the symmetry-broken phase. The red curve represents the

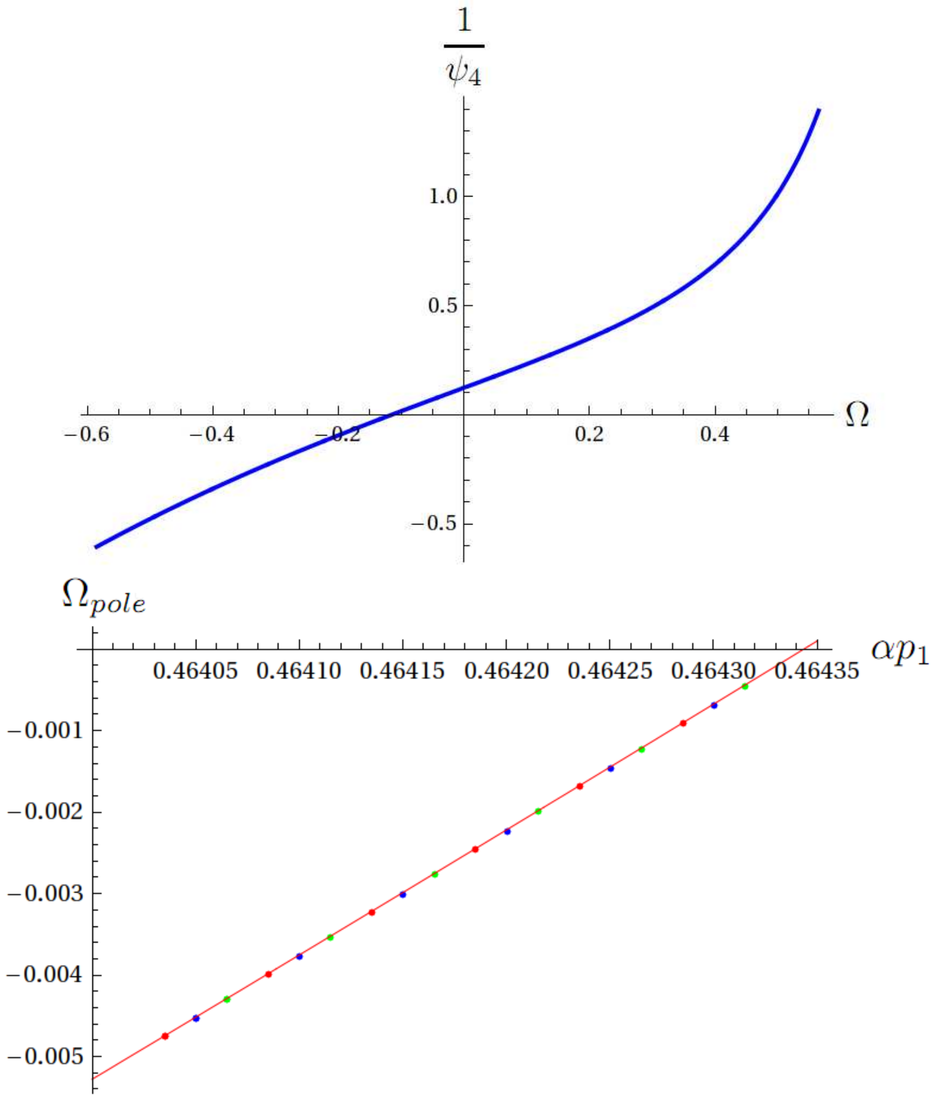

Figure 7.8: Top: Plot of ψ_4^{-1} versus Ω, with $p_1 = 0.455$ and $Q = 10^{-5}$ held fixed. Bottom: Position of the pole in $\psi_4(\Omega, Q)$ in the hydrodynamic limit as a function of $\alpha p_1 \sim t$. The blue dots are the numerical results for $Q = 10^{-5}$, the green dots are for $Q = 10^{-7}$, and the red dots are for $Q = 10^{-9}$. The red line is the best linear fit to the red dots.

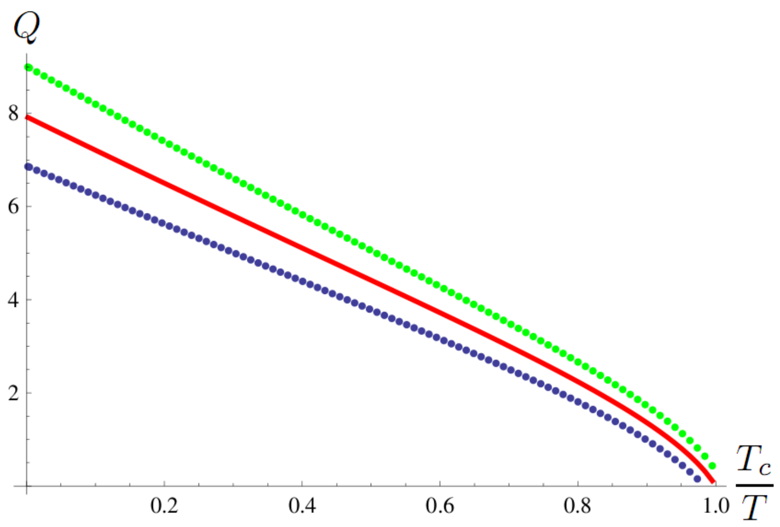

Figure 7.9: Plot of Q vs T_c/T in the symmetry-broken phase. The red curve represents the modes at the threshold $(\hat{\omega}, \hat{q}) = (0, \hat{q}_c)$. The green/blue dots are the stable/unstable modes with $\hat{\omega} = -0.1i$ and $\hat{\omega} = 0.1i$ respectively.

modes at the threshold $(\hat{\omega}, \hat{q}) = (0, \hat{q}_c)$. The green/blue dots are the stable/unstable modes with $\hat{\omega} = -0.1i$ and $\hat{\omega} = 0.1i$ respectively. The existence of the unstable modes (blue dots) represents a genuine instability in the system. This is interesting because it verifies that the Exotic Model is a counter example to the Correlated Stability Conjecture given in [26]. Related counter-examples are studied in [21].

Chapter 8

Conclusions

Let us summarize and discuss the results and conclusion of this thesis. By considering AdS gravity in four and five dimensions, we calculated the thermodynamics of dual CFTs in three and four dimensions, respectively. We verified that the results are consistent with the laws of thermodynamics. We computed the dispersion relation of fluctuations of AdS_4 and concluded that the background solutions are stable. Then we considered the Exotic Model, which is essentially a model of black holes in AdS_4 with scalar hair. We computed the thermodynamics of this model and found an exotic type of second-order phase transition, where the symmetry-broken phase occurs above the critical temperature. Certain condensed matter systems exhibit this behaviour [15]. It would be interesting to study whether our model may have some applications in such condensed matter systems. We computed the speed of sound and bulk-to-shear viscosity ratio by considering the dispersion relation of small fluctuations of the background. We found that the bulk-to-shear viscosity ratio diverges at the phase transition in the symmetry-broken phase.

We interpreted of our physical parameters in terms of the language of ferromagnetism in order to study the critical phenomena associated with the phase transition from the symmetry-broken phase to the symmetric phase. We developed techniques to compute the critical exponents of our theory, and we found that the exponents are not of the mean-field type as expected in the large N limit. The positivity of the speed of sound squared suggested that all phases of this model are thermodynamically stable. On the other hand, by calculating the dynamical critical exponent, we discovered that this model has a genuine Gregory-Laflamme instability, and is thus classically unstable. This model is a counter-example to the Correlated Stability Conjecture.

The majority of the calculations done here had a fixed coupling $g = -100$. We argued from the numerics that as $g \to \infty$, the symmetry-broken phases approach the symmetric one. It would be interesting to show analytically that no phase crossing occurs in this limit. If such phase crossing does occur, then the symmetry-broken phases would be thermodynamically preferrable (i.e. lower free energy), and there could be interesting critical behaviour associated with such phase crossing.

The model here is phenomenological, and little is known about the dual field theory. Nevertheless, it is a simple example of the AdS/CFT correspondence in a non-conformal setting. The techniques used in this thesis are quite general and can be straightforwardly applied to other AdS/CFT models. For example, virtually all of the calcuations done here have also been applied to the celebrated $\mathcal{N} = 4$ SYM theory [5], and the $\mathcal{N} = 2^*$ model [9], which is among the best approximations to strongly coupled QCD and a description of the strongly coupled quark-gluon plasma created at the Relativistic Heavy Ion Collider [19].

Appendix A

Hawking temperature of AdS black holes

Here we will calculate the Hawking temperature of a metric of the form

$$ds^2 = -c_1(r)^2 dt^2 + c_2(r)^2 d\mathbf{x}^2 + c_3(r)^2 dr^2. \tag{A.1}$$

Changing to the new radial coordinate and imaginary time

$$y = \frac{c_1}{c_2}, \qquad t = it_E \tag{A.2}$$

we can write the metric as

$$\begin{aligned} ds^2 &= y^2 c_2(y)^2 dt_E^2 + c_2(y)^2 d\mathbf{x}^2 + c_3(y)^2 \left(\frac{dy}{dr}\right)^{-2} dy^2 \\ &= c_3^2 \left(\frac{dy}{dr}\right)^{-2} \left[dy^2 + y^2 \frac{c_2^2}{c_3^2} \left(\frac{dy}{dr}\right)^2 dt_E^2\right] + c_2^2 d\mathbf{x}^2 \\ &= c_3^2 \left(\frac{dy}{dr}\right)^{-2} \left[dy^2 + y^2 d\left(\frac{c_2^2}{c_3^2} \left(\frac{dy}{dr}\right) t_E\right)^2\right] + c_2^2 d\mathbf{x}^2 \\ &= c_3^2 \left(\frac{dy}{dr}\right)^{-2} \left[dy^2 + y^2 dt'^2\right] + c_2^2 d\mathbf{x}^2, \end{aligned} \tag{A.3}$$

where

$$t' = \frac{c_2}{c_3} \left(\frac{dy}{dr}\right) t_E. \tag{A.4}$$

In (A.3), the quantity in square brackets has the form of the metric for a two-dimensional plane in polar coordinates, where y plays the role of the radial co-ordinate, and t' plays the role of the angular coordinate. If we expand the metric to leading order about the horizon $y = 0$, we must insist that the coordinate t' be periodic in the sense as the angular coordinate in $\mathbb{R}^2$. That is

$$t' = t' + 2\pi. \tag{A.5}$$

Otherwise, the metric in square brackes would be that of a cone, and we would have a conical singularity at the horizon $y = 0$. Thus we have

$$t_E = t_E + \frac{2\pi}{\frac{c_2}{c_3}\left(\frac{dy}{dr}\right)}. \tag{A.6}$$

We identify the period of imaginary time with the temperature T as

$$t_E = t_E + \frac{1}{T}, \quad \text{so} \quad T = \frac{1}{2\pi} \left(\frac{c_2}{c_3} \frac{dy}{dr}\right)\bigg|_{horizon} \tag{A.7}$$

Appendix B

Numerical shooting method

B.1 Input

Here we outline the algorithm of our numerical shooting method. In the typical scheme, we have a set of n ordinary second-order differential equations of the form $b_j\left(g''_{B_i}(x), g'_{B_i}(x), g_{B_i}(x), x\right) = 0$, which govern the set of fields that we will denote $\{g_{B_i}\}$, $i = 1...n$. We have chosen the independent variable x such that the domain is mapped to $x \in [0, 1]$, where $x = 0$ corresponds to the AdS boundary and $x = 1$ corresponds to the black hole horizon. We generate another set of n equations of the form $h_j\left(g''_{H_i}(y), g'_{H_i}(y), g_{H_i}(y), x\right) = 0$ using the variable $y = 1 - x$, such that $y = 0$ ($y = 1$) corresponds to the horizon (boundary). Note that the domain is $y \in [0, 1]$, and that g_{H_i} is the same field as g_{B_i} but expressed in terms of the y variable. The labels B and H serve only to distinguish the field g_{H_i} and g_{B_i} as separate functions in our numerical method. Arranging the equations in a vector, this looks like,

$$\mathbf{f} = \begin{bmatrix} b_1\left(g''_{B_i}, g'_{B_i}, g_{B_i}, x\right) \\ \vdots \\ b_n\left(g''_{B_i}, g'_{B_i}, g_{B_i}, x\right) \\ h_1\left(g''_{H_i}, g'_{H_i}, g_{H_i}, y\right) \\ \vdots \\ h_n\left(g''_{H_i}, g'_{H_i}, g_{H_i}, y\right) \end{bmatrix} = 0, \qquad i = 1...n \tag{B.1}$$

We can produce series solutions for the fields g_{B_i} (g_{H_i}) about $x = 0$ ($y = 0$) to arbitrary order[1]. This typically generates $4n$ integration constants, some of which we can immediately fix using boundary conditions and other physical conditions. At the very least, $2n$ integration constants should remain unfixed[2]. We can arrange the integration constants c_j as

$$\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_{2n} \end{bmatrix} = 0. \tag{B.2}$$

[1]Usually just a few terms is sufficient.

[2]If more than $2n$ constants remain, then we repeat this algorithm for each set of values for the remaining constants in which we are interested

And we can arrange the series solutions as

$$\mathbf{e} = \begin{bmatrix} G_{B_1}(\mathbf{c},\mathbf{x}) \\ \vdots \\ G_{B_n}(\mathbf{c},\mathbf{x}) \\ G_{H_1}(\mathbf{c},\mathbf{y}) \\ \vdots \\ G_{H_n}(\mathbf{c},\mathbf{y}) \end{bmatrix}, \qquad \mathbf{e}' = \begin{bmatrix} \partial_x G_{B_1}(\mathbf{c},\mathbf{x}) \\ \vdots \\ \partial_x G_{B_n}(\mathbf{c},\mathbf{x}) \\ \partial_y G_{H_1}(\mathbf{c},\mathbf{y}) \\ \vdots \\ \partial_y G_{H_n}(\mathbf{c},\mathbf{y}) \end{bmatrix}. \tag{B.3}$$

where G_{B_i} (G_{H_i}) is the series solution for g_{B_i} (g_{H_i}).

B.2 Algorithm

Now we make an initial guess for the constants in $\mathbf{c}$. Call this guess $\mathbf{c}_0$. In order to numerically solve the system (B.1), we need to specify an initial condition for $g_{B_i}(x = 0)$ and $g'_{B_i}(x = 0)$ for the boundary equations b_j, and likewise for $g_{H_i}(y = 0)$ and $g'_{H_i}(y = 0)$ for the horizon equations h_j. In practice, we choose the initial value for $x, y = \epsilon$ to be small, but not zero. Then our initial conditions become

$$\begin{aligned} g_{B_i}(x = 0) &\approx G_{B_i}(\mathbf{c}_0, \epsilon) \qquad & g'_{B_i}(x = 0) &\approx \partial_x G_{B_i}(\mathbf{c}_0, \epsilon) \\ g_{H_i}(y = 0) &\approx G_{H_i}(\mathbf{c}_0, \epsilon) \qquad & g'_{H_i}(y = 0) &\approx \partial_y G_{H_i}(\mathbf{c}_0, \epsilon) \end{aligned} \tag{B.4}$$

Next, we feed our equations (B.1) and our initial conditions (B.4) into a numerical ODE solver and integrate them over $x, y \in [0, 1/2]$. Notice that this integrates the boundary (horizon) equations b_j (h_j) from the boundary $x = 0$ (horizon $y = 0$) in to some point in the bulk given by $x = y = 1/2$. We define the error Δ_e to be

$$\Delta_e = \|\mathbf{v}(\mathbf{c}_0)\|, \quad \text{where} \quad \mathbf{v} = \begin{pmatrix} g_{B_1}\left(x = \frac{1}{2}\right) - g_{H_1}\left(y = \frac{1}{2}\right) \\ \vdots \\ g_{B_n}\left(x = \frac{1}{2}\right) - g_{H_n}\left(y = \frac{1}{2}\right) \\ \partial_x g_{B_1}\left(x = \frac{1}{2}\right) + \partial_y g_{H_1}\left(y = \frac{1}{2}\right) \\ \vdots \\ \partial_x g_{B_n}\left(x = \frac{1}{2}\right) + \partial_y g_{H_n}\left(y = \frac{1}{2}\right) \end{pmatrix}. \tag{B.5}$$

The vector $\mathbf{v}$ encodes the discontinuity in the fields and their derivatives at the point $x = y = 1/2$. Notice that error vector is a function of the integration constants, i.e. $\mathbf{v} = \mathbf{v}(\mathbf{c})$, since changing the values in $\mathbf{c}$ will change the values in $\mathbf{v}$. What we have calculated so far is $\mathbf{v}(\mathbf{c}_0)$. Since g_{B_i} and g_{H_i} are actually the same funtion, the correct values of $\mathbf{c}$ should lead to a smooth solution for the fields everywhere, and thus have $\mathbf{v} = 0$. Now we slightly vary our guesses for the constant[3], i.e.

$$\mathbf{c}_0 \to \mathbf{c}_0 + \Delta\mathbf{c}_0, \tag{B.6}$$

and we likewise compute the new error vector $\mathbf{v}(\mathbf{c}_0 + \Delta\mathbf{c}_0)$. At this point we have values populating both vectors, $\mathbf{v}(\mathbf{c}_0)$ and $\mathbf{v}(\mathbf{c}_0 + \Delta\mathbf{c}_0)$. Taylor series gives

$$\mathbf{v}(\mathbf{c}_0 + \Delta\mathbf{c}_0) = \mathbf{v}(\mathbf{c}_0) + V\Delta\mathbf{c}_0 + \mathcal{O}\left(\Delta\mathbf{c}_0^2\right), \tag{B.7}$$

where V is a matrix with elements given by

$$V_{i,j} = \frac{\partial v_i}{\partial c_j}, \quad i, j = 1...2n. \tag{B.8}$$

[3]In our codes, we actually vary one constant at a time, solving the system anew for each varied constant.

We can neglect terms with $\mathcal{O}\left(\Delta\mathbf{c}_0^2\right)$ if we choose $\Delta\mathbf{c}_0 << 1$. Our ultimate goal is to have a vanishing error vector. Thus, enforcing that

$$\mathbf{v}\left(\mathbf{c}_0 + \Delta\mathbf{c}_0\right) = 0, \tag{B.9}$$

equation (B.7) tells us that we should have chosen

$$\Delta\mathbf{c}_0 = -V^{-1}\mathbf{v}\left(\mathbf{c}_0\right). \tag{B.10}$$

Now we restart the algorithm with an initial guess of

$$\mathbf{c}_0 \rightarrow \mathbf{c}_0 - V^{-1}\mathbf{v}\left(\mathbf{c}_0\right), \tag{B.11}$$

and continue this way until the error Δ_e converges to a sufficiently small value.

Appendix C

Numerical code

Here we present the Mathematica code that was used to solve the system in section 5.6. In this example we have six equations: three boundary equations in terms of x, and three horizon equations in terms of y. There are seven integration constants that must be fixed: $\{p_1, p_2, \chi_4, a_0, a_1, p_0, c_0\}$. So we have one extra constant. We will apply our shooting method for various values of p_1 to compute the remaining six constants[1] as functions of p_1.

The following code handles the input to the algorithm:

```
>c2ha = alpha*(a0 + a1*xi^2 + a2*xi^4)/(1 - xi^2)^(1/3);
>phiha = p0 + P1*xi^2 + P2*xi^4;
>chiha = c0 + C1*xi^2 + C2*xi^4;
>dc2ha = D[c2ha, xi];
>dphiha = D[phiha, xi];
>dchiha = D[chiha, xi];
>c2ba = (alpha + A1*xi^(2/3) + A2*xi + A3*xi^(4/3))/(2*xi - xi^2)^(1/3);
>phiba = p1*xi^(1/3) + p2*xi^(2/3) + p3*xi + p4*xi^(4/3);
>chiba = chi4*xi^(4/3) + c1*xi^2 + c2*xi^(7/3) + c3*xi^(8/3);
```

[1] That is, they are constant with respect to x and y.

```
>dc2ba = D[c2ba, xi];
>dphiba = D[phiba, xi];
>dchiba = D[chiba, xi];
>c2eqnh = (1/4)*c2h[x]*(chih’[x])ˆ2 + (1/4)*c2h[x]*(phih’[x])ˆ2 + c2h”[x]
- 4*(c2h’[x])ˆ2/c2h[x] - (c2h’[x])/x;
>phieqnh = phih”[x] + (phih’[x])/x - phih[x]*(-8*(c2h’[x])*c2h[x] - 12*(c2h’[x])ˆ2*x +
(phih’[x])ˆ2*x*c2h[x]ˆ2 + (chih’[x])ˆ2*x*c2h[x]ˆ2)*(-1 + g*chih[x]ˆ2)/(c2h[x]ˆ2*x*(-6 -
phih[x]ˆ2 + 2*chih[x]ˆ2 + g*phih[x]ˆ2*chih[x]ˆ2));
>chieqnh = chih”[x] + (chih’[x])/x - chih[x]*(-8*(c2h’[x])*c2h[x] - 12*(c2h’[x])ˆ2*x +
(phih’[x])ˆ2*x*c2h[x]ˆ2 + (chih’[x])ˆ2*x*c2h[x]ˆ2)*(2 + g*phih[x]ˆ2)/(c2h[x]ˆ2*x*(-6 -
phih[x]ˆ2 + 2*chih[x]ˆ2 + g*phih[x]ˆ2*chih[x]ˆ2));
>c2eqnb = c2b”[x] - (c2b’[x])/(-1 + x) - (1/4)*(-(chib’[x])ˆ2*c2b[x]ˆ2 - (phib’[x])ˆ2*c2b[x]ˆ2
+ 16*(c2b’[x])ˆ2)/c2b[x];
>phieqnb = phib”[x] + (phib’[x])/(-1 + x) - phib[x]*(-8*(c2b’[x])*c2b[x] + 12*(c2b’[x])ˆ2
- 12*(c2b’[x])ˆ2*x - (phib’[x])ˆ2*c2b[x]ˆ2 + (phib’[x])ˆ2*c2b[x]ˆ2*x - (chib’[x])ˆ2*c2b[x]ˆ2
+ (chib’[x])ˆ2*c2b[x]ˆ2*x)*(g*chib[x]ˆ2 - 1)/(c2b[x]ˆ2*(-1 + x)*(-6 - phib[x]ˆ2 + 2*chib[x]ˆ2
+ g*phib[x]ˆ2*chib[x]ˆ2));
>chieqnb = chib”[x] + (chib’[x])/(-1 + x) - chib[x]*(-8*(c2b’[x])*c2b[x] + 12*(c2b’[x])ˆ2
- 12*(c2b’[x])ˆ2*x - (phib’[x])ˆ2*c2b[x]ˆ2 + (phib’[x])ˆ2*c2b[x]ˆ2*x - (chib’[x])ˆ2*c2b[x]ˆ2
+ (chib’[x])ˆ2*c2b[x]ˆ2*x)*(2 + g*phib[x]ˆ2)/(c2b[x]ˆ2*(-1 + x)*(-6 - phib[x]ˆ2 + 2*chib[x]ˆ2
+g*phib[x]ˆ2*chib[x]ˆ2));
>P2 = (1/9)*p0*a0ˆ6*g*c0*C1 - (1/9)*p0*a0ˆ6 + (1/9)*p0*a0ˆ6*g*c0ˆ2 - (1/3)*p0*a0ˆ5*a1
+ (1/3)*p0*a0ˆ5*a1*g*c0ˆ2 - (1/18)*P1*a0ˆ6 + (1/18)*P1*a0ˆ6*g*c0ˆ2;
>C2 = (1/9)*c0*a0ˆ6*g*p0*P1 + (1/9)*c0*a0ˆ6*g*p0ˆ2 + (2/9)*c0*a0ˆ6
+ (1/3)*c0*a0ˆ5*a1*g*p0ˆ2 + (2/3)*c0*a0ˆ5*a1 + (1/18)*C1*a0ˆ6*g*p0ˆ2 + (1/9)*C1*a0ˆ6;
>a2 = -(1/216)*(36*a0*a1 + 14*a0ˆ8*c0ˆ2 + 42*a0ˆ2 - 54*a1ˆ2 + 12*a0ˆ8*c0*C1 +
6*a0ˆ8*g*p0ˆ2*c0*C1 + 6*a0ˆ8*g*p0*P1*c0ˆ2 - 6*a0ˆ8*p0*P1 + 21*a0ˆ7*a1*g*p0ˆ2*c0ˆ2
- 126*a0ˆ7*a1 + 42*a0ˆ7*a1*c0ˆ2 - 21*a0ˆ7*a1*p0ˆ2 + 7*a0ˆ8*g*p0ˆ2*c0ˆ2 - 42*a0ˆ8 -
```

```
7*a0^8*p0^2)/a0;
>P1 = -(2/9)*p0*a0^6 + (2/9)*p0*a0^6*g*c0^2;
>C1 = (2/9)*c0*a0^6*g*p0^2 + (4/9)*c0*a0^6;
>A3 = -(1/28)*alpha*p2^2 - (167/22400)*alpha*p1^4;
>A2 = -(1/18)*alpha*p1*p2;
>A1 = -(1/40)*alpha*p1^2;
>c3 = (1/25200)*chi4*(-600*p2^2 - 81*p1^4 + 90*p1^4*g + 1400*g*p2^2 + 200*g^2*p1^4);
>c2 = (1/18)*chi4*(3*g*p1*p2 + 12 - p2*p1);
>c1 = (1/70)*chi4*p1^2*(10*g - 3);
>p3 = (3/20)*p1^3;
```

The following code implements the algorithm:

```
>xs = 10^-8; xf = 0.5; result = ; step = 0.0001; orgstep = 0.01; err = Infinity; g = -100; alpha = 1;
>result = p1s = 0, p2s = 0, chi4s = 0, a0s = 1, a1s = 0, p0s = 0, c0s = 0, err = 0
>Do[p1s = p1s + 0.01; chi4s = 0.01; c0s = 0.01;
Do[
soln = NDSolve[c2eqnh == 0, c2eqnb == 0, phieqnh == 0, phieqnb == 0, chieqnh == 0, chieqnb == 0, c2h[xi] == c2ha, c2h'[xi] == dc2ha, c2b[xi] == c2ba, c2b'[xi] == dc2ba, phih[xi] == phiha, phih'[xi] == dphiha, phib[xi] == phiba, phib'[xi] == dphiba, chih[xi] == chiha, chih'[xi] == dchiha, chib[xi] == chiba, chib'[xi] == dchiba /. xi -> xs /. p1 -> p1s /. p2 -> p2s /. chi4 -> chi4s /. a0 -> a0s /. a1 -> a1s /. p0 -> p0s /. c0 -> c0s, c2h, c2b, phih, phib, chih, chib, x, xs, xf, WorkingPrecision -> 30];
v = (c2b[xf] - c2h[xf], c2h'[xf] + c2b'[xf], phib[xf] - phih[xf], phih'[xf] + phib'[xf], chib[xf] - chih[xf], chih'[xf] + chib'[xf] /. soln)[[1]];
soln2 = NDSolve[c2eqnh == 0, c2eqnb == 0, phieqnh == 0, phieqnb == 0, chieqnh == 0, chieqnb == 0, c2h[xi] == c2ha, c2h'[xi] == dc2ha, c2b[xi] == c2ba, c2b'[xi] == dc2ba, phih[xi] == phiha, phih'[xi] == dphiha, phib[xi] == phiba, phib'[xi] == dphiba, chih[xi] == chiha, chih'[xi] == dchiha, chib[xi] == chiba, chib'[xi] == dchiba /. xi -> xs /. p1 ->
```

```
p1s /. p2 -> p2s*(1 + step) /. chi4 -> chi4s /. a0 -> a0s /. a1 -> a1s /. p0 -> p0s /. c0 ->
c0s, c2h, c2b, phih, phib, chih, chib, x, xs, xf, WorkingPrecision -> 30];
v2 = (c2b[xf] - c2h[xf], c2h'[xf] + c2b'[xf], phib[xf] - phih[xf], phih'[xf] + phib'[xf],
chib[xf] - chih[xf], chih'[xf] + chib'[xf] /. soln2)[[1]];
dvdp2 = (v2 - v)/(p2s*step);
soln3 = NDSolve[c2eqnh == 0, c2eqnb == 0, phieqnh == 0, phieqnb == 0, chieqnh == 0,
chieqnb == 0, c2h[xi] == c2ha, c2h'[xi] == dc2ha, c2b[xi] == c2ba, c2b'[xi] == dc2ba,
phih[xi] == phiha, phih'[xi] == dphiha, phib[xi] == phiba, phib'[xi] == dphiba, chih[xi]
== chiha, chih'[xi] == dchiha, chib[xi] == chiba, chib'[xi] == dchiba /. xi -> xs /. p1 ->
p1s /. p2 -> p2s /. chi4 -> chi4s*(1 + step) /. a0 -> a0s /. a1 -> a1s /. p0 -> p0s /. c0 ->
c0s, c2h, c2b, phih, phib, chih, chib, x, xs, xf, WorkingPrecision -> 30];
v3 = (c2b[xf] - c2h[xf], c2h'[xf] + c2b'[xf], phib[xf] - phih[xf], phih'[xf] + phib'[xf],
chib[xf] - chih[xf], chih'[xf] + chib'[xf] /. soln3)[[1]];
dvdchi4 = (v3 - v)/(chi4s*step);
soln4 = NDSolve[c2eqnh == 0, c2eqnb == 0, phieqnh == 0, phieqnb == 0, chieqnh == 0,
chieqnb == 0, c2h[xi] == c2ha, c2h'[xi] == dc2ha, c2b[xi] == c2ba, c2b'[xi] == dc2ba,
phih[xi] == phiha, phih'[xi] == dphiha, phib[xi] == phiba, phib'[xi] == dphiba, chih[xi]
== chiha, chih'[xi] == dchiha, chib[xi] == chiba, chib'[xi] == dchiba /. xi -> xs /. p1 ->
p1s /. p2 -> p2s /. chi4 -> chi4s /. a0 -> a0s*(1 + step) /. a1 -> a1s /. p0 -> p0s /. c0 ->
c0s, c2h, c2b, phih, phib, chih, chib, x, xs, xf, WorkingPrecision -> 30];
v4 = (c2b[xf] - c2h[xf], c2h'[xf] + c2b'[xf], phib[xf] - phih[xf], phih'[xf] + phib'[xf],
chib[xf] - chih[xf], chih'[xf] + chib'[xf] /. soln4)[[1]];
dvda0 = (v4 - v)/(a0s*step);
soln5 = NDSolve[c2eqnh == 0, c2eqnb == 0, phieqnh == 0, phieqnb == 0, chieqnh == 0,
chieqnb == 0, c2h[xi] == c2ha, c2h'[xi] == dc2ha, c2b[xi] == c2ba, c2b'[xi] == dc2ba,
phih[xi] == phiha, phih'[xi] == dphiha, phib[xi] == phiba, phib'[xi] == dphiba, chih[xi]
== chiha, chih'[xi] == dchiha, chib[xi] == chiba, chib'[xi] == dchiba /. xi -> xs /. p1 ->
p1s /. p2 -> p2s /. chi4 -> chi4s /. a0 -> a0s /. a1 -> a1s*(1 + step) /. p0 -> p0s /. c0 ->
```

c0s, c2h, c2b, phih, phib, chih, chib, x, xs, xf, WorkingPrecision -> 30];

v5 = (c2b[xf] - c2h[xf], c2h'[xf] + c2b'[xf], phib[xf] - phih[xf], phih'[xf] + phib'[xf], chib[xf] - chih[xf], chih'[xf] + chib'[xf] /. soln5)[[1]];

dvda1 = (v5 - v)/(a1s*step);

soln6 = NDSolve[c2eqnh == 0, c2eqnb == 0, phieqnh == 0, phieqnb == 0, chieqnh == 0, chieqnb == 0, c2h[xi] == c2ha, c2h'[xi] == dc2ha, c2b[xi] == c2ba, c2b'[xi] == dc2ba, phih[xi] == phiha, phih'[xi] == dphiha, phib[xi] == phiba, phib'[xi] == dphiba, chih[xi] == chiha, chih'[xi] == dchiha, chib[xi] == chiba, chib'[xi] == dchiba /. xi -> xs /. p1 -> p1s /. p2 -> p2s /. chi4 -> chi4s /. a0 -> a0s /. a1 -> a1s /. p0 -> p0s*(1 + step) /. c0 -> c0s, c2h, c2b, phih, phib, chih, chib, x, xs, xf, WorkingPrecision -> 30];

v6 = (c2b[xf] - c2h[xf], c2h'[xf] + c2b'[xf], phib[xf] - phih[xf], phih'[xf] + phib'[xf], chib[xf] - chih[xf], chih'[xf] + chib'[xf] /. soln6)[[1]];

dvdp0 = (v6 - v)/(p0s*step);

soln7 = NDSolve[c2eqnh == 0, c2eqnb == 0, phieqnh == 0, phieqnb == 0, chieqnh == 0, chieqnb == 0, c2h[xi] == c2ha, c2h'[xi] == dc2ha, c2b[xi] == c2ba, c2b'[xi] == dc2ba, phih[xi] == phiha, phih'[xi] == dphiha, phib[xi] == phiba, phib'[xi] == dphiba, chih[xi] == chiha, chih'[xi] == dchiha, chib[xi] == chiba, chib'[xi] == dchiba /. xi -> xs /. p1 -> p1s /. p2 -> p2s /. chi4 -> chi4s /. a0 -> a0s /. a1 -> a1s /. p0 -> p0s /. c0 -> c0s*(1 + step), c2h, c2b, phih, phib, chih, chib, x, xs, xf, WorkingPrecision -> 30];

v7 = (c2b[xf] - c2h[xf], c2h'[xf] + c2b'[xf], phib[xf] - phih[xf], phih'[xf] + phib'[xf], chib[xf] - chih[xf], chih'[xf] + chib'[xf] /. soln7)[[1]];

dvdc0 = (v7 - v)/(c0s*step);

deltas = -Transpose[Inverse[dvdp2, dvdchi4, dvda0, dvda1, dvdp0, dvdc0]].v[[1]], v[[2]], v[[3]], v[[4]], v[[5]], v[[6]];

p2s = p2s + deltas[[1, 1]]; chi4s = chi4s + deltas[[2, 1]]; a0s = a0s + deltas[[3, 1]]; a1s = a1s + deltas[[4, 1]]; p0s = p0s + deltas[[5, 1]]; c0s = c0s + deltas[[6, 1]];

soln = NDSolve[c2eqnh == 0, c2eqnb == 0, phieqnh == 0, phieqnb == 0, chieqnh == 0, chieqnb == 0, c2h[xi] == c2ha, c2h'[xi] == dc2ha, c2b[xi] == c2ba, c2b'[xi] == dc2ba, phih[xi] == phiha, phih’[xi] == dphiha, phib[xi] == phiba, phib’[xi] == dphiba, chih[xi] == chiha, chih’[xi] == dchiha, chib[xi] == chiba, chib’[xi] == dchiba /. xi -> xs /. p1 -> p1s /. p2 -> p2s /. chi4 -> chi4s /. a0 -> a0s /. a1 -> a1s /. p0 -> p0s /. c0 -> c0s, c2h, c2b, phih, phib, chih, chib, x, xs, xf, WorkingPrecision -> 40];

v = (c2b[xf] - c2h[xf], c2h’[xf] + c2b’[xf], phib[xf] - phih[xf], phih’[xf] + phib’[xf], chib[xf] - chih[xf], chih’[xf] + chib’[xf] /. soln)[[1]];

err = Norm[v]; Print[err];, l, 1, 8];

result = Append[result, p1s, p2s, chi4s, a0s, a1s, p0s, c0s, err]; Export[”output.dat”, result];, q, 1, 100]

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