USRI Research Output

Under the Supervision of Dr. Lyle Muller and Dr. Ján Mináč

Lewis Glabush and Nathan Kershaw

University of Western Ontario

Overview

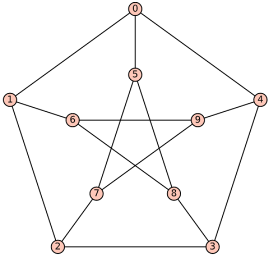
Our topic of study was Spectral Graph Theory. We studied the algabraic methods used to study the properties of graphs (networks) and became familiar with the applications of network analysis.
We spend a significant amount of time studying the way virus's spread on networks, with particular applications to Covid-19.
We also investegated the relationship between graph spectra and structural properties.

Spectral Graph Theory

Spectral Graph theory studies graphs (networks), by relating them to mathematical objects called matrices.

For a Graph $G = (V, E)$ with $|V| = N$, the **adjacency Matrix** is the $N \times N$ matrix A with

$$A_{ij} = \begin{cases} 1 \text{ If there is an edge between vertices } i \text{ and } j \\ 0 \text{ Otherwise} \end{cases}$$




```
[0 1 0 0 1 1 0 0 0 0]
[1 0 1 0 0 0 1 0 0 0]
[0 1 0 1 0 0 0 1 0 0]
[0 0 1 0 1 0 0 0 1 0]
[1 0 0 1 0 0 0 0 0 1]
[1 0 0 0 0 0 0 1 1 0]
[0 1 0 0 0 0 0 0 1 1]
[0 0 1 0 0 1 0 0 0 1]
[0 0 0 1 0 1 1 0 0 0]
[0 0 0 0 1 0 1 1 0 0]
```

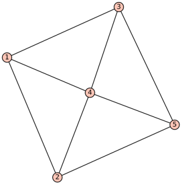
Graph Spectra

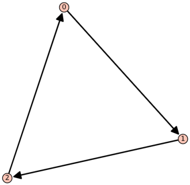
An **eigenvalue** λ of a matrix A over a field F is a constant in F, such that there exists a non-zero vector $\mathbf{v}$, with $A\mathbf{v} = \lambda\mathbf{v}$.

Equivalently, eigenvalues are the solutions to the characteristic polynomial of an $n \times n$ matrix given by

$$p_A(\lambda) = det(\lambda I - A) \text{ where } I \text{ is the } n \times n \text{ identity matrix.}$$

The **(adjacency) spectrum** of a graph G is the set of eigenvalues of the associated adjacency matrix A.



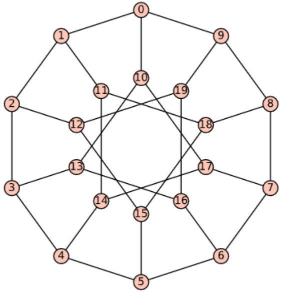


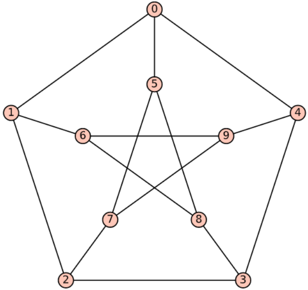
$[-2, 1-\sqrt{5}, 0, 0, 1+\sqrt{5}]$ $\quad$ $[1, -\frac{1}{2} - \sqrt{\frac{3}{2}}i, -\frac{1}{2} + \sqrt{\frac{3}{2}}i]$

Some Results

A Graph is said to be **regular** if each vertex has the same degree, meaning that each vertex has the same number of edges connected to it. A graph is **k-regular** if each vertex has degree k.

The largest eigenvalue λ_n of a graph is equal to the largest degree if and only if the graph is λ_n-regular.





The graphs shown above are both 3-regular and both have largest eigenvalue 3.

Some Results

A **walk** is a sequence of adjacent vertices. Let S_k denote the number of walks of length k from a vertex v_i to a vertex v_j then

$$S_k = (A^k)_{ij} \text{ where } A \text{ is the adjacency matrix of } G.$$

Undirected simple graphs always have diagonalizable matrices, and so for such graphs

$$s_k = \sum_{i=1}^{n} \lambda_i^k$$

Furthemore

1. The number of vertices of a graph is equal to the number of eigenvalues, multiplied by their respective algebraic multiplicities.
2. The number of edges on a graph is equal to $\frac{s_2}{2}$.
3. The average degree is equal to $\frac{s_2}{n}$
4. The number of triangles is equal to $\frac{s_3}{6}$

Graph Isomorphism

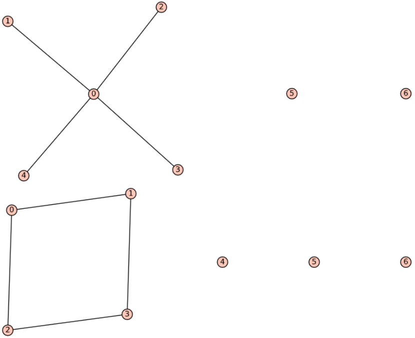
Two Graphs are **isomorphic** if there is a bijection between the vertex sets of each graph, that preserves edges. Say V_1 denotes the set of vertices of G_1, and V_2 represents the set of vertices of G_2, then $G_1 \cong G_2$ iff there exists a bijection

$\Phi : V_1 \to V_2$ such that $\Phi(v_1)$ and $\Phi(v_2)$ are adjacent iff v_1 and v_2 are.

Two graphs are isomorphic if they have the same adjacency matrix under some labelling of the vertices. If $|V| = n$, there are $n!$ such labelings, so it can be difficult to check. However, we can immediately tell if two graphs are non-isomorphic if they have different spectra.

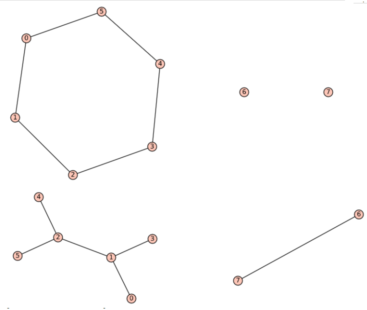
Isospectral Graphs

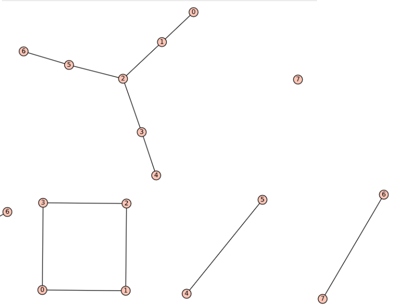
Two graphs are **isospectral** if they have the same spectrum.
Of particular interest are **isospectral non-isomorphic graphs**. A pair of isospectral non-isomorphic graphs is called a PING. The existence of a PING highlights structural properties of a graph that are not determined by the graph spectrum. Consider the following:



A Family of Non-isomorphic Isospectral Graphs

It had been conjectured that the spectrum of a graph could be used to determine the graph up to isomorphism. This was, of course, disproven by the construction of non-isomorphic isospectral graphs.





Applications

Graphs are ubiquitous data structures, that come up in computer science, chemistry, machine learning, virulogy, neuroscience, and many other areas.

One application of particular interest was studying Covid-19 through graphs. Contact tracing networks can be represented with graphs.

Spectral graph theory can be used to determine clusters on graphs, which can help identify at risk populations, once a certain individual tests positive without breaching privacy.

The epidemic threshold can also be determined using spectral graph theory, as it relates to the eigenvectors of contact tracing networks.

Further Topics

We are interested in continuing our study of spectral graph theory and investigating certain subjects. We would like to study the density of non-isomorphic isospectral graphs in the set of all graphs with a certain number of edges and vertices. We believe that walk structure becomes more complicated as graphs grow, and since graphs are isospectral if and only if they have the same walk structure, we believe that we can prove that as you increase the number of edges and vertices, the proportion of graphs which have non-isomorphic isospectral associates drops to 0.

We are also interested in studying the properties that characterize families of non-isomorphic isospectral graphs, which will require a more precise understanding of walk structure.