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FIRE PERFORMANCE OF RC FRAMES USING SECTIONAL ANALYSIS

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Abstract

Global behavior of RC structures during fire events is usually predicted using complex thermal nonlinear numerical simulations. However, these simulations are computationally expensive, which limit their use by design engineers. A practical approach to track the fire performance of RC frames during fire exposure is proposed and validated in this paper. A previously developed simple heat transfer technique is used to calculate an average 1D temperature distribution for heated RC sections. Consequently, flexural and axial stiffnesses as well as the unrestrained thermal deformation are evaluated using sectional analysis. Their expressions are mathematically derived based on a number of rational assumptions. The proposed method can be easily applied using available commercial linear structural analysis software to predict the fire performance of RC framed structures. Additional experimental and analytical work is required to validate the proposed method in non-standard fire scenarios.

Keywords: Concrete; Fire; Elevated temperatures; Sectional analysis; Flexural stiffness; Axial stiffness; Frames; Thermal Restraint.

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1. Introduction

Fire initiates when combustible materials ignite. Then, it spreads horizontally and vertically depending on the compartment boundaries [1]. A temperature gradient is generated through exposed RC elements. These elevated temperatures cause the element's stiffness to degrade and produce thermal deformations [2]. Structural fire safety of RC structures is currently evaluated based on the fire ratings of single elements, i.e. columns, beams, walls, and slabs [3]. However, the overall behavior of the structure during a fire should be assessed to ensure the safety of the occupants and the fire fighters during evacuation.

Fire testing is the most reliable approach to assess the fire endurance of a structure but its use for concrete frames is very limited [3]. This is mainly because of its cost, which makes it unsuitable for regular design. Finite Element (FE) tools are very powerful and capable of analyzing RC structures during fire events [5]. Drawbacks of using the FE method including: the need for a comprehensive computer program, the difficulty to comprehend its results and to identify potential modeling errors, and the long running time make it impractical for design engineers. To the best of the author's knowledge, simplified methods to analyze RC frames during fire exposure do not exist [6], [7], [8].

This paper provides engineers with a practical approach to predict the fire response of statically determinate or indeterminate RC frames. The proposed approach extends the work done by El-Fitiany and Youssef [9], [10][9]. In this work, they extended a simplified heat transfer method to convert the two-dimensional (2D) temperature distribution to an average one-dimensional (1D) temperature distribution and predicts the flexural behavior of the heated section at different axial load levels (λ) [11]. This paper provides the derivation of closed form formulations for concrete

1 stiffness, flexural and axial, and the steps needed to apply these formulations to analyze RC frames
2 during fire exposure. The proposed method eliminates the need to divide a fire exposed section
3 into finite elements or layers to conduct heat transfer and nonlinear stress analyses. The proposed
4 method is validated for standard fires by comparing its predictions with experimental and analytical
5 results by others.

6 Fig. 1a shows a 300 mm square concrete cross-section exposed to fire from three sides. Fig.
7 1a shows the elevated temperature contours within the heated cross-section after 1 hr of ASTM-
8 E119 fire exposure. The low thermal conductivity of concrete results in a steep temperature
9 distribution near the heated faces and almost a constant temperature at the core of the heated section
10 [10]. Fig. 1b shows the average temperature (T_{avg}) across the section width (b). As shown in Fig.
11 1b, T_{avg} is variable within distance z from the bottom face and constant within $h - z$ distance,
12 where h is the cross-section height. Thus, the concrete mechanical properties become variable, i.e.
13 non-linear, near the bottom heated face and constant at the inner concrete core. This concept is the
14 basis of the proposed method to predict the fire performance of concrete frames during fire
15 exposure.

17 2. Proposed Method

18 El-Fitiany and Youssef [7],[10] presented a sectional analysis technique to predict the behavior
19 of heated RC cross-sections while accounting for temperature gradient, strain nonlinearity, and
20 material degradation. Fig. 2a and 2b show the T_{avg} distribution for a RC section subjected to fire
21 from three faces. The free thermal strain (ϵ_{th}) can be predicted using T_{avg} , Fig. 2c. In order to
22 retain the section linearity, a self-induced strain (ϵ_{st}) is generated to maintain the section linearity
23 by converting the nonlinear to an equivalent linear strain ($\overline{\epsilon_{th}}$) as shown in Figs. 2d and 2e. $\overline{\epsilon_{th}}$ is

defined by the the value of the center axial strain (ε_i) and the curvature (ψ_i). The values of ε_i and ψ_i represent the unrestrained thermal deformation, i.e. elongation and curvature, of the heated section. Alternatively, $\overline{\varepsilon_{th}}$ can be defined using the factors $\overline{z}_3$ and $\overline{z}_4$ for the mathematical expressions presented in this paper. ε_{st} is considered as a residual strain and is added to the instantaneous stress-related strain (ε_c). The transient creep strain (ε_{tr}) is implicitly included in the concrete stress-strain relationship as explained later in this paper. As shown in Fig. 3, the total strain (ε) is then calculated by adding ($\varepsilon_c + \varepsilon_{tr} + \varepsilon_{st}$) to the equivalent linear strain ($\overline{\varepsilon_{th}}$). The effective flexural stiffness (EI_{eff}) and axial stiffness (EA_{eff}) can be calculated using the applied axial force (P_{app}) and flexural moment (M_{app}) and the corresponding total strain (ε).

The results of the sectional analysis is used to predict the global behavior of RC frames during fire exposure as explained later in this paper. Fig. 4 summarizes the main steps of the proposed method. For a given fire duration, the structural performance of RC frames can be predicted by:

- 1) determining an equivalent one-dimensional average temperature distribution for the fire exposed beams and columns cross-sections,
- 2) identifying the needed constitutive thermal and mechanical models for the heated elements,
- 3) predicting the self-induced strain and unrestrained thermal deformations for the heated elements,
- 4) evaluating the effective, i.e. secant, flexural and axial stiffnesses of the heated elements based on the applied axial forces (P_{app}) and flexural moments (M_{app}),
- 5) Analyzing the fire-exposed RC frame under the applied loads. The effective flexural and axial stiffnesses obtained in step 4 are utilized in this step. The moments and axial forces are redistributed based on the assigned stiffness values. Secondary straining actions are calculated and added to the primary moments and axial forces.

6) Recalculating the flexural and axial stiffnesses in step 4 for the revised flexural moments and axial forces obtained in step 5.

Steps 4 to 6 are repeated until the change in the obtained axial forces and moments is less than an assumed tolerance. The following sections explain each of these steps in detail.

3. Average temperature distribution

A simplified method to calculate the temperature distribution within a fire exposed concrete section was presented by El-Fitiany and Youssef [10]. In this method, the heated section is divided into regions, Fig. 1a, based on the number of fire exposure sides and the heat gradient extent (z) from each fire-exposed face. The average temperature is then calculated for each region, e.g. T_{avg1} and T_{avg2} . The weighted average one-dimensional temperature distribution (T_{avg}) can be then plotted along the section height as shown in Fig. 1a. The resulting T_{avg} profile consists of constant and nonlinear temperature distribution zones. T_{avg} is calculated at the two boundaries of the variable T_{avg} distribution only (i.e. $y = 0.0$ and $y = z$ in Fig. 1b). The nonlinear portion of the predicted T_{avg} profile in Fig. 1a is fitted by Eq. (1) which will be used in the proposed method.

$$T_{avg} = z_1 \cdot e^{(z_2 \cdot y)} \quad (1)$$

$$z_1 = T_{avg}(y=0.0) ; \text{ and } z_2 = \frac{\ln\left[\frac{T_{avg}(y=z)}{z_1}\right]}{z}$$

where

y vertical coordinate for any point within the heated section, origin is located at the bottom left of the section,

z height of the nonlinear T_{avg} zone,

z_1, z_2 constants of T_{avg} exponential fitting equation,

1 $T_{avg (y=0.0)}$ average temperature (°C) at the bottom of the section, and

2 $T_{avg (y=z)}$ average temperature (°C) at the end of the nonlinear zone.

3

4 **4. Concrete and steel constitutive relationships**

5 Fire temperature reduces the mechanical properties of concrete and steel. It also induces
6 new strains, i.e. thermal and transient creep strains, which significantly affect the constitutive
7 relationships of concrete and steel at elevated temperatures. The following sub-sections provide a
8 brief summary of the concrete and steel models used in this paper.

9

10 *4.1. Concrete Strains*

11 Figs. 2 and 3 show the different strain components for a RC section exposed to fire from three
12 sides. The positive and negative signs represent compression and tension strains, respectively. The
13 total concrete strain at elevated temperatures (ε) is composed of four terms: equivalent linear
14 thermal strain ($\overline{\varepsilon_{th}}$), self-induced thermal strains (ε_{st}), and instantaneous stress-related strain (ε_c),
15 transient creep strain (ε_{tr}). The terms ε_{st} , ε_c , and ε_{tr} are lumped into an equivalent mechanical
16 strain ε_{cT} as shown in Eq. (2) and Fig. 2.

$$17 \quad \varepsilon = \overline{\varepsilon_{th}} + (\varepsilon_{st} + \varepsilon_c + \varepsilon_{tr}) = \overline{\varepsilon_{th}} + \varepsilon_{cT} \quad (2)$$

18

19 *4.2. Thermal strains*

20 The free thermal strain (ε_{th}) is a strain resulting from fire temperature and it is calculated
21 using T_{avg} . The thermal strain of steel bars is calculated based on the concrete temperature at their
22 locations. ε_{th} can be predicted using the Eurocode 2 model [12], Eq. (3).

23 For siliceous concrete:

$$\begin{aligned} \varepsilon_{th} &= B_1 + B_2(T_{avg} - 20) + B_3(T_{avg} - 20)^3 \quad 20^\circ\text{C} < T_{avg} \leq 720^\circ\text{C} \\ &= B_4 \quad 720^\circ\text{C} < T_{avg} \end{aligned} \quad (3a)$$

where $B_1 = -1.8 \times 10^{-4}$, $B_2 = 9 \times 10^{-6}$, $B_3 = 2.3 \times 10^{-11}$, and $B_4 = 14 \times 10^{-3}$

For carbonate concrete:

$$\begin{aligned} \varepsilon_{th} &= B_1 + B_2(T_{avg} - 20) + B_3(T_{avg} - 20)^3 \quad 20^\circ\text{C} < T_{avg} \leq 824^\circ\text{C} \\ &= B_4 \quad 824^\circ\text{C} < T_{avg} \end{aligned} \quad (3b)$$

where $B_1 = -1.2 \times 10^{-4}$, $B_2 = 6 \times 10^{-6}$, $B_3 = 1.4 \times 10^{-11}$, and $B_4 = 12 \times 10^{-3}$

4.3. Concrete strain at peak stress

The value of the instantaneous stress-related strain (ε_c) at the peak compressive stress (f'_{cT}), i.e. ε_{oT} , defines the stress-strain relationship during fire exposure [2]. For loaded concrete, the effect of elevated temperatures on ε_{oT} is negligible [2]. On the other hand, fire temperature increases concrete ductility by inducing transient creep strain (ε_{tr}). Fig. 5 shows the variation of ($\varepsilon_{oT} + \varepsilon_{tr}$) with fire temperature as proposed by Eurocode 2. The shown values of ($\varepsilon_{oT} + \varepsilon_{tr}$) are consistent with Terro's model [13] within its validated temperature range, i.e. up to 600°C . A linear relationship, Eq. (4), is chosen to represent the Eurocode 2 recommendation. Such a relationship allows reaching a closed form solution for concrete internal forces. This approximation has a minor effect on the behavior of heated elements as shown later in section 8.2 [10].

$$\varepsilon_{oT} + \varepsilon_{tr} = 2.52 \times 10^{-5} T_{avg} \quad 80^\circ\text{C} < T_{avg} \leq 1200^\circ\text{C} \quad (4)$$

4.4. Concrete ultimate strain

Concrete ultimate strain is the strain at which concrete crushing occurs. Elevated temperatures increase this strain [2]. Eurocode 2 [12] proposes a linear relationship between ε_{uT}

and T_{avg} , Fig. 1. This relationship can be represented by Eq. (5). ε_{uT} is defined in Eurocode 2 as the strain corresponding to zero compression stress. The difference between ε_{uT} and $\varepsilon_{oT} + \varepsilon_{tr}$ is constant and equal to 0.02.

$$\varepsilon_{uT} = 2.52 \times 10^{-5} T_{avg} + \Delta\varepsilon = \varepsilon_{oT} + \varepsilon_{tr} + 0.02 \quad (5)$$

4.5. Maximum concrete strain

The maximum concrete strain is defined as the strain corresponding to the maximum moment resistance. This strain is usually less than ε_{uT} . Elbahy et al. [15] studied the variation of this strain with the axial load level, λ , at ambient temperature. At elevated temperatures, a parametric study was conducted by El-Fitiany and Youssef [10] to evaluate the compression strain ($\varepsilon_{cT\ max}$) corresponding to the flexural capacity at different λ values [10]. Based on this parametric study, ε_{max} is given by Eq. (6).

$$\varepsilon_{cT\ max} = (\varepsilon_{oT} + \varepsilon_{tr}) + 0.005 \quad (6)$$

4.6. Concrete compressive strength

Concrete compressive strength experiences significant degradation at elevated temperatures. Eurocode 2 predicts the reduced compressive strength (f'_{cT}) for siliceous and carbonate concretes as a ratio from its ambient value (f'_c) [12]. The reduction in f'_{cT} for siliceous concrete is fitted by a polynomial equation, Eq. (7), as it allows reaching closed form solution for flexural stiffness. Eq. (7) can be conservatively applied for carbonate aggregate concrete as shown in section 8.3 of this paper [10].

$$f'_{cT}/f'_c = 1.76 \times 10^{-9} T_{avg}^3 - 3 \times 10^{-6} T_{avg}^2 + 2.5 \times 10^{-4} T_{avg} + 1.00 \quad (7)$$

where T_{avg} is the average temperature, in °C, calculated in previous section

4.7. Concrete stress-strain relationship

The relationship between the compression stress, f_{cT} , and the corresponding mechanical strain, ε_{cT} , at elevated temperatures was studied by a number of researchers. Among the available models, Youssef and Moftah [2] proposed a stress-strain model that includes a simplified representation of transient creep strains (ε_{tr}) and better matches the available experimental data. It is also consistent with the stress-strain relationship recommended by Eurocode 2 [12]. Youssef and Moftah's concrete stress-strain relationship, up to the peak stress (f'_{cT}), is given by Eq. (8a).

A general and simple approach to estimate the $f_{cT} - \varepsilon_{cT}$ descending branch is proposed by Eurocode 2 [12] and represented by Eq. (8b). The Eurocode 2 descending curve is adopted in this study due to its simplicity and ease of implementation in the proposed method. Fig. 6 shows the application of Eq. (8) at three average temperatures ($T_{avg} = 200, 400, \text{ and } 600^\circ\text{C}$). Eqs. (4) and (7) are used to calculate $(\varepsilon_{oT} + \varepsilon_{tr})$ and f'_{cT}/f'_c , respectively.

$$f_{cT} = f'_{cT} \left[2 \left(\frac{\varepsilon_{cT}}{\varepsilon_{oT} + \varepsilon_{tr}} \right) - \left(\frac{\varepsilon_{cT}}{\varepsilon_{oT} + \varepsilon_{tr}} \right)^2 \right] \varepsilon_{cT} \leq (\varepsilon_{oT} + \varepsilon_{tr}) \quad (8a)$$

$$= f'_{cT} \left[\frac{\varepsilon_{uT} - \varepsilon_{cT}}{0.02} \right] (\varepsilon_{oT} + \varepsilon_{tr}) < \varepsilon_{cT} \leq \varepsilon_{uT} \quad (8b)$$

4.8. Steel stress-strain relationships

Lie et al.'s model **Error! Reference source not found.** is used to predict the reduced yield strength of reinforcing bars (f_{yT}), Eq. (9), and the stress-strain ($f_{sT} - \varepsilon_{sT}$) relationship, Eq. (10).

$$f_{yT} = \left[1 + \frac{T}{900 \ln(T/1750)} \right] f_y \quad 0 < T \leq 600^\circ\text{C} \quad (9a)$$

$$= \left[\frac{340 - 0.34 T}{T - 240} \right] f_y \quad 600 < T \leq 1000^\circ\text{C} \quad (9b)$$

$$f_{sT} = \frac{f[T, 0.001]}{0.001} \varepsilon_{sT} \varepsilon_{sT} \leq \varepsilon_p \quad (10a)$$

$$f_{sT} = \frac{f[T, 0.001]}{0.001} \varepsilon_p + f[T, (\varepsilon_{sT} - \varepsilon_p + 0.001)] - f[T, 0.001] \varepsilon_{sT} > \varepsilon_p \quad (10b)$$

$$\varepsilon_p = 4 \times 10^{-6} f_y \quad (10c)$$

$$f[T, 0.001] = (50 - 0.04 T) [1 - e^{(-30 + 0.03 T) \sqrt{0.001}}] \times 6.9 \quad (10d)$$

$$f[T, (\varepsilon_{sT} - \varepsilon_p + 0.001)] = (50 - 0.04 T) \left[1 - e^{(-30 + 0.03 T) \sqrt{(\varepsilon_{sT} - \varepsilon_p + 0.001)}} \right] \times 6.9 \quad (10e)$$

5

6 **5. Prediction of the Unrestrained Deformation for a Heated Section**

7 Fig. 2e shows the difference between the equivalent linear ($\overline{\varepsilon_{th}}$) and nonlinear (ε_{th}) thermal
8 strains, which represent the self-induced thermal strain (ε_{st}). While ignoring ε_{st} results in slightly
9 underestimating the flexural capacity of fire-exposed sections [10], its value should be assigned as
10 initial strains for the heated concrete and steel to properly predict the axial and flexural stiffness
11 values. The following sections present a simplified approach to calculate ε_{st} using the predicted
12 T_{avg} distribution and material models presented earlier in this paper.

13

14 *5.1. Self-induced strain (ε_{st}) profile*

15 The average temperature profile (T_{avg}), Eq. (1), is a function of the distance y . Utilizing the
16 predicted T_{avg} distribution, ($\varepsilon_{oT} + \varepsilon_{tr}$) distribution can be evaluated using Eq. (8). Figs. 4b and 4c
17 show these distributions for a concrete section exposed to fire from three faces. The difference
18 between $\overline{\varepsilon_{th}}$ and ε_{th} represents the self-induced thermal strain (ε_{st}), Eq. (11).

$$\varepsilon_{st} = \overline{\varepsilon_{th}} - \varepsilon_{th} = (\overline{z}_3 y + \overline{z}_4) - \varepsilon_{th} \quad (11)$$

20

21 where $\overline{z}_3$ and $\overline{z}_4$ are constants to define the linear variation of $\overline{\varepsilon_{th}}$ in y direction, Fig. 3c. ε_{th} is
22 calculated using Eq. (3).

23

5.2. Calculation of concrete internal force and its location due to ε_{st}

Fig. 3f shows the calculated distribution of the self-induced strain ε_{st} . The positive and negative signs represent compression and tensile stresses, respectively. The tensile capacity of heated concrete is neglected [7]. Concrete compression stresses $(f_{cT})_{avg}$ can be estimated using Eq. (8a) and the corresponding self-induced strain ε_{st} . $(f_{cT})_{avg}$ represents the average compressive stresses across the section width (b). These stresses are integrated over the section height to calculate the internal compression force in concrete (C_{th}) and its location (y), Eqs. (12) and (13).

$$C_{th} = \int_{y=y_1}^{y_2} (f_{cT})_{avg} \cdot b \cdot dy \quad (12)$$

$$C_{th} \cdot y = \int_{y=y_1}^{y_2} (f_{cT})_{avg} \cdot b \cdot y \cdot dy \quad (13)$$

Where b and h are the concrete section width and depth (m), respectively, y_1 and y_2 are the boundaries of the compression stresses zone (m), and $(f_{cT})_{avg}$ is the average concrete compressive stress (kPa) at different y values.

To allow reaching a closed form solution, the following substitutions are made in Eqs. (12) and (13): [1] $(f_{cT})_{avg} = \text{Eq. (8a)}$, [2] $(\varepsilon_{oT} + \varepsilon_{tr}) = \text{Eq. (4)}$, [3] $f'_{cT} = \text{Eq. (7)}$, [4] $T_{avg} = \text{either a constant value or Eq. (1)}$, and [5] $\varepsilon_{cT} = \text{Eq. (11)}$.

As concrete stresses corresponding to ε_{st} are low compared to f'_{cT} and to simplify the closed form solution, the quadratic term in concrete constitutive stress-strain relationship, Eq. (8a), is neglected when integrating Eqs. (12) and (13). A parametric study is conducted later in this paper, section 8.4, to evaluate the error associated with this approximation. The closed form solutions of Eqs. (12) and (13) are given by Eqs. (16) to (21), which are shown in Appendix I.

Fig. 4d shows the internal concrete force (C_{th}) due to the self-induced strain (ε_{st}) distribution shown in Fig. 4c. For variable T_{avg} distribution, two equations are given for the

compressive force in concrete, Eqs. (16) and (18). They allow evaluating $C_{th\ L(v)}$ and $C_{th\ N(v)}$ for linear and nonlinear ε_{st} distributions, respectively. The centroids of these compressive forces are given by Eqs. (17) and (19). For constant T_{avg} , the compressive force in concrete, $C_{th\ L(c)}$, can be evaluated using Eq. (20) for linear ε_{st} distribution. The centroid of this force is given by Eq. (21). The total internal compressive force due to thermal incompatibility, C_{th} , is calculated by summing two or more components as shown in Fig. 4d. The assumed $\overline{\varepsilon_{th}}$, i.e. $\overline{z}_3$ and $\overline{z}_4$ constants, is correct if concrete and steel reinforcement internal forces due to ε_{st} are in equilibrium.

6. Evaluation of the Flexural and Axial Stiffnesses for a Heated Section

Fig. 8a shows the applied axial force (P_{app}) and flexural positive and negative moments (M_{app}) on a RC section exposed to fire from three faces. The use of sectional analysis to evaluate the flexural and axial stiffnesses for this section involves the following steps:

1) A linear mechanical strain ($\varepsilon_c + \varepsilon_{tr}$) is assumed as shown in Fig. 8d. Utilizing the predicted T_{avg} distribution, ($\varepsilon_{oT} + \varepsilon_{tr}$) and ε_{uT} distributions can be evaluated using Eqs. (4) and (5), respectively. Fig. 8d shows these distributions for the concrete section subjected to fire from three faces.

2) The self-induced thermal strains (ε_{st}), calculated in the previous section, are assigned as initial strains for the concrete and steel to model the corresponding self-induced self-equilibrating thermal stresses. The terms ε_{st} , ε_c , and ε_{tr} are lumped into an equivalent mechanical strain ε_{cT} , Eq. (2). Fig. 8d shows the modified mechanical strain after superimposing the self-induced thermal strains. As shown in Fig. 8c, ε_{st} is divided into a series of straight lines. These lines can be then superimposed with ε_{cT} distribution, Eq. (14). This simplification allows using Eqs. (22) to (29)

after modifying z_3 and z_4 only to account for ε_{st} . The modified ε_{cT} distribution consists of different zones based on ε_{st} idealization. Internal concrete and steel forces in each zone are calculated based on z_3 and z_4 for each zone.

$$\varepsilon_{cT} = z_3 y + z_4 \quad (14)$$

where z_3 and z_4 are constants to define ε_{cT} variation in y direction.

3) Average concrete compressive stresses $(f_{cT})_{avg}$ that correspond to ε_{cT} profile can be estimated using Eq. (8). These stresses are integrated over the section area to calculate the internal compression force in concrete (C_c) and its location (y), Eqs. (12) and (13).

4) The following substitutions are made in Eqs. (12) and (13) to allow reaching a closed form solution: (1) $(f_{cT})_{avg} = \text{Eq. (8)}$, (2) $(\varepsilon_{oT} + \varepsilon_{tr}) = \text{Eq. (4)}$, (3) $f'_{cT} = \text{Eq. (7)}$, (4) $\varepsilon_{uT} = \text{Eq. (5)}$, (5) $T_{avg} = \text{either a constant value or Eq. (1)}$, and (6) $\varepsilon_{cT} = \text{Eq. (14)}$. The solutions are given by Eqs. (22) to (29), which are shown in the Appendix. For values of $\varepsilon_{cT} \leq (\varepsilon_{oT} + \varepsilon_{tr})$, two equations are given for the compressive force in concrete, Eqs. (22) and (26). They allow evaluating $C_{c o(v)}$ and $C_{c o(c)}$ for variable and constant T_{avg} distributions, respectively. The centroids of these compressive forces are given by Eqs. (23) and (27). For values of $\varepsilon_{cT} > (\varepsilon_{oT} + \varepsilon_{tr})$, the compressive force in concrete, $C_{c u(v)}$ or $C_{c u(c)}$, can be evaluated using Eqs. (24) and (28) for variable or constant T_{avg} distributions. The centroids of these forces are given by Eqs. (25) and (29).

Figs. 8d and 8f show two potential mechanical strain (ε_{cT}) distributions for positive and negative flexural moments, respectively. Figs. 8e and 8f show the corresponding internal concrete compression stresses and forces. Concrete tensile strength is neglected in the analysis. The

magnitude and location of $C_{co(c)}$, Fig. 8e, are evaluated using Eqs. (26) and (27), respectively. The magnitude and location of $C_{cu(c)}$ are evaluated using Eqs. (28) and (29), respectively. The magnitude and location of $C_{co(v)}$, Fig. 8g, are evaluated using Eqs. (22) and (23), respectively. The magnitude and location of $C_{cu(v)}$ are evaluated using Eqs. (24) and (25), respectively. The total internal compression force $C_{co(c)}$ is calculated by summing two or more components shown in Figs. 8e and 8g.

5) To satisfy equilibrium between the calculated internal forces and the external loads, i.e. P_{app} and M_{app} , iterations are executed by changing the values of the assumed $(\varepsilon_c + \varepsilon_c)$ in step 1. The final $(\varepsilon_c + \varepsilon_c)$ can be represented by its mid-height value (ε_{cT}) and slope (ψ_{cT}) .

6) The effective flexural stiffness (EI_{eff}) can be then calculated as the ratio between M_{app} and ψ_{cT} . The corresponding effective axial stiffness (EA_{eff}) equals to P_{app} divided by the center axial strain (ε_{cT}) .

7. Prediction of the Behaviour of RC Frames during Fire Exposure

As shown in Fig. 9, heating RC sections from the bottom face and the two sides cause the bottom concrete fibers to thermally expand more than the top concrete fibers and results in ψ_i . The acting moment induces a mechanical curvature (ψ_{cT}), which is either added to or deducted from ψ_i . As shown in Fig. 9a, a positive (sagging) moment induces a curvature that adds to the initial curvature. For negative (hogging) moments, compression stresses are applied on the bottom fibers. Curvature caused by these stresses opposes the initial curvature, Fig. 9b. Such a moment-curvature diagram is similar to that of a prestressed concrete section. While the initial curvature in such a section is caused by prestressing, it results from the thermal expansion in a fire-exposed section

[9].

For a specific fire duration, the effect of thermal strain on the $M-\psi$ relationship is not governed by M_{app} . It represents the unrestrained/free thermal expansion of the unloaded concrete element and results in shifting the $M-\psi$ and diagram by a value ψ_i , Fig. 9. Consequently, the total curvature (ψ) is the sum of the unrestrained thermal curvature (ψ_i) and the mechanical curvature (ψ_{cT}) and can be expressed in terms of the effective stiffness (EI_{eff}) as follows.

$$\psi = \psi_i + M_{app}/EI_{eff} \quad (15a)$$

Similarly, the total center axial strain (ε) is the sum of the unrestrained thermal strain (ε_i) and the center mechanical strain (ε_{cT}) and can be expressed in terms of the effective axial stiffness (EA_{eff}) as follows.

$$\varepsilon = \varepsilon_i + P_{app}/EA_{eff} \quad (15b)$$

Fire-exposed RC frames/subassemblies can modeled, utilizing any linear elastic FE software, and analyzed using the degraded flexural and axial stiffnesses calculated in section 6 of this paper. The effect of thermal expansion is considered by modeling ε_i and ψ_i , calculated in section 5, as an induced deformation for all fire-exposed RC members. The FE analysis is repeated for the fire-exposed frames till the external moments and axial forces are properly redistributed between the frame elements and results in the same flexural and axial stiffness input values. This approach for analyzing RC frames during fire exposure is illustrated and validated in section 9.

8. Error Analysis

The error associated with proposed method approximations are evaluated in the following sections.

8.1. Average Temperatures

The error corresponding to using T_{avg} for strength calculations is assessed by analyzing the sections shown in Table 1 at different axial load levels ($\lambda = 0.2 - 0.5$). All the sections are assumed exposed to ASTM-E119 standard fire from four faces. Comparisons between the analytical effective flexural and axial stiffnesses (EI_{eff} and EA_{eff}) obtained by considering the actual concrete strength and the strength calculated using T_{avg} are shown in Fig. 10. Using T_{avg} for stress calculations has a negligible effect on the flexural and axial stiffnesses of the examined concrete sections.

8.2. Concrete strain at peak stress

To evaluate the error associated with using Eq. (4) to define the concrete strain at peak stress ($\varepsilon_{oT} + \varepsilon_{tr}$), the effective flexural and axial stiffnesses (EI_{eff} and EA_{eff}) of the four-face heated sections shown in Table 1 are calculated, at different axial load levels ($\lambda = 0.2 - 0.5$), up to 4 hrs of ASTM-E119 standard fire exposure. Fig. 11 shows that this approximation has a minor effect on the flexural and axial stiffness predictions calculated using sectional analysis method.

8.3. Concrete compressive strength

For $T_{avg} \leq 900$ °C, Eq. (7) results in coefficient of variations of 0.067 and 0.195 for siliceous and carbonate concrete, respectively. The equation can be conservatively applied for carbonate aggregate concrete [10].

8.4. Effect of ignoring concrete nonlinearity on $\overline{\varepsilon_{th}}$

A parametric study is conducted in to evaluate the error in predicting $\overline{\varepsilon_{th}}$ due to ignoring

the quadratic term in Eq. 8a. The unrestrained thermal parameters ε_i and ψ_i are predicted using the sectional analysis method for the RC sections shown in Table 1. All the sections are subjected to ASTM-E119 standard fire exposure up to 4 *hrs*. The studied sections are unevenly heated, i.e. from three faces, to generate ψ_i during fire exposure. ε_i and ψ_i are calculated using Eq. (8a) as presented. Their values are recalculated ignoring the quadratic term only in Eq. (8a), i.e. assuming a linear constitutive stress-strain relationship between f'_{cT} and $(\varepsilon_{oT} + \varepsilon_{tr})$. Fig. 12 shows a comparison between the correct ε_i and ψ_i and the approximate ε_i and ψ_i for all the RC sections. As shown in the figure, the effect of ignoring the concrete nonlinearity is negligible for ε_i and results in a minor difference for ψ_i .

9. Validation Case 1 (Fang et al., 2012)

The proposed method is validated by comparing its predictions with experimental work by Fang et al. [4]. Fig. 13 shows a full-scale specimen that was exposed to a 3 *hrs* standard ISO 834 fire. The top 150 *mm* of the beam section was shielded by thick ceramic fiber to simulate the effect of floor slab. The outer 250 *mm* of the column section was also shielded as the joint was insulated by thick ceramic fiber during fire exposure. The column has dimensions of 500 *mm* by 500 *mm* and a height of 2860 *mm* and is reinforced with 12 – 22 *mm* bars. The full length of the column was exposed to fire. The beam has dimensions of 400 *mm* by 500 *mm* and a fire exposed length of 5750 *mm*. It is reinforced with 4 – 22 *mm* and 2 – 22 *mm* bottom and top bars, respectively, and additional top 2 – 25 *mm* bars at the beam end, i.e. SEC 2-2 in Fig. 13. The yield strength of the 22 *mm* and 25 *mm* reinforcing bars is 453 *MPa* and 411 *MPa*, respectively. The 28 days compressive strength of the siliceous concrete was 34.47 *MPa*. Prior to test, the column was axially loaded by 1727 *kN* and the beam was subjected to vertical loads P_1 of 78 and P_2 of 49 *kN*. All

loads were maintained during the fire test and the variation of the deflection at P_1 and P_2 as well as the horizontal displacement and rotation at the roller support were continuously monitored during the whole test. These values are predicted using the proposed method in the following sections. Beam SEC 2-2 will be considered to provide sample calculations. The effect of elevated temperatures on the shear capacity, bond loss between steel bars and concrete, and concrete spalling were not considered in the analysis.

9.1. Predicting Average Temperature Distribution

The steps involved in applying Wickstrom's method to calculate the temperature distribution within the fire exposed beam section are given below. Due to the insulation, the concrete beam and column are assumed to be subjected to ISO 834 heating from three sides only; Left (L), Right (R), and Bottom (B) faces. The ISO 834 fire temperature (T_f) is calculated at fire duration of $t = 3.0 \text{ hrs}$ as 1090°C [10]. The ratio between the surface temperature and the fire temperature (n_w) can be estimated as 0.98 [10]. The temperature rise in steel bars is calculated using n_y at top of the section equals to zero, Table 2. To account for the effect of insulation, the temperature of top steel bars is limited to 100°C . This temperature was recorded during the fire test. The average temperature rise in concrete is calculated using the following steps:

1) z value of 0.183 for t of 3.0 hrs is calculated and is used to define the region boundaries $R1$ and $R2$ as shown in Fig. 14a [10].

2) The average temperatures, across the section width (b), are calculated for each region $R1$ and $R2$.

for $R1$: $T_{avg\ 1} = 383 + 690 n_y$, and

for $R2$: $T_{avg\ 2} = 1064 n_y$

3) The ambient temperature (34 °C) is added to the calculated average temperatures. Weighted average temperatures for t of 3.0 hrs are then calculated at the two boundaries of the variable T_{avg} distribution, Fig. 14b. For 3.0 hrs fire exposure, $T_{avg}(y = 0.5 m) = 1107$ °C and $T_{avg}(y = 0.317 m) = 384$ °C.

4) The constants (z_1 and z_2) of Eq. (1) are evaluated using values of T_{avg} at y_1 of 0.317 m and y_2 of 0.500 m for t of 3.0 hrs. The equation representing the average temperature distribution over the section height is:

$$T_{avg} = 68.4 e^{(5.57 - y)} \text{ for } 0.317 \leq y \leq 0.500 m$$

The average temperature distribution, calculated using Eq. (1), is shown in Fig. 14b.

9.2.Evaluation of Unrestrained Thermal Parameters

The unrestrained thermal parameters, ε_i and ψ_i , for the RC beam section at $t = 3$ hrs are calculated in this section.

1) The predicted T_{avg} distribution is used to plot ε_{th} distribution along y direction using Eq. (3), Fig. 14c.

An equivalent linear thermal strain ($\overline{\varepsilon_{th}}$) distribution is assumed as shown in Fig. 14c, i.e. $\overline{z}_3 = -0.018$ and $\overline{z}_4 = -0.010$. Fig. 14d shows the difference between $\overline{\varepsilon_{th}}$ and ε_{th} which represents the self-induced thermal strain (ε_{st}) for concrete and steel. Concrete compression forces and corresponding moments are calculated using expressions provided in Appendix I as follows:

a. For $y = 0.423 \rightarrow 0.500 m$ (constant ε_{th} and variable temperature)

$$\text{Eqs. (16) and (17)} \rightarrow C_{th L (v)} = -26,397 N, C_{th L (v)} \cdot y = -1.58 kN.m (y = 0.060 m)$$

b. For $y = 0.373 \rightarrow 0.423 m$ (variable ε_{th} and variable temperature)

Eqs. (18) and (19) $\rightarrow C_{th\ N(v)} = -46,699\ N$, $C_{th\ N(v)} \cdot y = -4.56\ kN.m$ ($y = 0.098\ m$)

c. For $y = 0.000 \rightarrow 0.212\ m$ (constant ε_{th} and constant temperature)

Eqs. (20) and (21) $\rightarrow C_{th\ L(c)} = -360,429\ N$, $C_{th\ L(c)} \cdot y = -154.70\ kN.m$ ($y = 0.429\ m$)

The self-induced thermal strain (ε_{st}) in steel bars is shown in Table 2. Fig. 14e shows that internal forces/moments in concrete and steel are in equilibrium, this means that the assumed $\overline{\varepsilon_{th}}$ is correct. ε_i and ψ_i corresponding to the assumed $\overline{\varepsilon_{th}}$ are -0.0057 and $-1.789 \times 10^{-5}\ 1/m$, respectively.

9.3. Evaluation of Degraded Flexural and Axial Stiffnesses

The analysis steps explained in the section 2 titled “Proposed Method” are conducted following a number of iterations. The BMD acting on the beam-column subassembly is shown Fig. 15c. The subassembly is divided into a number of segments based on the applied loads, fire exposure conditions, and reinforcement configuration. A sectional analysis is conducted for a section in each segment using the maximum applied axial force and flexural moment in this segment. The temperature of steel bars is considered uniform along the beam length because of the high thermal conductivity of steel material [1]. Therefore, the reduction in EI_{eff} and EA_{eff} for the unexposed beam end (SEC 4-4) are calculated based on heated steel bars and full concrete strength, i.e. at ambient temperature. The required steps to calculate EI_{eff} and EA_{eff} for the RC beam (SEC 2-2) are explained below.

1) a mechanical strain (ε_{cT}) distribution is assumed as shown by the heavy line in Fig. 16c (z_3 and z_4 of Eq. (14) are equal to 0.080 and 0.023 , respectively). The line does not intersect the concrete ($\varepsilon_{oT} + \varepsilon_{tr}$) curve, which indicates that concrete crushing does not occur at this mechanical strain.

2) ε_{st} is added to ε_{cT} as initial strains for concrete and steel. For the constant temperature zone in Fig. 16b ($0.000 < y < 0.317 \text{ m}$), ε_{st} is linear and can be added to ε_{cT} as shown in Figs. 14d and 16b. For the variable temperature zone ($y > 0.317 \text{ m}$), a linear ε_{st} distribution is assumed between $y = 0.317 \text{ m}$ and $y = 0.500 \text{ m}$. The modified ε_{cT} distribution is plotted as dashed lines in Fig. 16c.

3) concrete compressive forces and corresponding centroids are calculated using expressions provided in Appendix II as follows:

a. For $y = 0.317 \rightarrow 0.500 \text{ m}$ (variable temperature and $\varepsilon_{cT} < (\varepsilon_{oT} + \varepsilon_{tr})$)

$$\text{Eqs. (22) and (23)} \rightarrow C_{c o(v)} = -517,203 \text{ N}, C_{c o(v)} \cdot y = -57.60 \text{ kN.m} \quad (y = 0.111 \text{ m})$$

b. For $y = 0.307 \rightarrow 0.317 \text{ m}$ (constant temperature and $\varepsilon_{cT} < (\varepsilon_{oT} + \varepsilon_{tr})$)

$$\text{Eqs. (26) and (27)} \rightarrow C_{c o(c)} = -26,515 \text{ N}, C_{c o(c)} \cdot y = -4.26 \text{ kN.m} \quad (y = 0.161 \text{ m})$$

4) the self-induced thermal strain (ε_{st}) in steel bars is added to the mechanical strain (ε_{sT}) as shown in Table 2. The corresponding steel stresses are calculated using Eq. (10) and given in Table 2.

5) the calculated concrete and steel forces are in equilibrium with external forces. The mechanical curvature and center strain, ψ_{cT} and ε_{cT} , are predicted after 3.0 hrs ISO 834 fire exposure as $8.01 \times 10^{-5} 1/m$ and -0.0029 , respectively.

6) the flexural stiffness is calculated by dividing the applied external moment over the calculated ψ_{cT} . The reduced (effective) flexural stiffness, (EI_{eff}) , for the heated beam (SEC 2-2) is $3.47 \times 10^{12} \text{ N.mm}^2$. Similarly, the axial stiffness (EA_{eff}) is calculated by dividing the applied external force over the calculated center ε_{cT} . The effect of fire temperature on the axial stiffness of

the column is found to be negligible and not considered in the analysis. The RC beam is axially unrestrained at the roller support and no axial forces are generated in it during fire exposure.

9.4. Modeling and Analysis of RC subassembly

The RC beam-column subassembly is modeled using SAP2000 software [17], Fig. 17. The insulated beam-column joint is modeled using rigid links. A linear temperature loading is applied on the beam-column subassembly to induce the calculated unrestrained thermal deformation. The temperature values are determined based on ε_i , ψ_i , coefficient of thermal expansion (defined in Sap2000), and section depth. The assigned temperature loading for the beam section based on 1.0×10^{-5} coefficient of thermal expansion is a 598 °C uniform temperature in addition to a 1564 °C/m gradient temperature in y direction. The reduced EI_{eff} and EA_{eff} are assigned to each segment based on the applied moment and axial force.

9.5. Predicting the Fire Performance of the Beam-Column Subassembly

The frame model is analyzed using SAP2000 [17] and the BMD are obtained. Fig. 15a shows the primary moment due to vertical loads. A secondary moment is generated in the beam-column subassembly due to the restrained rotation at the beam-column joint, Fig. 15b. The total BMD is the summation of the primary and the secondary moments as shown in Fig. 15c.

The total BMD is used to re-evaluate EI_{eff} and EA_{eff} in the next iteration. This procedure is repeated till convergence for EI_{eff} and EA_{eff} is achieved and correct secondary moments are obtained. The convergence of the analyzed RC subassembly is achieved after three iterations.

Fig. 17 shows the deflected shape of the subassembly at 3 hrs ISO 834 fire exposure. The predicted beam deflection reasonably matches the measured deflection data experimentally.

The recorded vertical deflection at P_2 during fire test was 62 mm and the calculated deflection by SAP2000 is 68 mm (10% error). The measured rotation (θ) at the roller support was 2.1° while SAP2000 estimate 2.2° (5% error). The measured displacement (Δ_1) at the roller support during fire test was 30 mm and the calculated Δ_1 by SAP2000 is 37 mm (23% error).

Although the proposed method reasonably predicts Δ_2 and θ , the difference in the roller support horizontal displacement (Δ_1) between the analytical and test results is due to overestimating the elevated temperature in the bottom steel bars in the beam which affects the thermal deformation of the beam.

10. Validation Case 2 (Iding et al., 1977)

Iding et al. [5] has analytically investigated the behavior of RC frames during fire exposure. Fig. 19 shows a schematic of a single bay RC frame analyzed using FIRES-RC II, a comprehensive FE software developed at University of California, Berkeley. The beam and column dimensions are $355\text{ mm} \times 711\text{ mm}$. The frame was exposed to a 1.0 hr of ASTM-E119 standard fire over its entire length while supporting the vertical loads shown in Fig. 19. The frame was analyzed assuming siliceous concrete and a compressive strength of 27.6 MPa . The yield strength of the reinforcing bars was 275.8 MPa .

This frame is analyzed similar to previous validation case. The effect of elevated temperatures on the shear capacity, bond loss between steel bars and concrete, and concrete spalling were not considered in the analysis. Figs. 20 and 21 show sample calculations for the frame's column. The total moments and axial forces are estimated using the proposed method in this chapter and compared with the results of FIRES-RC II FE software, Fig. 22. The frame is analyzed using SAP2000 [17] and the predicted deformed shape is shown in Fig. 22. A good match is found

1 between the proposed method and the nonlinear FIRES-RC II FE software. The difference in
2 deformations can be due to using different material models.

4 **11. Summary and Conclusions**

5 The overall behavior of RC framed structures exposed to fire is studied in this paper. A
6 practical approach based on superimposing the effects of thermal expansion and material
7 degradation is introduced. The nonlinear thermal expansion is converted to an equivalent uniform
8 thermal distribution, which can be represented by the unrestrained thermal axial strain ε_i and
9 curvature ψ_i . The degradation effect in material strength is considered by accounting for the
10 reduction in the effective flexural and axial stiffnesses, EI_{eff} and EA_{eff} , respectively.
11 Mathematical expressions are derived to calculate internal compression force and corresponding
12 moment for the heated concrete. Structural engineers can use these expressions to predict the
13 unrestrained thermal deformation and the effective flexural and axial stiffnesses, i.e. EI_{eff} and
14 EA_{eff} .

15 The proposed method is validated by comparing its results with two case studies: a RC
16 beam-column subassembly tested during ISO834 fire and a single storey RC frame exposed to
17 ASTM E-119 fire. A good agreement is found between the experimental data and the results of
18 the proposed method for both case studies. Although the same concept of the proposed method can
19 be applied during natural fires, more experimental addressing RC frames subjected to non standard
20 fires is required to confirm its applicability.

21 **12. Acknowledgments**

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- 1
- 2
- 3

13. Nomenclature

A_1	factor used in calculating internal concrete force, equals to $e^{y_1 z_2}$
A_2	factor used in calculating internal concrete force, equals to $e^{y_2 z_2}$
b	column width in x direction
C_c	internal compression force in concrete
$C_{c o(v)}$	concrete compression force at $\varepsilon_{cT} \leq (\varepsilon_{oT} + \varepsilon_{tr})$ for variable T_{avg} distribution
$C_{c o(v)} \cdot y$	concrete moment about x axis at $\varepsilon_{cT} \leq (\varepsilon_{oT} + \varepsilon_{tr})$ for variable T_{avg} distribution
$C_{c u(v)}$	concrete compression forces at $\varepsilon_{cT} > (\varepsilon_{oT} + \varepsilon_{tr})$ for variable T_{avg} distribution
$C_{c u(v)} \cdot y$	concrete moment about x axis at $\varepsilon_{cT} > (\varepsilon_{oT} + \varepsilon_{tr})$ for variable T_{avg} distribution
$C_{c o(c)}$	concrete compression force corresponding to $\varepsilon_{cT} \leq (\varepsilon_{oT} + \varepsilon_{tr})$ for constant T_{avg}
$C_{c o(c)} \cdot y$	concrete moment about x axis at $\varepsilon_{cT} \leq (\varepsilon_{oT} + \varepsilon_{tr})$ for constant T_{avg}
$C_{c u(c)}$	concrete compression force corresponding to $\varepsilon_{cT} > (\varepsilon_{oT} + \varepsilon_{tr})$ for constant T_{avg}
$C_{c u(c)} \cdot y$	concrete moment about x axis at $\varepsilon_{cT} > (\varepsilon_{oT} + \varepsilon_{tr})$ for constant T_{avg}
f'_c	compressive strength for concrete at ambient temperature
f_y	yield strength of steel bars at ambient temperature
f'_{cT}	reduced compressive strength at elevated temperatures
f_{cT}	compression stress in heated concrete
f_{yT}	reduced yield strength of reinforcing bars at elevated temperatures
f_{sT}	compression or tension stress in heated steel bars
$(f_{cT})_{avg}$	average concrete compressive stresses
h	cross-section height
M_{app}	flexural moment
n_w	ratio between the surface temperature and the fire temperature
n_x and n_y	ratios between the internal and surface temperatures due to heating in the x and y directions, respectively
P_{app}	axial load
t	fire duration
t^*	equivalent fire duration assuming ISO 834 standard fire
T	temperature in degree Celsius [$1^\circ\text{F} = 1.8^\circ\text{C} + 32$]
T_{xy}	temperature rise at any point located at (x, y)
T_{avg}	algebraic average distribution along the section height
$T_{avg 1}$	average temperature for regions affected by heating from either left or right
$T_{avg 2}$	average temperature for regions not affected by heating from left or right
$T_{avg 3}$	average temperature due to heating from the left and right sides simultaneously
T_f	fire temperature
$T_f(\text{ISO})$	ISO 834 standard fire temperature at a modified fire duration t^*
x, y	horizontal and vertical coordinates for any point within the column/beam section, origin located at bottom left of the section
y_1, y_2	boundaries of internal concrete compression force measured in y direction
z	boundary of fire affected regions
z_1, z_2	constants of average temperature fitting equation, Eq. (1)
z_3, z_4	constants defining the linear variation of ε_{cT} in y direction, Eq. (14)
$\bar{z}_3, \bar{z}_4$	constants defining the linear variation of $\overline{\varepsilon_{th}}$ in y direction, Eq. (11)

1		
2	ε	total concrete strain at elevated temperatures
3	ε_{th}	unrestrained thermal strain of concrete
4	ε_{tr}	transient creep strain in concrete
5	ε_c	instantaneous stress-related strain
6	ε_{cT}	equivalent mechanical strain in concrete during fire exposure
7	$\overline{\varepsilon_{th}}$	equivalent linear thermal strain
8	ε_i	unrestrained center thermal axial strain
9	ε_{st}	self induced thermal strains
10	ε_{sT}	equivalent mechanical strain in steel during fire exposure
11	ε_{oT}	value of ε_c at peak stress f'_{cT}
12	ε_{uT}	ultimate compressive strain of concrete
13	$\Delta\varepsilon$	difference between ε_{uT} and $(\varepsilon_{oT} + \varepsilon_{tr})$ equals to 0.02
14	ψ_i	unrestrained thermal curvature
15	λ	axial or flexural load level
16	Γ	compartment time factor
17		
18		

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Table (1) –Parametric study cases

Col #	b (mm)	h (mm)	f'_c (MPa)	f_y (MPa)	ρ % (A_g)
C1	305	305	36.1	443.7	2.1
C2	400	400	30.0	400.0	1.5
C3	600	600	40.0	400.0	1.5
C4	400	700	50.0	400.0	1.0
C5	500	700	25.0	400.0	1.0

* all columns are analyzed up to 4 hrs of standard
ASTM-E119 fire exposure

Table (2)–Calculation of steel internal forces

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
A_s	x	y	T_{xy}	ε_{sT}	ε_{st}	$\varepsilon_{sT} + \varepsilon_{st}$	$\frac{f_{yT}}{f_y}$	$f[T, 0.001]$	$f[\begin{matrix} T, \\ (\varepsilon_{sT} - \varepsilon_p + 0.001) \end{matrix}]$	f_{sT}	P_s
(mm ²)	(mm)	(mm)	(°C)					(MPa)	(MPa)	(MPa)	(kN)
			Eq. (2a)	Eq. (20)	Eq. (15)		Eq. (13)	Eq. (14d)	Eq. (14e)	Eq. (14a,b)	$f_{sT} \times 4$
774	----	439	100	-0.018	-0.002	-0.019	0.96	182	309	-458	-354,202
1000	----	439	100	-0.018	-0.002	-0.019	0.96	182	309	-458	-457,625
774	61	61	712	0.012	0.009	0.014	0.21	35	93	122	94,131
774	154	61	494	0.012	-0.009	0.010	0.57	80	160	225	173,979

a) temperature contours

b) T_{avg} dist.

Fig. 1. Heat gradient after 1 hr ASTM-E119

Fig. 2. Sectional analysis approach for heated RC sections

Fig. 3. Sectional analysis approach for heated RC sections

Fig. 4. Proposed method for analysis of RC frames during fire exposure

Fig. 5. Variation of $(\varepsilon_{oT} + \varepsilon_{tr})$ and ε_{uT} at elevated temperatures

Fig. 6. Concrete stress-strain relationships at different elevated temperatures

Fig. 7. Calculation of concrete internal forces corresponding to ϵ_{st}

a) applied forces and moments b) T_{avg} dist. c) self-induced thermal strain (ϵ_{st}) d) mechanical strain (ϵ_{cT}) e) internal concrete stresses (f_{cT})_{avg} or forces

a) applied forces and moments b) T_{avg} dist. f) mechanical strain (ϵ_{cT}) g) internal concrete stresses (f_{cT})_{avg} or forces

Fig. 8. Calculation of concrete internal forces corresponding to ϵ_{cT}

Fig. 9. (M)-(ψ) diagrams for RC sections during fire

EI_{eff} using actual concrete strength $\times 10^{12}$ (N.mm)

a) flexural stiffness EI_{eff}

EA_{eff} using actual concrete strength $\times 10^8$ (N.mm)

b) axial stiffness EA_{eff}

Fig. 10. Effect of using T_{avg} on the stiffness of fire-exposed sections

EI_{eff} using Eurocode model $\times 10^{12}$ (N.mm)

a) flexural stiffness EI_{eff}

EA_{eff} using Eurocode model $\times 10^8$ (N.mm)

b) axial stiffness EA_{eff}

Fig. 11. Effect of $(\epsilon_{oT} + \epsilon_{tr})$ model on the stiffness of fire-exposed sections

Fig. 12. Effect of concrete nonlinearity on thermal parameters ϵ_i and ψ_i

a) heated section

b) T_{avg} dist.

c) equivalent $\overline{\epsilon}_{th}$ dist.

a) heated section

d) self induced thermal strain (ϵ_{st})

e) internal force equilibrium

Fig. 14. Thermal Deformation of beam SEC 2-2
[Dimensions in m, forces in N]

Fig. 15. Moment redistribution after 3 hrs ISO-834 fire exposure
[Moments are in $kN \cdot m$]

Fig. 16. Mechanical stress analysis of beam SEC 2-2
[Dimensions in mm , forces in N , moments in $N \cdot mm$]

Fig. 19. Thermal Deformation of column SEC 1-1
[Dimensions in m, forces in N]

Fig. 20. Mechanical stress analysis of column SEC 1-1
[Dimensions in m, forces in N, moments in N.mm]

Fig. 21. External forces/moments after 1 hr ASTM-E119 fire exposure
[Axial forces in kN , moments in $kN.m$]

Fig. 22. RC frame deflection after 1 hr ASTM-E119 fire exposure

Appendix I

$C_{c\ o(v)\ L}$, $C_{c\ o(v)\ L} \cdot y$, $C_{c\ o(v)\ L}$, and $C_{c\ o(v)\ L} \cdot y$ are calculated from Appendix II using $\bar{z}_3$ and $\bar{z}_4$ instead of z_3 and z_4 , respectively.

$$C_{th\ L(v)} = C_{c\ o(v)\ L} + \left(\frac{1}{30.465} \frac{b\ f'_c}{z_2^2 z_1} \right) [+ (A_2^2 - A_1^2) \times 1065\ z_1^3 (z_2 \varepsilon_{th}) \quad (16)$$

$$+ (A_2 - A_1) \times 3630767\ z_1^2 (-z_2 \varepsilon_{th})$$

$$+ \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \times 1210.981 \times 10^9 (-z_2 \varepsilon_{th})$$

$$- (y_2 - y_1) \times 3031.623 \times 10^5 z_1 (-\varepsilon_{th}) z_2^2]$$

$$C_{th\ L(v)} \cdot y = C_{c\ o(v)\ L} \cdot y + \left(\frac{1}{9.1395} \frac{b\ f'_c}{z_2^3 z_1} \right) [\quad (17)$$

$$+ (A_2^2 y_2 - A_1^2 y_1) \times 3195 \times 10^{-4} z_1^3 z_2 (\varepsilon_{th} z_2)$$

$$+ (A_2^2 - A_1^2) \times 15975 \times 10^{-5} z_1^3 (-\varepsilon_{th} z_2)$$

$$+ (A_2 y_2 - A_1 y_1) \times 1.08923 \times 10^3 z_1^2 z_2 (-\varepsilon_{th} z_2)$$

$$+ \left(\frac{y_2}{A_2} - \frac{y_1}{A_1} \right) \times 3.632943 \times 10^8 (-\varepsilon_{th} z_2^2)$$

$$+ \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \times 3.632943 \times 10^8 (-\varepsilon_{th} z_2)$$

$$+ (A_2 - A_1) \times 1.08923 \times 10^3 (z_1^2 \varepsilon_{th} z_2)$$

$$- (y_2^2 - y_1^2) \times 4.547435 \times 10^4 (-z_1 \varepsilon_{th} z_2^3)]$$

$$C_{th\,N(v)} = C_{c\,o(v)\,L} + \left(\frac{1}{30.465} \frac{b}{z_2^2 z_1} f'_c \right) [\quad \quad \quad] \quad (18)$$

$$+ (A_2^5 - A_1^5) \times 426 z_1^6 z_2 B_3$$

$$- (A_2^4 - A_1^4) \times 907700 z_1^5 z_2 B_3$$

$$+ (A_2^3 - A_1^3) \times 710 z_1^4 z_2 (B_2 + 142328.64 B_3)$$

$$+ (A_2^2 - A_1^2) \times 1065 z_1^3 z_2 (B_1 + 568535211.3 z_1^3 B_3 -$$

$$1704.58227 z_1^3 B_2)$$

$$+ (A_2 - A_1) \times 3630767 z_1^2 (-z_2 B_1 + 83.498 z_2 B_2)$$

$$+ \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \times 1210.981 \times 10^9 (-z_2 B_1)$$

$$- (y_2 - y_1) \times 3031.623 \times 10^5 z_1 z_2^2 (-B_1 - 3994.5 B_2)]$$

$$C_{th\ N(v)} \cdot y = C_{c\ o(v)\ L} \cdot y + \left(\frac{1}{9.1395} \frac{b\ f'_c}{z_2^3 z_1} \right) [\quad (19)$$

$$\begin{aligned} & + (A_2^5 y_2 - A_1^5 y_1) \times 1278 \times 10^{-4} z_1^6 B_3 z_2^2 \\ & - (A_2^5 - A_1^5) \times 2556 \times 10^{-5} z_1^6 B_3 z_2 \\ & - (A_2^4 y_2 - A_1^4 y_1) \times 272.308 z_1^5 B_3 z_2^2 \\ & + (A_2^4 - A_1^4) \times 68.077 z_1^5 B_3 z_2 \\ & + (A_2^3 y_2 - A_1^3 y_1) \times 30316.67 z_1^4 z_2^2 (7.026 \times 10^{-6} B_2 + B_3) \\ & - (A_2^3 - A_1^3) \times 10105.33 z_1^4 z_2 (7.026 \times 10^{-6} B_2 + B_3) \\ & + (A_2^2 y_2 - A_1^2 y_1) \times 181647333 z_1^3 z_2^2 (B_3 - 3 \times 10^{-6} B_2 + 1.7589 \times 10^{-9} B_1) \\ & + (A_2^2 - A_1^2) \times 15975 \times 10^{-5} z_1^3 (-B_1 z_2 + 1704.59 B_2 z_2 - 568536 \times 10^3 B_3 z_2) \\ & + (A_2 y_2 - A_1 y_1) \times 1.08923 \times 10^3 z_1^2 z_2 (-B_1 z_2 + 83.5 B_2 z_2) \\ & + \left(\frac{y_2}{A_2} - \frac{y_1}{A_1} \right) \times 3.63294 \times 10^8 (-B_1 z_2^2) \\ & + \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \times 3.63294 \times 10^8 (-B_1 z_2) \\ & + (A_2 - A_1) \times 1.08923 \times 10^3 (z_1^2 B_1 z_2 - 83.5 z_1^2 B_2 z_2) \\ & - (y_2^2 - y_1^2) \times 4.547435 \times 10^4 (-z_1 B_1 z_2^3 - 3994.5 z_1 B_2 z_2^3)] \end{aligned}$$

$$C_{th\ L(c)} = (-1 \times 10^6 b\ f'_{cT}) \left[\left(\frac{1}{2} \frac{z_3}{(\varepsilon_{oT} + \varepsilon_{tr})} \right) (y_2^2 - y_1^2) + \left(\frac{z_4 - \varepsilon_{th}}{(\varepsilon_{oT} + \varepsilon_{tr})} \right) (y_2 - y_1) \right] \quad (20)$$

$$C_{th\ L(c)} \cdot y = (-1 \times 10^3 b\ f'_{cT}) \left[\frac{1}{3} \left(\frac{z_3}{(\varepsilon_{oT} + \varepsilon_{tr})} \right) (y_2^3 - y_1^3) + \frac{1}{2} \left(\frac{z_4 - \varepsilon_{th}}{(\varepsilon_{oT} + \varepsilon_{tr})} \right) (y_2^2 - y_1^2) \right] \quad (21)$$

Appendix II

$$C_{c\ o(v)} = 2\ C_{c\ o(v)\ L} - C_{c\ o(v)\ N} \quad (22)$$

$$C_{c\ o(v)\ L} = \left(\frac{1}{30.465} \frac{b}{z_2^2 z_1} \frac{f'_c}{z_1} \right) [\quad (22a)$$

$$\begin{aligned} & - (A_2^2 y_2 - A_1^2 y_1) \times 1065\ z_1^3 z_3 z_2 \quad + (A_2 y_2 - A_1 y_1) \times 3630767\ z_1^2 z_3 z_2 \\ & + (A_2^2 - A_1^2) \times 1065\ z_1^3 (0.5\ z_3 - z_4 z_2) \quad + (A_2 - A_1) \times 3630767\ z_1^2 (z_4 z_2 - z_3) \\ & + \left(\frac{y_2}{A_2} - \frac{y_1}{A_1} \right) \times 1210.981 \times 10^9 z_3 z_2 \\ & + \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \times 1210.981 \times 10^9 (z_3 + z_4 z_2) \\ & - (y_2^2 - y_1^2) \times 1515.812 \times 10^5 z_1 z_3 z_2^2 \\ & - (y_2 - y_1) \times 3031.623 \times 10^5 z_1 z_4 z_2^2] \end{aligned}$$

$$\begin{aligned}
C_{c\ o(v)\ N} = & \left(\frac{5.21026\ b\ f'_c}{z_2^3 z_1^2} \right) [\\
& + \left(\frac{y_2^2}{A_2^2} - \frac{y_1^2}{A_1^2} \right) \times 1.5137262 \times 10^{14} z_3^2 z_2^2 \\
& + \left(\frac{y_2}{A_2^2} - \frac{y_1}{A_1^2} \right) \times 3.0274525 \times 10^{14} (z_3 z_4 z_2^2 + 0.5 z_3^2 z_2) \\
& + \left(\frac{1}{A_2^2} - \frac{1}{A_1^2} \right) \times 1.5137262 \times 10^{14} (z_3 z_4 z_2 + 0.5 z_3^2 + z_4^2 z_2^2) \\
& + \left(\frac{y_2^2}{A_2} - \frac{y_1^2}{A_1} \right) \times 7.579058 \times 10^{10} z_1 z_3^2 z_2^2 \\
& + \left(\frac{y_2}{A_2} - \frac{y_1}{A_1} \right) \times 1.5158116 \times 10^{11} (z_1 z_3^2 z_2 + z_1 z_3 z_4 z_2^2) \\
& + \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \times 1.5158116 \times 10^{11} (z_1 z_3 z_4 z_2 + 0.5 z_1 z_4^2 z_2^2 + z_1 z_3^2) \\
& - (A_2 y_2^2 - A_1 y_1^2) \times 532.5016 \times 10^3 z_1^3 z_3^2 z_2^2 \\
& + (A_2 y_2 - A_1 y_1) \times 1065.0032 \times 10^3 (z_1^3 z_3^2 z_2 - z_1^3 z_3 z_4 z_2^2) \\
& + (A_2 - A_1) \times 1065.0032 \times 10^3 (z_1^3 z_3 z_4 z_2 - 0.5 z_1^3 z_4^2 z_2^2 - z_1^3 z_3^2) \\
& + (y_2^3 - y_1^3) \times 3.025639 \times 10^8 z_1^2 z_3^2 z_2^3 \\
& + (y_2^2 - y_1^2) \times 9.0769173 \times 10^8 z_1^2 z_3 z_2^3 z_4 \\
& + (y_2 - y_1) \times 9.0769173 \times 10^8 z_1^2 z_4^2 z_2^3]
\end{aligned}
\tag{22b}$$

$$C_{c\ o(v)} \cdot y = 2 C_{c\ o(v)\ L} \cdot y - C_{c\ o(v)\ N} \cdot y \quad (23)$$

$$C_{c\ o(v)\ L} \cdot y = \left(\frac{1}{9.1395} \frac{b}{z_2^3 z_1} f'_c \right) [\quad (23a)$$

$$\begin{aligned} & -(A_2^2 y_2^2 - A_1^2 y_1^2) \times 3195 \times 10^{-4} z_1^3 z_3 z_2^2 \\ & + (A_2 y_2^2 - A_1 y_1^2) \times 1.08923 \times 10^3 z_1^2 z_3 z_2^2 \\ & + (A_2^2 y_2 - A_1^2 y_1) \times 3195 \times 10^{-4} z_1^3 z_2 (z_3 - z_4 z_2) \\ & + (A_2^2 - A_1^2) \times 15975 \times 10^{-5} z_1^3 (z_4 z_2 - z_3) \\ & + (A_2 y_2 - A_1 y_1) \times 1.08923 \times 10^3 z_1^2 z_2 (-2 z_3 + z_4 z_2) \\ & + \left(\frac{y_2^2}{A_2} - \frac{y_1^2}{A_1} \right) \times 3.632943 \times 10^8 z_3 z_2^2 \\ & + \left(\frac{y_2}{A_2} - \frac{y_1}{A_1} \right) \times 3.632943 \times 10^8 (2 z_3 z_2 + z_4 z_2^2) \\ & + \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \times 3.632943 \times 10^8 (2 z_3 + z_4 z_2) \\ & + (A_2 - A_1) \times 1.08923 \times 10^3 (2 z_1^2 z_3 - z_1^2 z_4 z_2) \\ & - (y_2^2 - y_1^2) \times 4.547435 \times 10^4 z_1 z_4 z_2^3 + \\ & - (y_2^3 - y_1^3) \times 3.031623 \times 10^4 z_1 z_3 z_2^3] \end{aligned}$$

$$\begin{aligned}
& C_{c\ o(v)\ N} \cdot \mathcal{Y} = \left(\frac{5.21026\ b\ f'_c}{z_2^4 z_1^2} \right) [\\
& \quad + (y_2^4 - y_1^4) \times 2.2692293 \times 10^5 z_3^2 z_2^4 z_1^2 \\
& \quad + \left(\frac{y_2^3}{A_2^2} - \frac{y_1^3}{A_1^2} \right) \times 1.5137262 \times 10^{11} z_3^2 z_2^3 \\
& \quad + (y_2^3 - y_1^3) \times 6.0512782 \times 10^5 z_3 z_2^4 z_4 z_1^2 \\
& \quad - (A_2 y_2^3 - A_1 y_1^3) \times 532.5016\ z_1^3 z_3^2 z_2^3 \\
& \quad + \left(\frac{y_2^3}{A_2} - \frac{y_1^3}{A_1} \right) \times 7.57906 \times 10^7 z_1 z_3^2 z_2^3 \\
& \quad + \left(\frac{y_2^2}{A_2^2} - \frac{y_1^2}{A_1^2} \right) \times 3.0274525 \times 10^{11} (z_3 z_4 z_2^3 + 0.75\ z_3^2 z_2^2) \\
& \quad + \left(\frac{y_2^2}{A_2} - \frac{y_1^2}{A_1} \right) \times 1.5158116 \times 10^5 (1.5\ z_1 z_3^2 z_2^2 + z_1 z_3 z_2^3 z_4) \\
& \quad + (A_2 y_2^2 - A_1 y_1^2) \times 1065.0032 (1.5\ z_1^3 z_3^2 z_2^2 - z_1^3 z_3 z_2^3 z_4) \\
& \quad + (y_2^2 - y_1^2) \times 4.5384586 \times 10^5 z_4^2 z_2^4 z_1^2 \\
& \quad + (A_2 y_2 - A_1 y_1) \times 532.5016 (4\ z_1^3 z_3 z_4 z_2^2 - 6\ z_1^3 z_3^2 z_2 - z_1^3 z_4^2 z_2^3) \\
& \quad + \left(\frac{y_2}{A_2^2} - \frac{y_1}{A_1^2} \right) \times 3.0274525 \times 10^{11} (z_3 z_4 z_2^2 + 0.75\ z_3^2 z_2 + 0.5\ z_4^2 z_2^3) \\
& \quad + \left(\frac{y_2}{A_2} - \frac{y_1}{A_1} \right) \times 0.757906 \times 10^8 (6\ z_1 z_3^2 z_2 + 4\ z_1 z_3 z_4 z_2^2 + z_1 z_4^2 z_2^3) \\
& \quad + \left(\frac{1}{A_2^2} - \frac{1}{A_1^2} \right) \times 0.756863 \times 10^{11} (1.5\ z_3^2 + z_4^2 z_2^2 + 2\ z_3 z_4 z_2) \\
& \quad + \left(\frac{1}{A_2} - \frac{1}{A_1} \right) 0.757906 \times 10^8 (4\ z_1 z_3 z_4 z_2 + z_1 z_4^2 z_2^2 + 6\ z_1 z_3^2) \\
& \quad + (A_2 - A_1) \times 532.5016 (-4\ z_1^3 z_3 z_4 z_2 + z_1^3 z_4^2 z_2^2 + 6\ z_1^3 z_3^2)]
\end{aligned}
\tag{23b}$$

$$\begin{aligned}
1 \quad C_{cu}(v) &= \left(\frac{1}{2.67609} \frac{b}{z_2^2} \frac{f'_c}{\Delta \varepsilon} \right) [\\
2 \quad &+ (y_2^2 - y_1^2) \times 1.3403194 \times 10^6 z_3 z_2^2 \\
3 \quad &+ (A_2 y_2 - A_1 y_1) \times 6.71083 \times 10^2 z_1 z_3 z_2 \\
4 \quad &+ (A_2^3 y_2 - A_1^3 y_1) \times 1.571667 \times 10^{-3} z_1^3 z_3 z_2 \\
5 \quad &- (A_2^2 y_2 - A_1^2 y_1) \times 4.01855 z_1^2 z_3 z_2 \\
6 \quad &- (A_2^4 - A_1^4) \times 2.97045 \times 10^{-8} z_1^4 z_2 \\
7 \quad &+ (A_2^3 - A_1^3) \times (-5.23889 \times 10^{-4} z_1^3 z_3 + 3.607829 \times 10^{-5} z_1^3 z_2 \\
8 \quad &+ 1.571667 \times 10^{-3} z_1^3 z_4 z_2) \\
9 \quad &+ (A_2^2 - A_1^2) \times 4.01855 (0.5 z_1^2 z_3 - z_1^2 z_4 z_2) \\
10 \quad &+ (A_2^2 - A_1^2) \times 7.191535 \times 10^{-2} z_1^2 z_2 \\
11 \quad &+ (A_2 - A_1) \times 6.71083 \times 10^2 (-z_1 z_3 + z_1 z_4 z_2) \\
12 \quad &- (A_2 - A_1) \times 80.97376 z_1 z_2] \\
13 \quad & \\
14 \quad & \\
15 \quad & \\
16 \quad & \\
17 \quad & \\
18 \quad & \\
19 \quad & \\
20 \quad & \\
21 \quad & \\
22 \quad &
\end{aligned} \tag{24}$$

$$\begin{aligned}
& C_{cu(v)} \cdot \mathcal{Y} = \left(\frac{1}{3.2113} \frac{-b}{z_2^3 \Delta \varepsilon} f'_c \right) [\\
& \quad -(y_2^3 - y_1^3) \times 1.0722555 \times 10^3 z_3 z_2^3 \\
& \quad + (y_2^2 - y_1^2) \times 32.16767 (-50 z_4 z_2^3 + z_2^3) \\
& \quad + (A_2^2 y_2^2 - A_1^2 y_1^2) \times 4.82226 \times 10^{-3} z_1^2 z_3 z_2^2 \\
& \quad - (A_2^3 y_2^2 - A_1^3 y_1^2) \times 1.886 \times 10^{-6} z_1^3 z_3 z_2^2 \\
& \quad - (A_2 y_2^2 - A_1 y_1^2) \times 0.8053 z_1 z_3 z_2^2 \\
& \quad + (A_2^2 y_2 - A_1^2 y_1) \times 4.82226 \times 10^{-3} (-z_1^2 z_3 z_2 + z_1^2 z_4 z_2^2) \\
& \quad - (A_2^2 y_2 - A_1^2 y_1) \times 8.62984 \times 10^{-5} z_1^2 z_2^2 \\
& \quad + (A_2^3 y_2 - A_1^3 y_1) \times 1.257334 \times 10^{-6} (z_1^3 z_3 z_2 - 1.5 z_1^3 z_4 z_2^2) \\
& \quad - (A_2^3 y_2 - A_1^3 y_1) \times 4.3294 \times 10^{-8} z_1^3 z_2^2 \\
& \quad + (A_2 y_2 - A_1 y_1) \times 0.8053 (-z_1 z_4 z_2^2 + 2 z_1 z_3 z_2) \\
& \quad + (A_2 y_2 - A_1 y_1) \times 9.7169 \times 10^{-2} z_1 z_2^2 \\
& \quad + (A_2^4 y_2 - A_1^4 y_1) \times 3.56454 \times 10^{-1} z_1^4 z_2^2 \\
& \quad + (A_2^3 - A_1^3) \times 4.19113 \times 10^{-7} (1.5 z_1^3 z_4 z_2 - z_1^3 z_3) \\
& \quad + (A_2^3 - A_1^3) \times 1.4431 \times 10^{-8} z_1^3 z_2 \\
& \quad + (A_2 - A_1) \times 0.8053 (-0.5 z_1 z_3 + z_1 z_4 z_2) \\
& \quad - (A_2 - A_1) \times 9.7169 \times 10^{-2} z_1 z_2 \\
& \quad + (A_2^2 - A_1^2) \times 2.4111 \times 10^{-3} (z_1^2 z_3 - z_1^2 z_4 z_2) \\
& \quad + (A_2^2 - A_1^2) \times 4.315 \times 10^{-5} z_1^2 z_2 \\
& \quad - (A_2^4 - A_1^4) \times 8.911 \times 10^{-12} z_1^4 z_2]
\end{aligned} \tag{25}$$

$$C_{c \ o \ (c)} = (-1 \times 10^6 \ b \ f'_{cT}) \left[-\frac{1}{3} \frac{z_3^2}{\varepsilon_{oT}^2} (y_2^3 - y_1^3) + \left(\frac{z_3}{\varepsilon_{oT}} - z_4 \frac{z_3}{\varepsilon_{oT}^2} \right) (y_2^2 - y_1^2) + \left(2 \frac{z_4}{\varepsilon_{oT}} - \frac{z_4^2}{\varepsilon_{oT}^2} \right) (y_2 - y_1) \right] \quad (26)$$

$$C_{c \ o \ (c)} \cdot y = (-1 \times 10^3 \ b \ f'_{cT}) \left[-\frac{1}{4} \frac{z_3^2}{\varepsilon_{oT}^2} (y_2^4 - y_1^4) + \frac{1}{3} \left(2 \frac{z_3}{\varepsilon_{oT}} - 2 z_4 \frac{z_3}{\varepsilon_{oT}^2} \right) (y_2^3 - y_1^3) + \left(\frac{z_4}{\varepsilon_{oT}} - \frac{z_4^2}{2 \varepsilon_{oT}^2} \right) (y_2^2 - y_1^2) \right] \quad (27)$$

$$C_{c \ u \ (c)} = \left(\frac{-1 \times 10^6 \ b \ f'_{cT}}{\Delta \varepsilon} \right) \left[\varepsilon_{uT} (y_2 - y_1) - \frac{1}{2} z_3 (y_2^2 - y_1^2) - z_4 (y_2 - y_1) \right] \quad (28)$$

$$C_{c \ u \ (c)} \cdot y = \left(\frac{-1 \times 10^3 \ b \ f'_{cT}}{\Delta \varepsilon} \right) \left[-\frac{1}{3} z_3 (y_2^3 - y_1^3) + \frac{1}{2} (\varepsilon_{uT} - z_4) (y_2^2 - y_1^2) \right] \quad (29)$$