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Multi-Trace Matrix Models from Noncommutative Geometry

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Abstract

Dirac ensembles are finite dimensional real spectral triples where the Dirac operator is allowed to vary within a suitable family of operators and is assumed to be random. The Dirac operator plays the role of a metric on a manifold in the noncommutative geometry context of spectral triples. Thus, integration over the set of Dirac operators within a Dirac ensemble, a crucial aspect of a theory of quantum gravity, is a noncommutative analog of integration over metrics.

Dirac ensembles are closely related to random matrix ensembles. In order to determine properties of specific Dirac ensembles, we use techniques from random matrix theory such as Schwinger-Dyson equations and the recently introduced bootstrapping. In particular, we determine the relations between the second moments of our models and parameters of the models. All the other moments can be represented in terms of the coupling constants and the second moments using the set of recursive relations called the Schwinger-Dyson equations. Additionally, explicit relations for higher mixed moments are found.

We also introduce a new technique, the moment-coefficient method, to solve multi-trace matrix models in the large N limit. This technique is compatible with several well-known approaches to solving single matrix ensembles. Using this technique, we study Dirac ensembles in the so called "double scaling limit". It is significant to note that, as predicted by conformal field theory, the asymptotics of the partition function of these models is used to construct a solution for the Painlevé I differential equation. Moreover, results of this thesis are also justified numerically by Monte Carlo Metropolis-Hastings simulations.

Keywords: Noncommutative Geometry, Finite Dimensional Spectral Triples, Random Matrix Theory, Dirac Ensembles, Schwinger-Dyson Equations, Multi-Trace Matrix Models, Bootstrapping Technique, Phase Transition, Double Scaling Limit, Monte Carlo Metropolis-Hastings Algorithm.

Summary for Lay Audience

The nature of a theory of spacetime in quantum gravity is constrained by the existence of the Planck length. In fact, using the Heisenberg uncertainty principle and Einstein's general relativity theory, it can be shown that spacetime cannot be a smooth manifold at Planck length since black holes can emerge in very small scales. Several options has been suggested to replace the conventional spacetime. One such suggestion is a noncommutative Riemannian manifold in the sense of spectral triples. In particular for finite dimensional spectral triples, the role of the metric is played by a Dirac operator.

A random matrix is a matrix whose entries are random variables. The main goal of this study is to find the probability distribution function of eigenvalues of certain random matrices that appear in some toy models of quantum gravity based on noncommutative geometry.

In this thesis, in order to find the moments of random Dirac operator numerically, we use the bootstrapping method. The bootstrapping method is based on a set of recursive relations called "Schwinger-Dyson equations", and some positivity constraints that are satisfied by the moments of eigenvalue distributions of such matrices. We are able to find higher moments of the model in terms of the second moment, and we calculate the second moment numerically using the bootstrapping method.

In order to solve matrix ensembles when the size of the matrix reaches infinity, we offer a brand-new technique called the moment-coefficient method. This method is compatible with several well-known methods for solving single matrix ensembles. In particular, it is used to analyze Dirac ensembles at the double scaling limit, which occurs when the model's order parameter approaches the critical value and the size of the matrix reaches infinity. We show that the so called free energy of our models (logarithm of the partition function) can be constructed using solutions of the Painlevé I differential equation. Additionally, Monte Carlo Metropolis-Hastings simulations are used to numerically support the outcomes of this thesis.

Co-Authorship

This thesis incorporates material that is result of joint research, as follows:

- Chapter 2 is based on the paper: H. Hessam, M. Khalkhali, N. Pagliaroli, and L. S. Verhoeven. From Noncommutative Geometry to Random Matrix Theory. J. Phys. A: Math. Theor. 55 413002, 2022.

  The first three authors played the main roles in the research.

  Under the supervision of professor Khalkhali, I solved a cubic Dirac ensemble by bootstrapping method. In addition, the results in bootstrapping the quartic Dirac ensemble, the spectral phase transition of models, and Liouville quantum gravity have been done by Nathan Pagliaroli and me.

- Chapter 3 is based on the paper: H. Hessam, M. Khalkhali, and N. Pagliaroli. Bootstrapping Dirac Ensembles. J. Phys. A: Math. Theor. 55 335204, 2022.

  The authors played equal roles in the research.

  Under the supervision of professor Khalkhali, Nathan and I found the Schwinger-Dyson equations of models. I proved that the dimension of the search space for a quartic Dirac ensemble is equal to one. Thereafter, we solved the model together.

- Chapter 4 is based on the paper: H. Hessam, M. Khalkhali, and N. Pagliaroli. Double scaling limits of Dirac ensembles and Liouville quantum gravity. arXiv:2204.14206, 2022.

  The authors played equal roles in the research.

  Under the supervision of professor Khalkhali, Nathan and I introduced a new technique to solve multi-trace single matrix models. Then, we prove the existence of the double scaling limit for Dirac ensembles and found the corresponding critical values for coupling constants in different models.

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Chapter 1

Introduction to Matrix Ensembles

1.1 Beginning of random matrix theory

In the introduction of three papers forming chapters 2, 3, and 4 of this dissertation, we shall give further background to random matrix theory. In this chapter, we shall recall some elementary results without much detail. Further explanatory remarks can be found in appendices A, B, and C at the end of this thesis.

A random matrix is a matrix-valued random variable. That is a matrix whose entries are random variables. Random matrices first appeared in statistics in 1928, when Wishart [12] generalized the chi-squared law to multivariate random variables. Let us consider $N \geq p$ random p-component vectors and form a rectangular matrix X with N rows and p columns.

$$X = (X_1, X_2, \cdots, X_N)^T, \quad X_i = (x_{i1}, x_{i2}, \cdots, x_{ip}), \quad x_{ij} \in \mathbb{R}. \tag{1.1}$$

The X_i's are independent, identically distributed (i.i.d.) random vectors with a multivariate normal distribution $N_p(0, V)$ with zero mean and a covariance matrix V of size p:

$$N_p(0, V)(X_i) = \frac{e^{-\frac{1}{2}\operatorname{Tr} X_i V^{-1} X_i^T}}{(2\pi)^{\frac{p}{2}} \sqrt{\det V}} \prod_{j=1}^{p} dx_{ij}. \tag{1.2}$$

We are interested in the properties of the correlation matrix

$$S = X^T X,$$

which is a symmetric (positive semi-definite) square matrix of size p. The space of such matrices

$$W_p(V, n) = \{X^T X \mid X \in \mathbb{R}^{n \times p}\} \tag{1.3}$$

equipped with the pushforward measure is called Wishart ensemble. The probability law for the correlation matrix S is given by the Wishart distribution:

$$P(M) = \frac{(\det M)^{\frac{N-p-1}{2}} e^{-\frac{1}{2}\operatorname{Tr} V^{-1}M}}{2^{\frac{1}{2}Np}(\det V)^{\frac{N}{2}}\Gamma_p(\frac{N}{2})}. \tag{1.4}$$

Later, in 1967, Marchenko and Pastur [6] worked on the large N behaviour of $p\times p$ matrices $S_N = X^T X$, where the entries of X are i.i.d. variables with a normal distribution $N(0,\sigma^2)$. S_N is a non-negative symmetric random matrix. So its eigenvalues are non-negative real random variables:

$$0 \leq \lambda_1(S_N) \leq \lambda_2(S_N) \leq \cdots \leq \lambda_p(S_N).$$

Define the empirical distribution of eigenvalues on $\mathbb{R}$ as follows:

$$\mu_{N,p}(S_N) = \frac{1}{p}\sum_{i=1}^{p} \delta\left(x - \frac{\lambda_i(S_N)}{\sqrt{N}}\right) \in \mathfrak{P}(\mathbb{R}), \tag{1.5}$$

where $\mathfrak{P}(\mathbb{R})$ is the set of Borel probability measure on $\mathbb{R}$. Equivalently, for any measurable subset $A \subseteq \mathbb{R}$:

$$\mu_{N,p}(A) = \frac{1}{p}\#\left\{ i \;\middle|\; \frac{\lambda_i}{\sqrt{N}} \in A \right\}. \tag{1.6}$$

The large N limit of this measure is called the spectral distribution of the model and was found by Vladimir Marchenko and Leonid Pastur.

Theorem 1.1.1 (Marchenko–Pastur 1967 [6]) *Let $p, N \to \infty$ such that $\frac{p}{N} \to \lambda \in (0,\infty)$. Then the empirical spectral density function $\mu_{N,p}$ converges weakly in distribution to μ, where*

$$\mu(x) = \begin{cases} \left(1 - \frac{1}{\lambda}\right)\delta_0 + \nu(x) & \text{if } \lambda > 1 \\ \nu(x) & \text{if } 0 \leq \lambda \leq 1 \end{cases} \tag{1.7}$$

and

$$\nu(x) = \frac{1}{2\pi\sigma^2}\frac{\sqrt{(b-x)(x-a)}}{\lambda x}\chi_{[a,b]} \tag{1.8}$$

with $a = \sigma^2(1-\sqrt{\lambda})^2$ and $b = \sigma^2(1+\sqrt{\lambda})^2$.

1.1.1 Gaussian and Wigner ensembles

In the middle of the 1950s, Wigner published a number of studies that introduced random matrices to physics [9, 10, 11]. Wigner modeled the Hamiltonians of heavy nuclei by large random Hermitian or symmetric matrices. He studied the statistical distribution of the variable

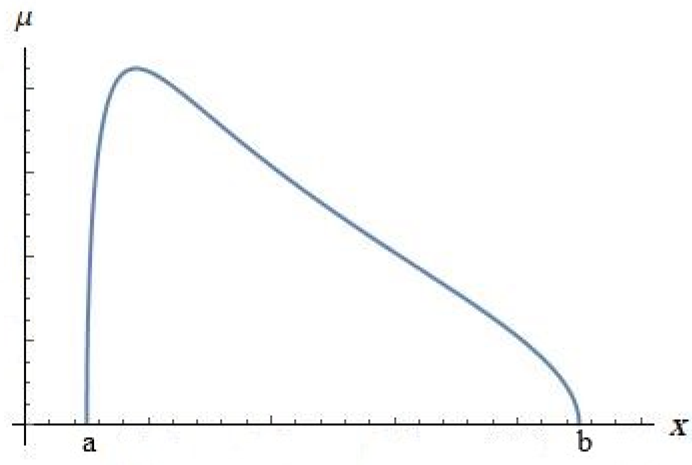

Figure 1.1: The graph of the Marchenko–Pastur law.

s (the distance between adjacent energy levels). If the energy levels were uncorrelated random numbers, the variable s would follow the Poisson distribution. But experimental observation showed the probability density is different, and very well approximated (within 1%) by the Wigner surmise:

$$P(s) = C_\beta s^\beta e^{-a_\beta s^2}, \tag{1.9}$$

where the parameter $\beta \in \{1, 2, 4\}$ is determined by the symmetries of the problem, and values of C_β and a_β are determined by:

$$\int_0^\infty P(s)ds = 1, \qquad \int_0^\infty sP(s)ds = 1. \tag{1.10}$$

Consider two independent families of i.i.d. real, or complex, valued random variables $\{Z_{i,j}\}_{1\le i<j\le N}$ with the normal distribution $N(0, 1)$ and $\{Y_i\}_{1\le i<N}$ with the normal distribution $N(0, 2)$. Now consider the symmetric, or Hermitian, $N \times N$ matrix X_N with entries:

$$X_N(i, j) = X_N^*(j, i) = \begin{cases} \frac{Z_{i,j}}{\sqrt{N}} & \text{if } i < j \\ \frac{Y_i}{\sqrt{N}} & \text{if } i = j. \end{cases} \tag{1.11}$$

Definition 1.1.1 *The space of such matrices equipped with the joint measure of the above random variables is called the Gaussian matrix ensemble.*

We can generalize the Gaussian ensembles by relaxing the underlying normal distribution restriction of its random variables.

Definition 1.1.2 *A random symmetric or Hermitian matrix with i.i.d. entries with mean* 0 *and variance* 1 *for off-diagonal elements and mean* 0 *and variance* 2 *for diagonal entries with the corresponding joint distribution is called Wigner ensemble.*

The Wigner surmise is also a very good approximation of the large size limit of the probability density for the distance between consecutive eigenvalues of Wigner matrices. In spite of its simplicity, this is actually a quite deep result. The probability of two eigenvalues being very close to each other is very small, i.e., the eigenvalues tend to repel each other (but not too much). It is like birds perching on an electric wire, or parked cars on a street.

The Wigner surmise for level spacing (the distribution of distances between adjacent eigenvalues) of symmetric Wigner matrices has the following form:

$$P(s) = \frac{1}{2}\pi s e^{-\frac{1}{4}\pi s^2}. \tag{1.12}$$

It is exact for 2 by 2 matrices and a good approximation for the actual distribution for real symmetric matrices of any dimension. Figure 1.2 is the sampling of difference of eigenvalues of matrices of size two and the Wigner surmise.

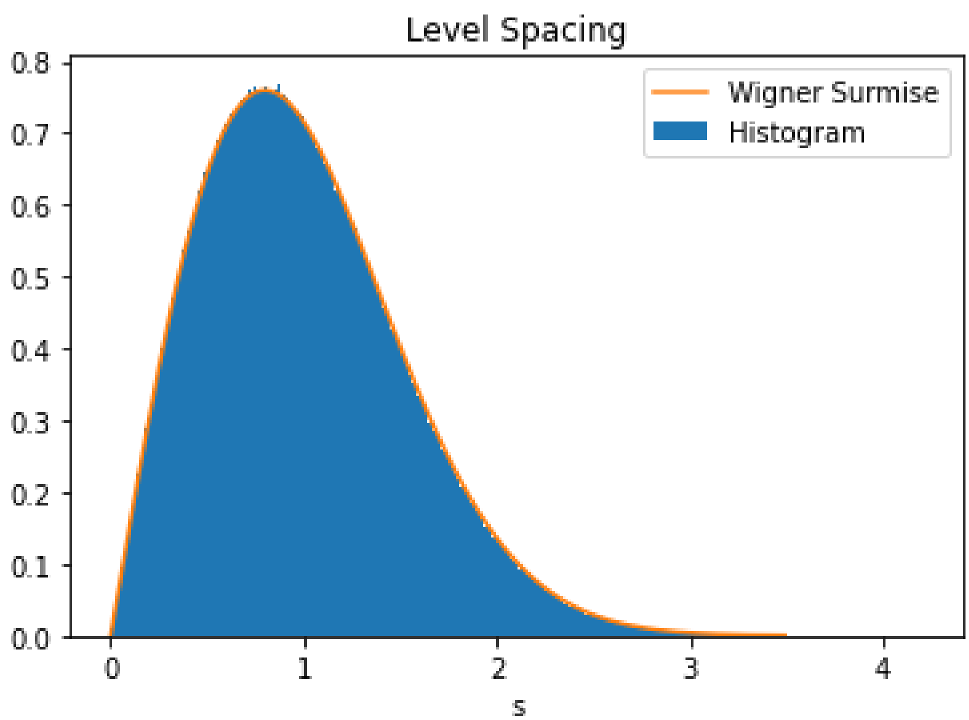

Figure 1.2: The histogram of level spacing of 4000000 symmetric matrices of size 2 with the proper normalization plotted with the Wigner surmise.

Recall that the empirical distribution of the eigenvalues is defined as:

$$\mu_N(X_N) = \frac{1}{p}\sum_{i=1}^{p} \delta\left(x - \lambda_i(X_N)\right) \in \mathfrak{P}(\mathbb{R}). \tag{1.13}$$

Note that since we have already normalized our matrices by the factor $\frac{1}{\sqrt{N}}$, we don't need to divide the eigenvalues by $\sqrt{N}$ in the empirical spectral distribution. Define the semicircle distribution (or law) as the probability distribution $\sigma(x)dx$ on $\mathbb{R}$ with the density function:

$$\sigma(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,\chi_{[-2,2]}. \tag{1.14}$$

Theorem 1.1.2 (Wigner 1955 [9]) *For a Wigner ensemble, the empirical measure μ_N converges weakly, in probability, to the semicircle distribution.*

We can say the Wigner's semicircle law is universal in the sense that the eigenvalue distribution of a symmetric or Hermitian matrix with i.i.d. entries, properly normalized, converges to the semicircle distribution regardless of the underlying distribution of the matrix entries. Figure 1.3 is the sampling of eigenvalues of symmetric matrices of size 1000 and the Wigner semicircle distribution.

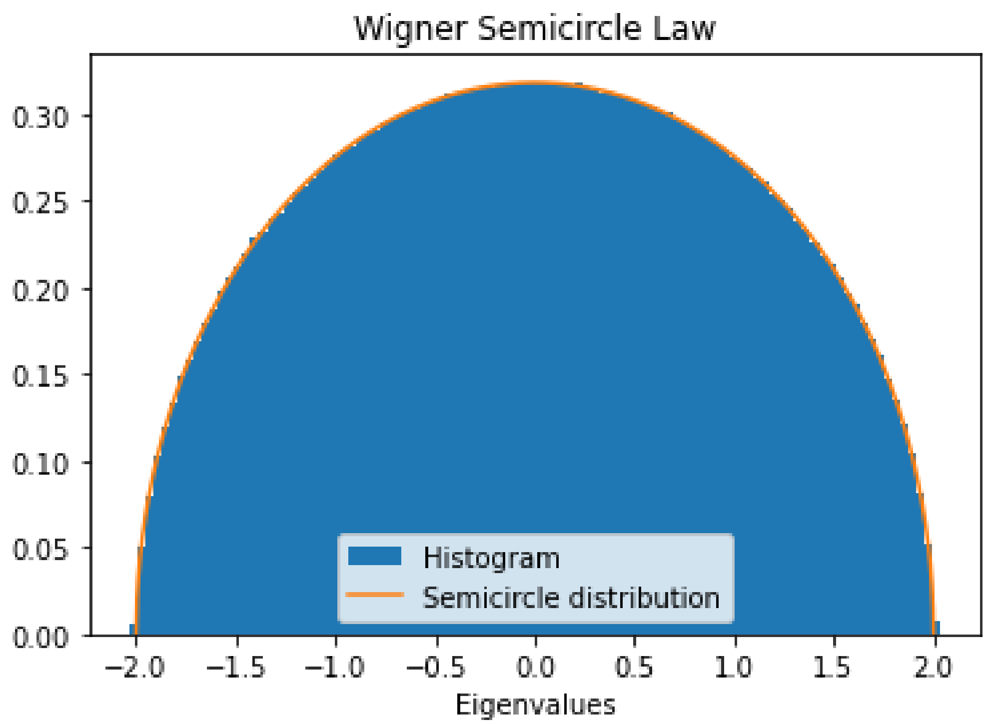

Figure 1.3: The histogram of the eigenvalues of 1000 symmetric matrices of size 1000 with the proper normalization compared with the Wigner semicircle distribution.

1.2 Unitary invariant matrix ensembles

There is a natural generalization of Gaussian ensembles called "*invariant ensembles*". A certain class of invariant ensembles, called Hermitian unitary invariant ensembles, play a role in many areas of mathematics and physics, such as orthogonal polynomials [2], KdV hierarchy [4], quantum gravity [5], Witten's conjecture [13] and many other interesting places. We will now define the Hermitian (unitary invariant) matrix models that this thesis is based on.

Definition 1.2.1 *A Hermitian unitary invariant matrix ensemble consists of:*

- *A unitary group U acting on a space of Hermitian matrices $\mathcal{H}$.*
- *A probability measure $d\mu$ on $\mathcal{H}$ that must be invariant under U and has the form of $e^{-\mathcal{V}(H)}dH$, where $\mathcal{V}(H)$ is the potential of the model.*

The partition function of the model is then defined as

$$\mathcal{Z} = \int_{\mathcal{H}} e^{-\mathcal{V}(H)} dH. \tag{1.15}$$

In general, the measure $d\mu = e^{-\mathcal{V}(H)}dH$ is not normalized, i.e., $\mathcal{Z} \neq 1$. However, we can define the expectation of a real, or complex values function f defined on $\mathcal{H}$ with respect to the ensemble $\mathcal{H}$ as:

$$\underbrace{\langle f(H)\rangle}_{physics\ notation} = \underbrace{\mathbb{E}\left(f\left(H\right)\right)}_{math\ notation} := \frac{1}{\mathcal{Z}}\int_{\mathcal{H}} f(H)\, d\mu(H). \tag{1.16}$$

Most of the models we have in this thesis are unitary invariant matrix models, and we will use the physics notation for expectation value.

Consider the space of $N \times N$ Hermitian matrices, denoted $\mathcal{H}_N$, and let $H \in \mathcal{H}_N$. Since H is a complex matrix, we may write the (i, j)-entry of the matrix H for $i \leq j$ as:

$$\begin{aligned} (H)_{ij} &:= H_{ij} = x_{ij} + i\, y_{ij} && 1 \leq i < j \leq N \\ (H)_{ii} &:= H_{ii} = x_{ii} && 1 \leq i \leq N, \end{aligned}$$

where x_{ij}, y_{ij}, and x_{ii} are real. Furthermore, the other entries can be written in terms of these elements. Therefore, the vector space $\mathcal{H}_N$ is isomorphic to $\mathbb{R}^{N^2}$ as a real subspace of vector space of complex matrices. Thus,

$$dH = \prod_{i=1}^{N} dx_{ii} \prod_{1\leq i<j\leq N} dx_{ij}\, dy_{ij} \tag{1.17}$$

forms the Lebesgue measure on $\mathcal{H}_N$. The unitary group $U(N)$ acts by conjugation on $\mathcal{H}_N$. Any $U(N)$ invariant probability measure $P(H)\, dH$ has the property that $P(H)$ must be invariant under the conjugate action of $U(N)$:

$$\mathcal{P}(UHU^*) = \mathcal{P}(H) \qquad \text{for } U \in U(N). \tag{1.18}$$

That follows that the probability law $P(H)\,dH$, can be written as:

$$P(H)\,dH = e^{-\mathcal{V}(H)}dH, \tag{1.19}$$

where $\mathcal{V}(H)$ is a trace polynomial function in H.

1.2.1 Gaussian unitary ensembles (GUE)

A special case of the Hermitian matrix model is the Gaussian unitary ensemble (GUE). It is described by the partition function:

$$\mathcal{Z}_{GUE} = \int_{\mathcal{H}_N} e^{-\frac{N}{2}\operatorname{Tr} H^2} dH. \tag{1.20}$$

Let H_{ij} denote the entry of $H \in \mathcal{H}_N$ in row i and column j. We get:

$$\begin{aligned}\operatorname{Tr} H^2 &= \sum_{i,j=1}^{N} H_{ij}H_{ji} = \sum_{i=1}^{N} H_{ii}^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{N} H_{ij}H_{ji} = \sum_{i=1}^{N} H_{ii}^2 + 2\sum_{1\le i<j\le N} H_{ij}\overline{H_{ij}}\\ &= \sum_{i=1}^{N} H_{ii}^2 + 2\sum_{1\le i<j\le N} |H_{ij}|^2 = \sum_{i=1}^{N} x_{ii}^2 + 2\sum_{1\le i<j\le N} (x_{ij}^2 + y_{ij}^2).\end{aligned} \tag{1.21}$$

It follows that x_{ij} and y_{ij} for $1 \le i < j \le N$, have the standard normal distribution $N(0,1)$, and x_{ii} for $1 \le i \le N$, has the normal distribution $N(0,2)$, and they are all independent. This means that the Gaussian unitary ensemble is indeed a Gaussian Wigner ensemble. In 1960, Porter and Rosenzweig proved the following theorem.

Theorem 1.2.1 (Porter and Rosenzweig [8]) *The only Wigner ensemble on the space of Hermitian matrices $\mathcal{H}_N$ which is also unitary invariant is the GUE.*

1.2.2 Single matrix models

We would like to consider potentials with additional terms to the Gaussian. We can write a joint probability distribution on the matrix entries:

$$d\mu_N = \frac{1}{\mathcal{Z}_N} e^{-N\operatorname{Tr} V(H)} dH, \tag{1.22}$$

where $V(H)$ is some polynomial in H and $\frac{1}{\mathcal{Z}_N}$ is the normalization factor that may differ from model to model. This is what we refer to as a single trace matrix model. By a multi-trace

The space of matrix ensembles

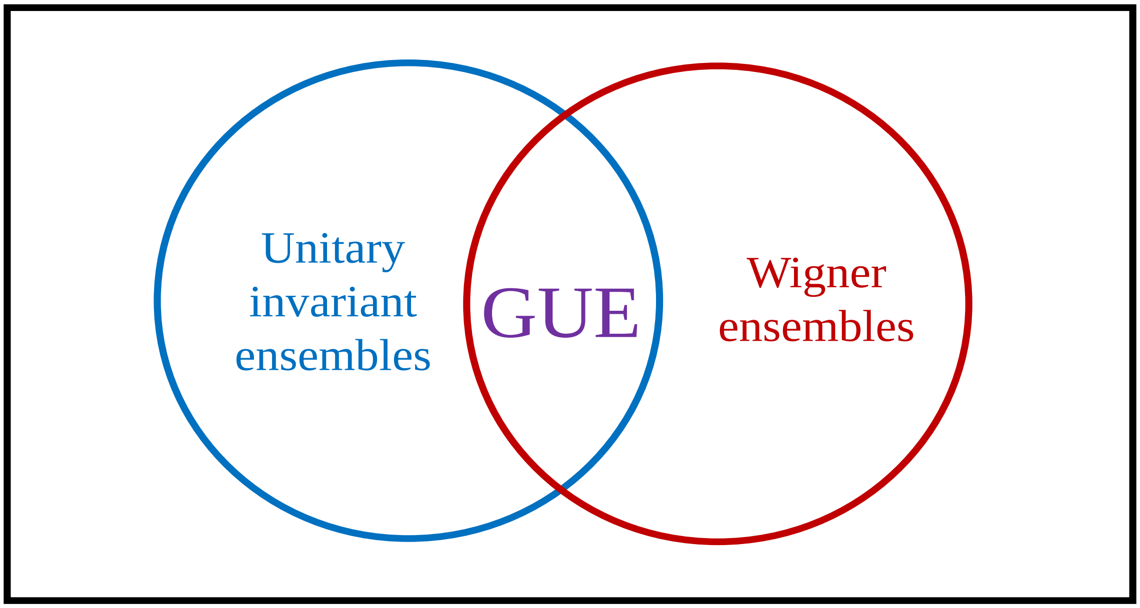

Figure 1.4: The Venn diagram of the space of matrix ensembles.

matrix model, we mean a matrix ensemble with the joint probability distribution function on the matrix entries:

$$d\mu_N = \frac{1}{\mathcal{Z}_N}\exp\left[-\sum_{g=0}^{\mathfrak{g}}\sum_{i=2}^{d}\sum_{p\vdash i} t_p^g\left(\frac{N}{t}\right)^{2-2g-|p|}\prod_{j\in p}\frac{\operatorname{Tr}H^j}{j}\right]dH.$$

the second sum is over partitions of i and the product is over addends of a partition. The t_p are coupling constants that allow us to fine tune parameters of these models. For example, let $d=3$ and $\mathfrak{g}=0$, then the potential becomes:

$$t_2\frac{N}{t}\operatorname{Tr}H^2 + t_{1,1}\operatorname{Tr}H\operatorname{Tr}H + t_3\frac{N}{t}\operatorname{Tr}H^3 + t_{1,2}\operatorname{Tr}H\operatorname{Tr}H^2 + t_{1,1,1}\left(\frac{N}{t}\right)^{-1}\operatorname{Tr}H\operatorname{Tr}H\operatorname{Tr}H.$$

If one sets all the coupling constants of nontrivial partitions equal to zero, we are left with a single trace model. When d is even and $t_d > 0$, it is not hard to see that the integral:

$$\mathcal{Z}_N := \int_{\mathcal{H}_N} d\mu_N,$$

known as the partition function, is convergent. Such an integral is called a *convergent matrix model*. However, regardless of the parity of d, one can define a *formal matrix integral*. To this end, let $t_2 = 1/2$, and define the formal sum:

$$\mathcal{Z}_{formal} := \sum_{g=0}^{\mathfrak{g}}\sum_{\substack{n_p^g=0\\ p\vdash i\\ 2\le i\le d}}^{\infty'}\int_{\mathcal{H}_N}\left(\prod_{g=0}^{\mathfrak{g}}\prod_{\substack{2\le i\le d\\ p\vdash i}}^{d'}\frac{(t_p^g)^{n_p^g}}{n_p^g!}\left(\prod_{j\in p}\frac{\operatorname{Tr}H^j}{j}\right)^{n_p^g}\right)d\mu_N^{GUE},$$

Informally one can think of this as Taylor expanding each exponential term of the integral and then swapping the order of integration and summation. What we are left with is a formal summation of Gaussian integrals.

Interestingly enough when physicists first worked with matrix integrals they did not even distinguish between convergent and formal models. This is because it is very often the case that the moments and cumulants of a formal matrix model and its convergent counterpart (if it exists) satisfy the same set of recursive relations, which under certain circumstances can be shown to have a unique solution. For more details on the relationship between formal and convergent matrix models see [3]. The simplest cases are single trace models with probability law:

$$\mathcal{P}(H) = \frac{1}{\mathcal{Z}} e^{-N \operatorname{Tr} \mathcal{V}(H)}, \tag{1.23}$$

where $\mathcal{V}(H)$ is a polynomial potential,

$$\mathcal{V}(x) = \frac{t_2}{2} x^2 + \frac{t_3}{3} x^3 + \cdots + \frac{t_n}{n} x^n. \tag{1.24}$$

and the partition function of the model is:

$$\mathcal{Z} = \int_{\mathcal{H}_N} e^{-N \operatorname{Tr} \mathcal{V}(H)} dH = \int_{\mathcal{H}_N} e^{-N \sum_{k=2}^{n} \frac{t_k}{k} \operatorname{Tr}(H^k)} dH. \tag{1.25}$$

1.2.3 Joint distribution of eigenvalues

A Hermitian matrix H has real eigenvalues, and can be diagonalized by a unitary transformation $U \in U(N)$.

$$H = U \Lambda U^{-1} \quad \text{with } \Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_N). \tag{1.26}$$

The diagonalization (1.26) of $H \in \mathcal{H}_N$ is not unique. In fact, H can be also diagonolized by U', where U' is U multiply by any elements of the following sets.

- Set of diagonal unitary matrices $U(1)^N \subset U(N)$.
- Set of permutation matrices $\mathcal{G}_N \subset U(N)$.

Therefore, one can show:

$$\mathcal{H}_N \simeq \frac{\frac{U(N)}{U(1)^N} \times \mathbb{R}^N}{\mathcal{G}_N}. \tag{1.27}$$

Using the diagonalization (1.26), the Lebesgue measure dH can be written in terms of the measure on Λ and the Haar measure on $U(N)$. Hence,

$$dH = |\Delta(\Lambda)|^2 \, d\Lambda \, dU_{Haar}, \tag{1.28}$$

where:

$$d\Lambda = \prod_{i=1}^{N} d\lambda_i \tag{1.29}$$

is the Lebesgue measure on $\mathbb{R}^N$, and the Jacobian is the square of the Vandermonde determinant,

$$|\Delta(\Lambda)|^2 = \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^2. \tag{1.30}$$

The partition function of the model (1.25) then becomes:

$$\mathcal{Z} = C_N \int_{\mathbb{R}^N} |\Delta(\Lambda)|^2 e^{-N\mathcal{V}(\Lambda)} d\Lambda. \tag{1.31}$$

This is commonly referred to as the Weyl integration formula [1]. Here, C_N is a constant and can be calculated using Selberg–Mehta integral [7] or using orthogonal polynomials, and it turns out that the prefactor is:

$$C_N = \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{k=1}^{N} k!}. \tag{1.32}$$

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Chapter 2

From Noncommutative Geometry to Random Matrix Theory

2.1 Introduction

In this paper we would like to give an overview of some of the recent developments on the intersection of noncommutative geometry, random matrix theory, and Euclidean quantum gravity. The existence of the Planck length puts restrictions on the nature of a theory of spacetime suitable for a quantum theory of gravity. In fact, a combination of the Heisenberg uncertainty principle and Einstein's general relativity shows that, due to the formation of black holes in small length scales, spacetime cannot be a smooth manifold. There have been several possibilities suggested for a replacement of classical spacetime. A noncommutative space, in the sense of spectral triples, is one such proposal [29]. Other possibilities include spin networks [76], random tensors [48], spin foams [69], and loop quantum gravity [77]. If the spectral triple is assumed to be finite one can use computer simulations and random matrix theory techniques to explore these models in detail.

Random matrices first appeared in physics in a series of papers by Wigner in the mid 50's [86, 87, 88], where Wigner modeled the Hamiltonians of heavy nuclei by large random Hermitian or symmetric matrices. Since then random matrices have found applications in many other areas of physics, in particular in models of two dimensional quantum gravity. This was first seen in the matrix integral used by Kontsevich to prove Witten's conjecture [89, 58]. Around the same time it was found that artifacts of two dimensional conformal field theory coupled with gravity, sometimes called Liouville quantum gravity (LQG), could be obtained from certain matrix models in the double scaling limit [33, 35, 10]. More recently it was discovered in [78, 81, 90, 67] that the genus expansion of partition functions in Jackiw–Teitelboim (JT) gravity can be computed using random matrix techniques. In particular, a process known as Topological Recursion, original developed in [38], was applied. In fact Topological Recursion can be used in all the cases stated above. We will discuss this method in Appendix B.3.

We are specifically interested in toy models of Euclidean quantum gravity where integration over metrics is replaced by integration over Dirac operators on a fixed finite noncommutative space, as proposed by Barrett and Glaser [5]. This scenario quickly leads to very interesting multi-trace multi-matrix models. As a rule such models are hard to analyze using standard methods. Yet the fact that they are obtained from specific potentials defined on the space of Dirac operators gives them a special structure and hence the possibility of analytic study.

The partition function of these models is of the form

$$Z = \int_{\mathcal{D}} e^{-S(D)} dD,$$

where the integration is carried out over the space $\mathcal{D}$ of Dirac operators. This can be justified by general principles of noncommutative geometry, starting from the fact that the metric structure of a smooth spin manifold is encoded in its Dirac operator [25, 28, 30, 66]. In particular Dirac operators are taken as dynamical variables and play the role of metric fields in gravity.

Such a matrix integral is not necessarily convergent, nor does it require a real valued action S. However, they may always be interpreted as formal matrix integrals, which are the generating functions of certain types of maps [8, 41]. The action functional S is, for models considered in this survey, alwas chosen in such a way that the partition function Z is absolutely convergent and finite. For example, we can choose $S(D) = \mathrm{Tr}(f(D))$ for a real polynomial f of even degree with a positive leading coefficient. For more details on formal and convergent matrix models see [37, 34].

There are several benefits Dirac ensembles have over the usual random matrix ensembles. A random matrix model can be interpreted as a zero-dimensional quantum field theory or, if the model is formal, it may be viewed as a discretized path integral from string theory, where maps act as discrete surfaces. A Dirac ensemble maintains these interpretations while also being formalized as a computable noncommutative path integral over metrics, represented as Dirac operators, which is a key feature of a theory of quantum gravity. Additionally Dirac ensembles have an interpretation as a random noncommutative space. The probability distribution on Dirac operators corresponds to a probability distribution on noncommutative geometries. This idea is first mentioned in [5] and explored in detail in [6], where geometric quantities are determined using the spectrum.

It is worth noting that the connections between matrix integrals, noncommutative geometry, and physics do not start with Dirac ensembles. In [49] path integrals over finite spectral triples are also studied. Additionally the Kontsevich model and some of its generalizations appear in noncommutative quantum field theory [80, 17, 18, 53]. In particular, it is conjectured that a quartic version of the Kontsevich model obeys a generalized version of Topological Recursion known as Blobbed Topological Recursion [14, 15]. For a review of these developments see [19].

For Dirac ensembles a connection was recently established with LQG [52]. The authors hope that one day a connection to JT gravity will also be established using similar methods. We will discuss this development and the relevant background.

This paper is organized as follows. In Section 1 we define Dirac ensembles, give examples of how they can be studied and how they give rise to interesting theories, such as Yang-Mills-Higgs fuzzy spaces. In Section 2 we review results on the spectral distributions and phase transitions in Dirac ensembles. In Section 3 we recall the bootstrap method as applied to Dirac ensembles. In Section 4 we formulate some natural open problems and directions for further study. Finally, in the Appendices we recall some aspects of random matrix theory, the Schwinger-Dyson equations and Topological Recursion.

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2.2 Random matrix models from spectral triples

In this section, before we introduce the key concept of a *Dirac ensemble*, we shall first briefly discuss the notions of *spectral triples* and in particular *fuzzy spectral triples* which are the base for the Dirac ensembles appearing in this survey. It may be helpful for readers unfamiliar with random matrix ensembles to review Appendices B.1 and B.2.

2.2.1 Fuzzy spectral triples

Spectral triples were introduced by Connes in [26] (see also [30]) and are defined by data $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is a unital, involutive, complex, associative algebra acting by bounded operators on a complex Hilbert space $\mathcal{H}$, and D is a self-adjoint (in general unbounded) operator acting on $\mathcal{H}$. This data is further required to satisfy certain finiteness and regularity conditions, which are automatically satisfied if $\mathcal{A}$ and $\mathcal{H}$ are finite dimensional. Since this will always be the case in this survey we omit these conditions here.

A *real spectral triple* is a spectral triple equipped with two additional operators J and γ called the *charge conjugation* and the *chirality operator*, where $J : \mathcal{H} \to \mathcal{H}$ is an anti-linear real structure, with the requirement that $[a, JbJ^{-1}] = 0$ for all $a, b \in \mathcal{A}$, and $\gamma : \mathcal{H} \to \mathcal{H}$ is a self-adjoint operator with $\gamma^2 = 1$. The data $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ is required to satisfy some further compatibility conditions between D, J, γ, and the representation of $\mathcal{A}$ which we will recall in Definition 2.2.1.

The overall idea is that a real spectral triple is a noncommutative analogue of a spin^c Riemannian manifold and its canonical Dirac operator. In fact, any closed spin^c Riemannian manifold M defines a real (commutative) spectral triple as follows. The algebra $\mathcal{A} = C^\infty(M)$ is the algebra of smooth complex valued functions on M. The Hilbert space consists of square integrable sections of the spinor bundle, with $\mathcal{A}$ acting as multiplication operators. The operator D is the Dirac operator of M acting on the spinors, and J and γ are the standard charge conjugation and chirality operators. The reconstruction theorem of Connnes states that, conversely, a commutative real spectral triple, i.e. a triple where $\mathcal{A}$ is commutative, satisfying some natural conditions is the spectral triple of a spin^c Riemannian manifold [28]. The reader can find further details and many interesting commutative and non-commutative examples of spectral triples in the book of Connes and Marcolli [30], as well as their applications to the standard model of elementary particles and quantum field theory in general.

A spectral triple is called *finite* if both $\mathcal{A}$ and $\mathcal{H}$ are finite dimensional vector spaces. In this paper we shall primarily consider a subclass of finite real spectral triples called *fuzzy spectral triples* or *fuzzy geometries* introduced and classified in their present form by Barrett in [4]. These should be thought of as spin^c Riemannian manifolds with a finite resolution or Plank length. It should be noted that important examples of fuzzy spectral triples like the fuzzy sphere [65, 32, 47, 4, 64] and fuzzy tori [61, 79, 7, 62] were defined and studied for their own interest before the concept of fuzzy geometry was coined. We should also mention that finite dimensional real spectral triples have been fully classified by Krajewski in [60]. Further references include [30, 66, 4, 84].

We will now specialize further to the class of fuzzy spectral triples. Let $\mathrm{C}\ell_{p,q}$, for nonnegative integers p and q, denote the real Clifford algebra associated to the vector space $\mathbb{R}^n$, $n = p + q$, and the pseudo-Euclidean metric η of signature (p, q) given by

$$\eta(v, v) = {v_1}^2 + \cdots + {v_p}^2 - {v_{p+1}}^2 - \cdots - {v_{p+q}}^2 , \quad v \in \mathbb{R}^n .$$

Let $\mathbb{C}\ell_n := \mathrm{C}\ell_{p,q} \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of $\mathrm{C}\ell_{p,q}$. Let $\{e_i\}_{i=1}^n$ denote the standard basis of $\mathbb{R}^n$. The *chirality element* $\Gamma \in \mathbb{C}\ell_n$ is defined by

$$\Gamma = \mathrm{i}^{\frac{1}{2}s(s+1)}\, e_1 e_2 \cdots e_n ,$$

where $s \equiv q - p \pmod 8$ is known as the KO-dimension. We denote by $V_{p,q}$ the unique (up to unitary equivalence) hermitian irreducible $\mathrm{C}\ell_{p,q}$-module, where for $n = p + q$ odd the chirality element Γ acts trivially on $V_{p,q}$. The module $V_{p,q}$ also comes with a charge conjugation operator $C : V_{p,q} \to V_{p,q}$ (see [66, 4] for details).

Definition 2.2.1 *A* f*uzzy spectral triple of type, or signature,* (p, q) *is a finite real spectral triple* $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ *where*

- $\mathcal{A} = \mathrm{M}_N(\mathbb{C})$ *is the algebra of complex* $N \times N$ *matrices,*
- $\mathcal{H} = V_{p,q} \otimes \mathrm{M}_N(\mathbb{C})$ *with the inner product*

$$\langle u \otimes A, v \otimes B\rangle = \langle u, v\rangle \ \mathrm{Tr}\,(AB^*)\ , \quad u, v \in V_{p,q}\ ,\ A, B \in \mathrm{M}_N(\mathbb{C}),$$

- *The action of* $\mathcal{A}$ *on* $\mathcal{H}$ *is defined by* $A \cdot (v \otimes B) = v \otimes (AB)$,
- *The charge conjugation operator is* $J(v \otimes A) = (Cv) \otimes A^*$,
- *The chirality operator is defined as* $\gamma(v \otimes A) = (\Gamma v) \otimes A$,
- *The Dirac operator* D *satisfies:*
  a) $D^* = D$,
  b) $D\gamma = -(-1)^s \gamma D$,
  c) $DJ = \epsilon' JD$ *, where* $\epsilon' = 1$ *for* $s = 0, 2, 3, 4, 6, 7$ *and* $\epsilon' = -1$ *for* $s = 1$ *or* 5,
  d) $[[D, a], JbJ^{-1}] = 0$ *for all* $a, b \in \mathcal{A}$.

The quantity $n = p + q$ *is called the dimension of the fuzzy spectral triple, the quantity* $s = q - p$ *is the KO-dimension.*

The main benefit of considering fuzzy spectral triples is that their Dirac operators can be expressed in terms of the gamma matrices γ^i (the image of e_i in the Clifford algebra), and commutators or anti-commutators with Hermitian or skew-Hermitian matrices. More precisely, Barrett proved in [4] that the Dirac operator of a fuzzy spectral triple is always of the form

$$D = \sum \gamma^I \otimes \{K_I, \cdot\}_{e_I} \tag{2.1}$$

where the sum is over increasingly ordered multi-indices I. If γ^I is Hermitian, $e_I = 1$ and $\{K_I, \cdot\}_{e_I} = \{H_I, \cdot\}$, where H_I is some Hermitian matrix. If γ^I is skew-Hermitian, $e_I = -1$ and $\{K_I, \cdot\}_{e_I} = [L_I, \cdot]$, where L_I is some skew-Hermitian matrix. This allows us to effectively parametrize the space of Dirac operators by matrices.

One prominent example of a fuzzy spectral triple is the fuzzy sphere [82, 85, 65, 47, 32]. Let J_1, J_2, J_3 be the standard skew-Hermitian generators of $\mathfrak{su}_2$ and denote the $2j+1$-dimensional irreducible representation of $\mathfrak{su}_2$ by (π_j, V_j). An initial definition of a Dirac operator for the fuzzy sphere is then defined on $\mathbb{C}^2 \otimes M_{2j+1}(\mathbb{C}) \cong \mathbb{C}^2 \otimes V_j \otimes V_j^*$ by

$$d = 1 \otimes 1 + \gamma^\mu \otimes \left[\pi_j(J_\mu), \cdot\right]$$

with $\gamma^\mu = i\sigma^\mu$ for the Pauli matrices σ^μ. This definition has two problems, it is of signature $(0,3)$ which has KO-dimension 3 rather than the desired 2, and it does not admit a grading [32]. Hence d cannot be the Dirac operator for a fuzzy geometry. Instead we form a Dirac operator of signature $(1,3)$, by $D = \sigma^1 \otimes d$ which admits a grading $\sigma^3 \otimes 1$. The real structure is given by $J(v \otimes a) = \sigma_2 \bar{v} \otimes a^*$.

2.2.2 Dirac ensembles

By a *Dirac ensemble* we mean a statistical ensemble of *fuzzy spectral triples* $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ where the "Fermion space" $(\mathcal{A}, \mathcal{H}, J, \gamma)$, is kept fixed but the Dirac operator D is a random variable with a given probability density. This a non-commutative analogue to a probability distribution on the space of metrics on a given manifold, as the Dirac operator encodes the metric structure while the algebra encodes the topology.

More precisely, let $\mathcal{D}$ denote the set of all possible Dirac operators $D : \mathcal{H} \to \mathcal{H}$ such that the quintuple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ satisfies the conditions a), b), c), and d) of definition 2.2.1. Clearly $\mathcal{D} \subset \mathrm{End}(\mathcal{H})$ is a real subspace, hence it is equipped with an inner product and thus a natural Lebesgue measure which we denote by dD. Given a choice of action functional $S : \mathcal{D} \to \mathbb{R}$, usually a polynomial whose choice is part of the data for a Dirac ensemble, the probability density on $\mathcal{D}$ is defined by

$$\frac{1}{Z} e^{-\operatorname{Tr} S(D)} dD$$

where

$$Z = \int_{\mathcal{D}} e^{-\operatorname{Tr} S(D)} dD$$

is the partition function of the model.

For fuzzy spectral triples the space of Dirac operators can be paramatrized by a combination of (skew)-Hermitian matrices or, by writing the skew-Hermitian matrices as i multiplied by a Hermitian one, purely by Hermitian matrices using equation (2.1). The probability density, and thus partition function, may then be written as a matrix integral of the form

$$Z = \int_{\mathcal{D}} e^{-\operatorname{Tr} S(D)} dD = \int_{\mathcal{H}_N^m} e^{-\tilde{S}(H_1, H_2, \ldots, H_m)} dH_1 \ldots dH_m, \tag{2.2}$$

where $\mathcal{H}_N$ is the space of Hermitian N by N matrices. For a given Dirac ensemble defined by $e^{-\operatorname{Tr} S(D)} dD$, we refer to the third term in equation (2.2) as the associated random matrix ensemble. We will see this correspondence in detail in Sections 2.2.3 and 2.2.4.

The matrix ensembles associated to Dirac ensembles are usually both multi-trace or multi-matrix. Since most of the results in random matrix theory are for single matrix and single trace models, this hints that the analytic study of Dirac ensembles as matrix integrals is quite difficult in general.

2.2.3 One dimensional Dirac ensembles

For fuzzy spectral triples of dimension $n = 1$ the only possible signatures are $(1,0)$ and $(0,1)$. In both case $\mathcal{H} = M_N(\mathbb{C})$ and the two possible choices for D are as follows [4, 5]:

Type (1, 0) The Dirac operator is the anticommutator with a Hermitian matrix H,

$$D = \{H, \cdot\}.$$

The trace of powers of D can be computed by

$$\operatorname{Tr} D^\ell = \sum_{k=0}^{\ell} \binom{\ell}{k} \operatorname{Tr} H^{\ell-k} \operatorname{Tr} H^k.$$

Type (0, 1) The Dirac operator is the commutator with a skew-Hermitian matrix L,

$$D = i[L, \cdot].$$

The trace of powers of D can be computed by

$$\operatorname{Tr} D^\ell = \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \operatorname{Tr} L^{\ell-k} \operatorname{Tr} L^k.$$

For the action functional we can consider a quartic potential

$$Z = \int_{\mathcal{D}} e^{-g \operatorname{Tr} D^2 - \operatorname{Tr} D^4} dD,$$

where the real parameter g is called a coupling constant. In type $(1,0)$ the integral is over the space $\mathcal{H}_N$ of Hermitian $N \times N$ matrices and the partition function in terms of H is

$$Z = \int_{\mathcal{D}} e^{-g \operatorname{Tr} D^2 - \operatorname{Tr} D^4} dD = \int_{\mathcal{H}_N} e^{-2N(g \operatorname{Tr} H^2 + \operatorname{Tr} H^4) - 2g(\operatorname{Tr} H)^2 - 8 \operatorname{Tr} H \operatorname{Tr} H^3 - 6(\operatorname{Tr} H^2)^2} dH.$$

The substitution of $\mathcal{H}_N$ for $\mathcal{D}$ is justified as the parametrization of Dirac operators by the Hermitian matrix H is bijective.

In type $(0,1)$ the integral is over the space $\mathcal{L}_N$ of skew-Hermitian $N \times N$ matrices, and the partition function is given by

$$Z = \int_{\mathcal{D}} e^{-g \operatorname{Tr} D^2 - \operatorname{Tr} D^4} dD = \int_{\mathcal{L}_N} e^{-2g(N \operatorname{Tr} L^2 - (\operatorname{Tr} L)^2) - \left(2N \operatorname{Tr} L^4 - 8 \operatorname{Tr} L \operatorname{Tr} L^3 + 6(\operatorname{Tr} L^2)^2\right)} dL.$$

Note that the kernel of the map $\mathcal{L}_N \to \mathcal{D}$ consists of the scalar matrices, which has lebesgue measure zero, justifying the substituion. We can write $L = iH$, for a Hermitian matrix H, to get

$$Z = \int_{\mathcal{D}} e^{-g \operatorname{Tr} D^2 - \operatorname{Tr} D^4} dD = i \int_{\mathcal{H}_N} e^{2g(N \operatorname{Tr} H^2 - (\operatorname{Tr} H)^2) + \left(-2N \operatorname{Tr} H^4 + 8 \operatorname{Tr} H \operatorname{Tr} H^3 - 6(\operatorname{Tr} H^2)^2\right)} dH.$$

In [56] it was shown that the two terms $(\operatorname{Tr} H)^2$ and $(\operatorname{Tr} H)\operatorname{Tr} H^3$ contribute nothing in the large N limit, giving us the same matrix integral as the above quartic type $(1,0)$ up to a factor of i. This idea extends to all type $(1,0)$ and $(0,1)$ Dirac ensembles with even potentials, that is, all such ensembles will have identical real eigenvalue density in the large N limit.

If one considers formal Dirac ensembles of types $(1,0)$ or $(0,1)$, it can be shown [3] that their partition functions and moments are the generating functions that count combinatorial objects know as stuffed maps in the sense of the work Borot and Shadrin in [14, 15]. Similar to the more common types of maps that arise in Hermitian matrix ensembles, the matrix integrals that generate stuffed maps obey a generalized form of Topological Recursion, called Blobbed Topological Recursion. Given the genus zero one-point and two-point generating functions one can recursively compute all higher order corrections. In particular, this (blobbed) topological recursion applies to $(1,0)$ or $(0,1)$ Dirac ensembles with multi-tracial potentials and was studied in [3].

2.2.4 Two dimensional Dirac ensembles

For two dimensional fuzzy geometries there are three options. In this case the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes M_N(\mathbb{C})$, where for $p+q=2$, $\mathbb{C}^2 \cong V_{p,q}$ is the space of spinors. The structure of the Dirac operator depends on the type as follows:

Type (2, 0) Let

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then,

$$D = \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes \{H_2, \cdot\},$$

where H_1 and H_2 are Hermitian matrices.

Type (1, 1) Let

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then,

$$D = \gamma^1 \otimes \{H, \cdot\} + \gamma^2 \otimes [L, \cdot],$$

where H is Hermitian and L is skew-Hermitian.

Type (0, 2) Let

$$\gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then,

$$D = \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot],$$

where L_1, L_2 are both skew-Hermitian.

The general structure of trace powers of Dirac operators ensembles of signature above 1 becomes quite complicated and has no obvious patterns. They were first studied in [70].

The associated matrix models also rise in complexity, take for example a type $(2,0)$ quartic potential

$$\begin{aligned} Z &= \int_{\mathcal{D}} e^{-\frac{t_2}{8}\operatorname{Tr} D^2 - \frac{t_4}{16}\operatorname{Tr} D^4} dD \\ &= \int_{\mathcal{H}_N^2} e^{\tilde{S}(H_1,H_2)} dH_1 dH_2, \end{aligned}$$

where

$$\begin{aligned} \tilde{S}(H_1,H_2) = -t_2(N\operatorname{Tr} H_1^2 + N\operatorname{Tr} H_2^2) - t_4\bigg(&\frac{1}{4}N\operatorname{Tr} H_1^4 + \frac{1}{4}N\operatorname{Tr} H_2^4 + N\operatorname{Tr} H_1^2H_2^2 \\ &-\frac{1}{2}N\operatorname{Tr} H_1H_2H_1H_2 + \frac{3}{4}(\operatorname{Tr} H_1^2)^2 + \frac{3}{4}(\operatorname{Tr} H_2^2)^2 + \frac{1}{2}\operatorname{Tr} H_1^2\operatorname{Tr} H_2^2\bigg) \end{aligned} \tag{2.3}$$

are the contributing terms in the large N limit. This ensemble is a bi-tracial two-matrix model. There are no known applicable analytic techniques from random matrix theory.

Note that similarly to the type $(0,1)$ geometry, Dirac ensembles with skew-Hermitian matrices can always be converted to ensembles of Hermitian matrices. So, in particular, studying models from the above geometries amounts to solving Hermitian two-matrix models.

2.2.5 Yang-Mills-Higgs Dirac ensembles

Dirac ensembles as defined so far describe only the metric structure of a fuzzy space. The space $V_{p,q}$ plays the role of a spinor space and $M_N(\mathbb{C})$ plays the role of L^2-functions on the manifold so that together they make a (trivial) spinor bundle. In order to include a gauge sector we can consider Yang-Mills-Higgs fuzzy spaces [72]. This approach is based on gauge theory on almost commutative manifolds [22, 23] (see also chapter 8 of [84]), and consists of introducing a finite spectral triple playing the role of an additional (trivial) vector bundle which will carry an analogue of a connection.

Concretely, let $M_f = (M_N(\mathbb{C}), \mathcal{H}_N, D_f, J, \gamma)$ be a fuzzy spectral triple in the sense of Definition 2.2.1 and let $F = (\mathcal{A}_F, \mathcal{H}_F, D_F, J_F, \gamma_F)$ be a finite spectral triple. This second spectral triple will be referred to as the *gauge* or *finite* spectral triple. If $\mathcal{A}_F = M_n(\mathbb{C})$ and $\mathcal{H}_F = M_n(\mathbb{C})$

the gauge triple is referred to as a Yang-Mills triple. The Gauge-Higgs fuzzy space, or Yang-Mills-Higgs fuzzy space if the gauge triple is of Yang-Mills type, is then the product spectral triple $M_f \times F$ given by

$$\left(M_N(\mathbb{C}) \otimes \mathcal{A}_F, \mathcal{H}_N \otimes \mathcal{H}_F, D_f \otimes 1 + \gamma \otimes D_F, J \otimes J_F, \gamma \otimes \gamma_F\right).$$

In this picture D_F is usually considered fixed and controls physical aspects of the gauge fields, while D_f will vary and describe the metric structure of the fuzzy space M_f. The smooth limit of the fuzzy Yang-Mills-Higgs triple is given by $N \to \infty$, while the size of the gauge sector n remains fixed. As in Equation 2.1 we can write $D_f = \sum \gamma^I \otimes \{K_I, \cdot\}_{e_I}$, the amplification $D_f \otimes 1$ can then be written as $\sum \gamma^I \otimes \{K_I \otimes 1_n, \cdot\}_{e_I}$ acting on $V_{p,q} \otimes M_N(\mathbb{C}) \otimes M_n(\mathbb{C})$.

The gauge theory enters this framework in the guise of inner fluctuations. These inner fluctuations [27] (see also chapter 6 of [84]) arise from Morita self-equivalences and are parametrised by the self-adjoint *Connes one-forms*, for a spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ these one-forms are given by

$$\Omega^1_D(A) = \left\{\sum a_i[D, b_i] \,\middle|\, a_i, b_i \in A\right\} \subset End(\mathcal{H})$$

where the sum is finite. For a self-adjoint $\omega \in \Omega^1_D(A)$, the *fluctuated Dirac operator* is given by $D_\omega = D + \omega + J\omega J^{-1}$.

For the fuzzy Yang-Mills-Higgs triple we can split the fluctuations into a fuzzy and a finite part, by $a[D, b] = a[D_f \otimes 1, b] + a[\gamma \otimes D_F, b]$ for $a, b \in M_N(\mathbb{C}) \otimes M_n(\mathbb{C})$. The fuzzy part of this, $a[D_f \otimes 1, b]$, can be parametrized by $\Omega^1_{D_f}(M_N(\mathbb{C})) \otimes M_n(\mathbb{C})$, while the finite part can, independently, be parametrized by $M_N(\mathbb{C}) \otimes \Omega^1_{D_F}(M_n(\mathbb{C}))$. The effect of the fuzzy part of the fluctuation, for $s \neq 1, 5$, on $D_f \otimes 1$ is to replace $K_I \otimes 1_n$ by $K_I \otimes 1_n + T_I$, with $T_I \in \Omega^1_{K_I}(M_N(\mathbb{C})) \otimes M_n(\mathbb{C})$ of the appropriate (skew-)adjointness. The finite part of the fluctuation does not affect the fuzzy section, since it carries the action of γ on the $V_{p,q}$ factor of $\mathcal{H}_N$. Instead, the finite part of the fluctuation together with D_F itself is gathered into one term $\Phi = 1_N \otimes D_F + \{\phi, \cdot\}_{\epsilon''}$ where ϵ'' depends on s and $\phi \in M_N(\mathbb{C}) \otimes \Omega^1_{D_F}(M_n(\mathbb{C}))$. Φ is suggestively called the Higgs potential.

To motivate the term Yang-Mills-Higgs spectral triple we will specialize to the 4-dimensional Riemannian case which has signature $(p, q) = (0, 4)$. In this case the fuzzy Dirac operator can be written

$$D_f = \sum_\mu \gamma^\mu \otimes [L_\mu, \cdot] + \gamma^{\hat{\mu}} \otimes \{X_\mu, \cdot\}.$$

Here $\gamma^{\hat{\mu}}$ is the increasingly ordered product of the gamma matrices for $V_{0,4}$ except γ^μ. The $(0, 4)$ fuzzy Dirac operator shows remarkable similarity to the Dirac operator on a commutative manifold, with $[L_\mu, \cdot]$ taking the place of ∂_μ and $\{X_\mu, \cdot\}$ taking the place of, appropriately symmetrized, Christoffel symbols. This leads us to say the fuzzy space is flat if all X_μ vanish.

The term Yang-Mills is further motivated by considering the gauge group [27, 30, 84] associated to a spectral triple (A, H, D, J, γ),

$$G(A, J) = \{uJuJ^{-1} \mid u \in U(A)\}.$$

For a Yang-Mills fuzzy geometry, this gauge group is $PU(N) \times PU(n)$ acting in the adjoint representations on $M_N(\mathbb{C}) \otimes M_n(\mathbb{C})$. The $PU(N)$ factor corresponds to symmetries of the base fuzzy geometry, while $PU(n)$ acts on the finite "vector bundle" part as a Yang-Mills gauge group.

When basing a Yang-Mills-Higgs fuzzy geometry on such a flat $(0, 4)$ fuzzy space the self-adjoint one-forms, and thus the inner fluctuations, can be parametrized as $\sum_\mu \gamma^\mu \otimes A_\mu$ with $A_\mu \in \Omega^1_{L_\mu}(M_N(\mathbb{C})) \otimes M_n(\mathbb{C})$ skew-adjoint. The effect of such a fluctuation on D_f is by replacing L_μ by $L_\mu + A_\mu$, making the action of inner fluctuations analogous to having a connection on the "vectorbundle" F. In this setting we define the field strength of a Dirac operator by $F_{\mu\nu} = \big[[L_\mu + A_\mu, \cdot], [L_\nu + A_\nu, \cdot]\big]$, mimicking the regular commutative definition of the field strength of a connection.

Considering the quartic potential $S(D) = g \operatorname{Tr} D^2 + \operatorname{Tr} D^4$ we have [72], for a fluctuated $(0, 4)$ Dirac operator,

$$S(D) = -2 \operatorname{Tr} F_{\mu\nu}F^{\mu\nu} + 4 \operatorname{Tr}(g\theta + \theta^2) + 4 \operatorname{Tr}(g\Phi^2 + \Phi^4) - 8 \operatorname{Tr}([L_\mu + A_\mu, \Phi][L^\mu + A^\mu, \Phi])$$

where $\theta = \eta^{\mu\nu}[L_\mu + A_\mu, \cdot] \circ [L_\nu + A_\nu, \cdot]$. Here θ is analogous to a Laplace operator and the trace is taken as operator on $M_N(\mathbb{C}) \otimes M_n(\mathbb{C})$. Each of these four tracial terms has an interpretation in terms of commutative Yang-Mills-Higgs theory:

- $\operatorname{Tr} F_{\mu\nu}F^{\mu\nu}$ is analogous to the classical Yang-Mills action,
- $\operatorname{Tr} g\theta + \theta^2$ contains geometric information similar to a Laplace operator through a type of heat-kernel expansion [6],
- $\operatorname{Tr} g\Phi^2 + \Phi^4$ represents, for appropriate values of g, the Higgs potential,
- $\operatorname{Tr}([L_\mu + A_\mu, \Phi][L^\mu + A^\mu, \Phi])$ is the coupling between the Yang-Mills connection and the Higgs field.

Hence we can include a Yang-Mills-Higgs action in a Dirac ensemble by taking the product with a finite spectral triple and considering inner fluctuations. The resulting Dirac ensemble has a space of Dirac operators $\mathcal{D}$ parametrized by, in the flat signature $(0, 4)$ case, the four skew-adjoint matrices L_μ as well as the inner fluctuations $A_\mu \in \Omega^1_{L_\mu}(M_N(\mathbb{C}))$ and $\phi \in M_N(\mathbb{C}) \otimes \Omega^1_{D_F}(M_n(\mathbb{C}))$. The physical D_F is considered fixed. It should be noted that the A_μ are not necessarily independent, as they originate from a single one-form $\omega \in \Omega^1_{D_f}(M_N(\mathbb{C}))$ further complicating any analytical investigation of these models.

Investigating Yang-Mills-Higgs ensembles numerically or for a lower dimensional fuzzy spectral triple would be very interesting. Another open direction is the addition of a fermionic term to the action in the Dirac ensemble of the form $\langle D\psi, \psi\rangle$ where $\psi \in \mathcal{H}_N \otimes \mathcal{H}_F$ plays the role of a fermion field and would be assigned a probability distribution together with the metric and gauge fields represented by D and its inner fluctuations.

2.3 Spectral statistics and phase transitions

In this section we discuss spectral statistics and phase transitions that have been studied for Dirac ensembles so far. In Section 2.1 we discuss general properties of the spectra of random Dirac ensembles. Next, in Section 2.2 we look at the phase transition in the large N limit of spectral density functions of Dirac ensembles and present a new result. In Section 2.3 we discuss the manifold-like behavior of spectra of Dirac ensembles at various phase transition points and attempts to make this idea more concrete. Finally, in Section 2.4 we show that in certain Dirac ensembles one can recover the critical exponents and partition functions of minimal models from Liouville quantum gravity. We recommend that readers unfamiliar with the spectral density functions and genus expansions of random matrix ensembles review Appendices B.1 and B.2.

2.3.1 Spectral statistics of Dirac ensembles

The Dirac ensembles of type $(1,0)$ and $(0,1)$ discussed in Sections 2.2.3 can be analyzed as bi-tracial matrix models using various standard random matrix techniques that are applicable to that setting. Consider a type $(1,0)$ or $(0,1)$ Dirac ensemble with a partition function of the form

$$Z = \int_{\mathcal{D}} e^{-\frac{t_2}{4}\operatorname{Tr} D^2 + \sum_{j=3}^{d} \frac{t_{2j}}{4j}\operatorname{Tr} D^{2j}} dD.$$

The spectra of H and L appearing in the associated matrix model can be computed, as we will see in subsequent sections. However, we are not just interested in the spectrum of H and L but also in the spectrum of D. It was first conjectured in [4] and later proven in [57] that if the limiting eigenvalue distribution, $\rho(x)$, of the associated random matrix ensemble exists then the limiting eigenvalue density function of D, $\rho_D(\mathrm{x})$, is given by the integral convolution of the random matrix spectral density function with itself i.e.

$$\rho_D(x) = \int_{\mathbb{R}} \rho(x-t)\rho(t)dt.$$

The relationship between spectral densities is far from clear for higher signature Dirac ensembles, even for the two dimensional ones from Section 2.2.4. Moreover, the associated matrix models of these Dirac ensembles are multi-trace and multi-matrix, of which little is known [37].

In the large N limit, there is a universality to the spectral density function of Dirac ensembles for any signature when the potential is Gaussian. In [4] a Gaussian potential

$$S(D) = \operatorname{Tr} D^2,$$

is investigated. When looking at the $(1,0)$ Dirac ensemble, the associated matrix potential becomes $2N \operatorname{Tr} H^2 + 2 \operatorname{Tr} H \operatorname{Tr} H$, where H is a Hermitian matrix. For the $(0,1)$ Dirac ensemble, this potential becomes $2N \operatorname{Tr} L^2 + 2 \operatorname{Tr} L \operatorname{Tr} L$ where L is skew-Hermitian. For these ensembles the numerics show that the distribution of the eigenvalues of H and L resembles Wigner's semicircular distribution as N increases. This suggests that the multi-trace term has little to no impact as N gets larger. Dirac ensembles of signatures $(2,0)$, $(1,1)$ and $(0,2)$ were also studied for small matrix size, where some of the above-mentioned results also apply. Furthermore, the eigenvalue density function of the Dirac operator was conjectured to be the integral convolution of Wigner's semicircular law with itself.

This was then proven in [57]. It was shown that in the large N limit the eigenvalue density function for D is universal, in the sense that for any signature (p,q) the limit is the same given the right scaling. Consider the partition function of the form

$$Z = \int_{\mathcal{D}} e^{-\frac{1}{2k} \operatorname{Tr} D^2} dD$$

where k is the dimension of $V_{p,q}$. Then the density function of D, for any signature (p,q), is in the large N limit given by

$$\rho_D(x) = \int_{\mathbb{R}} \rho_W(x-t)\rho_W(t)dt,$$

where

$$\rho_W(x) = \frac{1}{2\pi}\sqrt{4-x^2}_{[-2,2]}$$

is Wigner's Semicircular Distribution [57].

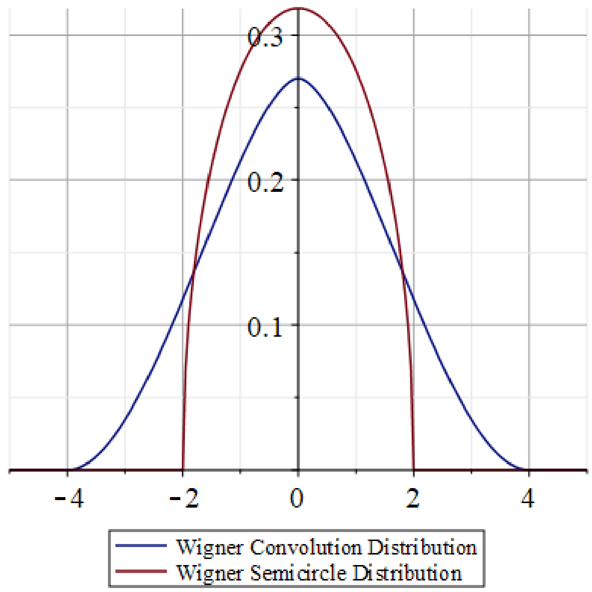


Figure 2.1: The Wigner semicircular distribution in blue compared to the Wigner convolution distribution in green.

2.3.2 Spectral phase transitions

In this subsection we will discuss results related to *spectral phase transitions*, which we define as a configuration of coupling constants where the number of connected components of the support of the large N eigenvalue density function of the random matrix ensemble changes. Like their random matrix model cousins, Dirac ensembles also exhibit phase transitions. This was first investigated numerically using Markov chain Monte Carlo methods in [5] and then in [44]. It was later proven rigorously in [56] that spectral phase transitions indeed occur in quartic type $(1,0)$ and $(0,1)$ Dirac ensembles. In this subsection we recall this quartic phase transition from [56] and we also present a new result proving a phase transition for a sextic Dirac ensemble. It is an interesting problem to analytically investigate phase transitions for Dirac ensembles of any type (p,q) and a potential of any order. We believe that these phase transitions do indeed exist, but we don't have a proof at hand. The existing results use what is known as the Coulomb gas method, the rigorous foundations of which are fully developed in [34]. We give a brief summary of this technique in Appendix B.1.

For Gaussian potentials, unsurprisingly, no phase transition exists. This motivated the study of a Landau-Ginzburg type quartic potential as studied in [5]. This potential is of the form

$$S(D) = g_2 \operatorname{Tr} D^2 + \operatorname{Tr} D^4,$$

and was studied numerically for different small signatures (p,q) for a matrix size of ten in [5].

It was discovered that, for many signatures, the spectrum of the Dirac operator displayed a single-cut distribution for certain values of g_2 which transitioned into a double-cut regime for different g_2. This suggests the existence of a spectral phase transition. Furthermore, it was noted that near the phase transition the spectrum of D asymptotically behaves like the Dirac operator on a two dimensional manifold i.e. $\rho_D(\lambda) \sim C_D|\lambda|$, as λ goes to infinity, where C_D is a constant.

Computing these eigenvalue density functions explicitly even in the large N limit is very difficult. Dirac ensembles of dimension higher than one with a potential more complicated than the Gaussian are multi-trace multi-matrix models about which little is known. Usual methods such as orthogonal polynomials cannot be applied because of the multi-trace terms, Weyl's integration formula cannot be applied because of the lack of unitary invariance, and the loop equations are too complicated for Topological Recursion. Furthermore, it is not of the form of a Harish-Chandra integral, and is too complicated for a characteristic expansion. For a review of these techniques see [40].

However, for Dirac ensembles of signature $(1, 0)$ or $(0, 1)$, several options are available. The multi-trace terms still prevents the use of orthogonal polynomials, but the Coulomb gas technique can be applied. This was done in [56] where the type $(1, 0)$ quartic model

$$Z = \int_{\mathcal{D}} e^{-g \operatorname{Tr} D^2 - \operatorname{Tr} D^4} dD = \int_{\mathcal{H}_N} e^{-2N(g \operatorname{Tr} H^2 + \operatorname{Tr} H^4) - 2g(\operatorname{Tr} H)^2 - 8 \operatorname{Tr} H \operatorname{Tr} H^3 - 6(\operatorname{Tr} H^2)^2} dH$$

was studied. We will recall those results here as an example. In our paper [56], the phase transition location is off due to a missing scalar factor. In this paper we will give the new correct value that we have been able to derive analytically and have also verified with our own Monte Carlo simulations. Furthermore, the relationship between the coupling constant and the support is given by slightly different equations in both the single cut and double cut solutions. It was shown in [56] that the quartic $(1, 0)$ and $(0, 1)$ ensembles have the same behavior in the large N limit. Using the Coulomb gas method as explained in Appendix B.1, we obtained the following explicit formula for the limiting eigenvalue density function of H: for $g > -4\sqrt{2}$,

$$\rho(x) = \frac{1}{\pi}(-4\gamma^2 + \frac{1}{2\gamma^2} + 4x^2)\sqrt{4\gamma^2 - x^2}_{[-2\gamma, 2\gamma]},$$

where the support $[-2\gamma, 2\gamma]$ can be found as a function of g as the root of

$$192\gamma^8 + 48\gamma^4 + 4g\gamma^2 - 1 = 0.$$

When $g = -4\sqrt{2}$ the spectral phase transition occurs. For $g < -4\sqrt{2}$ the new limiting eigenvalue density function was found to be

$$\rho(x) = \frac{2}{\pi}|x|\sqrt{(x^2 - a^2)(b^2 - x^2)}_{[-a,-b] \cup [b,a]},$$

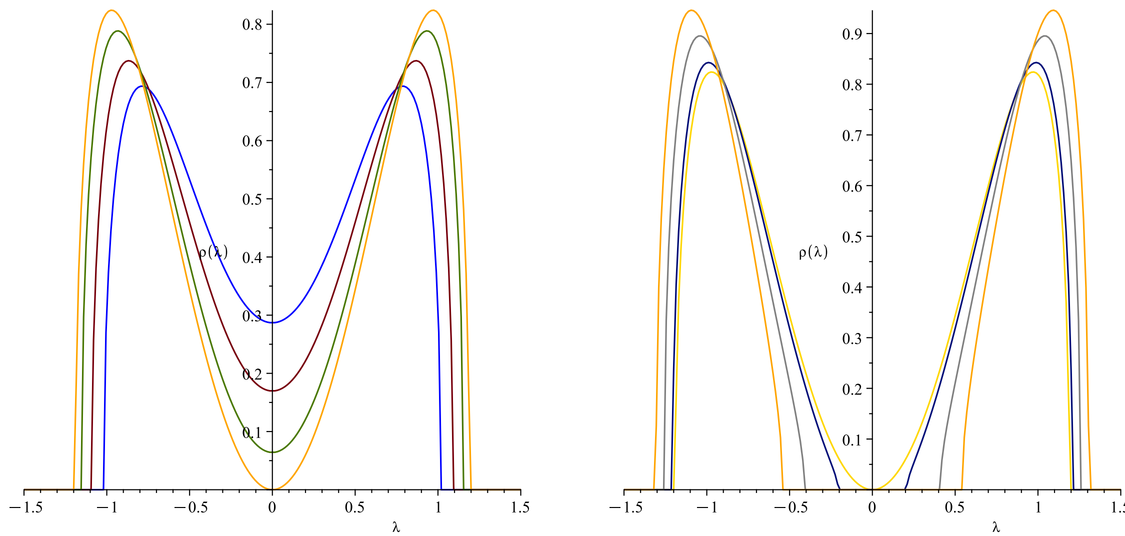


Figure 2.2: The eigenvalue density functions for type $(1, 0)$ and $(0, 1)$ quartic matrix models in the large N limit. The colors in the above figures correspond to different values of g as follows: blue is $g = -3$, red is $g = -4$, green is $g = -5$, yellow is $g = -4\sqrt{2}$, navy is $g = -6$, gray is $g = -7$, and orange is $g = -8$.

where the support $[-a, -b] \cup [b, a]$ can be found in terms of g via the equations

$$a^2 = -\frac{1}{8}g + \frac{1}{\sqrt{2}},$$

and

$$b^2 = -\frac{1}{8}g - \frac{1}{\sqrt{2}}.$$

These results are plotted in Figure 2.2.

As discussed in the previous section, the eigenvalue density function of the Dirac operator in the large N limit is the convolution of the density functions of H. See Figure 2.3.

We note that the techniques presented in [56] in fact apply to any even potential Dirac ensembles of signature $(1, 0)$ or $(0, 1)$ and any convergent bi-tracial single matrix model. For example, let us consider the following type $(1, 0)$ sextic model

$$Z = \int_{\mathcal{D}} e^{-\frac{g}{2}\operatorname{Tr} D^2 - \frac{1}{6}\operatorname{Tr} D^6} dD.$$

Once again employing the Coulomb gas method results in the following explicit formula for the limiting eigenvalue density function:

$$\rho(x) = \frac{-40\,\gamma^{12} + 20\,\gamma^{10}x^2 + 10\,\gamma^8 x^4 + 50\,\gamma^{10} - 50\,\gamma^8 x^2 + 24\,\gamma^6 - 12\,\gamma^4 x^2 - \gamma^2 x^4 - 1}{20\,\gamma^8\pi - 2\,\pi\,\gamma^2}\sqrt{4\,\gamma^2 - x^2}_{[-2\gamma, 2\gamma]}$$

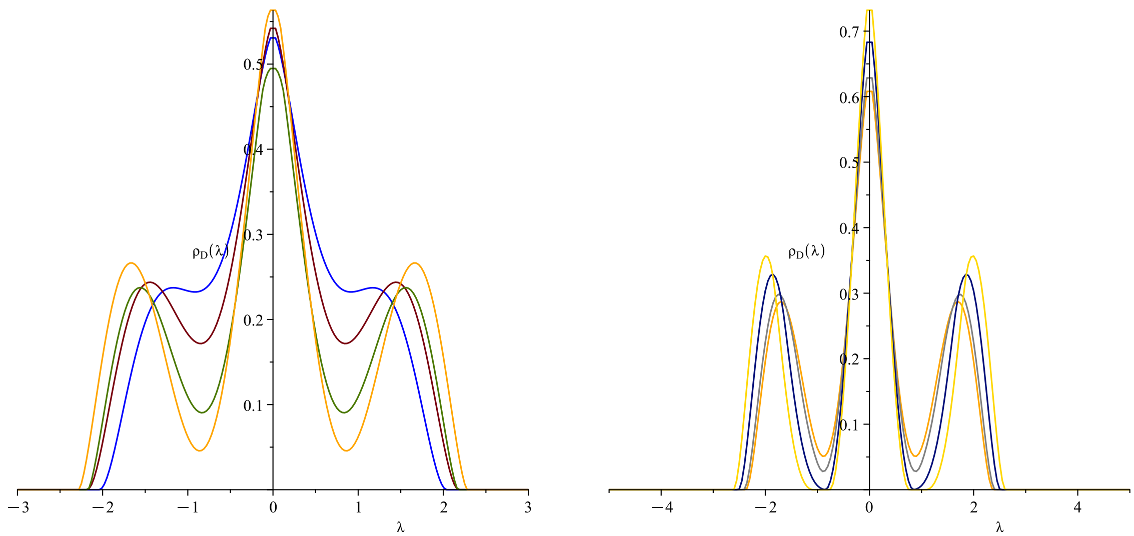


Figure 2.3: The eigenvalue density functions of the $(1, 0)$ and $(0, 1)$ quartic Dirac ensembles in the large N limit. The colors in the above figures correspond to different values of g as follows: blue is $g = -3$, red is $g = -4$, green is $g = -5$, yellow is $g = -4\sqrt{2}$, navy is $g = -6$, gray is $g = -7$, and cyan is $g = -8$.

where the support $[-2\gamma, 2\gamma]$ can be found as a function of g by (numerically) solving the equation

$$1 = 10\,\frac{\gamma^2\left(60\,\gamma^{16} - 130\,\gamma^{14} - 26\,\gamma^{10} - 22\,\gamma^8 + \gamma^6 g - 5\,\gamma^4 - g/10\right)}{10\,\gamma^6 - 1}.$$

Alternatively one can use the Lagrange inversion formula to find a perturbative expansion of γ in terms of g. This model also exhibits at least one spectral phase transition and in general the plots are very similar to the quartic model. However, it is the belief of the authors that it might exhibit another phase transition, but further analysis is required. For details of this analysis, which is identical to the formal case, see the sextic example in [52].

2.3.3 Spectral geometry of fuzzy spaces

In this subsection we shall very briefly give an overview of some of the results obtained in [6]. A simple, yet interesting, model of a discrete and finite noncommutative spin Riemannian manifold is a fuzzy spectral triple as we defined in Section 1. Given the important role that spectral geometry has played in global analysis, geometry, and topology of manifolds, as well as in noncommutative geometry, it is natural to ask to what extent its ideas can be extended to fuzzy spectral triples.

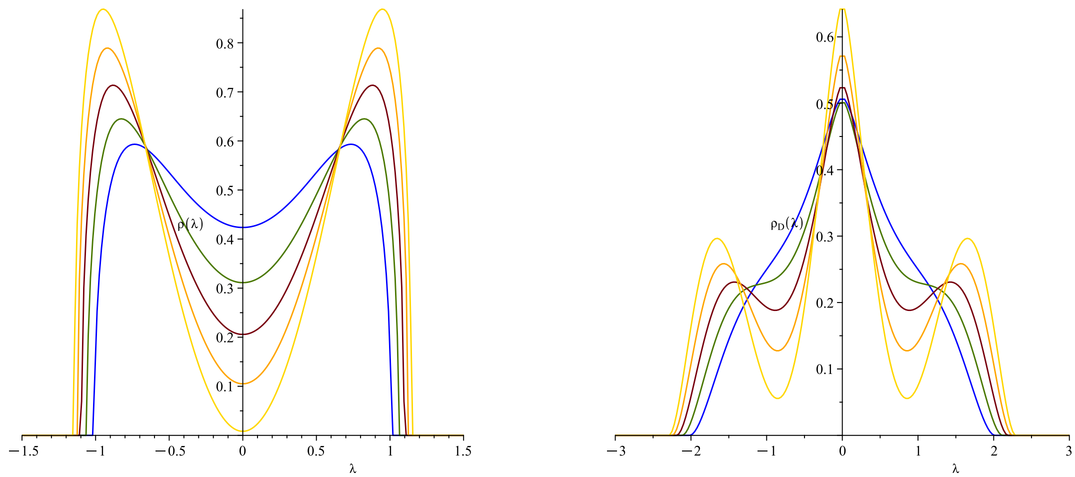


Figure 2.4: The eigenvalue density functions for type $(1,0)$ sextic matrix model (left) and for the corresponding Dirac ensemble (right) in the large N limit. The colors in the above figures correspond to different values of g as follows: blue is $g = -1.5$, green is $g = -4$, red is $g = -6.5$, orange is $g = -9$, and yellow is $g = -11.5$.

However, many of the results from spectral geometry are based on asymptotics of the spectrum $(\lambda_n)_{n\in\mathbb{N}}$ of the Dirac operator D (ordered such that $|\lambda_{n+1}| \geq |\lambda_n|$). Since the spectrum of the Dirac operator of a finite spectral triple is finite there is an obvious problem in directly generalizing such asymptotic results from spectral geometry to fuzzy spaces. For example, the first major result of spectral geometry is the celebrated Weyl's asymptotic law, according to which the volume and dimension of a compact d-dimensional Riemanian manifold can be recovered from its Dirac spectrum:

$$\mathrm{Vol}(M) = \lim_{n\to\infty} \frac{n\,(4\pi)^{d/2}\,\Gamma(1+d/2)}{k\,|\lambda_n|^d},$$

where k is the rank of the spinor bundle of M and $\Gamma(s) = \int_0^\infty e^{-t}\frac{t^s}{t}dt$ is the Gamma function.

In [6] the authors define, and numerically investigate, quantities computed from the spectrum that recover the usual asymptotic properties of dimension and volume for infinite spectra, but are also applicable to finite spectra. One such quantity is the spectral variance which is defined as

$$v_s(t) = 2t^2\left(\frac{\sum_i \lambda_i^4 e^{-\lambda_i^2 t}}{\sum_i e^{-\lambda_i^2 t}} - \left(\frac{\sum_i \lambda_i^2 e^{-\lambda_i^2 t}}{\sum_i e^{-\lambda_i^2 t}}\right)^2\right).$$

If computed for the spectrum of a manifold one obtains $\lim_{t\to 0} v_s(t) = \dim(M)$, which follows from Weyl's law. This quantity is also sensible for finite spectra, and gives the expected result of two for the fuzzy sphere and tori for appropriate values of t [6]. The presence of the parameter t should be thought of as an energy scale, with t small corresponding to high energies and small wavelengths and t large corresponding to low energies and large wavelengths. For fuzzy spaces $\lim_{t\to 0} v_s(t) = 0$, as fuzzy spaces have a minimum wavelength, or Planck length, built in.

Another way to access asymptotic information in the spectrum (λ_n) is by the spectral zeta function

$$\zeta_D(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2s}},$$

which turns out to be useful even in the finite case. It is well-known that Weyl's Law is equivalent to the fact that the volume of M can be expressed in terms of the residue of the spectral zeta function $\zeta_D(s)$ at its top pole $s = d/2$:

$$\text{Vol}(M) = \frac{(4\pi)^{d/2}}{k} \text{Res}_{s=d/2}(\zeta_D(s)\Gamma(s)).$$

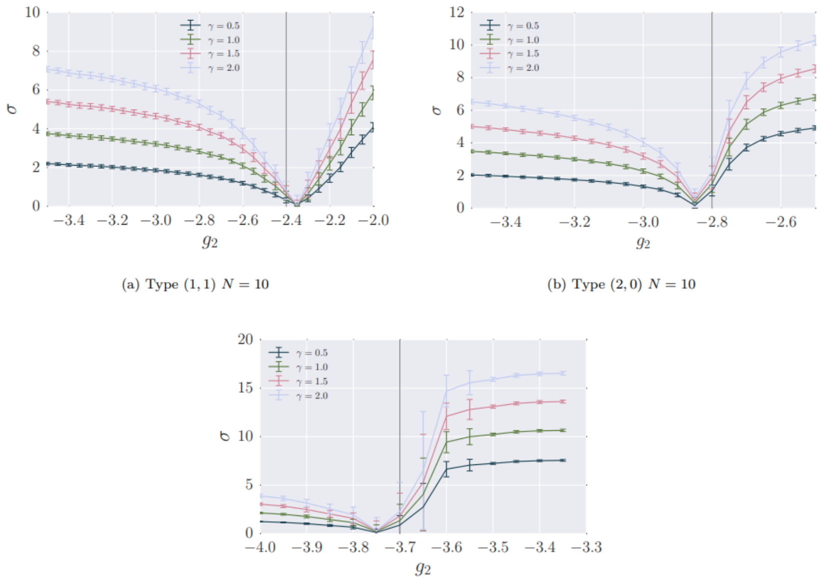
One can use the spectral zeta function to define a notion of distance between fuzzy spectral triples and even between fuzzy spectral triples and manifolds. One possibility for a notion of distance between metric spaces is the Gromov-Hausdorff distance [75], but in [31] the authors define a distance notion more useful in this situation, based on the spectra of Riemannian manifolds. Let D_1 and D_2 be two Dirac operators with $\zeta_{D_1}(s)$ and $\zeta_{D_2}(s)$ their spectral zeta functions. Then the distance between geometries is defined to be

$$\sigma(D_1, D_2) = \sup_{\gamma \le s \le \gamma+1} \left| \log\left(\frac{\zeta_{D_1}(s)}{\zeta_{D_2}(s)} \right) \right| \tag{2.4}$$

for some interval $[\gamma, \gamma + 1]$ where all poles lie below γ. For Dirac operators on compact spin manifolds it was found in [31] that this is indeed a metric, in particular $\sigma(D_1, D_2) = 0$ if and only if the spectra are the same.

In [6] this idea is adapted to define a distance between (random) fuzzy spectral triples. For example for each fuzzy sphere with matrix size N we have a Dirac operator of size N. This spectrum is the same as the Dirac operator on the spin bundle of the 2-sphere tensored with $\mathbb{C}^2$ but with a cut-off. As N goes to infinity the spectral zeta function of the fuzzy sphere converges to the spectral zeta function of the sphere uniformly on the interval $[\gamma, \gamma + 1]$ for any $\gamma > 1$. Thus, $\sigma(D_N, D_{S^2})$ goes to zero as N goes to infinity. Note that when considering truncated spectra, pointwise convergence of zeta functions is not a sufficient condition for $\sigma(D_1, D_2) = 0$, uniform convergence is needed. For more on fuzzy spaces and truncated spectral triples see [45, 46].

One of the most remarkable results in [6] is that, when using the spectral distance to compare random spectra sampled from each of the quartic Dirac ensembles of types $(1, 1)$, $(2, 0)$, and $(1, 3)$ to the fuzzy sphere for various values of the coupling constant, near the spectral phase transition the spectral distance σ tends to zero. See Figure 2.5. The authors of [6] found further numerical evidence that, near spectral phase transitions, Dirac ensembles display manifold-like behavior, but we are still far from proving such a conjecture rigorously.



(a) Type $(1,1)$ $N = 10$

(b) Type $(2,0)$ $N = 10$

(c) Type $(1,3)$ $N = 8$

Figure 2.5: The spectral distance metric (2.4) is plotted between various type (p,q) quartic Dirac ensembles of various matrix sizes and the fuzzy sphere [6]. Note that the spectral distance goes to zero near where phase transitions (denoted by a vertical line) were found to exist in [5, 44].

2.3.4 Liouville quantum gravity

It has long been known that random matrix theory has connections to two dimensional quantum gravity. These range from the famous Kontsevich model [89, 58], to the newly discovered connections to Jackiw-Teitelboim (JT) gravity [78, 81, 90, 67]. Of particular interest however, is the connection to conformal field theory. Heuristically, physicists knew in the 80's and 90's from the asymptotics of convergent matrix integrals (found using orthogonal polynomials) that certain matrix ensembles have critical exponents corresponding to models in conformal field theories coupled to gravity [21, 36].

The idea is that matrix models count maps which can be thought of as discretized Riemann surfaces. If the coupling constants of the models are tweaked such that the number of polygons that form these maps goes to infinity, one would in essence be counting Riemann surfaces, which are also counted in conformal field theories in two dimensions. Later in [10], this idea was made precise with formal matrix models. These formal models often have the same asymptotics as their convergent matrix model counterparts (if such a counterpart exists). We will briefly describe the idea behind these critical points here.

Suppose a formal random matrix model's partition function Z has the following genus expansion

$$\log Z = \sum_{g\geq 0} N^{2-2g} F_g,$$

where the F_g are the generating functions of certain types of maps (specified by the model) with no boundaries [41], see Appendix B.2 for details. Random matrix models often have critical points where the coupling constants of the model are such that the number of polygons in the maps goes to infinity. In such cases it is proven that the F_g have asymptotic expansions around these points, which have critical exponents corresponding to a minimal model in conformal field theory [35, 10]. Additionally, these matrix model asymptotics satisfy a partial differential equation that is also satisfied by the corresponding minimal model. For example, the quartic Hermitian matrix model corresponds to the $(3, 2)$ minimal model, also referred to as pure gravity, and its partition function satisfies Painlevé I.

How this relates to Dirac ensembles is not as obvious at a first glance. However, for several models examined in [52] we prove that single Hermitian matrix models are hidden in Dirac ensembles. In particular the cubic, quartic, and sextic Dirac ensembles of type $(1, 0)$ contain the cubic, quartic, and sextic Hermitian matrix models, respectively, in their phase space. This is nontrivial because in a Dirac ensemble coupling constants are not attached to specific bi-tracial terms. Instead many single and bi-tracial terms share the same coupling constants, so one cannot easily turn off all bi-tracial terms by setting certain coupling constants to zero.

Consider for example the quartic Dirac ensemble from Section 2.2.3. As alluded to, one can fine-tune its coupling constants such that we recover a quartic Hermitian matrix model in the large N limit. For a full proof see [52].

The single trace quartic model

$$\int_{\mathcal{H}_N} e^{-\frac{N}{2}\operatorname{Tr} H^2 - \frac{t_4}{4} N \operatorname{Tr} H^4} dH \tag{2.5}$$

has a critical point at $t_4 = -1/12$ which we will refer to as t_c [41]. The singular parts of the F_g are algebraic for all genus g, except for $g = 1$, in which case it is logarithmic. In particular they are

$$\operatorname{sing}(F_g) = C_g\,(t_4 - t_c)^{5(1-g)/2}$$

and

$$\operatorname{sing}(F_1) = C_1\,\log(t_4 - t_c)$$

for $g \neq 1$. These expansions are often referred to as a "double scaling limit". We may define the generating series

$$u(y) = \sum_{g=0}^{\infty} C_g y^{5(1-g)/2}.$$

Then $u''(y)$ satisfies the Painlevé I equation, i.e.

$$y = (u''(y))^2 - \frac{1}{3}u^{(4)}(y).$$

We know that the critical point of the quartic Hermitian model is $t_4 = -1/12$ [41], and that the quartic Dirac ensemble

$$Z = \int_{\mathcal{D}} e^{-\frac{t_2}{4}\operatorname{Tr} D^2 - \frac{t_4}{8}\operatorname{Tr} D^4} dD \tag{2.6}$$

contains it i.e. for a certain choice of coupling constants you recover the quartic Hermitian model. In particular, this happens when $t_2 = 4/3$ and $t_4 = -1/12$ [52]. Figure 2.6 allows one to visualize several phenomena of the model in a phase diagram.

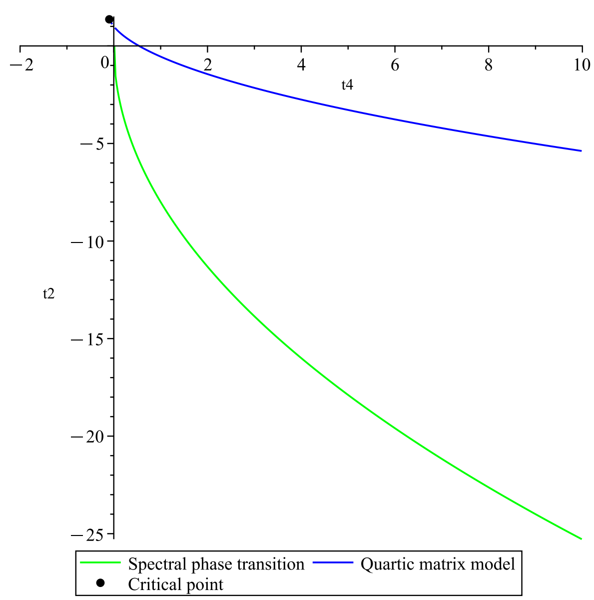


Figure 2.6: The phase diagram of the quartic Dirac ensemble [52]. The y-axis is t_2 and the x-axis is t_4.

In this diagram we have the curve where a spectral phase transition occurs, this is a generalization of the phenomenon in Section 2.3.2 with two coupling constants. The equation of this curve is

$$t_2 = -8\sqrt{t_4}.$$

Below the critical (green) curve the spectrum of the Dirac ensemble is in a 3-cut phase and above it 1-cut, as seen in Section 2.3.2.

The diagram also shows for what values of (t_2, t_4) the Dirac ensemble in equation (2.6) has the quartic Hermitian matrix model, equation (2.5), as the associated matrix model if the coupling constants lie on the blue curve with equation:

$$t_2 = -\frac{(1+12t_4)^{3/2} - 4 - 144t_4 + (36t_4+3)\sqrt{1+12t_4}}{72t_4}.$$

These equations are derived in [52] by solving the loop equations. Note that these curves do not intersect because the quartic Hermitian matrix model as stated in equation (2.5) does not have a spectral phase transition. The reason is that there is no coupling constant in front the $\operatorname{Tr} H^2$, which is required to reach the phase transition.

Hermitian multi-matrix models are also associated with a much wider class of minimal models than single matrix Hermitian matrix models [33, 35]. Thus it would be interesting to investigate numerically if higher signature Dirac ensembles are also associated to minimal models. The question also remains unproven as to which $(1, 0)$ or $(0, 1)$ Dirac ensembles correspond to minimal models. Additionally, there might also be other critical points in Dirac ensembles besides those mentioned or other interesting critical phenomena.

The authors would like to emphasize that none of the connections between conformal field theory and random matrix theory mentioned here are new. The point is rather to lend credit to a noncommutative theory of quantum gravity where integrating over finite Dirac operators in place of metrics can be used to recover other toy models of quantum gravity.

2.4 Bootstrapping the loop equations

Bootstrapping was introduced in elementary particle physics as part of the S-matrix program by Geoffrey Chew in the early 1960's. The idea was to use any consistency conditions available to compute various correlation functions of interest and especially to formulate a theory of strong interactions. The mantra was "particles pull themselves up by their own bootstraps". But, after an initial success, the idea stalled in producing viable new results and predictions. Meanwhile a competing theory, the standard model of elementary particles, was created based on the theory of quarks and gauge theory, which could indeed successfully account for experimental data. As a result, bootstrap methods were nearly forgotten for a long time. In recent years, however, there has been a revival of the bootstrap idea mostly thanks to the success of the conformal bootstrap program by Rattazzi, Rychkov, and collaborators in 2008 in understanding phase transition and critical phenomena in dimensions bigger than two [73]. In two dimensions, the conformal bootstrap was demonstrated to work in 1983 by Belavin, Polyakov and Zamolodchikov [8].

In the context of random matrix theory and related fields, bootstrapping recently emerged in several works, first by Anderson and Kruczenski in the context of lattice gauge theory [2]. In a random matrix setting bootstrapping was first used by Lin [63], then in our paper [51], and also by Kazakov and Zheng [55]. Bootstrapping has also recently been applied to matrix quantum mechanics as well [50, 11, 9]. In the following section we present a new example that deals with a cubic Dirac ensemble as well as recalling our results from [51]. To readers unfamiliar the with Schwinger-Dyson equations in the context of random matrix theory we recommend reviewing Appendix B.3.

In another interesting new development, bootstrapping is now used in computing the spectrum of Einstein and hyperbolic manifolds in [12, 13]. The eigenvalues of the Laplace-Beltrami operator, as well as the integrals of their eigenfunctions, satisfy certain positivity conditions that imply bounds on both quantities. One wonders whether an extension of these ideas to some classes of spectral triples is possible.

2.4.1 The cubic type $(1,0)$ Dirac ensemble

We will start with a brief overview of how bootstrapping works for Dirac ensembles. The large N limits of higher moments of random matrix models satisfy an infinite system of nonlinear equations, which was first derived by Migdal [68]. These so-called *loop equations* are consequences of Schwinger-Dyson equations and the factorization property of moments at large N limits. In general the loop equations are not restrictive enough to fully determine the moments. However, as we shall explain later in this section, one can bring to bear some positivity constraints on moments to further restrict the set of possible solutions to the loop equations. The process of further narrowing the search space by adding certain extra positivity constraints is called *bootstrapping*. Further positivity constraints are obtained from the fact that our matrix models stem from Dirac operators of spectral triples. This extra positivity is quite useful and is missing in general matrix models. By narrowing down the search space one can sometimes recover the values of the initial moments. From there the loop equations can be used, in theory, to find any moment.

We will now give a novel example of the bootstrap method by applying it to a cubic Dirac ensemble. Using the bootstrap technique we are able to find a relationship between the coupling constant of the model, the first moment and from there higher moments. Let us consider a type $(1,0)$ cubic Dirac ensemble with the partition function

$$Z = \int_{\mathcal{D}} e^{-\frac{1}{4}\operatorname{Tr} D^2 - \frac{g}{6}\operatorname{Tr} D^3} dD.$$

This integral is obviously not convergent but understood perturbatively, i.e. as a formal matrix model. The associated matrix model is a bi-tracial Hermitian single-matrix model with the potential

$$\tilde{S}(H) = \frac{1}{2}\left(N \operatorname{Tr} H^2 + (\operatorname{Tr} H)^2\right) + \frac{g}{3}\left(N \operatorname{Tr} H^3 + 3 \operatorname{Tr} H^2 \operatorname{Tr} H\right).$$

The loop equations of the model are then given by:

$$\sum_{k=0}^{\ell-1} m_k m_{\ell-k-1} = m_{\ell+1} + m_1 m_\ell + g\left(m_{\ell+2} + 2m_1 m_{\ell+1} + m_2 m_\ell\right) \quad \text{for } \ell \in \mathbb{N} \cup \{0\}. \tag{2.7}$$

In particular we get the following loop equation for $\ell = 0$:

$$g\left(m_2 + m_1^2\right) + m_1 = 0.$$

It can be shown that by having the first moment m_1, we can recursively calculate higher moments using the loop equations. We refer to such a situation as the dimension of the search space being one. If the loop equations required n moments or higher moments to determine all other (higher) moments, we would say the search space has dimension n.

The existence of an eigenvalue density function $\rho(x)$, which is a probability density function, gives us constraints on moments in the following way. Take a real polynomial $f(x) = \sum_{j=1}^{k} c_j x^j$. Then by the non-negativity of the integral we have

$$\int_{\mathbb{R}} f(x)^2 \rho(x) dx = \sum_{i,j=1}^{k} \int_{\mathbb{R}} c_i c_j x^{i+j} \rho(x) dx = \sum_{i,j=1}^{k} c_i c_j m_{i+j} \geq 0,$$

for all real values of the c_i. This shows that the quadratic form $\sum_{i,j=1}^{k} c_i c_j m_{i+j}$ is positive semi-definite. Since this holds for all $k = 1, 2, ...$ and it can be expressed nicely in terms of the positive semi-definiteness of the Hankel matrix of moments

$$\begin{bmatrix} m_0 & m_1 & m_2 & m_3 & \cdots \\ m_1 & m_2 & m_3 & m_4 & \cdots \\ m_2 & m_3 & m_4 & m_5 & \cdots \\ m_3 & m_4 & m_5 & m_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \geq 0.$$

As a side remark we should add that Hamburger's Theorem says that positivity of this matrix is a necessary and sufficient condition for a sequence of real numbers $m_0, m_1, m_2, ...$ to be the moments of a probability distribution, see page 145 of [74].

The positive semi-definiteness of the Hankel matrix tells us, in particular, that every leading principal minor is greater than or equal to zero. This gives us countably many inequalities, or constraints, involving the moments. Combining this observation with the fact that in this particular model every moment can be written in terms of m_1, thanks to the structure of the loop equations (2.7), we obtain an infinite number of nonlinear constraints on m_1. Using semidefinite programming to find the region satisfied by these constraints gives us Figure 2.7.

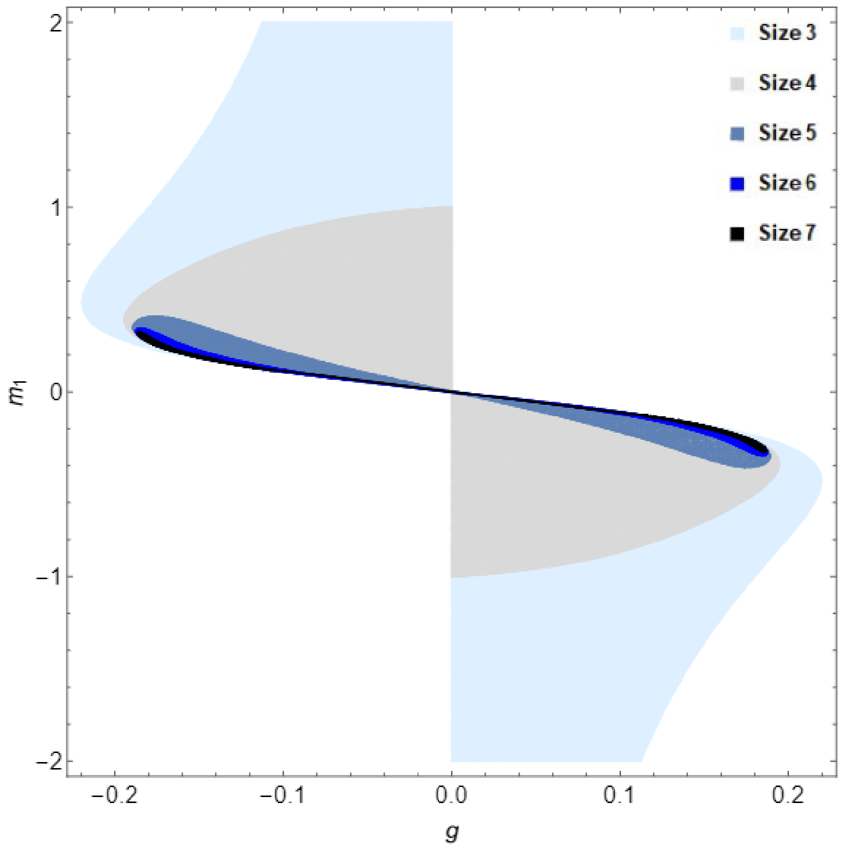


Figure 2.7: The constraints on the relation between g and m_1 for the (1,0) cubic model found by bootstrapping. Each colour corresponds to a different number of constraints derived from positivity of principal minors. The solution space narrows as the number of constraints increases. Notice that in this example increasing the number of constraints shows that there exists a nonlinear relationship between g and m_1. Additionally the analytic solution from [52] is plotted for comparison.

It is worth noting that positivity constraints can be applied to both the Hankel matrix of moments of the matrix ensemble and the Hankel matrix of moments of the Dirac ensemble. As mentioned earlier the moments of the Dirac operator are defined as

$$d_\ell = \lim_{N\to\infty} \left\langle \frac{1}{N^2} \operatorname{Tr} D^\ell \right\rangle = \lim_{N\to\infty} \frac{1}{N^2} \frac{1}{Z} \int_{\mathcal{D}} \operatorname{Tr} D^\ell e^{-\frac{1}{4}\operatorname{Tr} D^2 - \frac{g}{6}\operatorname{Tr} D^3} dD,$$

so by the same argument as before we have

$$\begin{bmatrix} 1 & d_1 & d_2 & d_3 & \cdots \\ d_1 & d_2 & d_3 & d_4 & \cdots \\ d_2 & d_3 & d_4 & d_5 & \cdots \\ d_3 & d_4 & d_5 & d_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \geq 0.$$

These additional constraints are a particular advantage that Dirac ensembles have over ordinary matrix ensembles when bootstrapping.

We emphasize that this model can be solved analytically [52], but is presented here as an insightful example of the bootstrap technique.

2.4.2 The quartic type $(2,0)$ Dirac ensemble

Bootstrapping can also be applied successfully to ensembles that are, to the best of our knowledge, unsolvable. To illustrate this let us consider the quartic action for a type $(2,0)$ ensemble, which appears in [5, 44, 6],

$$Z = \int_{\mathcal{D}} e^{-g \operatorname{Tr} D^2 - \operatorname{Tr} D^4} dD,$$

where the associated matrix potential is given by equation (2.3). We will summarize the results of [51] to show the effectiveness of bootstrapping for this model. Note that the action of this model is symmetric under the transformations

$$\begin{gathered} D \to -D, \\ H_1 \to -H_1, \\ H_2 \to -H_2, \\ \text{and} \\ H_1 \leftrightarrow H_2. \end{gathered}$$

This greatly simplifies the loop equations. In particular all the odd moments and odd higher moments are zero i.e. any moment of a word in H_1 and H_2 containing either an odd number of H_1 or H_2 is zero. After considering these symmetries the following terms of the potential are the ones that contribute to the loop equations in the large N limit

$$\begin{aligned} &g(4N \operatorname{Tr} H_1^2 + 4N \operatorname{Tr} H_2^2) + 4N \operatorname{Tr} H_1^4 + 4N \operatorname{Tr} H_2^4 \\ &+ 16N \operatorname{Tr} H_1^2 H_2^2 - 8N \operatorname{Tr} H_1 H_2 H_1 H_2 + 12(\operatorname{Tr} H_1^2)^2 \\ &+ 12(\operatorname{Tr} H_2^2)^2 + 8 \operatorname{Tr} H_1^2 \operatorname{Tr} H_2^2. \end{aligned}$$

We use the following loop equations for bootstrapping:

$$\sum_{k=0}^{\ell-1} m_k m_{\ell-k-1} = (8g + 64m_2)m_{\ell+1} + 16m_{\ell+3} - 16m_{\ell,1,1,1} + 32m_{\ell+1,2}$$

in the large N limit where we denote mixed moments as

$$m_{a,b,c,d} = \lim_{N\to\infty} \frac{1}{N}\langle \mathrm{Tr}\, H_1^a H_2^b H_1^c H_2^d \rangle.$$

Positivity constraints for mixed moments can be derived but require a slightly more general setting.

Unlike the approach in [63], we considered the moments of all words. By doing this we were able to prove in [51] that the search space for this model has dimension one. The results are displayed in Figure 2.8. One interesting feature is that the relation between m_2 and g appears to be linear for values of g below the phase transition [44]. This is also what was observed analytically in the type $(1,0)$ quartic in [56].

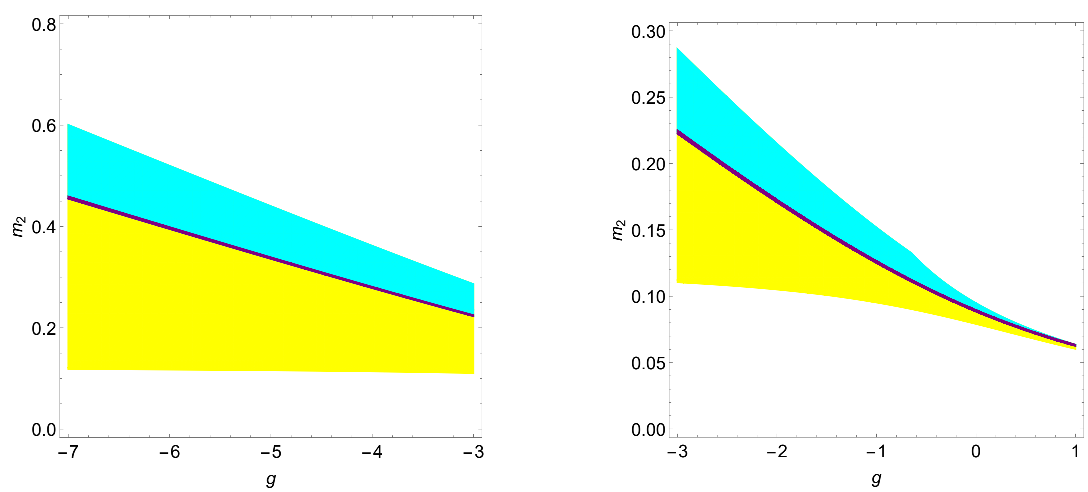


Figure 2.8: The search space region for the (2,0) quartic Dirac ensemble m. Note that near the phase transition found in [5] the relationship between the coupling constant g and m_2 appears to change from non-linear to linear [51].

By generating all the loop equations using *Maple* we found the following remarkable formulas for moments in terms of g and the second moment m_2:

$$m_4 = -\frac{1}{8}gm_2 + \frac{1}{64},$$

$$m_{2,2} = -\frac{1}{8}gm_2 - m_2^2 + \frac{1}{64},$$

$$
\begin{aligned}
m_{1,1,1,1} &= \frac{gm_2}{8} + 2m_2^2 - \frac{1}{64},\\
m_6 &= \frac{g^2 m_2}{64} - \frac{g}{512} - \frac{g{m_2}^2}{8} + \frac{3\,m_2}{64} - \frac{5\,{m_2}^3}{4},\\
m_{4,2} &= \frac{g^2 m_2}{64} + \frac{g{m_2}^2}{8} - \frac{g}{512} - \frac{{m_2}^3}{4} + \frac{m_2}{64},\\
m_{3,1,1,1} &= -\frac{g^2 m_2}{64} - \frac{3\,g{m_2}^2}{8} - \frac{7\,{m_2}^3}{4} + \frac{g}{512} + \frac{m_2}{64},\\
m_{2,1,2,1} &= \frac{g^2 m_2}{64} + \frac{3\,g{m_2}^2}{8} - \frac{g}{512} + \frac{11\,{m_2}^3}{4} - \frac{m_2}{64},
\end{aligned}
$$

$$
m_8 = -\frac{gm_2}{64} + \frac{{m_2}^4}{4} + \frac{g^2}{4096} + \frac{{m_2}^2}{256} + \frac{3}{4096} - \frac{g^3 m_2}{512} + \frac{3\,g^2{m_2}^2}{64} + \frac{g{m_2}^3}{2}.
$$

Note that the trace powers and therefore moments of this Dirac ensemble do not have a clear formula. These trace powers were studied closely in [70]. With the above formulas and those borrowed from [70] we have

$$
\begin{aligned}
d_2 &= 8\,m_2,\\
d_4 &= -4\,gm_2 + \frac{1}{2},\\
d_6 &= -160\,{m_2}^3 - 16\,g{m_2}^2 + 6\,m_2 + 2\,g^2 m_2 - \frac{1}{4}g.
\end{aligned}
$$

2.5 Summary and outlook

In this paper we gave an overview of the recent efforts to utilize random matrix theory techniques to give insight into toy models of Euclidean quantum gravity suggested by noncommutative geometry and initially proposed in [5]. We saw that type $(1,0)$ or $(0,1)$ Dirac ensembles can be analyzed analytically using the Coulomb gas technique [56] and with the Blobbed Topological Recursion of stuffed maps [3, 57]. However, for ensembles with dimension higher than one no known analytic techniques of random matrix theory seem to apply. Instead they may be examined at finite matrix size using Monte Carlo simulations [5, 44] or at large matrix size using bootstrap techniques [63]. Most recently it was discovered that certain Dirac ensembles are dual to minimal models in conformal field theory [52]. It is worth investigating if this is true for more types and potentials. Additionally one wonders if connections to other theories of quantum gravity are possible, such as the recent connection found between random matrix theory, Topological Recursion, and JT gravity [78, 81, 90, 67].

One naturally wants a coupling of these models with fermions and gauge fields. In noncommutative geometry there is a finite spectral triple F of the standard model of elementary particles [30, 84]. In their work, Chamseddine, Connes and Marcolli [23] consider spectral triples of the form $X \times F$, where X is a Riemannian manifold which represents the gravitational sector. Using the spectral action principle [22] and heat kernel expansion they were able to obtain the Lagrangian of the standard model coupled with gravity. For a recent account and survey see [24]. Instead of a manifold, we can take a noncommutative space and a finite real spectral triple as a good first approximation to that. We should mention that the initial steps in this direction has already been taken in [43], and also in [72]. Especially in the latter work the kinematics of coupling the gravitational field, in the context of finite spectral triples, with the Yang-Mills-Higgs field of the standard model is worked out. What remains to be done is to choose a suitable potential S to go in the path integral, to calculate various quantities of interest, and also to study the large N and double scaling limits of these quantities. We hope to come back to this project in the near future.

An alternative approach to path integral quantization is the BV formalism. In the context of gauge theory on spectral triples this has been studied in [54] and the BV formalism is applied directly to Dirac ensembles in [42].

Finally let us suggest some open questions and problems in this line of research:

- Investigate the limiting eigenvalue distribution of Dirac ensembles with more complicated potentials. This could be done numerically for any type or analytically for types $(1,0)$ and $(0,1)$ using the techniques outlined in [56].

- Do one dimensional Dirac ensembles have critical points other than those that appear when the coupling constants are tuned to become a single trace model [52]?

- Do these new critical points have double scaling limits of correlation functions that obey Blobbed Topological Recursion? This is seen in the single trace case [41].

- Are there minimal models associated with higher dimensional Dirac ensembles? This could be investigated by looking at critical exponents using Monte Carlo simulations [44] or perhaps using the functional renormalization group [71].

- Can one find and make rigorous a connection between Dirac ensembles at phase transitions and the spectra of two dimensional manifolds, along the lines of [6]?

- Is there a connection between Dirac ensembles and the recent work in noncommutative QFT [80, 17, 18, 53]? One of the goals of this series of papers is to prove that the correlation functions of the quartic Kontsevich model satisfy Blobbed Topological Recursion [14]. As proved in [3], certain Dirac ensembles also satisfy it. For a review see [19].

- Apply the Batalin-Vilkovisky (BV) formalism of [42] to other Dirac ensembles.
- Investigate a possible relationship between Dirac ensembles and one-loop corrections to the spectral action [83].
- Can we extend the Yang-Mills-Higgs theory to more general Dirac ensembles, coupling gravity with the standard model? [72].
- Investigate the consequences of adding a Fermionic term to the action of Dirac ensembles.

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Chapter 3

Bootstrapping Dirac Ensembles

3.1 Introduction

In this paper we use the bootstrap method to find the moments of certain *multi-trace* and *multi-matrix* random matrix models inspired by noncommutative geometry. These large N limit moments satisfy an infinite system of nonlinear equations, due to Migdal [30], called *loop equations*. The loop equations are consequences of Schwinger-Dyson equations (SDE's). The SDE's put constraints on the moments and these constraints help to narrow the search for moments, but this is usually not enough. The process of further narrowing the search space by adding certain extra *positivity constraints* is called *bootstrapping*. This idea was recently used by Anderson and Kruczenski in the context of lattice gauge theory [1] . Then in a random matrix setting it was used by Lin [28], and will be employed throughout this paper as well. We will see that further positivity constraints are obtained from the fact that our matrix ensembles originate from Dirac ensemble. This is an added feature that is absent in standard matrix models. By narrowing down the search space one can sometimes recover the values of the initial moments. From there the loop equations can be used, in theory, to find any moment. Using the bootstrap technique we are able to find the relationships between the coupling constant of the model and the second moment. From there all other moments can be expressed in terms of the coupling constant and the second moment, allowing them to be computed explicitly. We also obtain explicit relations for higher mixed moments.

By a *Dirac ensemble* we mean an statistical ensemble of *finite real spectral triples* where the Fermion space is kept fixed, but the Dirac operator is allowed to be random subject to constraints of a real spectral triple. Technical definitions will be given further below. Such ensembles were first defined by Barrett and Glaser [5] with the goal of building toy models of Euclidean quantum gravity over a finite noncommutative space. They studied these models via computer simulation, in particular by Markov chain Monte Carlo methods. They also indicated phase transition and multi-cut regimes in their spectral distribution. Quite significantly they also noticed, numerically and for particular models, that at the phase transition point the limiting spectral distribution of their models resembles the Dirac eigenvalue distribution of a round sphere. That is they have a manifold behaviour at phase transition. One expects all such models to have a manifold type behaviour at phase transition points, but we are still far from proving this attractive conjecture rigorously. We shall indicate the main reason for this further below in this introduction. In [2, 3, 33, 34, 35, 25, 26, 18], formal and analytic aspects of these models and their generalizations are studied through topological recursion techniques as well as standard random matrix theory methods.

In these models the integration over metrics is replaced with integration over Dirac operators,

$$Z = \int_{metrics} e^{-S(g)} D(g) \quad \Rightarrow \quad \mathcal{Z} = \int_{Diracs} e^{-S(D)} dD. \tag{3.1}$$

This can be justified by general principles of noncommutative geometry to be explained below. In particular Dirac operators are taken as dynamical variables and play the role of metric fields in gravity. It is a feature of these models that the moduli space of Dirac operators are typically finite dimensional vector spaces. The action functional S is chosen in such a way that the partition function $\mathcal{Z}$ is absolutely convergent and finite. For example, $S(D) = \mathrm{Tr}(f(D))$ for a real polynomial f of even degree with a positive leading coefficient. Note that, in general, these matrix integrals are not necessarily convergent, or even need not have a real valued potential. However, they may always be interpreted as formal matrix integrals, which are the generating functions of certain types of maps [17]. We mention that the models considered by Barrett and Glaser and in this paper are always convergent. For more details on formal and convergent matrix models see [16]. For a treatment of these integrals when convergent we refer the reader to [25].

The backbone of a spectral triple is the data $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an involutive complex algebra acting by bounded operators on a Hilbert space $\mathcal{H}$, and D is a self-adjoint (in general unbounded) operator acting on $\mathcal{H}$. This data is required to satisfy some regularity conditions. For finite spectral triples these conditions are automatically satisfied. A real spectral triple is equipped with two extra operators J and γ, the charge conjugation and the chirality operator. Finite dimensional real spectral triples have been fully classified by Krajewski in [27]. Other references include [15, 29, 4, 37]. In this paper we work exclusively with finite dimensional real spectral triples introduced by Barrett in [4]. Such finite real spectral triples represent a noncommutative finite set equipped with a metric.

Using the classifications of finite spectral triples and their Dirac operators one can express the Barrett-Glaser models as multi-trace and multi-matrix random matrix models [4, 5] (cf. also [2, 3, 33, 34, 35]). Since most of the results in random matrix theory is for single matrix and single trace models, this shows that the analytic study of these models as convergent matrix integrals is quite difficult in general. The scarcity of analytic tools is one of the reasons we are still far from a rigorous proof of the manifold type behaviour of these models at phase transition. The first analytic treatment of these models was carried out in [25] where the phase transition for type $(1, 0)$ and $(0, 1)$ models was rigorously proved. In a recent paper the large N limit spectral density function of Gaussian Dirac ensembles is obtained and shown to be given by the convolution of the Wigner semicircle law with itself [26]. We also mention that in [33, 34] the algebraic structure of the action functional of these models is further analyzed and linked to free probability theory. This gives more hope for analytic treatment of these models in general.

One of the motivations behind the introduction of these models in [5] is the well known observation that a combination of the Heisenberg uncertainty principle and Einstein's general relativity will force the spacetime to lose its classical nature as a pseudo-Riemannian manifold at Planck length. This is essentially due to formation of black holes when we probe the space at Planck length. There are many proposals as to what should replace the classical spacetime. A noncommutative space in the sense of spectral triples is an attractive proposal since the metric, as a necessary dynamical variable of a theory of gravity, is already encoded by the Dirac operator. Furthermore, assuming the spectral triple to be finite, allows computer simulation as well as methods of random matrix theory to be applied and various scenarios to be tested. There is also a more ambitious hope of coupling these models with fermions, as they appear in the finite spectral triple F of the standard model of elementary particles [15, 37]. This will be along the lines of the work of Chamseddine, Connes and Marcolli [12] where they consider spectral triples of the form $X \times F$, where X is a Riemannian manifold which represents the gravitational sector. Using the spectral action principle [10] and heat kernel expansion they were able to obtain the standard model Lagrangian coupled with gravity. For a recent account and survey see [11]. Again instead of a manifold, one needs to use a noncommutative space and a finite real spectral triple is a good first approximation to that. We should mention that the initial steps in this direction has already been taken in [18], and also in [35]. Specially in the latter work the kinematics of coupling the gravitational field, in the context of finite spectral triples, with the Yang-Mills-Higgs field of the standard model is worked out. What remains to be done, and that is a tall order indeed, is to choose a suitable potential S to go in the path integral (3.1), and to calculate various quantities of interest and also study the large N limits of them. We hope to come back to this project in the near future.

The idea of replacing metrics by Dirac operators is justified by general principles of noncommutative geometry as we explain next. For example the distance formula of Connes [13]

$$d(p,q) = \mathrm{Sup}\{|f(p) - f(q)|;\ \|[D,f]\| \leq 1\},$$

shows that the geodesic distance on a Riemannian spin manifold can be recovered from the action of its Dirac operator D on the Hilbert space of spinors. This role of the Euclidean Dirac operator as a selfadjoint elliptic operator can be abstracted and cast in the notion of a (real) spectral triple which simultaneously encodes the data of a Riemannian metric, a spin structure, and a Dirac operator on a commutative or noncommutative space [13]. A deep result, the *reconstruction theorem* of Connes [14], shows that a spin Riemannian manifold can be fully recovered from a commutative real spectral triple satisfying some natural conditions. For this reason, real spectral triples can be regarded as noncommutative spin Riemannian manifolds, and the space of their compatible Dirac operators as the space of Riemannian metrics.

Finite real spectral triples of interest in this paper are characterized by a pair of non-negative integers (p,q) [4, 5]. These integers count the number of gamma matrices that square to one and minus one, respectively. As p and q increase, the corresponding Dirac ensemble gets more and more complicated as a multi-matrix and multi-trace matrix model. The signature of the model $s = p + q$ determines the multiplicity of the matrix model. Consider for example the case when $p = 1$ and $q = 0$. The Dirac operator can then be expressed as $D = H \otimes I + I \otimes H$ and the partition function (3.1) reduces to

$$\mathcal{Z} = \int_{\mathcal{H}_N} e^{-\tilde{S}(H)} dH,$$

where $\tilde{S}$ is some new potential function in H and dH is the Lebesgue measure on the space $\mathcal{H}_N$ of $N \times N$ Hermitian matrices. Note that even in this case $\tilde{S}$ is a multi-trace function. We shall see more examples later in this paper. In this paper we are mainly concerned with finding the moments of these models in the large N limit i.e. when the matrix size approaches infinity. In the above mentioned $(1,0)$ ensemble they are defined as

$$m_k = \lim_{N\to\infty} \langle \frac{1}{N} \operatorname{Tr} H^k \rangle = \lim_{N\to\infty} \frac{1}{N} \frac{1}{Z} \int_{\mathcal{H}_N} \operatorname{Tr} H^k e^{-\tilde{S}(H)} dH.$$

Mixed moments are also defined in a similar manner.

Here is a brief outline of the contents of this paper. In Section two we derive and discuss the Schwinger-Dyson equations, the loop equations, as well as the crucial idea of mixed moment factorization in the large N limit. We will then explain the positivity constraints on moments in both single and multi-matrix models. In Section three we will compare the numerical results from bootstrapping to the analytic solution for signature one models obtained in [25]. In Section four we will compare features that we found using the bootstrap method for signature two models with those obtained in [5] via Monte Carlo simulation. In the Appendix A we briefly explain how the factorization of mixed moments is obtained from their genus expansion.

We would like to warmly thank the referees for several very useful comments and suggestions that we believe led to a better exposition of this paper.

3.2 The bootstrap method

3.2.1 The loop equations

In general, the partition function of a Dirac ensemble is a multi-trace and multi-matrix model of the form

$$\mathcal{Z} := \int_{\mathcal{H}_N^m} e^{-\tilde{S}(H_1,H_2,\ldots,H_m)} dH_1 \ldots dH_m,$$

where $\tilde{S}$ is the trace of a polynomial in traces of the m variables $H_1, \ldots, H_m$ and their products with suitable powers of N in the coefficients. This integral is over the Cartesian product of m copies of the space $\mathcal{H}_N$ of Hermitian $N \times N$ matrices and the integration is with respect to the Lebesgue measure in each matrix variable. Such an integral can be considered either as a formal or convergent matrix integral. Note that both types of models satisfy the same SDE's. The SDE's relate the moments of the model in some word W in the alphabet of matrix variables $\{H_1, H_2, ..., H_m\}$, defined as expectation values

$$\langle \frac{1}{N} \operatorname{Tr} W \rangle := \frac{1}{N} \frac{1}{Z} \int_{\mathcal{H}_N^m} \operatorname{Tr} W e^{-\tilde{S}(H_1,H_2,\ldots,H_m)} dH_1 \ldots dH_m.$$

The SDE's are a common technique used in Random Matrix Theory [17, 21] and can be derived in the following manner. We shall be very brief. Take a word W as before and consider the following relation

$$\sum_{i,j=1}^{N} \int_{\mathcal{H}_N^m} \frac{\partial}{\partial (H_q)_{ij}} \left(W_{ij}\, e^{-\tilde{S}(H_1,H_2,\ldots,H_m)} \right) dH_1 \ldots dH_m = 0,$$

where W_{ij} denotes the (i, j)-entry of the product of matrices that make up the word W. This relation easily follows from the Stokes' theorem. The use of the product rule in the left hand side results in the Schwinger-Dyson equations. For example, when $m = 1$, $W = H_1^{\ell}$, and $\tilde{S}(H_1) = \frac{N}{2} \operatorname{Tr} H_1^2$, the above equation generates the following relations

$$\sum_{k=0}^{\ell-1} \langle \operatorname{Tr} H_1^{\ell-1-k} \operatorname{Tr} H_1^{k} \rangle = \langle N \operatorname{Tr} H_1^{\ell} \tilde{S}'(H_1) \rangle = \langle N \operatorname{Tr} H_1^{\ell+1} \rangle.$$

For a finite N the SDE's put some constraints on moments and in general do not determine the moments. However, if the large N limits of moments exist, then the SDE's simplify dramatically[1]. This is a consequence of the factorization property of the large N limits of mixed moments. In particular, when $m = 1$, it is well known that the following factorization holds in the large N limit

$$\langle \operatorname{Tr} H_1^a \operatorname{Tr} H_1^b \rangle = \langle \operatorname{Tr} H_1^a \rangle \langle \operatorname{Tr} H_1^b \rangle,$$

for any positive integers a and b. This is true for single trace single matrix convergent models [32], and for formal multi-trace single matrix models. This factorization holds also in some formal multi-matrix models ($m > 1$), in particular when those models have a genus expansion. See the appendix B for a detailed explanation. One key assumption we will make for the signature two models is that this factorization does hold. This is proved in [26].

One key fact to notice is that the loop equations are relations for generating higher order moments from lower order ones. In particular one might wonder what is the minimum number of generating moments one needs to generate all of the rest. Using the terminology from [28] we will refer to this collection of moments as the *search space*. What we will see in the following sections is that the search space of the matrix ensembles studied here are of only one dimension.

3.2.2 Bootstrapping models with positivity

The positivity constraints on moments of a single matrix model in the large N limit are related to the *Hamburger moment problem*, as it was noticed by Lin [28]. However, positivity and bootstrapping was already used in the context of solving the loop equations for lattice gauge theory by Anderson and Kruczenski [1]. Bootstrapping has recently been used in several other works on matrix models including [24, 23], and for matrix quantum mechanics [22, 8, 7].

The Hamburger moment problem can be formulated as follows: given a sequence $(m_0, m_1, m_2, \dots)$ of real numbers, one asks if there is a positive Borel measure μ on the real line so that m_k is the kth moment of μ, that is

$$m_k = \int_{\mathbb{R}} x^k d\mu(x), \quad k = 0, 1, 2, \dots.$$

It is known that a necessary and sufficient condition for the existence of μ is that the *Hankel*

[1]The large N limit is understood differently for formal and convergent models, we again refer the reader to [16].

matrix of the moments

$$\mathcal{M} = \begin{bmatrix} m_0 & m_1 & m_2 & m_3 & \cdots \\ m_1 & m_2 & m_3 & m_4 & \cdots \\ m_2 & m_3 & m_4 & m_5 & \cdots \\ m_3 & m_4 & m_5 & m_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

is a positive matrix. That is

$$\sum_{j,k\geq 0} m_{j+k} c_j \overline{c_k} \geq 0,$$

for all sequences (c_j) of complex numbers with finite support. In fact checking the necessity of this condition is quite easy. Take the polynomial $f(x) = \sum c_j x^j$. Then the positivity of the integral $\int_{\mathbb{R}} f(x)\overline{f(x)}dx$ immediately implies the positivity of the Hankel matrix. For a proof of the sufficiency of the condition see page 145 of [36].

In our context $m_0 = 1$,

$$m_k = \lim_{N\to\infty} \frac{1}{N}\langle \mathrm{Tr} H^k \rangle, \quad k = 1, 2, \ldots,$$

and $d\mu(x) = \rho(x)dx$, where $\rho(x)$ is the limiting spectral density function.

These positivity constraints can be applied to both the Hankel matrix of moments of the matrix ensemble and the Hankel matrix of moments of the Dirac ensemble defined as

$$\mathcal{D} = \begin{bmatrix} 1 & d_1 & d_2 & d_3 & \cdots \\ d_1 & d_2 & d_3 & d_4 & \cdots \\ d_2 & d_3 & d_4 & d_5 & \cdots \\ d_3 & d_4 & d_5 & d_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where

$$d_\ell = \lim_{N\to\infty} \langle \frac{1}{N^2} \mathrm{Tr}\, D^\ell \rangle = \lim_{N\to\infty} \frac{1}{N^2} \frac{1}{Z} \int_{\mathcal{G}} \mathrm{Tr}\, D^\ell e^{-S(D)} dD.$$

These additional constraints are one advantage that Dirac ensembles have over matrix ensembles when bootstrapping.

There is a generalization of the (univariate) Hamburger moment problem to the non-commutative, multivariate case [9]. To discuss this generalization that we need in this paper we require the following definition.

Definition 3.2.1 *The sequence of the real number $\{m_w\}_{w\in W}$ indexed by word w, where W is the space of the words formed by Hermitian matrices $H_1, H_2, \cdots, H_n$, is called a tracial sequence, if $m_w = m_u$ whenever w and u are cyclic equivalent.*

The truncated moment problem asks for necessary and sufficient conditions for a tracial sequence to be a sequence of the moments of some non-commutative, multivariate distribution.

Consider the large N limit of a multi-trace multi-matrix model of the following form

$$\mathcal{Z} := \int_{\mathcal{H}_N^m} e^{-S(H_1,H_2,\ldots,H_m)} dH_1 \ldots dH_m.$$

The (infinite) tracial moment matrix $\mathcal{M}(m)$ of a tracial sequence $m = \{m_w\}$ indexed by words is defined by the symmetric matrix

$$\mathcal{M}(m) = (m_{w*u})_{w,u}.$$

The necessary, but not sufficient, condition for a sequence of $\{m_w\}_{w\in W}$ is positive semi-definiteness of the tracial moment matrix [9].

For instance, with tracial sequence $m_\emptyset = 1, m_A, m_B, m_{AA}, m_{AB}, m_{BB}, \ldots$, we can enforce positivity of the sub-matrix of $\mathcal{M}$, defined as

$$\begin{bmatrix} 1 & m_A & m_B & m_{AA} & m_{AB} & m_{BB} \\ m_A & m_{AA} & m_{AB} & m_{AAA} & m_{AAB} & m_{ABB} \\ m_B & m_{BA} & m_{BB} & m_{BAA} & m_{BAB} & m_{BBB} \\ m_{AA} & m_{AAA} & m_{AAB} & m_{AAAA} & m_{AAAB} & m_{AABB} \\ m_{AB} & m_{BAA} & m_{BAB} & m_{BAAA} & m_{BAAB} & m_{BABB} \\ m_{BB} & m_{BBA} & m_{BBB} & m_{BBAA} & m_{BBAB} & m_{BBBB} \end{bmatrix}.$$

3.2.3 The algorithm

A *Python* script is first used to generate the loop equations for all possible words up to a given order. This order is dependent on how many loop equations it takes to deduce the dimension of the search space. Once we have found the search space and generated a sufficient number of loop equations we compute all moments, assemble them into the matrices outlined above, then check for positivity of the matrix and various submatrices. This process is done for both matrix moments and Dirac moments. From here *Mathematica* is able to numerically find the region in which its corresponding positivity constraints are satisfied. We increase the number of constraints until a satisfactorily small region is found.

The manner in which our algorithm differs from [28] is twofold. First, in the multi-matrix models loop equations are generated for all possible words. One might expect that this would hinder us from finding the search space by introducing more moments. In fact, we found that this was helpful in finding the search space. Secondly, we are working with Dirac ensembles which have both matrix moments and Dirac moments, allowing us to derive more positivity constraints than if we were working with just a matrix model.

3.3 One matrix Dirac ensembles

Consider real finite spectral triples $(A, \mathcal{H}, D)$ where the algebra is $A = M_N(\mathbb{C})$ and the Hilbert space is $\mathcal{H} = \mathbb{C} \otimes M_N(\mathbb{C})$. The two signature one noncommutative geometries from [5] are

1. Type (1, 0) with
$$\gamma^1 = 1,$$
$$D = \{H, \cdot\},$$
where H is a Hermitian matrix.

2. Type (0, 1) with
$$\gamma^1 = -i,$$
$$D = \gamma^1 \otimes [L, \cdot],$$
where L is a skew-Hermitian matrix.

For each geometry we define a quartic action and a partition function
$$\int_{\mathcal{G}} e^{-\left(g \operatorname{Tr} D^2 + \operatorname{Tr} D^4\right)} dD.$$

The trace powers in the action for type $(1, 0)$ can be written in terms of H as
$$\operatorname{Tr} D^2 = 2N \operatorname{Tr} H^2 + 2 \operatorname{Tr} H \operatorname{Tr} H,$$
and
$$\operatorname{Tr} D^4 = 2N \operatorname{Tr} H^4 + 8 \operatorname{Tr} H \operatorname{Tr} H^3 + 6 \operatorname{Tr} H^2 \operatorname{Tr} H^2.$$
Similarly for type (0, 1) we have
$$\operatorname{Tr} D^2 = -2N \operatorname{Tr} L^2 + 2 \operatorname{Tr} L \operatorname{Tr} L,$$
and
$$\operatorname{Tr} D^4 = 2N \operatorname{Tr} L^4 - 8 \operatorname{Tr} L \operatorname{Tr} L^3 + 6 \operatorname{Tr} L^2 \operatorname{Tr} L^2.$$

As one can now see, these models are single matrix, multi-trace random matrix models. They have been found to share many similar properties, such as a genus expansion [2], with single trace single matrix models. Furthermore, many techniques to analyze them have been extended from single trace models such as the coloumb gas method [25] and blobbed topological recursion [3].

For the quartic potential, in the large N limit, when L is replaced with iH for some Hermitian matrix H, these are the same matrix and Dirac ensembles [25]. Furthermore, it is not hard to see that the trace of ℓ-th power of the Dirac operators is given by

$$\sum_{k=0}^{\ell} \binom{\ell}{k} \operatorname{Tr} H^{\ell-k} \operatorname{Tr} H^{k}.$$

Hence, by the factorization theorem the Dirac moments in the large N limit are

$$d_\ell = \lim_{N\to\infty} \frac{1}{N^2} \langle \operatorname{Tr} D^\ell \rangle = \sum_{k=0}^{\ell} \binom{\ell}{k} m_{\ell-k} m_k,$$

and the general loop equations of this model are as follows

$$\sum_{k=0}^{\ell-1} m_k m_{\ell-k-1} = g\,(4m_{\ell+1} + 4m_1 m_\ell) + 8m_{\ell+3} + 8m_3 m_\ell + 24 m_1 m_{\ell+2} + 24 m_2 m_{\ell+1}.$$

The odd moments are zero since we are taking the integral of odd functions. This simplifies the loop equations to the following form

$$m_{2\ell+2} = \frac{1}{8} \sum_{k=0}^{2\ell-2} m_k m_{2\ell-k-2} - \frac{1}{2} g m_{2\ell} - 3 m_2 m_{2\ell}.$$

It is clear from the above recursion that once we have m_2, we can find all moments of the model. Hence, the search space has dimension one. Using various positivity constraints on the above loop equations, we were able to approximate m_2 with respect to g.

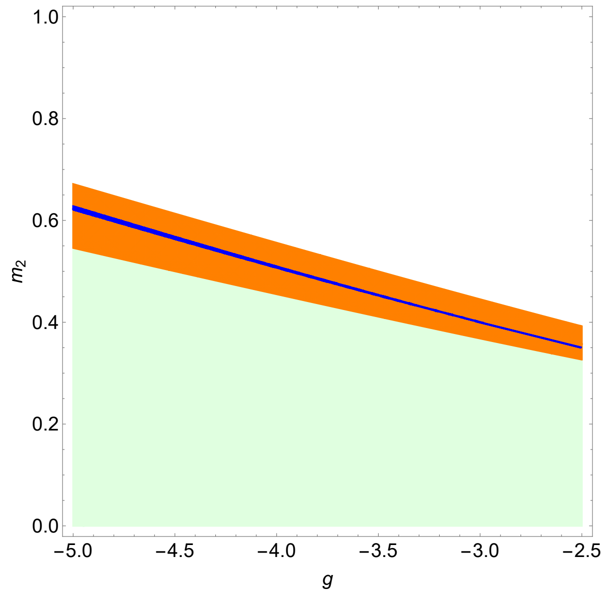

Figure 3.1: The approximate relation between m_2 and g, with g varying from −5 to −2.5. The different coloured regions denote different constraints applied.

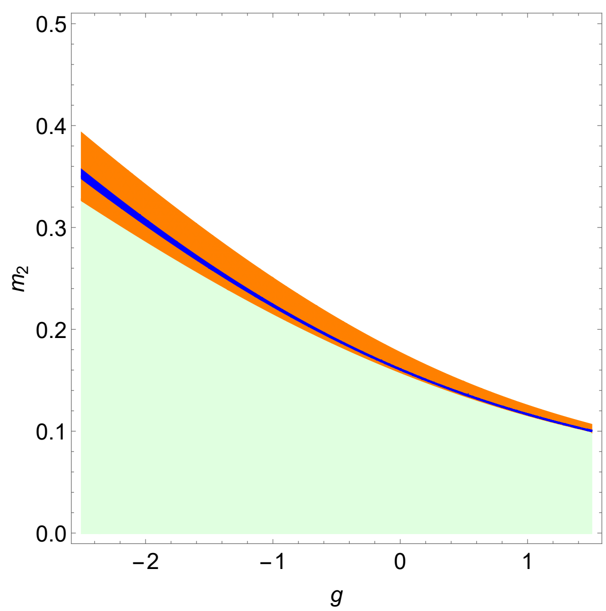

Figure 3.2: The approximate relation between m_2 and g, with g varying from −2.5 to 1.5.

The relationship found is remarkably similar (after re-scaling the appropriate factor of a half) to the analytic relationship found in [25]. In the analytic solution, for g below the critical value, the relationship is linear, which is clearly visible in the bootstrap solution. For values of g above the critical point, the curve is also very similar to its analytic counterpart.

3.4 Two matrix Dirac ensembles

Consider real finite spectral triples $(A, \mathcal{H}, D)$ where the algebra is $A = M_N(\mathbb{C})$ and the Hilbert space is $\mathcal{H} = \mathbb{C}^2 \otimes M_N(\mathbb{C})$. The three signature two noncommutative geometries from [5] are characterized by their Dirac operators D as follows:

1. Type (2, 0): Let
$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
Then,
$$D = \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes \{H_2, \cdot\},$$
where H_1 and H_2 are Hermitian matrices.

2. Type (1,1): Let
$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
Then,
$$D = \gamma^1 \otimes \{H, \cdot\} + \gamma^2 \otimes [L, \cdot],$$
where H is Hermitian and L is skew-Hermitian.

3. Type (0,2); Let
$$\gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
Then
$$D = \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot],$$
where L_1, L_2 are both skew-Hermitian.

Now for each geometry we consider a quartic action and the partition function

$$\mathcal{Z} = \int_{\mathcal{G}} e^{-\left(g \operatorname{Tr} D^2 + \operatorname{Tr} D^4\right)} dD.$$

We shall substitute $L = iH$, where H is Hermitian, for each skew-hermitian matrix L in the above geometries. The result is that all three matrix models have the same matrix action in the large N limit. To see the explicit potentials we refer the reader to the appendix of [5]. Hence, each of these matrix models is identical in this sense, allowing the following results to apply to all three. However, we should not confuse a random matrix ensemble with its Dirac ensemble; their eigenvalues are certainly related but their relationship depends on the geometry. Since the action of this model is even, the odd moments are zero. This means that any moment of a word containing either an odd number of H_1 or H_2 is zero. The following terms are the only ones that contribute to the loop equations in the large N limit

$$\operatorname{Tr} D^2 = 4N \operatorname{Tr} H_1^2 + 4N \operatorname{Tr} H_2^2,$$

and

$$\begin{aligned}\operatorname{Tr} D^4 &= 4N \operatorname{Tr} H_1^4 + 4N \operatorname{Tr} H_2^4 + 16N \operatorname{Tr} H_1^2 H_2^2 - 8N \operatorname{Tr} H_1 H_2 H_1 H_2 \\ &\quad + 12(\operatorname{Tr} H_1^2)^2 + 12(\operatorname{Tr} H_2^2)^2 + 8 \operatorname{Tr} H_1^2 \operatorname{Tr} H_2^2.\end{aligned}$$

Let us consider the words H_1^ℓ for $\ell \geq 1$. The loop equations of this model with respect to these words come from

$$\sum_{i,j=1}^{N} \int_{\mathcal{H}_N^2} \frac{\partial}{\partial (H_1)_{ij}} (H_1^\ell)_{ij}\, e^{-\left(g \operatorname{Tr} D^2 + \operatorname{Tr} D^4\right)} dH_1 dH_2 = 0,$$

giving us

$$\sum_{k=0}^{\ell-1} m_k m_{\ell-k-1} = (8g + 64m_2) m_{\ell+1} + 16 m_{\ell+3} - 16 m_{\ell,1,1,1} + 32 m_{\ell+1,2}, \tag{3.2}$$

in the large N limit. For more details on the loop equations see Appendix A. Here, we use the notation

$$m_{a,b,c,d} = \lim_{N\to\infty} \frac{1}{N} \langle \operatorname{Tr} H_1^a H_2^b H_1^c H_2^d \rangle.$$

When $\ell \leq 7$, in the left hand side there is no term that is a product of moments that come from degree four words or higher. For example $m_{2,2} m_4$ cannot be found. Thus, loop equations of words with length less than 7 can be seen as a system of linear equations and that can be solved in terms of g and m_2.

Proposition 3.4.1 *The number of non-trivial moments (up to cyclic permutation and symmetry) that appear in the loop equations of words with length $\ell \geq 9$ is less than the number of non-zero loop equations.*

Proof Denote by W the set of all words with length $\ell + 3$. Note that the degree of new moments that appear in equation (3.2) is $\ell + 3$. The set W is acted on by $A = \mathbb{Z}/(\ell + 3)\mathbb{Z}$, which shifts letters. Then it follows from Burnside's lemma that

$$|W/A| = \frac{1}{\ell+3} \sum_{a\in A} |W^a| \leq \frac{1}{\ell+3} \left(2^{\ell+3} + \sum_{j=1}^{l+3} 2^{\min(j, l+3-j)} \right) \leq \frac{1}{\ell+3} \left(2^{\ell+3} + 2^{\frac{\ell+7}{2}} \right),$$

where W^a denotes the set of words left invariant by $a \in A$. Considering the symmetry property and vanishing of odd moments, we will get

$$\#\,\text{new moments} \leq \frac{1}{\ell+3}\left(2^{\ell+1} + 2^{\frac{\ell+3}{2}}\right). \tag{3.3}$$

We may have similar loop equations for two different (up to cyclic symmetry property) words, but it is not hard to see that it is never the case that more than half of them are identical. Using symmetry property we have

$$\#\,\text{new loop equations} \geq 2^{\ell-2}. \tag{3.4}$$

It follows from equations (3.3) and (3.4) that for $\ell \geq 9$

$$\#\,\text{new loop equations} \geq \#\text{new moments}.$$

Corollary 3.4.2 *The dimension of the search space of the above model is* 1.

Proof Inductively we can substitute the lower moments into the new loop equations and solve the system of linear equations in terms of g and m_2. By proposition 3.4.1, the number of new moments is less than the number of nonzero loop equations for a given g. Thus the dimension of the search space is 1.

By generating all the loop equations for words up to order ten in *Python* and then using *Maple* we found some remarkable formulas for moments up to order eight strictly in terms of g and the second moment m_2. Here are a selection of them:

$$m_4 = -\frac{1}{8}gm_2 + \frac{1}{64},$$

$$m_{2,2} = -\frac{1}{8}gm_2 - m_2^2 + \frac{1}{64},$$

$$m_{1,1,1,1} = \frac{gm_2}{8} + 2m_2^2 - \frac{1}{64},$$

$$m_6 = \frac{g^2m_2}{64} - \frac{g}{512} - \frac{gm_2{}^2}{8} + \frac{3\,m_2}{64} - \frac{5\,m_2{}^3}{4},$$

$$m_{4,2} = \frac{g^2m_2}{64} + \frac{gm_2{}^2}{8} - \frac{g}{512} - \frac{m_2{}^3}{4} + \frac{m_2}{64},$$

$$m_{3,1,1,1} = -\frac{g^2m_2}{64} - \frac{3\,gm_2{}^2}{8} - \frac{7\,m_2{}^3}{4} + \frac{g}{512} + \frac{m_2}{64},$$

$$m_{2,1,2,1} = \frac{g^2m_2}{64} + \frac{3\,gm_2{}^2}{8} - \frac{g}{512} + \frac{11\,m_2{}^3}{4} - \frac{m_2}{64},$$

$$m_8 = -\frac{gm_2}{64} + \frac{m_2{}^4}{4} + \frac{g^2}{4096} + \frac{m_2{}^2}{256} + \frac{3}{4096} - \frac{g^3 m_2}{512} + \frac{3\,g^2 m_2{}^2}{64} + \frac{gm_2{}^3}{2}.$$

Using only the explicit formulas in terms of g and m_2 and some associated positivity constraints we were able to find the following regions in the solution space using Mathematica.

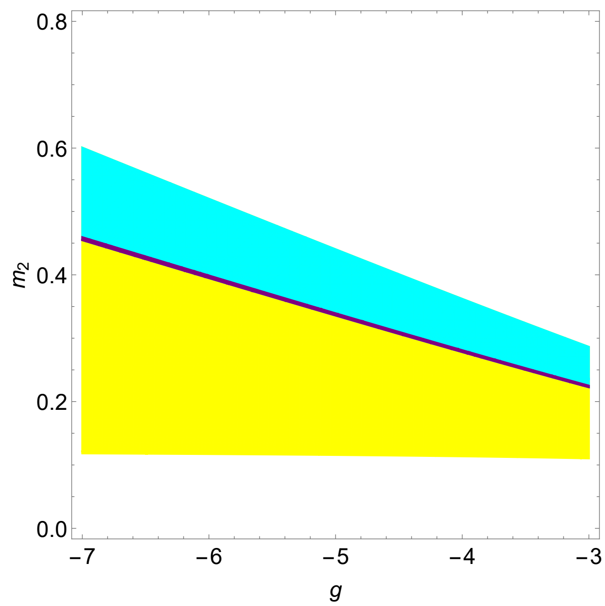

Figure 3.3: The search space region for the (2,0) quartic model where the relationship between g and m_2 becomes linear.

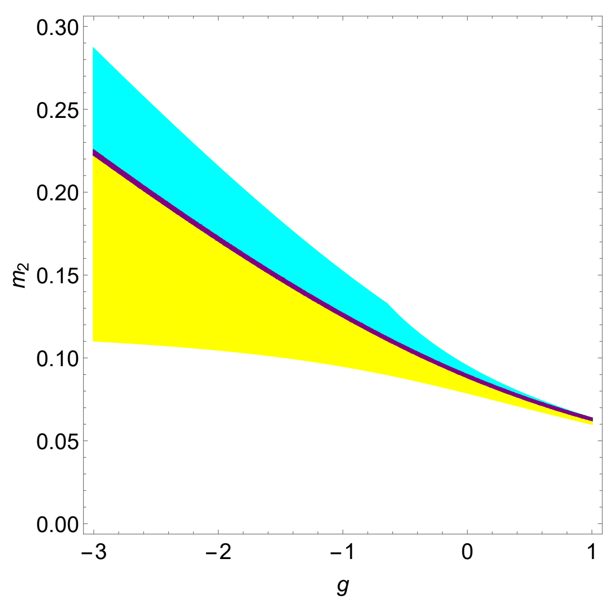

Figure 3.4: The search space region for the (2,0) quartic model where the relationship between g and m_2 is nonlinear.

Note that

$$\lim_{N\to\infty} \frac{1}{N^2}\langle \mathrm{Tr}\, D^2\rangle = 8 \lim_{N\to\infty} \frac{1}{N}\langle \mathrm{Tr}\, H_1^2\rangle.$$

With this factor of 8 taken into account the figure is remarkably similar to the one computed for ten by ten matrices in [19]. Unlike with the signature one models, no explicit formula is known for the moments of the Dirac ensemble of signature two models. However, they can be computed with much effort using the combinatorics of chord diagrams; see [33]. With the above formulas and those from [33] we have computed the first three nonzero moments of the signature two Dirac ensembles in terms of m_2 and g:

$$\begin{aligned} d_2 &= 8\, m_2, \\ d_4 &= -4\, g m_2 + \frac{1}{2}, \\ d_6 &= -160\, {m_2}^3 - 16\, g {m_2}^2 + 6\, m_2 + 2\, g^2 m_2 - \frac{1}{4} g. \end{aligned}$$

It is also worth noting that the relation between m_2 and g appears to be linear for values of g roughly below the phase transition [19]. This is precisely what happened for the signature one model analyzed in [25].

While in [28] the size of the search space is estimated for both single and multi-matrix models, the multi-matrix model from the (2,0) quartic geometry, despite its complexity, had a search space dimension of one! It is now worth noting that our technique differs from Lin's mainly in that when we look for the search space we examine the loop equations generated by all words up to a given order. A smaller search space dimension here is particularly counter intuitive since using more words means introducing more complicated new moments. We believe this is an artifact of this particular model.

3.5 Summary and outlook

In this paper Dirac ensembles and their associated matrix models are analyzed using the bootstrap technique. What is found is in very close agreement with simulation results of [19] and analytic treatment of [25]. What is particularly interesting is that the relationship between the coupling constant and the second moment of the signature two matrix ensemble is linear after the phase transition. This linear relationship was also found analytically for the signature one matrix models in a similar range of the coupling constant [25]. This finding suggests that there may be a deep relationship between the multi-matrix models studied here and the single matrix models.

It is hoped that the computation of moments found here will be used to learn more about these models. Known analytic results do not extend to geometries with signature of two or higher and Monte Carlo simulations are severely limited by matrix size. Hence, bootstrapping offers a new opportunity to study Dirac and random matrix ensembles suggested by noncommutative geometry.

We hope to apply the bootstrap method to more complicated geometries such as (0,3), studied in [19]. The methods outlined in this paper should in theory work for any higher signature geometry. It would also be interesting if one could estimate the supports of the limiting eigenvalue density functions. This would allow one to reconstruct the eigenvalue distributions of both the Dirac and the random matrix ensembles.

Additionally the formulas found for the Dirac moments seem to exhibit some a pattern. If one could find the loop equations strictly in terms of Dirac moments for any geometry, this would be an impactful step towards a better understanding of these models.

In [5] it was speculated that the $(2,0)$ model (among others) exhibited manifold-like behaviour near the phase transition. What is meant by this is that the spectrum visually has similarities with the spectrum of the Dirac operator on S^2. Its spectral density function is of the form $|\lambda|$. Evidence to support this was found in [6]. The work in [20] is promising if it can be applied to these models. Another even more recent approach is the utilization of Functional Renormalization Group techniques on these models found in [34].

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Chapter 4

Double Scaling Limits of Dirac Ensembles

4.1 Introduction

Attempts to construct theories of Euclidean quantum gravity typically involve a partition function where the integration is over possible topologies or metrics and matter fields. However, these integrals are nonrenormalizable. Some approaches often used in physics involve discretizing the space, metrics, or topologies. Discrete approximations of physical theories have often found much success, such as in lattice gauge theory [31]. An alternative approach comes through noncommutative geometry. One may approximate commutative spaces by replacing the algebra of commutative functions on that space by a corresponding algebra of matrices. This was first done with the fuzzy sphere [33]. Such constructions exist for other spaces; for an example see the construction of the complex projective plane in [25].

As mentioned earlier, one would like to construct a path integral over all metrics (and eventually over matter fields), but when a space is "fuzzified" its metric loses its meaning. Alternatively, in the framework of spectral triples the role of a metric is played by the Dirac operator, as in Conne's distance formula [14]. Barrett first suggested in [4] that a toy model for finite noncommutative quantum gravity could be constructed as a well-defined matrix integral over an appropriate space of Dirac operators. We refer to these models as *Dirac ensembles*. The goal was that in some appropriate limit Dirac ensembles might connect to some understood physics, thus validating this random fuzzy approximation. The resulting partition functions of these models are matrix integrals, allowing one to use techniques from random matrix theory. In this paper we find that certain Dirac ensembles are dual to minimal models from conformal field theory coupled to gravity in the double scaling limit.

In random matrix theory the expectation values of observables can typically be written as a formal summation organized by genus, called the genus expansion, which was first discovered by t' Hooft [38]. Taking the large N limit of matrix models is equivalent to counting various types of planar maps [7]. As proven in [29], Dirac ensembles of any dimension have a genus expansion. In particular, the leading order term, found by a large N limit of appropriately scaled quantities, amounts to counting strictly planar maps. In the late 80's and 90's physicists found artifacts of conformal field theory in the large N limit of matrix models when coupling constants were tuned to specific critical values [12, 20, 16]. These critical values occur when the genus expansion terms of the log of the partition function fail to be smooth. This is analogous to how critical values are found in statistical mechanics.

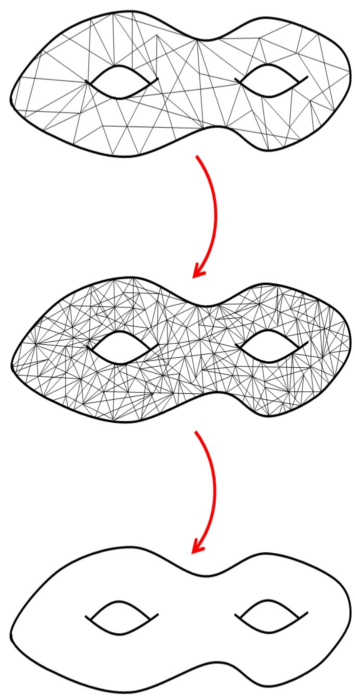

Figure 4.1: Intuitively if one fine tunes coupling constants of matrix models such that the number of polygons in maps goes to infinity, maps are replaced by smooth surfaces.

We give some intuition for this connection. Formal random matrix models are formal summations of Gaussian integrals, which via Feynman graph techniques can be realized as the weighted generating functions of maps [11]. A map is a type of embedding of a graph onto a two dimensional surface. Maps can be seen as a method of discretizing surfaces, and when the number of polygons that make up the map becomes very large, one would expect that one should be able to find a connection to partition functions that sum over surfaces. The number of polygons in maps can in fact grow by tuning the coupling constants to critical values [6]. Note that these critical values are not the same as those discussed in [5, 28].

The authors wish to emphasize that none of these connections between quantum gravity and random matrices are new, but our goal is to reframe them as a toy noncommutative theory of quantum gravity. We prove in this paper that single trace matrix ensembles emerge from Dirac ensembles when the coupling constants are tuned appropriately. As we will discuss, this is highly non-obvious because the coupling constants of bi-tracial and single trace terms are shared. This allows one to recover random matrix models that are dual to minimal models of conformal field theory, an old and well studied connection in physics [6].

In Section 2 we give a brief introduction to Dirac ensembles. The Schwinger-Dyson equations and stuffed maps are developed in Section 3. The resolvent function is computed for some examples of Dirac ensembles in Section 4. Section 5 explains how one uses blobbed topological recursion to compute higher order and genus correlation functions. Section 6 highlights the main result of this paper. In this section the critical points of the examples in Section 5 are discussed as well as the connection to Liouville conformal field theory. In Section 7 we summarize these results and future projects.

4.2 Dirac ensembles

A Dirac ensemble is defined by a fixed finite dimensional real spectral triple, in which the Dirac operator is allowed to be randomly selected subject to some consistency conditions. Let us briefly recall that a finite real spectral triple is a quintuple $(\mathcal{A}, \mathcal{H}, D, \Gamma, J)$ where $\mathcal{A}$ is a finite dimensional complex involutive algebra acting on the Hilbert space $\mathcal{H}$. The Dirac operator is a self-adjoint operator from $\mathcal{H} \to \mathcal{H}$. The two extra operators Γ and J, known as the charge conjugation and the chirality operator play no active role in our analysis so will not be discussed here. For a more detailed explanation see [15, 4, 39, 34].

The partition function of a Dirac ensemble is given by Barrett and Glaser [5] as

$$Z = \int_{\mathcal{D}} e^{-\operatorname{Tr} S(D)} dD, \tag{4.1}$$

where $\mathcal{D}$ is the space of possible Dirac operators, which form a real vector space, and dD is the Lebesgue measure on this space. In general, the Dirac operator can be expressed in terms of gamma matrices tensored with commutators and anti-commutators of Hermitian and skew Hermitian matrices. More precisely, in [5] it was found that for any fuzzy spectral triple, the Dirac operator is of the form

$$D = \sum \gamma^I \otimes \{K_I, \cdot\}_{e_I}$$

where the sum is over increasingly ordered multi-indices, and the following rules apply:

- If γ^I is Hermitian, $\{K_I, \cdot\}_{e_I} = \{H_I, \cdot\}$, where H_I is some Hermitian matrix and $\{\cdot\}$ is the anti-commutator.
- If γ^I is skew-Hermitian, $\{K_I, \cdot\}_{e_I} = [L_I, \cdot]$, where L_I is some skew-Hermitian matrix and $[\cdot]$ is the commutator.

Note that the H_I and L_I are free variables. As a result, integral (4.1) can be expressed as an integral over the Cartesian product of the spaces of $N \times N$ Hermitian matrices, $\mathcal{H}_N$, and skew-Hermitian matrices $\mathcal{L}_N$:

$$Z = \int e^{-\operatorname{Tr} S(D)} dD,$$

where dD is the Lebesgue measure on the product space of finitely many spaces of Hermitian and spaces of skew-Hermitian matrices. In particular when $p + q$ is equal to one or two we have

$$dD = \prod_{\ell,r=1}^{p,q} dH_\ell dL_r = \prod_{\ell,r=1}^{p,q} \prod_{i=1}^{N} dH_{r_{ii}} d\mathrm{Im}(L_{\ell_{ii}}) \prod_{i<j} d\,\mathrm{Re}(H_{\ell_{ij}}) d\,\mathrm{Im}(H_{\ell_{ij}})\, d\,\mathrm{Re}(L_{r_{ij}}) d\,\mathrm{Im}(L_{r_{ij}}).$$

The choice of $S(D)$ is left open. However, we are particularly interested in cases where S is a polynomial in D. Note that the use of (p, q) for the KO-dimension of fuzzy spectral triples has no relationship with the integers (p, q) that are used in Kac's table for minimal models in conformal field theory.

A skew Hermitian matrix can be written as a Hermitian matrix multiplied by the complex unit. Furthermore, since $S(D)$ is a polynomial, and thus the potential is a trace polynomial of the Hermitian and skew-Hermitian matrices seen in D, any integral of the above form can be written strictly as a Hermitian multi-matrix integral. These objects are interesting purely from a random matrix perspective, of which very little is known in general. Relatively recently some universal properties were established in [29].

Unlike the usual matrix model, in addition to the spectra of the H's we have the spectrum of D to study. As one might guess, there is a deep and not well understood relationship between them. The spectrum of the Dirac operator itself is not fully understood but displays some universal behaviors as seen in [29] and spectral phase transitions as seen in [5, 28, 24]. We are interested only in the simplest cases for now, partly because of the lack of analytical tools needed to study multi-matrix models.

We will consider a one dimensional Dirac ensemble of type $(1, 0)$. We emphasize that one could also work with $(0, 1)$ just as easily, such as in [28]. The type $(1, 0)$ signifies that the associated Clifford module of the fuzzy spectral triple has a signature of one. Such a Dirac ensemble consist of a finite real spectral triple of the form $(M_N(\mathbb{C}), M_N(\mathbb{C}), D)$ where

$$D = H \otimes I + I \otimes H,$$

and H is some Hermitian $N \times N$ matrix sampled from the probability distribution

$$e^{-\operatorname{Tr} S(D)} dH.$$

The function $S(D)$ is some polynomial in D and

$$dH = \prod_{i=1}^{N} dH_{ii} \prod_{i<j} d\operatorname{Re}(H_{ij}) d\operatorname{Im}(H_{ij}).$$

It is not hard to see that trace powers of D can be written as follows

$$\operatorname{Tr} D^{\ell} = \sum_{k=0}^{\ell} \binom{\ell}{k} \operatorname{Tr} H^{\ell-k} \operatorname{Tr} H^{k}. \tag{4.2}$$

Thus, the integral

$$Z = \int_{\mathcal{H}_N} e^{-\operatorname{Tr} S(D)} dH$$

is not just a matrix integral, but more specifically a bi-tracial matrix integral. We will discuss this further in the next section.

Note that there are in general two ways one may define a matrix integral. If the degree of the potential d is even and the leading term has a positive coefficient, then the integrand is a positive rapidly decaying function on $\mathcal{H}_N$. This allows the use of Fubini's theorem and other results of integral calculus. Thus, it may be interpreted as an N^2 dimensional convergent real integral. We call such an integral a *convergent matrix model*. Alternatively, regardless of the degree of the S or values of the coupling constant we can define a formal matrix integral as a formal sum by power series expanding all non-Gaussian terms in the integrand and swapping the order of integration and summation. Such formal sums of Gaussian integrals can be evaluated termwise using Wick's theorem [32, 22]. Graphically, the coefficients of this formal sum can be realized as a weighted generating function counting stuffed maps [8].

We can define the matrix moments and higher moments of this ensemble as

$$\mathcal{T}_{\ell} := \langle \frac{1}{N} \operatorname{Tr} H^{\ell} \rangle = \frac{1}{N} \frac{1}{Z} \int_{\mathcal{H}_N} \operatorname{Tr} H^{\ell} \, e^{-\operatorname{Tr} V(D)} dH$$

and

$$\langle \frac{1}{N^n} \operatorname{Tr} H^{\ell_1} \operatorname{Tr} H^{\ell_2} ... \operatorname{Tr} H^{\ell_n} \rangle := \frac{1}{N^n} \frac{1}{Z} \int_{\mathcal{H}_N} \operatorname{Tr} H^{\ell_1} \operatorname{Tr} H^{\ell_2} ... \operatorname{Tr} H^{\ell_n} \, e^{-\operatorname{Tr} V(D)} dH.$$

One can further define joint cumulants of higher moments using the classical moment-cumulant relations. For details see chapter one of [22]. These cumulants are denoted $\langle \frac{1}{N^n} \operatorname{Tr} H^{\ell_1} \operatorname{Tr} H^{\ell_2} ... \operatorname{Tr} H^{\ell_n} \rangle$ and are the generating functions of connected maps in the sense that the embedded graph is connected.

The Dirac moments can be computed from matrix moments in this case via formula (4.2). Higher order moments (which were not defined above) can be obtained in the usual manner from higher order cumulants. As proven in [3], the matrix moments and cumulants have a genus expansion, i.e.

$$\mathcal{T}_{\ell_1,\ldots,\ell_q} = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g-q} \mathcal{T}^{g}_{\ell_1,\ldots,\ell_q}$$

where t is a continuous formal parameter strictly greater than zero. The terms of the genus expansion can be put into generating functions of the form

$$W_k^g(x_1, x_2, \ldots x_k) = \sum_{\ell_1,\ell_2,\ldots,\ell_k=0}^{\infty} \frac{\mathcal{T}^{g}_{\ell_1,\ell_2,\ldots\ell_k}}{x_1^{\ell_1+1} x_1^{\ell_2+1} \ldots x_1^{\ell_1+1}}.$$

The terms of this so-called genus expansion can be computed recursively using a process called blobbed topological recursion [3]. Blobbed topological recursion is a generalization of the similar process of topological recursion [21] which has gained much interest in the last two decades. For a review we refer the reader to [10].

In Section 4.3.2, we will derive a set of recursive equations that can relate cumulants and moments (as well as their generating functions) called the Schwinger-Dyson equations. It is a well-known practice in random matrix literature to compute W_1^0 using resolvent techniques. The beauty of (blobbed) topological recursion is that given W_1^0 and W_2^0 (which is often in some sense universal), one can compute any W_k^g recursively by decreasing Euler characteristic $-\chi = 2g - 2 - n$.

For single trace matrix models this process is well studied and formalized, see [22]. However, for multi-trace matrix models we are unaware of any similar reference. In this paper we will explicitly show how to compute W_1^0 and show that W_2^0 has a universal form for bi-tracial matrix models.

4.3 Bi-tracial matrix integrals

In this section we will set the groundwork for analyzing bi-tracial matrix models.

4.3.1 Stuffed maps

In this paper we are strictly interested in bi-tracial matrix models since they are the ones of interest for Dirac ensembles. However, one would expect that this analysis can be extended to higher trace multiplicity. Consider the following matrix integral over the space of Hermitian matrices

$$Z = \int_{\mathcal{H}_N} e^{-S(H)} dH, \tag{4.3}$$

where the potential is a bi-tracial polynomial

$$S(H) = \frac{N}{2t} \operatorname{Tr} H^2 + \sum_{i=3}^{d} \frac{N}{t} \frac{t_i}{i} \operatorname{Tr} H^i + \sum_{i,j=1}^{d} \frac{t_{i,j}}{ij} \operatorname{Tr} H^i \operatorname{Tr} H^j$$

where the t_i's and $t_{i,j}$'s are coupling constants such that $t_{i,j} = t_{j,i}$.

It was first discovered in [7] that the moments and cumulants of the random Hermitian matrix ensembles coincide with the generating functions of maps. The maps of interest for bi-tracial matrix models are called stuffed maps and were first studied in [8] by Borot, and subsequently in [9]. Note that stuffed maps are a direct generalization of the maps in [22] used for single trace matrix models, thus all the following definitions simplify to the types of maps first considered in [7]. We now define the building blocks of stuffed maps.

An elementary 2-cell of topology (k, h) is a connected oriented surface of genus h with k boundaries. For example, a 2-cell with topology $(1, 0)$ is a disc. These 2-cells can be "glued" together by pairing edges of the perimeter to form a surface with an embedded graph. The resulting surface is referred to as a *stuffed map* of topology (n, g) with perimeters $(\ell_1, .., \ell_n)$. It is an orientable connected surface with boundaries of lengths $\ell_1, .., \ell_n$. For more on stuffed maps see [8].

We are interested in enumerating these maps. More specifically, we want to count stuffed maps that are glued from 2-cells with the topology of discs and cylinders. This comes from the fact that the Dirac ensembles of interest are bi-tracial matrix models with appropriate N-powers as coefficients [8]. We shall refer to these maps as *unstable* stuffed maps. We define $\mathbb{SM}_k^g(v)$ as the set of all unstable stuffed maps of genus g, with v vertices and k boundaries. It was proven in [29] that this set is finite, allowing us to define the following formal series:

$$\mathcal{T}^g_{\ell_1,\ldots,\ell_k} = \sum_{v=1}^{\infty} t^v \sum_{\Sigma \in \mathbb{SM}_k^g(v)} \prod_{i=1}^{d} t_i^{n_i(\Sigma)} \prod_{i,j=0}^{d} t_{i,j}^{n_{i,j}(\Sigma)} \frac{1}{|\mathrm{Aut}(\Sigma)|} \prod_{q=1}^{k} \delta_{\ell_q(\Sigma),\ell_q},$$

where $n_{ij}(\Sigma)$ is the number of 2-cells with boundaries of lengths i and j used in the gluing of the map Σ. It turns out that these formal series are precisely the genus expansion terms mentioned in the previous section, that is

$$\mathcal{T}_{\ell_1,\ldots,\ell_k} = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g-k} \mathcal{T}^g_{\ell_1,\ldots,\ell_k}$$

and

$$W_k(x_1, .., x_k) = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g-k} W_k^g(x_1, ..., x_k) = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g-k} \sum_{\ell=0}^{\infty} \frac{\mathcal{T}_k^g}{x_1^{\ell_1+1} \ldots x_k^{\ell_k+1}}.$$

For a detailed explanation and proof of this fact see [8].

4.3.2 The Schwinger-Dyson equations

The Schwinger-Dyson equations (SDE's) provide a powerful method to analyze random matrix models. They were first introduced by Migdal in [35]. In our context they are the consequence of matrix integrals of the total derivative vanishing. The SDE's of the bi-tracial matrix model (4.3) can be derived as follows. Consider the following relation,

$$\sum_{i,j}\int_{\mathcal{H}_N}\frac{\partial}{\partial H_{ij}}\left(\left(H^{\ell_1}\right)_{ij}\prod_{m=2}^{n}\operatorname{Tr}H^{\ell_m}e^{-S(H)}\right)dH=0, \tag{4.4}$$

where $\left(H^{\ell_1}\right)_{ij}$ is the ij-th entry of the matrix power H^{ℓ_1}, and the partial derivative $\frac{\partial}{\partial H_{ij}}$ satisfies

$$\frac{\partial}{\partial H_{ij}}\left(H_{pq}\right)=\delta_{ip}\delta_{jq}. \tag{4.5}$$

One can prove the following properties,

$$\frac{\partial}{\partial H_{ij}}\left(H^{\ell_1}\right)_{ij}=\sum_{k=0}^{\ell_1-1}\left(H^{k}\right)_{ii}\left(H^{\ell_1-k-1}\right)_{jj} \tag{4.6}$$

and

$$\frac{\partial}{\partial H_{ij}}\left(\prod_{m=2}^{n}\operatorname{Tr}H^{\ell_m}\right)=\sum_{r=2}^{n}\ell_r\left(H^{\ell_r}\right)_{ji}\prod_{\substack{m=2\\ m\neq r}}^{n}\operatorname{Tr}H^{\ell_m}. \tag{4.7}$$

Now, using the Leibniz product rule in (4.4) and the relations (4.6) and (4.7), we find that

$$\begin{aligned}
&\sum_{k=0}^{\ell_1-1}\langle\operatorname{Tr}H^{k}\operatorname{Tr}H^{\ell_1-k-1}\prod_{m=2}^{n}\operatorname{Tr}H^{\ell_m}\rangle+\sum_{r=2}^{n}\ell_r\langle\operatorname{Tr}H^{\ell_1+\ell_r-1}\prod_{\substack{m=2,\\ m\neq r}}^{n}\operatorname{Tr}H^{\ell_m}\rangle\\
&=\Bigg\langle\Bigg(\frac{N}{t}\operatorname{Tr}H^{\ell_1+1}+\sum_{i=3}^{d}\frac{N}{t}t_i\operatorname{Tr}H^{\ell_1+i-1}\\
&+\sum_{i,j=1}^{d}\frac{t_{i,j}}{ij}\Big(i\operatorname{Tr}H^{\ell_1+i-1}\operatorname{Tr}H^{j}+j\operatorname{Tr}H^{i}\operatorname{Tr}H^{\ell_1+j-1}\Big)\Bigg)\prod_{m=2}^{n}\operatorname{Tr}H^{\ell_m}\Bigg\rangle.
\end{aligned}$$

Note the model (4.3) can be considered as a formal matrix integral or a convergent matrix integral. Either way its moments will satisfy the nonlinear recursive relationship above. In the case of formal integrals, the equations are valid only order by order in t. Applying the genus expansions of moments and cumulants, then collecting terms of the same N/t powers, one finds the following.

Theorem 4.3.1 *The Schwinger-Dyson equations for the bi-tracial matrix model (4.3) are :*

$$\sum_{k=0}^{\ell_1-1}\left(\sum_{h=0}^{g}\sum_{J\subset L}\mathcal{T}^{(h)}_{k,J}\mathcal{T}^{(g-h)}_{\ell_1-k-1,L\backslash J}+\mathcal{T}^{(g-1)}_{k,\ell_1-k-1,L}\right)+\sum_{m=2}^{n}\ell_r\mathcal{T}^{(g)}_{\ell_1+\ell_r-1,L\backslash\{r\}}$$
$$=\mathcal{T}^{(g)}_{\ell_1+1,L}+\sum_{i=3}^{d}t_i\mathcal{T}^{(g)}_{\ell_1+i-1,L}$$
$$+\sum_{i,j=1}^{d}\frac{t_{i,j}}{ij}\left(\sum_{h=0}^{g}\sum_{J\subset L}\left(i\mathcal{T}^{(h)}_{\ell_1+i-1,J}\mathcal{T}^{(g-h)}_{j,L\backslash J}+j\mathcal{T}^{(h)}_{\ell_1+j-1,J}\mathcal{T}^{(g-h)}_{i,L\backslash J}\right)+i\mathcal{T}^{(g-1)}_{\ell_1+i-1,j,L}+j\mathcal{T}^{(g-1)}_{\ell_1+j-1,i,L}\right),\qquad(4.8)$$

where $L=\{\ell_2,\ell_3,\cdots,\ell_n\}$.

The coefficient of N/t, when the number of boundaries n is equal to 1, is the main object of study in the following section.

4.3.3 The spectral curve

When $g=0$ and $n=1$, equation (4.8) becomes

$$\sum_{k=0}^{\ell-1}\mathcal{T}^0_k\mathcal{T}^0_{\ell-k-1}=\mathcal{T}^0_{\ell+1}+\sum_{i=3}^{d}t_i\mathcal{T}^0_{\ell+i-1}+\sum_{i,j=1}^{d}\frac{t_{i,j}}{ij}\left(i\mathcal{T}^0_{\ell+i-1}\mathcal{T}^0_j+j\mathcal{T}^0_{\ell+j-1}\mathcal{T}^0_i\right).$$

This is significantly more complicated than the single trace case. To simplify the notation we write it as

$$\sum_{k=0}^{\ell-1}\mathcal{T}^0_k\mathcal{T}^0_{\ell-k-1}=\tilde{t}_2\mathcal{T}^0_{\ell+1}+\tilde{t}_{1,1}\mathcal{T}^0_{\ell}+\sum_{i=3}^{d}\tilde{t}_i\mathcal{T}^0_{\ell+i-1},\qquad(4.9)$$

where the bi-tracial terms become a combination of moments and coupling constants t_j, all inside the new $\tilde{t}_j$'s as follows.

$$\tilde{t}_k=\sum_{i=2}^{d}\delta_{i,k}t_k+2\sum_{i=1}^{d}\frac{t_{i,k}}{i}\mathcal{T}^0_i.\qquad(4.10)$$

Next we multiply equation (4.9) by $1/x^{\ell+1}$ and sum from $\ell=0$ to infinity. The resulting equation is

$$\sum_{\ell=0}^{\infty}\sum_{k=0}^{\ell-1}\frac{\mathcal{T}^0_k\mathcal{T}^0_{\ell-k-1}}{x^{\ell+1}}=\sum_{\ell=0}^{\infty}\tilde{t}_2\frac{\mathcal{T}^0_{\ell+1}}{x^{\ell+1}}+\sum_{\ell=0}^{\infty}\tilde{t}_{1,1}\frac{\mathcal{T}^0_{\ell}}{x^{\ell+1}}+\sum_{\ell=0}^{\infty}\sum_{i=3}^{d}\tilde{t}_i\frac{\mathcal{T}^0_{\ell+i-1}}{x^{\ell+1}}.$$

Then the equation can be written as

$$W^0_1(x)^2=S'(x)W^0_1(x)-P^0_1(x),\qquad(4.11)$$

where

$$W_1^0(x) = \sum_{\ell=0}^{\infty} \frac{\mathcal{T}_\ell^0}{x^{\ell+1}},$$

$$P_1^0(x) = t + \sum_{j=2}^{d} \sum_{t=0}^{j-2} \tilde{t}_j \mathcal{T}_{j-\ell-2}^0 x^\ell,$$

and

$$S(x) = \frac{1}{2}\tilde{t}_2 x^2 + \tilde{t}_{1,1} x + \sum_{j=3}^{d} \frac{\tilde{t}_j}{j} x^j.$$

By solving the quadratic equation (4.11), one can find the resolvent function

$$W_1^0(x) = \frac{1}{2}\left(S'(x) - \sqrt{S'(x)^2 - 4P_1^0(x)}\right). \tag{4.12}$$

Equation (4.11) is commonly referred to as the *spectral curve*. The solution $W_1^0(x)$, called the resolvent, can be used to give us the limiting eigenvalue density function of the random matrix associated to the given model. It is well-known that this relationship is given by the Stieltjes transform

$$\begin{aligned} W_1^0(x) &= \sum_{\ell=0}^{\infty} \frac{\mathcal{T}_\ell^0}{x^{\ell+1}} = \sum_{\ell=0}^{\infty} \frac{\lim_{N\to\infty}\langle \frac{1}{N} \operatorname{Tr} H^\ell \rangle}{x^{\ell+1}} \\ &= \lim_{N\to\infty} \langle \frac{1}{N} \operatorname{Tr} \sum_{\ell=0}^{\infty} \frac{H^\ell}{x^{\ell+1}} \rangle = \lim_{N\to\infty} \frac{1}{N} \langle \operatorname{Tr} \frac{1}{x-H} \rangle \\ &= \text{p.v.} \int_{\text{supp}\rho} \frac{\rho(y)}{x-y} dy, \end{aligned}$$

where we define $\rho(y)$ as the limiting eigenvalue density function of the model. Thus, applying the inverse Stieltjes transform tells us that

$$\rho(x) = -\frac{1}{\pi t} \Im W_1^0(x).$$

Consider for example the case that all coupling cosntants are zero except for $t = 1$. The resulting model is the Gaussian Unitary Ensemble and

$$W_1^0(x) = \frac{1}{2}(x - \sqrt{x^2 - 4}).$$

One can also derive that $\text{supp}\rho = [-2, 2]$. So

$$\rho(x) = \frac{1}{2\pi} \sqrt{4 - x^2}_{[-2,2]},$$

which is Wigner's semicircle distribution.

The following lemma for unstable stuffed maps first appeared in [29] and is a generalization of a famous result by the same name discussed in [22]. Ideally one would like to know how the discriminant in equation (4.12) factors. This will determine the support of the limiting spectral distribution.

Lemma 4.3.2 (1-Cut Brown's Lemma) *There exists formal power series α and γ^2, as well as a polynomial $M(x)$, such that*

$$\alpha = O(t), \qquad \gamma^2 = t + O(t^2), \qquad M(x) = \frac{S'(x)}{x} + O(t),$$

and

$$S'(x)^2 - 4P_1^0(x) = (M(x))^2(x-a)(x-b), \tag{4.13}$$

with $a = \alpha + 2\gamma$ and $b = \alpha - 2\gamma$.

Brown's lemma tells us that the models we are interested in will always have a single cut solution. In general it is possible to find the number of connected components of support of ρ. This will depend on ranges of the coupling constants as well as the degree and structure of $S(x)$.

4.4 The resolvent

In this section we will focus on finding the resolvent $W_1^0(x)$ for several formal bi-tracial matrix models that are type $(1,0)$ Dirac ensembles.

4.4.1 The Zhukovsky transform

The form of the discriminant in equation (4.13) motivates the use of the Zhukovsky transform

$$x(z) = \frac{a+b}{2} + \frac{a-b}{4}\left(z + \frac{1}{z}\right)$$

with an inverse

$$z = \frac{1}{2\gamma}\left(x - \alpha + \sqrt{(x-\alpha)^2 - 4\gamma^2}\right).$$

The Zhukovsky transform maps the x-plane minus a line segment to the exterior of the unit disk in the z-plane. It also has the following useful identity

$$\sqrt{(x(z)-a)(x(z)-b)} = \frac{a-b}{4}\left(z - \frac{1}{z}\right).$$

We now borrow Theorem 3.1.1 from [22], which applies to our models as well.

Theorem 4.4.1 *[22] For the formal power series α and γ^2 as mentioned above, we have the expansions*

$$S'(x(z)) = \sum_{k=0}^{d-1} u_k(z^k + z^{-k})$$

and

$$W_1^0(x(z)) = \sum_{k=0}^{d-1} u_k z^{-k},$$

with $u_0 = 0$ and $u_1 = t/\gamma$.

Notice that the theorem implies that since we have $S(x)$, in theory we should be able to compute the coefficients of $W_1^0(x(z))$ and then transform back to get $W_1^0(x)$. Computing the coefficients u_k is much more involved in our models than in the single trace cases seen in [22]. We will show how to find them in various examples.

4.4.2 Moments

In general one may apply Lagrange's inversion formula to compute the moments and the support of $\rho(x)$ for a given model. To see this one first observes that moments can be extracted from the resolvent generating function via the following contour integral

$$\mathcal{T}_\ell^0 = -\frac{1}{2\pi i}\oint_C x^\ell W_1^0(x)dx.$$

We then apply the Zhukovsky transform to get

$$\mathcal{T}_\ell^0 = -\frac{1}{2\pi i}\oint_C x(z)^\ell W_1^0(x(z))x'(z)dz.$$

Expanding one can find the general formula.

Corollary 4.4.2 *[22] The ℓ-th moment of $\rho(x)dx$ is*

$$\mathcal{T}_\ell^0 = \sum_{i+j<\ell, i<j<i+d} \frac{(j-i)\ell!}{(i+1)!(j+1)!(\ell-1-i-j)!}\alpha^{\ell-1-i-j}\gamma^{i+j+2}u_{j-i},$$

where the u_{j-i}'s are the coefficients from theorem 4.4.1 that are determined by the potential.

This formula looks the same as for single trace models, but it is important to note that the u_k's contain other moments of the model because of the bi-tracial terms. Because of this, this formula now gives us a system of nonlinear equations to solve for moments.

As a side note, if one wishes to compute γ or α, the Lagrange inversion formula can be applied to recover them in terms of the coupling constants. Just like in the single trace case even though we treated moments as just formal series, they are in fact algebraic functions of α and γ, which are algebraic functions of the coupling constants. Thus, they have a finite number of singularities and are convergent in some sufficiently small disc. These singularities are a beautiful artifact of these models that is fundamental for the connection to Liouville quantum gravity, as we will see. The characterization of these singularities is not quite understood for multi-trace models in general, and may be an interesting phenomenon to study.

4.4.3 The quartic model

Consider the potential

$$S(D) = \frac{t_2}{4}\operatorname{Tr} D^2 + \frac{t_4}{8}\operatorname{Tr} D^4.$$

The potential can be written in terms of the Hermitian matrix H:

$$S(H) = \frac{N}{2}t_2 \operatorname{Tr} H^2 + \frac{N}{4}t_4 \operatorname{Tr} H^4 + \frac{3}{4}t_4 \operatorname{Tr} H^2 \operatorname{Tr} H^2.$$

The derivative of the potential in the spectral curve (4.11) is given by

$$S'(x) = (t_2 + 3\mathcal{T}_2^0 t_4)x + t_4 x^3.$$

For this model $\mathcal{T}_\ell^0$ counts the number of planar gluings of quandrangles and 2-cells with two boundaries of lengths two, with one polygon boundary of length ℓ. If ℓ is odd no such gluings exist so the generating function is zero. Hence, we focus on even values of ℓ.

We next transform the resolvent and find that

$$W_1^0(x(z)) = \frac{1}{\gamma z} + t_4\gamma^3\frac{1}{z^3}.$$

Transforming back, we arrive at

$$W_1^0(x) = \frac{1}{2}\left((t_2 + 3\mathcal{T}_2^0 t_4)x + t_4x^3 - t_4(x^2 - \gamma^2 + \frac{t}{t_4\gamma^2})\sqrt{x^2 - 4\gamma^2}\right).$$

Thus, the limiting eigenvalue density function is

$$\rho(x) = -\frac{1}{\pi t}\Im W_1^0(x) = \frac{1}{2\pi}\left(t_4(x^2 - \gamma^2) + \frac{1}{\gamma^2}\right)\sqrt{4\gamma^2 - x^2}_{[-2\gamma,2\gamma]}.$$

One would like to be able to find the value of γ in terms of the coupling constants. To do this, we start by finding the transformed coefficients of S' :

$$S'(x(z)) = (t_2 + 3\mathcal{T}_2^0 t_4)(\alpha + \gamma(z + 1/z)) + t_4(\alpha + \gamma(z + 1/z))^3.$$

Expanding, we find that we may write the coefficients of $S'(x(z))$ in Theorem 4.4.1 as follows:

$$\begin{aligned} u_0 &= \alpha(t_2 + 3\mathcal{T}_2^0 t_4) + t_4(\alpha^3 + 6\alpha\gamma^2) \\ u_1 &= (t_2 + 3\mathcal{T}_2^0 t_4)\gamma + t_4(3\alpha^2\gamma + 3\gamma^3) \\ u_2 &= 3t_4\alpha\gamma^2 \\ u_3 &= t_4\gamma^3. \end{aligned}$$

Using the same theorem, we deduce that

$$0 = u_0 = \alpha((t_2 + 3\mathcal{T}_2^0 t_4) + t_4(\alpha^2 + 6\gamma^2))$$

and

$$\frac{t}{\gamma} = u_1 = \gamma((t_2 + 3\mathcal{T}_2^0 t_4) + t_4(3\alpha^2 + 3\gamma^2)).$$

By the one-cut lemma as t goes to zero, so does α and $\mathcal{T}_2^0$ by definition. Thus the factor $((t_2 + 3\mathcal{T}_2^0 t_4)) + t_4(\alpha^2 + 6\gamma^2))$ is nonzero when $t_2 \neq 0$ order by order in t. Hence $\alpha = 0$ in order for this equation to hold on each term of the formal sum. This reduces the second equation to

$$3t_4\gamma^4 + (t_2 + 3\mathcal{T}_2^0 t_4)\gamma^2 - 1 = 0$$

Using the formula for the second moment from Corollary 4.4.2, we have

$$3\, t_4^2\gamma^8 + 6t_4\gamma^4 + t_2\gamma^2 - 1 = 0.$$

Thus we are left with an equation that relates γ to the coupling constants, which can be solved symbolically using software such as Maple or Mathematica.

Note that if we travel along the curve $t_2 = 1 - 3\mathcal{T}_2^0 t_4$ the derivative of the potential curve becomes

$$S'(x) = x - t_4 x^3,$$

which is the same as the quartic Hermitian matrix model

$$\int_{\mathcal{H}_N} e^{-\frac{N}{2}\operatorname{Tr} H^2 - \frac{N}{4} t_4 \operatorname{Tr} H^4} dH.$$

See chapter 3 of [22] for details about this model. Thus, both models at these specifications have the same resolvent $W_1^0(x)$. This shall be important in later sections. We shall make similar conclusions about the rest of the models examined in the following two sections.

4.4.4 The hexic model

Consider the potential

$$S(D) = \frac{t_2}{4}\,\mathrm{Tr}\,D^2 + \frac{t_4}{8}\,\mathrm{Tr}\,D^4 + \frac{t_6}{12}\,\mathrm{Tr}\,D^6.$$

For the spectral curve (4.11) we get

$$S(x) = \frac{t_2\,x^2}{2} + t_4\left(\frac{x^4}{4} + \frac{3\,\mathcal{T}_2^0\,x^2}{2}\right) + t_6\left(\frac{x^6}{6} + \frac{5\,\mathcal{T}_2^0\,x^4}{2} + \frac{5\,\mathcal{T}_4^0\,x^2}{2}\right),$$

and thus

$$S'(x) = t_6\,x^5 + \left(10\,\mathcal{T}_2^0\,t_6 + t_4\right)x^3 + \left(5\,\mathcal{T}_4^0\,t_6 + 3\,\mathcal{T}_2^0\,t_4 + t_2\right)x.$$

For this model, if ℓ is odd the moment is zero by the same $H \to -H$ symmetry as in the quartic case. We then apply the Zhukovsky transform to obtain

$$\alpha^5 t_6 + (t_4 + t_6(20\gamma^2 + 10m_2))\alpha^3 + (t_2 + t_4(6\gamma^2 + 3m_2) + t_6(30\gamma^4 + 60\gamma^2 m_2 + 5m_4))\alpha = u_0 = 0.$$

By the same argument as in the quartic case, we can deduce $\alpha = 0$. Thus, we find

$$\begin{aligned}
\frac{1}{\gamma} &= u_1 = t_2\gamma + t_4(3\gamma^3 + 3\gamma m_2) + t_6(10\gamma^5 + 30\gamma^3 m_2 + 5\gamma m_4)\\
u_2 &= 0\\
u_3 &= t_4\gamma^3 + t_6(5\gamma^5 + 10\gamma^3 m_2)\\
u_4 &= 0\\
u_5 &= t_6\gamma^5.
\end{aligned}$$

Using the Corollary 4.4.2, we find that

$$\begin{aligned}
m_2 &= -\frac{\gamma^2\left(5\,t_6\,\gamma^6 + t_4\,\gamma^4 + 1\right)}{10\,t_6\,\gamma^6 - 1}\\
m_4 &= 2\,\gamma^4 + 3\,\gamma^5\left(t_4\,\gamma^3 + t_6\left(5\,\gamma^5 - 10\,\frac{\gamma^5\left(5\,t_6\,\gamma^6 + t_4\,\gamma^4 + 1\right)}{10\,t_6\,\gamma^6 - 1}\right)\right) + t_6\,\gamma^{10}.
\end{aligned}$$

This gives the limiting eigenvalue density function as

$$\begin{aligned}
\rho(x) = -\frac{1}{\pi t}\Im W_1^0(x) = \frac{1}{2\pi}\Bigg(t_6\,x^4 + \frac{x^2}{\gamma}\left(\frac{t_4\,\gamma^3 + t_6\left(5\,\gamma^5 + 10\,\gamma^3 m_2\right)}{\gamma^2} - 3\,t_6\,\gamma^3\right)\\
+\frac{\gamma^{-1} - t_4\,\gamma^3 - t_6\left(5\,\gamma^5 + 10\,\gamma^3 m_2\right) + t_6\,\gamma^5}{\gamma}\Bigg)\sqrt{4\gamma^2 - x^2}_{[-2\gamma,2\gamma]},
\end{aligned} \tag{4.14}$$

where γ can be found as the solution to the $1/\gamma = u_1$ relation for given t_2, t_4, and t_6.

Similarly as in the quartic ensemble, when one travels along the hypersurface $5\,\mathcal{T}_4^0\,t_6 + 3\,\mathcal{T}_2^0\,t_4 + t_2 = 1$, we have that the derivative of the potential becomes

$$S'(x) = t_6\,x^5 + \left(10\,\mathcal{T}_2^0\,t_6 + t_4\right)x^3 + x.$$

Next, one may set $10\,\mathcal{T}_2^0\,t_6 + t_4$ equal to some desired value to reduce the derivative of the potential, and therefore $W_1^0(x)$, to being the same as the hexic Hermitian matrix model

$$\int_{\mathcal{H}_N} e^{-\frac{N}{2}\operatorname{Tr}H^2-\frac{N}{4}t_4\operatorname{Tr}H^4-\frac{N}{6}t_6\operatorname{Tr}H^6}\,dH.$$

4.4.5 The cubic model

Consider the potential

$$S(D) = \frac{t_2}{4}\operatorname{Tr}D^2 + \frac{t_3}{6}\operatorname{Tr}D^3,$$

which becomes

$$V(x) = \frac{t_2\left(2\,m_1\,x + x^2\right)}{2} + \frac{t_3\left(3\,m_1\,x^2 + x^3 + 3\,xm_2\right)}{3}$$

in the spectral curve equation (4.11). Thus,

$$V'(x) = t_3\,x^2 + (2\,t_3\,m_1 + t_2)\,x + t_2\,m_1 + t_3\,m_2.$$

We then apply the Zhukovsky transform and find that

$$\begin{aligned}
0 = u_0 &= \frac{t_2\ (2\,m_1 + 2\,a)}{2} + \frac{t_3\left(3\,\alpha^2 + 6\,\alpha m_1 + 6\,\gamma^2 + 3\,m_2\right)}{3}\\
\frac{1}{\gamma} &= t_2\,\gamma + \frac{t_3\ (6\,\gamma\,\alpha + 6\,m_1\,\gamma)}{3}\\
u_2 &= t_3\gamma^2 z^2.
\end{aligned}$$

Using Corollary 4.4.2, we may compute the moments of interest

$$\begin{aligned}
m_1 &= \alpha + \gamma^4 t_3\\
m_2 &= \alpha^2 + \gamma^2 + 2\alpha\gamma^4 t_3.
\end{aligned}$$

This gives the limiting eigenvalue density function

$$\rho(x) = -\frac{1}{\pi t}\Im W_1^0(x) = \frac{1}{\pi}\left(\frac{1}{\gamma^2} + t_3(x-\alpha)\right)\sqrt{(x-\alpha-2\gamma)(\alpha-2\gamma-x)}_{[\alpha-2\gamma,\,\alpha+2\gamma]},$$

where γ and α can be found as the solution to the above relations for given t_2 and t_3.

When one travels along $2\,t_3\,m_1 + t_2 = 1$ and $t_2\,m_1 + t_3\,m_2 = 0$, the derivative of the potential is the same as the cubic Hermitian matrix model

$$\int_{\mathcal{H}_N} e^{-\frac{N}{2}\operatorname{Tr} H^2 - \frac{N}{3} t_3 \operatorname{Tr} H^3} dH.$$

Thus, in these circumstances both models have the same $W_1^0(x)$.

4.5 (Blobbed) topological recursion

Roughly speaking, and in our context of matrix models, topological recursion works as follows. Using the resolvent technique one first defines a complex curve (Riemann surface) Σ, called the spectral curve of the model. One then constructs a sequence of symmetric mermorphic differential forms $\omega_{g,n}(z_1, \ldots, z_n) dz_1 ... dz_n$ of degree n for $g \geq 0, n \geq 1$, on n-fold Cartesian products of Σ. Topological recursion works by induction on the Euler characteristic $2 - 2g - n$ of a surface of genus g with n boundaries. It gives an inductive formula for all $\omega_{g,n}$, starting with the first two forms $\omega_{0,1}$ and $\omega_{0,2}$:

$$\begin{aligned}\omega_{g,n+1}(I, z) = \sum_{\beta_i} \operatorname{Res}_{q \to \beta_i} K_i(z, q) \Big(& \omega_{g-1,n+2}\,(I, q, \sigma_i(q)) \\ + & \sum_{\substack{g_1+g_2=g \\ I_1 \uplus I_2 = I \\ (g_1, I_1) \neq (0,\emptyset) \neq (g_2, I_2)}} \omega_{g_1,|I_1|+1}\,(I_1, q)\, \omega_{g_2,|I_2|+1}\,(I_2, \sigma_i(q)) \Big),\end{aligned}$$

where $I = \{z_1, ..., z_n\}$ and β_i are the ramification points of the ramified covering x, defined via $dx(\beta_i) = 0$. The recursive kernel $K_i(z, q)$ is constructed from the initial data. In the special case when Σ is the projective line, $\omega_{0,2}$ is the *Bergmann kernel*

$$\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}.$$

As previously mentioned, it was first seen in [8] that multi-trace matrix models correspond to the generating function of stuffed maps. Such generating functions obeys a generalization of topological recursion [8], known as blobbed topological recursion. Recall the generating functions $W_k^g(x_1, .., x_k)$ of unstable stuffed maps defined in Section 4.3.1. As discussed these generating functions are precisely the resolvent functions' genus expansion terms for multi-trace matrix models. In our case we are dealing with bi-tracial models, whose genus expansion is proven rigorously in [29] which corresponds to unstable stuffed maps i.e. stuffed maps involving 2-cells with the topology of discs and cylinders. Given the initial generating functions

$W_1^0(x)$ and $W_2^0(x_1, x_2)$, which count connected unstable stuffed maps with one and two boundaries respectively, one can use a recursive formula to compute all higher genus and boundary generating functions [8]. In particular for unstable stuffed maps, blobbed topological recursion reduces to the usual topological recursion. If more than the product of two traces occurs in the potential, then blobbed topological recursion is required. These computations are all done in Zhukovsky space.

4.5.1 Unstable stuffed maps of genus zero with two boundaries

We would like to use blobbed recursion to compute higher order correlation functions of Dirac ensembles. In the previous section we showed how to compute $W_1^0(x)$ with various examples. The next step is to find $W_2^0(x_1, x_2)$, which for many important models, in Zhukovsky space, is universal. For example this is true in single trace Hermitian matrix models [22] and in the multi-matrix model seen in [13]. As far as the authors of this paper can tell, the form of this function has not been established for multi-trace matrix models, despite efforts in this direction given in [3]. For our bi-tracial models the proof of the universal form of $W_2^0(x_1, x_2)$ is the same as the proof in Section 3.2 of [22]. That is

$$W_2^0(x(z_1), x(z_2))x'(z_1)x'(z_2)dz_1dz_2 = \left(\frac{1}{(z_1 - z_2)^2} - \frac{x'(z_1)x'(z_2)}{(x(z_1) - x(z_2))^2}\right)dz_1dz_2. \tag{4.15}$$

Notice the appearance of the Bergman kernel.

If one wishes to compute the second order mixed moments, one needs to compute the residue

$$\mathcal{T}_{\ell_1,\ell_2} = \mathrm{Res}_{x_1\to\infty}\mathrm{Res}_{x_2\to\infty}x_1^{\ell_1}x_2^{\ell_2}W_2^0(x_1, x_2)dx_1dx_2.$$

To simplify this calculation one may apply the Zhukovsky transform in both variables and see that the second term on the right hand side of equation (4.15) contributes nothing to the residue.

In summary, we have established that for our bi-tracial matrix models $W_2^0(x_1, x_2)$ has the same universal form as the single trace matrix models mentioned above.

4.5.2 Single trace models hidden in Dirac ensembles

Now with $W_1^0(x)$ and $W_2^0(x_1, x_2)$ we may compute higher order correlation functions of our Dirac ensembles. This is proven as Theorem 9.1 in [3]. However, we shall not explicitly compute them here and instead have a different goal in mind.

As noted in the previous section, the coupling constants of the quartic, cubic, and hexic Dirac ensembles can be tuned to be the same as their respective Hermitian matrix model counterparts for certain values. When this is the case, $W_1^0(x)$ is the same, and in particular α and γ are as well. Thus, combining this fact with the universal form of $W_2^0(x_1, x_2)$, via topological recursion all higher order genus expansion terms will be identical. Hence, we have proven that these single trace models hide in the above mentioned Dirac ensembles, at least for certain values of the coupling constants.

In [6] and chapter 5 of [22] it is proven that single trace Hermitian matrix models have interesting behavior at certain critical points. For the quartic model this occurs at $t_4 = -\frac{1}{12}$, the hexic at $t_4 = -\frac{1}{9}$ and $t_6 = \frac{1}{270}$, and the cubic at $t_3 = -\frac{1}{2}\, 3^{-3/4}$. These points can be found as the locations of cusps of the spectral curve. See chapter 5 of [22] for details. Fortunately in the above mentioned Dirac ensembles all these critical points can be recovered. Their importance will be discussed in the following section.

4.6 The double scaling limit and 2D quantum gravity

In this section we will first review an old connection between random matrix theory and two dimensional quantum gravity. We will then discuss how these exact same connections hold for specific examples of Dirac ensembles of dimension one.

4.6.1 Large maps

As discussed in Section 4.3.1, formal matrix integrals count maps which are essentially polygonizations of Riemann surfaces. Intuitively, as the number of 2-cells that make up a map increases, the polygonization should give a better approximation of the underlying surface. Thus, our goal is to fine-tune the coupling constants of the model to some critical point that will cause the number of polygons that make up maps to go to infinity.

We emphasize that this is not a new idea. It was know on a heuristic level to physicists in the 80's and 90's [12, 20] for asymptotic quantities of random matrix models. Physicists predicted a connection to Liouville conformal field theory coupled to gravity, and it was indeed proven using the KPZ formula [11], that many models from both theories have the same critical exponents. Correlation functions of certain conformal field theories have the symmetry of representations of conformal groups. Such infinite representations in two dimensions were classified in Kac's table [23], and distinguished by two integers (p, q). For each such integer pair of a so-called minimal model the generating functions must satisfy a partial differential equation. It was proved in [6] that formal single matrix models have an associated $(2m + 1, 2)$ minimal model whose generating functions in the double scaling limit satisfy the associated partial differential equation and whose critical exponents match the minimal model. For example, the $(3, 2)$ minimal model is referred to as pure gravity. Its generating functions satisfy Painlevé I. Both the quartic and cubic matrix models are associated with this minimal model. For a more detailed description see Chapter 5 of [22].

This idea is particularly useful in the context of Dirac ensembles because it provides a direct link between path integrals over metrics of finite noncommutative geometries to two dimensional quantum gravity coupled to conformal field theories. We will describe this idea here. Consider the genus expansion of the partition function for some Dirac ensemble [29]

$$\log Z := \sum_{g=0}^{\infty} N^{2-2g} F_g,$$

where

$$F_g = \sum_{v=1}^{\infty} t^v \sum_{\Sigma \in \mathbb{SM}_k^g(v)} \prod_{i,j=1}^{d} t_{i,j}^{n_{i,j}(\Sigma)} \frac{1}{|\mathrm{Aut}(\Sigma)|}.$$

In general, F_g is a function of the coupling constants with some algebraic or logarithmic singularities [22]. These F_g's can be computed from the W_k^0's found using topological recursion. For details see Section 3.4 of [22]. As discussed in the previous section, for specific values of coupling constants the quartic, cubic, and hexic Dirac ensembles are precisely single trace Hermitian matrix models. Furthermore, all higher order correlation functions W_k^0 are the same in these ranges of coupling constants. Thus, all F_g are as well.

As the number of vertices gets very large, the behavior of the F_g's is going to be dominated by their singularities closest to zero. Thus, the behavior of unstable stuffed maps with a very large number of vertices is going to be controlled by the behavior of their generating functions near where its derivative diverges.

Now consider the quartic Dirac ensemble as an example. We can fine tune its coupling constants such that we recover the quartic Hermitian matrix model. As seen in [22], the explicit forms of the F_g's of the quartic matrix model imply that near a critical point, the series has an expansion as the sum of regular and singular functions. With $g \neq 1$ at the critical point $t_4 = 1/12$, the singular part of the expansion looks like

$$\mathrm{sing}(F_g) = C_g\,(t_4 - t_c)^{5(1-g)/2}$$

for some constant C_g. When $g = 1$,

$$\mathrm{sing}(F_1) = C_1\, \log(t_4 - t_c).$$

If one defines the new series

$$u(y) = \sum_{g=0}^{\infty} C_g y^{5(1-g)/2},$$

then $u''(y)$ satisfies the Painlevé I equation

$$y = (u''(y))^2 - \frac{1}{3}u^{(4)}(y).$$

Note that, even in the double scaling limit, the generating functions of large maps can be computed using topological recursion [22]. Furthermore, for general single trace matrix models, we can construct the following formal series:

$$\sum_{g\geq 0} N^{2-2g}\tilde{F}_g,$$

where $\tilde{F}_g$ are the leading terms in the asymptotic expansions of F_g. This series is a formal Tau-function of some reduction of the KdV hierarchy, and can be obtained from Liouville conformal field theory when coupled with gravity. For more details see Section 5.4 of [22]. Thus, if we can fine-tune a Dirac ensemble to the critical points of a single trace matrix model, the same result follows. In particular, we know that the critical point of the quartic Hermitian matrix model is $t_4 = -1/12$, and considering the analysis done in Section 4.4.3 we have the following phase diagram for the quartic Dirac ensemble.

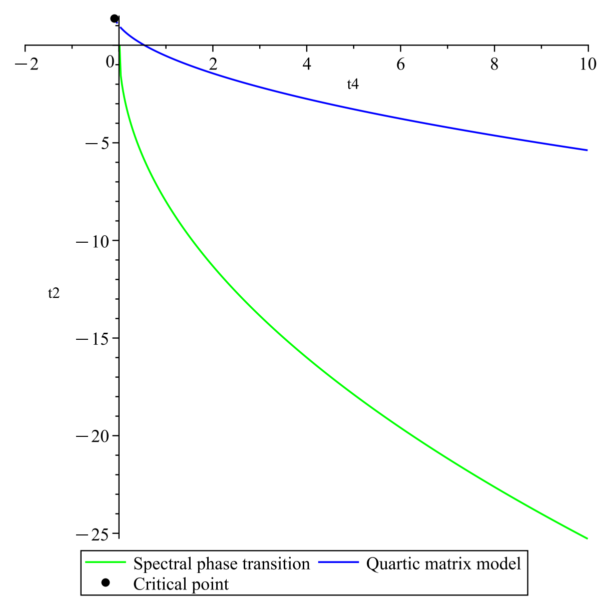

Figure 4.2: The phase diagram of the quartic Dirac ensemble.

In the above diagram we have the curve where the spectral phase transition happens. This is precisely when the constant t_2 is chosen in terms of t_4 as

$$t_2 = -8\sqrt{t_4}.$$

This can be found by finding when $\rho(0) = 0$, just as in [28].

One can also conclude that the quartic matrix model appears when

$$t_2 = -\frac{(1+12t_4)^{3/2} - 4 - 144t_4 + (36t4+3)\sqrt{1+12t_4}}{72t_4}.$$

In the quartic matrix model the critical point occurs at $t_4 = -\frac{1}{12}$. In the quartic Dirac ensemble this corresponds to the point $t_4 = -\frac{1}{12}$ and $t_2 = \frac{4}{3}$. Note that the spectral phase transition curve does not cross the quartic matrix model curve. This is because, in the quartic matrix model considered,

$$\int_{\mathcal{H}_N} e^{-\frac{N}{2}\operatorname{Tr} H^2 - \frac{c_4}{4} N \operatorname{Tr} H^4} dH,$$

there is no spectral phase transition. A coupling constant in front of the Gaussian term is required for this phenomenon to occur.

Remark Ultimately we are solving the loop equations, which are the same in this case regardless of whether the model is considered formal or convergent. Which interpretation we can choose is dependent upon which quadrants of Figure 4.2 we are considering. In the first and fourth quadrants t_4 is negative in the potential, so the model can always be seen as convergent since the quartic term dominates the Gaussian one at the limits of the integral. In quadrants one and two the model can always be viewed as formal since the Gaussian integrals in the definition of a formal integral are always convergent. In the third quadrant, however, the model is neither formal nor convergent, as the integrals diverge, but so do all the Gaussian integrals in the formal definition.

A similar analysis may be done for the cubic Dirac ensemble. However, no spectral phase transition occurs in this model. See Figure 4.3. The critical point is at $t_3 = -\frac{1}{2}3^{-3/4}$ and

$$t_2 = \frac{\sqrt{3}}{216} + \frac{1}{3} + \frac{3168\sqrt{3} + \left(411915\sqrt{3} + 418608 + 648\sqrt{239311 + 18512\sqrt{3}}\right)^{\frac{2}{3}} + 5043}{216\sqrt[3]{411915\sqrt{3} + 418608 + 648\sqrt{239311 + 18512\sqrt{3}}}}$$
$$\approx 1.297.$$

In general it was shown in [12] that a matrix model of the form

$$\int_{\mathcal{H}_N} e^{-\frac{N}{2}\operatorname{Tr} H^2 - \sum_{j\geq}^{d} \frac{c_j}{j} N \operatorname{Tr} H^j} dH$$

is associated with a $(2m+1, 2)$ minimal model in the double scaling limit. In particular, we can deduce then that the quartic, cubic, and hexic Dirac ensembles of type $(1, 0)$ are associated

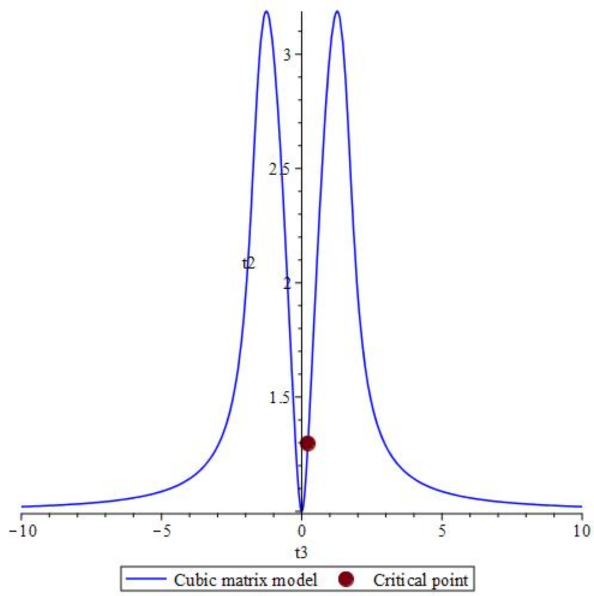

Figure 4.3: The phase diagram of the cubic Dirac ensemble.

with the $(3, 2)$, $(3, 2)$, and $(5, 2)$ minimal models. In general for single matrix models, if the potential of the model is degree d and it is odd then it is associated with the $(d, 2)$ minimal model. If the degree is even, then it is associated with the $(d - 1, 2)$ minimal model. We expect the same relationship holds for Dirac ensembles of the above mentioned form.

4.7 Conclusion and outlook

In this paper we analyze the quartic, cubic, and hexic Dirac ensembles of type $(1, 0)$ as formal matrix integrals. Their resolvent functions W_1^0 and limiting eigenvalue distributions are found explicitly. The cylinder amplitude W_2^0 is discussed to have a universal form. Thus, via the process of blobbed topological recursion one may compute all higher genus and boundary correlation functions. During this analysis, it is found that by fine-tuning the coupling constants of these models one can recover critical phenomena seen in certain Hermitian matrix models. In particular in the double scaling limit we find the critical exponents associated with minimal models from conformal field theories. Additionally, for these models the genus expansion terms of the log of the partition function satisfy the same differential equations as the partition functions of the corresponding minimal model in the double scaling limit. We hope to prove rigorously in future work that most Dirac ensembles of type $(1, 0)$ and $(0, 1)$ have a corresponding minimal model in the double scaling limit. Further, it would be interesting to construct multicritical matrix models [27, 1, 2] from Dirac ensembles.

Essentially, we have recovered conformal field theories coupled to gravity from toy models of quantum gravity on noncommutative spaces. We would like to extend this connection to Dirac operators of higher dimension. This seems likely considering that similar connections have been made for multi-matrix models [16]. Without any known analytic techniques to study matrix models seen in higher dimensional Dirac ensembles we can only aim for numerical evidence. It may be possible to deduce the critical values of Dirac ensembles using the bootstraps technique [26]. In random matrix models the range of coupling constants on which the model is defined determines the critical values. For example the quartic matrix model has solutions when $-1/12 < t_4$ in the large N limit. Thus, bootstrapping seems like a likely candidate to use to find these critical points. However, determining these critical exponents might be better suited to Monte Carlo simulations. This was explored in some sense in [24]. Note that the finite size of N might strongly affect these values. Another possibility is through functional renormalization group techniques [37]. We hope to explore these ideas in future works.

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Appendix A

Markov Chain Monte Carlo Simulation

In this appendix, we simulate a random matrix model using the method of Markov chain Monte Carlo (MCMC) simulation. By doing this we are able to verify results obtained in chapters 2,3, and 4 by numerical simulation. We discuss the Monte Carlo method of sampling and its benefits in numerical computations. We study the step size and autocorrelation of the MCMC samples with an example.

A.1 Method of MCMC in random matrix theory

Consider a random Hermitian matrix model with the partition function:

$$\mathcal{Z} = \int_{\mathcal{H}_N} e^{-V(H)} dH. \tag{A.1}$$

The expectation of a real or complex-valued function f on $\mathcal{H}_N$ is defined to be:

$$\langle f(H) \rangle = \frac{1}{\mathcal{Z}} \int_{\mathcal{H}_N} f(H) e^{-V(H)} dH. \tag{A.2}$$

The model can be approximated by a discrete ensemble $\{H_1, \cdots, H_n\}$, and the expectation can be estimated by averaging of the samples:

$$\langle f(H) \rangle_n = \frac{\sum_{j=1}^{n} f(H_j) e^{-V(H_j)}}{\sum_{j=1}^{n} e^{-V(H_j)}}, \tag{A.3}$$

i.e.,

$$\langle f(H) \rangle = \lim_{n\to\infty} \langle f(H) \rangle_n. \tag{A.4}$$

This can be done by using a Markov Chain Monte Carlo simulation. In a MCMC algorithm, the Hermitian matrix H_i is generated with a probability distribution such that

$$Pr(X = H_i) = \frac{e^{-V(H_i)}}{\sum_{j=1}^{n} e^{-V(H_j)}}. \tag{A.5}$$

In this case,

$$\langle f(H) \rangle_n = \sum_{i=1}^{n} f(H_i) Pr(H_i) \ . \tag{A.6}$$

A.2 Metropolis-Hastings algorithm in MCMC simulation

The Metropolis–Hastings algorithm is an MCMC method for obtaining a sequence of random samples from a probability distribution. It is a powerful tool to generate a discrete measure to approximate a matrix ensemble. The algorithm is as follows.

- Initialization (step 0): Produce a complex matrix with entries randomly chosen in the complex range $[-1-i, 1+i]$, i.e., the real and imaginary parts of each entries of the matrix is generated by uniformly distribution from $[-1, 1]$. Then we take the average of the matrix with its conjugate to make a Hermitian matrix. This matrix becomes the initial state x_0 of the algorithm.

- Iteration: Then for each iteration t:

  - Generate: Add a random Hermitian matrix with entries in the complex range

    $$[(-1-i)\ell, (1+i)\ell],$$

    to the previous state to get a random candidate state x'. Note that ℓ is the length of the steps and is kept fixed during the simulation.
  - Acceptance ratio: Calculate the ratio $r = \frac{e^{-V(x')}}{e^{-V(x_t)}} = e^{V(x_t)-V(x')}$.
  - Accept or reject:
    * Generate a uniform random number $u \in [0, 1]$.
    * If $r \geq u$, accept the candidate by setting $x_{t+1} = x'$.
    * If $r < u$, reject the candidate and set $x_{t+1} = x_t$.

With adjusting the length of the steps, the acceptance rate can be about 50% which is a desired rate to get a proper simulation [2]. After a sufficient number of moves, the probability distribution for x_j becomes independent of the initial state x_0. Only the states generated after reaching independence are representative of the probability distribution and can be used to measure observables. The histogram of their eigenvalues approximates the probability distribution of the model. For more details we refer the readers to [1, 3].

A.3 MCMC for a quartic bi-tracial matrix ensemble

Suppose the potential of a Hermitian matrix model is:

$$V(H) = N\frac{t_2}{2}\operatorname{Tr} H^2 + N\frac{t_4}{4}\operatorname{Tr} H^4 + \frac{t_{2,2}}{4}(\operatorname{Tr} H^2)(\operatorname{Tr} H^2).$$

We applied a Markov chain Monte Carlo simulation to get the following diagram.

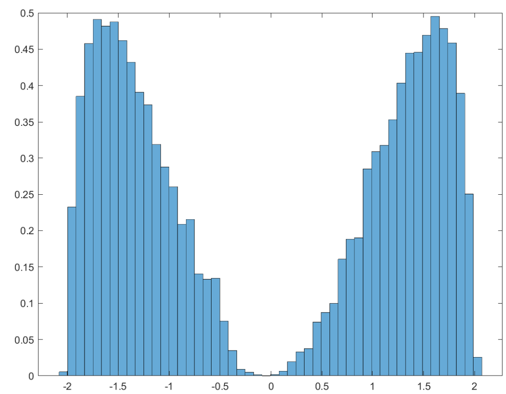

Figure A.1: The eigenvalue distribution of 50000 Hermitian matrices of size 30, at the phase transition, with the coupling constants $t_2 = -4$, $t_4 = 1$, and $t_{2,2} = 1$.

By using the MCMC algorithm, not only a good approximation of the moments of the model can be found (A.4), but also the phase transition can be achieved. The following diagram is the autocorrelation of the simulation we had.

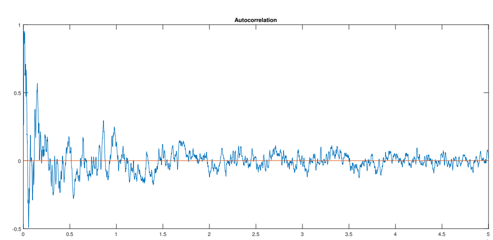

Figure A.2: Decay of the autocorrelation of the minimum eigenvalue for the model with coupling constants $t_2 = -4$, $t_4 = 1$, and $t_{2,2} = 1$. The horizontal axis is Monte Carlo time. After around 15000 steps, states become almost independent with respect to the initial state.

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Appendix B

Convergent and Formal Matrix Models

B.1 Convergent unitary invariant matrix ensembles

In general, a random matrix is a matrix valued random variable. In particular we are interested in random Hermitian and skew-Hermitian matrices. The simplest example of interest is what is called the Gaussian Unitary Ensemble (GUE). The joint distribution on the entries of a GUE matrix is given by

$$\frac{1}{Z_N^G} e^{-\frac{N}{2}\operatorname{Tr} H^2} dH,$$

where Z_N^G is the normalization constant and dH is the Lebesgue measure of the real N^2-dimensional vector space of $N \times N$ Hermitian matrices:

$$dH = \prod_i d\mathrm{Re}H_{ii} \prod_{i<j} d\mathrm{Re}H_{ij} d\mathrm{Im}H_{ij}.$$

The GUE gets its name from being invariant under unitary transformations on H. A much wider class of random matrix ensembles have this property and are referred to as (unitarily) invariant ensembles. Traditionally, the invariant ensembles of interest are given by measure a of the form

$$\frac{1}{Z_N} e^{-\frac{N}{2}\operatorname{Tr} H^2 - \sum_{j=3}^{d} \frac{Nt_j}{j} \operatorname{Tr} H^j} dH.$$

where d is some even integer greater than three, and the t_j's are some real coupling constants.

There are several quantities of interest when analyzing these ensembles. The partition function Z_N can often be computed using orthogonal polynomials [7]. However, when studying Dirac ensembles another type of invariant ensemble is considered, one where this technique is not applicable. Consider convergent matrix models of the form

$$Z_N = \int_{\mathcal{H}_N} e^{-\frac{N}{2}\operatorname{Tr} H^2 - \sum_{j=3}^{d} \frac{Nt_j}{j}\operatorname{Tr} H^j - \sum_{i,j=1}^{d} \frac{t_{i,j}}{i+j}\operatorname{Tr} H^i \operatorname{Tr} H^j} dH. \tag{B.1}$$

Such models are called bi-tracial.

Another important quantity of interest are moments which we define as

$$\langle \frac{1}{N}\operatorname{Tr} H^\ell \rangle := \frac{1}{N}\frac{1}{Z}\int_{\mathcal{H}_N} \operatorname{Tr} H^\ell e^{-\frac{N}{2}\operatorname{Tr} H^2 - \sum_{j=3}^{d} \frac{Nt_j}{j}\operatorname{Tr} H^j - \sum_{i,j=1}^{d} \frac{t_{i,j}}{i+j}\operatorname{Tr} H^i \operatorname{Tr} H^j} dH$$

for $\ell \geq 0$. In practice, given these moments one can find a unique corresponding probability distribution with compact support. This quantity is often difficult to compute for finite N. However, calculations simplify if one considers computing the limit of these moments as N goes to infinity, which we refer to as the large N limit. This sequence of moments, in practice, also has a unique corresponding probability distribution, which is called the *limiting eigenvalue distribution*. In particular for convergent integrals of the form of (B.1) it can be expressed as $d\mu(x) = \rho(x)dx$, where ρ is a continuous function, referred to as the *limiting eigenvalue density function*. For example, in the case of the GUE, the limiting eigenvalue distribution is the celebrated Wigner semicircular distribution.

Even though the direct computation of Z_N for finite N seems out of reach, one may still compute the limiting eigenvalue distribution in the large N limit using Coulomb gas techniques, which we will now review. The first step to compute this distribution is to apply Weyl's integration formula [1] to reduce the N^2-dimensional integral Z_N to an integral over its N eigenvalues:

$$\begin{aligned} Z_N = C_N \int_{\mathbb{R}^N} \exp\Bigg\{ -\frac{N}{2}\sum_{k=1}^{N} \lambda_k^2 - \sum_{j>2}^{d} \frac{Nt_j}{j}\left(\sum_{k=1}^{N} \lambda_k^j\right) \\ - \sum_{i,j=1}^{d} \frac{t_{i,j}}{i+j}\left(\sum_{k=1}^{N}\lambda_k^i\right)\left(\sum_{s=1}^{N}\lambda_s^j\right)\Bigg\} \prod_{k<s}(\lambda_k - \lambda_s)^2 \prod_{k=1}^{N} d\lambda_k \\ =: C_N \int_{\mathbb{R}^N} \exp\Bigg\{ -\sum_{i,j=1}^{N} Q(\lambda_i, \lambda_j) + 2\sum_{i<j} \log|\lambda_i - \lambda_j| \Bigg\} d\lambda_k \end{aligned}$$

for some constant C_N. Notice that the Jacobian from the change of variables gave us the square of the famous Vandermonde determinant in the integrand.

For certain potentials, the leading contribution of the integral is going to come from the set of eigenvalues that maximizes the integrand, we denote such a set $\{\lambda_i^*\}_{i=1}^N$. One can often show that such a point in $\mathbb{R}^N$ is unique, allowing us to apply Laplace's method. Furthermore, we may construct the normalized counting measure of eigenvalues:

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i^*}. \tag{B.2}$$

Using the results in chapter six of [7] one can show that for convergent integrals like (B.1), the measure (B.2) converges in the vague topology to the limiting eigenvalue distribution of the ensemble. The limiting eigenvalue distribution of (B.1) can be found as the unique measure μ that minimizes the following functional:

$$I(\mu) = \int_{\mathbb{R}} \int_{\mathbb{R}} (Q(x, y) - \log|x - y|)\, d\mu(x) d\mu(y).$$

With knowledge of μ one may compute the large N moments of a random matrix ensemble as

$$\lim_{N\to\infty} \frac{1}{N} \langle \operatorname{Tr} H^\ell \rangle = \int x^\ell d\mu(x).$$

Additionally, though not as obvious how, one can also compute the free energy

$$\lim_{N\to\infty} \frac{1}{N^2} \ln Z_N$$

from knowledge of the spectral density ρ. For more details see chapter 6 of [7].

Define the *resolvent* moment generating function as

$$W_1^0(x) = \lim_{N\to\infty} \left\langle \operatorname{Tr} \frac{1}{x - H} \right\rangle = \sum_{\ell=0}^{\infty} \frac{\lim_{N\to\infty} \langle \operatorname{Tr} H^\ell \rangle}{x^{\ell+1}}.$$

The resolvent is in fact the Stieltjes transform of the limiting eigenvalue density function:

$$W_1^0(x) = \int_{\operatorname{supp}\rho} \frac{\rho(y) dy}{x - y}.$$

Thus if one can compute the resolvent, then in theory one can invert the Stieltjes transform to find ρ.

Spectral phase transitions are common phenomena involving the limiting eigenvalue distribution and occurs when the number of connected components of the support changes as one changes the values of the coupling constants. We refer to Section 2.3.2 for examples.

For later use we will also define higher moments as

$$\langle \operatorname{Tr} H^{\ell_1} \operatorname{Tr} H^{\ell_2} ... \operatorname{Tr} H^{\ell_k} \rangle = \int_{\mathcal{H}_N} \operatorname{Tr} H^{\ell_1} \operatorname{Tr} H^{\ell_2} ... \operatorname{Tr} H^{\ell_k} e^{-\frac{N}{2} \operatorname{Tr} H^2 - \sum_{j=3}^{d} \frac{N t_j}{j} \operatorname{Tr} H^j} dH.$$

In practice however, it is often easier to compute the cumulants, denote with a subscript c, instead of the moments, which are related by the moment-cumulants, see relations chapter 1.2.5 of [13].

B.2 Formal matrix models

We will now discuss a very different but deeply related type of matrix model. Informally, given an expression for a matrix integral that may or may not be convergent, one defines a *formal matrix integral* by expanding the non-Gaussian terms in a power series and exchanging the order of integration and summation. For example, consider the expression

$$\int_{\mathcal{H}_N} e^{-\frac{N}{2t}\operatorname{Tr}H^2-\frac{t_4 N}{4t}\operatorname{Tr}H^4}\,dH.$$

A formal matrix integral corresponding to this expression may be defined as

$$Z_{\mathrm{quad}} := \sum_{n=0}^{\infty}\frac{N^n t_4^n}{(4t)^n n!}\int_{\mathcal{H}_N}(\operatorname{Tr}H^4)^n e^{-\frac{N}{2}\operatorname{Tr}H^2}\,dH.$$

In general, and as can be seen in the preceding example, formal matrix integrals are weighted formal summations of GUE moments. Note that these series are typically divergent and should be understood as formal power series in t, t_4 and other coupling constants.

GUE moments have a combinatorial interpretation as sums over *maps*. A map is a graph embedded in a Riemann surface that comes from gluing edges of various polygons in particular the complement of a map on a surface is a disjoint union of open discs. For a more detailed definition see [13]. For example consider Figure B.1. This leads to formal matrix integrals being the weighted generating functions of maps [6, 13].

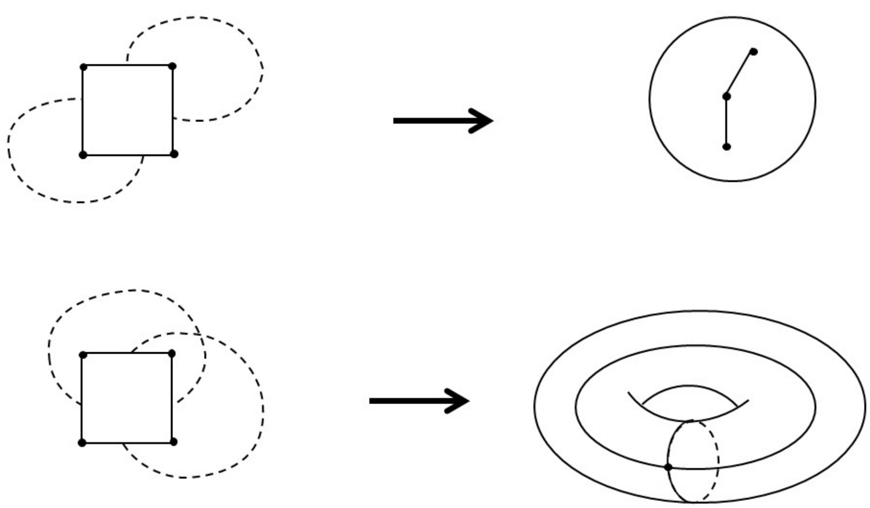

Figure B.1: The only two possible maps resulting from gluing an unmarked quandrangle.

Consider the previous example. One can show that

$$Z_{\mathrm{quad}} = \sum_{g\geq 0}\left(\frac{N}{t}\right)^{2-2g}\left[\sum_{v=0}^{\infty} t^{v}\sum_{\Sigma\in\mathcal{M}^{g}(v)}\frac{t_4^{n_4(\Sigma)}}{|\mathrm{Aut}(\Sigma)|}\right]$$

where, for the quartic model, $\mathcal{M}^g(v)$ is the set of maps with genus g and v vertices formed from gluing quadrangles together and $n_4(\Sigma)$ is the number of quadrangles glued to form the map Σ. One can show that for fixed g and v, $\mathcal{M}^g(v)$ is a finite set, thus the coefficients of the series

$$F_g := \sum_{v=0}^{\infty} t^v \sum_{\Sigma \in \mathcal{M}^g(v)} \frac{t_4^{n_4(\Sigma)}}{|\mathrm{Aut}(\Sigma)|} \tag{B.3}$$

are finite. This ensures that Z_{quad} is a well-defined formal series [13].

In general, the consequence of a term of the form

$$\frac{t_j N}{jt} \operatorname{Tr} H^j$$

in the potential is that one adds j-gons to the set of polygons that may be used to glue maps. Thus, a formal matrix integral of the form

$$Z_N = \int_{\mathcal{H}_N} e^{-\frac{N}{2t} \operatorname{Tr} H^2 - \sum_{j=3}^{d} \frac{N t_j}{jt} \operatorname{Tr} H^j} dH$$

is in fact the formal power series

$$Z_N = \sum_{g \geq 0} \left(\frac{N}{t}\right)^{2-2g} \left[\sum_{v=0}^{\infty} t^v \sum_{\Sigma \in \mathcal{M}^g(v)} \frac{t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \ldots t_d^{n_d(\Sigma)}}{|\mathrm{Aut}(\Sigma)|} \right], \tag{B.4}$$

where $\mathcal{M}^g(v)$ is the set of maps with genus g and v vertices formed from gluing: triangles, quadrangles, ..., and d-gons. For $1 \leq j \leq d$, $n_j(\Sigma)$ is the number of j-gons used to glue the map Σ. This formal summation is once again well-defined because the set $\mathcal{M}^g(v)$ is finite.

Note that formal matrix integrals like (B.4) are organized by the genus of the maps. Such an expression is called a large N expansion or *genus expansion*. Remarkably, convergent matrix integrals often also have a genus expansion whose coefficients coincide to leading order with those of its formal counterpart. For discussions on the relationship between formal and convergent matrix models see [9].

We will now extend these ideas to bi-tracial formal matrix models. A 2-cell of topology (k, g) is a connected oriented genus g surface with k boundaries. They generalize the usual polygons seen in the perturbative expansion of single trace matrix models. In particular, for bi-tracial matrix models, the consequence of a term of the form

$$\frac{t_j N}{(i+j)t} \operatorname{Tr} H^i \operatorname{Tr} H^j$$

in the potential is that one adds 2-cells with the topology of a cylinder with boundaries of lengths i and j to the set of polygons to glue. Maps glued from 2-cells with topologies other than a disc are referred to as *stuffed maps* and were studied in detail in [3, 4]. For example see Figure B.2.

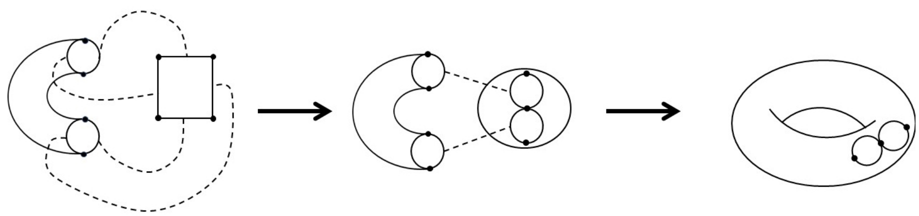

Figure B.2: A gluing resulting from a quandrangle and a 2-cell with the topology of a cylinder and boundaries of lengths two.

Some other important aspect of formal matrix integrals are their moments, higher moments, and cumulants. Consider the following formal matrix integral

$$\int_{\mathcal{H}_N} e^{-\frac{N}{2t}\operatorname{Tr} H^2-\sum_{j=3}^{d}\frac{Nt_j}{jt}\operatorname{Tr} H^j} dH.$$

Its moments are the formal series that result from considering the integral of $\operatorname{Tr} H^\ell$ then power series expanding all non-Gaussian terms and swapping the order of integration and summation. One can show that these moments, just like the partition function, have a genus expansion:

$$\langle \operatorname{Tr} H^\ell \rangle = \sum_{g\geq 0}\left(\frac{N}{t}\right)^{1-2g}\left[\sum_{v=0}^{\infty} t^v \sum_{\Sigma\in\mathcal{M}_\ell^g(v)} \frac{t_3^{n_3(\Sigma)}t_4^{n_4(\Sigma)}\ldots t_d^{n_d(\Sigma)}}{|\mathrm{Aut}(\Sigma)|}\right] := \sum_{g\geq 0}\left(\frac{N}{t}\right)^{1-2g}\mathcal{T}_\ell^g$$

where $\mathcal{M}_\ell^g(v)$ denotes the set of connected maps of genus g, with v vertices, glued from triangles, quandrangles, ..., d-gons, and one distinguished ℓ-gon with a rooted edge, which is called a *boundary*. By rooted edge we mean that one edge is distinct and has a direction that orients the polygon. Notice that maps in $\mathcal{M}_\ell^g(v)$ are connected, where by definition a map is called connected if it is draw on a connected Riemann surface. Note that if $\ell = 0$, then each $\mathcal{T}_0^g$ is precisely F_g from equation (B.3).

Next one wants to consider higher moments which have a similar genus expansion

$$\langle \operatorname{Tr} H^{\ell_1}\operatorname{Tr} H^{\ell_2}\cdots\operatorname{Tr} H^{\ell_k}\rangle = \sum_{g\geq 0}\left(\frac{N}{t}\right)^{2-2g-k}\left[\sum_{v=0}^{\infty} t^v \sum_{\Sigma\in\mathcal{M}_{\ell_1,\ldots\ell_k}^g(v)} \frac{t_3^{n_3(\Sigma)}t_4^{n_4(\Sigma)}\ldots t_d^{n_d(\Sigma)}}{|\mathrm{Aut}(\Sigma)|}\right],$$

where $\mathcal{M}_{\ell_1,\ldots,\ell_k}^g(v)$ denotes the set of not necessarily connected maps of genus g, with v vertices, glued from triangles, quandrangles, ..., d-gons, one boundary of length ℓ_1, ..., one boundary of length ℓ_k. Notice that this time the set is not necessarily connected. Counting connected maps is much easier than counting disconnected ones, thus in practice one computes the sum over connected maps, denoted by $\langle \operatorname{Tr} H^{\ell_1} \operatorname{Tr} H^{\ell_2} \cdots \operatorname{Tr} H^{\ell_k} \rangle_c$. Remarkably, the connected sums are precisely the cumulants from classical multivariate probability theory. In particular one can recover $\langle \operatorname{Tr} H^{\ell_1} \operatorname{Tr} H^{\ell_2} \cdots \operatorname{Tr} H^{\ell_k} \rangle$ using the moment-cumulant relations. For a discussion of this see chapter 1.2.5 of [13]. We will see in the next section that one can compute the coefficients of the genus expansions of moments and cumulants recursively.

In formal matrix integrals we have seen that the partition function, moments, and cumulants are well-defined formal series. It is often the case that these converge in some small multi-disc while also having (usually algebraic) singular behaviour for certain values of the coupling constants. At these singularities, the formal series has an asymptotic expansion. The exponent of the leading order term in this asymptotic expansion is usually referred to as a *critical exponent*. Let $A(x)$ be a formal series with a singularity at x_c, then the critical exponent of

$$A(x) \sim C(x - x_c)^a + ...$$

is a.

This idea is common in statistical mechanics to describe quantities near the singularities of phase transitions and are in some sense universal. In random matrix theory they are often used to find connections to areas of physics, such as conformal field theory [8]. Some examples of this are given in Section 2.3.4 for the genus expansion terms of the natural logarithm of the partition function.

B.3 The Schwinger-Dyson equations and topological recursion

A common tool used to analyze both formal and convergent random matrix integrals are Schwinger-Dyson equations (SDE's): an infinite system of recursive equations between moments and cumulants. They were first introduced by Migdal in [16]. These equations have a straightforward derivation. Consider a matrix integral, either formal or convergent, of the form

$$\int_{\mathcal{H}_N} e^{-S(H)} dH,$$

where $S(H)$ is some multi-tracial polynomial of powers of H. Using Stoke's formula, it follows that

$$\sum_{i,j=1}^{N} \int_{\mathcal{H}_N} \frac{\partial}{\partial H_{ij}} \left((H^\ell)_{ij} e^{-S(H)} \right) dH = 0$$

for any $\ell \geq 0$. Applying the product rule to the integrand we find that

$$\sum_{k=0}^{\ell-1} \langle \operatorname{Tr} H^{\ell-1-k} \operatorname{Tr} H^k \rangle - \langle \operatorname{Tr} H^\ell S'(H) \rangle = 0.$$

Suppose for example that $S(H) = N/2 \operatorname{Tr} H^2 + Nt_4/4 \operatorname{Tr} H^4$. Then this equations becomes

$$\sum_{k=0}^{\ell-1} \langle \operatorname{Tr} H^{\ell-1-k} \operatorname{Tr} H^k \rangle = N \langle \operatorname{Tr} H^{\ell+1} \rangle + N \langle \operatorname{Tr} H^{\ell+4} \rangle. \tag{B.5}$$

Often finding the solutions for these equations at finite N is quite difficult. By applying the genus expansion of moments, provided that it exists, one can derive equations for each order of N that relate the coefficients of the genus expansions. Continuing the quartic example, let

$$\langle \operatorname{Tr} H^\ell \rangle := \sum_{g \geq 0} N^{1-2g} \mathcal{T}_\ell^g.$$

Collecting like terms of the genus expansion of the moments in the Schwinger-Dyson equations, the leading order SDE's in N that come from (B.5), often called the *loop equations*, are

$$\sum_{k=0}^{\ell-1} \mathcal{T}_{\ell-1-k}^0 \mathcal{T}_k^0 = \mathcal{T}_{\ell+1}^0 + \mathcal{T}_{\ell+4}^0, \tag{B.6}$$

for $\ell \geq 0$. This equation is much simpler to solve due to the disappearance of the higher moments. In particular, in the case of a Gaussian potential, all odd moments are zero, let $\ell = 2n + 1$ for $k \geq 0$ then equation (B.6) becomes

$$\sum_{k=0}^{2n} \mathcal{T}_{2n-2k}^0 \mathcal{T}_{2k}^0 = \mathcal{T}_{2n+2}^0$$

with $\mathcal{T}_0^0 = 1$. The solution is $\mathcal{T}_{2n}^0 = C_n$ the nth Catalan numbers, which are the leading order terms in the genus expansion of GUE moments.

This is the main reason to introduce the genus expansion is that it allows access to simpler Schwinger-Dyson equations by restricting to leading order. However, it is possible to recover lower order contributions through a process called Topological Recursion which we will outline now.

Note that this process can be repeated for higher order moments by considering

$$\sum_{i,j=1}^{N} \int_{\mathcal{H}_N} \frac{\partial}{\partial H_{ij}} \left((H^{\ell_1})_{ij} \prod_{q=2}^{m} \operatorname{Tr} H^{\ell_q} e^{-S(H)} \right) dH = 0.$$

Denote the genus expansion terms of general cumulants as

$$\langle \operatorname{Tr} H^{\ell_1} \operatorname{Tr} H^{\ell_2} \cdots \operatorname{Tr} H^{\ell_n} \rangle_c := \sum_{g \geq 0} N^\chi \mathcal{T}_{\ell_1, \ell_2, \ldots, \ell_n}^g,$$

where $\chi = 2 - 2g - n$ is the Euler characteristics of the maps in that term of the expansion. Then, for the example of the quartic potential from before, one can find that

$$2\sum_{k=0}^{\ell_1-1}\mathcal{T}^0_{\ell_1-1-k}\mathcal{T}^0_{k,\ell_2}+\ell_2\mathcal{T}^0_{\ell_1-1+\ell_2}=\mathcal{T}^0_{\ell_1+1,\ell_2}+\mathcal{T}^0_{\ell+4,\ell_2},$$

for $\ell_1, \ell_2 \geq 1$. From the terms in the product on the left-hand side this equation relies on the solution to equation (B.6). Similarly, each equation for a given Euler characteristic relies on the solutions of the higher Euler characteristic equations. For details on the SDE's see [12, 13] or in the case that there is a multi-trace potential [3, 2, 14].

In summary the SDE's are an unwieldy infinite set of recursive equations between terms of the genus expansion of moments or cumulants. However, in [10] a method was outlined to streamline this process, called *Topological Recursion*. This process has been generalized [11, 15, 5] from its original use in matrix integrals. We start by defining the following generating functions of the genus expansion terms of moments and cumulants:

$$W^g_n(x_1, x_2, ..., x_n) = \sum_{\ell_1,\ell_2,...,\ell_n}\frac{\mathcal{T}^g_{\ell_1,\ell_2,...\ell_n}}{x_1^{\ell_1+1}x_2^{\ell_2+1}\cdots x_n^{\ell_n+1}}.$$

One can write the SDE's in terms of these new generating functions as a means of collecting terms. In single or bi-tracial single Hermitian matrix models Topological Recursion allows one to compute any W^g_n from just the information in the resolvent W^0_1 and the *cylinder amplitude* W^0_2. The cylinder amplitude is often in some sense universal. This means that the only fundamental information needed to compute any W^g_n is contained in the resolvent. Once one has sufficient W^g_n's it is possible to compute the genus expansion terms of the partition function directly. For details on this process see chapter 3 of [13]. For single matrix integrals with higher trace multiplicities than two one needs a generalized process called Blobbed Topological Recursion [3, 4].

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Appendix C

Factorization of Mixed Moments and Loop Equations

C.1 Factorization of mixed moments

Suppose a formal model is given as described in chapter two,

$$\mathcal{Z} := \int_{\mathcal{H}_N^m} e^{-\tilde{S}(H_1,H_2,\ldots,H_m)} dH_1 \ldots dH_m,$$

and further assume that the moments of such a model have a genus expansion, i.e.

$$\langle \operatorname{Tr} H_p^{\ell} \rangle = \sum_{g \geq 0} N^{1-2g} \, \mathcal{T}_{\ell}^{g},$$

and

$$\begin{aligned} \langle \operatorname{Tr} H_p^{\ell_1} \operatorname{Tr} H_q^{\ell_2} \rangle &= \langle \operatorname{Tr} H_p^{\ell_1} \operatorname{Tr} H_q^{\ell_2} \rangle_c + \langle \operatorname{Tr} H_p^{\ell_1} \rangle \langle \operatorname{Tr} H_q^{\ell_2} \rangle \\ &= \sum_{n \geq 0} N^{-2g} \mathcal{T}_{\ell_1,\ell_2}^{g} + \left(\sum_{n \geq 0} N^{1-2g} \mathcal{T}_{\ell_1}^{g} \right) \left(\sum_{n \geq 0} N^{1-2g} \mathcal{T}_{\ell_2}^{g} \right). \end{aligned}$$

Here the coefficients are a formal series that counts the number of genus g maps with a boundary of length ℓ. The subscript c denotes the sum over connected maps with two boundaries of lengths ℓ_1 and ℓ_2. Thus, taking the large N limit, we obtain

$$\lim_{N \to \infty} \frac{1}{N} \langle \operatorname{Tr} H_q^{\ell} \rangle = \mathcal{T}_{\ell}^{0},$$

and

$$\lim_{N \to \infty} \frac{1}{N^2} \langle \operatorname{Tr} H_p^{\ell_1} \operatorname{Tr} H_q^{\ell_2} \rangle = \mathcal{T}_{\ell_1}^{0} \mathcal{T}_{\ell_2}^{0} = \lim_{N \to \infty} \frac{1}{N^2} \langle \operatorname{Tr} H_p^{\ell_1} \rangle \langle \operatorname{Tr} H_q^{\ell_2} \rangle.$$

All the models studied in this thesis do indeed have a genus expansion, hence the above factorization holds.

C.2 Loop equations for a quartic type (2, 0) Dirac ensemble

Consider two matrix Dirac ensemble of type (2, 0) with the following partition function.

$$\mathcal{Z} = \int_{\mathcal{G}} e^{-\left(g \operatorname{Tr} D^2 + \operatorname{Tr} D^4\right)} dD,$$

where

$$\operatorname{Tr} D^2 = 4N \operatorname{Tr} H_1^2 + 4N \operatorname{Tr} H_2^2 + 4(\operatorname{Tr} H_1)^2 + 4(\operatorname{Tr} H_2)^2,$$

and

$$\begin{aligned}\operatorname{Tr} D^4 = {} & 4N \operatorname{Tr} H_1^4 + 4N \operatorname{Tr} H_2^4 + 16N \operatorname{Tr} H_1^2 H_2^2 - 8N \operatorname{Tr} H_1 H_2 H_1 H_2 + 16 \operatorname{Tr} H_1 \operatorname{Tr} H_1^3 \\ & + 16 \operatorname{Tr} H_1 \operatorname{Tr} H_2^2 H_1 + 16 \operatorname{Tr} H_2 \operatorname{Tr} H_2^3 + 16 \operatorname{Tr} H_2 \operatorname{Tr} H_1^2 H_2 + 16(\operatorname{Tr} H_1 H_2)^2 \\ & + 12(\operatorname{Tr} H_1^2)^2 + 12(\operatorname{Tr} H_2^2)^2 + 8 \operatorname{Tr} H_1^2 \operatorname{Tr} H_2^2.\end{aligned}$$

Using the factorization theorem, the loop equations of this model for the words with length seven or lower are the following relations. Note that all the loop equations that come from the even degree words are trivial since the model is even and odd moments are zero.

$$A:\ 1 = 8g\, m_2 + 16m_4 - 16m_{1,1,1,1} + 16m_{2,2} + 16m_{2,2} + 64m_2\, m_2$$

$$\begin{aligned}AAA &:\ 2m_2 = 8g\, m_4 + 16m_6 - 16m_{3,1,1,1} + 16m_{4,2} + 16m_{4,2} + 64m_2\, m_4 \\ ABB &:\ m_2 = 8g\, m_{2,2} + 16m_{4,2} - 16m_{3,1,1,1} + 16m_{2,1,2,1} + 16m_{4,2} + 64m_2\, m_{2,2} \\ BAB &:\ 0 = 8g\, m_{1,1,1,1} + 16m_{3,1,1,1} - 16m_{2,1,2,1} + 16m_{3,1,1,1} + 16m_{3,1,1,1} + 64m_2\, m_{1,1,1,1} \\ BBA &:\ m_2 = 8g\, m_{2,2} + 16m_{4,2} - 16m_{3,1,1,1} + 16m_{4,2} + 16m_{2,1,2,1} + 64m_2\, m_{2,2}\end{aligned}$$

$$\begin{aligned}AAAAA &:\ m_2^2 + 2m_4 = 8g\, m_6 + 16m_8 - 16m_{5,1,1,1} + 16m_{6,2} + 16m_{6,2} + 64m_2\, m_6 \\ AAABB &:\ m_{2,2} + m_2^2 = 8g\, m_{4,2} + 16m_{6,2} - 16m_{3,3,1,1} + 16m_{3,2,1,2} + 16m_{4,4} + 64m_2\, m_{4,2} \\ ABBBB &:\ m_4 = 8g\, m_{4,2} + 16m_{4,4} - 16m_{5,1,1,1} + 16m_{4,1,2,1} + 16m_{6,2} + 64m_2\, m_{4,2} \\ BAAAB &:\ 0 = 8g\, m_{3,1,1,1} + 16m_{3,1,3,1} - 16m_{3,2,1,2} + 16m_{3,3,1,1} + 16m_{3,3,1,1} + 64m_2\, m_{3,1,1,1} \\ BABBB &:\ 0 = 8g\, m_{3,1,1,1} + 16m_{3,3,1,1} - 16m_{4,1,2,1} + 16m_{3,1,3,1} + 16m_{5,1,1,1} + 64m_2\, m_{3,1,1,1} \\ BBABB &:\ m_2^2 = 8g\, m_{2,1,2,1} + 16m_{3,2,1,2} - 16m_{3,1,3,1} + 16m_{4,1,2,1} + 16m_{4,1,2,1} + 64m_2\, m_{2,1,2,1} \\ BBBAB &:\ 0 = 8g\, m_{3,1,1,1} + 16m_{3,1,1,3} - 16m_{4,1,2,1} + 16m_{5,1,1,1} + 16m_{3,1,3,1} + 64m_2\, m_{3,1,1,1} \\ BBBBA &:\ m_4 = 8g\, m_{4,2} + 16m_{4,4} - 16m_{5,1,1,1} + 16m_{6,2} + 16m_{4,1,2,1} + 64m_2\, m_{4,2}\end{aligned}$$

$AAAAAAA: \; 2m_2\,m_4 + 2m_6 = 8g\,m_8 + 16m_{10} - 16m_{7,1,1,1} + 16m_{8,2} + 16m_{8,2} + 64m_2\,m_8$

$AAAAABB: \; m_{4,2} + m_4\,m_2 + m_2\,m_{2,2} = 8g\,m_{6,2} + 16m_{8,2} - 16m_{5,3,1,1} + 16m_{5,2,1,2} + 16m_{6,4} + 64m_2\,m_{6,2}$

$AAABBBB: \; m_2\,m_4 + m_{4,2} = 8g\,m_{4,4} + 16m_{6,4} - 16m_{5,1,1,3} + 16m_{4,1,2,3} + 16m_{6,4} + 64m_2\,m_{4,4}$

$ABBBBBB: \; m_6 = 8g\,m_{6,2} + 16m_{6,4} - 16m_{7,1,1,1} + 16m_{6,1,2,1} + 16m_{8,2} + 64m_2\,m_{6,2}$

$BAAAAAB: \; 0 = 8g\,m_{5,1,1,1} + 16m_{5,1,3,1} - 16m_{5,2,1,2} + 16m_{5,1,1,3} + 16m_{5,3,1,1} + 64m_2\,m_{5,1,1,1}$

$BAAABBB: \; 0 = 8g\,m_{3,3,1,1} + 16m_{3,3,3,1} - 16m_{4,1,2,3} + 16m_{3,3,3,1} + 16m_{5,1,1,3} + 64m_2\,m_{3,3,1,1}$

$BABBBBB: \; 0 = 8g\,m_{5,1,1,1} + 16m_{5,3,1,1} - 16m_{6,1,2,1} + 16m_{5,1,3,1} + 16m_{7,1,1,1} + 64m_2\,m_{5,1,1,1}$

$BBAAAAA: \; m_{2,2}\,m_2 + m_{4,2} + m_2\,m_4 = 8g\,m_{6,2} + 16m_{8,2} - 16m_{5,1,1,3} + 16m_{6,4} + 16m_{5,2,1,2} + 64m_2\,m_{6,2}$

$BBAAABB: \; 2m_2\,m_{2,2} = 8g\,m_{3,2,1,2} + 16m_{3,2,3,2} - 16m_{3,3,3,1} + 16m_{4,3,2,1} + 16m_{4,1,2,3} + 64m_2\,m_{3,2,1,2}$

$BBABBBB: \; m_2\,m_4 = 8g\,m_{4,1,2,1} + 16m_{4,3,2,1} - 16m_{5,1,3,1} + 16m_{4,1,4,1} + 16m_{6,1,2,1} + 64m_2\,m_{4,1,2,1}$

$BBBAAAB: \; 0 = 8g\,m_{3,3,1,1} + 16m_{3,3,1,3} - 16m_{4,3,2,1} + 16m_{5,3,1,1} + 16m_{3,3,3,1} + 64m_2\,m_{3,3,1,1}$

$BBBABBB: \; 0 = 8g\,m_{3,1,3,1} + 16m_{3,1,3,3} - 16m_{4,1,4,1} + 16m_{5,1,3,1} + 16m_{5,1,3,1} + 64m_2\,m_{3,1,3,1}$

$BBBBAAA: \; m_{4,2} + m_4\,m_2 = 8g\,m_{4,4} + 16m_{6,4} - 16m_{5,3,1,1} + 16m_{6,4} + 16m_{4,3,2,1} + 64m_2\,m_{4,4}$

$BBBBABB: \; m_4\,m_2 = 8g\,m_{4,1,2,1} + 16m_{4,1,2,3} - 16m_{5,1,3,1} + 16m_{6,1,2,1} + 16m_{4,1,4,1} + 64m_2\,m_{4,1,2,1}$

$BBBBBAB: \; 0 = 8g\,m_{5,1,1,1} + 16m_{5,1,1,3} - 16m_{6,1,2,1} + 16m_{7,1,1,1} + 16m_{5,1,3,1} + 64m_2\,m_{5,1,1,1}$

$BBBBBBA: \; m_6 = 8g\,m_{6,2} + 16m_{6,4} - 16m_{7,1,1,1} + 16m_{8,2} + 16m_{6,1,2,1} + 64m_2\,m_{6,2}$

Curriculum Vitae

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Education:

• Ph.D., Mathematics, University of Western Ontario, Canada. (2019-2022)
• M.Sc., Pure Mathematics, Sharif University of Technology, Iran. (2011-2013)
• B.Sc., Pure Mathematics, Amirkabir University of Technology, Iran. (2006-2011)

Publications:

• **Hamed Hessam**, Masoud Khalkhali, Nathan Pagliaroli; *Double scaling limits of Dirac ensembles and Liouville quantum gravity*. arXiv: 2204.14206 (2022).

• **Hamed Hessam**, Masoud Khalkhali, Nathan Pagliaroli; *From Noncommutative Geometry to Random Matrix Theory*. arXiv:2204.14216 (2022).

• **Hamed Hessam**, Masoud Khalkhali, Nathan Pagliaroli; *Bootstrapping Dirac Ensembles*. Phys. A: Math. Theor. 55 015203 2022.

• Hamid-Reza Fanai, **Hamed Hessam**; *On 2-Killing and conformal vector fields on Riemannian manifolds*. International Mathematical Forum, Vol. 12, 2017, no. 18, 869-878 (2017).

Honours and Awards:

• The 1st Rank in national university entrance exam for M.Sc. degree among more than 12000 participants in 2011.
• The 2nd Prize in the 16th International Mathematics Competition (IMC 2009).
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Teaching Experience:

- Jan 2019 - Dec 2022, Teaching Assistant, Western University,Canada.
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