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Descent Systems, Eulerian Polynomials and Toric Varieties

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Abstract

We consider the permutation group in $n+1$ letters, S_{n+1}, which corresponds to the finite Weyl group of type A_n. S_{n+1} is generated by the set of simple reflections which corresponds to the set S of transpositions $s_1 = (12)\cdots, s_n = (n\ n+1)$. Let $J \subset S$. Using the theory of reductive algebraic monoids and representation theory of semisimple algebraic groups we associate to J a projective toric variety denoted by $X(J)$. We are interested in computing the Betti numbers of the projective variety $X(J)$ when it is a rationally smooth variety i.e., $X(J)$ has sufficiently mild singularities. Betti numbers are an important topological invariant of a space which help us measure the number of holes or cuts present in that space and classify spaces with the same topological structure. Moreover we define the Poincaré polynomial of a topological space as the generating function of its Betti numbers, via the polynomial whose coefficients are the Betti numbers.

We associate to $X(J)$ a simple polytope P_λ obtained as the convex hull of the S_{n+1}-orbit of a weight λ fixed by only the simple reflections $J = \{s_n, s_{n-1}, \cdots, s_{n-k+1}\}$ for some k with respect to the A_n root lattice. The Betti numbers of the variety $X(J)$ can be computed using the h-polynomial associated to the polytope P_λ. It is well-known that the Eulerian polynomials, which count permutations in S_{n+1} by their number of descents, give the h-polynomial of the simple polytopes known as permutohedra. Therefore the Eulerian polynomials give the Betti numbers for certain smooth toric varieties associated with the permutohedra.

In this thesis we derive a recurrence formula for the h-polynomials of a family of polytopes generalizing this. When $J = \{s_{n+k+1}, s_{n-k+2}, \cdots, s_n\}$, $1 \leq k \leq n$ and (W, S) is Weyl group of type A_n, we obtain a formula for the Poincaré polynomial of $X(J)$, in terms of Eulerian polynomials. Furthermore, we compute explicitly the Poincaré polynomial of $X(J)$ using the method of descent systems, in the case of a (W, S) finite Weyl group of type A_n, and J being a combinatorially smooth of the following forms: (1) $J = \{s_1, s_4, s_5, \cdots, s_n\} \subset S$ and (2) $J = \{s_4, s_5, \cdots, s_n\} \subset S$.

Contents

Introduction

Toric varieties and their cohomology have played an increasingly important role in studying the combinatorics of convex polytopes. They started around 1980 with Stanley's spectacular proof of the necessity of McMullen's conditions (characterizing the face numbers of a simple polytope) using the cohomology of rationally smooth projective toric varieties. This connection between the topology of toric varieties and the combinatorial geometry of convex polytopes is of interest to us.

Let (W, S) be a finite Weyl group of type A_n. Let J be any proper subset of S. Associated with J is a certain projective toric variety $X(J)$. We would like to calculate the Betti numbers of $X(J)$ when J is combinatorially smooth, *i.e.*, $X(J)$ is a rationally smooth variety using the h–polynomial of a simple polytope P_λ associated to an irreducible representation of a semisimple algebraic group.

The most basic combinatorial data of a d-dimensional convex polytope are the numbers f_i of i-dimensional faces encoded in the face polynomial $f(t) := \sum_{i=0}^{d} f_i t^i$. For simple polytopes, *i.e.*, where each vertex lies on exactly d edges, the possible f-polynomials are expressed in terms of the h-polynomials $h(t) = f(t-1) = \sum_{i=0}^{d} h_i t^i$ where h_i are strictly positive and satisfy the symmetry relation $h_i = h_{d-i}$. When a polytope P is rational, *i.e.*, all its vertices have rational coordinates with respect to some lattice, we associate to it a toric variety X_P using the normal fan construction. It turns out that the Poincaré polynomial of X_P coincides $h(t^2)$.

Let X be a complex algebraic variety of dimension n. Then X is *rationally smooth* at x if there is a neighbourhood U of x in the complex topology such that, for any

$y \in U,\ H^m(X, X \setminus \{y\}) = 0,\ m \neq 2n$ and $H^{2n}(X, X \setminus \{y\}) = \mathbb{Q}$. Here H^* denotes the cohomology of X with rational coefficients.

Danilov [11] proved that X_P is rationally smooth if and only if the polytope P is simple. We refer to J as *combinatorially smooth* if $X(J)$ is rationally smooth.

Consider a semisimple algebraic group G_0 with maximal torus T_0 and an irreducible representation ρ_λ of G_0 with the highest weight $\lambda \in X(T_0) \otimes \mathbb{Q}$. Consider the action of W on the vector space spanned by the simple roots of G_0 and take the convex hull of the W-orbit of λ, $P_\lambda = \mathrm{Conv}(W.\lambda) \subset X(T_0) \otimes \mathbb{Q}$. Using the inner normal fan construction associated to the polytope P_λ [15], we obtain a projective toric variety $X(J)$. The terminology is justified since $X(J)$ depends only on $J = \{s \in S \mid s(\lambda) = \lambda\}$.

In [32] Renner finds necessary and sufficient conditions for the polytope P_λ to be simple using the theory of algebraic monoids that he developed along with Putcha since 1980. For each Weyl group (W, S), Renner gives a classification of all $J \subseteq S$, such that $X(J)$ is rationally smooth. See Corollary 3.5 in [32].

When (W, S) is a finite Weyl group of type A_n and $J = \emptyset$, the polytope P_λ is a permutahedron. The Betti numbers of $X(\emptyset)$ are given by the Eulerian numbers. In [4], Brenti studies the descent polynomials (*i.e.*, the Poincaré polynomials of $X(\emptyset)$) as analogues of the Eulerian polynomials. When $J \neq \emptyset$, the weight λ is allowed to lie on certain reflecting hyperplanes. Of course, the orbit of a point in the complement of the arrangement is just the ordinary permutahedron. Whether the Poincare polynomial of $X(J)$ can be expressed in terms of the Eulerian numbers in this case, is an interesting question. we answer this question by computing the Poincaré polynomial of $X(J)$ when (W, S) is a finite Weyl group of type A_n and $J = \{s_{n-k+1}, \cdots, s_n\} \subseteq S$ is combinatorially smooth, where $s_k = (k,\ k+1) \in S_n$, $1 \leq k \leq n$.

According to Renner's classification, the subset $J \subset S$ is combinatorially smooth of type A_n if it has one of the following forms:

1. $J = \emptyset$,

2. $J = \{s_1, \cdots, s_i\}$ where $1 \leq i \leq n$,

3. $J = \{s_j, \cdots, s_n\}$ where $1 < j \leq n$,

4. $J = \{s_1, \cdots, s_i, s_j, \cdots, s_n\}$ where $1 \leq i, j \leq n$ and $j - i \geq 3$.

One needs to investigate further to see whether our new technique can provide answer for all types of combinatorially smooth sets J.

Our first result deals with the case when the highest weight λ is fixed only by the reflection $s_n = (n, n+1) \in S_n$. We obtain the following characterization of the h-polynomial of $X(J)$ in terms of the Eulerian polynomials.

[**Theorem 17, page 91**] Let $J = \{s_n\} \subset S$. Then J is combinatorially smooth of type A_n and the h-polynomial of $X(J)$ is given by

$$h(t) = E_{n+1}(t) - \binom{n+1}{2} tE_{n-1}(t).$$

Then we generalize the computations to the case of $J = \{s_{n-k+1}, \cdots, s_n\} \subseteq S$ for $1 \leq k \leq n$. Our main result is a recursive relation for the Poincaré polynomial of $X(J)$ in terms of the $(n-k)$-Eulerian polynomials.

The following result has been accepted for publication in *Communication in Algebra journal*, March 2011.

[**Theorem 19, page 100**] Let $J(k,n) = \{s_{n-k+1}, s_{n-k+2}, \cdots, s_n\} \subseteq S$, $1 \leq k \leq n$ and let $h_k(t)$ denote the h-polynomial of the n–dimensional variety $X(J(k,n))$. Then $J(k,n)$ is combinatorially smooth and the following recurrence relation holds:

$$h_k(t) = h_{k-1}(t) - \binom{n+1}{k+1} (t^k + t^{k-1} + \cdots + t)E_{n-k}(t).$$

where $J(0,n) = \emptyset$ and $h_0 = E_{n+1}$ the $(n+1)$–Eulerian polynomial.

Finally, the recurrence relation is illustrated for $J(n-1,n) = \{s_2, s_3, s_4, \cdots, s_n\}$ and $J(n-2,n) = \{s_3, s_4, \cdots, s_n\}$ where the h-polynomial of $X(J(n-2,n))$ and

$X(J(n-1,n))$ are computed in [32] and [33].

An interesting investigation has been done for computing explicitly the Poincaré polynomial of $X(J)$. Renner describes the Poincare polynomial of $X(J)$ when $X(J)$ is rationally smooth in terms of the poset $(W^J, \leq, \{\nu_s\})$ [33]. He computes explicitly the coefficients of the Poincare polynomial of $X(J)$ for the following types of $J \subset S$:

1. (W,S) is Weyl group of type A_n and $J = \{s_2, s_3, \cdots, s_n\} \subset S = \{s_1, s_2, \cdots, s_n\}$ where $s_i = (i\ i+1) \in S_{n+1}$.

2. (W,S) is Weyl group of type A_n and $J = \{s_3, s_4, \cdots, s_n\} \subset S$

3. (W,S) is Weyl group of type B_l and $J = \{s_1, s_2, \cdots, s_{l-1}\} \subset S$

Inspired by these results we are interested in computing explicitly the Poincaré polynomial of $X(J)$ in two interesting cases of $J \subseteq S$ combinatorially smooth. We obtained the following results:

[**Theorem 13, page 45**] Let (W,S) be the Weyl group of type A_n and $J = \{s_1, s_4, s_5, \cdots, s_n\}$, $J \subset S$ such that $X(J)$ is rationally smooth. Then the Poincaré polynomial of $X(J)$ is given by the following formula:

$$P(X(J),t) = \sum_w t^{\nu(w)} = 1 + c(n,1)t^2 + \cdots + c(n,n-1)t^{2(n-1)} + t^{2n},$$

where $c(n,1) = c(n,n-1) = n+2$ and for $2 \leq i \leq n-2$ we have $c(n,i) = n+2+\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$.

[**Theorem 14, page 63**] Let (W,S) be of type A_n and $J = \{s_4, s_5, \cdots, s_n\} \subset S = \{s_1, \cdots s_n\}$ such that $X(J)$ is rationally smooth. The Poincaré polynomial of $X(J)$ is:

$$P(X(J),t) = 1 + d(n,1)t^2 + \cdots + d(n,n-1)t^{2(n-1)} + t^{2n}. \tag{1}$$

where $d(n,1) = d(n,n-1) = n+2+\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$ and $d(n,i) = n+2+n(n+1)$ for $2 \leq i \leq n-2$.

This thesis is structured as follows. In Chapter 1 we introduce the $\mathcal{J}$-irreducible monoids of type J and the cross section lattice associated to them. In Chapter 2 we explain how to construct the projective toric variety $X(J)$ associated to a reductive monoid of type J. We discuss about its BB-cell decomposition and introduce the h-polynomial associated to a simple polytope. In Proposition 6 we give a formula for the number of i-dimensional faces of the polytope P_λ corresponding to the variety $X(J)$. This gives us a good handle of the h-polynomial of $X(J)$, which can be expressed in terms of the subsets of S. An interesting interplay between the geometry of $X(J)$ and the combinatorics of finite sets isillustrated in Proposition 7. In Chapter 3 we discuss about Descent systems, introduced by Renner in [36]. Moreover, using descent systems, Renner gives in Theorem 12 a formula for the Poincaré polynomial of $X(J)$. He then computes explicitly the Poincaré polynomial of $X(J)$ in two cases of type A_n and one case of type B_n. Chapter 3 culminates with our first new results. We compute explicitly the Poincaré polynomial of $X(J)$ when (W, S) is Weyl group of type A_n in the following two cases: $J = \{s_1, s_4, s_5, \cdots, s_n\}$ and $J = \{s_4, s_5, \cdots, s_n\}$. Chapter 4 contains the main results of the thesis. We introduce Eulerian polynomials and prove our main result in Theorem 19.

We conclude this section with an example that illustrates the recurrence formula obtained in Theorem 19. Chapter 5 is a survey based on paper [36], on Betti numbers of irreducible representations, with the purpose of showing the reader how the h–polynomial from toric geometry and the length polynomial from the theory of projective homogeneous spaces fit together in a more general. These results can be found in details in the work of Renner [31], [36] and [37].

Chapter 1

Algebraic monoids

1.1 Cross section lattice

In this chapter, we introduce the terminology and the basic results (without proofs) related to the cross section lattice associated to a toric variety using the theory of algebraic monoids developed by Renner and Putcha. For references, we cite the texts [24], [30], [32], [33].

A *linear algebraic monoid* is an affine algebraic variety together with an associative morphism and an identity element.

Let M be a reductive algebraic monoid which is an irreducible (as variety) monoid with a reductive unit group G. Consider $B \subset G$ a Borel subgroup of G and $T \subset B$ a maximal torus of G. We denote $\overline{T}$ the Zariski closure of T in M. Since $GM \subset M$ and $MG \subset G$, the group $G \times G$ acts on M by $(g,h).a = gah^{-1}$ for $g, h \in G$ and $a \in M$. Let $G\diagdown M \diagup G$ denote the set of orbits $\mathcal{O} = GaG$ for this action.

Example 1. Let $M = M_n$ then $G = GL_n$. If $a, b \in M$ then $GaG = GbG$ if and only if $\text{rank}(a) = \text{rank}(b)$. Thus there is a bijection between the set of $G \times G$ orbits and the set $\{0, 1, \cdots, n\}$ given by $GaG \to \text{rank}(a)$. In particular the number of $G \times G$ orbits is finite.

Example 2. Let $M \subset M_{n+1}$ consist of all matrices

$$X = \begin{pmatrix} x & x_1 & x_2 & \cdots & x_n \\ 0 & x & 0 & \cdots & 0 \\ 0 & 0 & x & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & x \end{pmatrix}$$

where $x, x_1, \cdots x_n \in k$. For simplicity we write $a = (x, x_1, \cdots x_n)$. The group G consists of those a with $x \neq 0$. The $G \times G$ orbits are G, $\{0 = (0, 0 \cdots 0)\}$ and orbits which contain matrices $(0, x_1, \cdots, x_n)$ with x_i not all zero.

Observe that $\{(0, x_1, \cdots, x_n)\}$ and $\{(0, y_1, y_2, \cdots, y_n)\}$ are in the same orbit if and only if there exists $c \in k^*$ with $y_i = cx_i$ for all i. Thus these orbits are in one to one correspondence with points in the projective space $\mathbb{P}^{n-1}(k)$. In this case the number of $G \times G$ orbits is infinite and M has no idempotents except 0 and 1.

The next results wer first observed by Putcha in [24]. For more details consult Theorem 4.5 in [30] as well.

Theorem 1. [24] Let M be a reductive monoid with zero. Let G be its unit group. Then the set of $G \times G$ orbits of M is finite and every $G \times G$ orbit contains an idempotent of M.

In the theory of reductive groups, a main guiding principle is to reduce problems, as far as possible, to a study of the Weyl group and its action on $E(\overline{T})$ where T is a maximal torus.

The Weyl group of G relative to T is by definition

$$W(G, T) = \frac{N_G(T)}{T}$$

where $N_G(T)$ is the normalizer of T in G.

Theorem 2. [38] Let G be a reductive group and let T be a maximal torus. Let $\phi = \phi(G, T)$ be the set of roots. If $\alpha \in \phi$ then $-\alpha \in \phi$ and there exists a reflection $s_\alpha \in W$ such that $s_\alpha \alpha = -\alpha$. The finite Weyl group

$$W = \langle s_\alpha \mid \alpha \in \phi \rangle,$$

is generated by the reflections s_α corresponding to the simple roots $\alpha \in \phi$. Furthermore, if α, $\beta \in \phi$ then there exists $n_{\beta,\alpha} \in \mathbb{Z}$ such that

$$s_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha.$$

Denote by S the set of reflections that correspond to the simple roots, $S = \{s_\alpha\}$, $\alpha \in \phi(G, T)$ simple roots such that S it generates the Weyl group.

Proposition 1. Let $I \subset S$. Define $W^I = \{w \in W \mid l(ws) > l(w) \text{ for all } s \in I\}$. Given $w \in W$ there is a unique $u \in W^I$ and a unique $v \in W_I$ such that $w = uv$. Moreover, u is the unique element of smallest length in the coset wW_I.

According to the preceding proposition, the set W^I contains all the coset representatives of minimal length in W/W_I,

$$W^I = \{t \in W \mid t \text{ has minimal length in } tW_I\}$$

Definition 1. Let (W, S) be a Coxeter group. Each element $w \in W$ can be written as a product of generators

$$w = s_1 s_2 \cdots s_k, \;\; s_i \in S.$$

If k is minimal among all such expressions for w, then k is called the length of w.

Consider the case of (W, S) finite Weyl group of type A_n. Then the following characterization of the set W^I with $I \subset S$, turn out to be extremely useful in our computations.

$$\begin{aligned} W^I &= \{t \in W \mid t \text{ has minimal length in } tW_I\} \\ &= \{w \in W |\ l(ws) = l(w) + 1 \text{ for any } s\ \in I\} \\ &= \{w \in S_{n+1} \mid w(i) < w(i+1) \text{ for any } (i\ i+1) \in I\} \end{aligned} \tag{1.1}$$

Next we introduce the notion of cross section lattice asssociated to a reductive monoid.

Definition 2. The *cross section lattice* of M relative to T and B is defined by the following rule:

$$\Lambda = \{e \in \overline{T} \mid e^2 = e,\ eB = eBe\},$$

The map $\lambda : \Lambda \longrightarrow 2^S$ given by: $\lambda : \Lambda \longrightarrow 2^S$, where $\lambda(e) = \{s \in S \mid se = es\}$ is called the *type map*. Moreover $\lambda(e) \subset S$ is the unique subset such that $P(e) = P_{\lambda(e)}$ where $P(e) = \{x \in G \mid xe = exe\}$ and $P_{\lambda(e)}$ is a parabolic subgroup of G.

The type map is the most important combinatorial invariant in the structure theory of reductive monoids. It is in some sense the monoid analogue of the Coxeter-Dynkin graph. It shows which $G \times G$-orbits are involved in the monoid, as well as how the monoid structure is built up from these orbits [25].

Example 3. The type map of $M_{n+1}(k)$. Let $M = M_{n+1}(k)$, $B = T_n(k)$, $T = D_n(k)$ and $S = \{s_1, \cdots s_n\}$ where $s_i = (i\ i+1)$. Then the cross section lattice is given by:

$$\Lambda = \{0, e_1, \cdots, e_{n+1}\}$$

where e_k is the rank k matrix for $k = 1, \cdots, n$:

$$\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

where I_k is the $k \times k$ identity matrix Let $i \geq 2$, then the type map λ is given by:

$$\lambda(e_i) = \{s_1, s_2, \cdots s_{i-1}\} \cup \{s_{i+1}, \cdots, s_n\}.$$

The $G \times G$-orbits of M turn out to be particularly important for the structure of the monoid M.

Proposition 2. [24] The following holds:

$$M = \bigsqcup_{e \in \Lambda} GeG.$$

where $GeG \subset \overline{GfG}$ if and only if $ef = e$.

There is a partial ordering on the $G \times G$-orbits described as follows: $GeG \prec GfG$ if and only if $GeG \subset \overline{GfG}$ if and only if $ef = e$.

Definition 3. A reductive monoid M with $0 \in M$ is called $\mathcal{J}$–irreducible if $M - \{0\}$ has exactly one minimal $G \times G$-orbit. See section 7.3 of [30] for a systematic discussion of the important class of reductive monoids and for a proof of the following result.

In order to understand the action of the Weyl group on the set $E(\overline{T})$ we consider the following: let $N = N_G(T)$, if $w \in N$ and $e \in E(\overline{T})$ then $wew^{-1} \in \overline{T}$. If $\overline{w} = wT \in W$ then wew^{-1} depends only on $\overline{w}$ so we may define $\overline{w}e\overline{w}^{-1} = wew^{-1}$. Thus W acts by conjugation on $E(\overline{T})$ and the cross section lattice Λ turn out to be the set of W-orbit representatives for this action.

Theorem 3. ([30]) Let M be a reductive monoid. The following are equivalent:

1. M is $\mathcal{J}$-irreducible.

2. There is an irreducible rational representation $\rho : M \to \mathrm{End}(V)$ which is finite as a morphism of algebraic varieties.

3. If $\overline{T} \subset M$ is the Zariski closure in M of a maximal torus $T \subset G$ then the Weyl group W of T acts transitively on the set of minimal nonzero idempotents of $\overline{T}$.

We notice that we can construct, up to finite morphisms, all $\mathcal{J}$-irreducible monoids from irreducible representations of a semisimple group. For details of the following construction see [32].

Let G_0 be semisimple and $\rho : G_0 \to \mathrm{GL}(V)$ be an irreducible representation. Define $M_1 \subset \mathrm{End}(V)$ as the Zariski closure of $\mathbb{C}^*\rho(G_0)$. Finally let $M(\rho)$ be the normalization of M_1. According to the previous theorem $M(\rho)$ is $\mathcal{J}$-irreducible.

By the results of [26] if M is $\mathcal{J}$-irreducible, there is a unique, minimal nonzero idempotent $e_1 \in E(\overline{T})$ such that $e_1B = e_1Be_1$, where B is the given Borel subgroup containing T.

Definition 4. If M is $\mathcal{J}$-irreducible we say that M is $\mathcal{J}$-irreducible of type J if

$$J = \{s \in S \mid se_1 = e_1s\},$$

where S is the set of reflections relative to T and B and e_1 minimal idempotent such that $e_1B = e_1Be_1$.

The set J can be determined in terms of any irreducible representation satisfying condition 2 of the previous theorem. Indeed, let $\lambda \in X(T)$ be any highest weight such that $\{s \in S \mid s(\lambda) = \lambda\} = J$. Then $M(\rho_\lambda)$ is $\mathcal{J}$–irreducible of type J where ρ_λ is the irreducible representation of G_0 with highest weight λ. The representation ρ_λ determines a representation of $M(\rho_\lambda)$ on V.

Next, we describe the $G \times G$–orbit structure of a $\mathcal{J}$–irreducible monoid of type $J \subset S$. The following result was first recorded in [26].

Theorem 4. ([26]) Let M be a $\mathcal{J}$– irreducible monoid of type $J \subset S$.

1. There is a canonical one-to-one order preserving correspondence between the set of $G \times G$– orbits acting on M and the set of W-orbits of the set of idempotents of $\overline{T}$. This set can be canonically identified with $\Lambda = \{e \in E(\overline{T}) \mid eB = eBe\}$.

2. $\Lambda - \{0\} \cong \{I \subset S \mid$ no connected component of I is contained entirely in $J\}$ in such a way that e corresponds to $I(e) \subset S$ if $I(e) = \{s \in S \mid se = es \neq e\}$.

3. If $e \in \Lambda - \{0\}$ corresponds to $I(e)$, as in 2 above, then $C_W(e) = \{w \in W \mid wew^{-1} = e\} = W_{I^*(e)}$ the parabolic subgroup generated by the set $I^*(e)$ where $I^*(e) = I \cup \{s \in J \mid st = ts$ for all t $\in$ I(e)$\}$

We observe that the cross section lattice Λ is related to the Coxeter-Dynkin diagram as follows: under the type map λ, to each $e \in \Lambda \setminus \{0\}$ it corresponds a subset in S, namely $I(e)$, such that no connected component of $I(e)$ is entirely contained in J.

Example 4. Consider $(W, S) = (S_5, \{s_1, s_2, s_3, s_4\})$ of type A_4 and $J = \{s_1, s_2, s_4\} \subset S$. Describe the cross section lattice $\Lambda - \{0\} \subset 2^S$.

Consider the *graph structure on* S:

$$\text{s and t are joined by an edge if st} \neq \text{ts}$$

The edges of the graph of S are: $\{s_1, s_2\}$, $\{s_2, s_3\}$, $\{s_3, s_4\}$. Then the cross section lattice is given by:

$$\Lambda - \{0\} \cong \{\{\emptyset\}, \{s_3\}, \{s_3, s_4\}, \{s_2, s_3\}, \{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \{s_1, s_2, s_3, s_4\}\}.$$

Chapter 2

The toric variety $X(J)$

2.1 Construction of $X(J)$

Let (W, S) be a finite Weyl group and let $W \subset GL(V)$ acting as a reflection group on the rational vector space V. Associated to $\lambda \in V$, there is a certain projective toric variety $X(J)$. In this chapter we analyze the T-orbit structure of $X(J)$, the BB-cell decomposition of $X(J)$ and then give a characterization of the h-polynomial of $X(J)$ in terms of the cross section lattice associated to $X(J)$.

Consider k an algebraically closed field, $k = \bar{k}$. An algebraic torus is an algebraic group T isomorphic to $(k^*)^m = k^* \times k^* \times \cdots \times k^*$.

Kempf, Mumford(1973) were among the first who studied varieties X containing T as an affine open dense subset such that the translations on T extend to an action of T on X.

Definition 5. A *toric variety* of dimension n is a normal variety X which contains a torus $T = (k^*)^n$ as a Zariski open subset in such a way that the natural action of T on itself given by the group structure extend to an action of T on X.

The following examples of toric varieties are among the most natural examples of toric varieties one can come up with.

1) $T = (\mathbb{C}^*)^n \subset \mathbb{C}^n$ under the natural inclusion.

2) $T = (\mathbb{C}^*)^n \subset P^n$ under the map

$$(t_1, \cdots, t_n) \to [t_1 : \cdots t_n : 1]$$

3) $X = V(xy - zw) \subset \mathbb{C}^4$ contains the Zariski open set $X \cap (\mathbb{C}^*)^4$. The map

$$(r, s, t) \to (r, s, t, rs/t)$$

induces a bijection $(\mathbb{C}^*)^3 \simeq X \cap (\mathbb{C}^*)^4$. Thus X contains a copy of the torus $T = (\mathbb{C}^*)^3$ as a Zariski open subset.

For the next results we consider an algebraically closed field $k = \bar{k}$. We define the *character group* as follows:

$$M = \{\chi : T \to k^* \mid \chi \text{ is a morphism and a group homomorphism}\}$$

And the *group of 1-parameter subgroups*

$$N = \{\lambda : k^* \to T \mid \lambda \text{ is a morphism and a group homomorphism}\}$$

Note that $M \simeq \mathbb{Z}^n$ where $m = (m_1, \cdots m_n) \in \mathbb{Z}^n$ gives

$$\chi^m(t_1, \cdots, t_n) = t_1^{m_1} \cdots t_n^{m_n}$$

and $N \simeq \mathbb{Z}^n$ where $u = (u_1, \cdots u_n) \in \mathbb{Z}^n$ gives $\lambda^u(t) = (t^{u_1}, \cdots, t^{u_n})$.

A rational polyhedral cone $\sigma \subset N_R = N \times \mathbb{R}$ is a cone generated by finitely many elements of N:

$$\sigma = \{\lambda_1 p_1 + \cdots + \lambda_s p_s \in N_R \mid \lambda_1, \cdots, \lambda_s \geq 0\}$$

where $p_1, \cdots, p_s \in N$.

If $\sigma \in N_R$ is a rational polyhedral cone its dual cone $\sigma' \in M_R = M \otimes \mathbb{R}$ is defined by

$$\sigma' = \{m \in M_R \mid \langle m, u\rangle \geq 0 \text{ for all u} \in \sigma\}.$$

Then consider the finitely generated semigroup algebra $k[\sigma' \cap M]$ consisting of linear combinations of characters χ^m, with multiplication given by $\chi^m \chi^{m'} = \chi^{m+m'}$.

Theorem 5. [10] Let $\sigma \in N_R \simeq R^n$ be a rational polyhedral cone. Then the irreducible affine variety

$$\mathrm{Spec}(k[\sigma' \cap M]) = X_\sigma$$

is a normal toric variety of dimension n associated to σ.

The next theorem describes all normal affine varieties that are toric.

Theorem 6. [10] Let X be an affine variety that is also toric variety. Then X is isomorphic to X_σ for some rational polyhedral cone σ if and only if X is normal.

Definition 6. [7] Let X be a complex algebraic variety of dimension n. Then X is rationally smooth at $x \in X$ if there is a neighbourhood U of x in the complex topology such that, for any $y \in U$,

$$H^m(X, X - \{y\}) = (0),\ m \neq 2n,$$

$$H^{2n}(X, X - \{y\}) = \mathbb{Q}.$$

Here H^* denotes the singular cohomology of X with rational coefficients.

Example 5. The curve $\{(x, y, z) \in k^3 \mid xz = y^2\}$ is rationally smooth at $\{(0, 0, 0\}$ and k is an algebraically closed field.

Consider the ring homomorphism $f : k[x, y, z] \to k[u, v]$ given by $f(x) = u^2$,

$f(y) = uv$, $f(z) = v^2$ whsoe kernel is the ideal $(xz - y^2)$. Hence

$$\frac{k[x,y,z]}{(xz - y^2)} \cong k[u^2, uv, v^2]$$

Let $G = \langle g \rangle$ be the finite group generated by the matrix

$$g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

The action of g on $k[u,v]$ is $g.u = -u$, $g.v = -v$. Thus the invariant ring is $k[u,v]^G = k[u^2, uv, v^2]$ and we obtain the isomorphism of varieties

$$\{(x,y,z) \in k^3 \mid xz = y^2\} \cong k^2/G,$$

where k^2/G is locally rationally smooth at origin.

For more details and proofs of the following results see [28]. Throughout this section we consider M a $\mathcal{J}$-irreducible monoid of type J and $\rho : M \to \mathrm{End}(V)$ an irreducible representation which is finite as a morphism. M has a reductive unit group G. Let $B \subset G$ be a Borel subgroup of G and $T \subset B$ a maximal torus of G. We let $\overline{T}$ denote the Zariski closure of T in M. The T–orbit of the unit element is open in its closure hence T is open in $\overline{T}$.

By theorem 5.4 of [30], $\overline{T}$ is a normal variety. Hence $\overline{T}$ is an affine toric variety. Let $E(\overline{T}) = \{e \in \overline{T} \mid e = e^2\}$. There is exactly one idempotent in each T-orbit on $\overline{T}$. For the proof of the following result see [24]

$$\overline{T} = \bigsqcup_{e \in E(\overline{T})} T.e$$

We call the *rank* of an idempotent $e \in \overline{T}$ the dimension of the T–orbit corresponding to e $i.e.$, $\mathrm{rank}(e) = \mathrm{dim} Te$.

Because $\overline{T}$ is an affine toric variety there is a rational polyhedral cone $\sigma \subset X(\overline{T}) \otimes \mathbb{Q}$ such that $X(\overline{T}) = \sigma \cap X(T)$ [20]. σ is the rational convex hull of a finite number of rays $R_i \subset \sigma$, $i = 1, \cdots s$, and for each i there is a unique character $\chi_i \in R_i \cap X(\overline{T})$ with minimal distance from origin.

Let $E_1(\overline{T}) = \{e \in \overline{T} \mid e^2 = e,\ \dim(e\overline{T}) = 1\}$. We consider the following set of characters $\{\chi_e\}_{e\in E_1} \subset X(\overline{T})$, where

$$\chi_e : \overline{T} \to k \simeq e\overline{T},\ \text{given by}\ x \to \chi_e(x) = ex \neq 0$$

for any $e \in E_1(\overline{T})$.

On each ray of the cone σ there is a unique character $\chi_e \in X(\overline{T})$, $e \in E_1(\overline{T})$, of minimal distance from the origin.

Proposition 3. [28] $\{\chi_e\}_{e\in E_1(\overline{T})}$ satisfy the following properties:

1. For all $\chi \in X(\overline{T})$, $\chi^m = \prod \chi_e^{m_e}$ for some m, $m_e \geq 0$.

2. $\chi_e \in I \setminus I^2$, where $I = \{f \in k[\overline{T}] \mid f(0) = 0\}$.

3. Let s be the cardinal of the set $\{e : m_e > 0\}$. Then s is minimal with properties (1) and (2). Furthermore, $\{\chi_e :\ m_e > 0\}$ is unique and is called the set of fundamental generators.

Thus on each ray of the cone σ there is a unique character $\chi_e \in X(\overline{T})$, $e \in E_1(\overline{T})$, of minimal distance from the origin.

This canonical relationship between $E_1(\overline{T})$ and the one dimensional faces of σ can be extended to yield a canonical bijection of lattices

$$\mathcal{F}(\sigma) \leftrightarrow E(\overline{T}) \tag{2.1}$$

where $\mathcal{F}(\sigma)$ denotes the set of faces of σ and $\overline{T}$ is the affine toric variety associated to the dual cone of σ, namely

$$\overline{T} = \mathrm{Spec}([\sigma' \cap \mathbb{Z}^n]).$$

For the rest of the section we consider $k = \mathbb{C}$. Next, we construct the projective toric variety associated to the graded ring $\mathbb{C}[\overline{T}])$, namely

$$[\overline{T} - \{0\}]/\mathbb{C}^* = \mathrm{Proj}(\mathbb{C}[\overline{T}])$$

The ring $\mathbb{C}[\overline{T}]$ is a graded ring: we consider the action of $\mathbb{C}^*$ on $\mathbb{C}[\overline{T}])$ given by:

$$\mathbb{C}^* \times \mathbb{C}[\overline{T}]) \to \mathbb{C}[\overline{T}])$$

$$(\lambda, f) \to (\lambda.f)(t) = f(\lambda t)$$

The d graded component is defined as follows: $\mathbb{C}([\overline{T}])_d = \{f \in \mathbb{C}[\overline{T}]) \mid \lambda.f = \lambda^d f\}$ and $\mathbb{C}[\overline{T}])_0 = \mathbb{C}$.

As a projective toric variety it corresponds to a polytope. We describe its associated polytope in the following way: let G_0 be the semisimple part of G (G_0 is the commutator subgroup $G_0 = (G, G)$), with maximal torus $T_0 = T \cap G_0$ and $\rho_\lambda = \rho_{|G_0}$ be the representation of G_0 that corresponds to the the highest weight $\lambda \in X(T_0)$. Let the Weyl group W act on the finite dimensional vector space $X(T_0) \otimes \mathbb{Q}$ by reflctions. Take the W-orbit of λ and consider its convex hull in $X(T_0) \otimes \mathbb{Q}$.

We obtain the polytope $P_\lambda = \mathrm{Conv}(W.\lambda) \subset X(T_0) \otimes \mathbb{Q}$. The face lattice $\mathcal{F}_\lambda$ depends only on $W_\lambda = \{w \in W \mid w(\lambda) = \lambda\} = W_J = \langle s \mid s \in J\rangle$ where $J = \{s \in S \mid s(\lambda) = \lambda\}$. Renner proved [28] that the highest weight λ is a rational multiple of any nonzero point on one of the rays of the cone $\sigma = \langle \chi_e \rangle_{e \in E_1(\overline{T})} \otimes \mathbb{Q}^+$. We obtain the polytope P_λ is in bijection with the convex hull of $\{\chi_e\}_{e \in E_1(\overline{T})}$.

From (2.1) we have the following lattice isomorphism between $E(\overline{T})$ and the face lattice of the polytope P_λ:

$$e \in E(\overline{T}) \leftrightarrow \mathcal{F}_e,$$

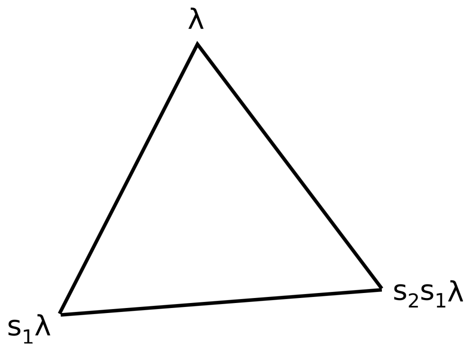
such that

$$\operatorname{rank}(e) = \dim(\mathcal{F}_e) + 1, \tag{2.2}$$

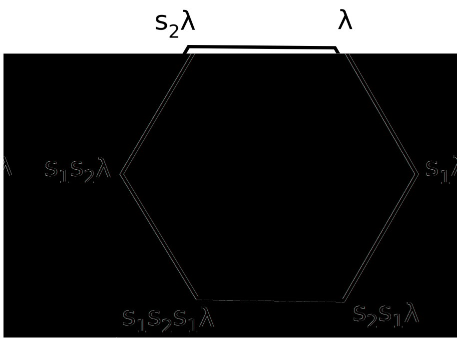
where $\mathcal{F}_e$ is a face of the polytope P_λ.

Next we present several examples of various polytopes P_λ when W is finite Weyl group of type A_2 and of type A_3.

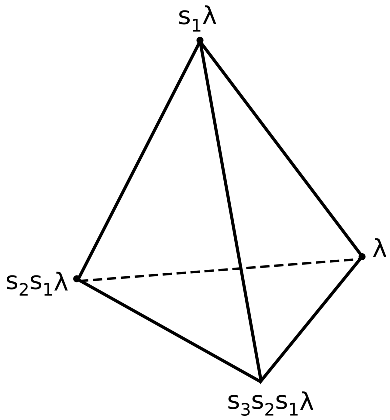
Example 6. A_2, $(W, S) = (S_3, S = \{s_1, s_2\})$ and $J = \{s_2\}$.



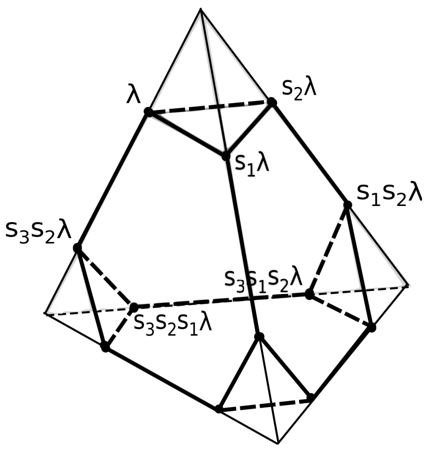
Example 7. A_2, $(W, S) = (S_3, \{s_1 = (12),\ s_2 = (23)\}),\ J = \emptyset$.



Example 8. A_3, $(W,S) = (S_4, \{s_1 = (12),\ s_2 = (23),\ s_3 = (34)\})$, $J = \{s_2, s_3\}$



Example 9. A_3, $J = \{s_3\} \subset S = \{s_1, s_2, s_3\}$.



The next corollary recorded without a proof in [32] is a key result in our computations of the h–polynomial of the polytope P_λ.

Corollary 1 (21, Corollary 1.3)**.** Let W be a Weyl group and let $r : W \to GL(V)$ be the usual reflection representation of W. Let $\mathcal{C} \subset V$ be the rational Weyl chamber and let $\lambda \in \mathcal{C}$. Assume that $J = \{s \in S \mid s(\lambda) = \lambda\}$. Then the set of orbits of W acting on the face lattice $\mathcal{F}_\lambda$ of $\mathcal{P}_\lambda$ is in one-to-one correspondence with the set $\{I \subseteq S \mid \text{no connected component of } I \text{ is contained entirely in } J\}$.

We conclude the W–orbit contains a representative face whose W–stabilizer/isotropy subgroup is the parabolic subgroup $W_{I_J^*}$, generated by the set

$$I_J^* = \{I\} \cup \{s \in J \mid st = ts \text{ for all t } \in I\}.$$

Next, let $S(J) = \{I \subset S \mid \text{no connected component of } I \text{ is contained entirely in } J\}$, $S(J) \subseteq \mathcal{P}(S)$, the power set of S. The following examples turn out to be extremely helpful for a better understanding of Corollary 1.

Example 10. Let $G = SL_4$ and $W = S_4$ where $S = \{s_1, s_2, s_3\}$.
For $J = \{s_2, s_3\}$ we have $S(J) = \{\emptyset, I_1, I_2, S\}$ where $I_1 = \{s_1\}$, $I_2 = \{s_1, s_2\}$. The subset $I \subseteq S$ corresponds to the unique face F of the polytope P_λ with $I = \{s \in S \mid s(F) = F \text{ and } s_{|F} \neq id\}$ whose relative interior F^0 has nonempty intersection with the Weyl chamber $\mathcal{C}$. To I_1 it corresponds an edge labeled I_1 and to I_2 it corresponds a triangle labeled I_2, both faces drawn in Figure 2.1.

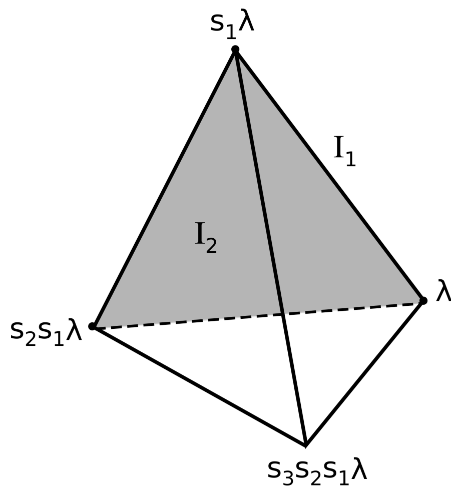


Figure 2.1: Tetrahedron

For $J = \{s_3\}$ we have $S(J) = \{\emptyset, I_1, I_2, I_3, I_4, S\}$, where $I_1 = \{s_1\}$, $I_2 = \{s_2\}$, $I_3 = \{s_1, s_2\}$ and $I_4 = \{s_2, s_3\}$. The corresponding faces to I_i, $1 \leq i \leq 4$ are drawn in Figure 2.2.

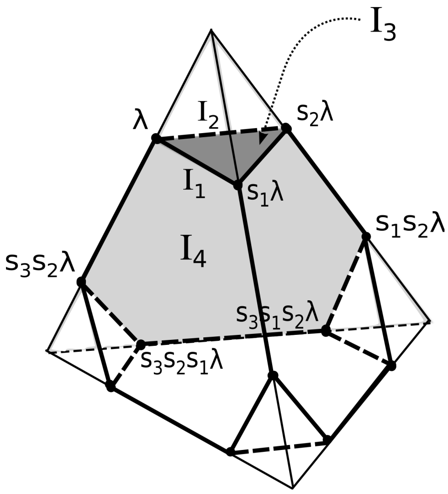


Figure 2.2: Truncated Tetrahedron

Next, we know that T is a Zariski open subset of $\overline{T}$. Hence the torus $T/\mathbb{C}^*$ is an open subset of $[\overline{T} - \{0\}]/\mathbb{C}^*$.

Our interest is in the projective toric variety denoted by $X(J)$, terminology justified since it depends only on $J = \{s \in S \mid s(\lambda) = \lambda\}$ and not on λ or M:

$$X(J) = \frac{\overline{T} - \{0\}}{\mathbb{C}^*} = \mathrm{Proj}[\mathbb{C}[\overline{T}]],$$

The T*–orbit structure* of $X(J)$ can be described as follows: for $e \in \overline{T}$ denote $[e] \in X(J)$. We have that the T–orbit of $[e]$ is of dimension $\mathrm{rank}(e) - 1$. Hence from (2.2) we associate uniquely to every k–dimensional T–orbit of $X(J)$ a k–dimensional face of the polytope P_λ.

The set of T–fixed points of $X(J)$, denoted by $X(J)^T$, corresponds to vertices of the polytope P_λ as $X(J)^T$ is in one-to-one correspondence with the set of one-dimensional T-orbits on $\overline{T}$. In order to see this consider $[x] \in X(J)$ for $x \in \overline{T}$. Then $[x]$ is a T-fixed point if and only if $[tx] = [x]$ for all $t \in T$. Hence $tx = \alpha x$ for $\alpha \in \mathbb{C}^*$ and for all $t \in T$. We conclude that $Tx = \mathbb{C}^* x$. We know that $x = te$ where $t \in T$ and $e \in E(\overline{T})$ such that $Tx = Te = k^* te$ and so $e \in E_1(\overline{T})$.

Remark 1. In general in order to obtain the variety $X(J)$ given a set $J \subseteq S$ we do the following: we choose a weight in the fundamental Weyl chamber corresponding to (W, S) such that $J = \{s \in S \mid s(\lambda) = \lambda\}$. Then consider the representation ρ of G_0 of highest weight λ, where G_o is a semimsimple algebraic group. Take the Zariski closure of $G = k^*\rho(G_0)$. A maximal torus inside G is $T = k^*\rho(T_0)$ where T_0 is a maximal torus of G_0. We considered k any algebraically closed field.

Define $X(J) = \overline{T} \setminus \{0\}/k^*$. It can be shown that $X(J)$ depends only on J and not on M or λ using the theory of algebraic monoids of type J. See [26] for more details.

Remark 2. Using the inner normal fan construction associated to the polytope P_λ (i.e., the fan is obtained by taking cones over faces of the dual polytope of P_λ and they satisfy a natural notion of ″*gluing*” affine varieties) we get the projective toric variety $X(J)$.

Next, we illustrate the construction of $X(J)$ in the following examples.

Proposition 4. Let $G = SL_4(\mathbb{C})$ with $S = \{s_1, s_2, s_3\}$ and $J = \{s_1, s_2\}$. The polytope P_λ is a tetrahedron and $X(J) = \mathbb{P}^3$.

Proof. In this case the simple roots are given by $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$, $\alpha_3 = \epsilon_3 - \epsilon_4$ where $\epsilon_i \in X(T_4)$ and $\epsilon_i(A) = t_i$ for $A = \mathrm{diag}(t_1, t_2, t_3, t_4)$.
We have the relation $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0$.
Next, consider the three dimensional vector space V, whose basis is given by ϵ_1, ϵ_2, ϵ_3. Let s_α, s_β, s_γ be reflections into planes orthogonal to α, β, γ in V such that the following holds:

1. $s_\alpha(\lambda) = \lambda - 2\frac{(\lambda,\alpha)}{(\gamma,\gamma)}\alpha$.

2. $s_1(\lambda) = \lambda$. Hence $(\lambda, \alpha_1) = 0$.

3. $s_2(\lambda) = \lambda$. Hence $(\lambda, \alpha_2) = 0$.

4. $(\epsilon_i, \epsilon_j) = \delta_{ij}$.

5. $s_i(\alpha_j) = \alpha_j - n_{\alpha_i,\alpha_j}\alpha_j$.

Using the above properties we can compute n_{α_i,α_j} which corresponds to the (i, j) entry of the Cartan matrix of A_3 given by

$$X = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

The convex hull of $W.\lambda$ in V, when $W = \langle s_1, s_2, s_3 \rangle$ is a tetrahedron with vertices given by: λ, $s_3(\lambda)$, $s_2s_3(\lambda)$, $s_1s_2s_3(\lambda)$.

First note that the following computations hold: $s_3(\lambda) = \lambda - \alpha_3$, $s_2s_3(\lambda) = \lambda - \alpha_2 - \alpha_3$ and $s_1s_2s_3(\lambda) = \lambda - \alpha_1 - \alpha_2 - \alpha_3$.

Next λ, $s_3(\lambda)$, $s_2s_3(\lambda)$, $s_1s_2s_3(\lambda)$ are affine independent $\Longleftrightarrow$ $s_3(\lambda)-\lambda$, $s_2s_3(\lambda)-\lambda$, $s_1s_2s_3(\lambda) - \lambda$ are linearly independent.

Using the inner normal fan construction associated to the tetrahedron P_λ we obtain the toric variety $X(J) = \mathbb{P}^3$. $\square$

2.) For $G = SL_n$, $S = \{s_1, s_2, \cdots, s_n\}$ and $J = \{s_2, \cdots, s_n\}$ we obtain the polytope P_λ as a $(n-1)$–simplex and $X(J) = \mathbb{P}^n$ using similar computations as in the previous example.

3.) For G any reductive algebraic group with ρ an irreducible representation of G then its highest weight λ belongs to the interior of the fundamental Weyl chamber if $J = \emptyset$ and the toric variety obtained is the toric variety associated to the Weyl chamber decomposition (the fan in this case is a family of cones given by the Weyl chambers) studied by Processi in [22]

2.2 Cell structure of $X(J)$

The BB-cell decomposition discovered by Bialynicki-Birula is the most commonly studied cell decomposition in algebraic geometry. If k^* acts on a smooth complete variety X, with finite fixed point set $F \subset X$, then $X = \sqcup_{\alpha \in F} X_\alpha$, where each BB–cell is defined as $X_\alpha = \{x \mid \lim_{t \to 0} tx = \alpha\}$ and it turn out that X_α is isomorphic to an affine space. The BB–cells make sense even if X is not smooth but in that case they don't behave so well.

In the case of the projective rationally smooth toric variety $X(J)$ Renner quantifies the BB–cells in terms of idempotents, $B \times B$-orbits and other monoid notions. In [33] Renner describes the BB-decomposition for an appropriate one-parameter subgroup of T in terms of the idempotents of $\overline{T}$. For proofs of the following results see [33].

Definition 7. Let e, $e' \in E_1(\overline{T})$. We say that $e < e'$ if $eBe' \neq 0$ and $e \neq e'$.

By the results of [32] the poset $(E_1(\overline{T}), \leq)$ is anti-isomorphic to the poset $(W^J, \leq)$.

Theorem 7. [33] Let M be a $\mathcal{J}$-irreducible reductive monoid of type $J \subset S$ with unit group G and connected center $Z \subset G$. Let $B_u \subset B$ be the subgroup of unipotent elements of B and $E_1 = E_1(\overline{T})$. Choose a one-parameter subgroup $\lambda : k^* \to T$ such that

1. $\lim_{t \to 0}(tut) = 1$ for all $\mathrm{u} \in \mathrm{B_u}$
2. $\{x \in \overline{T} \setminus \{0\} \mid \lambda(t)x \in Zx \text{ for all } t \in k\} = \bigcup_{e \in E_1(\overline{T})} eT$.

Let

$$X(J) = \sqcup_{e \in E_1} X(J)(e)$$

be the BB-decomposition of $X(J)$ relative to λ. Then $X(J)(e) = \sqcup_{f \in \rho_e} T[f]$ where

$$\rho_e = \{f \in E(\overline{T}) \mid ef = e \text{ and } e'f = 0 \text{ for all } e' > e\}$$

Theorem 8. [33] The following are equivalent:

1. $[f] \in X(J)(e)$

2. $fe' = 0$ for all $e' > e$ and $fe = e$.

In the same paper Renner studies the case when $X(J)$ is rationally smooth toric variety and recovers the dimension of the BB-cells $\{X(J)(e) \subset X(J) \mid e \in E_1\}$ from the Bruhat poset $(W^J, \leq, \{\nu_s\})$. Here $\nu_s(w)$ is the cardinal of the ascent set associated with $s \in S \setminus J$ and $w \in W^J$.

Assume now that Y is a rationally smooth projective toric variety with the torus action $T \times Y \to Y$. Let $F \subset Y$ be the set of T-fixed points. Choose a 1-psg $\lambda : k^* \to T$ so that Y has BB-decomposition

$$Y = \bigsqcup_{\alpha \in F} Y_\alpha$$

Proposition 5. [33] Let $U_\alpha = \{x \in Y \mid \alpha \in \overline{Tx}\}$. Then Y_α is the closure in U_α of a T-orbit. In particular Y_α is irreducible.

Proof. T acts on U_α and $\alpha \in U_\alpha$ is the unique fixed point of this action. $U_\alpha \subset Y$ is rationally smooth as Y is rationally smooth. Thus there exists a finite dominant flat T-equivariant morphism

$$p_\alpha : U_\alpha \to k^n$$

where $n = \dim(Y)$ and k^n has the usual structure of an affine toric variety for the n-torus. Thus we may write $p_\alpha(x) = (x_1, \cdots x_n)$ and $\lambda(t)(x_1, \ldots x_n) = (t^{a_1}x_1, \cdots, t^{a_n}x_n)$. Thus by definition of the BB-cell,

$$Y_\alpha = \{x \in U_\alpha \mid x_i = 0 \text{ if } a_i < 0\} = p_\alpha^{-1}(\{(x_1, \cdots x_n) \in k^n \mid x_i = 0 \text{ if } a_i < 0\})$$

This is the closure of a T-orbit in U_α since p_α induces a bijection on T-orbits. $\square$

We return to the situation where $X(J)$ comes from a Weyl group (W, S). We assume also that $X(J)$ is rationally smooth.

Let

$$\widehat{X}(J)(e) = \{y \in \overline{T} \mid [y] \in X(J)(e)\}.$$

Consider X to be the cone on $X(J)$, and if $e \in E_1(X)$ let $\mathcal{U}_e = \{x \in X \mid ex \neq 0\}$ and $E_2(\mathcal{U}_e) = E_2(X) \cap E(\mathcal{U}_e)$, wherfe $E_2(X)$ is the set of two-dimensional T-orbits of X. Notice that $\widehat{X}(J)(e) \subset \mathcal{U}_e$.

Proposition 5 turn out to be extremely useful in obtaining the following result.

Theorem 9. [33] Let $e \in E_1$ and let $\mathcal{U} = \cup_{e'>e}\mathcal{U}_{e'}$.

Then

$$\widehat{X}(J)(e) = \mathcal{U}_e \setminus \mathcal{U} = f\mathcal{U}_e$$

where $f \in E(X)$ is the unique smallest idempotent with $fh = h$ for all $h \in E_2(\mathcal{U}_e) - A$. In particular, $\dim(\widehat{X}(J)(e)) = |E_2(\mathcal{U}_e) - A| + 1 = |S| - |A| + 1$. In this case, $A = \{g \in E_2(X) \mid ge = e \text{ and } ge' = e' \text{ for some } e' > e\}$.

The next formula calculates the dimension of the BB-cells corresponding to the BB-decomposition of $X(J)$.

For $e \in E_1(X)$, let

$$\Gamma(e) = \{g \in E_2(X) \mid ge = e, \text{ and } ge' = e' \text{ for some } e' < e\}$$

Notice that

$$\Gamma(e) = E_2(\mathcal{U}_e) \setminus A$$

where A is defined as in Theorem 9.

Theorem 10. [33] Assume $X(J)$ is rationally smooth. For $e \in E_1$ recall that

$$X(J)(e) = \{[x] \in X(J) \text{ such that } ex \neq 0 \text{ and } e'x = 0 \text{ for all } e' > e\}$$

and as above let $\widehat{X}(J)(e) = \{y \in X \mid [y] \in X(J)(e)\}$. Then

$$\widehat{X}(J)(e) = \mathcal{U}_e \setminus \mathcal{U} = f\mathcal{U}_e$$

as in Theorem 9 and $\dim(X(J)(e) = |\Gamma(e)|$.

From the previous theorem we can recover the dimensions of the BB-cells $\{X(J)(e) \subset X(J) \mid e \in E_1\}$ from $(W^J, \leq, \{\nu_s\})$. If $\nu(w) = \sum_s \nu(s)$ then from part 3 of Theorem 2.23 in [32] we obtain that $\nu(w) = |\Gamma(e)|$

Remark 3. The following table was taken from [33] and it provides the reader with a summary-translation between $X(J)$ and the Bruhat poset jargon. Let

$$\Lambda^\times = \{I \subset S \mid \text{no component of I is contained in J}\}$$

and for $I \in \Lambda^\times$ let $I^* = I \cup \{t \in J \mid ts = st \text{ forall } s \in I\ \}$.

For each $w \in W^J$ the BB-cell C_w is defined as:

$$C_w = \bigsqcup_{A \in A \in \mathcal{O}(w)} A$$

where $\mathcal{O}(w) = \{A \subset X(J) \mid A = Tx \text{ for some } x \in X(J),\ w(x_0) \in \bar{A} \text{ and } v(x_0) \notin \bar{A} \text{ if } v < w\}$ A T-orbit $A \subseteq X(J)$ is in C_w if and only if any one-dimensional T-orbit of $\bar{A}$ has $w(x_0)$ in its closure.

Renner showed in [32] that $(W^J, \leq)$ is isomorphic to the poset $(E_1(\overline{T}), \leq)$. If $w \in W$ and $x_0 \in X(J)^T$ is the element corresponding to $e_0 \in \Lambda_1$ then wx_0 is the element of $X(J)^T$ corresponding to wx_0w^{-1}. We have that W^J is canonically identified with the set of fixed points $X(J)^T$ of T acting on $X(J)$.

The set of one-dimensional T-orbits $\mathcal{O}_1(X(J))$ of $X(J)$ is identified with $\{(u, v) \in W^J \times W^J \mid u < v \text{ and } u^{-1}v \in S^J W_J\}$. If $(u, v) \in W^J \times W^J$ and $u^{-1}v \in S^J W_J$ then either $v < u$ or else $u < v$. The question of whether $v < w$ or $w < v$, is coded in the "descent system" (W^J, S^J).

$X(J)$ **jargon**	W^J **jargon**
$x_0 \in X(J)^T$	$1 \in W^J$
$x = w(x_0) \in X(J)^T$	$w \in W^J$
The T-orbit $A \subset X$ with $\overline{A} \cap X^T = W_I x_0$	$I \in \Lambda^\times$
The set of T-orbits (on $X(J)$) of dim $= 1$	$\{(u, v) \in W^J \times W^J$ such that $u < v$ and $u^{-1}v \in S^J W_J\}$
The set of T-orbits of dim $= 1$ with $x_0 \in \overline{A}$	$S^J = (W_J(S \setminus J)W_J) \cap W^J$
The set of T-orbits of *dim* $= 1$ in C_w	$A^J(w) = \{r \in S^J \mid w < wr\}$
The set of T-orbits on $X(T)$	$\{(w, I) \mid I \in \Lambda^\times,\ w < ws$ if $s \in I^*\}$

2.3 h-polynomial of $X(J)$

This section is a brief discussion of the h-polynomial associated to a simple polytope. For details and proofs of the following statements consult [42]. A convex n-dimensional polytope $\mathcal{P}$ is called *simple* if exactly $n-1$- codimension-one faces meet at each vertex.

Definition 8. Let $\mathcal{P}$ be a simple n-polytope. Denote by f_i the number of codimension $(i+1)$–faces of $\mathcal{P}$ where $i = -1, 0, \cdots, n-1$. The integer vector $(f_0, f_1, \cdots, f_{n-1})$ is called the *f–vector* of $\mathcal{P}$. We also put $f_1 = 1$ as $\mathcal{P}$ itself is a face of codimension zero. The *h–vector* of $\mathcal{P}$ is the integer vector $(h_0, h_1, \cdots, h_n)$ defined from the equation

$$\sum_{i=0}^{n} h_i t^{n-i} = \sum_{i=-1}^{n-1} f_i (t-1)^{n-i-1}. \tag{2.3}$$

The f–vector and h–vector carry the same information about the polytope and determine each other by means of linear relations, namely

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i}{k-i} f_{i-1}, \text{ for } k = 0, \cdots, n,$$

and

$$f_j = \sum_{i=0}^{j} \binom{j}{i} h_i, \text{ for } j = 0, \cdots, n.$$

Proposition 6. (Dehn-Sommerville Equations). If $\mathcal{P}$ is a simple n-polytope then

$$\sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} f_j = f_i, \text{ for } i = 0, \cdots, n.$$

Equivalentely,

$$h_i = h_{n-i} \text{ for } 0 \leq i \leq n.$$

Furthermore, the equation $h_0 = h_n$ is equivalent to the Euler-Poincaré formula,

$$\sum_{i=0}^{n} (-1)^i f_i = 1.$$

Moreover when P_λ is a simple integral polytope, the cohomology ring of the toric variety $X(J)$ over $\mathbb{Q}$ has the form:

$$[?]\ H^*(X(J);\mathbb{Q}) = H^0(X(J);\mathbb{Q}) \oplus H^2(X(J);\mathbb{Q}) \oplus \cdots \oplus H^{2d-1}(X(J);\mathbb{Q}),$$

where $\dim H^{2i}(X(J);\mathbb{Q}) = h_i$ that is, the $2i$-th Betti numbers of $X(J)$ are the same as h_i while the $(2i+1)$-th Betti numbers of $X(J)$ are zero.

Hence the Poincaré polynomial of $X(J)$ is expressed in terms of the h-polynomial:

$$P(X(J), t) = \sum_i (-1)^i b_i t^i = \sum_i (-1)^{2i} b_{2i} t^{2i} = \sum_i h_i t^{2i} = h(t^2)$$

The *Poincaré duality* holds when $X(J)$ is rationally smooth:

$$\dim H^q(X(J);\mathbb{Q}) = \dim H^{2d-q}(X(J);\mathbb{Q}).$$

Next, using the theory of algebraic monoids and cross section lattice associated to an algebraic monoid we present a formula for calculating the number of i-dimensional faces of the polytope P_λ. This formula appeared in [17] and it turns out to be extremely useful for our computations.

Proposition 7. [17] Let $n = \dim(P_\lambda)$. The number of i-dimensional faces of P_λ is:

$$f_i = \sum_{e \in \Lambda_i} \frac{|W|}{|W_{I^*(e)}|}$$

where $\Lambda_i = \{e \in \Lambda \mid \operatorname{rank}(e) = \dim(Te) = i + 1\}$ and $0 \leq i \leq n$.

Proof. Let $\mathcal{F}_i$ be the set of all i–dimensional faces of the polytope P_λ. We know that $W.\mathcal{F}_i = \mathcal{F}_i$ as the Weyl group permutes the i dimensional faces of P_λ.

We use the lattice isomorphism between $E(\overline{T})$ and the face lattice of the polytope P_λ obtained in (2.2) where the action of Weyl group W is on $E(\overline{T})$ is given by conjugation. Then for any $e \in E(\overline{T})$, the isotropy group of e is the centralizer of e in W, namely $W_{I^*(e)}$ according to Theorem 4. Hence we get:

$$f_i = |\mathcal{F}_i| = |W.\mathcal{F}_i| = \sum_{e \in \Lambda_i} W.e = \sum_{e \in \Lambda_i} \frac{|W|}{|W_{I^*(e)}|}$$

$\square$

Let $S(J) = \{I \subset S \mid \text{no connected component of } I \text{ is contained entirely in } J\}$. From Theorem 8 we have that $\Lambda \setminus \{0\} \cong S(J)$. Define for any $I \in S(J)$,

$$I_J^* = I \cup \{s \in J \mid st = ts \text{ for all } t \in I\}.$$

Proposition 8. The h–polynomial of $X(J)$ can be expressed in terms of the cross section lattice Λ as follows:

$$h(t) = \sum_{I \in S(J)} \frac{|W|}{|W_{I_J^*}|}(t-1)^{|I|}.$$

Proof. According to Theorem 8, to each element $e \in \Lambda \setminus \{0\}$ corresponds uniquely to a subset of S, denoted by $I(e)$ such that $I(e) = \{s \in S \mid se = es \neq e\}$ and $\mathrm{rank}(e) = |I(e)| + 1$. We associate to $I(e)$ the following set

$$I_J^*(e) = I(e) \cup \{s \in J \mid st = ts \text{ for all } t \in I(e)\}.$$

Under the correspondence 2.2 we have that $\mathrm{rank}(e) = \dim \mathcal{F}_e + 1$ where $\mathcal{F}_e$ is the face of the polytope P_λ that corresponds uniquely to $e \in E(\overline{T}) \setminus \{0\}$. We know that the h–polynomial of $X(J)$ is defined in terms of the f-polynomial, *i.e.*, $h(t) = \sum_{i=0}^{n} f_i(t-1)^i$, where f_i is the number of i-dimensional faces of the polytope P_λ. Using the preceding proposition and the fact that

$$\Lambda = \bigsqcup_{i=0}^{n} \Lambda_i,$$

we conclude that the h-polynomial is given by the following formula:

$$\begin{aligned} h(t) &= \sum_{i=0}^{n} f_i(t-1)^i = \sum_{i=0}^{n} \sum_{e \in \Lambda_i} \frac{|W|}{|W_{I^*(e)}|}(t-1)^i \\ &= \sum_{i=0}^{n} \sum_{e \in \Lambda_i} \frac{|W|}{|W_{I^*(e)}|}(t-1)^{\mathrm{rank}(e)-1} \\ &= \sum_{i=0}^{n} \sum_{e \in \Lambda_i} \frac{|W|}{|W_{I^*(e)}|}(t-1)^{|I(e)|} \\ &= \sum_{e \in \Lambda \setminus \{0\}} \frac{|W|}{|W_{I_J^*(e)}|}(t-1)^{|I(e)|} \end{aligned}$$

To simplify the notation in the preceding formula we replace for every $e \in \Lambda \setminus \{0\}$ the corresponding set $I(e) \in S(J)$ by $I \in S(J)$. We know from Theorem 8 that $\Lambda \setminus \{0\} \cong S(J)$ hence, this yields the desired formula. $\square$

Chapter 3

Betti numbers of X(J) in terms of Descent Systems

3.1 Descent systems

The notion of descent systems for algebraic monoids was introduced by Renner. For a systematic discussion on descent systems see [32].

I would refer to (1.1) for the definition of W^J, when $J \subset S$.

Definition 9. Let (W, S) be a Weyl group and let $J \subset S$ be a proper subset. Define the *descent system associated with* $J \subset S$ as:

$$S^J = (W_J(S \setminus J)W_J) \cap W^J$$

We refer to (W^J, S^J) as the descent system associated with $J \subset S$. We say J is *combinatorially smooth* if $\mathcal{P}_\lambda$ is a simple polytope.

An important result was obtained by Danilov in [11], namely $\mathcal{P}_\lambda$ is a simple polytope if and only if $X(J)$ is rationally smooth. Renner showed the following holds:

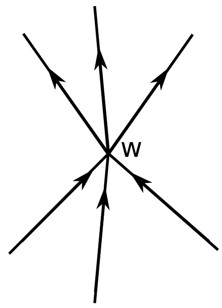


Figure 3.1: Descent system

Proposition 9. [32] Let (W^J, S^J) be the descent system associated with $J \subset S$. The following are equivalent:

1. J is combinatorially smooth.

2. $|S^J| = |S|$.

3. $X(J)$ is rationally smooth.

Definition 10. Let $w \in W^J$ and $S^J_s = W_J s W_J \cap W^J$. Define

$D^J_s(w) = \{r \in S^J_s \mid wrc < w$ in the Bruhat order for some $c \in W_J\}$.

$A^J_s(w) = \{r \in S^J_s \mid w < wr$ in the Bruhat order$\}$.

We refer to $D^J(w) = \bigsqcup_{s\in S\setminus J} D^J_s(w)$ as the descent set of w relative to J, and $A^J(w) = \bigsqcup_{s\in S\setminus J} A^J_s(w)$ as the ascent set of w relative to J. We have for $w \in W^J$ that $S^J = D^J(w) \sqcup A^J(w)$.

Remark 4. Notice that $wrc < w$ for some $c \in W_J$ if and only if $(wr)_0 < w$, where $(wr)_0 \in wrW_J$ is the element of minimal length in wrW_J.

Notice that at every vertex w of the polytope P_λ there are S^J- number of edges. And some edges are ascent edges, they correspond to the ascent set associated to that vertex and some other edges correspond to the descent set associated to that vertex as in Figure 3.1.

Proposition 10. [32] Let $u, v \in W^J$ be such that $u^{-1}v \in S^J W_J$. In particular $u \neq v$. Then either $u < v$ or $v > u$ in the Bruhat order $<$ on W^J.

Theorem 11. [32] Assume $J \subset S$ is combinatorially smooth. Then:

1. $S^J = \bigsqcup S^J_s$.

2. Let $s \in S \setminus J$. In case $st = ts$ for all $t \in J$, $S^J_s = \{s\}$. Otherwise,

$$S^J_s = \{s, t_1 s, t_2 t_1 s, \cdots, t_m \cdots t_1 s\}$$

   where $C_s = \{t_1, t_2 \cdots, t_m\}$ is the connected component of J attached to s, $st_1 \neq t_1 s$ and $t_i t_{i+1} \neq t_{i+1} t_1$ for $i = 1, \cdots, m-1$.

The following remark, which summarizes how $\mathcal{J}$–irreducible monoids are involved here, is taken from [32]. See Remark 2.24 of [32].

Remark 5. Let $E = E(\overline{T})$ be the set of idempotents of $\overline{T}$ and let $E_i = \{f \in E \mid \dim(fT) = i\} \subset E$. We have $e_1 \in E_1 = E_1(\overline{T})$ the unique element such that $e_1 B = e_1 B e_1$. For $e \in E_1$ let $v \in W^J$ be the unique element such that $e = v e_1 v^{-1}$. We write $e = e_v$. For $e, f \in E$ we write $e \sim f$ if there exists $w \in W$ such that $wew^{-1} = f$. If $s \in S \setminus J$, let $g_s \in E_2$ be the unique idempotent such that $g_s s = s g_s \neq g_s$ and $g_s B = g_s B g_s$. Let $\Lambda^\times = \{I \subset S \mid \text{no component of I is contained in J}\}$ and for $I \in \Lambda^\times$ let $I^* = I \cup \{t \in J \mid ts = st \text{ for all } s \in I\ \}$.

In the following table Renner provides a summary-translation between the monoid jargon and the Bruhat poset jargon.

The picture here is this: The subset $W^J \subset W$ is canonically identified with the subset of vertices of the rational polytope P_λ. Evidently $(E_1, \leq)$ and $(W^J, \leq)$ are anti-isomorphic as posets. Furthermore the set of edges Edj$(P_\lambda$ of P_λ is canonically identified with $E_2 = E_2(\overline{T})$. If $g(v, w) = g(w, v) \in$ Edj$(P_\lambda$ is the edge of P_λ joining the distinct vertices $v, w \in W^J$ then either $v < w$ or else $w < v$. Given $v \in W^J$, with edges $\text{edj}(v) = \{g \in E_2 \mid g = g(v, w) \text{ for some } w \in W^J\}$, the question whether $v < w$ or $w < v$ is coded in the descent system (W^J, S^J).

Reductive Monoid Jargon	**Bruhat Order Jargon**
$e_1 \in \Lambda_1 = \{e_1\}$	$1 \in W^J$
$e = e_v \in E_1$	$v \in W^J$ with $e = ve_1v^{-1}$
$e_v \leq e_w$ in E_1, i.e. $e_vBe_w \neq 0$	$w \leq v$ in W^J
$E_2 = \{g \in E \mid dim(gT) = 2\}$	$\{(u, v) \in W^J \times W^J\}$ $u < v$ and $u^{-1}v \in S^JW_J\}$
$\{g \in E_2 \mid gB = gBg\}$	$S \setminus J$
$\{g \in E_2 \mid ge_1 = e_1 \}$	$S^J = (W_J(S \setminus J)W_J) \cap W^J$
$\{g \in E_2 \mid ge_1 = e_1, g \sim g_s\}$	$S^J_s = (W_JsW_J) \cap W^J$
$E_2(e_w) = \{g \in E_2 \mid ge_w = e_w\}$	$\{v \in W^J \mid w^{-1}v \in S^JW_J\}$
$\Gamma(e_w) = \{g \in E_2(e_w) \mid ge' = e'$ for some $e' < e_w\}$	$A^J(w) = \{r \in S^J \mid w < wr\}$
$\Gamma_s(e_w) = \Gamma(e_w) \cap \{g \in E_2 \mid g \sim g_s \}$	$A^J_s(w) = \{r \in S^J_s \mid w < wr\}$
$E(\overline{T}) \setminus \{0\}$	$\{(w, I) \mid I \in \Lambda^{\times},\ w < ws$ if $s \in I^*\}$

3.2 Betti numbers of $X(J)$. Known examples

In the previous chapter we have seen how the structure and the dimensions of the BB-cells of $X(J)$ ca be described in terms of the descent system (W^S, S^J). The notion of the descent system turn out to be extremely useful in the study of the variety $X(J)$ and the main theorem of this chapter is Renner's description of the Poincare polynomial of $X(J)$ in terms of the augmented poset $(W^J, \leq, \{\nu_s\})$. By definition, $(W^J, \leq)$ is the usual Bruhat poset (which is canonically isomorphic to the poset $(E_1, \leq)$) and $\nu_s(w) = |A^J_s(w)|$, where $A^J_s(w)$ is the ascent set associated with $s \in S \setminus J$. Renner illustrates his new method with several examples: two are of type A_n where $J = \{s_2, \cdots, s_n\}$ and $J = \{s_3, \cdots, s_n\}$. A third example is (W, S) of type B_l, where $J = \{s_1, \cdots, s_{l-1}\}$.

Theorem 12. [33] Assume $X(J)$ is rationally smooth. Then the Poincare polynomial of $X(J)$ is

$$P(X(J), t) = \sum_{w \in W^J} t^{2\nu(w)}.$$

Proof. From Proposition 5 the map p_α induces a bijection on T-orbits so that the T-orbit structure on the BB-cell of dimension d is the same as the T_d-orbit structure on k^d. Here T_d is the set of invertible diagonal $d \times d$- matrices and the action of T_d

on k^d is given by multiplication. In order to determine the h-polynomial of a BB-cell of dimension d we need to find the number of codimension (i+1) T_d-orbits of k^d. A codimension $(i+1)$ T_d-orbit is of the following form: $\{\alpha_1, \alpha_2, \cdots \alpha_{n-i-1}, 0, 0, \cdots 0)\}$. The number of all such T_d -orbits is $\begin{pmatrix} d \\ i+1 \end{pmatrix}$.

Hence the h-polynomial of a d-dimensional BB-cell is given by:

$$h(e) = (t-1)^d + \begin{pmatrix} d \\ 1 \end{pmatrix} (t-1)^{d-1} + \cdots + 1 = t^d$$

By Theorem 10 there is a BB-cell $X(J)(e)$ for each $e \in E_1$ whose dimension turn out to be equal to $d = \dim X(J)(e) = |\Gamma(e)| = |A^J(w)| = \nu(w)$ where $w \in W^J$.

We proved that $h(e) = t^{\nu(w)}$, where $w \in W^J$ corresponds uniquely to $e \in E_1$. But $X(J) = \sqcup_{e \in E_1} X(J)(e)$, and so the h-polynomial of $X(J)$ is given by

$$h(t) = \sum_{e \in E_1} h(e) = \sum_{w \in W^J} t^{\nu(w)}.$$

□

Next, we illustrate the previous theorem with several examples studied by Renner in [33].

Example 11. [33] Assume that $J = \emptyset$ and let $X = X(\emptyset)$. We want to compute

$$P(X, t) = \sum_{e \in E_1} t^{2\nu(e)}$$

In this case $W^J = W$ and $S^J = S$. In this case $W \simeq E_1$ via $w \to e_w$ if $e_w = w e_1 w^{-1}$ where $e_1 \in E_1(\overline{T})$ is the unique element such that $e_1 B = e_1 B e_1$.

By the results of [33] we have:

$$\Gamma(e_w) \simeq \{s \in S \mid l(w) < l(ws)\}$$

Thus $\nu(e_w) = |\{s \in S \mid l(w) < l(ws)\}| = |S| - |D(w)|$ where

$$D(w) = \{s \in S \mid l(w) > l(ws)\}.$$

We let $d(w) = |D(w)|$. By Poincaré duality $\sum_{w \in W} t^{\nu(w)} = \sum_{w \in W} t^{2d(w)}$.

By theorem 7.2.1 of [3] we have (taking into account the doubling of degrees) that

$$P(X,t) = \sum_{I \subset S} t^{2|S \setminus I|}(t^2 - 1)^{|I|}|W^I|$$

where $W^I = \{w \in W \mid D(w) \subset S \setminus I\}$. This sum is called the **Eulerian polynomial** of W.

Next, in the case when (W, S) is the Coxeter group of type A_{n-1}, define the Eulerian numbers to be $E(n,k) = |\{w \in S_n \mid D(w) = k+1\}|$. Thus, for the associated variety X,

$$P(X,t) = \sum_{k=-1}^{n-2} E(n,k) t^{2(k+1)}$$

Similar formulas can be derived for the Coxeter groups of type B and D.

Example 12. [33] In this example we list the Poincaré polynomials associated with combinatorially smooth polyhedra of type A_3. Here $S = \{s_1, s_2, s_3\}$ with $s_1 s_2 \neq s_2 s_1$ and $s_2 s_3 \neq s_3 s_2$.

J	Associated Polyhedron	Poincaré Polynomial of $X(J)$
$\{s_1, s_2\}$	tetrahedron	$1 + t^2 + t^4 + t^6$
$\{s_1\}$	truncated tetrahedron	$1 + 5t^2 + 5t^4 + t^6$
$\{s_2, s_3\}$	tetrahedron	$1 + t^2 + t^4 + t^6$
$\{s_3\}$	truncated tetrahedron	$1 + 5t^2 + 5t^4 + t^6$
ϕ	permutahedron	$1 + 11t^2 + 11t^4 + t^6$

Example 13. In this example we list the Poincaré polynomials associated with combinatorially smooth polyhedra of type C_3. Here $S = \{s_1, s_2, s_3\}$ with $s_1 s_2 \neq s_2 s_1$ and

$s_2s_3 \neq s_3s_2$. $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$ and α_3 is the long simple root.

J	Associated Polyhedron	Poincaré Polynomial of $X(J)$
$\{s_1, s_2\}$	cube	$1 + 3t^2 + 3t^4 + t^6$
$\{s_1\}$	truncated cube	$1 + 11t^2 + 11t^4 + t^6$
$\{s_3\}$	truncated octahedron	$1 + 11t^2 + 11t^4 + t^6$
ϕ	rhombitruncated cuboctahedron	$1 + 23t^2 + 23t^4 + t^6$

Example 14. [33] In this example we discuss the Poincaré polynomial of $X(J)$ where $(W, S) = (S_{n+1}, \{s_1, s_2, \cdots, s_n\})$ is the Weyl group of type A_n $(n \geq 2)$ and $J = \{s_3, s_4, \cdots s_n\}$. Renner illustrates in [33] the computation of $P(X(J), t)$ using the structure of (W^J, S^J).

First we need to determine the set W^J. We know that $|W^J| = \frac{|S_{n+1}|}{|S_{n-1}|} = n(n+1)$. The following relations are true:

$$S_{n+1} = \bigsqcup_{i=1}^{n} (s_i \cdots s_1) S_n \cup \mathrm{id}\, S_n$$

$$S_n = \bigsqcup_{j=2}^{n} (s_j \cdots s_2) S_{n-1} \cup \mathrm{id}\, S_{n-1}$$

Hence

$$\begin{aligned} S_{n+1} \;\; = \;\; & \textstyle\bigsqcup_{i=1}^{n} \bigsqcup_{j=2}^{n} (s_i \cdots s_1)(s_j \cdots s_2) S_{n-1} \cup \bigsqcup_{i=1}^{n} (s_i \cdots s_1) S_{n-1} \cup \\ & \textstyle\bigsqcup_{j=2}^{n} (s_j \cdots s_2) S_{n-1} \cup \mathrm{id}\, S_{n-1}. \end{aligned}$$

The above calculation shows that

$$W^J = \{(s_p \cdots s_1)(s_q \cdots s_2)\} \cup \{s_p \cdots s_1\} \cup \{s_q \cdots s_2\}$$

where $1 \leq p \leq n$ and $2 \leq q \leq n$. Furthermore, by Theorem 4.2 of [32] we have

$$S_n^J = \{s_1, s_2, s_3s_2, s_4s_3s_2, \cdots s_ns_{n-1} \cdots s_3s_2\} = S_{n-1}^J \cup \{s_ns_{n-1} \cdots s_3s_2\}$$

Notice also that

$$W_n^J = W_{n-1}^J \cup \{(s_n \cdots s_1)(s_p \cdots s_2)\} \cup \{s_n \cdots s_1)\} \cup \{s_q \cdots s_1)(s_n \cdots s_2)\} \cup \{s_n \cdots s_2\}, \tag{3.1}$$

where $2 \leq p \leq n$.

The following proposition records the computation of the ascent sets of $w \in W^J$, $A^J(w) = \{r \in S^J \mid w < wr\}$.

Proposition 11. [33] Let (W, S) and $J \subset S$ be as above.

1. If $w \in W_{n-1}^J$ then $s_n \cdots s_1 \in A_n^J(w)$. Thus
   $A_n^J(w) = A_{n-1}^J(w) \cup \{s_n \cdots s_2\}$.

2. $A_n^J(s_n \cdots s_1) = \{s_2, s_3 s_2, ..., s_n s_{n-1} \cdots s_2\}$
   $A_n^J(s_n \cdots s_1 s_2) = \{s_3 s_2, s_4 s_3 s_2, ..., s_n s_{n-1} \cdots s_2\}$
   $A_n^J(s_n \cdots s_1 s_3 s_2) = \{s_4 s_3 s_2, s_5 s_4 s_3 s_2, ..., s_n s_{n-1} \cdots s_2\}$
   $\vdots$
   $A_n^J((s_n \cdots s_1)(s_{n-1} s_{n-2} \cdots s_2)) = \{s_n s_{n-1} \cdots s_2\}$
   $A_n^J((s_n \cdots s_1)(s_n s_{n-1} \cdots s_2)) = \phi$.

3. $A_n^J((s_p \cdots s_1)(s_n \cdots s_2)) = \{s_1\}$ if $1 \leq p < n$.
   $A_n^J(s_n \cdots s_2) = \{s_1\}$.

We omit the proof as it can be found in great details in [33]. Now we have all the necessary information to determine the Poincaré polynomial of $X(J)$ using Theorem 16.

Corollary 2. [33] Let $(W_n, S_n) =< s_1, s_2, ..., s_n > (n \geq 2)$, where $S_n = \{s_1, s_2, ..., s_n\}$. As above, we also let $J = \{s_3, s_4, ..., s_n\} \subset S_n$ and $X_n(J)$ the associated torus embedding. Then

$$P(X_n(J), t) = t^{2n} + (n+2)t^{2(n-1)} + (n+2)t^{2(n-2)} + ... + (n+2)t^4 + (n+2)t^2 + 1.$$

Proof. We use induction on n. When $S = \{s_1, s_2, \cdots, s_n\}$ we denote the variety $X(J)$ by $X_n(J)$ and $X_{n-1}(J)$ when $S = \{s_1, s_2, \cdots, s_{n-1}\}$.

Assume

$$P(X_{n-1}(J), t) = t^{2(n-1)} + (n+1)t^{2(n-2)} + \cdots + (n+1)t^2 + 1.$$

Using the relation 3.2 we can prove the following:

$$P(X_n(J), t) = \sum_{w \in W_n^J} t^{2\nu_n(w)} = \sum_{w \in W_{n-1}^J} t^{2\nu_n(w)} + \sum_{w \in W_n^J \setminus W_{n-1}^J} t^{2\nu_n(w)}.$$

From Proposition 11 the following relations hold:

$$\nu_n(w) = \nu_{n-1}(w) + 1.$$

$$|A_n^J(s_n \cdots s_1)| = n - 1.$$

$$A_n^J(s_n \cdots s_1 s_2)| = n - 2.$$

$$A_n^J(s_n \cdots s_1)(s_{n-1} \cdots s_2)| = 1.$$

$$A_n^J(s_n \cdots s_1)(s_n \cdots s_2)| = 0.$$

$$A_n^J(s_q \cdots s_1)(s_n \cdots s_2)| = 1 \text{ for} 1 \leq q < n.$$

$$A_n^J(s_n \cdots s_2)| = 1.$$

Hence we have

$$\begin{aligned} P(X_n(J), t) &= \sum_{w \in W_{n-1}^J} t^{2(\nu_{n-1}(w)+1)} + \sum_{p=2}^{n} t^{2\nu_n((s_n \cdots s_1)(s_p \cdots s_2))} + t^{2(\nu_n(s_n \cdots s_1)}. \\ &+ t^{2\nu_n(s_n \cdots s_2)} + \sum_{q=1}^{n-1} t^{2\nu_n(s_q \cdots s_1)(s_n \cdots s_2)}. \\ &= t^2 P(X_{n-1}(J), t) + t^{2(n-2)} + \cdots + t^2 + t^{2(n-1)} + \cdots t^{2(n-1)} + 1 \end{aligned}$$

$$
\begin{aligned}
&= t^2(t^{2(n-1)} + (n+1)t^{2(n-2)} + \cdots + (n+1)t^2 + 1) \\
&+ (t^{2(n-1)} + \cdots + t^2 + 1) + nt^2. \\
&= t^{2n} + (n+2)t^{2(n-1)} + \cdots + (n+2)t^4 + (n+2)t^2 + 1.
\end{aligned}
$$

□

Example 15. [33] In this example we consider the root system of type B_l. Let E be a real vector space with orthonormal basis $\{\epsilon_1, ..., \epsilon_l\}$. Then

$\Phi^+ = \{\epsilon_i - \epsilon_j \mid i < j\} \cup \{\epsilon_i + \epsilon_j \mid i \neq j\} \cup \{\epsilon_i\}$, and

$\Delta = \{\epsilon_1 - \epsilon_2, ..., \epsilon_{l-1} - \epsilon_l, \epsilon_l\} = \{\alpha_1, ..., \alpha_l\}$.

Let $S = \{s_1, s_2, \cdots s_{l-1}, s_l\}$ be the corresponding set of simple reflections. Here we consider the case $J = \{s_1, \cdots, s_{l-1}\}$.

We first calculate $W^J = \{w \in W \mid w(\alpha_i) \in \Phi^+ \text{ for all } 1 \leq i \leq l-1\}$. We obtain the following:

$$W^J \simeq \{1 \leq i_1 < i_2 < \cdots < i_k \leq l\},$$

via

$$w(\epsilon_v) = \epsilon_{i_v} \text{ for } 1 \leq v \leq k$$

and

$$w(\epsilon_{k+v} = -\epsilon_{j_v} \text{ for } 1 \leq v \leq l-k$$

where $l \geq j_1 > j_2 > \cdots > j_{l-k} \geq 1$ (so that $\{1, ..., l\} = \{i_1, i_2, ..., i_k\} \sqcup \{j_1, j_2, ..., j_{l-k}\}$)

After a more rigorous computation we obtain the ascent sets for each $w \in W^J$ of the following form:

$$A^J(w) = \{s_k \cdots s_l, \cdots, s_1 \cdots s_l\} = \{r \in S^J \mid w < wr\}.$$

Thus we obtain

$$\nu(w) = |\{j \mid w(\epsilon_v) = \epsilon_j \text{ for some } v\}|.$$

We can use this information to calculate the Poincaré polynomial of $X(J)$. We obtain that:

$$P(X(J),t) = \sum_{w \in W^J} t^{2\nu(w)} = \sum_{A \subset \{1,\ldots,l\}} t^{2|A|} = (1+t^2)^l.$$

Example 16. [32] Let (W,S) be the Weyl group of type A_n and let $J = \{s_2, \cdots, s_n\} \subset S$ combinatorially smooth. One checks that

$$W^J = \{1, s_1, s_2s_1, s_3s_2s_1, \cdots, s_ns_{n-1}\cdots s_2s_1\},$$

and $S^J = W^J \setminus \{1\}$. Notice that

$$1 < s_1 < s_2s_1 < \cdots s_ns_{n-1}\cdots s_1.$$

We compute the ascent sets corresponding to each $w \in W^J$ using the following calculation:

$$(s_j \cdots s_1)(s_1) = [s_j \cdots s_2],$$

$$(s_j \cdots s_1)(s_i \cdots s_1) = (s_{i-1} \cdots s_1)[s_j \cdots s_2] \text{ if } 1 < i \leq j, \text{ and}$$

$$(s_j \cdots s_1)(s_i \cdots s_1) = (s_i \cdots s_1)[s_{j+1} \cdots s_2] \text{ if } i > j \geq 1$$

We conclude from this that

$$A^J(s_j \cdots s_1) = \{s_m \cdots s_1 \mid m > j\}$$

and the corresponding Poincaré polynomial of $X(J)$ is given by the following formula:

$$P(X(J),t) = \sum_{w \in W^J} t^{\nu(w)} = \sum_{i=1}^{n} t^i.$$

3.3 Two new examples

Renner describes the Poincaré polynomial of $X(J)$ when $X(J)$ is rationally smooth in terms of the poset $(W^J, \leq, \{\nu_s\})$. In Examples 9, 10 and 11, he computes the coefficients of the Poincaré polynomial of $X(J)$ using the method of descent systems (S^J, W^J). Inspired by these results we are interested in computing explicitly the Poincaré polynomial of $X(J)$ in two interesting cases of $J \subseteq S$ combinatorially smooth. Let (W, S) be Weyl group of type A_n, with $W = S_{n+1}$, $S = \{s_1, s_2, \cdots s_n\}$, $s_i = (i\ i+1)$ and consider $J \subset S$ of the following forms:

1. $J = \{s_1, s_4, s_5, \cdots s_n\} \subset S = \{s_1, s_2, \cdots s_n\}$.

2. $J = \{s_4, s_5, \cdots, s_n\} \subset S = \{s_1, s_2, \cdots, s_n\}$.

In order to compute the Poincaré polynomial of $X(J)$, we could follow two approaches: one using Li's method of computing the h-polynomial of $X(J)$ from the f-vector and another one using Renner's method of descent systems.

In both cases listed above we compute the Poincaré polynomial of $X(J)$ using Theorem 12, which relies on determining all elements of W^J and computing their corresponding ascent sets. We introduce a new method for finding all elements of W^J, different then the method used in the previous examples.

Then using Remark 4 we compute for each $w \in W^J$ its corresponding ascent set $A(w)$ by considering the products wr where $r \in S^J$ and then writing $wr = (wr)_0 c$ where $(wr)_0 \in W^J$ and $c \in W_J$. According to Proposition 10, if $(wr)_0^{-1} w \in S^J W_J$ then $w > (wr)_0$ or $w < (wr)_0$ and this amounts to comparing their corresponding lengths, $l(w)$ and $l(wr)_0$.

Nicole Lemire suggested me a different method in proving our main results of this section. I have included a second proof for each result, proves given by Nicole Lemire.

The first main result of this section is given in the following theorem.

Theorem 13. Let (W,S) be the Weyl group of type A_n and $J=\{s_1,s_4,s_5,\cdots,s_n\}$, $J\subset S$ such that $X(J)$ is rationally smooth. Then the Poincaré polynomial of $X(J)$ is given by the following formula:

$$P(X(J),t)=\sum_w t^{\nu(w)}=1+c(n,1)t^2+\cdots+c(n,n-1)t^{2(n-1)}+t^{2n},$$

where $c(n,1)=c(n,n-1)=n+2$ and for $2\le i\le n-2$ we have $c(n,i)=n+2+\begin{pmatrix} n+1\\ 2\end{pmatrix}$.

Proof. Let

$$a_i=s_is_{i-1}\cdots s_1=(i+1\ i\cdots 1)\in S_{n+1} \text{ for } 0\le i\le n$$

$$b_j=s_js_{j-1}\cdots s_2=(j+1\ j\cdots 2)\in S_{n+1} \text{ for } 1\le j\le n$$

$$c_k=s_ks_{k-1}\cdots s_3=(k+1\ k\cdots 3)\in S_{n+1} \text{ for } 2\le k\le n$$

where $a_0=b_1=c_2=\mathrm{id}\in S_{n+1}$.

Let $S_3=\{\mathrm{id},a_2b_2,a_1,a_2,b_2,a_1b_2\}\subset S_{n+1}$, $X=\{\mathrm{id},a_1b_2,b_2\}\subset S_{n+1}$ and $Y=\{\mathrm{id},a_2,a_1\}\subset S_{n+1}$.

Lemma 1. Then

$$W^J=\{\sigma b_{i-1}\mid\sigma\in S_3\}_{4\le i\le n+1}\cup\{\sigma c_{i-1}\mid\sigma\in X\}_{4\le i\le n+1}\cup\{\sigma b_{i-1}c_{j-1}\mid\sigma\in Y\}_{4\le i<j\le n+1}$$

$$\cup\{\sigma b_{i-1}c_j\mid\sigma\in Y\}_{4\le j<i\le n+1}\cup\{\sigma b_{i-1}c_{j-1}\mid\sigma\in Y\}_{4\le i<j\le n+1}\cup\{\sigma a_{i-1}b_{j-1}\mid\sigma\in Y\}_{4\le i<j\le n+1}$$

$$\cup\{a_{i-1}b_{j-1}c_{k-1}\}_{4\le i<j<k\le n+1}\cup\{a_{i-1}b_{j-1}c_{k+1}\}_{4\le k<i<j\le n+1}\cup\{a_{i-1}b_{j-1}c_k\}_{4\le i<k<j\le n+1}$$

Proof. We know that

$$W^J\cong\frac{W}{W_J}=\frac{S_{n+1}}{\langle s_1,s_4,s_5\cdots,s_n\rangle}=\frac{S_{n+1}}{\langle s_4,s_5\cdots,s_n\rangle\times\langle s_1\rangle}.$$

Hence,

$$|W^J| = \frac{(n+1)!}{2(n-2)!} = \frac{(n-1)n(n+1)}{2}.$$

From 3.2 we have

$$W^J = \{w \in W \mid w(4) < w(5) < \cdots < w(n) \text{ and } w(1) < w(2)\}.$$

Next, we fix $w \in W^J$ and define the set E_w as follows:

$$E_w = \{i : \ 4 \leq i \leq n+1 \mid w(i) \in \{1, 2, 3\}\}.$$

The following are the only possible values for the cardinality of E_w.

1. $|E_w| = 0$. In this case there is no element in the set $\{4, 5, \cdots, n+1\}$ whose image in the permutation w belongs to the set $\{1, 2, 3\}$. We have $4 \leq w(i) < w(i+1) \leq n+1$ for $4 \leq i \leq n+1$ as $w \in W^J$. Hence $w(i) = i$ for $4 \leq i \leq n+1$ and $w(1), w(2), w(3) \in \{1, 2, 3\}$ with $w(1) < w(2)$.

   Therefore, w has one of the following forms that correspond to the cases:

   (a) $w = id$.

   (b) $w = s_1 s_2 = a_1 b_2$.

   (c) $w = s_2 = b_2$.

2. $|E_w| = 1$. In this case there exists a unique element $t \in \{4, \cdots, n+1\}$ such that $w(t) \in \{1, 2, 3\}$. We know that $w(l) < w(l+1)$ for $l \in \{4, \cdots, n\}$. Hence $t = 4$. Then there exists i such that for $5 \leq j \leq i$, $w(j) = j-1$ and for $j > i$, $w(j) = j$. We either have $w(1) = i$, or $w(2) = i$ or $w(3) = i$. The following cases hold:

(a) $w(2) = i$ and w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & i & i+1 & \dots & n+1 \\ w(1) & i & w(3) & w(4) & 4 & \dots & i-1 & i+1 & \dots & n+1 \end{pmatrix}$$

The effect of multiplying $w \in S_{n+1}$ on the right by the transposition s_i is that of interchanging the places $w(i)$ and $w(i+1)$ in the permutation w. Our goal is to find $s_i \in S$ such that $w \prod_i s_i = id$.

Consider

$$\sigma = w s_2 s_3 \cdots s_{i-1},$$

with $\sigma(i) \in \{1, 2, 3\}$ for $i \in \{1, 2, 3\}$ and $\sigma(i) = i$ for $i \in \{4, \cdots, n+1\}$. This corresponds to the following six cases:

i. $\sigma s_2 s_1 = id \implies w = s_1 s_2 s_{i-1} s_{i-2} \cdots s_3 s_2 = a_1 b_2 b_{i-1}$.

ii. $\sigma s_2 = id \implies w = s_2 s_{i-1} s_{i-2} \cdots s_3 s_2 = b_2 b_{i-1}$.

iii. $\sigma = id \implies w = s_{i-1} \cdots s_3 s_2 = b_{i-1}$.

iv. $\sigma s_1 s_2 = id \implies w = s_2 s_1 s_{i-1} \cdots s_2 = a_2 b_{i-1}$.

v. $\sigma s_1 = id \implies w = s_1 s_{i-1} \cdots s_2 = a_1 b_{i-1}$.

vi. $\sigma s_2 s_1 s_2 = id \implies w = s_2 s_1 s_2 s_{i-1} \cdots s_2 = a_2 b_2 b_{i-1}$

(b) $w(3) = i$ and w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & i & i+1 & \dots & n+1 \\ w(1) & w(2) & i & w(4) & 4 & \dots & i-1 & i+1 & \dots & n+1 \end{pmatrix}$$

Consider

$$\sigma = w s_3 s_4 \cdots s_{i-1}.$$

We have $\sigma(1) = w(1)$, $\sigma(2) = w(2)$ and $\sigma(3) = w(4) \in \{1, 2, 3\}$, with $w(1) < w(2)$. Therefore w has one of the following forms that correspond to the cases:

i. $\sigma s_2 s_1 = id \implies w = s_1 s_2 s_{i-1} s_{i-2} \cdots s_3 = a_1 b_2 c_{i-1}$

ii. $\sigma s_2 = id \implies w = s_2 s_{i-1} s_{i-2} \cdots s_3 = b_2 c_{i-1}$.

iii. $\sigma = id \implies w = s_{i-1} s_{i-2} \cdots s_3 = c_{i-1}$.

3. $|E_w| = 2$. In this case there exist two elements, $t_1, t_2 \in \{4, 5 \cdots, s+1\}$ such that their images in the permutation w is an element of the set $\{1, 2, 3\}$. We know $w(l) < w(l+1)$ for $l \in \{4, 5, \cdots, n\}$. Hence $t_1 = 4$, $t_2 = 5$. Then there exists i, j, $6 \leq i, j \leq n+1$ such that for $6 \leq l \leq i+1$ we have $w(l) = l+2$, for $i+2 \leq l \leq j$ we have $w(l) = l+1$ and for $j < l \leq n+1$ we have $w(l) = l$. This corresponds to the following cases:

   (a) $w(2) = i$, $w(3) = j$ for some $4 \leq i < j \leq n+1$ and w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & i+1 & i+2 & \dots \\ w(1) & i & j & w(4) & w(5) & 4 & \dots & i-1 & i+1 & \dots \end{pmatrix}$$

$$\begin{matrix} \dots & j & j+1 & \dots & n+1 \\ \dots & j-1 & j+1 & \dots & n+1 \end{matrix}\Bigg)$$

   Consider

$$\sigma = w s_3 s_4 \cdots s_{j-1} s_2 s_3 \cdots s_{i-1}.$$

   We have $\sigma(1) = w(1)$, $\sigma(2) = w(4)$ and $\sigma(3) = w(5) \in \{1, 2, 3\}$ with $w(4) < w(5)$. Hence w has one of the following forms that correspond to the cases:

   i. $\sigma = id \implies w = s_{i-1} s_{i-2} \cdots s_2 s_{j-1} s_{j-2} \cdots s_4 s_3 = b_{i-1} c_{j-1}$.

   ii. $\sigma s_1 = id \implies w = s_1 s_{i-1} s_{i-2} \cdots s_2 s_{j-1} s_{j-2} \cdots s_4 s_3 = a_1 b_{i-1} c_{j-1}$.

   iii. $\sigma s_1 s_2 = id \implies w = s_2 s_1 s_{i-1} s_{i-2} \cdots s_2 s_{j-1} s_{j-2} \cdots s_4 s_3 = a_2 b_{i-1} c_{j-1}$.

   (b) $w(2) = i$, $w(3) = j$ for some $4 \leq j < i \leq n+1$ and w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & j+1 & j+2 & \dots \\ w(1) & i & j & w(4) & w(5) & 4 & \dots & j-1 & j+1 & \dots \end{pmatrix}$$

$$\begin{matrix} \dots & i & i+1 & \dots & n+1 \\ \dots & i-1 & i+1 & \dots & n+1 \end{matrix}\Bigg)$$

Consider

$$\sigma = ws_3 \cdots s_{j+1}s_2s_3 \cdots s_{i-1}.$$

We have $\sigma(1) = w(1)$, $\sigma(2) = w(4)$ and $\sigma(3) = w(5) \in \{1, 2, 3\}$ with $w(4) < w(5)$. Hence w has one of the following forms that correspond to the cases:

i. $\sigma = id \implies w = s_{i-1} \cdots s_2 s_{j+1} \cdots s_3 = b_{i-1}c_{j+1}$.

ii. $\sigma s_1 = id \implies w = s_1 s_{i-1} \cdots s_2 s_{j+1} \cdots s_3 = a_1 b_{i-1} c_{j+1}$

iii. $\sigma s_1 s_2 = id \implies w = s_2 s_1 s_{i-1} \cdots s_2 s_{j+1} \cdots s_3 = a_2 b_{i-1} c_{j+1}$

(c) $w(1) = i$, $w(2) = j$ for some $4 \leq i < j \leq n+1$ and w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & i+1 & i+2 & \dots \\ i & j & w(3) & w(4) & w(5) & 4 & \dots & j-1 & j+1 & \dots \end{pmatrix}$$

$$\begin{matrix} \dots & j & j+1 & \dots & n+1 \\ \dots & j-1 & j+1 & \dots & n+1 \end{matrix}\Bigg)$$

Consider

$$\sigma = ws_2s_3 \cdots s_{j-1}s_1s_2 \cdots s_{i-1}.$$

We have $\sigma(1) = w(3)$, $\sigma(2) = w(4)$ and $\sigma(3) = w(5) \in \{1, 2, 3\}$ with

$w(4) < w(5)$. Hence w has one of the following forms that correspond to the cases:

i. $\sigma = id \implies w = s_{i-1} \cdots s_1 s_{j-1} \cdots s_2 = a_{i-1} b_{j-1}$.

ii. $\sigma s_1 = id \implies w = s_1 s_{i-1} \cdots s_1 s_{j-1} \cdots s_2 = a_1 a_{i-1} b_{j-1}$.

iii. $\sigma s_1 s_2 = id \implies w = s_2 s_1 s_{i-1} \cdots s_1 s_{j-1} \cdots s_2 = a_2 a_{i-1} b_{j-1}$.

4. $|E_w| = 3$. In this case there exist $t_1, t_2, t_3 \in \{4, 5, \cdots, n+1\}$ such that $w(t_1), w(t_2) \in \{1, 2, 3\}$. We can assume without loss of generality, $w(t_1) < w(t_2) < w(t_3)$. We know $w(l) < w(l+1)$ for $l \in \{4, 5, \cdots, n\}$. Hence $t_1 = 4$, $t_2 = 5$, $t_3 = 6$. Then there exist $i, j, k \in \{4, 5, \cdots, n+1\}$ such that $w(1) = i$, $w(2) = j$ and $w(3) = k$. This corresponds to the following cases:

   (a) $4 \leq i < j < k \leq n+1$ and w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & i & i+1 & i+2 & i+3 & \dots \\ i & j & k & 1 & 2 & \dots & i-3 & i-2 & i-1 & i+1 & \dots \end{pmatrix.$$
$$\begin{matrix.\begin{matrix} \dots & j & j+1 & j+2 & \dots & k & k+1 & \dots & n+1 \\ \dots & j-2 & j-1 & j+1 & \dots & k-1 & k+1 & \dots & n+1 \end{matrix}\right)$$

   Direct computation shows that $w s_3 s_4 \cdots s_{k-1} s_2 s_3 \cdots s_{j-1} s_1 s_2 \cdots s_{i-1} = id$. Hence $w = s_{i-1} \cdots s_1 s_{j-1} \cdots s_2 s_{k-1} \cdots s_3 = a_{i-1} b_{j-1} c_{k-1}$

   (b) $4 \leq k < i < j \leq n+1$ and w is represented by the following matrix:

$$w = \left(\begin{matrix} 1 & 2 & 3 & 4 & 5 & \dots & k & k+1 & k+2 & k+3 & \dots \\ i & j & k & 1 & 2 & \dots & k-3 & k-2 & k-1 & k+1 & \dots \end{matrix}\right.$$
$$\left.\begin{matrix} \dots & i & i+1 & i+2 & \dots & j & j+1 & \dots & n+1 \\ \dots & i-2 & i-1 & i+1 & \dots & j-1 & j+1 & \dots & n+1 \end{matrix}\right)$$

Direct computation shows that $ws_3 \cdots s_{k+1}s_2 \cdots s_{j-1}s_1s_2 \cdots s_{i-1} = id$,

Hence $w = s_{i-1} \cdots s_1 s_{j-1} \cdots s_2 s_{k+1} \cdots s_3 = a_{i-1}b_{j-1}c_{k+1}$

(c) $4 \leq i < k < j \leq n+1$ where w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & i & i+1 & i+2 & i+3 & \dots \\ i & j & k & 1 & 2 & \dots & i-3 & i-2 & i-1 & i+1 & \dots \end{pmatrix.$$

$$\begin{matrix} \dots & k & k+1 & k+2 & \dots & j & j+1 & \dots & n+1 \\ \dots & k-2 & k-1 & k+1 & \dots & j-1 & j+1 & \dots & n+1 \end{pmatrix}$$

Direct computation shows that $ws_3 \cdots s_k s_2 \cdots s_{j-1}s_1 \cdots s_{i-1} = id$.

Hence $w = s_{i-1} \cdots s_1 s_{j-1} \cdots s_2 s_k \cdots s_3 = a_{i-1}b_{j-1}c_k$

Let $\mathcal{F}$ be the set of all products in $s_1, s_2 \cdots s_n$ obtained above. Hence $\mathcal{F} \subseteq W^J$. The cardinal of $\mathcal{F}$ can be computed as follows:

$$|\{s_{i-1} \cdots s_2 : \ 4 \leq i \leq n+1\}| = n-2.$$

$$|\{s_{i-1} \cdots s_2 s_{j-1} \cdots s_3 : \ 4 \leq i < j \leq n+1\}| = \binom{n-2}{2}.$$

$$|\{s_{i-1} \cdots s_1 s_{j-1} \cdots s_2 s_{k-1} \cdots s_3 : \ 4 \leq i < j < k \leq n+1\}| = \binom{n-2}{3}.$$

Thus:

$$|\mathcal{F}| = 3 + 9(n-2) + 9\binom{n-2}{2} + 3\binom{n-2}{3} = \frac{(n-1)n(n+1)}{2} = |W^J|.$$

We conclude that $\mathcal{F} = W^J$. $\square$

Proposition 12. 1. We calculate the ascent sets for each $w \in W^J$ as follows:

(a) $A(id) = S^J$.

(b) $A(a_1b_2) = A(a_1b_{i-1}) = A(a_2b_{i-1}) = A(a_{s-1}b_{t-1}) = \{c_3, c_4, \cdots, c_n\}$, where $4 \leq s < t \leq n+1$, $4 \leq i \leq n+1$.

(c) $A(b_2) = A(b_{i-1}) = \{a_1b_2, c_3, \cdots, c_n\}$, $4 \leq i \leq n+1$.

(d) $A(a_1b_2b_{i-1}) = A(b_2b_{i-1}) = \{a_1b_2, c_4, \cdots, c_n\}$, $4 \leq i \leq n+1$.

(e) $A(a_2b_2b_{i-1}) = A(a_1a_{s-1}b_{t-1}) = \{c_4, \cdots, c_n\}$, $4 \leq s < t \leq n+1$, $4 \leq i \leq n+1$.

(f) $A(c_{i-1}) = A(a_1b_2c_{i-1}) = A(b_2c_{i-1}) = \{a_1b_2, b_2, c_p \mid p \geq i\}$, $4 \leq i \leq n+1$.

(g) $A(b_{i-1}c_{j-1}) = A(a_2b_{i-1}c_{j-1}) = A(a_1b_{i-1}c_{j-1}) = \{a_1b_2, b_2, c_p \mid p \geq j\}$, where $4 \leq i < j \leq n+1$.

(h) $A(b_{i-1}c_j) = A(a_1b_{i-1}c_j) = A(a_2b_{i-1}c_j) = \{a_1b_2, c_p \mid p \geq j+1\}$, where $4 \leq j < i \leq n+1$.

(i) $A(a_2a_{i-1}b_{j-1}) = \{c_5, \cdots, c_n\}$, $4 \leq i < j \leq n+1$.

(j) $A(a_{i-1}b_{j-1}c_{k-1}) = \{a_1b_2, b_2, c_p \mid p \geq k\}$, $4 \leq k \leq n$.

(k) $A(a_{i-1}b_{j-1}c_{k+1}) = \{c_p \mid p \geq k+2\}$, $4 \leq k \leq n-2$.

(l) $A(a_{i-1}b_{j-1}c_k) = \{b_2, c_p \mid p \geq k+2\}$, $4 \leq k \leq n-1$

2. Let $\nu(w) = |A(w)|$. Then for each $w \in W^J$ we determine the component polynomial in the Poincaré polynomial as follows:

(a) Let $w = \text{id}$ then $t^{\nu(w)} = t^n$.

(b) Let $w = a_1b_2$ then $t^{\nu(w)} = t^{n-2}$.

(c) Let $X = \{a_1b_{i-1}, a_2b_{i-1} :\ 4 \leq i \leq n+1\}$ then $\sum_{w\in X} t^{\nu(w)} = 2(n-2)t^{n-2}$.

(d) Let $X = \{a_{s-1}b_{t-1} :\ 4 \leq s < t \leq n+1\}$, then

$$\sum_{w\in X} t^{\nu(w)} = \binom{n-2}{2} t^{n-2}.$$

(e) Let $w = b_2$ then $t^{\nu(w)} = t^{n-1}$.

(f) Let $X = \{b_{i-1} :\ 4 \leq i \leq n+1\}$, then $\sum_{w \in X} t^{\nu(w)} = (n-2)t^{n-1}$.

(g) Let $X = \{a_1 b_2 b_{i-1}, b_2 b_{i-1} :\ 4 \leq i \leq n+1\}$, then $\sum_{w \in X} t^{\nu(w)} = 2(n-2)t^{n-2}$.

(h) Let $X = \{a_2 b_2 b_{i-1}\ 4 \leq i \leq n+1\}$ then $\sum_{w \in X} t^{\nu(w)} = (n-2)t^{n-3}$.

(i) Let $X = \{a_1 a_{s-1} b_{t-1} :\ 4 \leq s < t \leq n+1\}$ then $\sum_{w \in X} t^{\nu(w)} = \begin{pmatrix} n-2 \\ 2 \end{pmatrix} t^{n-3}$.

(j) Let $X = \{c_{i-1}, a_1 b_2 c_{i-1}, b_2 c_{i-1}\}$, $4 \leq i \leq n+1$, then

$$\sum_{w \in X} t^{\nu(w)} = 3 \sum_{s=1}^{n-2} t^{s+1}.$$

(k) Let $X = \{b_{i-1} c_{j-1}, a_2 b_{i-1} c_{j-1}, a_1 b_{i-1} c_{j-1} :\ 4 \leq i < j \leq n+1\}$ then

$$\sum_{w \in X} t^{\nu(w)} = 3(\sum_{s=1}^{n-3} (n-s-2) t^{s+1}).$$

(l) Let $X = \{b_{i-1} c_j, a_2 b_{i-1} c_j, a_1 b_{i-1} c_j :\ 4 \leq j < i \leq n+1\}$ then

$$\sum_{w \in X} t^{\nu(w)} = 3(\sum_{s=1}^{n-3} s t^s).$$

(m) Let $X = \{a_2 a_{i-1} b_{j-1} :\ 4 \leq i < j \leq n+1\}$ then $\sum_{w \in X} t^{\nu(w)} = \begin{pmatrix} n-2 \\ 2 \end{pmatrix} t^{n-4}$.

(n) Let $X = \{a_{i-1} b_{j-1} c_{k-1} :\ 4 \leq i < j < k \leq n+1\}$, then

$$\sum_{w \in X} t^{\nu(w)} = \sum_{s=1}^{n-4} \begin{pmatrix} n-s-2 \\ 2 \end{pmatrix} t^{s+1}.$$

(o) Let $X = \{a_{i-1}b_{j-1}c_{k+1} :\ 4 \leq k < i < j \leq n+1\}$ then

$$\sum_{w \in X} t^{\nu(w)} = \sum_{s=1}^{n-4} \binom{s+1}{2} t^{s-1}.$$

(p) Let $X = \{a_{i-1}b_{j-1}c_k :\ 4 \leq i < k < j\}$ then

$$\sum_{w \in X} t^{\nu(w)} = \sum_{s=1}^{n-4} s(n-s-3)t^s.$$

We explain the computation in details for only some representative types of elements $w \in W^J$. For the rest of the elements the computation is similar.

Consider $S \setminus J = \{s_2, s_3\}$ hence using Theorem 11 we obtain

$$S^J = \{s_2, s_1s_2, s_3, s_4s_3, \cdots s_n \cdots s_3\}.$$

1. For $w = id$, we have $A(w) = S^J$ and the corresponding component polynomial is $t^{\nu(w)} = t^n$.

   Proof. For any $r \in S^J$ we have $wr = r \in W^J$, hence $w = id < wr = r$ in the Bruhat order on W^J and in this case $A(w) = S^J$. Then $\nu(w) = |A(w)| = n$ and $t^{\nu(w)} = t^n$. $\square$

2. For $w = s_1s_2 = a_1b_2$ we have $A(w) = \{s_3, s_4s_3, \cdots, s_n \cdots s_3\}$ and the corresponding component polynomial is $t^{\nu(w)} = t^{n-2}$.

   Proof. Let $r = s_2 \in S^J$ then $wr = s_1s_2s_2 = s_1 \notin W^J$ and $wrW_J = s_1W_J = W_J$. Therefore $(wr)_0 = id$ and $(wr)_0^{-1}w = s_2 \in s^JW_J$. We have that $l((wr)_0) < l(w)$, hence $r \notin A(w)$.

   Let $r = s_1s_2 \in S^J$ then $wr = s_1s_2s_1s_2 = s_2s_1 \notin W^J$ and $wrW_J = s_2s_1W_J = s_2W_J$. Therefore $(wr)_0 = s_2$ and $(wr)_0^{-1}w = s_2s_1s_2 = s_1s_2s_1 \in S^JW_J$. We have that $l(wr)_0)) = 1 < l(w) = 2$, hence $r \notin A(w)$.

Let $r = s_{i-1} \cdots s_3 \in S^J$ for $4 \leq i \leq n$ and $wr = s_1 s_2 s_{i-1} \cdots s_3 \in W^J$. Therefore $(wr)_0 = wr$. We have that $l(w) < l(wr)$, hence $r \in A(w)$.

We obtain that the ascent set corresponding to w is of the following form: $A(w) = \{s_3, s_4 s_3, \cdots, s_n \cdots s_3\}$, and $\nu(w) = |A(w)| = n - 2$. The component polynomial is given by $t^{\nu(w)} = t^{n-2}$. □

3. For $w = b_2 = s_2$ we have $A(w) = \{s_1 s_2, s_3, s_4 s_3, \cdots, s_n \cdots s_3\}$ and the corresponding component polynomial is $t^{\nu(w)} = t^{n-1}$.

   Proof. Let $r = s_2 \in S^J$ then $wr = id \in W^J$ and $wr < w$ in the Bruhat order on W^J, hence $r \notin A(w)$.

   Let $r = s_1 s_2 \in S^J$ then $wr = s_1 s_2 s_1 \notin W^J$ and $wrW_J = s_1 s_2 W_J$. Therefore $(wr)_0 = s_1 s_2$ and $(wr)_0^{-1} w = s_2 s_1 s_2 = s_1 s_2 s_1 \in S^J W_J$. We have that $l(w) = 1 < l((wr)_0) = 2$, hence $r \in A(w)$.

   Let $r = s_{i-1} \cdots s_3$ for $4 \leq i \leq n$ and $wr = s_2 s_{i-1} \cdots s_3 \in W^J$. Therefore $(wr)_0 = wr$. We have that $l(wr)_0 > l(w)$, hence $r \in A(w)$.

   We obtain the ascent set corresponding to w is of the following form:

$$A(w) = \{s_1 s_2, s_3, s_4 s_3, \cdots, s_n \cdots s_3\},$$

   and $\nu(w) = |A(w)| = n - 1$. Then the component polynomial is given by $t^{\nu(w)} = t^{n-1}$. □

4. For $w = b_{i-2} = s_{i-1} \cdots s_2 : 4 \leq i \leq n{+}1$ we have $A(w) = \{s_1 s_2, s_3, s_4 s_3, \cdots, s_n \cdots s_3\}$ and for all $w = s_{i-1} \cdots s_2 : \; 4 \leq i \leq n+1$ the corresponding component polynomial is $t^{\nu(w)} = (n-2)t^{n-1}$.

   Proof. Let $r = s_2 \in S^J$ then $wr = s_{i-1} \cdots s_3 \in W^J$ and $(wr)_0 = wr$. We have that $(wr)_0^{-1} wr = s_2 \in S^J W_J$ and $l(wr) = i - 3$ and $l(w) = i - 2$, hence $r \notin A(w)$.

Let $r = s_1s_2 \in S^J$ then $wr = s_{i-1}\cdots s_3s_1s_2s_1 \notin W^J$. In order to determine $(wr)_0$ we do the following computation: $wrW_J = s_{i-1}\cdots s_3s_1s_2W_J = s_1s_{i-1}\cdots s_3s_2W_J$, hence $(wr)_0 = s_1s_{i-1}\cdots s_2$. Using the braid relations, we obtain that $(wr)_0^{-1}(wr) = s_1s_2s_1 \in S^JW_J$. We have that $l((wr)_0) = i-1$ and $l(w) = i-2$, hence $r \in A(w)$.

Let $r = s_3 \in S^J$ then $wr = s_{i-1}\cdots s_3s_2s_3 = s_2s_{i-1}\cdots s_2 \in W^J$. We have that $l(wr) = i-1 > l(w)$, hence $r \in A(w)$.

Let $r = s_p\cdots s_3 \in S^J$ for $p < i$ then $wr = s_{i-1}\cdots s_2s_p\cdots s_3 \in W^J$. Therefore $(wr)_0 = wr$. We have that $l(wr) = i+p-4 > l(w) = i-2$, hence $r \in A(w)$.

Let $r = s_p\cdots s_3 \in S^J$ for $i \leq p$ then $wr = s_{i-1}\cdots s_2s_p\cdots s_3 \in W^J$. Therefore $(wr)_0 = wr$. We have that $l(wr) = i+p-4 > l(w) = i-2$, hence $r \in A(w)$.

We obtain the ascent set of corresponding to w is of the following form

$$A(w) = \{s_1s_2, s_3, s_4s_3, \cdots, s_n\cdots s_3\},$$

and $\nu(w) = |A(w)| = n-1$. Then the component polynomial is given by $t^w = \sum_{i=4}^{n+1} t^{n-1} = (n-2)t^{n-1}$. □

5. For $w = c_{i-1} = s_{i-1}\cdots s_3 : 4 \leq i \leq n+1$ we have $A(w) = \{s_1s_2, s_2, s_p\cdots s_3 : p \geq i\}$ and the corresponding component polynomial is

$$t^{\nu(w)} = t^{n-1} + t^{n-2} + \cdots + t^3 + t^2.$$

Proof. Let $r = s_2 \in S^J$ then $wr = s_{i-1}\cdots s_3s_2 \in W^J$. We have that $l(wr) = i-2 > l(w) = i-3$, hence $r \in A(w)$.

Let $r = s_1s_2 \in S^J$ then $wr = s_{i-1}\cdots s_3s_1s_2 = s_1s_{i-1}\cdots s_2 \in W^J$. We have that $l(wr) = i-1 > i-3 = l(w)$, hence $r \in A(w)$.

Let $r = s_p\cdots s_3 \in S^J$ for $p \geq i$ then using braid relations we have $wr =$

$s_{i-1}\cdots s_3 s_p\cdots s_3 = s_p\cdots s_3(s_i\cdots s_4)$. Therefore $(wr)_0 = s_p\cdots s_3 \in W^J$ for $4 \leq i \leq n$. We have that $l(wr)_0 > l(w)$, hence $r \in A(w)$. Let $r = s_p\cdots s_3 \in S^J$ for $p < i$ then using braid relations we have $wr = s_{p-1}\cdots s_3(s_{i-1}\cdots s_4)$. Therefore $(wr)_0 = s_{p-1}\cdots s_3 \in W^J$. We have that $l(wr)_0) < l(w)$, hence $r \notin A(w)$.

We obtain the ascent set corresponding to w is of the following form:

$$A(w) = \{s_1 s_2, s_2, s_p\cdots s_3 : p \geq i\},$$

and $\nu(w) = |A(w)| = n-i+3$. Then the corresponding component polynomial is given by $\sum_{i=4}^{n+1} t^{n-i+3} = t^{n-1} + t^{n-2} + \cdots + t^3 + t^2$. □

6. For $w = b_{i-2}c_{j-1} = s_{i-1}\cdots s_2 s_{j-1}\cdots s_3 : 4 \leq i < j \leq n+1$ we have $A(w) = \{s_2, s_1 s_2, s_p\cdots s_3 :\ p \geq j\}$ and the corresponding component polynomial is

$$t^{\nu(w)} = t^{n-2} + 2t^{n-3} + 3t^{n-4} + \cdots + (n-3)t^2.$$

Proof. Let $r = s_1 s_2 \in S^J$ then $wr = s_{i-1}\cdots s_2 s_{j-1}\cdots s_3 s_1 s_2 = s_{i-1}\cdots s_1 s_{j-1}\cdots s_2$, $wr \in W^J$. Therefore $(wr)_0 = wr$. We have $l(wr) = i+j-3 > l(w) = i+j-5$, hence $r \in A(w)$.

Let $r = s_2 \in S^J$ then $wr = s_{j-1}\cdots s_3 s_2(s_i\cdots s_3) \in W^J$.

Therefore $(wr)_0 = wr$. We have $l(wr) = i+j-4 > l(w) = i+j-5$, hence $r \in A(w)$.

Let $r = s_p\cdots s_3 \in S^J$ for $p \geq j$ then using braid relations we obtain $wr = s_{i-1}\cdots s_2 s_p\cdots s_3(s_j\cdots s_4)$ and $wrW_J = s_{i-1}\cdots s_2 s_p\cdots s_3 W_J$.

Therefore $(wr)_0 = s_{i-1}\cdots s_2 s_p\cdots s_3 \in W^J$. We have that $l(wr)_0 > l(w)$, hence $r \in A(w)$.

Let $r = s_p\cdots s_3 \in S^J$ for $i < p < j$ then using braid relations we have $wr = s_{i-1}\cdots s_2 s_{p-1}\cdots s_3(s_{j-1}\cdots s_4)$ and $wrW_J = s_{i-1}\cdots s_2 s_{p-1}\cdots s_3 W_J$.

Therefore $(wr)_0 = s_{i-1} \cdots s_2 s_{p-1} \cdots s_3 \in W^J$. We have that $l(wr)_0) < l(w)$, hence $r \notin A(w)$.

Let $r = s_p \cdots s_3 \in S^J$ for $p \leq i < j$ then using braid relations we have $wr = s_{i-1} \cdots s_2 s_{p-1} \cdots s_3 (s_{j-1} \cdots s_4)$ and $wrW_J = s_{i-1} \cdots s_2 s_{p-1} \cdots s_3 W_J$.

Therefore $(wr)_0 = s_{i-1} \cdots s_2 s_{p-1} \cdots s_3 \in W^J$. We have that $l(wr)_0 = i+p-5 < l(w) = i+j-5$, hence $r \notin A(w)$. We obtain the ascent set corresponding to w is of the following form:

$$A(w) = \{s_2, s_1 s_2, s_p \cdots s_3 : p \geq j\}$$

and $\nu(w) = |A(w)| = n + 3 - j$. The component polynomial is given by

$$t^{\nu(w)} = \sum_{i=4}^{n} \sum_{j \geq i+1} t^{n+3-j} = t^{n-2} + 2t^{n-3} + \cdots + (n-3)t^2.$$

$\square$

7. For $w = a_{i-1} b_{j-1} = s_{i-1} \cdots s_1 s_{j-1} \cdots s_2 : 4 \leq i < j \leq n+1$, we have

$$A(w) = \{s_3, s_4 s_3, \cdots, s_n \cdots s_3\}$$

and the corresponding component polynomial is given by

$$t^{\nu(w)} = \binom{n-2}{2} t^{n-2}.$$

Proof. Let $r = s_2 \in S^J$, then $wr = s_{i-1} \cdots s_1 s_{j-1} \cdots s_3 = s_{i-1} \cdots s_2 s_{j-1} \cdots s_3 s_1$. We have $wrW_J = s_{i-1} \cdots s_2 s_{j-1} \cdots s_3 s_1 W_J$. Hence $(wr)_0 = s_{i-1} \cdots s_2 s_{j-1} \cdots s_3 \in W^J$. We know $(wr)_0^{-1} w = s_1 s_2 \in S^J W_J$ and $l(wr)_0 < l(w)$, hence $r \notin A(w)$.

Let $r = s_1s_2 \in S^J$, then using braid relations we obtain

$$\begin{aligned} wr &= s_{i-1}\cdots s_1s_{j-1}\cdots s_2s_1s_2 \\ &= s_{i-1}\cdots s_2s_{j-1}\cdots s_2s_1 \\ &= s_{j-1}\cdots s_2s_i\cdots s_3s_1 \end{aligned}$$

and $wrW_J = s_{j-1}\cdots s_2s_i\cdots s_3W_J$.

Therefore $(wr)_0 = s_{j-1}\cdots s_2s_i\cdots s_3 \in W^J$ and $(wr)_0^{-1}w = s_1s_2s_1 \in S^JW_J$. We have that $l(wr)_0 = i+j-4 < l(w) = i+j-3$, hence $r \notin A(w)$.

Let $r = s_p\cdots s_3 \in S^J$ for $i < j \leq p$ then $wr = s_{i-1}\cdots s_1s_{j-1}\cdots s_2s_p\cdots s_3 \in W^J$. Therefore $(wr)_0 = wr$. We have that $l(wr) > l(w)$, hence $r \in A(w)$.

Let $r = s_p\cdots s_3$ for $i < p < j$ then $wr = s_{i-1}\cdots s_1s_{j-1}\cdots s_2s_p\cdots s_3 \in W^J$. Therefore $(wr)_0 = wr$. We have that $l(wr) > l(w)$, hence $r \in A(w)$.

Let $r = s_p\cdots s_3$ for $p \leq i < j$ then $wr = s_{i-1}\cdots s_1s_{j-1}\cdots s_2s_p\cdots s_3 \in W^J$. Therefore $(wr)_0 = wr$. We have that $l(wr) > l(w)$, hence $r \in A(w)$.

We obtain the ascent set corresponding to w is of the following form:

$A(w) = \{s_3, s_4s_3, \cdots, s_n\cdots s_3\}$ and $\nu(w) = |A(w)| = n-2$. Then the component polynomial is of the following form:

$$t^{\nu(w)} = \sum_{4\leq i<j\leq n} t^{n-2} = \binom{n-2}{2} t^{n-2}.$$

$\square$

8. For $w = a_{i-1}b_{j-1}c_{k-1} = s_{i-1}\cdots s_1s_{j-1}\cdots s_2s_{k-1}\cdots s_3 : 4 \leq i < j < k \leq n+1$, we have

$$A(w) = \{s_1s_2, s_2, s_p\cdots s_3 :\ p \geq k\}.$$

When $k = n+1$, $w = s_{i-1}\cdots s_1s_{j-1}\cdots s_2s_n\cdots s_3$, $A(w) = \{s_1s_2, s_2\}$ and the corresponding component polynomial is given by

$$t^{\nu(w)} = t^{n-3} + \binom{3}{2} t^{n-4} + \cdots + \binom{n-4}{2} t^3 + \binom{n-3}{2} t^2.$$

Proof. Let $r = s_1 s_2 \in S^J$ then using braid relations we obtain

$$\begin{aligned} wr &= s_{i-1}\cdots s_1 s_{j-1}\cdots s_2 s_{k-1}\cdots s_3 s_1 s_2 \\ &= s_{i-1}\cdots s_1 s_{j-1}\cdots s_1 s_{k-1}\cdots s_2 \\ &= s_{j-1}\cdots s_1 s_i\cdots s_2 s_{k-1}\cdots s_2 \\ &= s_{j-1}\cdots s_1 s_{k-1}\cdots s_2 s_{i+1}\cdots s_3. \end{aligned}$$

Hence $wr \in W^J$ and $(wr)_0 = wr$. We know that $l(wr) > l(w)$, hence $r \in A(w)$.

Let $r = s_2 \in S^J$ then using braid relations we have

$$\begin{aligned} wr &= s_{i-1}\cdots s_1 s_{j-1}\cdots s_2 s_{k-1}\cdots s_3 s_2 \\ &= s_{i-1}\cdots s_1 s_{k-1}\cdots s_2 s_j\cdots s_3. \end{aligned}$$

Hence $wr \in W^J$ and $(wr)_0 = wr$. We know that $l(wr) > l(w)$, therefore $r \in A(w)$. Let $r = s_p\cdots s_3 \in S^J$ for $i < j < k \leq p$ then using braid relations we obtain

$$\begin{aligned} wr &= s_{i-1}\cdots s_1 s_{j-1}\cdots s_2 s_{k-1}\cdots s_3 s_p\cdots s_3 \\ &= s_{i-1}\cdots s_1 s_{j-1}\cdots s_2 s_p\cdots s_3 (s_k\cdots s_4) \end{aligned}$$

and $(wr)_0 = s_{i-1}\cdots s_1 s_{j-1}\cdots s_2 s_p\cdots s_3 \in W^J$.

We know $(wr)_0^{-1} w = s_{k-1}\cdots s_3 (s_4\cdots s_{p+1}) \in S^J W_J$ and $l((wr)_0) = p+i+j-5 > l(w) = k+i+j-6$, hence $r \in A(w)$.

Let $r = s_p\cdots s_3 \in S^J$ for $i < j < p < k$ then using braid relations we obtain

$$\begin{aligned} wr &= s_{i-1}\cdots s_1 s_{j-1}\cdots s_2 s_{k-1}\cdots s_3 s_p\cdots s_3 \\ &= s_{i-1}\cdots s_1 s_{j-1}\cdots s_2 s_{p-1}\cdots s_3 (s_{k-1}\cdots s_4) \end{aligned}$$

and $(wr)_0 = s_{i-1}\cdots s_1 s_{j-1}\cdots s_2 s_{p-1}\cdots s_3 \in W^J$.

We know $(wr)^{-1}w = s_{k-1}\cdots s_3 s_4\cdots s_p \in S^J W_J$ and $l((wr)_0) = i+j+p-7 < i+j+k-6 = l(w)$, hence $r \notin A(w)$. Let $r = s_p\cdots s_3 \in S^J$ for $i < p \leq j < k$ then using braid relations we obtain

$$\begin{aligned} wr &= s_{i-1}\cdots s_1 s_{j-1}\cdots s_2 s_{k-1}\cdots s_3 s_p\cdots s_3 \\ &= s_{i-1}\cdots s_1 s_{j-1}\cdots s_2 s_{p-1}\cdots s_3(s_{k-1}\cdots s_4) \end{aligned}$$

and $(wr)_0 = s_{i-1}\cdots s_1 s_{j-1}\cdots s_2 s_{p-1}\cdots s_3 \in W^J$. We have that $(wr)^{-1}w = s_{k-1}\cdots s_3 s_4\cdots s_p \in S^J W_J$ and $l((wr)_0) = i+j+p-7 < i+j+k-6 = l(w)$, hence $r \notin A(w)$.

Let $r = s_p\cdots s_3 \in S^J$ for $p \leq i < j < k$ then using braid relations we have $wr = s_{i-1}\cdots s_1 s_{j-1}\cdots s_2 s_{k-1}\cdots s_3 s_p\cdots s_3 = s_{i-1}\cdots s_1 s_{j-1}\cdots s_2 s_{p-1}\cdots s_3(s_{k-1}\cdots s_4)$ and $(wr)_0 = s_{i-1}\cdots s_1 s_{j-1}\cdots s_2 s_{p-1}\cdots s_3 \in W^J$.

We know $(wr)^{-1}w = s_{k-1}\cdots s_3 s_4\cdots s_p \in S^J W_J$ and $l((wr)_0) = i+j+p-7 < i+j+k-6 = l(w)$. Hence $r \notin A(w)$.

We obtain the ascent set corresponding to w of the following form: $A(w) = \{s_1s_2, s_2, s_p\cdots s_3 : n \geq p \geq k\}$, and $\nu(w) = n-k+3$. Next, we determine the component polynomial corresponding to w:

$$\begin{aligned} t^{\nu(w)} &= \sum_{4<i<j<k\leq n+1} t^{n-k+1} = \sum_{i=4}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} t^{n-k+1} \\ &= \sum_{i=4}^{n-2}\Big(\sum_{k=i+2}^{n} t^{n-k+3} + \sum_{k=i+3}^{n} t^{n-k+3} + \cdots + \sum_{k=n-1}^{n} t^{n-k+3} + t^3\Big) \\ &= \sum_{i=4}^{n-2}(t^{n-i+1} + 2t^{n-i} + 3t^{n-i-1} + \cdots + (n-i+1)t^3) \\ &= t^{n-3} + (1+2)t^{n-4} + (1+2+3)t^{n-5} + \cdots + (1+2+\cdots+n-5)t^3 \end{aligned}$$

$$t^{\nu(w)} = \sum_{l=4}^{n-2} \binom{l-2}{2} t^{n-l+1}.$$

For $k = n+1$, $w = s_{i-1}\cdots s_{j-1}\cdots s_2 s_n \cdots s_3$, $A(w) = \{s_1 s_2, s_2\}$ hence $\nu(w) = 2$ and the corresponding component polynomial is given by

$$\binom{n-3}{2} t^2. \tag{3.2}$$

From 3.2 and 3.3, the corresponding component polynomial of w is:

$$t^{\nu(w)} = \sum_{l=4}^{n-1} \binom{l-2}{2} t^{n-l+1}.$$

□

Using the computations done in the previous proposition we obtain the coefficients of the Poincaré polynomial denoted by $c(n,i)$ as follows:

For $i = 1$ we have $c(n,1) = 1 + n - 2 + 3 = n + 2$.

For $i = 2$ we have

$$c(n,2) = 1 + 4(n-2) + 6 + \binom{n-2}{2} = n + 2 + \binom{n+1}{2}.$$

For $i = 3$ we have

$$c(n,3) = n + 8 + 3(n-3) + \binom{n-2}{2} = n + 2 + \binom{n+1}{2}.$$

For $i = 4$ we have

$$c(n,4) = 15 + 4(n-4) + \binom{n-2}{2} = n + 2 + \binom{n+1}{2}.$$

For $i \geq 5$

$$c(n,i) = 3 + 3(i-1) + 3(n-i) + \binom{n-i+2}{2} + \binom{i-1}{2} + (i-3)(n-i))$$

$$= n + 2 + \binom{n+1}{2}. \qquad \square$$

Similarly we can compute explicitely the Betti numbers of $X(J)$ in the case of (W,S) Weyl group of type A_n and $J = \{s_4, s_5, \cdots, s_n\}$.

Theorem 14. Let (W,S) be of type A_n and $J = \{s_4, s_5, \cdots, s_n\} \subset S = \{s_1, \cdots s_n\}$ such that $X(J)$ is rationally smooth. The Poincaré polynomial of $X(J)$ is:

$$P(X(J),t) = 1 + d(n,1)t^2 + \cdots + d(n,n-1)t^{2(n-1)} + t^{2n}. \tag{3.3}$$

where $d(n,1) = d(n,n-1) = n + 2 + \binom{n+1}{2}$ and $d(n,i) = n + 2 + n(n+1)$ for $2 \leq i \leq n-2$.

Recall the previous notation: Let

$$a_i = s_i s_{i-1} \cdots s_1 = (i+1\ i \cdots 1) \in S_{n+1} \text{ for } 0 \leq i \leq n$$

$$b_j = s_j s_{j-1} \cdots s_2 = (j+1\ j \cdots 2) \in S_{n+1} \text{ for } 1 \leq j \leq n$$

$$c_k = s_k s_{k-1} \cdots s_3 = (k+1\ k \cdots 3) \in S_{n+1} \text{ for } 2 \leq k \leq n$$

where $a_0 = b_1 = c_2 = \mathrm{id} \in S_{n+1}$.

Let $S_3 = \{\mathrm{id}, a_1, b_2, a_1 b_2, a_2, a_2 b_2\}$ and $Y = \{\mathrm{id}, a_2, a_1\} \subset S_{n+1}$.

Lemma 2. Then

$$W^J = \{\sigma \in S_3\} \cup \{\sigma b_{i-1} \mid \sigma \in S_3\}_{4\leq i\leq n} \cup \{\sigma c_{i-1} \mid \sigma \in S_3\}_{4\leq i\leq n+1} \cup$$

$$\cup \{\sigma a_{i-1} \mid \sigma \in S_3\}_{4\leq i\leq n+1} \cup \{\sigma b_{i-1} c_{j-1} \mid \sigma \in Y\}_{4\leq i<j\leq n+1} \cup \{\sigma b_{j-1} c_i \mid \sigma \in Y\}_{4\leq i<j\leq n+1} \cup$$

$$\cup \{\sigma a_{i-1} b_{j-1}\ \sigma \in Y\}_{4\leq i<j\leq n+1} \cup \{\sigma a_{j-1} c_i \mid \sigma \in Y\}_{4\leq i<j\leq n+1} \cup \{\sigma a_{j-1} c_i \mid \sigma \in S_3\}_{4\leq i<j\leq n+1} \cup$$

$$\cup \{\sigma a_{i-1} c_{j-1} \mid \sigma \in S_3\}_{4\leq i<j\leq n+1} \cup \{a_{i-1} b_{j-1} c_{k-1}\}_{4\leq i<j<k\leq n+1} \cup \{a_{i-1} b_{j-1} c_{k+1}\}_{4\leq k<i<j\leq n+1} \cup$$

$$\cup \{a_{i-1} b_j c_{k-1}\}_{4\leq j<i<k\leq n+1} \cup \{a_{i-1} b_j c_k\}_{4\leq j<k<i\leq n+1} \cup \{a_{i-1} b_{j-1} c_k\}_{4\leq i<k<j\leq n+1}$$

$$\cup \{a_{i-1} b_j c_{k+1}\}_{4\leq k<j<i\leq n+1}$$

Proof. We have $W^J \cong \frac{W}{W_J} = \frac{S_{n+1}}{\langle s_4, s_5 \cdots, s_n \rangle} = \frac{S_{n+1}}{S_{n-2}}$. Hence

$$|W^J| = \frac{(n+1)!}{2(n-2)!} = \frac{(n-1)n(n+1)}{2}.$$

From 3.2 we obtain:

$$W^J = \{w \in W \mid w(4) < w(5) < \cdots < w(n)\}.$$

Next, we fix $w \in W^J$ and define the set E_w as follows:

$$E_w = w^{-1}(\{1,2,3\}) \cap \{4,5,\cdots,n+1\} = \{i :\ 4 \leq i \leq n+1 \mid w(i) \in \{1,2,3\}\}.$$

The following are the only possible values for the cardinality of E_w.

1. $|E_w| = 0$. In this case there is no element in the set $\{4,5,\cdots,n+1\}$ whose image in the permutation w belongs to the set $\{1,2,3\}$. We have $4 \leq w(i) < w(l+1) \leq n+1$ for $4 \leq i \leq n+1$ as $w \in W^J$. Hence $w(i) = i$ for $4 \leq i \leq n+1$ and $w(1), w(2), w(3) \in \{1,2,3\}$. This corresponds to the following six cases:

   (a) $w = id$.

(b) $w = s_1 s_2$.

(c) $w = s_2$.

(d) $w = s_2 s_1$.

(e) $w = s_1$.

(f) $w = s_2 s_1 s_2$.

2. $|E_w| = 1$. In this case there exists a unique element $t \in \{4, \cdots, n+1\}$ such that $w(t) \in \{1, 2, 3\}$. We know that $w(l) < w(l+1)$ for $l \in \{4, \cdots, n\}$. Hence $t = 4$. Then there exists i such that for $5 \leq l \leq i$, $w(j) = j - 1$ and for $j > i$, $w(j) = j$. For $5 \leq i \leq n+1$ we have either $w(1) = i$, or $w(2) = i$ or $w(3) = i$. The following cases hold:

   (a) $w(2) = i$ and $w(1)$, $w(3)$, $w(4) \in \{1, 2, 3\}$, so that w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & i & i+1 & \ldots & n+1 \\ w(1) & i & w(3) & w(4) & 4 & \ldots & i-1 & i+1 & \ldots & n+1 \end{pmatrix}$$

   Consider

$$\sigma = w s_2 s_3 \cdots s_{i-1}$$

   where $\sigma(1) = w(1)$, $\sigma(2) = w(3)$, and $\sigma(3) = w(4) \in \{1, 2, 3\}$. This corresponds to the following six cases:

   i. $\sigma s_2 s_1 = id \implies w = s_1 s_2 s_{i-1} \cdots s_2$.

   ii. $\sigma s_2 = id \implies w = s_2 s_{i-1} \cdots s_2$.

   iii. $\sigma = id \implies w = s_{i-1} \cdots s_2$.

   iv. $\sigma s_1 s_2 = id \implies w = s_2 s_1 s_{i-1} \cdots s_2$.

   v. $\sigma s_1 = id \implies w = s_1 s_{i-1} \cdots s_2$.

   vi. $\sigma s_2 s_1 s_2 = id \implies w = s_2 s_1 s_2 s_{i-1} \cdots s_2$.

(b) $w(3) = i$ and $w(1), w(2), w(4) \in \{1, 2, 3\}$, so that w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & i & i+1 & \dots & n+1 \\ w(1) & w(2) & i & w(4) & 4 & \dots & i-1 & i+1 & \dots & n+1 \end{pmatrix}$$

Consider

$$\sigma = ws_3s_4 \cdots s_{i-1},$$

where $\sigma(1) = w(1)$, $\sigma(2) = w(2)$, and $\sigma(3) = w(4) \in \{1, 2, 3\}$. Therefore w has one of the following forms that correspond to the cases:

i. $\sigma s_2s_1 = id \implies w = s_1s_2s_{i-1} \cdots s_3$.

ii. $\sigma s_2 = id \implies w = s_2s_{i-1} \cdots s_3$.

iii. $\sigma = id \implies w = s_{i-1} \cdots s_3$.

iv. $\sigma s_1 = id \implies w = s_1s_{i-1} \cdots s_3$.

v. $\sigma s_1s_2 = id \implies w = s_2s_1s_{i-1} \cdots s_3$.

vi. $\sigma s_2s_1s_2 = id \implies w = s_2s_1s_2s_{i-1} \cdots s_3$.

(c) $w(1) = i$ for some $4 \leq i \leq n+1$, and $w(2), w(3), w(4) \in \{1, 2, 3\}$ so that w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & i & i+1 & \dots & n+1 \\ i & w(2) & w(3) & w(4) & 4 & \dots & i-1 & i+1 & \dots & n+1 \end{pmatrix}$$

Consider

$$\sigma = ws_1s_2 \cdots s_{i-1},$$

where $\sigma(1) = w(2)$, $\sigma(2) = w(3)$, and $\sigma(3) = w(4) \in \{1, 2, 3\}$. Therefore w has one of the following forms that correspond to the cases:

i. $\sigma s_2s_1 = id \implies w = s_1s_2s_{i-1} \cdots s_1$.

ii. $\sigma s_2 = id \implies w = s_2s_{i-1} \cdots s_1$.

iii. $\sigma = id \implies w = s_{i-1} \cdots s_1$

iv. $\sigma s_1 = id \implies w = s_1 s_{i-1} \cdots s_1$.

v. $\sigma s_1 s_2 = id \implies w = s_2 s_1 s_{i-1} \cdots s_1$.

vi. $\sigma s_2 s_1 s_2 = id \implies w = s_2 s_1 s_2 s_{i-1} \cdots s_1$.

3. $|E_w| = 2$. In this case there exist two elements, $t_1, t_2 \in \{4, 5 \cdots, n+1\}$ such that their images in the permutation w is any element in the set $\{1, 2, 3\}$. We know $w(l) < w(l+1)$ for $l \in \{4, 5, \cdots, n\}$. Hence $t_1 = 4$, $t_2 = 5$. Then there exists i, j, $6 \leq i, j \leq n+1$ such that for $6 \leq j \leq i+1$, $w(l) = l+2$, for $i+2 \leq l \leq j$, $w(l) = l+1$ and for $j < l \leq n+1$, $w(l) = l$. This corresponds to the following cases:

(a) $w(2) = i$, $w(3) = j$ and $w(1)$, $w(4)$, $w(5) \in \{1, 2, 3\}$. We have w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & i+1 & i+2 & \dots \\ w(1) & i & j & w(4) & w(5) & 4 & \dots & i-1 & i+1 & \dots \end{pmatrix}$$

$$\begin{matrix} \dots & j & j+1 & \dots & n+1 \\ \dots & j-1 & j+1 & \dots & n+1 \end{matrix}\Bigg)$$

Consider

$$\sigma = w s_3 \cdots s_{j-1} s_2 \cdots s_{i-1},$$

where $\sigma(1) = w(1)$, $\sigma(2) = w(4)$, $\sigma(3) = w(5)$ and $w(4) < w(5)$. This corresponds to the following cases:

i. $\sigma = id \implies w = s_{i-1} \cdots s_2 s_{j-1} \cdots s_3$.

ii. $\sigma s_1 = id \implies w = s_1 s_{i-1} \cdots s_2 s_{j-1} \cdots s_3$.

iii. $\sigma s_1 s_2 = id \implies w = s_2 s_1 s_{i-1} \cdots s_2 s_{j-1} \cdots s_3$.

(b) $w(2) = j$, $w(3) = i$, so that w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & i+1 & i+2 & \dots \\ w(1) & j & i & w(4) & w(5) & 4 & \dots & i-1 & i+1 & \dots \end{pmatrix}$$

$$\begin{pmatrix} \dots & j & j+1 & \dots & n+1 \\ \dots & j-1 & j+1 & \dots & n+1 \end{pmatrix}$$

Consider

$$\sigma = ws_3 \cdots s_i s_2 \cdots s_{j-1},$$

where $\sigma(1) = w(1)$, $\sigma(2) = w(4)$, $\sigma(3) = w(5)$ and $w(4) < w(5)$. This corresponds to the following cases:

i. $\sigma = id \implies w = s_{j-1} \cdots s_2 s_i \cdots s_3$.

ii. $\sigma s_1 = id \implies w = s_1 s_{j-1} \cdots s_2 s_i \cdots s_3$.

iii. $\sigma s_1 s_2 = id \implies w = s_2 s_1 s_{j-1} \cdots s_2 s_i \cdots s_3$.

(c) $w(1) = i$, $w(2) = j$ and $w(3), w(4), w(5) \in \{1, 2, 3\}$, so that w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & i+1 & i+2 & \dots \\ i & j & w(3) & w(4) & w(5) & 4 & \dots & i-1 & i+1 & \dots \end{pmatrix}$$

$$\begin{pmatrix} \dots & j & j+1 & \dots & n+1 \\ \dots & j-1 & j+1 & \dots & n+1 \end{pmatrix}$$

Consider

$$\sigma = ws_2 \cdots s_{j-1} s_1 \cdots s_{i-1},$$

where $\sigma(1) = w(3)$, $\sigma(2) = w(4)$, $\sigma(3) = w(5)$ and $w(4) < w(5)$. This corresponds to the following cases:

i. $\sigma = id \implies w = s_{i-1}\cdots s_1 s_{j-1}\cdots s_2.$

ii. $\sigma s_1 = id \implies w = s_1 s_{i-1}\cdots s_1 s_{j-1}\cdots s_2.$

iii. $\sigma s_1 s_2 = id \implies w = s_2 s_1 s_{i-1}\cdots s_1 s_{j-1}\cdots s_2.$

(d) $w(1) = i$, $w(3) = j$ and $w(2), w(4), w(5) \in \{1, 2, 3\}$, so that w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & i+1 & i+2 & \dots \\ i & w(2) & j & w(4) & w(5) & 4 & \dots & i-1 & i+1 & \dots \end{pmatrix}$$

$$\left. \begin{matrix} \dots & j & j+1 & \dots & n+1 \\ \dots & j-1 & j+1 & \dots & n+1 \end{matrix} \right)$$

Consider

$$\sigma = w s_3 \cdots s_{j-1} s_1 \cdots s_{i-1},$$

where $\sigma(1) = w(2)$, $\sigma(2) = w(4)$, $\sigma(3) = w(5)$ and $w(4) < w(5)$. This corresponds to the following cases:

i. $\sigma = id \implies w = s_{i-1}\cdots s_1 s_{j-1}\cdots s_3.$

ii. $\sigma s_1 = id \implies w = s_1 s_{i-1}\cdots s_1 s_{j-1}\cdots s_3.$

iii. $\sigma s_1 s_2 = id \implies w = s_2 s_1 s_{i-1}\cdots s_1 s_{j-1}\cdots s_3.$

(e) $w(1) = j$, $w(2) = i$ and $w(3), w(4), w(5) \in \{1, 2, 3\}$, so that w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & i+1 & i+2 & \dots \\ j & i & w(3) & w(4) & w(5) & 4 & \dots & i-1 & i+1 & \dots \end{pmatrix}$$

$$\left. \begin{matrix} \dots & j & j+1 & \dots & n+1 \\ \dots & j-1 & j+1 & \dots & n+1 \end{matrix} \right)$$

Consider

$$\sigma = ws_2 \cdots s_i s_1 \cdots s_{j-1},$$

where $\sigma(1) = w(3)$, $\sigma(2) = w(4)$, $\sigma(3) = w(5)$ and $w(4) < w(5)$. This corresponds to the following cases:

i. $\sigma = id \implies w = s_{j-1} \cdots s_1 s_i \cdots s_2$.

ii. $\sigma s_1 = id \implies w = s_1 s_{j-1} \cdots s_1 s_i \cdots s_2$.

iii. $\sigma s_1 s_2 = id \implies w = s_2 s_1 s_{j-1} \cdots s_1 s_i \cdots s_2$.

(f) $w(1) = j$, $w(3) = i$ and $w(2), w(4), w(5) \in \{1, 2, 3\}$, so that w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & i+1 & i+2 & \dots \\ j & w(2) & i & w(4) & w(5) & 4 & \dots & i-1 & i+1 & \dots \end{pmatrix.$$
$$\begin{matrix} \dots & j & j+1 & \dots & n+1 \\ \dots & j-1 & j+1 & \dots & n+1 \end{pmatrix}$$

Consider

$$\sigma = ws_3 \cdots s_i s_1 \cdots s_{j-1},$$

where $\sigma(1) = w(3)$, $\sigma(2) = w(4)$, $\sigma(3) = w(5)$ and $w(4) < w(5)$. This corresponds to the following cases:

i. $\sigma = id \implies w = s_{j-1} \cdots s_1 s_i \cdots s_3$.

ii. $\sigma s_1 = id \implies w = s_1 s_{j-1} \cdots s_1 s_i \cdots s_3$.

iii. $\sigma s_1 s_2 = id \implies w = s_2 s_1 s_{j-1} \cdots s_1 s_i \cdots s_3$.

4. $|E_w| = 3$. In this case there exist $t_1, t_2, t_3 \in \{4, 5, \cdots, n+1\}$ such that $w(t_1), w(t_2) \in \{1, 2, 3\}$. We can assume without loss of generality, $w(t_1) < w(t_2) < w(t_3)$. We know $w(l) < w(l+1)$ for $l \in \{4, 5, \cdots, n\}$. Hence $t_1 = 4$, $t_2 = 5$, $t_3 = 6$. Then there exist $i, j, k \in \{4, 5, \cdots, n+1\}$ such that $w(1) = i$,

$w(2) = j$ and $w(3) = k$. This corresponds to the following cases:

(a) $4 \leq i < j < k \leq n+1$ and w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & i & i+1 & i+2 & i+3 & \dots \\ i & j & k & 1 & 2 & \dots & i-3 & i-2 & i-1 & i+1 & \dots \end{pmatrix.$$
$$\begin{matrix} \dots & j & j+1 & j+2 & \dots & k & k+1 & \dots & n+1 \\ \dots & j-2 & j-1 & j+1 & \dots & k-1 & k+1 & \dots & n+1 \end{pmatrix}$$

Direct computation shows that $ws_3s_4 \cdots s_{k-1}s_2s_3 \cdots s_{j-1}s_1s_2 \cdots s_{i-1} = id$.

Hence $w = s_{i-1} \cdots s_1 s_{j-1} \cdots s_2 s_{k-1} \cdots s_3$.

(b) $4 \leq k < i < j \leq n+1$ and w is represented by:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & k & k+1 & k+2 & k+3 & \dots \\ i & j & k & 1 & 2 & \dots & k-3 & k-2 & k-1 & k+1 & \dots \end{matrix}$$
$$\begin{matrix} \dots & i & i+1 & i+2 & \dots & j & j+1 & \dots & n+1 \\ \dots & i-2 & i-1 & i+1 & \dots & j-1 & j+1 & \dots & n+1 \end{pmatrix}$$

Direct computation shows $ws_3 \cdots s_{k+1}s_2 \cdots s_{j-1}s_1s_2 \cdots s_{i-1} = id$.

Hence $w = s_{i-1} \cdots s_1 s_{j-1} \cdots s_2 s_{k+1} \cdots s_3$.

(c) $4 \leq j < i < k \leq n+1$ where w is represented by:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & j & j+1 & j+2 & j+3 & \dots \\ i & j & k & 1 & 2 & \dots & j-3 & j-2 & j-1 & j+1 & \dots \end{matrix}$$
$$\begin{matrix} \dots & i & i+1 & i+2 & \dots & k & k+1 & \dots & n+1 \\ \dots & i-2 & i-1 & i+1 & \dots & k-1 & k+1 & \dots & n+1 \end{pmatrix}$$

Direct computation shows $ws_3 \cdots s_{k-1}s_2 \cdots s_{j+1}s_1 \cdots s_{i-1} = id$.

Hence $w = s_{i-1} \cdots s_1 s_j \cdots s_2 s_{k-1} \cdots s_3$.

(d) $4 \leq j < k < i \leq n+1$ where w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & j & j+1 & j+2 & j+3 & \dots \\ i & j & k & 1 & 2 & \dots & j-3 & j-2 & j-1 & j+1 & \dots \end{matrix}$$

$$\begin{matrix} \dots & k & k+1 & k+2 & \dots & i & i+1 & \dots & n+1 \\ \dots & k-2 & k-1 & k+1 & \dots & i-1 & i+1 & \dots & n+1 \end{pmatrix}$$

Direct computation shows $ws_3 \cdots s_k s_2 \cdots s_j s_1 \cdots s_{i-1} = id$.

Hence $w = s_{i-1} \cdots s_1 s_{j+1} \cdots s_2 s_k \cdots s_3$.

(e) $4 \leq i < k < j \leq n+1$ where w is represented by the following matrix:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & i & i+1 & i+2 & i+3 & \dots \\ i & j & k & 1 & 2 & \dots & i-3 & i-2 & i-1 & i+1 & \dots \end{matrix}$$

$$\begin{matrix} \dots & k & k+1 & k+2 & \dots & j & j+1 & \dots & n+1 \\ \dots & k-2 & k-1 & k+1 & \dots & j-1 & j+1 & \dots & n+1 \end{pmatrix}$$

Direct computation shows $ws_3 \cdots s_k s_2 \cdots s_{j-1} s_1 \cdots s_{i-1} = id$.

Hence $w = s_{i-1} \cdots s_1 s_{j-1} \cdots s_2 s_k \cdots s_3$.

(f) $4 \leq k < j < i \leq n+1$ where w is represented by:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & k & k+1 & k+2 & k+3 & \dots \\ i & j & k & 1 & 2 & \dots & k-3 & k-2 & k-1 & k+1 & \dots \end{matrix}$$

$$\begin{matrix} \dots & j & j+1 & j+2 & \dots & i & i+1 & \dots & n+1 \\ \dots & j-2 & j-1 & j+1 & \dots & i-1 & i+1 & \dots & n+1 \end{pmatrix}$$

Direct computation shows $ws_3 \cdots s_{k+1} s_2 \cdots s_j s_1 \cdots s_{i-1} = id$.

Hence $w = s_{i-1}\cdots s_1 s_j \cdots s_2 s_{k+1}\cdots s_3$.

□

Next, we compute for each element $w \in W^J$ its corresponding ascent set $A(w)$ in a similar fashion as in Theorem 13. We summarize the computation in the following proposition and we omit its proof because it does not carry anything new other then repeating the previous sort of computations.

Proposition 13. 1. We calculate the ascent sets for each $w \in W^J$ as follows:

(a) $A(\mathrm{id}) = S^J$.

(b) $A(\mathrm{id}) = S^J$, $A(a_1) = A(a_2) = \{b_2, c_3, c_4, \cdots, s_n\}$.

(c) $A(b_2) = A(a_1b_2) = A(b_{i-1}) = A(a_1b_{i-2}) = A(a_2b_{i-1}) = \{a_1, c_3, c_4, \cdots, s_n\}$.

(d) $A(a_2b_2) = \{c_3, c_4, \cdots, c_n\}$.

(e) $A(a_1b_2b_{i-1}) = A(a_2b_2b_{i-1}) = A(b_2b_{i-1}) = \{a_1, c_4, \cdots, c_n\}$.

(f) $A(c_{i-1}) = A(b_2c_{i-3}) = A(a_1b_2c_{i-1}) = \{a_1, b_2, c_p :\ p \geq i\}$.

(g) $A(a_1c_{i-1}) = A(a_2c_{i-1}) = A(a_2b_2c_{i-1}) = \{b_2, c_p :\ p \geq i\}$.

(h) $A(a_{i-1}) = \{b_2, c_3, \cdots, c_n\}$.

(i) $A(a_1b_{i-1}) = A(a_2a_{i-1}) = \{c_3, c_4, \cdots, c_n\}$.

(j) $Aa_1b_2a_{i-1}) = A(b_2a_{i-1}) = \{b_2, c_4, \cdots, c_n\}$.

(k) $A(a_2b_2a_{i-1}) = \{c_4, c_5, \cdots, c_n\}$.

(l) $A(b_{i-1}c_{j-1}) = A(a_2a_{i-1}c_{j-1}) = A(a_1a_{i-2}c_{j-1}) = \{a_1, b_2, c_p :\ p \geq j\}$

(m) $A(b_{j-1}c_i) = A(a_2b_{j-1}c_i) = A(a_1b_{j-1}c_i) = \{a_1, c_p : p \geq i+1\}$.

(n) $A(a_{i-1}b_{j-1}) = \{a_1, c_3, c_4, \cdots, c_n\}$.

(o) $A(a_2a_{i-1}b_{j-1}) = \{a_1, c_5, \cdots, c_n\}$.

(p) $A(a_1a_{i-1}b_{j-1}) = \{a_1, c_4, c_5, \cdots, c_n\}$.

(q) $A(a_2a_{j-1}b_i) = \{c_5, c_4, \cdots, c_n\}$, $A(a_1a_{j-1}b_i) = \{c_4, c_5, \cdots, c_n\}$.

(r) $A(a_{j-1}b_i) = \{c_3, c_4, \cdots, c_n\}$.

(s) $A(a_{j-1}c_i) = A(a_2a_{j-1}c_i) = A(a_1a_{j-1}c_i) = \{b_2, c_p : \ p \geq i+1\}$.

(t) $A(a_{i-1}c_{j-1}) = A(a_2a_{i-1}c_{j-1}) = A(a_1a_{i-1}c_{j-1}) = \{b_2, c_p : \ p \geq j\}$.

(u) $A(a_{i-1}b_{j-1}c_{k-1}) = \{a_1, b_2, c_p : p \geq k\}$, $4 \leq i < j < k\}$.

(v) $A(a_{i-1}b_{j-1}c_{k+1}) = \{a_1, c_p : p \geq k+2\}$, $4k < i < j \leq n+1$.

(w) $A(a_{i-1}b_jc_{k-1}) = \{b_2, c_p : \ p \geq k\}$, $4 \leq j < i < k \leq n+1$.

(x) $A(a_{i-1}b_{j-1}c_k) = \{a_1, c_p : \ p \geq k+1\}$, $4 \leq i < k < j \leq n+1$.

(y) $A(a_{i-1}b_jc_{k+1}) = \{c_p : \ p \geq k+2\}$, $4 \leq k < j < i \leq n+1$.

(z) $A(a_{i-1}b_jc_k) = \{b_2, c_p : \ p \geq k+1\}$, $4 \leq j < k < i \leq n+1\}$.

2. Let $\nu(w) = |A(w)|$. then for each $w \in W^J$ we determine the component polynomial in the Poincaré polynomial as follows:

(a) Let $w = \text{id}$ then $t^{\nu(w)} = t^n$.

(b) Let $X = \{a_1, b_2, a_1a_2, a_2\}$ then $\sum_{w\in X} t^{\nu(w)} = 3t^{n-1}$.

(c) Let $w = a_2b_2$ then $t^{\nu(w)} = t^{n-2}$.

(d) Let $X = \{b_{i-1}, a_1b_{i-1}, a_2b_{i-1}, a_{i-1}\}$ then $t^{\nu(w)} = 4(n-2)t^{n-1}$.

(e) $X = \{a_1b_2b_{i-1}, a_2b_2b_{i-1}, b_2a_{i-1}, a_1a_{i-1}, b_2a_{i-1}, a_2a_{i-1}, a_1b_2a_{i-1}, \ 4 \leq i \leq n+1\}$ then $t^{\nu(w)} = 7(n-2)t^{n-2}$.

(f) Let $X = \{c_{i-1}, b_2c_{i-1}, a_1b_2c_{i-1}, \ 4 \leq i \leq n+1\}$ then $t^{\nu(w)} = 3\sum_{s=1}^{n-2} t^{s+1}$.

(g) Let $X = \{a_1c_{i-1}, a_2c_{i-1}a_2b_2c_{i-1}, \ 4 \leq i \leq n+1\}$ then $t^{\nu(w)} = 3\sum_{s=1}^{n-2} t^s$.

(h) Let $w = \{a_2b_2a_{i-1}, \ 4 \leq i \leq n+1\}$ then $t^{\nu(w)} = (n-2)t^{n-3}$.

(i) Let $X = \{b_{i-1}c_{j-1}, a_2b_{i-1}c_{j-1}, a_1b_{i-1}c_{j-1} \ 4 \leq i < j \leq n+1\}$ then

$$\sum_{w\in X} t^{\nu(w)} = 3\sum_{s=1}^{n-3}(n-s-2)t^{s+1}.$$

(j) Let $X = \{b_{j-1}c_i, a_2b_{j-1}c_i, a_1b_{j-1}c_i, a_{j-1}c_i, a_2a_{j-1}c_i, a_1a_{j-1}c_i :\ 4 \leq i < j \leq n+1\}$ then

$$\sum_{w \in W} t^{\nu(w)} = 6 \sum_{s=1}^{n-3} st^s.$$

(k) Let $w = a_{i-1}b_{j-1}$, $4 \leq i < j \leq n+1$ then $t^{\nu(w)} = \begin{pmatrix} n-2 \\ 2 \end{pmatrix} t^{n-1}$.

(l) Let $X = \{a_2a_{i-1}b_{j-1}, a_1a_{j-1}b_i, 4 \leq i < j \leq n{+}1$ then $t^{\nu(w)} = \begin{pmatrix} n-2 \\ 2 \end{pmatrix} t^{n-3}$

(m) Let $w = a_1a_{i-1}b_{j-1}, a_{j-1}c_i$, $4 \leq i < j \leq n+1$ then $t^{\nu(w)} = \begin{pmatrix} n-2 \\ 2 \end{pmatrix} t^{n-2}$.

(n) Let $w = a_2a_{j-1}b_i$, $4 \leq i < j \leq n+1$ then $t^{\nu(w)} = \begin{pmatrix} n-2 \\ 2 \end{pmatrix} t^{n-4}$.

(o) Let $X = \{a_{i-1}c_{j-1}, a_2a_{i-1}c_{j-1}, a_1a_{i-1}c_{j-1}\}$, $4 \leq i < j \leq n+1$ then

$$\sum_{w \in X} t^{\nu(w)} = 3 \sum_{s=1}^{n-3} (n-s-2)t^s.$$

(p) Let $w = a_{i-1}b_{j-1}c_{k-1}$, $4 \leq i < j < k \leq n+1$ then

$$t^{\nu(w)} = \sum_{s=1}^{n-4} \begin{pmatrix} n-s-2 \\ 2 \end{pmatrix} t^{s+1}.$$

(q) Let $w = a_{i-1}b_{j-1}c_{k+1}$, $4 \leq k < i < j \leq n+1$ then

$$t^{\nu(w)} = \sum_{s=1}^{n-4} \begin{pmatrix} s+1 \\ 2 \end{pmatrix} t^s.$$

(r) Let $w = a_{i-1}b_jc_{k-1}$, $4 \leq j < i < k \leq n+1$ then

$$t^{\nu(w)} = \sum_{s=1}^{n-4} \binom{n-s-2}{2} t^s.$$

(s) Let $w = a_{i-1}b_{j-1}c_k$, $4 \leq i < k < j \leq n+1$ then

$$t^{\nu(w)} = \sum_{s=1}^{n-4} (n-s-3)st^s.$$

(t) Let $w = a_{i-1}b_{j-1}c_{k-1}$, $4 \leq k < j < i \leq n+1$ then

$$t^{\nu(w)} = \sum_{s=0}^{n-4} \binom{s+2}{2} t^s.$$

(u) let $w = a_{i-1}b_jc_k$, $4 \leq j < k < i \leq n+1$ then

$$t^{\nu(w)} = \sum_{s=1}^{n-4} \binom{n-s-2}{2} t^{s+1}.$$

We can easily identify the coefficients of the Poincaré polynomial denoted by $d(n,i)$, in the expression $\sum_{w \in W^J} t^{\nu(w)}$ using the computations done in the previous proposition. The following holds:

For $i = 1$ we have $d(n,1) = 7 + 4(n-2) + \binom{n-2}{2} = n + 2 + \binom{n+1}{2}$.

For $i = 2$ we have $d(n,2) = 7(n-2) + 10 + (n-2)(n-3) = n + 2 + n(n+1)$

For $i = 3$ we have $d(n,3) = 16 + n - 2 + 6(n-3) + (n-2)(n-3) = n + 2 + n(n+1)$.

For $i = 4$ we have

$$\begin{aligned} d(n,4) &= 22 + 8(n-4) + \binom{n-2}{2} + \binom{n-3}{2} \\ &= n + 2 + n(n+1). \end{aligned}$$

For $i = 5$ we have

$$\begin{aligned} d(n,5) &= 24 + 10(n-5) + \binom{4}{2} + \binom{n-4}{2} + \binom{3}{2} + \binom{n-3}{2}. \\ &= n + 2 + n(n+1). \end{aligned}$$

For $6 \le i \le n-2$ we have

$$\begin{aligned} d(n,i) &= 2(i-3)(n-i) + 6(i-2) + 6(n-i) + 3(i-1) + 6 \\ &+ \binom{n-i+2}{2} + \binom{i-2}{2} + \binom{n-i+1}{2} + \binom{i-1}{2}. \\ &= n^2 + 2n + 2 = n + 2 + n(n+1). \end{aligned}$$

For $i = n-1$ we have

$$d(n,n-1) = 7 + 4(n-2) + \binom{n-2}{2} = n + 2 + \binom{n+1}{2}.$$

Therefore the Poincaré polynomial is obtained as follows:

$$P(X(J),t) = \sum_w t^{\nu(w)} = 1 + d(n,1)t^2 + \cdots + d(n,n-1)t^{2n-2} + t^{2n},$$

where $d(n,1) = d(n,n-1) = n+2+\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$ and $d(n,i) = n+2+n(n+1)$ for $i = 2, \cdots, n-2$.

Another approach was suggested to me by Nicole Lemire and with her permission I would like to include it in this document.

$second\ proof\ of\ Theorem$ 14 $given\ by\ NicoleLemire$

We have

$$W^J \cong \frac{W}{W_J} = \frac{S_{n+1}}{S_{n-2}} = \frac{\langle s_1, s_2, \cdots, s_n \rangle}{\langle s_4, s_5 \cdots, s_n \rangle} = \frac{\langle s_1, s_2, \cdots, s_n \rangle / \langle s_2, s_3, \cdots, s_n \rangle}{\langle s_2, s_3, \cdots, s_n \rangle / \langle s_3, s_4, \cdots, s_n \rangle}$$

A set of coset representatives for S_n in S_{n+1} is given by $\{\mathrm{id}, s_1, s_2s_1, \cdots, s_ns_{n-1}\cdots s_1\}$.

Let $a_i = (1\ 2 \cdots i\ i+1) \in S_{n+1}$ for $0 \leq i \leq n$, then

$$S_{n+1} = \bigsqcup_{i=0}^{n} a_i S_n. \tag{3.4}$$

A set of coset representatives for S_{n-1} in S_n is given by $\{\mathrm{id}, s_2, s_3s_2, \cdots, s_ns_{n-1}\cdots s_2\}$.

Let $b_j = (2\ 3 \cdots j\ j+1) \in S_{n+1}$ for $1 \leq j \leq n-1$, then

$$S_n = \bigsqcup_{j=1}^{n} b_j S_{n-1}. \tag{3.5}$$

A set of coset representatives for S_{n-2} in S_{n-1} is given by $\{\mathrm{id}, s_3, s_4s_3, \cdots, s_ns_{n-1}\cdots s_3\}$.

Let $c_k = (3\ 4 \cdots j\ j+1) \in S_{n+1}$ for $2 \leq j \leq n$, then

$$S_{n-1} = \bigsqcup_{j=2}^{n} c_k S_{n-2}. \tag{3.6}$$

From (3.5), (3.6), (3.7) we obtain a list of coset representatives for S_{n-2} in S_{n+1} given by $\{a_i b_j c_k\}_{i=0,\ j=1,\ k=2}$ where

$$S_{n+1} = \bigsqcup_{i=0,\ j=1\ k=2}^{n} a_i b_j c_k S_{n-2} \tag{3.7}$$

Recall the definition of $W^J = \{w \in S_{n+1} \mid w(r) < w(r+1),\ 4 \le r \le n\}$

Next we want to show that $\{a_i b_j c_k\}_{i=0,\ j=1,\ k=2}$ is a set of coset representatives of S_{n-2} in S_{n+1} of minimal length hence $W^J = \{a_i b_j c_k\}_{i,j,k}$.

We consider the case of $i \ge j \ge k$ (the other cases are treated similarly) and obtain that $a_i b_j c_k(r) = r - 3$ if $4 \le r \le k+1$, $a_i b_j c_k(r) = r - 2$ if $k+1 < r \le j+1$, $a_i b_j c_k(r) = r - 1$ if $j + 1 < r \le i + 1$ and $a_i b_j c_k(r) = r$ if $i + 1 < r \le n + 1$.

Hence $a_i b_j c_k(r) < a_i b_j c_k(r+1)$ for $4 \le r \le n$. We obtain $\{a_i b_j c_k\}_{i,j,k} \subseteq W^J$ and $|\{a_i b_j c_k\}_{i,j,k}| = (n-1)n(n+1)$ therefore $W^J = \{a_i b_j c_k\}_{i=0,\ j=1,\ k=2}$.

Next, notice that $S^J = \{s_1, s_2, s_3, \cdots, s_n \cdots s_3\} = \{a_1, b_2, c_3, \cdots c_n\}$.

Let $d_l = (l+1\ l \cdots 4)$, $3 \le l \le n$. Then using braid relations we obtain that

$$c_i c_j = c_{j-1} d_i,\ 3 \le j \le i \le n$$

$$c_i c_j = c_j d_{i+1},\ 3 \le i < j \le n,\ c_2 c_j = c_j$$

and similarly

$$b_i b_j = b_{j-1} c_i,\ 2 \le j \le i \le n$$

$$b_i b_j = b_j c_{i+1},\ 2 \le i < j \le n,\ b_1 b_j = b_j$$

$$a_i a_j = a_{j-1} b_i,\ 1 \le j \le i \le n$$

$$a_i a_j = a_j b_{i+1},\ 1 \le i < j \le n,\ a_0 b_j = a_j$$

Next we prove that $c_l \in A_n^J(a_i b_j c_k)$ if and only if $k < l$.

Here we use Proposition 10 to compare two elements in the Bruhat ordering on W^J. Notice that for $w = a_i b_j c_k \in W^J$ and $r = c_l \in S^J$, if $(wr)^{-1}w \in S^J W_J$ then $w < wr$ in the Bruhat ordering on W^J if and only if $l(w) < l(wr)$ according to Proposition 10.

Let $w = a_i b_j c_k$ then $wr = a_i b_j c_k c_l = a_i b_j c_{l-1} d_k$ for $l \leq k$ and $wr = a_i b_j c_l d_{k+1}$ for $k < l$. Let w_0 be the minimal coset representative corresponding to wrW_J.

Hence $wrW_J = a_i b_j c_{l-1} W_J = w_0 W_J$ for $l \leq k$ and $wrW_J = a_i b_j c_l W_J = w_0 W_j$ for $k < l$. Using braid relations we obtain:

$$w^{-1}w_0 = s_{l-1} \cdots s_3 (s_4 \cdots s_k) \in S^J W_J \text{ for } l \leq k$$

$$w^{-1}w_0 = s_l \cdots s_3 (s_4 \cdots s_k) \in S^J W_J \text{ for } k < l.$$

Next we compare in Bruhat ordering w and w_0 according to Proposition 10 knowing that $l(w) = l(a_i b_j c_k) > l(w_0) = l(a_i b_j c_{l-1})$ for $l \leq k$ and $l(w) = l(a_i b_j c_k) < l(w_0) = l(a_i b_j c_l)$ for $k < l$.

Since $a_i b_j c_k s_2 = a_i b_j c_k$, and $a_i b_j c_k s_1 = a_i b_j s_1 c_k = a_i a_j c_k$ similarly it can be proved that $s_1 \in A_n^J(a_i b_j c_k)$ if and only if $i < j$ and $s_2 \in A_n^J(a_i b_j c_k)$ if and only if $j < k$. So

Case I: $0 \leq i < j < k \leq n$

$$A_n^J(a_i b_j c_k) = \{s_1, s_2, c_{k+1}, \cdots, c_n\}$$

then $\nu_n^J(a_i b_j c_k) = n - k + 2$ and for fixed $2 \leq k \leq n$ we have

$$|\{a_i b_j c_k :\ 0 \leq i < j < k\}| = \binom{k}{2}.$$

Case II: $2 \leq k \leq j \leq i \leq n$

$$A_n^J(a_i b_j c_k) = \{c_{k+1}, \cdots, c_n\}.$$

Then $\nu_n^J(a_i b_j c_k) = n - k$ and for fixed $2 \leq k \leq n$ we have

$$|\{a_i b_j c_k : \ k \leq j \leq i \leq n\}| = \frac{(n-k+1)(n-k+2)}{2}$$

Case III: $0 \leq i < j \leq n$ and $2 \leq k \leq j \leq n$

$$A_n^J(a_i b_j c_k) = \{s_1, c_{k+1}, \cdots, c_n\}$$

Then $\nu_n^J(a_i b_j c_k) = n - k + 1$ and for a fixed $2 \leq k \leq n$

$$|\{a_i b_j c_k : \ 0 \leq i < j, \ k \leq j \leq n\}| = \sum_{r=k}^{n} r = \frac{n(n+1)}{2} - \frac{k(k-1)}{2}$$

Case IV: $1 \leq j \leq i \leq n$ and $1 \leq j < k \leq n$

$$A_n^J(a_i b_j c_k) = \{s_2, c_{k+1}, \cdots, c_n\}$$

then $\nu_n^J(a_i b_j c_k) = n - k + 1$ and for fixed $2 \leq k \leq n$

$$|\{a_i b_j c_k : \ 1 \leq j \leq i \leq n, \ j < k\}| = \sum_{r=n-k+2}^{n} r = \frac{n(n+1)}{2} - \frac{(n-k+1)(n-k+2)}{2}$$

To find $\sum_{w \in W^J} t^{2\nu_n^J(w)}$ we only need to find $c_r = |\{w \in W^J : \ \nu_n(w) = r\}|$ for each $0 \leq r \leq n$.

Let $r = 0$. Then $\nu_n(a_i b_j c_k) = 0$ is only possible in Case II. In this case, $k = n$ and $k \leq j \leq i \leq n$ implies that $a_n b_n c_n$ is the only such element and $c_0 = 1$.

Let $r = n$. Then $\nu_n(a_i b_j c_k) = n$ is only possible in Case I. In this case, $k = 2$ and $0 \leq i < j < 2$ implies that $a_0 b_1 c_2 =$ is the only such element and $c_n = 1$.

Let $r = 1$. Then $\nu_n(a_i b_j c_k) = 1$ is only possible in Cases II, III and IV. For Case II, $k = n-1$ and for Cases II and III, $k = n$. So
$c_1 = \dfrac{2(3)}{2} + \dfrac{n(n+1)}{2} - \dfrac{1(2)}{2} + \dfrac{n(n+1)}{2} - \dfrac{n(n-1)}{2} = \dfrac{n(n+1)}{2} + n + 2.$

Let $r = n-1$. Then $\nu_n(a_i b_j c_k) = 1$ is only possible in Cases I, III and IV. For Case I, $k = 3$ and for Case II and III, $k = 2$. So
$c_{n-1} = \dfrac{2(3)}{2} + \dfrac{n(n+1)}{2} - \dfrac{1(2)}{2} + \dfrac{n(n+1)}{2} - \dfrac{n(n-1)}{2} - \dfrac{r(r-1)}{2} = \dfrac{n(n+1)}{2} +$
$n + 2.$

Let $2 \leq r \leq n-2$. Then $\nu_n(a_i b_j c_k) = r$ is possible in all cases. For Case I, $n - k + 2 = r$ implies $k = n - r + 2$, for Case II, $n - k = r$ implies $k = n - r$ and for Case III and IV, $n - k + 1 = r$ implies $k = n - r + 1$. So
$c_r = \dfrac{(n-r+2)(n-r+1)}{2} + \dfrac{(r+1)(r+2)}{2} + \dfrac{n(n+1)}{2} - \dfrac{(n-r+1)(n-r)}{2} + \dfrac{n(n+1)}{2}$
$= n(n+1) + (r+1) + (n-r+1) = n(n+1) + n + 2.$

We have obtained that

$$P(X(J), t) = t^{2n} + 1 + (\frac{n(n+1)}{2} + n + 2)(t^{2(n-1)} + 1) + (n(n+1) + n + 2)(\sum_{k=2}^{n-2} t^{2k}).$$

Lemire's method turn out to be efficient for proving Theorem 13 as well. Next, I would like to present a second proof of Theorem 13, suggested to me by Nicole Lemire.

second proof of Theorem 13 *given by Nicole Lemire*

Let $J = \{s_1, s_4, s_5, \cdots, s_n\}$. Then

$$W^J \cong \frac{S_{n+1}}{\langle s_1 \rangle \times \langle s_4, s_5, \cdots, s_n \rangle} \cong \frac{S_{n+1}}{S_2 \times S_{n-1}}.$$

Consider the set $Y = \{y_{ij} + y_{ji} : 1 \leq i < j \leq n+1\}$ is an S_{n+1}–set with the S_{n+1}-action given by: $\sigma(y_{ij} + y_{ji}) = y_{\sigma(i),\sigma(j)} + y_{\sigma(j),\sigma(i)}$, $\sigma \in S_{n+1}$. The action of S_{n+1} on the set Y is a tranzitive action and the stabiliser subgroup of $y_{12} + y_{21}$ is

$S_{n-1} \times S_2 = \langle s_1, s_3, \cdots, s_n \rangle$. The cardinal of Y is given by $n(n+1)/2$ and note that

$$y_{ij} + y_{ji} = a_i b_j (y_{12} + y_{21}).$$

Hence Y is a set of coset representatives of $S_2 \times S_{n-1}$ in S_{n+1}.

$$S_{n+1} = \bigsqcup_{0 \leq i < j \leq n} a_i b_j (S_{n-1} \times S_2) \tag{3.8}$$

We have

$$S_{n-1} \times S_2 = \bigsqcup_{k=2}^{n} c_k (S_{n-2} \times S_2 \tag{3.9}$$

Then from (3.9) and (3.10) we obtain that $\{a_i b_j c_k : \ 0 \leq i < j \leq n, 2 \leq k \leq n\}$ is a set of coset representatives of $S_{n-2} \times S_2$ in S_{n+1}.

$$S_{n+1} = \bigsqcup_{0 \leq i < j \leq n, 2 \leq k \leq n} a_i b_j c_k (S_{n-2} \times S_2).$$

We need to check that $\{a_i b_j c_k\}_{0 \leq i < j \leq n, 2 \leq k \leq n}$ is a set of coset representatives of minimal length, therefore we need to check that $a_i b_j c_k(r) < a_i b_j c_k(r+1)$ for $4 \leq r \leq n$. We already checked above that $a_i b_j c_k(r) < a_i b_j c_k(r+1)$ for $4 \leq r \leq n$ and for all $0 \leq i \leq n$, $1 \leq j \leq n$, $2 \leq k \leq n$ so it is certainly true for our subset. We only need $a_i b_j c_k(1) < a_i b_j c_k(2)$ which is true since for $i < j$, $a_i b_j c_k(1) = i$ and $a_i b_j c_k(2) = j$. So

$$W^J = \{a_i b_j c_k : \ 0 \leq i < j \leq n, \ 2 \leq k \leq n\}$$

Now $S^J = \{s_1 s_2, s_2, s_3, \cdots, s_n \cdots s_3\} = \{a_1 b_2, b_2, c_3, \cdots, s_n\}$. As above, $c_l \in A_n^J(a_i b_j c_k)$ if and only if $k < l$ (using Proposition 10) and $s_2 \in A_n^J(a_i b_j c_k)$ if and only if $j < k$.

We only have to figure out when $s_1 s_2 \in A_n^J(a_i b_j c_k)$ if $0 \leq i < j \leq n$ and $2 \leq k \leq n$.

$$a_i b_j c_k s_1 s_2 = a_i b_j c_k s_2 = a_i a_j b_k = a_j b_{i+1} b_k$$

So if $i+1<k$, $a_ib_jc_ks_1s_2 = a_jb_kc_{i+2}$ and if $i+1 \leq k$, $a_ib_jc_ks_1s_2 = a_jb_{k-1}c_{i+1}$. This means that the length goes up if $i+1<k$ and stays constant if $i+1 \geq k$. So $s_1s_2 \in A_n^J(a_ib_jc_k)$ if and only if $i+1<k$.

Note that $i<j$ implies that $j<k$ and $i+1<k$. So it is not possible to have s_2 in an ascent set but not to have s_1s_2.

Case I: $0 \leq i < j < k \leq n$

$$A_n^J(a_ib_jc_k) = \{s_1s_2, s_2, c_{k+1}, \cdots, s_n\}.$$

Then $\nu_n^J(a_ib_jc_k) = n-k+2$ and for fixed $2 \leq k \leq n$

$$|\{a_ib_jc_k : \ 0 \leq i < j < k\}| = \binom{k}{2}$$

Case II: $2 \leq k \leq i+1 \leq j \leq n$

$$A_n^J(a_ib_jc_k) = \{c_{k+1}, \cdots s_n\}$$

Then $\nu_n^J(a_ib_jc_k) = n-k$ and for a fixed $2 \leq k \leq n$

$$|\{a_ib_jc_k : \ k \leq i+1 \leq n\}| = \frac{(n-k+2)(n-k+1)}{2}$$

Case III: $1 \leq i+1 < k \leq j \leq n$

$$A_n^J(a_ib_jc_k) = \{s_1s_2, c_{k+1}, \cdots, c_n\}$$

then $\nu_n^J(a_ib_jc_k) = n-k+1$ and for a fixed $2 \leq k \leq n$

$$|\{a_ib_jc_k : \ 1 \leq i+1 < k \leq j \leq n\}| = (k-1)(n-k+1)$$

To find $\sum_{w \in W^J} t^{2\nu_n^J(w)}$ we need only to find $c_r = |\{w \in W^J : \nu_n(w) = r\}|$ for each $0 \leq r \leq n$.

Let $r = 0$. then $\nu_n(a_i b_j c_k) = 0$ is only possible in Case II. In this case, $k = n$ and $k \leq i + 1 \leq j \leq n$ implies that $a_{n-1} b_n c_n$ is the only such element and $c_0 = 1$.

Let $r = n$. Then $\nu_n(a_i b_j c_k) = n$ is only possible in Case I. In this case, $k = 2$ and $0 \leq i < j < 2$ implies that $a_0 b_1 c_2$ is the only such element and $c_n = 1$.

Let $r = 1$. then $\nu_n(a_i b_j c_k) = 1$ is only possible in Case II and III. For Case II, $k = n - 1$ and for Case III, $k = n$. So $c_1 = \frac{2(3)}{2} + n - 1 = n + 2$.

Let $r = n - 1$. Then $\nu_n(a_i b_j c_k) = 1$ is only possible in Case I and III. For Case I, $k = 3$ and for Case III, $k = 2$. So $c_{n-1} = \frac{2(3)}{2} + (2 - 1)(n - 2 + 1) = n + 2$.

Let $2 \leq r \leq n - 2$. Then $\nu_n(a_i b_j c_k) = r$ is possible in all cases. For Case I, $n - k + 2 = r$ implies $k = n - r + 2$, for Case II, $n - k = r$ implies $k = n - r$ and for Case III, $n - k + 1 = r$ implies $k = n - r + 1$.

Therefore

$$c_r = \frac{(n - r + 2)(n - r + 1)}{2} + \frac{(r + 1)(r + 2)}{2} + r(n - r) = \frac{(n(n + 1)}{2} + n + 2.$$

and the Poincaré polynomial is given by

$$P(X(J), t) = t^{2n} + 1 + (n + 2)(t^{2(n-1)} + t^2) + (\frac{n(n + 1)}{2} + n + 2)(\sum_{k=2}^{n-2} t^{2k}).$$

Chapter 4

Betti numbers of X(J) in terms of Eulerian polynomials

4.1 Eulerian polynomials

In this section we introduce Eulerian polynomials and discuss the unifying ideas between the cross section lattice, Eulerian polynomials and h-polynomials.

The Eulerian polynomial of a finite Weyl group (W, S) of type A_n records, for each $k \in \{1, \cdots, n\}$, the number of elements $w \in S_{n+1}$ with an ascent set of size k. More results on this topic can be found in [16] and [3].

Let $\sigma \in S_{n+1}$, $\sigma(i) = p_i$, $i = 1, \cdots, n+1$. Define the ascent set of σ

$$A(\sigma) = \{i \mid 1 \leq i < n : \; p_i < p_{i+1}\}.$$

It turns out that

$$i \in A(\sigma) \iff l(\sigma s_i) = l(\sigma) + 1,$$

for $s_i = (i \; i+1) \in S_{n+1}$. We define Eulerian polynomials when (W, S) is finite Weyl group of type A_n as follows: Let

$$E(n,i) = |\{\sigma \in S_{n+1} \mid |A(\sigma) = i\}|$$

be the Eulerian numbers. Define the $(n+1)$–Eulerian polynomials to be:

$$E_{n+1}(t) = \sum_{i=0}^{n} E(n+1,i)t^i = \sum_{\sigma \in S_{n+1}} t^{|a(\sigma)|}, \text{ with } a(\sigma) = |A(\sigma)|.$$

An important property of the Eulerian polynomials is that these polynomials are palindromic polynomials, namely:

$$E(n,i) = E(n, n-1-i), \text{ for any } i = 0, \cdots, n-1.$$

It follows that the Eulerian numbers can be generated by the well-known recurrence:

$$E(m+1,n) = (n+1)E(m,n) + (m-n+1)E(m,n-1), \ m,n \geq 1$$

Remark 6. An alternative definition of the Eulerian polynomials is using the geometric series:

$$\sum_{k \geq 1} x^k = \frac{x}{1-x}$$

Differentiating repeatedly and multiplying by x, we obtain:

$$\sum_{k \geq 1} kx^k = \frac{x}{(1-x)^2}(1)$$

$$\sum_{k \geq 1} k^2x^k = \frac{x}{(1-x)^3}(1+x)$$

$$\sum_{k \geq 1} k^3x^k = \frac{x}{(1-x)^4}(1+4x+x^2)$$

$$\cdots$$

$$\sum_{k\geq 1} k^m x^k = \frac{x}{(1-x)^{m+1}} (\sum_{n=o}^{m-1} E(m,n)x^n)$$

It turn out that

$$E_m(t) = \sum_{n=0}^{m-1} E(m,n)t^n$$

is the *m–Eulerian polynomial.* One can see why the two definitions are the same by checking both polynomials satisfy the same recurrence relation.

We list the first few Eulerian polynomials:

m	$E_m(t)$
1	1
2	$1+t$
3	$1+4t+t^2$
4	$1+11t+11t^2+t^3$
5	$1+26t+66t^2+26t^3+t^4$
6	$1+57t+302t^2+302t^3+57t^4+t^5$
7	$1+120t+1191t^2+2416t^3+1191t^4+120t^5+t^6$
8	$1+247t+4293t^2+15619t^3+15619t^4+4293t^5+247t^6+t^7$
9	$1+502t+14608t^2+88234t^3+156190t^4+88234t^5+14608t^6+502t^7+t^8$

Definition 11. A permutahedron $\mathcal{P}_{n-1} \in \mathbb{R}^n$ is the convex hull in $\mathbb{R}^n$ of the set

$$\{(p_1, p_2, \cdots, p_n) \in \mathbb{R}^n \mid \sigma(i) = p_i, \sigma \in S_n\}.$$

It is a known fact that the Betti numbers of the toric variety $X(\emptyset)$ associated to a permutahedron are the Eulerian numbers. See Example 11.

For the next result we present a proof due to Renner.

Theorem 15. Let $E_n(t)$ be the n–th Eulerian polynomial, let $h_{n-1}(t)$ be the h-

polynomial of the permutahedron $\mathcal{P}_{n-1}$. Then

$$E_n(t) = h_{n-1}(t).$$

Proof. We first study the structure of the $(n-1)$–permutahedron:

1. The vertices of $\mathcal{P}_{n-1}$ correspond uniquely to the permutation group S_n.

2. The edges of $\mathcal{P}_{n-1}$ correspond uniquely to pairs (w, ws) with $w \in S_n$ and $s \in S$ such that $l(ws) = l(w) + 1$.

3. For any face $F \subset \mathcal{P}_{n-1}$ of the permutahedron there is a unique vertex $\sigma \in F$ such that all edges of F are ascent of σ. (We say F likes σ).

4. From 3, the set of faces $\mathcal{F}$ of $\mathcal{P}_{n-1}$ is identified with

$$\{(\sigma, I) \mid I \subset A(\sigma), \sigma \in S_n\}.$$

5. Given $\sigma \in S_n$ there is a largest face F that likes σ. Furthermore, $\dim(F) = |A(\sigma)|$.

6. The number of faces of dimension i that like σ is $\begin{pmatrix} k \\ i \end{pmatrix}$ where $k = |A(\sigma)|$. Indeed, each one correspond to a subset of $A(\sigma)$ with i elements.

By definition,

$$\begin{aligned} h_n(t) &= \sum_{i=0}^{n-1} f_i(t-1)^i. \\ &= \sum_{\sigma \in S_n} (\sum_{F \text{ likes } \sigma} (t-1)^{\dim F}. \\ &= \sum_{\sigma \in S_n} (1 + k(t-1) + \begin{pmatrix} k \\ 2 \end{pmatrix} (t-1)^2 + \cdots + (t-1)^n). \end{aligned}$$

$$\begin{aligned} &= \sum_{\sigma \in S_n} (1 + (t-1)^{a(\sigma)}. \\ &= \sum_{\sigma \in S_n} t^{a(\sigma)}. \\ &= E_{n+1}. \end{aligned}$$

□

As a consequence of Proposition 8 and Theorem 19 we obtain a characterization of the $(n+1)$–Eulerian polynomial in terms of the subsets $I \subseteq S$. Recall that S is the minimal set of reflections that generate the permutation group S_{n+1}, $S = \{s_1, \cdots s_n\}$ with $s_i = (i\ i+1) \in S_{n+1}$.

Theorem 16. Let E_{n+1} be the $(n+1)$-Eulerian polynomial. The following identity holds:

$$E_{n+1}(t) = \sum_{I \subseteq S} \frac{(n+1)!}{|W_I|} (t-1)^{|I|}.$$

Proof. Consider $(W, S) = (S_{n+1}, S)$ finite Weyl group of type A_n and let $J = \emptyset$. In this case, the highest weight λ is in the interior of the fundamental Weyl chamber. By applying reflections $s_i = (i,\ i+1) \in S_{n+1}$ about the hyperplanes orthogonal to the simple roots we permute i and $i+1$ coordinates of λ. The polytope P_λ given by the convex hull of the W-orbit of λ turns out to be an n permutahedron (see [21]). From Theorem 19 and Proposition 7 we have that

$$h(t) = E_{n+1} = \sum_{I \in S(J)} \frac{|W|}{|W_{I_J^*}|} (t-1)^{|I|}.$$

where $I_J^* = I$, $S(J) = \mathcal{P}(S)$ the power set of S, when $J = \emptyset$. □

4.2 New results

Our aim in this section is to obtain a recursive formula for the Betti numbers of the rationally smooth toric variety $X(J)$ in terms of the Eulerian numbers when (W, S) is finite Weyl group of type A_n and $J = \{s_{n-k+1}, \cdots, s_n\} \subseteq S$, for $1 \leq k \leq n-1$. In section 2.4 we computed the Poincaré polynomial of $X(J)$ using the method of descent system (W^J, S^J) in the case of $J = \{s_4, s_5, \cdots, s_n\}$, and (W, S) of type A_n.

In this section we consider the general case of $J(k, n) = \{s_{n-k+1}, \cdots, s_n\}$, $1 \leq k \leq n$ and instead of calculating the Poincaré polynomial of $X(J(k, n))$ using descent systems we use the combinatorics of simple polytopes together with the theory of algebraic monoids to obtain a recurrence in terms of the $(n-k)$–Eulerian polynomials.

Finally, the recurrence is illustrated for $k = n-2$. For the remainder of this section we specialize the discussion to the case only of type A_n, where $W = S_{n+1}$, $S = \{s_1, \cdots, s_n\}$, and $s_i = (i\ i+1) \in S_{n+1}$. We choose $J \subseteq S$ such that J is combinatorially smooth.

Theorem 17. Let $J = \{s_n\}$. Then J is combinatorially smooth of type A_n and the h-polynomial of $X(J)$ is given by

$$h(t) = E_{n+1}(t) - \begin{pmatrix} n+1 \\ 2 \end{pmatrix} tE_{n-1}(t).$$

Proof. From Corollary 3.5 [32] we obtain that $J = \{s_n\}$ is combinatorially smooth. Let M be a $\mathcal{J}$-irreducible monoid of type $J = \{s_n\}$. We associate to M the cross section lattice denoted by $\Lambda(1)$. From Theorem 4 we have that $\Lambda(1) \setminus \{0\} \cong S(J(1, n))$, where $S(J(1, n)) = \{I \subseteq S \mid \text{no connected component of } I \text{ is contained entirely in } J\}$. Notice that

$$S(J(1, n)) \subseteq \mathcal{P}(S)$$

where $\mathcal{P}(S)$ denotes the power set of S.

One of the main ideas in proving this result is to partition $S(J(1,n))$ into disjoint sets according to the following rule:

$$S(J(1,n)) = \{A \subseteq S \mid s_{n-1} \in A\} \sqcup \{A \subset S \mid s_{n-1} \notin A, s_n \notin A\}.$$

We observe for $A \in S(J(1,n))$ such that $s_{n-1} \notin A$ we must have $s_n \notin A$, since $\{s_n\}$ is a connected component of A contained entirely in J, and this contradicts the definition of $S(J(1,n))$. We are now in a position to introduce notation for the two disjoint subsets of $S(J(1,n))$. Let

$$M_0 = \{A \subseteq S \mid s_{n-1} \in A\} \subseteq \mathcal{P}(S),$$

and let

$$M_1 = \{A \subseteq S \mid s_{n-1} \notin A, s_n \notin A\} = \mathcal{P}(S \setminus \{s_{n-1}, s_n\}).$$

Then the h-polynomial of $X(J)$ is computed using Proposition 7:

$$h(t) = \sum_{A \in M_0} \frac{(n+1)!}{|W_{A^*}|}(t-1)^{|A|} + \sum_{A \in M_1} \frac{(n+1)!}{|W_{A^*}|}(t-1)^{|A|}.$$

Next, we determine $A^* = A \cup \{s \in J \mid st = ts \text{ for any } t \in A\}$ for any $A \in S(J(1,n)$.

For $A \in M_0$ we have $A^* = A$ and $W_{A^*} = W_A$. For $A \in M_1$ we have $A^* = A \cup \{s_n\}$ and W_{A^*} is the subgroup generated by $A \cup \{s_n\}$ namely, $W_{A^*} = W_A \times S_2$. It follows that the h-polynomial of $X(J)$ is given by:

$$\begin{aligned} h(t) &= \sum_{A \in M_0} \frac{(n+1)!}{|W_A|}(t-1)^{|A|} + \sum_{A \in M_1} \frac{(n+1)!}{|W_A \times S_2|}(t-1)^{|A|} \\ &= \sum_{A \in M_0} \frac{(n+1)!}{|W_A|}(t-1)^{|A|} + \frac{n(n+1)}{2} \sum_{A \in M_1} \frac{(n-1)!}{|W_A|}(t-1)^{|A|}. \end{aligned} \tag{4.1}$$

Using Theorem 16 we are able to express the $(n-1)$–Eulerian polynomial in terms of the elements of M_1. We obtain the following:

$$E_{n-1}(t) = \sum_{A \in M_1} \frac{(n-1)!}{|W_A|}(t-1)^{|A|}.$$

By 4.1, this implies that the h-polynomial of $X(J)$ equals:

$$h(t) = \sum_{A \in M_0} \frac{(n+1)!}{|W_A|}(t-1)^{|A|} + \frac{n(n+1)}{2} E_{n-1}(t). \tag{4.2}$$

Next, we let $N_0 = \{A \subseteq S \mid s_{n-1} \notin A\}$ and apply Theorem 16 to the $(n+1)$–Eulerian polynomial. The following identity is obtained:

$$\begin{aligned} E_{n+1}(t) &= \sum_{A \subseteq S} \frac{(n+1)!}{|W_A|}(t-1)^{|A|} \\ &= \sum_{A \in M_0} \frac{(n+1)!}{|W_A|}(t-1)^{|A|} + \sum_{A \in N_0} \frac{(n+1)!}{|W_A|}(t-1)^{|A|}. \end{aligned} \tag{4.3}$$

Next we expand the set N_0 in the following way:

$$N_0 = \{A \subseteq S \mid s_{n-1} \notin A, s_n \notin A\} \sqcup \{A \subset S \mid s_{n-1} \notin A, s_n \in A\} \subseteq \mathcal{P}(S)$$

Let $N_1 = \{A \subseteq S \mid s_{n-1} \notin A, s_n \in A\} = \{A' \cup \{s_n\} \mid A' \subseteq \{s_1, s_2, \cdots, s_{n-2}\}\}$, such that $N_0 = M_1 \sqcup N_1$. Direct computation shows that the subgroup generated by any $A \in N_1$ can be expressed as follows: for $A \in N_1$ then there exists $A' \in M_1$ such that $A = A' \cup \{s_n\}$ and $W_A = W_{A'} \times S_2$. Therefore we evaluate the following summand of $E_{n+1}(t)$ as follows:

$$\begin{aligned} \sum_{A \in N_0} \frac{(n+1)!}{|W_A|}(t-1)^{|A|} &= \sum_{A \in M_1} \frac{(n+1)!}{|W_A|}(t-1)^{|A|} + \sum_{A \in N_1} \frac{(n+1)!}{|W_A|}(t-1)^{|A|}. \\ &= \sum_{A \in M_1} \frac{(n+1)!}{|W_A|}(t-1)^{|A|} + \sum_{A' \in M_1} \frac{(n+1)!}{|W_{A'}||S_2|}(t-1)^{|A'|+1}. \end{aligned}$$

$$\begin{aligned}
&= \sum_{A \in M_1} \frac{(n+1)!}{|W_A|}(t-1)^{|A|} \\
&\quad + \frac{n(n+1)}{2}(t-1) \sum_{A' \in M_1} \frac{(n-1)!}{|W_{A'}|}(t-1)^{|A'|}. \\
&= n(n+1)E_{n-1}(t) + \frac{n(n+1)}{2}(t-1)E_{n-1}(t). \\
&= \frac{n(n+1)}{2}(t+1)E_{n-1}(t).
\end{aligned} \tag{4.4}$$

From 4.3 and 4.4 we obtain the first summand of the h-polynomial:

$$\sum_{A \in M_0} \frac{(n+1)!}{|W_A|}(t-1)^{|A|} = E_{n+1}(t) - \frac{n(n+1)}{2}(t+1)E_{n-1}(t). \tag{4.5}$$

This by 4.2 and 4.5 implies the following relation holds:

$$\begin{aligned}
h(t) &= E_{n+1}(t) - \frac{n(n+1)}{2}(t+1)E_{n-1}(t) + \frac{n(n+1)}{2}E_{n-1}(t) \\
&= E_{n+1}(t) - \binom{n+1}{2} tE_{n-1}(t).
\end{aligned}$$

$\square$

Corollary 3. The Poincaré polynomial of $X(J)$ with $J = \{s_n\}$ is given by:

$$P(t) = E_{n+1}(t^2) - \binom{n+1}{2} t^2 E_{n-1}(t^2). \tag{4.6}$$

Proof. We use the relation between the h-polynomial and the Poincaré polynomial recorded in [41], namely $h(t^2) = P(t)$. $\square$

Theorem 18. Let $J(2,n) = \{s_{n-1}, s_n\}$ and $J(1,n) = \{s_n\}$. Denote the h-polynomial of $X(J(2,n))$ by $h_2(t)$ and of $X(J(1,n))$ by $h_1(t)$. Then the following recurrence relation holds:

$$h_2(t) = h_1(t) - \binom{n+1}{3}(t^2+t)E_{n-2}(t).$$

Proof. Let M be a $\mathcal{J}$-irreducible monoid of type $J(2,n)$. We associate to M the cross section lattice denoted by $\Lambda(2)$. From Theorem 4 we have $\Lambda(2) \setminus \{0\} \cong S(J(2,n))$, where $S(J(2,n)) = \{I \subseteq S \mid \text{no connected component of } I \text{ is contained entirely in } J\}$. We obtain a good handle on the set $S(J(2,n))$ by partitioning it into disjoint sets according to the following rule:

$$\begin{aligned} S(J(2,n)) \quad = \quad & \{A \subseteq S \mid s_{n-2} \in A, s_{n-1} \in A, s_n \in A\} \sqcup \\ & \{A \subseteq S \mid s_{n-2} \in A, s_{n-1} \in A, s_n \notin A\} \sqcup \\ & \{A \subseteq S \mid s_{n-2} \in A, s_{n-1} \notin A, s_n \notin A\} \sqcup \\ & \{A \subseteq S \mid s_{n-2} \notin A, s_{n-1} \notin A, s_n \notin A\}. \end{aligned}$$

We are now in a position to introduce notation for the disjoint subsets of $S(J(2,n))$. Let

$$M_0 = \{A \subseteq S \mid s_{n-2} \in A, s_{n-1} \in A, s_n \in A\} \subseteq \mathcal{P}(S).$$

$$M_1 = \{A \subseteq S \mid s_{n-2} \in A, s_{n-1} \in A, s_n \notin A\} \subseteq \mathcal{P}(S).$$

$$M_2 = \{A \subseteq S \mid s_{n-2} \in A, s_{n-1} \notin A, s_n \notin A\} \subseteq \mathcal{P}(S).$$

$$M_3 = \{A \subseteq S \mid s_{n-2} \notin A, s_{n-1} \notin A, s_n \notin A\} \subseteq \mathcal{P}(S).$$

Then we have

$$S(J(2,n)) = \bigsqcup_{i=0}^{3} M_i.$$

Using Corollary 8 the h-polynomial of $X(J(2,n))$ is given by:

$$h_2(t) = \sum_{i=0}^{3} \sum_{A \in M_i} \frac{(n+1)!}{|W_{A^*}|}(t-1)^{|A|}.$$

Next, we compute for each $A \in S(J(2,n))$, the corresponding A^* defined by

$$A^* = A \cup \{s \in J(2,n) \mid st = ts \text{ for any t} \in A\}.$$

Let $A \in M_0 \cup M_1$ then we have $A^* = A$ and $W_{A^*} = W_A$.

Let $A \in M_2$ then we have $A^* = A \cup \{s_n\}$ and $W_{A^*} = W_A \times S_2$.

Let $A \in M_3$ then we have $A^* = A \cup \{s_{n-1}, s_n\}$ and $W_{A^*} = W_A \times S_3$.

Thus the h-polynomial of $X(J(2,n))$ is given by:

$$h_2(t) = \sum_{i=0}^{3} \sum_{A \in M_i} \frac{(n+1)!}{i! \times |W_A|}(t-1)^{|A|}. \tag{4.7}$$

Consider $J(1,n) = \{s_n\}$ then $\Lambda(1) \setminus \{0\} \cong S(J(1,n))$. We partition $S(J(1,n))$ into disjoint sets whose elements may contain s_{n-1} or not contain s_{n-1}, and obtain:

$$\begin{aligned} S(J(1,n)) &= \{A \subseteq S \mid s_{n-1} \in A, s_n \in A\} \sqcup \{A \subseteq S \mid s_{n-1} \in A, s_n \notin A\} \sqcup \\ &\quad \{A \subseteq S \mid s_{n-1} \notin A, s_n \notin A\}. \end{aligned}$$

An important step towards proving our result is finding a relation between $S(J(1,n))$ and $S(J(2,n))$. In order to achieve this we use a rather elementary idea which turns out to be extremely efficient for our computations. We expand $S(J(1,n))$ in the following way:

$$\begin{aligned} S(J(1,n)) &= \{A \subseteq S \mid s_{n-2} \in A, s_{n-1} \in A, s_n \in A\} \sqcup \\ &\quad \{A \subseteq S \mid s_{n-2} \notin A, s_{n-1} \in A, s_n \in A\} \sqcup \\ &\quad \{A \subseteq S \mid s_{n-2} \in A, s_{n-1} \in A, s_n \notin A\} \sqcup \\ &\quad \{A \subseteq S \mid s_{n-2} \notin A, s_{n-1} \in A, s_n \notin A\} \sqcup \\ &\quad \{A \subseteq S \mid s_{n-2} \notin A, s_{n-1} \notin A, s_n \notin A\} \sqcup \\ &\quad \{A \subseteq S \mid s_{n-2} \in A, s_{n-1} \notin A, s_n \notin A\}. \end{aligned}$$

We introduce the following notations:

$$N_0 = \{A \subseteq S \mid s_{n-2} \notin A, s_{n-1} \in A, s_n \in A\} \subseteq \mathcal{P}(S).$$

$$N_1 = \{A \subseteq S \mid s_{n-2} \notin A, s_{n-1} \in A, s_n \notin A\} \subseteq \mathcal{P}(S).$$

Hence the desired relation is:

$$S(J(1,n)) = S(J(2,n)) \sqcup N_0 \sqcup N_1. \tag{4.8}$$

Next, we compute $A^* = A \cup \{s \in J(1,n) \mid st = ts \text{ for any t} \in A\}$ for $A \in M_i$, where $i = 0,1,2,3$ and $A \in N_i : i = 0,1$.

For $A \in M_0 \cup M_1 : A^* = A$ and $W_{A^*} = W_A$. For $A \in M_2 \cup M_3 : A^* = A \cup \{s_n\}$ and $W_{A^*} = W_A \times S_2$. For $A \in N_i : A^* = A$ where $i = 0,1$, and $W_{A^*} = W_A$.

Furthermore the h-polynomial of $X(J(1,n))$ is given by:

$$\begin{aligned} h_1(t) &= \sum_{i=0}^{2} \sum_{A \in M_i} \frac{(n+1)!}{i! \times |W_A|}(t-1)^{|A|} + \sum_{i=0}^{1} \sum_{A \in N_i} \frac{(n+1)!}{|W_A|}(t-1)^{|A|} \\ &\quad + \sum_{A \in M_3} \frac{(n+1)!}{2!|W_A|}(t-1)^{|A|}. \end{aligned} \tag{4.9}$$

By 4.7 and 4.9, this implies:

$$h_1(t) - h_2(t) = \sum_{i=0}^{1} \sum_{A \in N_i} \frac{(n+1)!}{|W_A|}(t-1)^{|A|} + (\frac{1}{2!} - \frac{1}{3!}) \sum_{A \in M_3} \frac{(n+1)!}{|W_A|}(t-1)^{|A|}. \tag{4.10}$$

In order to compute the subgroups generated by $A \in N_0 \cup N_1$, we observe that the following relations hold:

$$\begin{aligned} N_0 &= \{A \subseteq S \mid s_{n-2} \notin A, s_{n-1} \in A, s_n \in A\} \\ &= \{A' \cup \{s_{n-1}, s_n\} \mid A' \subseteq \{s_1, s_2, \cdots, s_{n-3}\}\}. \end{aligned}$$

Thus for $A \in N_0$ there exists $A' \subseteq \{s_1, s_2, \cdots s_{n-3}\}$ such that $A = A' \cup \{s_{n-1}, s_n\}$ and

$$W_A = W_{A'} \times S_3. \tag{4.11}$$

Similarly, we have:

$$\begin{aligned} N_1 &= \{A \subseteq S \mid s_{n-2} \notin A, s_{n-1} \in A, s_n \notin A\} \\ &= \{A' \cup \{s_{n-1}\} \mid A' \subset \{s_1, s_2, \cdots, s_{n-3}\}\} \end{aligned}.$$

Thus for $A \in N_1$ there exists $A' \subseteq \{s_1, s_2, \cdots s_n\}$ such that $A = A' \cup \{s_n\}$ and

$$W_A = W_{A'} \times S_2. \tag{4.12}$$

From 4.10, 4.11, 4.12 we obtain the following relation:

$$\begin{aligned} h_1(t) - h_2(t) &= \sum_{A' \in M_3} \frac{(n+1)!}{3!|W_{A'}|}(t-1)^{|A'|+2} + \sum_{A' \in M_3} \frac{(n+1)!}{2!|W_{A'}|}(t-1)^{|A'|+1} \\ &\quad + \sum_{A \in M_3} (\frac{1}{2!} - \frac{1}{3!}) \frac{(n+1)!}{|W_A|}(t-1)^{|A|} \\ &= \frac{(n-1)n(n+1)}{3!}(t-1)^2 \sum_{A' \in M_3} \frac{(n-2)!}{|W_{A'}|}(t-1)^{|A'|} \end{aligned}$$

$$+\frac{(n-1)n(n+1)}{2!}(t-1)\sum_{A'\in M_3}\frac{(n-2)!}{|W_{A'}|}(t-1)^{|A'|}$$
$$+\frac{(n-1)n(n+1)}{3}\sum_{A\in M_3}\frac{(n+1)!}{|W_A|}(t-1)^{|A|}.$$

By definition $M_3 = \mathcal{P}(S \setminus \{s_n, s_{n-1}, s_{n-2}\})$ and using Theorem 16 we are able to express the $(n-2)$–Eulerian polynomial in terms of the elements of M_3. We obtain the following identity:

$$E_{n-2}(t) = \sum_{A\in M_3}\frac{(n-2)!}{|W_A|}(t-1)^{|A|}. \tag{4.13}$$

By 4.13, this implies the following recurrence relation:

$$\begin{aligned} h_1(t) - h_2(t) &= \binom{n+1}{3}(t-1)^2E_{n-2}(t) + \frac{(n-1)n(n+1)}{2!}(t-1)E_{n-2}(t) \\ &\quad + \frac{(n-1)n(n+1)}{3}E_{n-2}(t). \\ &= \binom{n+1}{3}(t^2+t)E_{n-2}(t). \end{aligned}$$

This concludes our proof and obtain the following recurrence formula:

$$h_2(t) = h_1(t) - \binom{n+1}{3}(t^2+t)E_{n-2}(t).$$

□

Corollary 4. Let $P_2(t)$ be the Poincaré polynomial of $X(J(2,n))$. Then the following formula holds:

$$P_2(t) = E_{n+1}(t^2) - \binom{n+1}{2}t^2E_{n-1}(t^2) - \binom{n+1}{3}(t^4+t^2)E_{n-2}(t^2). \tag{4.14}$$

Proof. From Theorem 18 we have the relation

$$P_2(t) = P_1(t) - \binom{n+1}{3}(t^4 + t^2)E_{n-2}(t^2),$$

where $P_1(t)$ denotes the Poincaré polynomial of $X(J(1,n))$. Using Corollary 3 we obtained the desired relation. $\square$

Next, we generalize the computation to the case of $J(k,n) = \{s_{n-k+1}, \cdots, s_n\}$, for $1 \leq k \leq n$. The main result of this section is a recurrence relation for the Poincaré polynomial of $X(J(k,n))$ in terms of the $(n-k)$– Eulerian polynomials.

Theorem 19. [14] Let $J(k,n) = \{s_{n-k+1}, s_{n-k+2}, \cdots, s_n\} \subseteq S$, $1 \leq k \leq n$ and let $h_k(t)$ denote the h-polynomial of the n–dimensional variety $X(J(k,n))$. Then $J(k,n)$ is combinatorially smooth and the following recurrence relation holds:

$$h_k(t) = h_{k-1}(t) - \binom{n+1}{k+1}(t^k + t^{k-1} + \cdots + t)E_{n-k}(t).$$

where $J(0,n) = \emptyset$ and $h_0 = E_{n+1}$ the $(n+1)$–Eulerian polynomial.

Proof. From Corollary 3.5 [32] we obtain that $J(k,n)$ is combinatorially smooth. Let M be a $\mathcal{J}$-irreducible monoid of type $J(k,n)$ and let $\Lambda(k)$ be the cross section lattice associated to M. From Theorem 4 we have that $\Lambda(k)\backslash\{0\} \cong S(J(k,n))$, where $S(J(k,n)) = \{I \subseteq S \mid \text{no connected component of } I \text{ is contained entirely in } J(k,n)\}$.

Next, consider for $0 \leq i \leq k+1$,

$$M_i = \{A \subseteq S \mid J(k+1,n) \setminus J(i,n) \subseteq A \subseteq S \setminus J(i,n)\} \subseteq \mathcal{P}(S).$$

So, in particular, $M_0 = \{A \subseteq S \mid J(k+1,n) \subseteq A\} \subseteq \mathcal{P}(S)$, and

$M_{k+1} = \{A \subseteq S \mid A \subseteq S \setminus J(k+1,n)\} \subseteq \mathcal{P}(S)$. Hence, we obtain

$$S(J(k,n)) = \bigsqcup_{i=0}^{k+1} M_i,\ 0 \leq i \leq k+1.$$

We associate to each $A \in S(J(k,n))$, $A_k^* = A \cup \{s \in J(k,n) \mid st = ts \text{ for any } t \in A\}$.

We compute the h-polynomial of $X(J(k,n)$ using Proposition 7 and obtain:

$$h_k(t) = \sum_{i=0}^{k+1} \sum_{A \in M_i} \frac{(n+1)!}{|W_{A_k^*}|}(t-1)^{|A|}. \tag{4.15}$$

Then for $A \in M_0 \cup M_1$ we have $A_k^* = A$ and $W_{A_k^*} = W_A$. For $A \in M_i$, $2 \leq i \leq k+1$ we have $A_k^* = A \cup J(i-1)$ and $W_{A_k^*} = W_A \times S_i$. Thus, the h-polynomial of $X(J(k,n))$ is given by:

$$h_k(t) = \sum_{i=0}^{k+1} \sum_{A \in M_i} \frac{(n+1)!}{i! \times |W_A|}(t-1)^{|A|}. \tag{4.16}$$

Consider $J(k-1,n) = \{s_{n-k+2}, \cdots, s_n\} \subset J(k,n)$. From Theorem 4 we obtain that $\Lambda(k-1) \setminus \{0\} \cong S(J(k-1,n))$. Let

$$S_i = \{A \subseteq S \mid J(k,n) \setminus J(i,n) \subseteq A \subseteq S \setminus J(i,n)\} \subseteq \mathcal{P}(S)$$

for $0 \leq i \leq k$. So, in particular, $S_0 = \{A \subseteq S \mid J(k,n) \subseteq A\} \subseteq \mathcal{P}(S)$ and $S_k = \{A \subset S \mid A \subseteq S \setminus J(k,n)\} \subseteq \mathcal{P}(S)$. Note that for each $0 \leq i \leq k$, we have

$$S_i \cap \{A \subseteq S \mid s_{n-k} \in A\} = M_i \text{ and}$$

$$S_i \cap \{A \subseteq S \mid s_{n-k} \notin A\} = \{A \subseteq S \mid J(k,n) \setminus J(i,n) \subseteq A \subseteq (S \setminus (J(i,n)) \cup \{s_{n-k}\}).$$

Let

$$\begin{aligned} N_i &= \{A \subseteq S \mid J(k,n) \setminus J(i,n) \subseteq A \subseteq S \setminus (J(i,n) \cup \{s_{n-k}\})\} \\ &= \{A' \cup (J(k,n) \setminus J(i,n)) \ |A' \subseteq S \setminus J(k+1,n)\}. \end{aligned}$$

Notice also that $N_k = M_{k+1}$. Then the following relation holds:

$$S(J(k-1,n)) = S(J(k,n)) \sqcup \bigsqcup_{i=0}^{k-1} N_i.$$

Next we compute $A^*_{k-1} = A^*_{J(k-1,n)}$ for all $A \in S(J(k-1,n)$. For $A \in M_0$ and $A \in M_1$, we have

$$A^*_{k-1} = A \text{ and } W_{A^*_{k-1}} = W_A.$$

For $A \in M_i : i = 2, \cdots, k$, we have

$$A^*_{k-1} = A \cup J(k-1,n) \text{ and } W_{A^*} = W_A \times S_i.$$

We know that $M_{k+1} = \mathcal{P}(S \setminus J(k+1,n))$. Hence for $A \in M_{k+1}$, we have

$$A^*_{k-1} = A \cup J(k-1,n) \text{ and } W_{A^*} = W_A \times S_k.$$

For $A \in N_0$ and $A \in N_1$, we have $A^*_{k-1} = A$ and $W_{A^*_{k-1}} = W_A$. For $A \in N_i$ where $i = 1, \cdots, k-1$, we have $A^*_{k-1} = A \cup J(i-1,n)$ and $W_{A^*_{k-1}} = W_A \times S_i$. Furthermore, the h-polynomial of $X(J(k-1,n))$ is given by:

$$\begin{aligned} h_{k-1}(t) \quad = \quad & \sum_{i=0}^{k} \sum_{A \in M_i} \frac{(n+1)!}{i! \times |W_A|}(t-1)^{|A|} + \sum_{i=0}^{k-1} \sum_{A \in N_i} \frac{(n+1)!}{|W_A| \times i!}(t-1)^{|A|} \\ & + \sum_{A \in M_{k+1}} \frac{|W|}{k! \times |W_A|}(t-1)^{|A|}. \end{aligned} \tag{4.17}$$

By 4.17 and 4.18, this implies that

$$h_{k-1}(t) - h_k(t) = \sum_{i=0}^{k-1} \sum_{A \in N_i} \frac{(n+1)!}{i! \times |W_A|}(t-1)^{|A|} + (\frac{1}{k!} - \frac{1}{(k+1)!}) \sum_{A \in M_{k+1}} \frac{(n+1)!}{|W_A|}. \tag{4.18}$$

The following relations hold: for $A \in N_i$ there exists $A' \in M_{k+1}$ such that

$$A = A' \cup (J(k,n) \setminus J(i,n)) \text{ and } W_A = W_{A'} \times S_{k-i+1}.$$

We use these relations in 4.19 and obtain the following:

$$\begin{aligned} h_{k-1}(t) - h_k(t) &= \sum_{i=0}^{k-1} \sum_{A' \in M_{k+1}} \frac{(n+1)!}{i!(k-i+1)!|W_{A'}|}(t-1)^{|A'|+k-i} \\ &\quad + \frac{(n+1)!}{(n-k)!}(\frac{1}{k!} - \frac{1}{(k+1)!}) \sum_{A \in M_{k+1}} \frac{(n-k)!}{|W_A|}(t-1)^{|A|} \end{aligned}$$

$$\begin{aligned} = \sum_{i=0}^{k-1} \frac{(n+1)!}{(n-k)!(k-i+1)!i!}(t-1)^{k-i} \sum_{A' \in M_{k+1}} \frac{(n-k)!}{|W_{A'}|}(t-1)^{|A'|} \\ + \frac{(n+1)!}{(n-k)!(k+1)!} k \sum_{A \in M_{k+1}} \frac{(n-k)!}{|W_A|}(t-1)^{A|}. \end{aligned}$$

Theorem 16 allows us to express the $(n-k)$-Eulerian polynomial in terms of the elements of M_{k+1}:

$$E_{n-k}(t) = \sum_{A \in M_{k+1}} \frac{(n-k)!}{|W_A|}(t-1)^{|A|}.$$

In the next formula we replace $(k-i+1)! \times i!$ by $\frac{1}{(k+1)!} \begin{pmatrix} k+1 \\ i \end{pmatrix}$ and obtain:

$$\begin{aligned} h_k(t) - h_{k-1}(t) &= \\ &\sum_{i=0}^{k-1} \frac{(n+1)!}{(n-k)!(k-i+1)! \times i!}(t-1)^{k-i} E_{n-k}(t) \\ &+ \frac{(n+1)!}{(n-k)!(k+1)!} k E_{n-k}(t) \end{aligned} \tag{4.19}$$

$$
\begin{aligned}
&= \left[\sum_{i=0}^{k-1} \frac{(n+1)!}{(n-k)!(k+1)!} \binom{k+1}{i} (t-1)^{k-i} + \right. \\
&\qquad \left. \frac{(n+1)!}{(n-k)!(k+1)!} k \right] E_{n-k}(t) \qquad (4.20) \\
&= \binom{n+1}{k+1} \left[\sum_{i=0}^{k-1} \binom{k+1}{i} (t-1)^{k-i} + k\right] E_{n-k}(t).
\end{aligned}
$$

We need now to show that

$$\sum_{i=0}^{k-1} \left(\binom{k+1}{i} (t-1)^{k-i} + k \right) = \sum_{i=1}^{k} t^i. \tag{4.21}$$

Let $f(t) = \sum_{i=0}^{k-1} \binom{k+1}{i} (t-1)^{k-i}$. Observe that

$$\sum_{i=0}^{k+1} \binom{k+1}{i} (t-1)^{k+1-i} = (t-1)f(t) + \binom{k+1}{k} (t-1) + \binom{k+1}{k+1} (t-1)^0$$

By the binomial theorem we have

$$\sum_{i=0}^{k+1} \binom{k+1}{i} (t-1)^{k+1-i} = t^{k+1} = ((t-1)+1)^{k+1}.$$

So

$$f(t) + k = \frac{t^{k+1} - (k+1)(t-1) - 1}{t-1} + k = \frac{t^{k+1} - t}{t-1} = \sum_{i=1}^{k} t^i.$$

The theorem now follows from 4.20 and 4.21. $\square$

Corollary 5. Let $J(k,n) = \{s_{n-k+1}, \cdots, s_n\} \subseteq S$, for $1 \leq k \leq n$. Then the Poincaré polynomial of $X(J(k,n))$ is given by the formula:

$$P(X(J(k,n),t) = E_{n+1}(t^2) - \sum_{i=1}^{k} \begin{pmatrix} n+1 \\ i+1 \end{pmatrix} (t^{2i} + \cdots + t^2)E_{n-i}(t^2)$$

Proof. We obtain from Theorem 17 that

$$h_k(t) = h_0 + \sum_{i=1}^{k} (h_i(t) - h_{i-1}(t)).$$

Hence

$$h_k(t) = E_{n+1}(t) - \sum_{i=1}^{k} \begin{pmatrix} n+1 \\ i+1 \end{pmatrix} (t^{2i} + \cdots + t^2)E_{n-i}(t).$$

Using the fact that $P(X(J),t) = h(t^2)$ we obtain the desired formula. □

We can exemplify the previous corollary with the following example.

Example 17. Let (W,S) Weyl group of type A_3 and let $J = \{s_2, s_3\}$. We have shown in Chapter 2 that $X(J) = \mathbb{CP}^3$.

According to previous corollary we have the following formula for the Poincaré polynomial of $\mathbb{CP}^3$:

$$P(\mathbb{CP}^3, t) = E_4(t^2) - \begin{pmatrix} 4 \\ 2 \end{pmatrix} t^2 E_2(t^2) - \begin{pmatrix} 4 \\ 3 \end{pmatrix} (t^4 + t^2)E_1(t^2)$$

$$= 1 + 11t^2 + 11t^4 + t^6 - 6t^2 - 6t^4 - 4t^4 - 4t^2 = t^6 + t^4 + t^2 + 1.$$

Next, we verify the recurrence formula obtained in Theorem 19 in some particular cases of k.

Example 18. Let $k = n-1$. Then $J(n-1,n) = \{s_2, s_3, \cdots, s_n\}$ is combinatorially smooth and the h-polynomial of $X(J(n-1,n))$ is computed in Example 4.3 [32]. It is given by the formula:

$$h_{n-1}(t) = 1 + t + t^2 + \cdots + t^n.$$

The recurrence formula obtained in Theorem 19 is equivalent to

$$h_{n-1} = h_{n-2}(t) - \begin{pmatrix} n+1 \\ n \end{pmatrix} (t^{n-1} + \cdots + t)E_1(t).$$

The h-polynomial of $X(J(n-2,n))$ is computed in Example 4.6 [33] and is given by the following formula:

$$h_{n-2}(t) = 1 + (n+2)t + (n+2)t^2 + .. + (n+2)t^{n-1} + t^n.$$

Hence

$$h_{n-2}(t) - h_{n-1}(t) = n(t^{n-1} + \cdots + t)$$

yields the desired relation.

Example 19. Let $k = n-3$. then $J(n-3,n) = \{s_4, \cdots, s_n\}$ is combinatorially smooth and the h polynomial of $X(J(n-3,n)$ is computed in Theorem 14. It is given by the following formula:

$$\begin{aligned} h_{n-3} &= 1 + (n+2+\frac{n(n+1)}{2})t + (n+2+n(n+1))t^2 + \cdots \\ &+ (n+2+n(n+1))t^{n-2} + (n+2+\frac{n(n+1)}{2})t^{n-1} + t^n. \end{aligned}$$

The h-polynomial of $X(J(n-2,n)$ is computed in Example 4.6 [33]. It is given by the formula:

$$h_{n-2}(t) = 1 + (n+2)t + (n+2)t^2 + .. + (n+2)t^{n-1} + t^n.$$

The recurrence formula obtained in Theorem 19 is equivalent to:

$$\begin{aligned}
h_{n-2}(t) &= h_{n-3}(t) - \binom{n+1}{n-1}(t^{n-2} + t^{n-3} + \cdots + t)E_2(t) \\
&= h_{n-3}(t) - \frac{n(n+1)}{2}(t^{n-2} + \cdots + t)(t+1) \\
&= h_{n-3}(t) - \frac{n(n+1)}{2}(t^{n-1} + 2t^{n-2} + 2t^{n-3} + \cdots + 2t^2 + t) \\
&= 1 + (n+2)t + (n+2)t^2 + \cdots + (n+2)t^{n-1} + t^n.
\end{aligned}$$

Next let (W, S) be the Weyl group of type A_{m-1}. Recall the definition of the Eulerian polynomials in terms of a power series given in Remark 5. Let

$$S_m(x) = \sum_{k \geq 1} k^m x^k = \frac{x}{(1-x)^{m+1}}(\sum_{n=0}^{m-1} E(m,n)x^n) = \frac{x}{(1-x)^{m+1}} E_m(x).$$

where $E_m(x)$ is the h-polynomial of the case $J = \emptyset$ and $|S| = m-1$.

Corollary 6. Then

$$S_m(x) - \sum_{i=1}^{k} \binom{m}{i+1} S_{m-i-1}(x)S_i(x) = \frac{x}{(1-x)^{m+1}} h_k(t)$$

where $h_k(x)$ is the h-polynomial of $X(J(k,n))$.

Proof. Let

$$h_1(x) = E_m(x) - \frac{m(m-1)}{2} x E_{m-2}(x)$$

be the h-polynomial for the case $J = \{s_{m-1}\}$ and $|S| = m-1$. Consider

$$B(x) = \frac{x}{(1-x)^{m+1}} h_1(x).$$

Then, we obtain

$$B(x) = S_m(x) - \frac{m(m-1)}{2} S_1(x) S_{m-2}(x).$$

Next, when $J = \{s_{m-2}, s_{m-1}, s_m\}$ and $|S| = m-1$ we get

$$h_2(x) = E_m(x) - \frac{(m-1)m}{2} x E_{m-2} - \frac{(m-2)(m-1)m}{3!} (x^2 + x) E_{m-3}(x)$$

and

$$B(x) = \frac{x}{(1-x)^{m+1}} h_2(x) =$$

$$= S_m(x) - \frac{m(m-1)}{2!} S_{m-2}(x) S_1(x) - \frac{(m-2)(m-1)m}{3!} S_{m-3}(x) S_2(x).$$

In general when $J = \{s_{m-k}, \cdots s_{m-1}\}$ and $|S| = m-1$ using the recurrence relation between the $h_k(x)$ and $h_{k-1}(x)$ polynomials we obtain the following formula:

$$B(x) = \frac{x}{(1-x)^{m+1}} h_k(x) = S_m(x) - \sum_{i=1}^{k} \begin{pmatrix} m \\ i+1 \end{pmatrix} S_{m-i-1}(x) S_i(x)$$

$\square$

Chapter 5

Betti Numbers of an Irreducible Representation: an overview

The goal of this chapter is to explain to the reader how the h–polynomial of $X(J)$ is needed to calculate the H–polynomial of certain embeddings of a semisimple group arising from irreducible representations. It is meant to be a survey based on several papers by Renner. We don't include proofs for the results we discuss here but rather try to unify ideas used in previous chapters in order to create a more general context where lots of interesting questions can be asked. For technical results I refer to [31], [36], [37].

Renner introduces the H–polynomial (Definition 12 bellow) of a reductive monoid M in terms of the set $B \times B$–orbits of M. Let M be a reductive monoid. We consider the $B \times B$–action on M as follows: $B \times B \times M \to M$, $(g, h, x) \to gxh^{-1}$. It turns out that there are a finite number of $B \times B$–orbits.

Any reductive group G has a Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwB,$$

where $W = N_G(T)/T$ is the Weyl group. By the results of [27] there is a perfect analogue for reductive monoids. Instead of W we use the *Renner monoid*

$$\mathcal{R} = \overline{N_G(T)}/T,$$

where $\overline{N_G(T)} \subseteq M$ is the Zariski closure of $N_G(T)$ in M. Since $xT = Tx$ for each $x \in \overline{N_G(T)}$, $\mathcal{R}$ is a monoid and for reductive monoids the corresponding Bruhat decomposition into $B \times B$–orbits is controlled by the Renner monoid:

$$M = \bigsqcup_{r \in \mathcal{R}} BrB.$$

The Renner monoid can be written as a disjoint union of $W \times W$–orbits, parametrized by the cross section lattice:

$$\mathcal{R} = \bigsqcup_{e \in \Lambda} WeW.$$

A reductive monoid M is called *semisimple* if it has a zero element and its unit group G has a one-dimensional center.

Definition 12. Let M be a semisimple monoid with monoid $\mathcal{R}$ of $B \times B$–orbits. Define $H(\mathcal{R})$, the *H–polynomial* of $\mathcal{R}$, as follows:

$$H(\mathcal{R}) = \sum_{x \in \mathcal{R}} (t-1)^{r(x)} t^{l(x)-r(x)}$$

where $r(x) = \dim(Tx)$ is the rank of x and $l(x) = \dim(BxB)$ is its length. We then let

$$H(M) = (t-1)^{-1}(H(\mathcal{R}) - 1).$$

$H(M)$ is called the H–polynomial of M.

The H–polynomial is a synthesis of the h–polynomial of a toric variety [39] and the Poincaré polynomial of a Schubert variety [1]. In the former case one collects summands of the form $(t-1)^a$ while in the later case one collects summands of the form t^b.

When M is semisimple we consider the induced action of $G \times G$ on the projective variety

$$\mathbb{P}(M) = [M \setminus \{0\}]/k^*$$

Remark 7. This H–polynomial is not the correct tool for investigating varieties with singularities that are not rationally smooth. In the case of Schubert varieties, and Kazhdan-Lusztig theory, the correct formulation incorporates a correction factor (via the KL-polynomial) that takes into account local intersection cohomology groups. In case the singularities of $\mathbb{P}(M)$ are rationally smooth, the polynomial $IP_X(t)$ of [5] agrees with the polynomial $H(M)$. However, in the absence of rationally smooth singularities, these local intersection cohomology groups may not be so well adapted to cellular decompositions.

5.1 Betti numbers and cellular decomposition

Clearly one can define a Poincaré polynomial for any reasonable cohomology theory. In [5] the authors compute the *intersection cohomology* polynomial Poincaré polynomial for a large class of $G \times G$–embeddings X of G. However it is known [13] that $IP_X(t) = P_X(t)$ in case X has rationally smooth singularities.

Theorem 20. [36] Let M be a semisimple algebraic monoid such that $M \setminus \{0\}$ is rationally smooth. Then

$$H(M)(t^2) = P_X(t)$$

where $X = [M \setminus \{0\}]/k^*$.

Next we describe how $\mathcal{R}$ decomposes naturally into a disjoint union of cells $\mathcal{C}_r$, $r \in \mathcal{R}_1$, where $\mathcal{R}_1$ is the set of rank-one elements of $\mathcal{R}$.

Definition 13. Define for $r \in \mathcal{R}_1$,

$$\mathcal{C}_r = \bigsqcup_{x \in C_r} BxB.$$

We refer to $\mathcal{C}_r$ as the *monoid cell* associated with $r \in \mathcal{R}_1$.

It turns out that the following holds:

$$M \setminus \{0\} = \bigsqcup_{r \in \mathcal{R}_1} \mathcal{C}_r.$$

5.2 The H-polynomial of a semisimple monoid

Next, suppose that M is a $\mathcal{J}$–irreducible of type $J \subset S$, and such that $M \setminus \{0\}$ is rationally smooth. There is a combinatorial object associated with this situation called the *augmented poset* $(E_1, \leq, \{\nu_s\})$. It turns out that this augmented poset encodes all the relevant information about the H–polynomial of M. The main result answers the following question posed in [34]. Let $r = (u, v) \in \mathcal{R}_1 \cong W^J \times W^J$, and let $\mathcal{C}_r \subset \mathbb{P}(M)$ be the corresponding cell. To calculate the H–polynomial of M one first need to answer the following technical question.

$$\text{What is } \dim(\mathcal{C}_\mathrm{r}) \text{ in terms of } (\mathrm{u}, \mathrm{v}),\ \mathrm{J},\ \text{and } (\mathrm{W}, \mathrm{S})?$$

Renner quantifies the dimension of $\mathcal{C}_r$ in terms of r and the descent system (W^J, S^J). This allows him to write the H–polynomial of M entirely in terms of (W^J, S^J).

The set up is the following. Let $J \subset S$ be combinatorially smooth and let $s \in S \setminus J$, $w \in W^J$. Then let

a) $\delta(s) = |C_s| + 1$, where C_s is the connected component of J attached to s.

b) $\nu_s(w) = |\{r \in S_s^J \mid w < wr\}| = |A_s^J(w)|$.

c) $w_o \in W^J$ be the longest element.

Theorem 21. [36] The H–polynomial $H(M)$ of M is given by

$$H(M) = (\sum_{w \in W^J} t^{l(w_0)-l(w)+m(w)})(\sum_{v \in W^J} t^{l(v)})$$

where $m(w) = \sum_{s \in S \setminus J} \delta(s)\nu_s(w)$ and $H(J) = \sum_{v \in W^J} t^{l(v)}$ is the H–polynomial of G/P_J.

The descent system is necessary to obtain a precise description of the quantity $m(v)$ in the above formula for $H(M)$.

Next, we illustrate the previous theorem with the following examples taken from [36].

Example 20. Let $M = M_{n+1}(k)$. Then M is $\mathcal{J}$–irreducible of type $J \subset S$, where $J = \{s_2, s_3, \cdots, s_n\}$ and (W, S) is of type A_n. In this examples

$S^J = \{s_1, s_2s_1, \cdots, s_n \cdots s_1\}$, $W^J = S^J \sqcup \{1\}$.

Write $a_i = s_i \cdots s_1$ if $i > 1$, and $a_0 = 1$. An elementary calculation yields $S \setminus J = \{s_1\}$,

$l(a_i) = i$,

$w_0 = s_n \cdots s_1$,

$\delta(s_1) = n$,

$\nu_{s_1}(a_i) = n - i$,

$P(J) = \sum_{i=0}^{n} t^{2i}$, and $X_M = \mathbb{P}^{(n+1)^2-1}(k)$.

We obtain the following formula for the H–polynomial:

$$H(M_{n+1}(k)) = (\sum_{i=0}^{n} t^{(n-i)(n+1)})(\sum_{i=0}^{n} t^i) = \sum_{i=0}^{(n+1)^2-1} t^i$$

Example 21. In this example we consider M a $\mathcal{J}$–irreducible of type $J \subset S$, where (W, S) is of type A_n and $J = \{s_3, s_4, \cdots, s_n\}$.

If $w \in W_n^J$ we write $w = a_p b_q$ where $a_p = s_p \cdots s_1$ $(1 \leq p \leq n)$ and $b_q = s_q \cdots s_2$

$(2 \leq q \leq n)$. Thus

$$W_n^J = \{a_p b_q \mid 0 \leq p \leq n \text{ and } 1 \leq q \leq n\}.$$

Now $S \setminus J = \{s_1, s_2\}$ so that $C_{s_1} = \emptyset$ and $C_{s_2} = \{s_3, \cdots, s_n\}$. Thus,

1) $\delta(s_1) = 1$, and

2) $\delta(s_2) = n - 2 + 1 = n - 1$.

3)$\nu_{s_1}(a_p b_q) = 1$ if $p < q$ and $\nu_{s_1}(a_p b_q) = 0$ if $p \leq q$.

4)$\nu_{s_2}(a_p b_q) = n - q$.

Thus, by definition,

1) $m(a_p b_q) = (n-1)(n-q) + 1$ if $p < q$ and

2) $m(a_p b_q) = (n-1)(n-q)$ if $p \leq q$.

Finally, $l(a_p b_q) = p + q - 1$, and $a_n b_n \in W^J$ is the longest element.

Thus, for $w = a_p b_q \in W^J$, we obtain

$$l(w_0) - l(w) + m(w) = n - p + n(n-q) + \epsilon$$

where $\epsilon = 1$ if $0 \leq p < q \leq n$, and $\epsilon = 0$ if $n \geq p \geq q \geq 1$. Thus

$$\sum_{w \in W^J} t^{l(w_0) - l(w) + m(w)} = \sum_{0 \leq p < q \leq n} t^{n-p+n(n-q)+1} + \sum_{n \geq p \geq q \geq 1} t^{n-p+n(n-q)}$$

The other factor is

$$H(J) = \sum_{w \in W^J} t^{l(w)} = \sum_{i=1}^{n} i(t^{i-1} + t^{2n-i})$$

Finally we obtain

$$H(M) = \sum_{0 \leq p < q \leq n} t^{n-p+n(n-q)+1} + \sum_{n \geq p \geq q \geq 1} t^{n-p+n(n-q)} + \sum_{i=1}^{n} i(t^{i-1} + t^{2n-i}).$$

In the last example we consider the root system of type B_l.

Example 22. Here we consider the case of $\mathcal{J}$–irreducible monoid of type $J = \{s_1, \cdots, s_{l-1}\}$. For related computations and a description of W^J see [35, Example 6.10]

We omit the relevant details of the computations made in this case. For a complete proof see [36]. The H–polynomial is given by the following formula:

$$H(M) = (\prod_{k=1}^{l}(1 + t^{k+l}))(\prod_{k=1}^{l}(1 + t^{k}))$$

where the factor $\prod_{k=1}^{l}(1+t^{k+l})$ is $H(G/P_J) = \sum_{v \in W^J} t^{l(v)}$ and the factor $\prod_{k=1}^{l}(1+t^{k})$ is $\sum_{w \in W^J} t^{l(w_0)-l(w)+m(w)}$

Bibliography

[1] S.Billey, V.Lakshmibai, *Singular Loci of Schubert varities* , Progress in Mathematics vol.182, Birkhauser, Boston, 2000.

[2] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Annals of Math. 98 (1973), 480-497.

[3] A. Bjorner, F. Brenti, *"Combinatorics of Coxeter Groups"*, Graduate Texts in Mathematics 231, Springer Verlag, New York, 2005.

[4] F. Brenti, *q-Eulerian poynomials arising from Coxeter groups*, European Journal of Combinatorics (1994)15, 417-441.

[5] M. Brion, R.Joshua, *Intersection Cohomology of Reductive Varieties*, J. Eur. Math. Soc 6(2004)), 465-481.

[6] M. Brion, *Local structure of algebraic monoids*, Semigroup Forum, to appear.

[7] M. Brion, *Rational smoothness and fixed points of torus actions*, Transformation Groups 4 (1999), 127-156.

[8] A. M. Cohen, *Eulerian polynomials of spherical type*, Münster J.Math, 1 (2008), 1-8.

[9] C. De Concini, C.Processi, *Complete symmetric varieties*, Springer Lecture Notes in Mathematics 996(1983), 1-44.

[10] D. Cox, *Minicourse on Toric varieties* (2008). Lectures presented at University of Buenos Aires, 2001.

[11] V.I. Danilov, *"The Geometry of Toric Varieties"*, Russian Mathematical Surveys 33 (1978), 97-154.

[12] I. Dolgachev, V. Lunts, *A character formula for the representation of a Weyl group on the cohomology of the associated toric variety*, Journal of Algebra 168 (1994), 741-772.

[13] F. D. Grosshans, *The variety of points which are not semi-stable*, Illinois Journal of Math. 26(1982), 138-148.

[14] L. M. Golubitsky, *Betti numbers of Toric varieties and Eulerian polynomials*, accepted for publication in Communications in Algebra journal, March 2011.

[15] W. Fulton, *Introduction to toric varieties*, Annals of Math., vol.131, 1980.

[16] F. Hirzebruch, *Eulerian polynomial*, Münster J.Math, 1 (2008), 9-14.

[17] Z.I. Zhou Li, *Poincare polynomial of certain toric varieties*. Unpublished, 2007.

[18] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, GTM vol. 9, Third Ed., Springer, 1980.

[19] J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics, vol.21. Springer-Verlag, Berlin-Heidelberg-New York, 1991.

[20] G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat, *Toroidal Embeddings*, Lecture Notes in Mathematics, vol. 339, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[21] J. Patera, A. Klimyk, *Orbit functions*, Symmetry, Integrabiblity and Geometry: Methods and Applications 2 (2006).

[22] C. Procesi, *The toric variety associated to Weyl chambers*, in "Mots", M. Lothaire, Ed., 153-161, Hermés, Paris, 1990.

[23] R. A. Proctor, *Bruhat lattices, plane partition generating functions and miniscule representations*, Europ. J. Combinatorics 5 (1984), 331-350.

[24] M. S. Putcha, *Linear Algebraic Monoids*, LNM, Cambridge University Press, 1988.

[25] M. S. Putcha, *Monoids on group with BN-pair*, Journal of Algebra 120 (1989), 139-169.

[26] M. S. Putcha, L. E. Renner, *The system of idempotents and lattice of $\mathfrak{J}$-classes of reductive algebraic monoids*, Journal of Algebra 116(1988), 385-399.

[27] L. E. Renner, *Analogue of the Bruhat decomposition for algebraic monoids*, Journal of Algebra 101(1986), 303-338.

[28] L. E. Renner, *Classification of semisimple varieties*, Journal of Algebra 122(1989), 275-287.

[29] L. E. Renner, *An explicit cell decomposition of the canonical compactification of an algebraic group*, Can. Math. Bull., 46(2003), 140-148.

[30] L. E. Renner, *"Linear Algebraic Monoids"*, Encyclopedia of Mathematical Sciences, vol. 134, Springer, 2005.

[31] L. E. Renner, *The H-polynomial of a Semisimple Monoid*, Journal of Algebra 319 (2008), 360-376.

[32] L. E. Renner, *Descent Systems for Bruhat Posets*, Journal of Algebraic Combinatorics 29 (2009), 413-435.

[33] L. E. Renner, *Weyl Groups, Descent Systems and Betti Numbers*, Rocky Mountain Journal of Mathematics, accepted September 2008.

[34] L. E. Renner, *Rationally Smooth Algebraic Monoids*, Semigroup Forum 78 (2009), 384-395.

[35] L. E. Renner, *Canonical Cellular Decomposition for $\mathfrak{J}$-irreducible monoids*, Communications in Algebra 38 (2010), 181-198.

[36] L. E. Renner, *H-polynomial of an Irreducible Representation*, Journal of Algebra, accepted June 2010.

[37] L. E. Renner, *The H-polynomial of a Group Embedding*, to appear.

[38] L. Solomon, *An introduction to reductive monoids*, "Semigroups, formal languages and groups", J. Fountain, Ed., Kluwer Academic Publishers, 1995, 295-352.

[39] T. A. Springer, *Intersection cohomology of $B \times B$ - orbit closures in group compactifications*, Journal of Algebra 258 (2002), 71-111.

[40] J. R. Stembridge, *Some permutation representations of Weyl groups associated with the cohomology of toric varieties*, Advances in Mathematics 106 (1994), 244-301.

[41] R. P. Stanley, *Generalized h-vectors, intersection cohomology of toric varieties, and related results*, Advanced Studies in Pure Mathematics 11 (1987), 187-213.

[42] R.P. Stanley, *Recent development in algebraic combinatorics*, Israel J. Math (2004), 317-340.

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