Portfolio Optimization Analysis in the Family of 4/2 Stochastic Volatility Models

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A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Statistics and Actuarial Sciences
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Abstract

Over the last two decades, trading of financial derivatives has increased significantly along with richer and more complex behaviour/trait in the underlying assets. The need for more advanced models to capture traits and behaviour of risky assets is crucial. In this spirit, the state-of-the-art 4/2 stochastic volatility model was recently proposed by Grasselli in 2017 and has gained great attention ever since. The 4/2 model is a superposition of a Heston (1/2) component and a 3/2 component, which is shown to be able to eliminate the limitations of these two individual models, bringing the best out of each other. Based on its success in describing stock dynamics and pricing options, the 4/2 stochastic volatility model is an ideal candidate for portfolio optimization. Hence, in this thesis, we focus on portfolio optimization problems under the 4/2 stochastic volatility class of models.

To highlight the 4/2 stochastic volatility model in portfolio optimization problems, five related and self-contained projects are conducted. We firstly investigate, in Chapter 2, portfolio optimization problems under the 4/2 stochastic volatility model within the framework of expected utility theory for a constant-relative-risk-averse (CRRA) investor in incomplete and complete markets. We postulate the market prices of risk are proportional to variance’s driver. By employing a dynamic programming approach, we formulate the corresponding Hamilton-Jacobi-Bellman (HJB) equations and solve them via an exponential-affine ansatz. Verification theorems are provided to ensure optimality. We find that the optimal strategy recommended by the 4/2 model depends on the levels of current volatility, a reasonable feature not reported in the existing literature. To present a meaningful empirical study, a full estimation is performed for the 4/2 model along with its embedded popular models (i.e. the 3/2 and 1/2). We compare the optimal recommendations from various models and illustrate the wealth-equivalent losses from classical sub-optimal strategies (i.e. as those produced by 1/2, 3/2 and Geometric Brownian motion). Given the fact that investors are not only risk-averse but also ambiguity-averse, we further take ambiguity-aversion into account and examine, in Chapter 3, a robust portfolio optimization problem under the setting described before. We determine the robust optimal strategy and the worst-case measure by allowing separate levels of uncertainty for variance and stock drivers. The impact of ambiguity aversion on the optimal strategy is studied under a realistic parametric set and viable ambiguity-aversion levels following a detection error analysis. The theoretical and numerical analyses confirm an inverse relation between absolute risky exposure and the level of ambiguity aversion. In particular, exposures to assets could decrease by 50% for reasonable ambiguity-aversion values. In Chapter 4, we incorporate a consumption decision into the most complete portfolio optimization problem described before, and employ the preferable proportion-to-volatility market price of risk in a new analysis of the 4/2 model. Due to the non-affine nature, the solution for the value function involves confluent hypergeometric functions similar to those needed for option pricing within the 4/2 model. The chapter explores all cases where closed-form solutions are available, in all combinations of settings: complete-incomplete, consumption or its absence, ambiguity-averse or its absence, leverage or its absence. In Chapter 5, we propose a multivariate 4/2 stochastic volatility model to capture advanced stylized facts in the behaviour of multiple assets, such as co-volatility movements and stochastic correlations among assets. The model is built as a linear combination of independent one-dimensional 4/2 processes, which keeps the number of parameters parsimonious.
In this new model, the conditional characteristic functions (c.f.) under historical and risk-neutral measures are derived in closed-form, which would allow for derivative pricing and risk management analysis (not conducted here). A rich portfolio optimization problem in a multivariate setting is presented which included risk-aversion and incomplete markets, conditions for verification and proper solutions are provided; we also study the co-volatility movements and highlight the importance of the $4/2$ underlying structure as part of a numerical analysis. In Chapter 6, we combine all we have learned about stochastic volatility and expected utility to reveal the largest class of stochastic volatility processes solvable in closed form within a larger family of utility functions, the HARA (hyperbolic absolute risk aversion). This chapter lists all solvable cases reported in the literature, adding many others thanks to a change of control technique. The extension to ambiguity-aversion analyses is also considered.

**Keywords:** 4/2 stochastic volatility model, CRRA utility, optimal portfolio choice, Heston’s model, 3/2 model, wealth equivalent losses, robust portfolio choice, ambiguity aversion, consumption, multivariate, stochastic correlation, co-volatility movements, HARA utility.
Summary for Lay Audience

Over the last two decades, the trading of financial products has increased significantly along with evidence of more complex behaviour in assets. The need for more advanced models to capture this complexity is crucial. In this spirit, the state-of-the-art 4/2 stochastic volatility model was recently proposed by Grasselli in 2017 and has gained great success in capturing various stylized facts of stock prices. Hence, in this thesis, we apply the 4/2 stochastic volatility class of models to the problem of finding the best allocation of wealth between a risky asset and a cash account. We find that the optimal strategy recommended by the 4/2 model depends on the levels of current volatility, an intuitive feature that investors may expect but not reported in the existing literature. Investors are uncertain about the probability distribution of the stock, so they would consider a set of alternative models when making investment decisions, this is called a robust portfolio optimization. Based on the optimal portfolio problems above, we extend the literature in two directions. On the one hand, the optimal allocations hinge upon the choices of market price of risk, i.e., the return in excess of the risk-free rate that investors demand as a compensation for taking/bearing risk, but there is no consensus in the existing literature. Hence, we consider a conventional choice of market price of risk, namely, proportional to volatility, solving the corresponding (robust) portfolio optimization problem. On the other hand, due to the importance of modelling the stochastic correlation implied by multivariate volatility, instead of constant correlation or single-factor stochastic volatility, we further consider a multivariate model with 4/2 structured counterparts in an optimal portfolio choice problem. Lastly, we combine all we have learned about stochastic volatility and expected utility to reveal the largest class of stochastic volatility processes solvable in closed form within a larger family of utility functions, the hyperbolic absolute risk aversion, which is more flexible in assessing risk aversion of investors and can be connected with yet another area of portfolio problems known as mean variance theory.
Co-Authorship Statement

Chapter 2 and chapter 3 of this thesis have already been published. The remaining chapters are now in manuscript form and submitted to journals. I hereby declare that the research results in this thesis are direct results of my efforts.


- The content of 4 is based on a paper, co-authored with my supervisor, Marcos Escobar-Anel. “Optimal consumption and robust portfolio choice for the 3/2 and 4/2 stochastic volatility models.”, submitted to Applied Mathematical Finance.


An integrated-article format is employed in line with Western’s thesis guidelines. Each chapter in this thesis stands on its own and can be read independently in a self-contained manner.

This research was conducted from 2020 to present under the supervision of Dr. Marcos Escobar-Anel at the University of Western Ontario.
Acknowledgements

First and foremost, I would like to express my sincere and deepest gratitude to my supervisor, Dr. Marcos Escobar-Anel, for his dedicated guidance, invaluable advice, constant encouragement, and unlimited support. I want to thank Dr. Escobar for the opportunity to undertake a summer project, for having me as a Master student, and for guiding me to this Ph.D. career. Dr. Escobar is exceptionally knowledgeable and very humble, who always appreciates my independent thinking and provides constructive feedback. He is the most fearless explorer in research and the most positive person in life, making him a great supervisor and a life mentor for me. I am truly grateful to Dr. Escobar for making my time at Western University much more enjoyable and meaningful, and keeping me motivated throughout the whole process.

I also would like to thank Dr. David Saunders from the University of Waterloo, Dr. Rogemar Mamon, Dr. Lars Stentoft, and Dr. Hao Yu, who served on my Ph.D. thesis committee, devoted their precious time, shared their expertise, and provided their insightful comments in the assessment of this thesis. It is my honor to have these knowledgeable professors witness this incredible journey.

In addition, I would like to thank all my colleagues and friends at Western University. Last but not least, I must thank my family for their love and support all the way along. Without them, this day would not be possible.
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Chapter 1

Introduction

1.1 Literature overview and research motivation

The development of the type of portfolio optimization problem treated in this thesis can be traced back to Merton (1969, 1971, 1975), who conceived the dynamic control formulation and solved it for the Black-Scholes, continuous-time, model under expected utility theory. Specifically, the investor aims at finding an optimal strategy that maximizes an investor’s expected utility, where the portfolio consists of risky assets and a risk free asset. Thanks to the simplicity of the Black-Scholes model, which assumes constant expected returns and volatility for the process of a stock’s price, a closed-form solution to the optimal investment allocation can be obtained. However, Merton’s setting fails in capturing many stylized facts reported in the last 50 years from empirical observations, such as volatility clustering, asymmetry of log return distribution, leverage effects, and stochastic expected returns, to mention a few. One of the best known facts is that constant volatility implies a horizontal implied volatility surface, while real data confirms that implied volatility surfaces display a “smile” or “skew” with respect to strike prices. Many scholars have developed richer models leading to challenges on solving the corresponding portfolio optimization problems.

One natural extension addressing non-constant implied volatilities appears with the introduction of stochastic volatility in modelling prices of risky assets. Heston (1993) proposed a stochastic volatility framework, known as the Heston model or the 1/2 model, by assuming the variance of the underlying asset follows a Cox-Ingersoll-Ross (CIR) process. The CIR process is a mean-reverting process, introduced by Cox et al. (1985). In particular, the Brownian motion of the underlying asset can be correlated to the randomness of the variance process, which accounts for the asymmetric distribution of log returns and the leverage effect. As expected, the Heston model is capable of reproducing the volatility “smile” with good accuracy. On the other hand, although the complexity of the portfolio optimization problem under the Heston model increases, the problem is still solvable, see Kraft (2005) and Zeng and Taksar (2013). In an important extension, Liu and Pan (2003) modeled the stock prices as jump-diffusion processes with Heston stochastic volatility, where investors can also invest in derivatives and thus consider the portfolio optimization problem in a so called complete market, this means that all sources of risk are hedged and taken advantage of in constructing the optimal portfolio. Nev-
Nevertheless, the Heston model would predict a flattened implied volatility skew when volatility is high due to a market crisis, and it requires a high volatility of volatility parameter to capture the “spikes” in the stock prices.

Heston (1997) and Platen (1997) independently proposed the \( \frac{3}{2} \) stochastic volatility model several years later to overcome some of the limitations described above. Instead of considering a CIR process, the \( \frac{3}{2} \) model takes an inverse CIR process such that the power of the diffusion part of the variance process is \( \frac{3}{2} \), which explains the name of the model. Similar to the Heston model, the mean-reverting speed parameter can reflect volatility clustering, while the correlation coefficient accounts for leverage and asymmetry in stock returns. On the other hand, one of the improvements of the \( \frac{3}{2} \) model is its ability in capturing extreme paths with “spikes” in instantaneous volatility, allowing stronger deviations for the variance, and predicting a steeper skew. With this being said, there also exist some disadvantages in the \( \frac{3}{2} \) model. For instance, the \( \frac{3}{2} \) model shows less flexibility in modelling small leverage effects than the Heston model. More details and comparisons between the Heston model and the \( \frac{3}{2} \) model can be found in Drimus (2012), which performed a calibration on the same data, i.e., S&P 500 options, for both models and provided some interesting insight. Given the enhancement achieved by the \( \frac{3}{2} \) model, Chacko and Viceira (2005) considered the optimal investment and consumption problem under the \( \frac{3}{2} \) model with an infinite horizon and an incomplete market, while Zeng and Taksar (2013) included the \( \frac{3}{2} \) model as one of their examples under expected utility theory and an incomplete market.

From the previous review on the Heston model and the \( \frac{3}{2} \) model, it is obvious both models have their benefits and also admit shortcomings. Hence, ideally we need them both to better explain stock behaviour. With this in mind, Grasselli (2017) proposed the so-called \( \frac{4}{2} \) stochastic volatility model in the context of option pricing. By combining the Heston component and the \( \frac{3}{2} \) component, the \( \frac{4}{2} \) model is designed to eliminate their limitations, bringing out the best of both components. As claimed in Grasselli (2017), these two components complement each other in the form of \( \frac{4}{2} \) model, and the flexibility of the \( \frac{4}{2} \) model can be seen in its ability of reproducing various shapes of implied volatility surfaces. Ever since Grasselli (2017), the \( \frac{4}{2} \) model has drawn great attention and has been widely applied in derivative pricing and calibration, i.e., Cui et al. (2017a); Zhu and Wang (2019); Lin et al. (2017); Cui et al. (2018); Escobar-Anel and Gong (2020) among others.

With the richness/flexibility of the \( \frac{4}{2} \) model in recovering both volatility “smile” and “skew”, and its success in pricing options, the superiority of the \( \frac{4}{2} \) model makes it a better candidate compared to the Heston or the \( \frac{3}{2} \) model, to model the underlying asset and derivatives. Undoubtedly, the solution to the portfolio optimization problem would be beneficial to investors. This motivates us to investigate portfolio optimization problems for the \( \frac{4}{2} \) stochastic model along with the embedded \( \frac{3}{2} \) model (the \( \frac{3}{2} \) model has been rarely explored) in the framework of expected utility theory (EUT) with finite horizon in both incomplete and complete markets. Specifically, we consider a constant relative risk averse (CRRA) investor, who aims at maximizing his/her terminal wealth in Chapter 2 of the thesis. We were first conducting a portfolio optimization problem of this kind in the context of the advanced \( \frac{4}{2} \) model. This problem is formulated and solved in Chapter 2.
Richer stochastic models are sought to capture more stylized facts or traits observed from time series of various assets. The concern/uncertainty about model mis-specification arises and leads to the second direction of extending the seminal work of Merton’s portfolio optimization formulation. This is known as ambiguity-averse analysis, and the associated portfolio optimization problem is called robust analysis. According to Ellsberg (1961) and Bossaerts et al. (2010), investors are not only risk-averse, but also ambiguity-averse. That is, investors are skeptical about the actual distribution of the underlying asset prices, and they may consider a set of alternative/plausible models to determine an investment strategy. This strategy is called a robust strategy, which would perform reasonably well even in a worst-case scenario. Maenhout (2004) obtained closed-form solutions by considering ambiguity-aversion in the diffusion of a GBM risky stock under expected utility theory for the first time. The analytical tractability and wealth independence are preserved by the methodology proposed in Maenhout (2004), which has been widely applied in robustness analysis under many different types of state variables. Ambiguity aversion on jumps and/or risky assets can be found in Branger and Larsen (2013); Liu et al. (2005); ambiguity aversion on interest rates and/or risky asset are studied in Flor and Larsen (2014); Chen et al. (2021). In particular, Escobar et al. (2015) related ambiguity to a univariate Heston stochastic volatility model with jumps, which was further extended by Bergen et al. (2018); Yang et al. (2020); Han and Wong (2020) to a multivariate model with ambiguity-aversion in correlation. As for 3/2 model, the work of Chacko and Viceira (2005) was extended by Faria and Correia-da Silva (2016) to the context of robustness. Inspired by the review above, we further consider a robust portfolio optimization problem for the 4/2 model in Chapter 3.

There is no doubt that investor consumption is a very important decision/control variable in portfolio optimization, it makes the problem more realistic and it could shed more light on the impact of stylized facts on portfolio problems. In an incomplete market, Chacko and Viceira (2005) considered the optimal investment-consumption problem under the 3/2 model in an infinite horizon with excess returns that are either constant or linear in variance. Kraft et al. (2013) studied the problem in a finite horizon under a general setting of consumption (known as Epstein-Zin preferences) with applications to the Heston and 3/2 models, where the market price of risk (MPR) was assumed to be proportional to the variance driver for both models. Indeed, in the absence of consumption, Kraft (2005) considered a portfolio optimization problem in two forms of market price of risk for the Heston model, proportional to volatility and constant, which led to excess returns that are proportional to variance and volatility respectively. In Chapters 2 and 3, we employed the form of market price of risk that is proportional to the drivers of the variance of assets’ return to reach the structure of Kraft (2005) and Chacko and Viceira (2005) but with applications to the 4/2 model. Actually, these specification of the form of excess return (or MPR) follow along the three types brought up in the work of Merton (1980). It seems that there is no consensus in the existing literature on the form of the excess return (or MPR). Motivated by this stream of the literature, we investigate the solvability of a portfolio optimization problem with excess return proportional to the 4/2 structured variance in Chapter 4, where we incorporate several ingredients of interest, such as market completeness, terminal wealth with/without consumption, and ambiguity aversion.
A further extension to the works above is to entertain a multi-dimensional model in the portfolio optimization problem. As reported by Christoffersen et al. (2009), multiple stochastic drivers are needed to explain asset behaviour, in addition to stochastic correlation, and stochastic joint movements of the volatility of multiple assets. Hence, we focus on a two-dimension and two-component model (extendable to any dimension and number of components), which relies on the structure of a linear combination of independent one-dimensional $4/2$ processes. In particular, the model fits right into the PCA-concept model studied in Escobar et al. (2017) with volatility components in a $4/2$ structure under a special parametrization. Furthermore, this model is able to create more flexibility for correlation between assets and correlation among variances (known as co-volatility movements) while maintaining the independent structure of the linear combination in the assets’ returns dynamics. The corresponding portfolio optimization problem is analyzed in Chapter 5.

In the setting of continuous-time models, dynamic programming is one of the main approaches for tackling the implied optimal control problem associated with the EUT setting. This is the approach that we have been using in this thesis. Finding the optimal control (e.g., the optimal proportion allocated to risky assets) involves solving a partial differential equation (PDE) that may not be tractable. Although numerical methods have advanced significantly in the past two decades, closed-form solutions are still desirable and convenient to gain a better understanding and interpretation of solutions. Liu (2007) made the most celebrated attempt at revealing a large family of solvable models in EUT for CRRA investors, presenting the class of exponential-quadratic value functions and its originating multivariate models. The author detailed the functional forms of the drifts and diffusion terms of asset prices, state variables (stochastic volatility, or stochastic short rate, or predictors of stock returns), and the correlation between asset prices and state variables that allows for such exponential-quadratic form of the value function. Importantly, the paper was not concerned with subclasses or particular cases, leaving the door open to exploring members of such large family in terms of, for example, the verification theorem, conditions for well-defined solutions, or further extensions in lower dimensions. In Chapter 6, we take advantage of a one-dimensional framework to list all solvable cases, extending the process in the work of Liu (2007) to richer families of models within a larger family of utilities (HARA). Moreover, we highlight the simplicity in finding closed-form solutions for these models in the presence of ambiguity-aversion.

1.2 Research objectives

The objectives of this thesis are comprised of the following points:

- Obtain the optimal allocation and value function for an investor who maximizes expected utility of terminal wealth under the $4/2$ stochastic volatility model and compare the investment recommendations with other embedded popular models.

- Conduct an estimation of the $4/2$ model from real data and perform a meaningful sensitivity analysis of optimal portfolio solutions on important parameters.

- Quantify wealth-equivalent loss to gain a deeper insight into the $4/2$ model and its suggestion on portfolio optimization via numerical analyses.
1.3 Structure of the thesis

This thesis consists of seven chapters. The current chapter (Chapter 1) gives a literature overview of the portfolio optimization problem and provides motivation for the various objectives of the thesis. The rest of the chapter briefly reviews five related projects (Chapters 2-6). Finally, Chapter 7 concludes this thesis.

In Chapter 2, we firstly introduce the 4/2 stochastic volatility model proposed in Grasselli (2017) and motivate our choice of market price of risk, i.e., proportional to the variance driver, in Section 2.2. Then the portfolio optimization problems are formulated and solved for the 4/2 model and the embedded 3/2 model (Heston, 1997) by maximizing the expected utility of terminal wealth for a CRRA investor and solving the corresponding Hamilton–Jacobi–Bellman (HJB) equations for complete and incomplete markets in Sections 2.3.1 and 2.3.2 respectively. Conditions for the verification theorems are provided in Section 2.4. Further, suboptimal strategies are considered and derived in Section 2.5. Procedures of estimation from real data and a meaningful empirical analysis are performed in Section 2.6.

In Chapter 3, we take ambiguity-aversion into account and consider the first optimal portfolio analysis under the 4/2 stochastic volatility model in a complete market setting. We determine the robust optimal strategy and the worst case measure by allowing separate levels of ambiguity aversion.

In Chapter 4, we propose a multivariate 4/2 stochastic model to capture the behaviour of multiple risky assets and their stochastic correlations. Find the characteristic function of the multivariate asset for derivative pricing. Study a portfolio optimization problem for a risk-averse investor in an incomplete market.

In Chapter 5, we provide the most up-to-date review of solvable models within EUT for a HARA investor, which also connects to pre-commitment solutions of Mean-Variance theory.

In Chapter 6, we use a simple change-of-control method to reveal a large family of fully solvable models, opening the door to far more complex and realistic configurations of diffusion and drift terms.

In Chapter 7, we conclude this thesis.
uncertainty for variance and stock drivers in Section 3.3. Also, technical conditions for well-defined solutions are detailed together with a Verification result in the same section. Some common suboptimal strategies are derived in Section 3.4 and a managerial numerical analysis is conducted in Section 3.5.

In Chapter 4, a new market price of risk, as a difference to previous chapters, is entertained in a (robust) optimal consumption and investment problem under the 4/2 stochastic volatility class of models, for both incomplete and complete markets. In particular, Section 4.3 investigates portfolio optimization under EUT, while Section 4.4 considers robust consumption portfolio optimization under EUT. Numerical studies are followed in Section 4.5.

In Chapter 5, we introduce a multivariate 4/2 stochastic covariance process generalizing the one-dimensional 4/2 counterparts presented in Grasselli (2017). The motivation of considering a multivariate model is explained in Section 5.1. Then we introduce the model and explore its properties in Sections 5.2 and 5.3. By employing similar MPR to Chapters 2 and 3, conditions for proper changes of measure and closed-form characteristic functions under risk-neutral and historical measures are provided, allowing for applications of the model to risk management and derivative pricing. Further, we apply the model to a portfolio optimization problem in incomplete markets in Section 5.4. Our analysis leads to closed-form solutions for the optimal allocations and value function. Conditions are provided for well-defined solutions together with a verification theorem. Numerical analysis is given in Section 5.5.

In Chapter 6, as a result of all the learning about stochastic volatility and expected utility in previous chapters, we reveal the largest class of stochastic volatility processes solvable in closed form for a HARA investor. Section 6.3 lists all solvable cases reported in the literature, adding many others thanks to a change of control technique.

Chapter 7 concludes this study and gives an outlook for possible future research.

1.4 Contributions of the thesis

For clarity, we list the main contributions of this thesis below:

- (Chapter 2) Analytical representations of the optimal investment strategy, optimal wealth and value functions are obtained. The interesting structure (i.e., variance-dependent) of the optimal strategy in the 4/2 and 3/2 models is highlighted. We are not aware of such an analysis on the 4/2 model in the existing literature.

- (Chapter 2) Conditions are provided for the existence of a risk-neutral measure, as are the sufficient conditions (i.e., verification theorem) for the fidelity and well-definedness of the value functions.

- (Chapter 2) A full estimation of four models — the GBM (Merton’s model), 1/2 (Heston’s model), 3/2 and 4/2 models — is conducted on S&P500 and VIX data.

- (Chapter 2) The empirical analysis demonstrates that the 1/2 component carries the larger weight in the 4/2 model which explains why the 4/2 behaviour leads to similar investment strategies as that of 1/2 model. Investments from the 3/2 model are the most conservative in high-variance settings (20% of Merton’s solution) and more aggressive in low-variance ones (e.g., at 85% the level of Merton’s solution).
1.4. Contributions of the Thesis

- (Chapter 2) Wealth-equivalent losses due to deviations from the 4/2 model reveal the largest losses for the 1/2 and GBM models (40% over 10 years). On the other hand, losses due to market incompleteness are harshest for the 1/2 model (60%), compared to 30% for the 3/2 model and 40% for the 4/2 model.

- (Chapter 3) We are the first to find analytical representations of the optimal investment strategy, worst case measure, optimal wealth, and value functions for the 4/2 stochastic volatility model. Conditions ensuring well-defined solutions are presented, and a verification result for complete markets is included. Moreover, the value functions of two popular embedded suboptimal strategies, i.e. the Heston 1/2 model and Merton GBM model, are produced facilitating a comparison to the optimal 4/2 setting. These value functions are of a non-affine nature.

- (Chapter 3) The impact of ambiguity aversion on the optimal trading strategy is studied under a realistic parametric set and viable ambiguity-aversion levels following a detection error analysis. The theoretical and numerical analyses confirm an inverse relation between absolute risky exposure and the level of ambiguity aversion. In particular, exposures could decrease by 50% for reasonable ambiguity-aversion values.

- (Chapter 3) Important suboptimal strategies are assessed in terms of wealth-equivalent loss (WEL) in our managerial analysis. The study demonstrates WELs of up to 20% when ignoring ambiguity, and 15% due to market incompleteness. As important, popular strategies such as those by Heston (1/2 model) and Merton (GBM model) could lead to WELs of up to 24% and 51%, respectively. Lastly, the single most important parameters regarding the impact of WELs are the market prices of risk (MPRs), i.e., with up to 70% WEL for stock MPR and 60% WEL for volatility MPR.

- (Chapter 4) We conduct the first risk-averse, expected utility analysis in the presence of consumption for the non-affine class of SV models known as 4/2, under the preferable setting of MPR proportional to variance (type I). Our closed-form solutions, see Propositions 4.3.1 and 4.3.2, are of a non-affine nature, requiring confluent hypergeometric functions. As a byproduct, we produce the very first closed-form portfolio analysis for the 3/2 model for finite horizons.

- (Chapter 4) We extend the solutions described above to an ambiguity-averse investor, leading to the very first related analyses for the 4/2 and 3/2 models, see Propositions 4.4.1 and 4.4.3. In all cases, we consider complete and incomplete markets, providing conditions for well-defined solutions under the assumption of existence.

- (Chapter 4) For a risk-averse investor, in a complete market, we illustrate the differences between the 4/2 model and the popular embedded cases of the 1/2 (Heston) and 3/2 models. On the one hand, the 4/2 and 1/2 models recommend similar levels of consumption and exposure. On the other hand, the 3/2 leads to 20% or higher levels of consumption and absolute exposures (see Figures 4.1 to 4.6). The difference in terms of exposures is exacerbated when considering an ambiguity-averse investor in a complete market. In such case, the 3/2 model performance could double absolute exposures compared to the 1/2 and 4/2 models (see Figure 4.12).
• (Chapter 4) In the case of incomplete markets for only risk-averse investors, there is more substantial difference among the models. In particular, \( \frac{3}{2} \) proposes roughly 20\% less consumption than the \( \frac{1}{2} \) and \( \frac{4}{2} \) models, while allocations for \( \frac{3}{2} \) are the most aggressive —approximately 35\% and 67\% higher than \( \frac{4}{2} \) and \( \frac{1}{2} \) models, respectively (see Figures 4.16 to 4.19).

• (Chapter 5) We introduce a stochastic covariance model driven by the newly developed \( \frac{4}{2} \) stochastic processes. The model boasts useful and important properties. A rich stochastic correlation dynamic that avoids extreme values for long periods, and stochastic co-volatility movements. We obtain conditions for valid changes of measure, the characteristic function (c.f.) and moment-generating function (m.g.f.) under risk-neutral and historical measures of joint and marginal asset prices are presented in closed form; this alongside the m.g.f. of prices conditional on terminal variances for simulation purposes.

• (Chapter 5) The multivariate process is applied to an EUT problem with a CRRA investor in an incomplete market. Closed-form solutions are obtained for the optimal allocation and value function. Conditions for well-defined solutions and a verification theorem under mild assumptions are presented.

• (Chapter 5) The numerical section illustrates the impact of risk-aversion levels, average stochastic correlation, average volatility of idiosyncratic component, average stochastic correlation of variances, and the new key parameters on portfolio allocations within the new model. The study shows that the stochastic correlation of variances has little impact on optimal allocations, far lesser than stochastic correlation among stocks or stochastic volatilities.

• (Chapter 6) We list all solvable models for a HARA expected utility problem for one asset, highlighting well-known stochastic volatility cases and models not yet implemented in the literature. These are called “base cases”.

• (Chapter 6) We demonstrate that exponential-polynomial expressions for the value function, with a polynomial of order 3 or above, do not lead to solvable models. Therefore, other solvable cases shall be explored using alternative families.

• (Chapter 6) Relying on the base cases, the technique of changing control is applied to reveal solvable models with more complex expressions for the drift and diffusion terms of the risky asset return.

• (Chapter 6) We demonstrate that if investors are also ambiguity averse and hence consider a robust portfolio choice, the closed-form solvability remains the same.
Chapter 2

Optimal investment strategy in the family of 4/2 stochastic volatility models.

2.1 Introduction

Building on the success of the 4/2 model in terms of pricing, we study the benefits of the 4/2 model from the perspective of dynamic portfolio optimization within expected utility theory. We consider a risk-averse investor as per a CRRA utility: one who allocates to a money market account (cash), a stock that follows a 4/2 model and possibly a financial derivative to complete the market. This work builds on extensive literature on portfolio optimization that traces back to the seminal work of Merton (1971). There have been numerous studies within EUT with quasi-analytical solutions; for instance, Kraft (2005) and Liu (2007) solved the problem for various cases where the volatility and expected returns of a stock depend on stochastic factors. Liu and Pan (2003) modeled stock prices as jump-diffusion processes with Heston-type stochastic volatility where the investor can also trade in derivatives, thereby completing the market, while Larsen and Munk (2012) developed a framework to compare optimal and suboptimal strategies in diverse contexts such as stochastic interest rates and variances. The work of Chacko and Viceira (2005) developed solutions for a consumption and investment problem with an infinite horizon in the context of the 3/2 model and incomplete markets. The authors considered two forms of excess return, constant and proportional to variance. Eeckhoff et al. (2010) studied the optimal investment decision from terminal wealth for a Heston model with stochastic mean reverting level (2 factors) in the context of complete markets using variance swaps. More recently, Zeng and Taksar (2013) presented a general class of stochastic volatility model with closed-form solutions for maximizing utility in an incomplete market. Their work included the Heston and the 3/2 models separately.

This chapter is organized as follows. Section 2.2 introduces the model and some properties under historical and risk-neutral measures. Section 2.3 derives the solutions based on the optimal strategy for complete and incomplete markets. The properties of these solutions, such as the well-definiteness of the value functions and the verification theorems, are presented in Section 2.4. Section 2.5 motivates several popular suboptimal investment strategies and how they perform in the 4/2 model; the expressions for wealth-equivalent losses are produced here. Section 2.6 starts with the estimation methodology utilized for all the underlying models, then
Chapter 2. Optimal investment strategy in the family of 4/2 stochastic volatility models.

a full analysis of the solutions for the incomplete and complete markets is performed. Section 2.7 presents the conclusion.

2.2 The model

In this section, we assume that the financial market consists of one risk-free and one risky asset (i.e., a stock), which can be traded continuously. Let all the stochastic processes introduced in this chapter be defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0,T]})\), where \(\{\mathcal{F}_t\}_{t \in [0,T]}\) is a right-continuous information filtration generated by standard Brownian motions (BM). The price process of the risk-free asset (money market) \(M_t\) evolves according to

\[
dM_t = M_r dt, \quad M_0 = 1,
\]

where the interest rate \(r\) is assumed to be constant.

The price process \(S_t\) of the risky asset follows the so-called 4/2 model (Grasselli, 2017):

\[
dS_t = S_t \left( \mu_t dt + \left( a \sqrt{v_t} + b \frac{\sqrt{v_t}}{v_t} \right) dW_t \right), \quad S(0) = S_0 > 0,
\]

\[
dv_t = \kappa (\theta - v_t) dt + \sigma \sqrt{v_t} dZ_{1t}, \quad v(0) = v_0 > 0,
\]

where \(v_t\) is the variance driver, which follows a Cox–Ingersoll–Ross (CIR) process with mean-reversion rate \(\kappa > 0\), long-run mean \(\theta > 0\) and volatility of volatility \(\sigma > 0\). The Feller condition, i.e., \(2\kappa \theta > \sigma^2\), is also imposed to keep the process \(v_t\) strictly positive. The two standard Brownian motions \(W_t\) and \(Z_{1t}\) are correlated with parameter \(\rho \in (-1, 1)\), hence for convenience we write \(dW_t = \rho dZ_{1t} + \sqrt{1 - \rho^2} dZ_{2t}\), where \(Z_{2t}\) is another standard BM, independent of \(Z_{1t}\).

The variance process implied by the model and the correlation between risky asset and stochastic volatility, i.e., "leverage effect", can be written as follows:

\[
z(v) = \left( a \sqrt{v_t} + b \frac{\sqrt{v_t}}{v_t} \right)^2 = a^2 v_t + b^2 + 2ab,
\]

\[
\text{Leverage} = \frac{\text{cov}(dS_t, d\sqrt{v_t})}{\sqrt{\text{var}(dS_t, d\sqrt{v_t})}} = \text{sign}(a - b) \rho.
\]

The parameters \(a\) and \(b\) are critical in the 4/2 model, and here we assume that they are both positive; i.e., \(a \geq 0\) and \(b \geq 0\), (for other specifications, see Cui et al. (2017b), Kirkby et al. (2017) and Lin et al. (2017)). It should be noted that the specification \(a = 1, b = 0\) corresponds to the celebrated Heston model (Heston, 1993), and the case \(a = 0, b = 1\) is known as the 3/2 model in the literature (Heston, 1997). A more convenient parametrization for \((a, b)\) that avoids the problem of parameter identification on the estimation is used in this chapter. Assuming that the initial variance of the stock is at its long-term value \(\theta\), (i.e., \(z_0 = \theta\)), we have

\[
E[z_\infty | z_0 = \theta] = a^2 \theta + 2ab + b^2 \left( \frac{2\kappa}{2\kappa \theta - \sigma^2} \right),
\]

where \(z_\infty\) is the long-term variance of the stock.
2.2. The model

which follows as expected for the CIR and inverse-CIR processes (Hurd and Yi, 2008). The
new parametrization would be
\[
\begin{aligned}
a &= cb \\
\mathbb{E}[Z_\omega | Z_0 = \theta] &= \theta \\
a &= cb
\end{aligned}
\]  
(2.7)

This not only allows \( c \) and \( \theta \) to control both \( a \) and \( b \), but also assigns meanings to \( \theta \) as the
long-term variance and \( c \) as the ratio of \( 1/2 \) to \( 3/2 \) models in the market.

To better understand the particular structure implied by the \( 4/2 \) model, let us derive the
first- and second-order derivatives of the variance process \( z(v) \) with respect to \( v \). We can see
that the variance reaches the minimum value \( 4ab \) at the point \( v' = \frac{b}{a} = \frac{1}{c} \). That is, the variance
is decreasing in \((0, \frac{1}{c}]\) and increasing in \((\frac{1}{c}, \infty)\). Similarly, the leverage has the same sign as \( \rho \) in the interval \((\frac{1}{c}, \infty)\), with a change in sign whenever the variance driver falls within the
interval \((0, \frac{1}{c}]\). The empirical section demonstrates that the interval \((0, \frac{1}{c}]\) is very unlikely to occur with the process \( v_t \).

With the aim of establishing historical and risk-neutral measures for our models, we subse-
duently define the idiosyncratic market prices of risk (i.e., the stock idiosyncratic risk and the
variance driver’s idiosyncratic risk), which are both proportional to \( \sqrt{v_t} \):
\[
\begin{aligned}
\lambda_1(v_t) &= \tilde{\lambda}_1 \sqrt{v_t} \\
\lambda_2(v_t) &= \tilde{\lambda}_2 \sqrt{v_t}
\end{aligned}
\]  
(2.8)

Here, \( \lambda_1(v_t) \) is the market price of the idiosyncratic variance driver risk (i.e., with respect to
\( Z_{1t} \)), and \( \tilde{\lambda}_1 \) is a constant; \( \lambda_2(v_t) \) is the explicit market price of the idiosyncratic stock risk (i.e.,
with respect to \( Z_{2t} \)), and \( \tilde{\lambda}_2 \) is a constant.

In this setting, the market price of the idiosyncratic variance driver risk is proportional to
its volatility. While the market price of the idiosyncratic stock risk follows a Heston-like and
a 3/2-like diffusion structures, i.e. \( \sqrt{v_t} \) as per Kraft (2005) and Chacko and Viceira (2005)
respectively. That is, both the Heston and the 3/2 component exhibit stochastic market price of
risk (MPR) given by the product of a constant and \( \sqrt{v} \). The rationale comes from the \( 4/2 \) model
as a superposition of these two types of diffusion terms (Grasselli, 2017). One should be aware
that the choice of MPR could jeopardize the existence of a risk-neutral measure. Therefore,
the drift process for the stock, \( \mu_t \), can be established as follows:
\[
\mu_t = r + \tilde{\lambda}_1 \rho (av_t + b) + \tilde{\lambda}_2 \sqrt{1 - \rho^2} (av_t + b) = r + \tilde{\lambda} (av_t + b).
\]  
(2.9)

Note that \( \tilde{\lambda} = \tilde{\lambda}_1 \rho + \tilde{\lambda}_2 \sqrt{1 - \rho^2} \) is constant and can be interpreted as a controller for the excess
return. An alternative interpretation of this change of measure for the stock price arises when
rewriting it in terms of the volatility of the stock: \( \tilde{\lambda} (av_t + b) = g(v_t) \left( a \sqrt{v_t} + \frac{b}{\sqrt{v}} \right) \). Here, the
factor \( g(v_t) = \tilde{\lambda} \sqrt{v_t} \) represents the stock’s market price of risk. Chernov and Ghysels (2000),
Bakshi and Kapadia (2003) and Escobar et al. (2015) found that variance risk is negatively
priced; i.e., \( \tilde{\lambda}_1 < 0 \). On the other hand, the stock excess return \( \tilde{\lambda} \) should be positive, as shown
by Ait-Sahalia and Kimmel (2007), and the leverage effect should be negative (\( \rho < 0 \)). This
means that \( \tilde{\lambda}_2 = \frac{\tilde{\lambda}_1 \rho}{\sqrt{1 - \rho^2}} \) can be either positive or negative.
The topic of viable changes of measure for the $4/2$ model is related to that of $3/2$ models. Some changes of measure and parametric setting can lead to local martingales and therefore a lack of risk-neutral measures (see Platen and Heath (2006), Baldeaux et al. (2015) and Grasselli (2017)). A very interesting discussion of possible market prices of risk in the $4/2$ model, given exchange rates of the Heston or $3/2$ type, can be found in Gnoatto et al. (2016). A general family of market prices of risk, as presented in Equation (2.10), and their viability for portfolio optimization is a rich topic that should be considered in future research linked to calibration exercises to facilitate interpretations. It is not difficult to find conditions on the parametric space to ensure the existence of a valid risk-neutral measure and quasi closed-form portfolio solutions. For simplicity, such general cases are left for future studies.

\[
\begin{align*}
\lambda_1(v_t) &= \bar{\lambda}_{11}a_v \sqrt{v_t} + \bar{\lambda}_{12} \frac{b_v}{\sqrt{v_t}}, \\
\lambda_2(v_t) &= \bar{\lambda}_{21}a_v \sqrt{v_t} + \bar{\lambda}_{22} \frac{b_v}{\sqrt{v_t}},
\end{align*}
\]

(2.10)

where $\bar{\lambda}_{11}$, $\bar{\lambda}_{12}$, $\bar{\lambda}_{21}$, and $\bar{\lambda}_{22}$ are constants that feature the market price of each risk factor or source.

Next, we explore the feasibility of changing measure under the market price of risk introduced in Equation (2.8). As noted by Grasselli (2017), a risk-neutral measure may not exist in the $4/2$ model, which is a feature inherited from having the $3/2$ model. This failure implies that the discounted asset price process may be a strict $Q$-local martingale, and not a true $Q$-martingale that is equivalent to the historical measure $P$.

In the next proposition we identify the parametric conditions needed for the existence of a valid risk-neutral measure $Q$.

**Proposition 2.2.1.** The change of measure is well defined for pricing purposes under the following conditions:

\[
\begin{align*}
\max \|\lambda_1\|, |\lambda_3| &< \frac{K}{\sigma}, \\
\sigma^2 &\leq 2\kappa\theta - 2|\sigma\rho b|, \\
\kappa + \sigma \rho a &> 0, \\
\kappa + \sigma \bar{\lambda}_1 &> 0.
\end{align*}
\]

(2.11)

(2.12)

(2.13)

(2.14)

See Appendix A.1 for the complete proof.

### 2.3 Problem formulation and solution

In this section we consider an investor who aims at maximizing utility from terminal wealth at time $T$ with a CRRA risk preference; i.e., a power utility function $u(x) = \frac{x^\gamma}{\gamma}$, $x \geq 0$, where $\gamma < 1$. First, we address the incomplete market case, which assumes that the investor cannot hedge all the risk coming from the stochastic variance. Second, we allow the investor to allocate to options, hence working on a complete market.
2.3. Problem formulation and solution

2.3.1 Incomplete market solution

We assume a risk-averse investor who allocates a proportion \( \pi_t \) of wealth to the stock, and the rest goes to the bank account. The wealth process for this investor under the historical measure evolves according to

\[
dX_t = X_t \left[ r + \pi_t \left( \lambda (av_t + b) \right) \right] dt + X_t \pi_t \left[ (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) (p dZ_{1t} + \sqrt{1 - \rho^2} dZ_{2t}) \right], \quad X(0) = x > 0,
\]

where \( x \) is the initial budget.

The goal/objective of the investor is to find an investment strategy that maximizes the terminal utility:

\[
J(x, v, t) = \sup_{u \in \mathcal{U}} \mathbb{E}_{x, v, t}[u(X_T)],
\]

where \( J(x, v, t) \) is the value function and \( \mathcal{U} \) denotes the space of admissible strategies (see Definition 2.3.2). According to the principles of dynamics programming, the HJB equation for such a problem should satisfy the following:

\[
0 = \sup_{\pi} \left\{ J_t + x \left( r + \pi \lambda (av + b) \right) J_x + \kappa (\theta - v) J_v 
\right. \\
+ \left. \frac{1}{2} x^2 \pi^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 J_{xx} + \frac{1}{2} \sigma^2 v J_{vv} + \pi x (av + b) \sigma J_{vx} \right\},
\]

with boundary condition \( J(x, v, T) = \frac{x^\gamma}{\gamma}. \) Here \( J_t, J_x, J_v, J_{xx}, J_{vv}, \) and \( J_{vx} \) are the first and second partial derivatives of function \( J \) with respect to \( t, x, \) and \( v. \) The next result summarizes the optimal strategy, terminal wealth and value function.

**Proposition 2.3.1.** A candidate solution to (2.16) is given by

\[
J(x, v, t) = \frac{x^\gamma}{\gamma} \exp \left\{ A(T - t) + B(T - t)v \right\},
\]

where \( \tau(t) = T - t, A(\tau) \) and \( B(\tau) \) are given by

\[
A(\tau) = \gamma \tau \tau + \frac{2 \theta k}{k_2} \ln \left( \frac{2 k_3 e^{k_1 \tau} - 1}{k_3 + (k_1 + k_3) (e^{k_1 \tau} - 1)} \right),
\]

\[
B(\tau) = \frac{k_0 (e^{k_1 \tau} - 1)}{2 k_3 + (k_1 + k_3) (e^{k_1 \tau} - 1)},
\]

with auxiliary parameters \( k_0 := \frac{\gamma^3}{1 - \gamma}, k_1 := (k - \lambda \gamma \rho), k_2 := (\sigma^2 + \gamma \gamma \rho^2) \) and \( k_3 := \sqrt{k_1^2 - k_0 k_2}. \) The optimal strategy is as follows:

\[
\pi^* = \frac{v}{av + b} \left[ \frac{\sigma B(\tau)}{(1 - \gamma)} + \frac{\lambda}{(1 - \gamma)} \right].
\]

\(^{\dagger}\)This candidate solution can be shown to be the solution in our verification theorem 2.4.3.
Here we assume the investor can also allocate to an option on the underlying risky asset. Let \( O_t = m(S_t, v_t, t) \) denote the price of the option. It can be shown that the option price evolves as

\[
\frac{dO_t}{O_t} = rdt + \frac{1}{\sqrt{v_t}} \left( m_S \rho S_t + m_v \frac{\sigma \sqrt{v_t}}{a \sqrt{\sqrt{v_t} + \frac{b \sqrt{v_t}}{v_t}}} \right) \left( a \sqrt{\sqrt{v_t} + \frac{b \sqrt{v_t}}{v_t}} \right) \left( \lambda_1 \sqrt{\sqrt{v_t}} dt + dZ_1 \right) + \frac{1}{\sqrt{v_t}} \left( m_S \sqrt{1 - \rho^2} S_t \right) \left( a \sqrt{\sqrt{v_t} + \frac{b \sqrt{v_t}}{v_t}} \right) \left( \lambda_2 \sqrt{\sqrt{v_t}} dt + dZ_2 \right),
\]

where \( m \) and \( \lambda_i \) are the parameters of the option, and \( dZ_1, dZ_2 \) are standard Brownian motions. 

### 2.3.2 Complete market solution

Here we assume the investor can also allocate to an option on the underlying risky asset. Let \( O_t = m(S_t, v_t, t) \) denote the price of the option. It can be shown that the option price evolves as
2.3. Problem formulation and solution

where $m_S = \frac{\partial m}{\partial S}$ and $m_v = \frac{\partial m}{\partial v}$ denote the partial derivatives of the option price function $m$ with respect to $S_t$ and $v_t$ (see Appendix A.3 for details). The dynamics of the option price follows

$$dO_t = O_t \left[ \left( r + K \frac{\partial E}{\partial v} + L \frac{\partial E}{\partial \lambda} \right) dt + KdZ_{1t} + LdZ_{2t} \right].$$

Together with Equation (2.2), they can be written in matrix form, that is,

$$dS_t = D[S_t, O_t] \left[ (\bar{\bar{\rho}} + \bar{\bar{G}}\bar{\bar{V}}) dt + \bar{\bar{G}}dZ_t \right],$$

where $D[S_t, O_t] = \text{diag}(S_t, O_t)$, $\bar{\bar{\rho}} = \begin{bmatrix} r \\ r \end{bmatrix}$, $\bar{\bar{G}} = \begin{bmatrix} \rho / K_1 & \sqrt{1 - \rho^2} \\ L_1 / \sqrt{V} \end{bmatrix}$, $\bar{\bar{V}}_t = (a \sqrt{V_t} + \frac{\eta}{V_t}) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\lambda_t = \sqrt{V_t} \begin{bmatrix} \bar{\bar{\lambda}}_1 \\ \bar{\bar{\lambda}}_2 \end{bmatrix}$, $dZ_t = \begin{bmatrix} dZ_{1t} \\ dZ_{2t} \end{bmatrix}$, with $Z_{1t} \perp Z_{2t}$.

Let $\bar{\eta}_t := (\pi^s_t, \pi^v_t)^T$ denote the fraction of wealth invested in the stock and the option respectively. Thereby, the left portion $(1 - \pi^s_t - \pi^v_t)$ is the fraction that is invested in the risk-free money market. The wealth process then evolves according to the stochastic differential equation (SDE):

$$\frac{dX_t}{X_t} = \left( r + \eta_t^T \bar{\bar{V}} \lambda_t \right) dt + \eta_t^T \bar{\bar{V}} dZ_t, \quad (2.22)$$

where $\eta_t = \bar{\bar{G}}^T \bar{\eta}_t$ is introduced as the new control variable, representing exposure to the underlying risk factors. The value function with exposure $\eta_t$ is defined as follows:

$$\bar{J}(x, v, t) = \sup_{\eta \in \mathcal{U}_t} \mathbb{E}_t, [u(X_T)],$$

where $\mathcal{U}_t$ denotes the space of admissible exposures.

**Definition 2.3.2.** $\eta$ is an admissible exposure if:

1) $\eta$ is progressively measurable; and

2) For all $(x_0, v_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $t \in [0, T]$, the SDE (2.22) has a pathwise unique solution $\{X_t^\eta\}_{t \in [0, T]}$ under the measure $\mathbb{Q}$ and

$$\mathbb{E}_t^{\mathbb{Q}} [u(X_T)] < \infty,$$

where $\mathbb{E}_t^{\mathbb{Q}}[.] = \mathbb{E}[., X_t = x, v_t = v]$ denotes the conditional expectation.

The corresponding HJB equation for the value function is given by:

$$0 = \bar{J}_t + \sup_{\eta \in \mathbb{R}} \left\{ \frac{1}{2} \eta^2 \eta^T \bar{\bar{V}} \eta \bar{\bar{J}}_{xx} + \eta^T \left( r + \eta^T \bar{\bar{V}} \lambda \right) \bar{\bar{J}}_x + \eta^T \bar{\bar{V}} \bar{\bar{R}}^T \Sigma \sqrt{\bar{\bar{V}}} \bar{\bar{J}}_{xv} + \frac{1}{2} \sigma^2 v \bar{\bar{J}}_{vv} + \kappa (\theta - v) \bar{\bar{J}}_v \right\}, \quad (2.23)$$

with boundary condition $\bar{J}(x, v, T) = \frac{u}{x}$, where $\bar{\bar{R}} = (1, 0)$ represents the correlation between the diffusion part of the stochastic volatility process (i.e., $dZ_{1t}$) and that of the stock (i.e., $[dZ_{1t}, dZ_{2t}]^T$). The next proposition provides the solution to this problem.
Proposition 2.3.3. A candidate solution to (2.23) is given by

\[
\bar{J}(x, v, t) = \frac{x^\gamma}{\gamma} \exp \left\{ \tilde{A}(T-t) + \tilde{B}(T-t)v \right\},
\]

with \( \tau(t) = T - t \); \( \tilde{A}(\tau) \) and \( \tilde{B}(\tau) \) are given by

\[
\tilde{A}(\tau) = \gamma \tau + \frac{2\theta k}{k_2} \ln \left( \frac{2\tilde{k}_3 e^{\frac{\gamma_1}{1-\gamma} \tau}}{2\tilde{k}_3 + (\tilde{k}_1 + \tilde{k}_3)(e^{\gamma_1 \tau} - 1)} \right),
\]

\[
\tilde{B}(\tau) = \frac{\tilde{k}_0 (e^{\gamma \tau} - 1)}{2k_3 + (\tilde{k}_1 + \tilde{k}_3)(e^{\gamma \tau} - 1)},
\]

where \( \tilde{k}_0 := \frac{\gamma_0 (\gamma_1 + 2\gamma_2)}{1-\gamma}, \tilde{k}_1 := (\gamma - \frac{\gamma_1 \sigma}{1-\gamma}), \tilde{k}_2 := \frac{\sigma^2}{1-\gamma} \) and \( \tilde{k}_3 := \sqrt{\tilde{k}_1 - \tilde{k}_2} \). The optimal exposures to the risk factors and strategies are:

\[
\eta^* = \left. \frac{\partial^2 \bar{J}}{\partial x^2} \right|_{x=v} = \frac{x^\gamma}{\gamma} \left( \frac{1}{1-\gamma} \right) \left( \sigma \tilde{B}(T-t) + \tilde{A}_1 \right),
\]

\[
\pi_i^S^* = \left( \frac{\gamma}{av + b} \right) \left( \frac{\gamma_2}{1-\gamma} \right) \tilde{A}_1 \frac{m_S}{O_i} \pi_i^{O^*},
\]

\[
\pi_i^O^* = \left( \frac{O_i}{(1-\gamma)m_v} \right) \left( \frac{1}{1-\gamma} \right) \left( \tilde{B}(T-t) + \tilde{A}_1 \sqrt{1-\rho^2 - \tilde{\lambda}_2 \rho} \right),
\]

where \( m_S \) and \( m_v \) are partial derivatives of \( O_i = m(F_i, t) \).

See Appendix A.4 for the complete proof.

As with the solution for the incomplete market (see Proposition 2.3.1), the optimal exposures to both risk factors (\( \eta^* \)) are equal to that of the Heston model (Escobar et al., 2015) times the factor \( y(v) = \frac{v}{av + b} \). A similar rationale as per the incomplete market case can be concluded and therefore the impact of the factor \( y(v) \) is critical to the aggressiveness of the strategies.

### 2.4 Properties of the solution

In this section, conditions on the parametric space are provided to ensure that the solutions \( J(x, v, t) \) and \( \bar{J}(x, v, t) \) are real-valued and finite. Verification theorems are also presented with their implied conditions on the model.

---

2This candidate solution can be shown to be the solution in our verification theorem 2.4.4.
2.4. Properties of the solution

2.4.1 Well-defined value functions

The first proposition describes conditions for a real-valued and finite value function in the incomplete market case, while the second proposition deals with the complete market case. The proofs are provided in Appendixes A.5 and A.6, respectively.

**Proposition 2.4.1.** The function \( J(x, v, t) \) is a well-defined solution to the HJB equation (2.16) if the parameters satisfy the following technical condition:

\[
\kappa^2 - 2\kappa\lambda \frac{\gamma}{1 - \gamma} \sigma^2 - \frac{\gamma}{1 - \gamma} \lambda^2 \sigma^2 > 0.
\]  

(2.30)

**Proposition 2.4.2.** The function \( \bar{J}(x, v, t) \) is a well-defined solution to the HJB equation (2.23) if the parameters satisfy the following technical condition:

\[
\kappa^2 + \frac{\gamma}{1 - \gamma} \left(-2\kappa\lambda_1 \sigma + \frac{\gamma}{1 - \gamma} \lambda_1^2 \sigma^2 - \frac{1}{1 - \gamma} (\lambda_1^2 + \lambda_2^2) \sigma^2\right) > 0.
\]  

(2.31)

2.4.2 Verification theorems

On occasion, the value function that solves the HJB equation may not be optimal for our original problem. To ensure the optimal strategy and confirm that the corresponding value function solves the optimal problem, we need a verification theorem. Next, we provide conditions for the incomplete market and complete market situations, respectively.

**Theorem 2.4.3.** Consider a function \( J(x, v, t) : [0, T] \times [0, \infty) \times [0, \infty) \to \mathbb{R} \), such that:

1) \( J \) is real-valued, finite, once continuously differentiable in \( t \) and twice continuously differentiable in \( x \) and \( v \); and

2) \( J \) satisfies Equations (2.16) and its terminal condition, with \( J(x, v, t) = \frac{\gamma}{\gamma - 1} h(v, t) \) for a positive function \( h(v, t) = e^{A(T-t)+B(T-t)v} \),

3) The function \( B(T-t) \) in Equation (2.19) satisfies the condition: For \( 0 < \bar{\lambda} \leq -\frac{(\gamma - 1)^2 - \gamma(\gamma - 2)\sigma^2}{\gamma p} B(T-t) \):

\[
-\frac{1}{2} \frac{\gamma^2}{(\gamma - 1)^2} \lambda^2 \geq -\frac{\kappa^2}{2\sigma^2}.
\]  

(2.32)

For \( \bar{\lambda} > -\frac{(\gamma - 1)^2 - \gamma(\gamma - 2)\sigma^2}{\gamma p} B(T-t) \):

\[
-\frac{1}{2} \left[ (1 - \frac{\gamma(\gamma - 2)}{(\gamma - 1)^2} \lambda^2) B^2(T) + 2 \frac{\gamma}{(\gamma - 1)^2} \sigma B(T) p \bar{\lambda} + (\frac{\gamma}{\gamma - 1})^2 \lambda^2 \right] \geq -\frac{\kappa^2}{2\sigma^2}.
\]  

(2.33)

Then \( \pi^* \) in Equation (2.20) is the optimal strategy and \( J \) in Equation (2.17) is the corresponding value function.
Chapter 2. Optimal investment strategy in the family of $4/2$ stochastic volatility models.

See Appendix A.7 for the complete proof.

The complete market case is presented next.

**Theorem 2.4.4.** Consider a function $\bar{J}(x, v, t) : [0, T] \times [0, \infty) \times [0, \infty) \to \mathbb{R}$, such that:
1) $\bar{J}$ is real-valued, finite, once continuously differentiable in $t$ and twice continuously differentiable in $x$ and $v$;
2) $\bar{J}$ satisfies Equations (2.23) and its terminal condition with $\bar{J}(x, v, T) = \frac{\sqrt{\bar{J}(v, t)}}{\gamma}$ for a positive function $\bar{h}(v, t)$; and
3) The function $\bar{B}(T - t)$ in Equation (2.26) satisfies the following conditions:
   For either $i) \lambda_1 < 0$ and $\gamma < 0$, or $ii) 0 > \lambda_1 \geq -\frac{\kappa B(t - L)}{\gamma}$ and $0 < \gamma < 1$:
   \[
   -\frac{1}{2} \left(1 - \frac{1}{(\gamma - 1)^2}\right) \left[\sigma^2 B^2(T) + 2\gamma \sigma B(T)\lambda_1 + \gamma^2 (\lambda_1^2 + \lambda_2^2)\right] \geq -\frac{\kappa^2}{2\sigma^2}. \tag{2.34}
   \]
   For $\lambda_1 < -\frac{\sigma B(t - L)}{\gamma}$ and $0 < \gamma < 1$:
   \[
   -\frac{1}{2} \left(1 - \frac{\gamma}{(\gamma - 1)^2}\right) (\lambda_1^2 + \lambda_2^2) \geq -\frac{\kappa^2}{2\sigma^2}. \tag{2.35}
   \]
Then $\eta'$ in Equation (2.27) is the optimal exposure of risk factors and $\bar{J}$ in Equation (2.24) is the corresponding value function.

See Appendix A.8 for the complete proof.

**2.5 Suboptimal analysis**

Due to the popularity, this section analyses the Merton and Heston solutions by measuring the wealth-equivalent utility loss incurred by a risk-averse investor who decides to use such strategies in the context of a $4/2$ model; i.e., these popular strategies are suboptimal and hence denoted by $\pi^{(s)}$ for $4/2$ investors.

Let us denote the value function of a risk-averse investor who follows the suboptimal strategy $\pi^{(s)}$ as $J^{(s)}(x, v, t)$. The wealth-equivalent utility loss $L$ is then defined as:

\[ J(x(1 - L), v, t) = J^{(s)}(x, v, t), \]

which represents the percentage loss of wealth when the investor follows a suboptimal strategy $\pi^{(s)}$.

In particular, the first two sections derive the general welfare loss of a suboptimal strategy for an incomplete and a complete market, respectively. In section 2.5.3, we delve into specific strategies; for instance, subsection 2.5.3.1 assumes a myopic investor who ignores the intertemporal component in the solution. Subsection 2.5.3.2 studies the embedded cases of the Heston (parameter $b = 0$) and $3/2$ (parameter $a = 0$) models for both complete and incomplete markets. Subsection 2.5.3.3 analyses the impact of failing to complete the market (not investing in derivatives), while the last section derives the wealth-equivalent losses of the most popular continuous-time strategy, the Merton solution.
2.5. Suboptimal analysis

2.5.1 Incomplete market: Suboptimal strategies

Let us denote the value function of a risk-averse investor who follows the suboptimal strategy \( \pi^{(s)} \) as \( J^{(s)}(x, v, t) \). According to the definition of the value function \( J \) in Equation (2.17) we have: \( J \geq J^{(s)} \), with equality when the strategy is optimal.

The function \( J^{(s)} \) should satisfy the partial differential equation (PDE):

\[
0 = J_t^{(s)} + x \left( r + \pi^{(s)} \lambda (av + b) \right) J_x^{(s)} + \kappa (\theta - v) J_v^{(s)} + \frac{1}{2} \sigma^2 v J_{vv}^{(s)} + \pi^{(s)} x (av + b) \sigma^2 J_{v}^{(s)},
\]

with boundary condition \( J^{(s)}(x, v, T) = \frac{x^\gamma}{\gamma} \). Next, we solve this equation, assuming convenient suboptimal strategies independent of wealth \( x \).

**Proposition 2.5.1.** Assume that \( \pi^{(s)} \) is a function of \((t, v)\); the solution to PDE (2.36) is then given by

\[
J^{(s)}(x, v, t) = \frac{x^\gamma}{\gamma} h^{(s)}(v, t),
\]

where the function \( h^{(s)}(v, t) \) satisfies the PDE:

\[
0 = h_t^{(s)} + \gamma \left( r + \pi^{(s)} \lambda (av + b) \right) h_x^{(s)} + \kappa (\theta - v) h_v^{(s)} + \frac{1}{2} \gamma \pi^{(s)} (av + b) \sigma^2 h_{v}^{(s)},
\]

with boundary condition \( h^{(s)}(v, T) = 1 \). Furthermore, the wealth-equivalent utility loss \( L \) in the incomplete market is

\[
L(v, t) = 1 - \left( \frac{h^{(s)}(v, t)}{h(v, t)} \right)^{\frac{1}{\gamma}}.
\]

The proof follows easily from substituting the ansatz for \( J^{(s)}(x, v, t) \) in Equation (2.36). The closed-form solvability of \( h^{(s)}(v, t) \) depends on the choice of \( \pi^{(s)} \) and is discussed in the upcoming sections.

2.5.2 Complete market: Suboptimal strategies

The value function (i.e., \( J^{(s)} \)) of a risk-averse investor who follows a suboptimal strategy \( \bar{\pi}^{(s)} \) in the complete market satisfies \( \bar{J} \geq \bar{J}^{(s)} \), where \( \bar{J} \) comes from Equation (2.24), with equality when the strategy is optimal. \( \bar{J}^{(s)} \) should satisfy

\[
0 = \bar{J}_t^{(s)} + \frac{1}{2} \sigma^2 \bar{J}_{vv}^{(s)} + \kappa (\theta - v) \bar{J}_v^{(s)} + x \left( r + (\eta^{(s)})^T \bar{V} \lambda \right) \bar{J}_x^{(s)} + \lambda (av + b) \bar{J}_v^{(s)} + \frac{1}{2} \sigma^2 \bar{J}_{v}^{(s)},
\]

where \( \eta^{(s)} \) is a function of \((t, v)\), and \( \bar{V} \) is a \( n \times 1 \) vector.

The proof follows easily from substituting the ansatz for \( J^{(s)}(x, v, t) \) in Equation (2.36). The closed-form solvability of \( h^{(s)}(v, t) \) depends on the choice of \( \pi^{(s)} \) and is discussed in the upcoming sections.
where \( \bar{R} = (1, 0) \), \( \bar{V}_t = (a \sqrt{V_t} + \frac{h}{\sqrt{V_t}}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), \( \bar{\lambda} = \sqrt{\bar{V}_t} \begin{bmatrix} \bar{\lambda}_1 \\ \bar{\lambda}_2 \end{bmatrix} \), \( \bar{G} = \begin{bmatrix} \rho & \sqrt{1 - \rho^2} \\ K_1 & L_1 \end{bmatrix} \), \( \bar{\pi}^{(s)} := \left( (\pi_t^{(s)})^S, (\pi_t^{(s)})^G \right)^T \), \( \bar{\eta}^{(s)} = \bar{G}^T \bar{\pi}^{(s)} = \begin{bmatrix} \bar{\eta}_1^{(s)} \\ \bar{\eta}_2^{(s)} \end{bmatrix} \) denote the new control variable corresponding to the suboptimal allocation and we have the boundary condition \( \bar{J}^{(s)}(x, v, T) = \frac{v^2}{\gamma} \).

**Proposition 2.5.2.** Assume that \( \bar{\eta}^{(s)} \) is a function of \((t, v)\); the solution to PDE (2.38) is then given by

\[
\bar{J}^{(s)}(x, v, t) = \frac{v^2}{\gamma} \bar{h}^{(s)}(v, t),
\]

with \( \bar{h}^{(s)}(v, t) \) satisfying the PDE:

\[
0 = \bar{h}_t^{(s)} + \frac{1}{2} \gamma (\eta^{(s)})^T \bar{V}^2 \eta^{(s)} (\gamma - 1) \bar{h}^{(s)} + \gamma (r + (\eta^{(s)})^T \bar{V} \bar{\lambda}) \bar{h}^{(s)} + \gamma \bar{\eta}^{(s)} + \frac{1}{2} \sigma^2 \bar{V} \eta^{(s)} + \kappa (\theta - v) \bar{h}^{(s)},
\]

with boundary condition \( \bar{h}^{(s)}(v, T) = 1 \). Furthermore, the wealth-equivalent utility loss \( \bar{L} \) in the complete market is

\[
\bar{L}(v, t) = 1 - \left( \frac{\bar{J}^{(s)}(v, t)}{\bar{h}(v, t)} \right)^{\frac{1}{\gamma}}.
\]

The proof follows from substituting the ansatz for \( \bar{J}^{(s)}(x, v, t) \) in Equation (2.38).

### 2.5.3 Important suboptimal strategies

In this section, we consider suboptimal strategies \( \pi^{(s)} \) for incomplete markets and suboptimal exposures \( \eta^{(s)} \) for complete markets of practical interest, many of which also lead to functions \( \bar{h}^{(s)}(v, t) \) and \( \bar{J}^{(s)}(v, t) \) that are of exponential-affine form and hence closed-form. This allows us to study the impact of such substandard investment allocations in detail in the numerical section.

#### 2.5.3.1 Ignoring intertemporal hedging

Investors who ignore the inter-temporal term fail to account for future movements in volatility and therefore follow a suboptimal strategy; these are also known as myopic investors. In this section, we consider the case where the investor uses the following strategy for the incomplete market:

\[
\pi^{(s)}(t) = \begin{bmatrix} v \\ av + b \end{bmatrix} \begin{bmatrix} \bar{\lambda}_1 \\ \bar{\lambda}_2 \end{bmatrix}.
\]

Similarly, for the complete market, we assume:

\[
\eta^{(s)}(t) = \begin{bmatrix} v \\ av + b \end{bmatrix} \begin{bmatrix} \frac{1}{1 - \gamma} \bar{\lambda}_1 \\ \frac{1}{1 - \gamma} \bar{\lambda}_2 \end{bmatrix}.
\]
In these two cases, we simply remove the time-dependent term from the optimal strategies in Equations (2.20) and (2.27), respectively. We provide the closed-form expressions for \( h^{(t)}(t, v) \) and \( \bar{h}^{(t)}(t, v) \) in the following corollaries.

**Corollary 2.5.3.** For incomplete markets, \( h^{(t)}(v, t) \) is in the exponential-affine form, such as \( h^{(t)}(v, t) = \exp(A^{(t)}(\tau) + B^{(t)}(\tau)v) \), where \( \tau(t) = T - t \) and

\[
A^{(t)}(\tau) = r\gamma t + \frac{2\theta}{k_2} \ln \left( \frac{2k_3 e^{\frac{-1}{1+\gamma}}}{2k_3 + (k_1 + k_3)(e^{k_1 \tau} - 1)} \right),
\]

\[
B^{(t)}(\tau) = \frac{k_0 (e^{k_1 \tau} - 1)}{2k_3 + (k_1 + k_3)(e^{k_1 \tau} - 1)},
\]

where
\[
k_0 := \frac{\gamma \lambda^2}{1 - \gamma}, k_1 := \left( \kappa - \frac{\gamma \lambda}{1 - \gamma} \sigma \rho \right), k_2 := \sigma^2, k_3 := \sqrt{k_1^2 - k_0 k_2}.
\]

Moreover, the wealth-equivalent loss can be represented as

\[
L(v, t) = 1 - \exp \left\{ \frac{1}{\gamma} \left( (A^{(t)}(T - t) - A(T - t)) + (B^{(t)}(T - t) - B(T - t)) v \right) \right\},
\]

where the functions \( A^{(t)}(T - t), A(T - t), B^{(t)}(T - t) \) and \( B(T - t) \) characterize the value function \( J^{(t)} \) (or \( J \)).

See Appendix A.9 for the complete proof.

**Corollary 2.5.4.** For complete markets, \( \bar{h}^{(t)}(v, t) \) is in the exponentially-affine form, such as \( \bar{h}^{(t)}(v, t) = \exp(\bar{A}^{(t)}(\tau) + \bar{B}^{(t)}(\tau)v) \), with time horizon \( \tau(t) = T - t \) and

\[
\bar{A}^{(t)}(\tau) = r\gamma t + \frac{2\theta}{k_2} \ln \left( \frac{2\bar{k}_3 e^{\frac{-1}{1+\gamma}}}{2\bar{k}_3 + (\bar{k}_1 + \bar{k}_3)(e^{\bar{k}_1 \tau} - 1)} \right),
\]

\[
\bar{B}^{(t)}(\tau) = \frac{\bar{k}_0 (e^{\bar{k}_1 \tau} - 1)}{2\bar{k}_3 + (\bar{k}_1 + \bar{k}_3)(e^{\bar{k}_1 \tau} - 1)},
\]

where
\[
\bar{k}_0 := \frac{\gamma \lambda^2}{1 - \gamma}, \bar{k}_1 := \left( \kappa - \frac{\gamma \lambda}{1 - \gamma} \sigma \rho \right), \bar{k}_2 := \sigma^2, \bar{k}_3 := \sqrt{\bar{k}_1^2 - \bar{k}_0 \bar{k}_2}.
\]

The wealth-equivalent loss is represented by

\[
\bar{L}(v, t) = 1 - \exp \left\{ \frac{1}{\gamma} \left( (\bar{A}^{(t)}(T - t) - \bar{A}(T - t)) + (\bar{B}^{(t)}(T - t) - B(T - t)) v \right) \right\},
\]

where the functions \( \bar{A}^{(t)}(T - t), \bar{A}(T - t), \bar{B}^{(t)}(T - t) \) and \( B(T - t) \) characterize the value function \( \bar{J}^{(t)} \) (or \( \bar{J} \)).

See Appendix A.10 for the complete proof.
Thus Corollary 2.5.5. If \( a = 0 \) and \( b = 1 \), then the 4/2 model produces the 3/2 model (Heston (1997)) with the strategy \( \pi^{(2)}_t = \nu \left[ \frac{\sigma_B(t)}{(1 - \gamma)} + \frac{1}{(1 - \gamma)} \right] \), while the case of \( b = 0 \) and \( a = 1 \) leads to the 1/2 model (Heston (1993) with \( \pi^H_t = \left[ \frac{\sigma_B(t)}{(1 - \gamma)} + \frac{1}{(1 - \gamma)} \right] \)). As Grasselli (2017) claims, the superposition of the CIR factor and the flipped CIR factor in the volatility is motivated by their different contributions in explaining some stylized facts and capturing various trends of the implied volatility surface.

Unfortunately, when \( \pi^{(2)}_t = \pi^H_t \) or \( \pi^{(2)}_t = \pi^{(2)}_t \) is assumed, there is no exponential-affine structure for the corresponding function \( h^{(x)}(v, t) \), hence we cannot produce the equivalent loss in closed form.

 Nonetheless, we envision a quick numerical procedure to compute these losses. The methodology starts by substituting the suboptimal strategy, \( \pi^H_t \) or \( \pi^{(2)}_t \), into the wealth process, i.e., Equation (2.15), producing \( X_t^{x^{(a)}} \). The value function can then be obtained via simulation as an expectation: \( J^{(x)}(x, t, v) = \mathbb{E}_{x,v,t} \left[ u(X_T^{x^{(a)}}) \right] \). Lastly, as \( J(x, v, t) \) denotes the value function under the optimal strategy (known in closed-form), we have

\[
J^{(x)}(x, v, t) = J(x(1 - L), v, t) = \frac{(x(1 - L))^\gamma}{\gamma} e^{A(\tau) + B(\tau)v}.
\]

The wealth losses can be computed as

\[
L = 1 - \left( \frac{\gamma}{x^\gamma} \frac{\mathbb{E}_{x,v,t} \left[ u(X_T^{x^{(a)}}) \right]}{e^{A(\tau) + B(\tau)v}} \right)^{\frac{1}{\gamma}}.
\]

(2.45)

The same idea can be applied to the complete market, leading to

\[
\bar{L} = 1 - \left( \frac{\gamma}{x^\gamma} \frac{\mathbb{E}_{x,v,t} \left[ u(X_T^{x^{(a)}}) \right]}{e^{\bar{A}(\tau) + \bar{B}(\tau)v}} \right)^{\frac{1}{\gamma}}.
\]

(2.46)

2.5.3.3 Ignoring derivatives trading

There is plenty of evidence that a risk-averse investor could benefit greatly from trading in derivatives to complete the market (see Da Fonseca et al. (2011), Liu and Pan (2003), Escobar et al. (2017)). Thus, in this subsection, we focus on the losses from no trading in options, i.e., \( \pi^D = 0 \).

The following corollary presents the closed-form representation for losses when an investor, in the setting of a 4/2 model, ignores trading in derivatives.

**Corollary 2.5.5.** Let \( \bar{\pi}^{(3)} = \left[ \begin{array}{c} \pi^{(3)}_t \\ 0 \end{array} \right] \); therefore, \( \eta^{(3)} = \pi^{(3)}_t \left[ \frac{\rho}{\sqrt{1 - \rho^2}} \right] \), where \( \pi^{(3)}_t = \left( \frac{v}{a + b} \right) \left( \frac{3}{(1 - \gamma)(1 - \rho^2)} \right) \).

Thus \( \tilde{h}^{(x)}(v, t) \) is exponential affine with the representation:

\[
\begin{align*}
\tilde{h}^{(x)}(t, v) & = \exp(\bar{A}^{(x)}(\tau) + \bar{B}^{(x)}(\tau)v), \\
\bar{A}^{(x)}(\tau) & = \frac{\rho}{\sqrt{1 - \rho^2}}, \\
\bar{B}^{(x)}(\tau) & = \frac{3}{(1 - \gamma)(1 - \rho^2)}. 
\end{align*}
\]
where \( \tau(t) = T - t \), and the functions \( \bar{A}(\tau) \) and \( \bar{B}(\tau) \) are given by

\[
\bar{A}(\tau) = \gamma \tau + \frac{2k}{k_2} \ln \left( \frac{2k_3 e^{A_1 t + A_2 \tau}}{2k_3 + (k_1 + k_3)(e^{A_3 \tau} - 1)} \right),
\]
\[
\bar{B}(\tau) = \frac{k_0 (e^{A_1 t + A_2 \tau} - 1)}{2k_3 + (k_1 + k_3)(e^{A_3 \tau} - 1)},
\]

with

\[
k_0 := \gamma \left( \frac{-\lambda_2^2}{1 - \rho^2} + \frac{2\lambda_2 (\rho \lambda_1 + \sqrt{1 - \rho^2})}{\sqrt{1 - \rho^2}} \right),
\]
\[
k_1 := \left( \kappa - \gamma \left( \frac{\rho \lambda_1 \sigma}{\sqrt{1 - \rho^2}} \right) \right),
\]
\[
k_2 := \sigma^2, k_3 := \sqrt{k_1 - k_0 k_2}.
\]

Moreover, the wealth-equivalent loss is:

\[
\bar{L}(v, t) = 1 - \exp \left\{ \frac{1}{\gamma} \left[ (\bar{A}^{(i)}(T - t) - \bar{A}(T - t)) + (\bar{B}^{(i)}(T - t) - \bar{B}(T - t))v \right] \right\},
\]

where the functions \( \bar{A}^{(i)}(T - t), \bar{A}(T - t), \bar{B}^{(i)}(T - t) \) and \( \bar{B}(T - t) \) characterize the value function \( \bar{f}^{(i)} \) (or \( \bar{J} \)).

See Appendix A.11 for the complete proof.

### 2.5.3.4 The loss from using Merton’s solution

Due to the importance of Merton’s solution, here we represent the wealth-equivalent losses in a 4/2 setting using this simple solution.

**Proposition 2.5.6.** Assume \( \pi_i^{(s)} = \frac{3}{(t-y)} \) in Equation (2.37), and consider the following conditions:

\[
a \pi \lambda + \frac{1}{2} \pi^2 \gamma (\gamma - 1) < \frac{k^2}{2\sigma^2}, \quad \frac{1}{2} \pi^2 b^2 \gamma (\gamma - 1) \leq \frac{(2k\theta - \sigma^2)^2}{8\sigma^2},
\]
\[
0 < \frac{1}{2\sigma^2} \left( 2k\theta + \sigma^2 + \sqrt{(2k\theta - \sigma^2)^2 + 8\sigma^2 \left( \frac{1}{2} \pi^2 b^2 \right)} \right), \quad 0 \geq -\frac{\sqrt{k^2 + 2\mu \sigma^2 + \kappa}}{\sigma^2}.
\]
Then, the solution of \( h^{(i)}(v, t) \) in Equation (2.37) can be written as

\[
h^{(i)}(v, t) = \exp \left\{ \left( ry + b \pi y \lambda + ab \pi^2 \gamma (y - 1) \right) (T - t) \right\} \\
\times E \left[ e^{(\alpha \pi y \lambda + \frac{1}{2} \sigma^2 \gamma (y - 1) r) t + \left( \frac{1}{2} \sigma^2 \gamma (y - 1) r \right) \frac{1}{\pi} t^2} dr \right] \\
= \exp \left\{ \left( ry + b \pi y \lambda + ab \pi^2 \gamma (y - 1) \right) (T - t) \right\} \times \left( \frac{\beta(T - t, v)}{2} \right)^{m+1} v^{-\frac{m}{\alpha^2}} (K(T - t))^{-\left(\frac{1}{2} + \frac{m}{\sigma^2}\right)} \\
\times e^{\frac{\kappa}{\sigma^2} (T - t)} - \sqrt{\kappa} v \cosh \left( \frac{\sqrt{\kappa}(T - t)}{2} \right) \frac{\Gamma(\frac{1}{2} + \frac{m}{\sigma^2} + \frac{\kappa}{\sigma^2})}{\Gamma(m + 1)} \times _1F_1 \left( \frac{1}{2} + \frac{m}{2} + \frac{\kappa}{\sigma^2}, m + 1, \frac{\beta(T - t, v)^2}{4(K(T - t))} \right),
\]

with

\[
m = \frac{1}{\sigma^2} \sqrt{(2 \kappa \theta - \sigma^2)^2 + 8 \sigma^2 v}, \quad A = \kappa^2 - 2 \left( a \pi y \lambda + \frac{1}{2} \sigma^2 \gamma (y - 1) \right) \sigma^2, \\
\beta(T - t, v) = \frac{2 \sqrt{A} v}{\sigma^2 \sinh \left( \frac{\sqrt{A}(T - t)}{2} \right)}, \quad K(T - t) = \frac{1}{\sigma^2} \left( \sqrt{A} \coth \left( \frac{\sqrt{A}(T - t)}{2} \right) + \kappa \right),
\]

where \( \Gamma \) and \( _1F_1 \) denote the gamma and hypergeometric confluent functions, respectively.

Moreover, the wealth-equivalent loss can be formulated as

\[
L(v, t) = 1 - \left( \frac{h^{(i)}(v, t)}{h(v, t)} \right)^{\frac{1}{4}},
\]

where \( h^{(i)}(v, t) \) and \( h(v, t) \) characterize the value function \( J^{(i)} \) (or \( J \)).

See Appendix A.12 for the complete proof.

### 2.6 Numerical analysis

This section is organized as follows: We first present a seven-step estimation methodology for the 4/2 model, which is later adapted to the 1/2, 3/2 and GBM models. Section 2.3.1 studies the solution for the incomplete market case in detail, paying particular attention to the sensitivities of the investment strategies to various parameters and the welfare-equivalent losses from suboptimal choices. A similar analysis is performed for the complete market in Section 2.3.2.

The data-set for our empirical study consists of the S&P 500 and its volatility index, as well as the VIX reported by the Chicago Board Options Exchange (CBOE), from January 2010 to the last day of December 2019. The S&P 500 has been regarded as the best single gauge of all U.S. stock exchanges, while the VIX is a real-time market index that measures the expectation...
of the market for the next 30 days. The latter is referred to as the “fear gauge”, as it provides an overview of market risk and investors’ sentiments. The VIX data are used to estimate the “volatility group” as volatility accommodating to each stochastic model (i.e., 4/2, Heston, 3/2, and Merton model), and the S&P 500 data are used to estimate the “drift group” as asset prices. It is worth mentioning that the relation between VIX and variance $\sigma_t$ was recently worked out for the 4/2 model by Lin et al. (2017). Due to the annualized nature of our parameters and the 21-day horizon for VIX options, the correction factor is negligible in many parametric cases, and the instantaneous variance can be approximated by VIX$^2$.

### 2.6.1 Estimation for the 4/2 model

In our model, there are seven parameters to be estimated: $r$, $\lambda$, $c$, $\rho$, $\kappa$, $\theta$, and $\sigma$. The parameters $c$, $\kappa$, $\theta$, and $\sigma$ are called “volatility group” parameters, while $r$, $\rho$ and $\lambda$ belong to the “drift group”. In both cases we proceed via regression using available data on the stock and its variance.

Our estimation approach consists of seven steps. The first five relate to the volatility group, while the remaining two pertain to drift parameters. The consistency of our methodology has been explored via simulations, which have shown that a wide range of parameters can be recovered.

As a first step, we estimate the minimum of the variance process $\{ z_t \}$, denoted by $M$, with $M = \min_{0 \leq t \leq T} z_t$. This minimum relates to our parameters; to see this, substitute $c$ into the variance $z_t$ (i.e., $z_t = b^2 (c^2 v + \frac{1}{v} + 2c)$), and we realize that the minimum of the process $\{ z_t \}$ occurs when $v = \frac{1}{c}$, hence

$$ M = \min z_t = 4cb^2 = \frac{4c\theta}{c^2\theta + 2c + \frac{2c}{2 \sigma^2}}. $$

Second, by regarding the observable variance $z_t$ (VIX$^2$) as a quadratic function of $v_t$, we can obtain two likely values for the underlying, denoted by $v^\pm_t$:

$$ cv^\pm_t = \left( z_t - \frac{M}{2} \right) \pm \sqrt{\left( \frac{M}{2} - z_t \right)^2 - \frac{M^2}{4}} \frac{2}{M}. $$

This means the time series of $cv^-_t$ and $cv^+_t$ (in short, of $cv^+_t$) can be estimated using observations of variance $z_t$ (VIX data) and the estimate of the minimum $M$.

Third, as our stochastic volatility $v_t$ follows a CIR process, the dynamic of $cv^+_t$ also follows CIR processes:

$$ \frac{d(cv)}{\sqrt{cv}} = \frac{cx\theta}{\beta_3} \frac{dt}{\sqrt{cv}} - \frac{\kappa}{\beta_2} dt \sqrt{cv} + \sqrt{cv} \sigma \, dZ_t. $$

We then estimate the coefficients $\beta_1$, $\beta_2$, and $\beta_3$ via an Euler discretization of the CIR (full truncation; see Lord et al. (2010)) using $\Delta t = \frac{T}{n}$, $t_i = i \Delta t$, $i = 0, \ldots, n$, combined with a linear regression, and we extract the error term $e_i = Z_{t_{i+1}} - Z_{t_i}$. This leads to two sets of coefficients: $(\hat{\beta}_1^+, \hat{\beta}_2^+, \hat{\beta}_3^+)$ and $(\hat{\beta}_1^-, \hat{\beta}_2^-, \hat{\beta}_3^-)$. 


In the fourth step, we use the two sets of equations to solve for the two sets of parameters: \( \Theta^+ = (\hat{\kappa}^+, \hat{\theta}^+, \hat{\sigma}^+, \hat{\epsilon}^+) \) and \( \Theta^- = (\hat{\kappa}^-, \hat{\theta}^-, \hat{\sigma}^-, \hat{\epsilon}^-) \),

\[
\beta_1 = c\kappa \theta, \beta_2 = -\kappa, \beta_3 = \sqrt{c\sigma}, M = 4c\frac{\theta}{c^2\theta + 2c + \frac{2\kappa}{2\kappa\theta - \sigma^2}} \implies \\
\hat{\epsilon} = \frac{4\hat{\beta}_1}{(-\frac{\hat{\beta}_1}{\beta_2}) + 2 + \frac{2\beta_1}{\beta_2}} (-\hat{\beta}_2 M), \hat{\kappa} = -\hat{\beta}_2, \hat{\theta} = \frac{\hat{\beta}_1}{-\hat{\epsilon} \hat{\beta}_2}, \hat{\sigma} = \frac{\hat{\beta}_3}{\sqrt{\hat{\epsilon}}}. 
\]

Fifth, since \( v_i \) is asymptotically a gamma random variable with parameters \( \left( \frac{2\theta}{\sigma^2}, \frac{\kappa}{\sigma^2} \right) \) and \( \frac{1}{v_i} \) is asymptotically inverse gamma with parameters \( \left( \frac{\kappa}{\sigma^2}, \frac{2\theta}{\sigma^2} \right) \), the first three moments are known. We design a selection criterion to pick one of the two candidate sets \( \Theta^+ \) and \( \Theta^- \) based on these moments. The criterion, denoted by \( F \), is represented as

\[
F = \left( \mu_1 - \frac{1}{n + 1} \sum_{i=0}^{n} (z_{i,t} - \hat{M}/2)^2 \right)^2 + \left( \mu_2 - \frac{1}{n + 1} \sum_{i=0}^{n} (z_{i,t} - \hat{M}/2)^2 \right)^2 + \left( \mu_3 - \frac{1}{n + 1} \sum_{i=0}^{n} (z_{i,t} - \hat{M}/2)^3 \right)^2, 
\]

(2.49)

where

\[
\mu_1 = \mathbb{E} \left( \hat{\epsilon}^2 v_i + \frac{\hat{b}_v^2}{v_i} \right) = \hat{\epsilon}^2 \theta + \frac{2\hat{\kappa} \hat{b}_v^2}{2\kappa \theta - \sigma^2},
\]

(2.50)

\[
\mu_2 = \mathbb{E} \left( \hat{\epsilon}^2 v_i + \frac{\hat{b}_v^2}{v_i} \right) = 2\hat{\epsilon}^4 \theta (2\kappa \theta + \sigma^2) + \frac{4\hat{\kappa}^2 \hat{b}_v^4}{(2\kappa \theta - \sigma^2)(2\kappa \theta - 2\sigma^2)} + 2\hat{\epsilon}^2 \hat{b}_v^2,
\]

(2.51)

\[
\mu_3 = \mathbb{E} \left( \hat{\epsilon}^2 v_i + \frac{\hat{b}_v^2}{v_i} \right) = 2\hat{\epsilon}^4 \theta v_i^3 + 3\hat{\epsilon}^4 \theta v_i^3 \left( \frac{\hat{b}_v^2}{v_i} \right) + 3\hat{\epsilon}^4 \theta v_i^3 \left( \frac{\hat{b}_v^2}{v_i} \right),
\]

(2.52)

and

\[
M = \frac{4}{\hat{\kappa}} \frac{(c\epsilon^+ + c\epsilon^-)}{\hat{\kappa}} = \frac{v_i - \hat{M}/2}{\hat{\epsilon}^2 v_i + \frac{1}{\hat{\epsilon}^2 v_i}}.
\]

Note that we need to impose the condition \( 2\kappa \theta \geq 3\sigma^2 \) to ensure that these moments of the inverse gamma distribution are well-defined. By substituting the two sets of candidate estimates \( \Theta^+ \) and \( \Theta^- \) into \( \mu_1, \mu_2, \) and \( \mu_3, \) we can then search for the parameters that minimize the objective function \( F \).

The sixth step targets the “drift group” parameters; we discretize the stock price process \( S_t \) and its stochastic volatility \( v_t \):

\[
\frac{S_{t+1} - S_t}{S_t (a \sqrt{v_t} + \frac{1}{\sqrt{v_t}})} - \frac{r \Delta t}{(a \sqrt{v_t} + \frac{1}{\sqrt{v_t}})} = \tilde{\lambda} \Delta t \sqrt{v_t} + \epsilon_t.
\]

We then perform a linear regression analysis with \( y_t = \frac{S_{t+1} - S_t}{S_t (a \sqrt{v_t} + \frac{1}{\sqrt{v_t}})} - \frac{r \Delta t}{(a \sqrt{v_t} + \frac{1}{\sqrt{v_t}})}, x_t = \sqrt{v_t} \Delta t, \) hence \( \beta_1 = \tilde{\lambda} \implies \epsilon_t = y_t - \beta_1 x_t. \)
2.6. Numerical analysis

The seventh and last step estimates $\rho$. For this, we use the residuals from steps three ($\epsilon_i^1$) and six ($\epsilon_i^2$) to obtain $\rho = \text{corr}(\epsilon_i^1, \epsilon_i^2)$.

The results of the estimation are presented below:

Table 2.1: Estimation from the VIX (4 January 2010 to 31 December 2019) with the 4/2 model.

<table>
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<tr>
<th></th>
<th>$\hat{\Theta}_0^-$</th>
<th>$\hat{\Theta}_0^+$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>7.3479</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
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<td>0.0328</td>
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<tr>
<td>$\hat{\sigma}$</td>
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<td>0.6612</td>
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<td>$\hat{b}$</td>
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<td>F-val</td>
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<td>1.4287e-06</td>
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<td>$\rho$</td>
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<td>$\hat{\lambda}$</td>
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<td>2.9428</td>
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<td>Theoretical leverage ($\nu_i = \theta$)</td>
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<td>-0.7689</td>
</tr>
<tr>
<td>Observed leverage</td>
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<td></td>
</tr>
<tr>
<td>$\beta_i$'s</td>
<td>(0.8535, -6.4665, 1.1088)</td>
<td>(94.5024, -7.3479, 13.0952)</td>
</tr>
<tr>
<td>$M$</td>
<td>0.0084</td>
<td></td>
</tr>
</tbody>
</table>

Both parameter sets $\hat{\Theta}_0^-$ and $\hat{\Theta}_0^+$ satisfy all the required conditions and lead to small values of the objective function $F$. The key observation is that the values of $\hat{c}$ are quite different for these two sets. We select the set with a lower value of $F$ (i.e., $\hat{\Theta}_0^+$) as the best estimate. It should be noted that further optimizations (e.g., minimizing $F$) do not change the conclusion, and they only slightly alter the value of the parameters.

2.6.1.1 Estimation for the 3/2, 1/2 and GBM models

These three models are particular cases of the 4/2 model. For consistency we use the same procedure described in the previous section to avoid unnecessary steps.

As $c \to 0$, for the 3/2 model, it follows that

$$a = \sqrt{\frac{c^2\theta}{c^2 + 2c + \frac{2c}{2\kappa - \sigma^2}}} \to 0, \quad b = \sqrt{\frac{\theta}{c^2 + 2c + \frac{2c}{2\kappa - \sigma^2}}} \to \sqrt{\frac{2\theta \kappa - \sigma^2}{2\kappa}}.$$

Here, the variance of the 3/2 model is given by $z_t = \frac{b}{\nu_t}$, hence the volatility for the 3/2 model can be solved explicitly with $\nu_t = \frac{b^2}{z_t}$. Since $b$ depends on the CIR parameters, we work with the observable $\frac{b}{\nu_t} = \frac{1}{z_t}$. As $\nu_t$ follows a CIR process, we define $dy_t := \frac{dy_t}{\nu_t}$ and obtain:

$$\frac{dy_t}{\sqrt{\nu_t}} = \kappa \theta \frac{d\nu_t}{\nu_t} = \frac{\sigma}{b} dZ_t,$$
Table 2.2: Estimates among the various models

<table>
<thead>
<tr>
<th></th>
<th>4/2 Model</th>
<th>3/2 Model</th>
<th>Heston</th>
<th>Merton</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{k}$</td>
<td>7.3479</td>
<td>6.9884</td>
<td>14.6290</td>
<td>-</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>0.0328</td>
<td>0.0323</td>
<td>0.0315</td>
<td>0.1686</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>0.6612</td>
<td>0.3760</td>
<td>0.5210</td>
<td>-</td>
</tr>
<tr>
<td>$\hat{a}$</td>
<td>0.9051</td>
<td>0</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>$\hat{b}$</td>
<td>0.0023</td>
<td>0.0268</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>$\hat{c}$</td>
<td>392.2363</td>
<td>0</td>
<td>$\infty$</td>
<td>-</td>
</tr>
<tr>
<td>$\hat{\rho}$</td>
<td>-0.7689</td>
<td>0.7905</td>
<td>-0.8129</td>
<td>-</td>
</tr>
<tr>
<td>$\hat{\lambda}$</td>
<td>2.9428</td>
<td>2.2362</td>
<td>3.0071</td>
<td>3.3431</td>
</tr>
</tbody>
</table>

Theoretical leverage ($v_t = \theta$) 

Where $\beta_1$, $\beta_2$, and $\beta_3$ are determined by least squares. We can then solve for $b$, $k$, $\theta$ and $\sigma$ using four equations:

\[
b = \sqrt{\frac{\theta(2\kappa\theta - \sigma^2)}{2\kappa}}, \quad \beta_1 = \frac{\kappa\theta}{b^2}, \quad \beta_2 = -\kappa, \quad \beta_3 = \sigma \frac{\theta}{b}
\]

\[
\hat{\theta} = -\frac{2\hat{\beta}_2}{2\hat{\beta}_1 - \hat{\beta}_3^2}, \quad \hat{k} = -\hat{\beta}_2, \quad \hat{b} = \sqrt{\frac{\hat{k}\theta}{\hat{\beta}_1}}, \quad \hat{\sigma} = \hat{b}\hat{\beta}_3.
\]

Steps five to seven are the same as in the 4/2 case.

In the 1/2 model (Heston), as $c \to \infty$,

\[
a = cb = \sqrt{\frac{\theta}{c^2\theta + 2c + \frac{2\kappa}{2\kappa\theta - \sigma^2}}} \to 1, \quad b = \sqrt{\frac{\theta}{c^2\theta + 2c + \frac{2\kappa}{2\kappa\theta - \sigma^2}}} \to 0.
\]

The variance of the Heston model is given by $z_t = v_t = VIX^2$. As $v_t$ follows a CIR process, we can easily proceed as before via a discretization and least squares estimation (steps three to seven) with $\beta_1 = \kappa\theta$, $\beta_2 = -\kappa$, $\beta_3 = \sigma$ ($\Rightarrow \hat{k} = -\hat{\beta}_2$, $\hat{\theta} = -\frac{\hat{\theta}}{\hat{\beta}_1}$, $\hat{\sigma} = \hat{b}\hat{\beta}_3$).

Lastly, for the GBM, the volatility is constant, i.e., $\sigma_M$. In order to adapt it to this model, we let $\sigma_M = \mathbb{E}[VIX]$ during the targeted period.

In addition, the technical condition (2.30), the change-of-measure condition in Proposition 2.2.1, and condition (2.32) or (2.33) in the verification theorem are all checked and satisfied by the estimates for the various models.

### 2.6.2 Incomplete market analysis

This section uses the estimates provided in Tables 2.1 and 2.2 to compare optimal strategies and suboptimal performances within the setting of incomplete markets. We set the investment horizon to $T = 4$ years and choose the level of risk aversion ($\gamma = -5$) in such a way that it leads to a reasonable investment from the perspective of Merton’s solution — i.e., no financial or insurance investment manager would permit high exposures to risky assets, therefore we try to ensure that $\pi < 1.5$ (i.e., 50% borrowing).
2.6. Numerical analysis

2.6.2.1 Optimal strategy in various models

The most interesting aspect of the 4/2 and 3/2 models is the influence of the underlying process $v_t$ on the optimal strategy ($\pi_t$) via the factor $y(v) = \sqrt{\frac{v}{\kappa + \frac{\lambda}{2}}}$ . Therefore, here we study how different variance levels impact investment strategies across the four models (i.e., including the GBM and 1/2 cases as well). We capture this by plotting the investment strategy at a future time $t$ ($\pi_t$) against possible values of the variance at time $t$, i.e., $z_t$ in an interval around its long-term mean ($\theta = \mathbb{E}[z_\infty | z_0 = \theta]$); this shows scenarios where the future variance is larger or smaller than the long-term (and current) variance. Figure 2.1 below shows the investment strategies suggested by the four models using the estimates from the previous section.

![Figure 2.1: $\pi_t$ versus $z_t$ in different models](image)

As we can observe from the picture, all stochastic volatility models recommend less aggressive strategies than the Merton model (GBM). In particular, the 3/2 model is the most aggressive when the future variance value is less than its long-term mean, and it allocates the least to the stock when variance is larger than the long-term mean. Interestingly, the 4/2 model performs very similarly to the Heston model except in scenarios of very low variance, where it is rather conservative. As explained earlier, these traits are strongly related to the structure of the market price of risk, which determines the ideal combination of variance and excess return for investors. More specifically, the excess return in the 3/2 model does not depend on the variance of the stock, hence large variances are a sign of poor performance. In contrast, the excess return in the 4/2 model increases in $v$. This explains the mild increase in risky asset allocation in this model when the variance increases.

2.6.2.2 Sensitivity analysis among models

This section presents a sensitivity analysis on the optimal strategies with respect to investment horizon $T$, correlation $\rho$, volatility of volatility $\sigma$, market prices of risk $\lambda$ and risk aversion. Unless otherwise specified, all parameters in the various models take their estimates from the last section. Moreover, the range allowed for the parameters is the maximum possible that ensures all the conditions — the Feller condition, technical condition (2.30), the change-of-measure condition in Proposition 2.2.1, and condition (2.32) or (2.33) in the verification theorem — are satisfied.
Chapter 2. Optimal investment strategy in the family of 4/2 stochastic volatility models.

The relationship between optimal strategy and investment horizon $T$ is presented in Figures 2.2 and 2.3. It can be seen that in the 1/2, 3/2 and 4/2 models, the optimal allocation increases with the investment horizon. If the variance at time $t$ is higher than the long-term expected variance level $\theta$, the optimal strategy in the 3/2 model suggests the most conservative allocation (20% that of the Merton model and roughly 50% of the 4/2 or 1/2 allocations) to the risky asset as the investment horizon changes. On the other hand, if at time $t$ the variance is lower than $\theta$, the optimal strategy in the 3/2 model indicates a rise in this allocation (twice the 4/2 and 1/2 exposures, still half of Merton’s recommendation) due to a less volatile market.

Figure 2.2: Strategy versus horizon for large variances: $\pi_t$ vs. $T$, $z_t > \theta$

Figure 2.3: Strategy versus horizon for small variances: $\pi_t$ vs. $T$, $z_t < \theta$

For the relationship between optimal allocation and correlation $\rho$, Figures 2.4 and 2.5 show that all the models make minor adjustments of allocations, except for the Merton model. As for the impact of volatility of volatility $\sigma$, Figures 2.6 and 2.7 demonstrate that only the 3/2 model displays significant sensitivity. In both cases, when the variance is larger or smaller than the long-term variance, the model recommends a significant decrease in exposure as the volatility of volatility increases.

Figure 2.4: Strategy versus leverage for large variances: $\pi_t$ vs. $\rho$, $z_t > \theta$

Figure 2.5: Strategy versus leverage for small variances: $\pi_t$ vs. $\rho$, $z_t < \theta$
2.6. Numerical analysis

The relationship between optimal allocation and $\lambda$ is shown in Figures 2.8 and 2.9. All the models indicate an increase in the risk premium. In particular, as variance is lower than the long-term variance, the optimal allocation in the 4/2 model behaves the most conservatively among the models for a moderate value of $\lambda$. On the other hand, when $\lambda$ is very high, the optimal strategy suggested by the 4/2 model is slightly more aggressive than that in the Heston model, but still conservative compared to the 3/2 and Merton models. However, when the variance is larger than the long-term variance, the 3/2 model behaves the most prudently among all the other models.

Note that under the combined effects of $a$, $b$ and $v$, the variance of the 4/2 model and the factor $\frac{\sigma^2}{a+b}$ remain stable, therefore the optimal strategy in the 4/2 model does not change very much. On the contrary, the parameter $b \left(\sqrt{\frac{2a^2-\sigma^2}{2a}}\right)$ in the 3/2 model decreases in $\sigma$, so its variance $\frac{\sigma^2}{v}$ becomes larger, while the factor $\frac{v}{b}$ from its optimal strategy is smaller. This shows how sensitive the 3/2 model is to changes in $v$, while the 4/2 model has the advantage of combining the Heston and 3/2 models to absorb some instability in the market and avoid frequent or major rebalancing in risky assets.

Lastly, for a risk-averse investor ($\gamma < 0$, to avoid the discontinuity at 0), we plot the components of the optimal strategy implied by the 4/2 model against the risk-aversion parameter.


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\( \gamma \), separating the myopic and inter-temporal components (see Figure 2.10). As the investor becomes less risk-averse (i.e., as \( \gamma \) becomes less negative), the inter-temporal hedge demand remains small and almost constant, while the myopic allocation increases substantially.

### 2.6.2.3 Analysis of wealth-equivalent losses

Here we study the wealth-equivalent losses in an incomplete market setting. We plot the losses, as compared to the 4/2 model, for investors who wrongly follow the Heston, 3/2 model and Merton models, respectively (see Section 2.5).

The wealth-equivalent losses as functions of \( \sigma \) and \( \rho \) are presented in Figures 2.11 and 2.12, respectively. These figures indicate losses of up to 20% in scenarios of large negative leverage or small (for the 3/2 model) and large (for the 1/2 or Heston model) volatility of volatility. All three models show a decreasing tendency, which is milder for the 3/2 model (up to 16%), with correlation \( \rho \). In addition, comparing the Heston and Merton models, the former causes slightly lower losses than the latter for a more negative correlation, while the latter suffers slightly less with a positive \( \rho \).

![Figure 2.11: wealth-equivalent loss versus \( \sigma \)](image1)

![Figure 2.12: wealth-equivalent loss versus \( \rho \)](image2)

The influence of the market price of risk \( \bar{\lambda} \) is represented in Figure 2.13. This is one of the most challenging parameters due to the difficulties in estimating it. All the models suffer increasing wealth losses as the value of \( \bar{\lambda} \) becomes larger, with the Merton sustaining the worst losses (close to 100% loss of initial wealth), while the 3/2 suffers less than the Heston or the Merton model. However, even in the 3/2 model, the wealth-equivalent loss followed by high risk premium can incur a loss of up to 95%. This states means that the higher the value of the risk premium, the larger the cost from mistakenly assuming the wrong model.

The impact of the investment horizon \( T \) on welfare loss is shown in Figure 2.14. As expected, the wealth-equivalent loss in all three models increases with the investment horizon. For a 10-year investment period, the pronounced increase in losses can be as large as 42% of the initial wealth. In particular, the 3/2 model suffers the least, the Merton model suffers the most, and the Heston lies in between. It is worth mentioning that, as the investment horizon increases, the 3/2 model widens the differences in its less losses compared to the other two models.
2.6. Numerical analysis

Figure 2.13: Wealth-equivalent loss versus $\lambda$

Figure 2.14: Wealth-equivalent loss versus $T$

Figure 2.15: Myopic Loss versus $\lambda$ in the incomplete market

Figure 2.15 illustrates the wealth-equivalent loss from following a myopic strategy in an incomplete market, i.e., Equation (2.43), as a function of $\lambda$. This is by far the most impactful parameter compared to $\sigma$ and $\rho$; losses of up to 23% can easily be observed when $\lambda$ increases.

2.6.3 Complete market analysis

In this section, we implement sensitivity analyses in the complete market case for the 4/2 model. As before, we assume that the risk-aversion parameter $\gamma = -5$, the investment horizon $T = 4$, and all the other parameters involved employ the estimation result of the 4/2 model unless specified otherwise.

We perform the analyses for the complete market in terms of direct optimal exposures to the risk factor. Note that the optimal strategy/allocation for the risky asset and the option can be computed numerically using the matrix $G$.

2.6.3.1 Optimal exposures in the complete market

Figures 2.16 and 2.17 show the optimal exposures to risk factors $Z_1$ and $Z_2$ respectively. Using the relationship $\rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2 = \lambda$, we can assess the individual impacts of the risk premiums $\lambda_1$ and $\lambda_2$. Note that the variance risk should be negatively priced (i.e., $\lambda_1$ is negative), while $\lambda_2$ can be positive or negative as long as the excess return determined by $\lambda$ remains positive. With this in mind, we choose a value of $\lambda_1$ in $(-6, -0.5)$ and keep both $\lambda$ and $\rho$ the same as the estimation result of the 4/2 model, yielding values of $\lambda_2$ in $(-1.8, 4)$.

Figure 2.16 presents the optimal exposure $\eta_1$ plotted against $\lambda_1$, while the relationship between the optimal exposure $\eta_2$ and $\lambda_2$ is displayed in Figure 2.17. It is worth mentioning that all the conditions are ensured to be satisfied over the considered range of values for the examined parameter. Both optimal exposures increase significantly with their risk premiums in an absolute sense. That is, when the risk premium is negative, the optimal exposure is also negative, and the investor would correspondingly take a short position in the risk factor.

2.6.3.2 Myopic strategy and ignoring derivatives in the complete market

By following the myopic strategy in the complete market, the wealth-equivalent loss can be determined via Equation (2.44). Similar to the incomplete market case, $\lambda$ is the most important parameter and hence our focus. The losses are displayed in Figure 2.18.
To ensure comparability, we keep $\bar{\lambda}$ in the same range as that of the incomplete market, i.e., $(1, 10)$, and investigate the change in myopic loss from $\bar{\lambda}$ in terms of different choices of risk premium $\bar{\lambda}_2$. On the one hand, the myopic loss changes faster for a negative risk premium of the other risk factor, i.e., $\bar{\lambda}_2 = -1.5$, and it can approach 100% when the excess return is large. On the other hand, no matter how much $\bar{\lambda}_2$ varies, being short-sighted will always hurt more than that in the incomplete market by causing much higher losses for the same excess return of the risky asset.

Figure 2.18 shows the wealth-equivalent loss from not trading in derivatives compared to trading in derivatives in the complete market — i.e., Equation (2.47). The loss resulting from ignoring options changes faster for a negative value of $\bar{\lambda}_2$. That is, as the other risk factor is also negatively priced, a larger loss can be incurred for the same excess return of the risky asset. This is also observed in the myopic loss in the complete market. On the other hand, not trading options leads to more losses than myopic action does. This implies that, if one can invest in derivatives but chooses not to, it would cause larger losses among relevant suboptimal strategies. Lastly, Figure 2.20 demonstrates that the $1/2$ model carries the largest loss for ignoring derivatives trading: up to 60% over 10 years (see also Liu and Pan (2003)). On the other hand, losses for not trading in options are the lowest in the $3/2$ model (up to 30%) and up to 40% in the $4/2$ model. Therefore, despite the investment similarities between the $1/2$ and $4/2$ models, the latter leads to smaller losses.
2.7 Conclusion

This chapter conducted the first portfolio optimization analysis for the 4/2 model family, which includes the 3/2 and 1/2 models. We found closed-form solutions to all the quantities of relevance — e.g., optimal strategy, optimal wealth and value functions — not only for the primal expected utility problem, but also for several subproblems associated with popular suboptimal strategies. The existence and well-definiteness of the solution, as well as a verification theorem were provided for a full theoretical analysis of the model.

Using data from the S&P500 and the VIX, we estimated all four relevant models (i.e., the GBM, 1/2, 3/2 and 4/2) and compared their solutions. We found the 3/2 model to be the least aggressive in terms of risk-asset exposure among all four models (i.e., only 20% of Merton’s recommendation), which is reasonable due to the structure of the market price of risk. In addition, 4/2 investments are similar in value (albeit stochastic) to 1/2 ones, except in low-variance scenarios where the former allocate less to stocks. Despite the similarities between the 1/2 and 4/2 models, a wealth-equivalent suboptimal analysis highlights the larger losses when failing to complete the market in the 1/2 (60% over 10 years) compared to the 4/2 model (40%), with the 3/2 carrying the lower losses from ignoring derivatives.
Chapter 3

Robust portfolio choice under the 4/2 stochastic volatility model.

3.1 Introduction

This chapter is the first studying an expected utility investor which is not only risk-averse as represented by a (CRRA) utility function, but also ambiguity-averse, under the state-of-the-art stochastic volatility model named the 4/2 model in Grasselli (2017). Next we provide a brief overview of relevant work on the 4/2 model as well as ambiguity-aversion analyses.

Robust optimal control and its corresponding HJBI equations led to closed-form solutions for the first time in the work of Maenhout (2004). The author considered ambiguity-aversion in the diffusion of a GBM risky stock under the EUT. This opened the door to many extensions studying the impact of robustness on key state variables. Liu et al. (2005) considered ambiguity in the jumps of state variable dynamics, while Branger and Larsen (2013) considered ambiguity in the diffusion and the jump of a risky asset. Flor and Larsen (2014) explored ambiguity in both interest rate and a risky asset. Escobar et al. (2015) studied ambiguity aversion in a Heston stochastic volatility model with jumps in a univariate setting, while Faria and Correia-da Silva (2016) similarly extended another stochastic volatility model (Chacko and Viceira, 2005) for robust analysis. The former was further extended a) by Bergen et al. (2018) in a multivariate stochastic covariance setting, b) by Yang et al. (2020) for multi-factor stochastic volatility, and c) by Han and Wong (2020) for ambiguity in correlations. Finally, Chen et al. (2021) analyzed robust portfolio problems for commodity markets with ambiguity in stochastic interest rates.

The chapter is organized as follows. Section 3.2 formulates the optimal portfolio problem. Thereafter, Section 3.3 presents the optimal solution and analyzes the general results for the complete market. In Section 3.4, losses from suboptimal strategies are derived. Section 3.5 presents a real world analysis for a manager, which involves optimal allocations, worst case measures, and wealth equivalent losses. Section 3.6 concludes. The most important proofs are provided in the section B.1 in Appendix B. Complementary proofs, supplementary figures, and a full analysis of the incomplete market case are presented in the remainder of Appendix B.
3.2 Problem formulation

Assume that the financial market consists of one risk-free asset and one risky asset (i.e., stock), which can be traded continuously. Let all the stochastic processes introduced in this chapter be defined in a complete probability space \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0,T]}\)) where \{\mathcal{F}_t\}_{t \in [0,T]} is a right-continuous information filtration generated by the involved standard Brownian motions. The price process of the risk-free asset (Money Market) \(M_t\), evolves according to

\[
dM_t = rM_t dt, \quad M_0 = 1,
\]

where the interest rate \(r\) is assumed to be constant.

The price process \(S_t\) of the risky asset follows the so-called 4/2 model:

\[
dS_t = S_t \left[ \mu_t dt + (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) dW_t \right], \quad S(0) = S_0 > 0,
\]

\[
dv_t = \kappa (\theta - v_t) dt + \sigma \sqrt{v_t} dZ_{1t}, \quad v(0) = v_0 > 0,
\]

where \(v_t\) is the variance driver, which follows a Cox–Ingersoll–Ross (CIR) process with mean-reversion rate \(\kappa > 0\), long-run mean \(\theta > 0\), and volatility of volatility \(\sigma > 0\). The Feller condition (i.e., \(2\kappa \theta \geq \sigma^2\)), is also imposed to keep the process \(v_t\) strictly positive. The standard Brownian motions (BM) \(W_t\) and \(Z_{1t}\) are correlated with parameter \(\rho \in (-1, 1)\); hence, for convenience, we will write

\[
dW_t = \rho dZ_{1t} + \sqrt{1 - \rho^2} dZ_{2t},
\]

where \(Z_{2t}\) is another standard BM, independent of \(Z_{1t}\).

With the aim of establishing historical and risk-neutral measures for our models, we define market prices of risk next:

\[
\begin{aligned}
\lambda_1(v_t) &= \tilde{\lambda}_1 \sqrt{v_t} \\
\lambda_2(v_t) &= \tilde{\lambda}_2 \sqrt{v_t}
\end{aligned}
\]

where \(\lambda_1(v_t)\) is the market price of the idiosyncratic variance driver risk (i.e., with respect to \(Z_{1t}\)), \(\tilde{\lambda}_1\) is a constant and \(\lambda_2(v_t)\) is explicitly the market price of idiosyncratic stock risk (i.e., with respect to \(Z_{2t}\)), with constant \(\tilde{\lambda}_2\). Therefore, the drift process for the stock \(\mu_t\) can be established as follows:

\[
\mu_t = r + \tilde{\lambda}_1 \rho (av_t + b) + \tilde{\lambda}_2 \sqrt{1 - \rho^2} (av_t + b) = r + \tilde{\lambda}(av_t + b),
\]

where \(\tilde{\lambda} = \tilde{\lambda}_1 \rho + \tilde{\lambda}_2 \sqrt{1 - \rho^2}\) is constant and can be interpreted as a controller for the excess return.

That is, the price process \(S_t\) of the risky asset can be rewritten as

\[
dS_t = S_t \left[ (r + \tilde{\lambda}(av_t + b)) dt + (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}})(\rho dZ_{1t} + \sqrt{1 - \rho^2} dZ_{2t}) \right].
\]
Next, we assume that the investor can also allocate to an option on the underlying. Let \( O_t = m(S_t, v_t, t) \) denote the price of the option. It can be shown that the option price evolves as

\[
\frac{dO_t}{O_t} = rd_t + \frac{1}{O_t} \left( m_S \rho S_t + m_v \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \right) \left( \hat{\lambda}_1 \sqrt{v_t} dt + dZ_1 \right) \\
+ \frac{1}{O_t} \left( m_S \sqrt{1 - \rho^2 S_t} \right) \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \left( \hat{\lambda}_2 \sqrt{v_t} dt + dZ_2 \right),
\]

where \( m_S = \frac{dm}{dS} \) and \( m_v = \frac{dm}{dv} \) denote the partial derivatives of the option price function \( m \) with respect to \( S_t \) and \( v_t \) (see details in complementary Appendix B.2.1). The model in equations (3.6), (3.3), and (3.7) is referred to as the reference model.

As noted by Grasselli (2017), a risk-neutral measure may not exist in the \( 4/2 \) model, which is a feature inherited from the \( 3/2 \) component. This failure means that the discounted asset price process may be a strict \( Q \)-local martingale, and not a true \( Q \)-martingale equivalent to the historical measure \( P \). We thus need the following result to ensure the feasibility of changing measure under the market price of risk introduced in equation (3.4) for the reference model.

**Proposition 3.2.1.** The change of measure is well-defined for pricing purposes under the following conditions:

\[
\max \left\{ |\hat{\lambda}_1|, |\hat{\lambda}_2| \right\} < \frac{\kappa}{\sigma}, \quad (3.8) \\
\sigma^2 \leq 2\kappa \theta - 2|\sigma \rho b|, \quad (3.9) \\
\kappa + \sigma \rho a > 0, \quad (3.10) \\
\kappa + \sigma \hat{\lambda}_1 > 0. \quad (3.11)
\]

See Cheng and Escobar (2021) for a proof.

Assume that our investor is uncertain about the probability distribution for the reference model and considers a set of plausible, alternative models when making investment decisions. Specifically, the investor is uncertain about the joint distribution of \( Z_{1t} \) and \( Z_{2t} \).

Let \( e := (e^v, e^S) \) be an \( \mathbb{R}^2 \)-valued, \( \mathcal{F}_t \)-progressively measurable process and define the Radon-Nikodym derivative process by

\[
\xi_t = \frac{d\mathbb{P}_e}{d\mathbb{P}} |\mathcal{F}_t| = \exp \left\{ - \int_0^t \left( \frac{(e^v)^2 + (e^S)^2}{2} d\tau + e^v e^S dZ_{1\tau} + e^v e^S dZ_{2\tau} \right) \right\}, \quad (3.12)
\]

According to Girsanov’s theorem, the process

\[
\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix} = \begin{bmatrix} \int_0^t e^v d\tau \\ \int_0^t e^S d\tau \end{bmatrix} + \begin{bmatrix} \hat{Z}_{11} \\ \hat{Z}_{21} \end{bmatrix}, \quad (3.13)
\]

is a Wiener process under probability measure \( \mathbb{P}_e \). Let \( e[0, T] \) denote the set of all \( \mathcal{F}_t \)-progressively measurable processes such that the process (3.12) is a well-defined Radon-Nikodym derivative.
3.2. Problem formulation

We consider an ambiguity-averse agent with CRRA utility who aims to maximize the expected wealth. The wealth process can also be presented in terms of exposures to the risk factors of the stock and its variance driver (i.e., $Z_{1t}$ and $Z_{2t}$, respectively).

The alternative model is then as follows:

$$
\frac{dS_t}{S_t} = \left[ r + \lambda (av_t + b) - \rho (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) \bar{e}_t^\varphi - \sqrt{1 - \rho^2} (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) \bar{e}_t^\psi \right] dt \\
+ (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) \nu dt + \sqrt{1 - \rho^2 dZ_{1t}},
$$

$$
dv_t = (\kappa (\theta - v_t) - \sigma \sqrt{v_t} \bar{e}_t^i) dt + \sigma \sqrt{v_t} d\bar{Z}_{1t}, \tag{3.14}
$$

$$
\frac{dO_t}{O_t} = r dt + \frac{1}{O_t} \left[ \left( m_S \rho S_t + m_v \frac{\sigma \sqrt{v_t}}{a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}} \right) \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \left( \bar{\lambda}_1 \sqrt{v_t} - \bar{e}_t^i \right) dt + d\bar{Z}_{1t} \right] \\
+ \frac{1}{O_t} \left( m_S \sqrt{1 - \rho^2} S_t \right) \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \left( \bar{\lambda}_2 \sqrt{v_t} - \bar{e}_t^i \right) dt + d\bar{Z}_{2t}. 
$$

Note that the alternative model setup allows for different levels of ambiguity in the stock and its variance driver. This can be reasoned by the fact that the investor could use different sources to obtain information about the probability laws of the risky asset price and its variance driver’s process.

Let $\bar{\pi}^s$ be the fraction of wealth invested in the stock, $\bar{\pi}^o$ be the fraction of wealth invested in the option, and $(1 - \bar{\pi}^s - \bar{\pi}^o)$ be the remaining portion of wealth invested in the money market. The wealth $X_t$ of the investor can be written as follows:

$$
\frac{dX_t}{X_t} = \bar{\pi}^s \frac{dS_t}{S_t} + \bar{\pi}^o \frac{dO_t}{O_t} + (1 - \bar{\pi}^s - \bar{\pi}^o) r dt. \tag{3.15}
$$

The wealth process can also be presented in terms of exposures to the risk factors $\bar{Z}_{1t}$ and $\bar{Z}_{2t}$:

$$
\frac{dX_t}{X_t} = \left[ r + \Theta^s \bar{\lambda}_1 (av_t + b) - \Theta^o \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \bar{e}_t^i + \Theta^s \bar{\lambda}_2 (av_t + b) - \Theta^o \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \bar{e}_t^i \right] dt \\
+ \Theta^s \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) d\bar{Z}_{1t} + \Theta^o \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) d\bar{Z}_{2t}, \tag{3.16}
$$

where

$$
[\Theta^s] = \begin{bmatrix}
\rho \\
\sqrt{1 - \rho^2}
\end{bmatrix} \frac{1}{\partial_t} \left( m_S \rho S_t + m_v \frac{\sigma \sqrt{v_t}}{a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}} \right) \begin{bmatrix}
\bar{\pi}^s \\
\bar{\pi}^o
\end{bmatrix}. \tag{3.17}
$$

We consider an ambiguity-averse agent with CRRA utility who aims to maximize the expected...
utility from their terminal wealth $X_T$. Define the reward function realized when choosing an alternative model specified by $\epsilon$ as

$$ w^\epsilon(x, v; t; \Pi) = \frac{1}{\gamma} \mathbb{E}^{\Pi}_{t,x,v} [(X_T)^\gamma], \quad (3.18) $$

with $\Pi = (\Theta^\epsilon, \Theta^s)$, and define the indirect utility function as

$$ \bar{J}(x, v, t) = \sup_{\Pi \in \mathcal{U}} \inf_{\epsilon \in \mathcal{E}^{x,t,T}} I(x, v, t) \quad (3.19) $$

where

$$ I(x, v, t) = w^\epsilon(x, v; t; \Pi) + \mathbb{E}^{\Pi}_{t,x,v} \left[ \int_t^T \left( \frac{(e^\epsilon)^2}{2\Phi_1(\tau, X_\tau, v_\tau)} + \frac{(e^s)^2}{2\Phi_2(\tau, X_\tau, v_\tau)} \right) d\tau \right]. $$

$\bar{J}(x, v, t)$ is the value function, and $\mathcal{U}$ denotes the space of admissible strategies. This is the set of feedback strategies satisfying the following conditions: (1) $(\Theta^\epsilon, \Theta^s) \in \mathcal{U}$ are $\mathcal{F}_t$ progressively measurable; (2) Under $\Pi$, the wealth equation, i.e., equation (3.16), has a path-wise unique solution; (3) The integrability conditions necessary for the expectation in equation (3.19) to be well defined are satisfied; (4) The wealth $X_t \geq 0$ a.s. for all $t \in [0, T]$. The space $e_t$ of $\mathcal{F}_t$-adapted processes $e_t = (e^\epsilon, e^s) \in \mathbb{R}^2$ is the set of perturbations. The last two terms are the penalty for deviation from the reference model, in the sense of the relative entropy that arises from diffusion risk.

The perturbations $e^\epsilon_i$ and $e^s_i$ in the penalty term are scaled by $\Phi_1$ and $\Phi_2$, respectively. That is, the larger the values of $\Phi_1$ and $\Phi_2$, the smaller the penalties for deviating from the reference model, which implies that the investor is more uncertain about the model. We follow Maenhout (2004), assuming

$$ \Phi_i = \frac{\phi_i}{\gamma \bar{J}(x, v, t)}, \quad i = 1, 2, \quad (3.20) $$

where $\phi_i > 0$ denotes the ambiguity-aversion parameters. In this specification, as Maenhout (2004) demonstrated, the optimal strategy would be independent of the current wealth level for a power utility investor. For ease of explanation, $\phi_1$ can be interpreted as ambiguity aversion regarding the volatility driver, while $\phi_2$ represents ambiguity about the stock process. Similar interpretations were made by Escobar et al. (2015) for stock and volatility in a univariate model, by Bergen et al. (2018) for stock and volatility in a multivariate setting, and by Flor and Larsen (2014) for bond and stock.

### 3.3 The optimal investment strategies

In this section, we find the optimal wealth exposures in the complete market. If we can find the optimal wealth exposures $\Theta^\epsilon$ and $\Theta^s$ to the fundamental risk factors $\tilde{Z}_{1t}$ and $\tilde{Z}_{2t}$, then the corresponding optimal wealth weights $\bar{\pi}^\epsilon$ and $\bar{\pi}^s$ can be obtained as well.
The value function $\tilde{J}(x, v, t)$ satisfies the HJBI equation:

$$
\sup_{\psi, g} \inf_{\omega, \mu} \left\{ \tilde{J}_t + x \left[ r + \Theta^\gamma \lambda_1 (av + b) - \Theta^\gamma \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) e^\gamma + \Theta^\gamma \lambda_2 (av + b) - \Theta^\gamma \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) e_i^\gamma \right] \tilde{J}_x 
+ \frac{1}{2} x^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \left[ (\Theta^\gamma)^2 + \left( \Theta^\gamma \right)^2 \right] \tilde{J}_{xx} + \left[ \kappa (\theta - v) - \sigma \sqrt{v} e_i^\gamma \right] \tilde{J}_v 
+ \frac{1}{2} \sigma^2 v \tilde{J}_{vv} + \sigma x (av + b) \Theta^\gamma \tilde{J}_x v + \left( \psi^\gamma \right)^2 + \left( \psi_i^\gamma \right)^2 \right\} = 0,
$$

(3.21)

with boundary condition $\tilde{J}(x, v, T) = \frac{\bar{V}}{v}$. Here, $\bar{J}_t$, $\bar{J}_x$, $\bar{J}_v$, $\bar{J}_{xx}$, $\bar{J}_{vv}$, and $\bar{J}_{xv}$ are the first and second partial derivatives of function $\tilde{J}$ with respect to $t$, $x$, and $v$. The next proposition presents the solution to the complete market case.

**Proposition 3.3.1.** The solution to (3.21) is given by

$$
\tilde{J}(x, v, t) = \frac{x^\gamma}{\gamma} \exp \left\{ \tilde{A}(T - t) + \tilde{B}(T - t) v \right\},
$$

(3.22)

where $\tau(t) = T - t$, $\tilde{A}(\tau)$, and $\tilde{B}(\tau)$ are given by

$$
\tilde{B}(\tau) = \frac{\bar{k}_0 (e^{\overline{\lambda}_1} - 1)}{2 \bar{k}_3 + (\bar{k}_1 + \bar{k}_3) (e^{\overline{\lambda}_1} - 1)}
$$

$$
\tilde{A}(\tau) = \frac{2 \theta k}{\bar{k}_2} \ln \left( \frac{2 \bar{k}_3 e^{\frac{1}{3} \overline{\lambda}_1} \tau}{2 \bar{k}_3 + (\bar{k}_1 + \bar{k}_3) (e^{\overline{\lambda}_1} - 1)} \right),
$$

(3.23)

with auxiliary parameters

$$
\bar{k}_0 = -\frac{\gamma \overline{\lambda}_1^2}{(\gamma - 1 - \phi_1)} - \frac{\gamma \overline{\lambda}_2^2}{(\gamma - 1 - \phi_2)}, \bar{k}_1 = -\frac{(\phi_1 - \gamma) \sigma \overline{\lambda}_1}{(\gamma - 1 - \phi_1)} + \kappa, \bar{k}_2 = -\frac{(\phi_1 - \gamma) \sigma^2}{\gamma (\gamma - 1 - \phi_1)}, \bar{k}_3 = \sqrt{\bar{k}_1^2 - \bar{k}_0 \bar{k}_2}.
$$

(3.24)

The optimal exposures are given as

$$
(\Theta^\gamma)^* = \frac{v}{av + b} \left( \frac{\phi_1}{\gamma - 1 - \phi_1} \bar{B}(\tau) - \frac{\overline{\lambda}_1}{(\gamma - 1 - \phi_1)} \right)
$$

$$
(\Theta^\gamma)^* = \frac{v}{av + b} \left( \frac{-\overline{\lambda}_2}{\gamma - 1 - \phi_2} \right).
$$

(3.25)

Thereby, the optimal allocation follows

$$
(\vec{\lambda}^\gamma)^* = -\left( \frac{v}{av + b} \right) \frac{\overline{\lambda}_2}{(\gamma - 1 - \phi_2) \sqrt{1 - \rho^2}} - \frac{m_s S_i (\vec{\lambda}^0)^*}{\Omega_i},
$$

(3.26)

$$
(\vec{\lambda}^0)^* = \frac{O_i}{m_v \sigma} \left( \frac{\phi_1}{\gamma - 1 - \phi_1} \sigma \bar{B}(\tau) - \frac{\overline{\lambda}_1}{\gamma - 1 - \phi_1} + \frac{\rho \overline{\lambda}_2}{\sqrt{1 - \rho^2} (\gamma - 1 - \phi_2)} \right),
$$

(3.27)
where \(m_S \) and \(m_v \) are partial derivatives of the option price function \(O_t = m(S_t, v_t, t)\).

The worst case measure is determined by

\[
(e^v)^* = \phi_1 \left( \frac{-\sigma}{\gamma (\gamma - 1 - \phi_1)} B(\tau) - \frac{\lambda_1}{(\gamma - 1 - \phi_1)} \right) \sqrt{v}
\]

\[
(e^S)^* = \phi_2 \left( \frac{-\lambda_2}{\gamma - 1 - \phi_2} \right) \sqrt{v}.
\]  

(3.28)

See Appendix B.1.1 for the proof.

### 3.3.1 Optimal exposures

The dependence of the optimal exposures \(\Theta^v\) and \(\Theta^S\) in equation (3.25) on the factor \(\frac{v}{avb}\) is in line with Cheng and Escobar (2021). This factor is related to the driver of the stock’s variance, and its magnitude relies on the weights of the 1/2 and 3/2 components as well as the instantaneous volatility driver \(v_t\). Consistent with the existing literature, the optimal wealth exposures are composed of a myopic component and an inter-temporal hedging component under stochastic opportunity sets (Escobar et al., 2015; Chacko and Viceira, 2005; Yang et al., 2020). The myopic component of the volatility risk exposure \(\Theta^v\) is proportional to the risk premium of the stochastic volatility driver \(v_t\), while it is inversely proportional to the ambiguity parameter \(\phi_1\). According to Chernov and Ghysels (2000) and Bakshi and Kapadia (2003), variance risk is negatively priced, and thus \(\lambda_1 < 0\). Therefore, the stochastic volatility risk exposure is negative. The myopic component of additional stock risk exposure \(\Theta^S\) is proportional to the risk premium of additional stock risk, while it is inversely proportional to the ambiguity parameter \(\phi_2\). Note that \(\lambda_2\) can be negative or positive. For example, if \(\lambda_2\) is negative, then the additional diffusive price risk exposure is negative. Apart from this, as the ambiguity level \(\phi_1\) (\(\phi_2\)) increases, the wealth exposed to the myopic component in the volatility risk (additional stock risk) becomes less aggressive.

On the other hand, the inter-temporal hedging component in the volatility risk exposure is time-dependent and fading out in an absolute sense as the investment horizon shrinks. In particular, given that \(\gamma < 1\) and \(\phi_i > 0\) for \(i = 1, 2\), the hedging demand for the stochastic volatility driver risk is negative regardless of the level of risk aversion or ambiguity aversion. Moreover, the hedging demand becomes more negative as the ambiguity level \(\phi_1\) increases, which can be easily verified. Nevertheless, as \(\phi_1\) increases, the wealth exposure in stochastic volatility risk would not remain necessarily positive as it would depend on the magnitude of the stochastic risk premium \(\lambda_1\).

### 3.3.2 The worst case measure

The worst case probability measure, implied by the change of measure, \(e^* = ((e^v)^*, (e^S)^*)\) in the complete market is shown in equation (3.28). They are proportional to \(\sqrt{v}\), which coincides with the form of market price of the risk described in equation (3.4). As a result, the model misspecification can be viewed as ambiguity about the mis-evaluated risk premiums. Therefore, instead of considering \(\lambda_1\) and \(\lambda_2\), an ambiguity-averse investor considers \(\lambda_1 - \frac{\sigma}{\gamma}\).
Proposition 3.3.2. For the optimal Radon-Nikodym derivative $\xi_t^*$ in the complete market to be a well-defined density, parameters should satisfy the following condition:

$$\sup_{0 < t < T} K(T - t) \leq \frac{\kappa^2}{\sigma^2}$$

(3.29)

where $K(T - t) = \phi_1^2 \left( \frac{\sigma^2}{\gamma (\gamma - 1 - \phi_1)} B^2(T - t) + \frac{2 \sigma \lambda_1}{\gamma (\gamma - 1 - \phi_1)} B(T - t) \right) + \left( \phi_1^2 \frac{\lambda_1^2}{\gamma (\gamma - 1 - \phi_1)} + \phi_2^2 \frac{\lambda_2^2}{\gamma (\gamma - 1 - \phi_1)} \right)$ with the function $B(T - t)$ from equation (3.23). Moreover, the function $\bar{J}(x, v, t)$ is well-defined if the parameters satisfy the following technical conditions:

for $\gamma < 0$, then $\bar{J}(x, v, t)$ is real-valued if

$$-\frac{(\phi_1 - \gamma) \sigma^2 \lambda_1}{(\gamma - 1 - \phi_1)} + \left( \frac{\gamma \lambda_1^2}{(\gamma - 1 - \phi_1)} + \frac{\gamma \lambda_2^2}{(\gamma - 1 - \phi_2)} \right) - \frac{(\phi_1 - \gamma) \sigma^2}{\gamma (\gamma - 1 - \phi_1)} > 0. \quad (3.30)$$

For $0 < \gamma < 1$, then $\bar{J}(x, v, t)$ is finite if

$$\left( \frac{\gamma \lambda_1^2}{(\gamma - 1 - \phi_1)} + \frac{\gamma \lambda_2^2}{(\gamma - 1 - \phi_2)} \right) - \frac{(\phi_1 - \gamma) \sigma^2}{\gamma (\gamma - 1 - \phi_1)} > 0. \quad (3.31)$$

See Appendix B.1.2 for the proof.

To ensure the optimal control is the unique solution and its associated value function solves the optimal problem, a verification result is provided in the next proposition. This result builds upon Kraft et al. (2013) (Appendix C), and the generalization to sup-inf problems in Pu and Zhang (2021) (Proposition 3.5 and theorem 4.2). For simplicity, we consider a convenient sub-class of admissible strategies defined as follows:

$$\mathcal{A}_t := \left\{ \left( \Theta^v, \Theta^s; e^v, e^s \right) : \left( \Theta^v, \Theta^s \right) \in \mathcal{U}_t, \ e = \left( \frac{e^1}{\sqrt{\nu}}, \frac{e^2}{\sqrt{\nu}} \right) \in \mathcal{V}_t \right\};$$

$$|\Theta_1|, |\Theta_2|, |e^1|, |e^2|$$ are bounded by $K^v, K^s, K^e$, $K^e$ respectively; $X_t \geq 0$ for $t \in [0, T]$.

(3.32)

where $K^v, K^s, K^e$, and $K^e$ are positive, finite real numbers.

Proposition 3.3.3. Assume the following six conditions, denoted (i), (ii), (H1), (H2), (H3) and (H4), hold:
(i) For any \((\Theta^a, \Theta^b) \in \mathcal{U}, a = (e^a, e^b) \in e_i,\) equations (3.16)-(3.21) lead to a unique strong solution \((x_t, \bar{J}_t)_{t \in [T]}:\)

(ii) There exist admissible \(e^* \in e_i\) and \(((\Theta^a)^*, (\Theta^b)^*) \in \mathcal{U},\) satisfying, for all \((x, v, t) \in \mathbb{R}^+ \times \mathbb{R} \times [t, T]:\)

\[
\bar{J}_t + x \left[ r + \Theta^a \bar{\lambda}_1 (av + b) - (\Theta^a)^* \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) e^a + (\Theta^b)^* \bar{\lambda}_2 (av + b) - (\Theta^b)^* \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) (e^b)^* \right] \bar{J}_x
+ \frac{1}{2} v^2 \left[ (\Theta^a)^* \right]^2 + \left( (\Theta^b)^* \right)^2 \bar{J}_{xx} + \left[ k(\theta - v) - \sigma \sqrt{v} (e^*) \right] \bar{J}_x,
\]

\[
+ \frac{1}{2} \sigma^2 v \bar{J}_{vv} + \sigma x (av + b)(\Theta^a)^* \bar{J}_{vx} + \frac{(e^*)^2}{2} \left( e^2 \right)_t = 0,
\]

\[(3.33)\]

(H1) \(q > 2\) such that \(q - 2\) is sufficiently small with \(\gamma < 0\) or \(0 < \phi_1 < \gamma < 1,\)

(H2) \(2|\gamma| \sigma K^v - K^{e^*} < \kappa,\)

(H3) \(-40 \sigma^2 \gamma^2 \left( \frac{1}{2} (K^v)^2 + (K^e)^2 \right) - \left( K^v \bar{\lambda}_1 - K^e \bar{\lambda}_2 - K^v K^{e^*} - K^e K^{e^*} \right) \leq \kappa^2,\)

(H4) \(4 \gamma^2 \sigma^2 \left( (K^v)^2 + (K^e)^2 \right) < \kappa^2,\)

where \(K^v, K^s, K^{e^*}, K^{e^*},\) and \(\kappa\) are defined as follows:

\[
K^v := \left| \frac{\bar{\lambda}_1}{\gamma - 1 - \phi_1} - \frac{\phi_1}{\gamma - 1 - \phi_1} \right|, 
\]

\[(3.34)\]

\[
K^s := \left| \frac{\bar{\lambda}_2}{\gamma - 1 - \phi_2} \right|, 
\]

\[(3.35)\]

\[
K^{e^*} := \phi_1 \left| \frac{-\sigma \bar{B}(T)}{\gamma(\gamma - 1 - \phi_1)} - \frac{\bar{\lambda}_1}{\gamma - 1 - \phi_1} \right|, 
\]

\[(3.36)\]

\[
K^{e^*} := \phi_2 \left| \frac{\bar{\lambda}_2}{\gamma - 1 - \phi_2} \right|, 
\]

\[(3.37)\]

\[
\kappa := \kappa - 2 |\gamma| \sigma K^v + K^{e^*}. 
\]

Then \(((\Theta^a)^*, (\Theta^b)^*, e^*)\) is the optimal control and \(\bar{J}\) is the value function of the optimization problem (3.21).

See Appendix B.1.3 for the proof.

### 3.4 Losses from suboptimal strategies

In this section, we analyze some suboptimal strategies with regard to the wealth-equivalent utility loss in a setting of a complete market. Specifically, we consider the suboptimal allocations from a) ignoring model uncertainty, b) not trading in derivatives to complete the market, and c) assuming either a 1/2 (Heston) model or a Merton (GBM) model.
The indirect utility function of an investor who follows a suboptimal strategy is denoted by \( \bar{J}^{(s)} \), with

\[
\bar{J}^{(s)}(x, v, t) = \inf_{\varepsilon \in \mathcal{U}(t, t)} \left\{ w^*(x, v, t; \varepsilon^{(s)}) + \mathbb{E}_t^w \left[ \int_t^T \left( \frac{(e^*_r)^2}{2\Phi_1(\tau, X_\tau, v_\tau)} + \frac{(e^*_e)^2}{2\Phi_2(\tau, X_\tau, v_\tau)} \right) d\tau \right] \right\},
\]

where \( \bar{J}^{(s)}(x, v, t) \) is the value function, and the suboptimal \( \Pi^{(s)} \) is in the space of admissible strategies \( \mathcal{U} \). In addition, the penalty terms that are scaled by \( \Phi_i \)'s are assumed to be

\[
\Phi_i = \frac{\phi_i}{\gamma^i \bar{J}^{(s)}(x, v, t)}, \quad i = 1, 2,
\]

with ambiguity-aversion parameters \( \phi_i > 0 \). This choice of penalty function ensures the suboptimal problem is embedded and compatible with the corresponding optimal problem. For suboptimal strategies that make different assumptions on ambiguity compared to the optimal strategy, parameters \( \phi_i, i = 1, 2 \) are different from the correct \( \phi_i, i = 1, 2 \), which are used in the optimal strategy in equation (3.25). By definition, the relation between the value function in a complete market \( J(x, v, t) \) and \( \bar{J}^{(s)}(x, v, t) \) is \( J \geq \bar{J}^{(s)} \), with equality only when the investment allocation is optimal. The value function \( \bar{J}^{(s)}(x, v, t) \) satisfies the robust HJB equation:

\[
\inf_{\varepsilon^+, \varepsilon^-} \left\{ \bar{J}^{(s)} + x \left[ r + \Theta^x \lambda_1(\alpha v + b) - \Theta^x \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) e^\gamma + \Theta^x \lambda_2(\alpha v + b) - \Theta^x \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) e^\gamma \right] \bar{J}^{(s)} \right\} \\
+ \frac{1}{2} \left[ a \sqrt{v} + \frac{b}{\sqrt{v}} \right]^2 \left[ (\Theta^x)^2 + (\Theta^x)^2 \right] \bar{J}^{(s)} + \left[ \sigma^2 \sqrt{v} \bar{J}^{(s)} + \frac{(e^\gamma)^2}{2\Phi_1} + \frac{(e^\gamma)^2}{2\Phi_2} \right] = 0,
\]

with boundary condition \( \bar{J}^{(s)}(v, T) = \frac{v^\gamma}{\gamma} \). The solution for this generic case is presented next.

**Proposition 3.4.1.** Assume that \( \Pi^{(s)} \) is a function of \( (v, t) \); a candidate solution to PDE (3.41) is then given by

\[
\bar{J}^{(s)}(x, v, t) = \frac{x^\gamma}{\gamma} \bar{h}^{(s)}(v, t),
\]

where the function \( \bar{h}^{(s)} \) satisfies the PDE:

\[
\bar{h}^{(s)} + \left[ r + \Theta^x \lambda_1(\alpha v + b) + \Theta^x \lambda_2(\alpha v + b) \right] \bar{h}^{(s)} \\
+ \frac{1}{2} \left[ a \sqrt{v} + \frac{b}{\sqrt{v}} \right]^2 \left[ (\Theta^x)^2 + (\Theta^x)^2 \right] \gamma(\gamma - 1) \bar{h}^{(s)} + \kappa(\theta - v) \bar{h}^{(s)} + \frac{1}{2} \sigma^2 \sqrt{v} \bar{h}^{(s)} + \sigma(\alpha v + b)\Theta^x \gamma \bar{h}^{(s)} \\
- \frac{\phi_1}{2} \left[ (\Theta)^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \gamma \bar{h}^{(s)} + \sigma^2 \sqrt{v} \frac{1}{\gamma} \left( \bar{h}^{(s)} \right)^2 + 2\Theta^x(\alpha v + b)\sigma \bar{h}^{(s)} \right] - \frac{\phi_2}{2} \left[ (\Theta^x)^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \gamma \bar{h}^{(s)} \right] = 0,
\]

with boundary condition \( \bar{h}^{(s)}(v, T) = 1 \).
Corollary 3.4.2. must be in the form of equation (3.25) but with averse investor. Denote the suboptimal strategy of ignoring model uncertainty as $\Pi^{(s)}(\cdot)$. The loss from ignoring model uncertainty is popular in the literature, see Flor and Larsen (2014); Larsen and Munk (2012). This measure is different from the certainty-equivalent wealth defined in Liu and Pan (2003), nonetheless Larsen and Munk (2012) shows a one-to-one correspondence between the two measures.

The specific solutions for the cases of interest are presented in the upcoming subsections.

### 3.4.1 The loss from ignoring model uncertainty

In this subsection, we compute the wealth loss from ignoring model uncertainty for an ambiguity-averse investor. Denote the suboptimal strategy of ignoring model uncertainty as $\Pi^{(s)}(\cdot)$; this must be in the form of equation (3.25) but with $\phi_1 = \phi_2 = 0$:

\[
\begin{align*}
(\Theta^y)^{(s)} & = \frac{v}{a v + b} \left( -\sigma B(\tau) - \bar{\lambda}_1 \right) \\
(\Theta^s)^{(s)} & = \frac{v}{a v + b} \left( -\bar{\lambda}_2 \right).
\end{align*}
\]  

(3.45)

**Corollary 3.4.2.** For complete markets, $\bar{h}^{(s)}(v, t)$ is in the exponential-affine form, such as $\bar{h}^{(s)}(v, t) = \exp(A^{(s)}(\tau) + B^{(s)}(\tau)v)$, where $\tau(t) = T - t$ and

\[
\begin{align*}
B^{(s)}(\tau) & = \frac{k_0^{(s)}}{2k_3^{(s)}} \left( e^{\bar{\lambda}_1\tau} - 1 \right) + \frac{k_1^{(s)}}{2k_3^{(s)}} \left( e^{\bar{\lambda}_2\tau} - 1 \right) \\
A^{(s)}(\tau) & = \gamma \tau + \frac{2\theta k}{k_2^{(s)}} \ln \left( \frac{2k_3^{(s)} e^{\bar{\lambda}_3\tau}}{2k_3^{(s)} + (k_1^{(s)} + k_3^{(s)})(e^{\bar{\lambda}_3\tau} - 1)} \right).
\end{align*}
\]  

(3.46)
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with

\[
\bar{k}^{(s)}_1 = \frac{\kappa - \sigma}{(y - 1)} \left( \frac{-\sigma \bar{B}(\bar{t}) - \bar{\lambda}_1}{(y - 1)} \right), \quad \bar{k}^{(s)}_2 = \sigma^2 (1 - \frac{\bar{\phi}_1}{\gamma}), \quad \bar{k}^{(s)}_3 = \sqrt{\left(\bar{k}^{(s)}_1\right)^2 - \bar{k}^{(s)}_0 \bar{k}^{(s)}_2}. 
\]

Moreover, the WEL follows

\[
\bar{L}^{(s)}(v, t) = 1 - \exp \left\{ \frac{1}{\gamma} \left[ \bar{A}^{(s)}(\tau) + \bar{B}^{(s)}(\tau) \right] \right\}, 
\]

where the functions \( \bar{A}(T - t) \) and \( \bar{B}(T - t) \) follow equation (3.23), which characterizes the value function \( \bar{J}(x, v, t) \) under the optimal strategy in the complete market, while the functions \( \bar{A}^{(s)}(T - t) \) and \( \bar{B}^{(s)}(T - t) \) follow equation (3.46), which characterizes the value function \( \bar{J}^{(s)}(x, v, t) \) under the suboptimal strategy. Furthermore, the worst case measure under the suboptimal strategy \( \Pi^{(s)} \) follows

\[
\left\{ \begin{array}{l}
\left( e^{v^{(s)}} \right) = \bar{\phi}_1 (\Theta^{(s)}(t) (a \sqrt{v} + b \sqrt{\gamma}) + \sigma \sqrt{\gamma} \bar{B}^{(s)}(t)) \\
\left( e^{v^{(s)}} \right) = \bar{\phi}_2 (\Theta^{(s)}(t) (a \sqrt{v} + b \sqrt{\gamma})) 
\end{array} \right.
\]

where \( (\Theta^{v})^{(s)} \) and \( (\Theta^{\delta})^{(s)} \) correspond to a suboptimal strategy \( \Pi^{(s)} \) in equation (3.45).

See Appendix B.2.3 for the complete proof.

3.4.2 The loss from not trading in derivatives

Denote the suboptimal strategy of no allocation in derivatives as \( \Pi^{(2)} \), and note that it is in the form of equation (3.17) with \( \bar{\pi}^\delta = 0 \), and \( \bar{\pi}^\rho = -\left( \frac{v}{av + b} \right) \frac{\lambda_2}{(y - 1 - \bar{\phi}_2) \sqrt{1 - \rho^2}} \) in equation (3.26):

\[
\left[ \begin{array}{c}
(\Theta^{v})^{(2)} \\
(\Theta^{\delta})^{(2)} 
\end{array} \right] = \left[ \begin{array}{c}
\rho \\
\sqrt{1 - \rho^2} 
\end{array} \right] \left[ \begin{array}{c}
\frac{1}{\sigma} \bar{m}_S t + m_1 \sqrt{\bar{v}} \\
\frac{1}{\sigma} \bar{m}_S \sqrt{1 - \rho^2} S_1 
\end{array} \right] \bar{\pi}^\rho = \left[ \begin{array}{c}
\rho \bar{\pi}^\rho \\
\sqrt{1 - \rho^2} \bar{\pi}^\rho 
\end{array} \right].
\]

\[
\left[ \begin{array}{c}
\bar{v}^\rho \\
\bar{v}^\rho 
\end{array} \right] = \left[ \begin{array}{c}
\frac{\rho \bar{\pi}^\rho}{\sqrt{1 - \rho^2} \bar{\pi}^\rho} \\
\frac{\rho \bar{\pi}^\rho}{\sqrt{1 - \rho^2} \bar{\pi}^\rho} 
\end{array} \right].
\]

Corollary 3.4.3. For complete markets, \( \bar{h}^{(s)}(v, t) \) is in the exponential-affine form, such that \( \bar{h}^{(s)}(v, t) = \exp(\bar{A}^{(s)}(\tau) + \bar{B}^{(s)}(\tau)v) \), where \( \tau(t) = T - t \) and

\[
\bar{B}^{(s)}(\tau) = \frac{\bar{\delta}_0}{2 \bar{k}_3} \left( e^{\left(\gamma\tau + \frac{2\theta_k}{\bar{k}_2}\right)} - 1 \right)
\]

\[
\bar{A}^{(s)}(\tau) = \gamma \tau + \frac{2\theta_k}{\bar{k}_2} \ln \left( \frac{e^{\gamma \tau} e^{\left(\gamma \tau + \frac{2\theta_k}{\bar{k}_2}\right)/2}}{2 \bar{k}_3 \left( e^{\left(\gamma \tau + \frac{2\theta_k}{\bar{k}_2}\right)/2} + \bar{k}_1 + \bar{k}_2 \left( e^{\left(\gamma \tau + \frac{2\theta_k}{\bar{k}_2}\right)/2} - 1 \right) \right)} \right),
\]

(3.51)
with

\[
\bar{\kappa}_0^{(s)} = -2 \left[ \frac{\rho \gamma \lambda_1 \lambda_2}{(1 - \rho^2)(1 - \phi_2)} + \frac{\gamma \lambda_1^2}{(1 - \phi_2)} \right] + \gamma(\gamma - 1) \left[ \frac{\rho^2 \lambda_2^2}{(1 - \rho^2)^2} + \frac{\lambda_1^2}{(1 - \phi_2)^2} \right] \\
- \phi_1 \left[ \frac{\rho \gamma \lambda_1 \lambda_2}{(1 - \rho^2)(1 - \phi_2)^2} \right] - \phi_2 \left[ \frac{\gamma \lambda_1^2}{(1 - \phi_2)^2} \right]
\]

\[
\bar{\kappa}_1^{(s)} = \kappa + (\gamma - \phi_1) \left( \frac{\sigma \rho \lambda_2}{(1 - \phi_2)(1 - \rho^2)} \right)
\]

\[
\bar{\kappa}_2^{(s)} = \sigma^2 (1 - \phi_1) \frac{(1 - \gamma)(\lambda_1)}{\gamma}
\]

\[
\bar{\kappa}_3^{(s)} = \sqrt{(\bar{\kappa}_1^{(s)})^2 - \bar{\kappa}_0^{(s)} \bar{\kappa}_2^{(s)}}.
\]

Moreover, the WEL follows

\[
L^{(s)}(\nu, t) = 1 - \exp \left\{ \frac{1}{\gamma} \left[ (\bar{A}^{(s)}(\tau) - \bar{A}(\tau)) + (\bar{B}^{(s)}(\tau) - \bar{B}(\tau)) \nu \right] \right\},
\]

where \(\bar{A}(T - t)\) and \(\bar{B}(T - t)\) follow equation (3.23), while \(\bar{A}^{(s)}(T - t)\) and \(\bar{B}^{(s)}(T - t)\) follow equation (3.51). Furthermore, the worst case measure under suboptimal strategy \(\Pi^{(s)}\) follows

\[
\begin{align*}
(e^\nu)^{(s)} &= \phi_1 \left[ (\Theta^\nu)^{(s)}(a \sqrt{\nu} + \frac{b}{\sqrt{\nu}}) + \sigma \sqrt{\nu} \bar{B}^{(s)}(\tau) \right] \\
(e^\delta)^{(s)} &= \phi_2 \left[ (\Theta^\delta)^{(s)}(a \sqrt{\nu} + \frac{b}{\sqrt{\nu}}) \right]
\end{align*}
\]

where \((\Theta^\nu)^{(s)}\) and \((\Theta^\delta)^{(s)}\) correspond to a suboptimal strategy \(\Pi^{(s)}\) in equation (3.50).

See Appendix B.2.4 for the complete proof.

### 3.4.3 The loss from the Heston strategy

Denote the suboptimal strategy of ignoring \(b\) (i.e., following the Heston strategy) in the complete market as \(\Pi^{(h)}\), which can be obtained from setting \(b = 0\) in equation (3.25).

**Proposition 3.4.4.** Assume Heston strategy \(\Pi^{(h)}\) in equation (3.42) with \(\phi_1 = 0\) (ignoring ambiguity about the volatility driver) and either satisfying \(\lambda_1 = 0\) or being myopic. Then consider the following conditions:

\[
- \frac{1}{2} \frac{(\lambda_2^2 \gamma)}{(1 - \phi_2)} > -\kappa^2 \frac{1}{2a^2} \frac{b^2 \lambda_2^2 \gamma}{2((1 - \phi_2))} + \frac{(2\kappa \theta - \sigma^2)^2}{8\sigma^2},
\]

\[
0 < \frac{1}{2\sigma^2} \left( 2\kappa \theta + \sigma^2 \sqrt{(2\kappa \theta - \sigma^2)^2 + 8\sigma^2 \gamma} \right), 0 \geq -\frac{\sqrt{\kappa^2 + 2\mu \sigma^2 + \kappa}}{\sigma^2}.
\]
The solution to $\bar{h}\est{3}$ in equation (3.42) can be written as

$$
\bar{h}\est{3}(v, t) = e^{\gamma(T-t)}q(\tau, v; \alpha, \lambda, \mu, \nu) \left( \frac{\beta(\tau, v)}{2} \right)^{m+1} v^{-\frac{\mu}{\sigma^2}} (\lambda + K(\tau))^{-\left(\frac{1}{2} + \frac{\sigma}{\sigma^2} + \frac{\mu}{\sigma^2}\right)}
\times e^{\frac{1}{\sigma^2}\left(\frac{1}{2}\theta^2 \tau - \sqrt{\nu} \coth(\nu + \nu)\right)^{(1/2 + \frac{\sigma}{\sigma^2} + \frac{\mu}{\sigma^2})}} \times \frac{1}{\Gamma(\tau - t)} \times _1F_1\left(\frac{1}{2} + m - \alpha + \frac{\mu}{\sigma^2} m + 1, \frac{\beta(\tau, v)^2}{4(\lambda + K(\tau))}\right),
$$

with

$$
m = \frac{1}{\sigma^2} \sqrt{(2\nu - \sigma^2)^2 + 8\nu^2 \frac{b^2 \lambda^2 \gamma}{2(1 - \phi_2) a^2}}, D = \kappa^2 + 2\left(1 - \frac{1}{2(1 - \phi_2) a^2}\right)\nu^2,
$$

$$
\beta(\tau, v) = \frac{2\nu}{\sigma^2 \sinh(\nu)}, K(\tau) = \frac{1}{\sigma^2} \left(\sqrt{\nu} \coth(\frac{\sqrt{\nu}}{2}) + \kappa\right),
$$

where $\Gamma$ and $\gamma$ denote the gamma and hypergeometric confluent function, respectively. Moreover, the WEL can be formulated as

$$
\bar{L}\est{3}(v, t) = 1 - \left(\frac{\bar{h}\est{3}(v, t)}{\bar{h}(v, t)}\right)^{\frac{1}{2}},
$$

where $\bar{L}\est{3}$ and $\bar{h}$ characterize the value function $\bar{J}\est{3}$ for suboptimal strategies $\Pi\est{3}$ and $\bar{J}$, respectively.

The proof can be found in the complementary Appendix B.2.5.

As stated in the previous proposition, the explicit solution to $\bar{h}\est{3}$ under the Heston strategy is only available in a special case. Moreover, when $\Pi\est{3}$ is assumed, there is no exponential-affine structure for the corresponding function $\bar{h}\est{3}(v, t)$; hence, we cannot produce the equivalent loss in closed form. Nonetheless, we can find an approximation. Given that $\bar{J} \geq \bar{J}\est{3}$, with equality only when the investment allocation is optimal and $\Phi_i = \frac{\delta_i}{\gamma B_{0}(y,v)}$, $i = 1, 2$, the penalty term from equation (3.39) can be bounded as follows:

$$
E_t^{\mathcal{P}} \left[ \int_t^T \left\{ \frac{(e^\delta_i)^2 \gamma^2 \bar{J}\est{3}(X_\tau; v_\tau, \tau)}{2\Phi_1} + \frac{(e^\delta_i)^2 \gamma^2 \bar{J}\est{3}(X_\tau; v_\tau, \tau)}{2\Phi_2} \right\} d\tau \right] 
\leq E_t^{\mathcal{P}} \left[ \int_t^T \left\{ \frac{(e^\delta_i)^2 X^2 e^{\delta_1(T-\tau)+B_1(T-\tau)v_\tau}}{2\Phi_1} + \frac{(e^\delta_i)^2 X^2 e^{\delta_1(T-\tau)+B_1(T-\tau)v_\tau}}{2\Phi_2} \right\} d\tau \right],
$$

where the functions $\bar{A}(T - t)$ and $\bar{B}(T - t)$ characterize the value function under the optimal strategy, namely, equation (3.23). That is, we can find an upper bound for the penalty term of the value function under the suboptimal strategy. We will show the upper bound of the penalty term numerically, and we will demonstrate that it is negligible. The methodology of estimating losses from following the Heston strategy therefore starts by substituting the
suboptimal strategy $\Pi^{(s)}$ into the wealth process in equation (3.16), hence producing $X_T^{W(s)}$. The value function in equation (3.39) can then be approximated via simulation as an expectation without the penalty term: $J(s)(x, t, v) = \mathbb{E}_{x,t,v} \left[ u(X_T^{W(s)}) \right]$. Lastly, as $J(x, v, t)$ denotes the value function under the optimal strategy (known in closed form), we have

$$J(x, v, t) = J(x(1 - L), v, t) = \frac{(x(1 - L))^\gamma}{\gamma} e^{\bar{A}(\tau) + B(\tau)v},$$

where $J$ is given in (3.22). The WEL can be computed as

$$\bar{L} = 1 - \left( \frac{\gamma}{\gamma} \mathbb{E}_{x,v,t} \left[ u(X_T^{W(s)}) \right] \right)^{\frac{1}{\gamma}}. \quad (3.60)$$

### 3.4.4 The loss from the Merton strategy

Due to the importance of Merton’s work, we study the suboptimal strategy of following Merton’s solution with ambiguity in this subsection. Denote the suboptimal strategy of following Merton’s solution in a 4/2 model setting as $\Pi^{(s)}$, which can be obtained by setting $\sigma = 0$ and $\rho = 0$ in equation (3.25), since there is no stochastic volatility:

$$(\Theta^v)^{(s)} = 0, (\Theta^s)^{(s)} = \frac{-\bar{\lambda}_M}{(\gamma - 1 - \phi_2)}, \quad (3.61)$$

where $\bar{\lambda}_M = \frac{\bar{\lambda}_M}{(\gamma - 1 - \phi_2)}$, with $\theta$ being the mean-reverting level of our stochastic volatility process. Thus, $\theta$ is constant, and so is $\bar{\lambda}_M$, which characterizes the market price of risk in Merton’s model.

**Proposition 3.4.5.** Assume $\Pi^{(s)}$ in equation (3.42), and consider the following conditions

$$\frac{\bar{\lambda}_M \bar{a} \gamma a}{(\gamma - 1 - \phi_2)} + \frac{1}{2} a^2 \frac{\bar{\lambda}_M^2 \gamma}{(\gamma - 1 - \phi_2)} > -\frac{k^2}{2} \frac{1}{2} b^2 \frac{\bar{\lambda}_M^2 \gamma}{(\gamma - 1 - \phi_2)} \geq \frac{(2k\theta - \sigma^2)^2}{8\sigma^2}, \quad (3.62)$$

$$0 < \frac{1}{2\sigma^2} (2k\theta + \sigma^2 \sqrt{(2k\theta - \sigma^2)^2 + 8\sigma^2 \bar{\lambda}_M^2}) \geq -\frac{\sqrt{k^2 + 2\mu\sigma^2 + \kappa}}{\sigma^2}.$$

Then, the solution to $\tilde{h}^{(s)}$ in equation (3.42) can be written as

$$\tilde{h}^{(s)}(v, t) = \left( \frac{\beta(\tau, v)}{2} \right)^{m+1} v^{-\frac{m}{2}} (\lambda + K(\tau))^{-(m+\frac{m+1}{2})} \times e^{\frac{\beta(\tau, v)\tau}{2}} \frac{\frac{1}{\gamma}}{\gamma(\tau-t)} + \gamma(\tau-t) \times \times \left( \frac{1}{2} + m - \alpha + \frac{k\theta}{\sigma^2}, m + 1, \frac{\beta(\tau, v)\tau}{4(\lambda + K(\tau))} \right). \quad (3.63)$$
3.5 Managerial example

<table>
<thead>
<tr>
<th>Table 3.1: Parameter values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{k}$</td>
</tr>
<tr>
<td>4/2 model</td>
</tr>
<tr>
<td>Heston model</td>
</tr>
</tbody>
</table>

with

$$
\begin{align*}
\bar{m} &= \frac{1}{\sigma^2} \sqrt{(2\kappa \theta - \sigma^2)^2 + 8\sigma^2 \left( \frac{1}{2} b^2 \frac{\lambda_M^2}{(\gamma - 1 - \phi_2)} \right)}, \\
D &= \kappa^2 + 2 \left( \frac{\lambda_M \tilde{\lambda} a \gamma a}{(\gamma - 1 - \phi_2)} + \frac{1}{2} a^2 \frac{\lambda_M^2}{(\gamma - 1 - \phi_2)} \right) \sigma^2, \\
\beta(\tau, \nu) &= \frac{2 \sqrt{Dv}}{\sigma^2 \sinh \left( \frac{\sqrt{Dv}}{2} \right)}, \\
K(\tau) &= \frac{1}{\sigma^2} \left( \sqrt{D} \coth \left( \frac{\sqrt{Dv}}{2} \right) + \kappa \right),
\end{align*}
$$

(3.64)

where $\Gamma$ and $\, _1F_1$ denote the gamma and hypergeometric confluent functions, respectively. Moreover, the WEL can be formulated as

$$
\bar{L}^{(v)}(v, t) = 1 - \left( \frac{\bar{h}^{(v)}(v, t)}{\bar{h}(v, t)} \right)^{\frac{1}{\alpha}},
$$

(3.65)

where $\bar{h}^{(v)}$ and $\bar{h}$ characterize the value function $\bar{J}^{(v)}$ for the suboptimal strategies $\Pi^{(v)}$ and $\bar{J}$, respectively.

The proof can be found in the complementary Appendix B.2.6.

3.5 Managerial example

In this section, we do a complete managerial analysis using real data estimates (Table 3.1) as reported in Cheng and Escobar (2021), which is based on S&P 500 and VIX data from January 2010 to the last day of December 2019. To report the practical implications of our findings for a manager, we also consider ambiguity-aversion parameters in a region recommended by Anderson et al. (2003).

Note that the variance risk is negatively priced (i.e., $\lambda_1 < 0$), while $\lambda_2$ can be negative or positive as long as the excess return determined by $\lambda$ remains positive. Thus, we set $\lambda_2 = 1.2$ and solve for $\lambda_1$ using $\lambda = \rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2$. In addition, we assume an investment horizon of $T = 10$ years and choose the level of risk aversion at $\gamma = -5$. The risk-free interest rate is assumed to be 2%. In addition, the plausible values of ambiguity-aversion parameters $\phi_1$ and $\phi_2$ are determined by the detection-error probability (DEP), which is introduced in Section 3.5.1.

3.5.1 Detection-error probability

To determine the plausible or reasonable values of the uncertainty-aversion parameters $\phi_1$ and $\phi_2$, we make use of the DEP, similarly to Anderson et al. (2003) and Maenhout (2006) et al.
According to the above-mentioned authors’ papers, the value of the ambiguity-avoidance parameters φ’s should be chosen so that the worst-case alternative model and the reference model are sufficiently difficult to distinguish based on a time series sample of finite length N. Specifically, high DEP means there is a high likelihood that the investor cannot tell the difference between the reference model and the alternative model. Hence, DEP should be set high enough so that investors have difficulty in deciding. Anderson et al. (2003) advocated that the appropriate DEP should be set at least at 10%.

DEP can be calculated via Fourier inversion of the conditional characteristic functions of the Radon-Nikodym derivatives between the reference model and the alternative model, i.e., \( \Xi_{1,t} := \frac{d\mathbb{P}}{d\mathbb{F}} |\mathcal{F}_t \) and \( \Xi_{2,t} := \frac{d\mathbb{P}^e}{d\mathbb{F}} |\mathcal{F}_t \). Consider the logarithms of \( \Xi_{1,t} \) and \( \Xi_{2,t} \), such that

\[
\begin{align*}
\xi_{1,t} &:= \ln \Xi_{1,t} = -\int_0^t \left( \frac{(e^\gamma)^2 + (e^\delta)^2}{2} d\tau + e^\gamma dZ_{1t} + e^\delta dZ_{2t} \right) \\
\xi_{2,t} &:= \ln \Xi_{2,t} = \int_0^t \left( \frac{(e^\gamma)^2 + (e^\delta)^2}{2} d\tau + e^\gamma dZ_{1t} + e^\delta dZ_{2t} \right). 
\end{align*}
\]

If the reference model \( \mathbb{P} \) is true, then investors would mistakenly follow the alternative model \( \mathbb{P}^e \) for a given finite time series with length N when \( \xi_{1,t} > 0 \). In contrast, if the alternative model is true, then investors would erroneously choose the reference model \( \mathbb{P} \) and reject the alternative model if \( \xi_{2,t} > 0 \). Note that \( \xi_{2,t} > 0 \) is equivalent to \( \xi_{1,t} < 0 \). The DEP is thus defined as

\[
\epsilon_N(\phi_1, \phi_2) = \frac{1}{2} \Pr(\xi_{1,N} > 0 | \mathbb{P}, \mathcal{F}_0) + \frac{1}{2} \Pr(\xi_{1,N} < 0 | \mathbb{P}^e, \mathcal{F}_0).
\]

The definition above explicitly indicates its dependence on the ambiguity-averse parameters φ’s: when φ is higher, it would be easier for the investor to distinguish between the reference model and the alternative model, so the DEP would be lower. See complementary Appendix B.2.7 for details about using Fourier inversion to find the DEP.

We determine plausible values of the ambiguity-averse parameters \( \phi_1 \) and \( \phi_2 \) by taking values or ranges that make the DEP reasonable (s.t., \( \epsilon_N \geq 10\% \)). From Figures 3.1a–3.1c, it can be seen that when \( \phi_1 \) and \( \phi_2 \) are less than 5, the DEP always stays above 0.1. Without violating all of the other conditions we imposed on the parameters, we consider \( \phi_1 \) and \( \phi_2 \) that do not exceed 5.

### 3.5.2 Optimal exposures

The worst case measure \( e^\gamma \) and \( e^\delta \) are proportional to \( \sqrt{\nu} \); hence, the ratios \( \frac{e^\gamma}{\nu} \) and \( \frac{e^\delta}{\nu} \) can be thought of as variations in the market price rates of volatility risk and stock risk, respectively. Note that these ratios are defined as \( q_\nu \) and \( q_\delta \) in the process of finding the DEP. Next, we explore the relation between \( q_\nu \) and \( \phi_1 \), as well as between \( q_\delta \) and \( \phi_2 \), in Figures 3.2a and 3.2b in order to gain a better understanding of the DEP. As illustrated in Figures 3.2a and 3.2b, \( \frac{e^\gamma}{\nu} \) is quite sensitive to \( \phi_1 \) and insensitive to \( \phi_2 \), while \( \frac{e^\delta}{\nu} \) is sensitive to \( \phi_2 \) and insensitive to \( \phi_1 \). To be more specific, if the investor’s preference for robustness increases, then the investor would
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Figure 3.1: Detection-error probabilities (DEP) as functions of ambiguity aversion parameters $\phi_1$ and $\phi_2$. Panel (a) displays DEP under complete market with $\phi_1 = \phi_2$. Panel (b) displays DEP under complete market with $\phi_2$ changes. Panel (c) displays DEP under complete market with $\phi_1$ changes.

Require the market price of the idiosyncratic variance driver risk $\hat{\lambda}_1$ to increase to $\hat{\lambda}_1 - \frac{\epsilon}{\sqrt{v}}$ and the market price of idiosyncratic stock risk $\hat{\lambda}_2$ to decrease to $\hat{\lambda}_2 - \frac{\epsilon}{\sqrt{v}}$. Note that if the investor is confident about the reference model, and if there is no uncertainty, then no adjustment on the market prices of risk is needed. These observations are consistent with Escobar et al. (2015) and Yang et al. (2020).

Figure 3.2: Adjustments $e^v/\sqrt{v}$ and $e^S/\sqrt{v}$ to parameters $\hat{\lambda}_1$ and $\hat{\lambda}_2$, respectively, caused by ambiguity aversion in a complete market. Panel (a) shows $e^v/\sqrt{v}$ vs. $\phi_1$. Panel (b) shows $e^S/\sqrt{v}$ vs. $\phi_2$.

In Figures 3.3a to 3.3c, we explore the optimal wealth exposures $\Theta^v$ and $\Theta^S$ in the complete market versus ambiguity-aversion parameters $\phi_1$ and $\phi_2$. The optimal wealth exposure $\Theta^v$ is more sensitive to the ambiguity-averse parameter $\phi_1$ and almost insensitive to the ambiguity parameter $\phi_2$, while the optimal wealth exposure $\Theta^S$ varies dramatically with $\phi_2$ and is independent of $\phi_1$. Furthermore, the wealth exposures to the variance risk driver and stock risk decline in an absolute sense with the investor’s increase in robustness. This more conservative portfolio choice is highly intuitive and rational for an ambiguity-averse investor.
Chapter 3. Robust portfolio choice under the $4/2$ stochastic volatility model.

3.5.3 Utility losses

In this section, we consider the wealth-equivalent utility losses incurred by investors who follow popular and important suboptimal strategies, which include ignoring model uncertainty ($L^{(s1)}$), failing to complete the market ($L^{(s2)}$), and following a Heston strategy ($L^{(s3)}$) or a Merton ($L^{(s4)}$) strategy. In all cases, the losses are estimated with different levels of ambiguity aversion.

Figures 3.4a and 3.4b demonstrate the losses as a function of the ambiguity-averse parameters $\phi_1$ and $\phi_2$, respectively. As expected and documented for stochastic volatility (Escobar et al., 2015), the WEL is more sensitive to the ambiguity in variance driver risk ($\phi_1$), whereas it is almost insensitive to ambiguity in additional stock risk ($\phi_2$). To be more specific, given a level of $\phi_2$, the WEL could rise to 18% as $\phi_1$ increases from 1 to 5. This higher sensitivity of loss to $\phi_1$ than to $\phi_2$ can be explained by Figures 3.3a and 3.3b, where the optimal strategy is more sensitive to the increase in ambiguity $\phi_1$.

Figures 3.5a and 3.5b depict the relationships between WEL incurred by not completing
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the market for different levels of ambiguity aversion $\phi_1$ and $\phi_2$, respectively. In particular, the loss decreases in $\phi_1$, since the optimal wealth exposure to stock $\Theta^S$ is unchanged and the optimal wealth exposure to variance driver $\Theta^\nu$ shrinks as $\phi_1$ rises. The total wealth exposure is consequently reduced, and the discrepancy between this suboptimal strategy and the optimal strategy shrinks in a complete market. Therefore, the wealth losses decrease in $\phi_1$. Moreover, as $\phi_2$ increases, the losses increase with a fixed $\phi_1$. In particular, when $\phi_1$ is larger, the slopes of the increase in loss become flatter. This is explained by the combined effects of a slight increase in the wealth exposure $\Theta^\nu$ and a decrease in the wealth exposure $\Theta^S$ as $\phi_2$ increases. See Escobar et al. (2015) for a similar phenomenon. Note that although the wealth exposure $\Theta^\nu$ only slightly changes with $\phi_2$, it is more aggressive than $\Theta^S$ (especially when $\phi_1$ is small), and its effects are dominant.

![Figure 3.5](image)

Figure 3.5: Wealth-equivalent loss (WEL) from not completing the market as function of ambiguity aversion parameters $\phi_1$ and $\phi_2$. Panel (a) illustrates WEL vs. $\phi_1$. Panel (b) illustrates WEL vs. $\phi_2$.

Given that the Heston model and Merton model are quite popular among practitioners and academics, we also investigate the wealth losses incurred by investors who follow their strategies. Figures 3.6a and 3.6b present approximated losses that could be incurred if an investor mistakenly follows a Heston strategy as a function of $\phi_1$ and $\phi_2$, respectively. The penalty term is shown to be negligible in Figures B.4a and B.4b in the appendix, which confirms the validity of the approximation. Furthermore, it can be seen that the losses are more sensitive to $\phi_1$ than to $\phi_2$ in a downward manner. Specifically, the WEL decreases from 23% to 12% as $\phi_1$ increases from 1 to 5 for $\phi_2 = 1$, while the losses decrease from 23% to 21% as $\phi_2$ increases from 1 to 5 with $\phi_1 = 1$. That is, ambiguity regarding variance plays a decisive role in the extent of the losses. Figures 3.7a and 3.7b illustrate the losses incurred from following a Merton strategy. As Merton assumes constant variance and hence does not entertain ambiguity $\phi_1$, the losses depend on $\phi_2$ only. In particular, the losses are persistently above 50% of the initial wealth as $\phi_2$ varies between 1 and 5. The losses incurred by the Merton strategy deteriorate significantly compared to the Heston strategy.

The uncertainty regarding risk premiums has been acknowledged and addressed in the literature—see Escobar et al. (2015) and Liu and Pan (2003) among others—and ambiguity regarding model misspecification can be regarded as adjusting risk premiums. We therefore
Figure 3.6: Approximated wealth-equivalent loss (WEL) from following the Heston strategy as a function of the ambiguity aversion parameters $\phi_1$ and $\phi_2$ in a complete market. Panel (a) reports approximated WEL from Heston strategy vs. $\phi_1$. Panel (b) reports approximated WEL from Heston strategy vs. $\phi_2$.

Figure 3.7: Wealth-equivalent loss (WEL) from following the Merton strategy as function of the ambiguity aversion parameters $\phi_1$ and $\phi_2$ in a complete market. Panel (a) displays WEL vs. $\phi_1$. Panel (b) reports WEL vs. $\phi_2$. 
also demonstrate the relationships between the WEL from ignoring ambiguity aversion and the levels of the stock price risk premium \( \lambda \) and stochastic volatility driver risk premium \( \lambda_1 \), see Figures 3.8a and 3.8b, respectively. Note that the ranges of parameters in these two figures are at the maximum capacity of not violating any of the conditions. It can be observed that the losses can be as large as 70% of the initial wealth as the stock price risk premium \( \lambda \) changes between 0 and 4, and up to 60% as the variance driver risk premium changes between -4 and -0.5. These results confirm the significance of the level of risk premiums and the role of ambiguity.

![Figure 3.8](image)

**Figure 3.8**: Wealth-equivalent loss (WEL) from ignoring model uncertainty as function of risk premiums. Panel (a) shows WEL vs. the stock price risk premium \( \lambda \). Panel (b) shows WEL vs. the stochastic volatility risk premium \( \lambda_1 \).

### 3.6 Conclusion

In this chapter, a portfolio choice problem for a risk-averse and ambiguity-averse investor under the 4/2 stochastic volatility model is investigated. We adopt a realistic setting of a complete market, i.e. investments in a money account, a stock, and a derivative where investors can have different levels of uncertainty with regards to the variance’s risk and stock risk drivers.

We not only find closed-form solutions for all the objects of interest, e.g. optimal allocation, worst case measure, value function, but also derive analytical solutions to important suboptimal strategies in the context of a 4/2 SV model. Given the practical validation of a 4/2 SV model in the literature, we empirically demonstrate that if managers adopt simpler models, like the Heston SV model or a GBM, then they could incur in large wealth-equivalent losses of up to 50% of the initial investment.
Chapter 4

Optimal consumption and robust portfolio choice for the $3/2$ and $4/2$ SV models.

4.1 Introduction

Dating back to the 1980s, in the seminal work of Merton (1980) (equations II.1–II.8), the excess return of a security, also known as its risk premium, has been prescribed as proportional to powers of the volatility. Specifically, three models were proposed, all presented in terms of the market price of risk (MPR)—that is, technically the ratio of the excess return and the volatility of the security. The first model, type I, assumed a MPR proportional to volatility (i.e., power $1/2$). This model implies that each risk factor earns a risk premium that is proportional to the variance of the factor’s return. The second and the third models (types II and III) postulate constant MPR (i.e., power 0 on variance) and constant excess return (i.e., power $-1/2$ on variance, inversely proportional to volatility), respectively. These models have been widely used in the literature; see Heston (1993); Bakshi et al. (1997), and Bates (2000) for examples involving stochastic volatility (SV), stochastic interest, and jumps.

The specification of MPRs plays a very important role in expected utility portfolio optimization. In this context, Kraft (2005) solved the portfolio optimization problem for MPR of types I and II, in a setting of CRRA (power) utility, in an incomplete market with finite horizon for the Heston model (aka the $1/2$ model). Chacko and Viceira (2005) considered the optimal investment and consumption problem in an incomplete market for the $3/2$ model of Heston (1997) with Epstein-Zin-Weil recursive utility and an infinite horizon, which implies a value function independent of time. In particular, the authors considered two forms of excess return—constant and linear in the variance (i.e., the MPRs of types I and III). Nevertheless, the exact solution is only available when the agent’s elasticity of intertemporal substitution is one with constant excess return (type III). For all other cases, the solutions are approximations.

This chapter presents the very first closed-form analysis for type I MPR on the recently proposed $4/2$ model (see Grasselli (2017)) with a finite horizon, consumption, and complete markets. It leads, as a by-product, to the first analysis of type I MPR on the $3/2$ model. We incorporate several ingredients of interest to practitioners in an EUT setting: complete and incomplete markets, consumption and terminal wealth, and ambiguity aversion.

Two recent studies have been conducted on the $4/2$ model under MPRs outside of the settings...
4.2 Model formulation

We assume the stochastic processes describing the financial market are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a right-continuous filtration $(\mathcal{F}_t)_{t \in [0,T]}$. The price process $S_t$ of the risky asset follows the so-called 4/2 model:

$$\frac{dS_t}{S_t} = \left( r + \rho \lambda_1 (a \sqrt{v_t} + b) + \sqrt{1 - \rho^2 \lambda_2 (a \sqrt{v_t} + b)^2} \right) dt + (a \sqrt{v_t} + b) dW_t, \quad (4.1)$$

$$dv_t = \kappa (\theta - v_t) dt + \sigma \sqrt{v_t} dZ_{1t}, \quad v(0) = v_0 > 0, \quad (4.2)$$

where $v_t$ is the variance driver, which follows a CIR process with mean-reversion rate $\kappa > 0$, long-run mean $\theta > 0$, and volatility of volatility $\sigma > 0$. The Feller condition (i.e., $2\kappa \theta \geq \sigma^2$) is also imposed to keep the process $v_t$ strictly positive. The standard Brownian motions (BM) $W_t$ in the dynamics of the risky asset $S_t$ and $Z_{1t}$ in the dynamics of variance driver $v_t$ are correlated with parameter $\rho \in (-1, 1)$. Thus, we will write $dW_t = \rho dZ_{1t} + \sqrt{1 - \rho^2} dZ_{2t}$, where $Z_{2t}$ is
another standard BM, independent of $Z_{t_1}$. The variance, denoted by $z_t = z(v_t)$ is given as follows:

$$z_t = \left( a \frac{\sqrt{v_t}}{v_t} + \frac{b}{\sqrt{v_t}} \right)^2 = a^2 v_t + \frac{b^2}{v_t} + 2ab.$$  \hspace{1cm} (4.3)

This setting implies market prices of risk with the following representation:

$$\begin{align*}
\lambda_1(v_t) &= \lambda_1 \left( a \frac{\sqrt{v_t}}{v_t} \right), \\
\lambda_2(v_t) &= \lambda_2 \left( a \frac{\sqrt{v_t}}{v_t} \right), \\
\lambda(v_t) &= \rho \lambda_1(v_t) + \sqrt{1 - \rho^2} \lambda_2(v_t) = \bar{\lambda} \left( a \frac{\sqrt{v_t}}{v_t} \right),
\end{align*}$$  \hspace{1cm} (4.4)

where $\bar{\lambda} = \left( \frac{\rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2}{2} \right)$, $a$ and $b$ are positive constants, $\lambda_1$ and $\lambda_2$ are constant. $\lambda_1(v_t)$ is the market price of variance risk, and $\lambda(v_t)$ is the market price of stock risk. Moreover, $\lambda_2(v_t)$ can be interpreted as the market price of stock idiosyncratic risk (i.e., with respect to $Z_{t_2}$). Note that in this form of market price of risk, the excess return of the risky asset is proportional to its variance as recommended in the economics literature; see Merton (1980) equation (II.6), type I.

As for the market price of variance risk $\lambda_1(v_t)$, we use Ito’s lemma to create the process of the variance:

$$dz_t = (...)dt + (a - \frac{b}{v_t}) \left( a \frac{\sqrt{v_t}}{v_t} + \frac{b}{\sqrt{v_t}} \right) dZ_{t_1}. $$  \hspace{1cm} (4.5)

Hence our choice of market price of variance risk is: $\bar{\lambda}_1 \left( a \frac{\sqrt{v_t}}{v_t} \right) = \lambda_1 \sqrt{v_t}$. That is, it is proportional to the volatility of the asset. This is similar to the proposal in Heston (1993).

Furthermore, we assume the investor can also allocate on a financial derivative on the underlying. Let $O_t = \phi(S_t, v_t, t)$ denote the price of the option. It can be shown that the option price evolves with the stochastic differential equation (SDE):

$$\begin{align*}
\frac{dO_t}{O_t} &= r dt + \frac{1}{O_t} \left[ m_S \rho_S + m_v \frac{\sigma}{\sqrt{v_t}} \right] \left( a \frac{\sqrt{v_t}}{v_t} + \frac{b}{\sqrt{v_t}} \right) \lambda_1(a \frac{\sqrt{v_t}}{v_t} + \frac{b}{\sqrt{v_t}}) dt + dZ_{t_1} \\
&\quad + \frac{1}{O_t} \left[ m_S \sqrt{1 - \rho^2} \right] \left( a \frac{\sqrt{v_t}}{v_t} + \frac{b}{\sqrt{v_t}} \right) \lambda_2(a \frac{\sqrt{v_t}}{v_t} + \frac{b}{\sqrt{v_t}}) dt + dZ_{t_2},
\end{align*}$$  \hspace{1cm} (4.6)

where $m_S = \frac{\partial \phi}{\partial S}$, and $m_v = \frac{\partial \phi}{\partial v}$ denote the partial derivatives of the option price function $\phi$ with respect to $S_t$ and $v_t$. Equations (4.1), (4.2), and (4.6) are considered as the reference model.

### 4.3 Portfolio optimization under EUT

We consider the investor exhibits CRRA utility for both intermediate consumption and terminal wealth with the same risk-aversion level $\gamma$. That is,

$$u(c) = \epsilon_1 \frac{c^\gamma}{\gamma}, \quad u(X_T) = \epsilon_2 \frac{X_T^\gamma}{\gamma},$$  \hspace{1cm} (4.7)
where coefficients $\epsilon_1$ and $\epsilon_2$ are non-negative. The ratio $\epsilon_1/\epsilon_2$ indicates the relative importance of intermediate consumption and terminal wealth, and it thus affects decision-making (optimal strategy). Without loss of generality, we can set $\epsilon_2 = 1$, and let $\epsilon_1$ determine the relative importance ratio.

The objective of the investor is to maximize their utility from intermediate consumption $c_t$ and terminal wealth $X_T$; therefore the reward functional for the investor is defined as follows:

$$w(x, v, t; \Theta, c) = \mathbb{E}_{x,v,t} \left[ \epsilon_1 \int_t^T e^{-\delta (T-t)} \frac{c_t^\gamma}{\gamma} dt + e^{-\delta (T-t)} \frac{X_T^\gamma}{\gamma} \right],$$

where $\delta$ is a discount rate, and the goal is

$$\bar{J}(x, v, t) = \sup_{(\Theta,c) \in \mathcal{U}} w(x, v, t; \Theta, c),$$

where $\bar{J}(x, v, t)$ is the value function and the space $\mathcal{U}$ of admissible controls $[\Theta, c]_{t \in [0,T]}$ with $\Theta \in \mathbb{R}$, $c \in \mathbb{R}^+$, is the set of feedback strategies that satisfy standard conditions (see Escobar et al. (2015)).

### 4.3.1 Complete market analysis

Let $\pi_t^S$ be the fraction of wealth invested in the stock, $\pi_t^O$ be the fraction of wealth invested in the option that follows (4.6), $(1 - \pi_t^S - \pi_t^O)$ be the remaining portion of wealth invested in the money account, and $c_t$ the consumption at time $t$. The wealth $X_t$ of the investor follows the SDE:

$$dX_t = X_t \left[ \pi_t^S dS_t + \pi_t^O dO_t + (1 - \pi_t^S - \pi_t^O) r dt \right] - c_t dt$$

$$= X_t \left[ r + \Theta^v \tilde{\lambda}_1 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 + \Theta^x \tilde{\lambda}_2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 \right] dt - c_t dt$$

$$+ X_t \left[ \Theta^v (a \sqrt{v} + \frac{b}{\sqrt{v}}) dZ_{1t} + \Theta^x (a \sqrt{v} + \frac{b}{\sqrt{v}}) dZ_{2t} \right],$$

where we have assumed the money market account evolves as $\frac{dM}{M} = r dt$, and

$$\begin{bmatrix} \Theta^v \\ \Theta^x \end{bmatrix} = \begin{bmatrix} \rho \\ \frac{1}{\sqrt{1-\rho^2}} \left( m_S \sqrt{1-\rho^2} \right) \end{bmatrix} \begin{bmatrix} \pi_t^S \\ \pi_t^O \end{bmatrix}.$$  

Under the Bellman principle, the value function satisfies the HJB equation:

$$\sup_{\theta^v, \theta^x, \sigma} \left\{ u(c) - \delta \bar{J} + \bar{J}_x + \left( x r + \Theta^v \tilde{\lambda}_1 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 + \Theta^x \tilde{\lambda}_2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 \right) \right\} = 0,$$

$$+ \frac{1}{2} \chi^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \left( (\Theta^v)^2 + (\Theta^x)^2 \right) + \frac{1}{2} \chi^2 (\Theta^v)^2 \bar{J}_{xx} + \kappa (\theta - v) \bar{J}_v + \frac{1}{2} \chi^2 \bar{J}_{xx} + \sigma_x (av + b) \Theta^v \bar{J}_{vx} = 0,$$

(4.12)
with boundary condition $\bar{J}(x, v, T) = \frac{\gamma}{\gamma}$. Here $\bar{J}_x$, $\bar{J}_v$, $\bar{J}_{xx}$, $\bar{J}_{vv}$, and $\bar{J}_{vv}$ are the first and second partial derivatives of function $\bar{J}$ with respect to $t$, $x$, and $v$.

We conjecture our value function can be represented as follows:

$$\bar{J}(x, v, t) = \frac{\gamma}{\gamma} (\bar{h}(t, v))^{1 - \gamma},$$  \hspace{1cm} \text{(4.13)}

where $\bar{h}(T, v) = 1$ for all $v$. This conjecture leads to the following PDE for $\bar{h}$:

$$\left(\epsilon_1\right)^{-\frac{1}{\gamma}} \bar{h}_t + \bar{h}_v + \left[ \frac{\gamma}{2} \frac{2}{\gamma - 1} \bar{a} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 - \frac{1}{2} \frac{2}{\gamma - 1} \gamma \bar{h} \right] \frac{\gamma}{1 - \gamma} \bar{h}$$

$$\begin{align*}
&+ \left[ \frac{\gamma}{\gamma - 1} + \gamma(\gamma - 1)\sigma(a v + b) \right] \bar{h}_v + \frac{1}{2} \sigma^2 v \bar{h}_{vv} = 0, \\
&\hspace{1cm} \text{where } \Gamma(v, t), V(v, t): \mathbb{R}_+ \times [0, T] \rightarrow \mathbb{R} \text{ are measurable functions, and} \\
&\hspace{1cm} \bar{k} = \left( - \frac{\gamma^2(\gamma - 2) - 1}{\gamma - 1} \sigma a \lambda_1 + \kappa \right), \\
&\hspace{1cm} \bar{k} \bar{\theta} = \left( \frac{\gamma^2(\gamma - 2) - 1}{\gamma - 1} \sigma b \lambda_1 + \kappa \theta \right). \\
\end{align*}$$  \hspace{1cm} \text{(4.14)}

Details of this calculation can be found in Appendix C.1. Next, we provide the solution to the HJB equation.

**Proposition 4.3.1 (4/2 model in complete market).** Let us define

$$\mu = \frac{1}{2} \frac{-\gamma}{(1 - \gamma)^2} \left( \lambda_1^2 + \lambda_2^2 \right) a^2,$$

$$\nu = \frac{1}{2} \frac{-\gamma}{(1 - \gamma)^2} \left( \lambda_1^2 + \lambda_2^2 \right) b^2.$$  \hspace{1cm} \text{(4.16)}

If the parameters satisfy the following three conditions:

$$\kappa \bar{\theta} \geq \frac{\sigma^2}{2}, \hspace{0.5cm} \mu > \frac{-\kappa^2}{2\sigma^2}, \hspace{0.5cm} \nu > \frac{(2\kappa \bar{\theta} - \sigma^2)^2}{8\sigma^2},$$  \hspace{1cm} \text{(4.17)}

then the candidate solution of the HJB equation (4.12) is well defined and has the representation (4.13), with

$$\bar{h}(v, t) = \left(\epsilon_1\right)^{-\frac{1}{\gamma}} \int_t^T g(v, \bar{\tau}) d\bar{\tau} + g(v, t),$$  \hspace{1cm} \text{(4.18)}

where $(\tau(t) = T - t \geq 0)$:

$$g(v, \tau) = \exp \left\{ \frac{\gamma}{1 - \gamma} \left( \frac{\lambda_1^2 a b}{\gamma - 1} - \frac{\lambda_2^2 a b}{\gamma - 1} \right) (T - t) \right\} \times q(T - t, v; \mu, \nu)$$  \hspace{1cm} \text{(4.19)}
4.3. Portfolio optimization under EUT

\[ q(\tau,v;\mu,\nu) = \left(\frac{\beta(\tau,v)}{2}\right)^{m+1} v^{-\frac{m}{2}} \sigma^2 K(\tau)^{-\frac{1}{2}} e^{\frac{1}{\sigma^2} \left(\tilde{\sigma} \nu - \sqrt{\nu} \coth \left(\frac{\nu \tau + \tilde{\nu}}{2}\right)\right)} \frac{\Gamma \left(\frac{1}{2} + \frac{m}{2} + \frac{\tilde{\nu}}{\sigma^2}\right)}{\Gamma(m+1)} \]

\times {}_1 F_1 \left(\frac{1}{2} + \frac{m}{2} + \frac{\tilde{\nu}}{\sigma^2}, m + 1, \beta(\tau,v)^2\right) \tag{4.20}

with

\[ m = \frac{1}{\sigma^2} \sqrt{(2k\tilde{\theta} - \sigma^2)^2 + 8\sigma^2 \nu}, D = \tilde{k}^2 + 2\mu \sigma^2, \]
\[ \beta(\tau,v) = \frac{2 \sqrt{D} \nu}{\sigma^2 \sinh \left(\frac{\sqrt{2} \nu}{2}\right)}, K(\tau) = \frac{1}{\sigma^2} \left(\sqrt{D} \coth \left(\frac{\sqrt{2} \nu \tau}{2}\right) + \tilde{k}\right). \tag{4.21} \]

Moreover, the optimal consumption–wealth ratio, and variance-stock exposures are given by

\[ \left(\frac{C}{\lambda}\right)^* = \tilde{h}^{-1}(e_1)^{\frac{1}{\gamma+1}}, \]
\[ (\Theta^*)^* = \frac{\sigma \sqrt{\nu}}{(a \sqrt{\nu} + \frac{b}{\sqrt{\nu}}) \tilde{h}} - \frac{\tilde{\lambda}_1}{(\gamma-1)}, \tag{4.22} \]
\[ (\Theta^*)^* = \frac{-\tilde{\lambda}_2}{\gamma - 1}. \]

See the proof in Appendix C.1.

It should be noted that in the case of no consumption, we can assume a simpler value function representation:

\[ \tilde{J}(x,v,t) = \frac{x^\gamma}{\gamma} \tilde{h}(v,t), \tag{4.23} \]

and solving the maximization problem in (4.12), we obtain

\[ 0 = r\gamma \tilde{h} + \tilde{h}_t + \frac{1}{2} \sigma^2 v \tilde{h}_{vv} + \kappa(\theta - v) \tilde{h}_v - \frac{1}{2} \sigma^2 \nu \left(\frac{\gamma}{\gamma - 1} \tilde{h}\right) - \frac{1}{2} (\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2) (a \sqrt{\nu} + \frac{b}{\sqrt{\nu}})^2 \left(\frac{\gamma}{\gamma - 1} \tilde{h}\right) - \sigma \tilde{\lambda}_1 (av + b) \left(\frac{\gamma}{\gamma - 1} \tilde{h}\right). \]

Nonetheless, as \( \gamma \neq 0 \), the nonlinear term \( \frac{\tilde{\lambda}_1^2}{\gamma} \) cannot be eliminated in the PDE for \( \tilde{h} \), thus rendering a closed-form solution impossible under the 4/2 SV model. Moreover, a closed-form solution is available in Liu and Pan (2003) (Example 1) for stock prices following the 1/2 SV model in a complete market without ambiguity.

4.3.2 Incomplete market analysis

Given the popularity of incomplete market situations, we briefly describe the solution for this setting next. Let \( \pi_t \) be the fraction of wealth invested in the stock and \( (1 - \pi_t) \) be the remaining portion of wealth invested in the money account, while \( c_t \) is the consumption at time \( t \). The
wealth $X_t$ of the investor follows

$$
dX_t = X_t \left[ r + \pi \left( \rho \lambda_t (a \sqrt{V_t} + \frac{b}{\sqrt{V_t}})^2 + \sqrt{1 - \rho^2} \lambda_t (a \sqrt{V_t} + \frac{b}{\sqrt{V_t}})^2 \right) \right] dt - \sigma \sqrt{V_t} dS_t.
$$

The value function $J(x, v, t)$ satisfies the HJB equation:

$$\sup_{\tilde{c}(c)} (u(c) - \delta J + \left( \left[ r + \pi \left( \rho \lambda_t (a \sqrt{V_t} + \frac{b}{\sqrt{V_t}})^2 + \sqrt{1 - \rho^2} \lambda_t (a \sqrt{V_t} + \frac{b}{\sqrt{V_t}})^2 \right) \right] - \tilde{c} \right) J_x$$

$$+ \frac{1}{2} \sigma^2 \left( a \sqrt{V_t} + \frac{b}{\sqrt{V_t}} \right)^2 \pi^2 J_{xx} + \kappa (\theta - v) J_v + \frac{1}{2} \sigma^2 J_{vv} + \sigma \lambda_t (a v + b) J_v = 0,$$

with boundary condition $J(x, v, T) = \frac{\zeta}{\gamma}$. Here, we conjecture $J(x, v, t) = \frac{\zeta}{\gamma} \left( h(t, v) \right)^{1-\gamma}$, with $h(T, v) = 1$ for all $v$. This leads to a PDE for $h$ as follows:

$$(1 - \gamma) (\xi^2) \gamma + \left( 1 - \gamma \right) h_t + \left[ r - \delta - \frac{1}{2} \left( \rho \lambda_t + \sqrt{1 - \rho^2} \lambda_t (a \sqrt{V_t} + \frac{b}{\sqrt{V_t}})^2 \right) \right] \gamma h$$

$$+ \left[ \kappa (\theta - v) (1 - \gamma) - \sigma \lambda_t (a v + b) \right] h_v + \frac{1}{2} \sigma^2 \gamma (1 - \gamma) h_v$$

$$+ \frac{1}{2} \sigma^2 \gamma (1 - \gamma) h_{vv} = 0.$$  

(4.26)

The coefficient of term $\frac{\xi^2}{\gamma}$ in Equation (4.26) cannot be made zero under any parametric setting; that is, $\frac{1}{2} \sigma^2 \gamma (1 - \gamma) \gamma - \frac{1}{2} \gamma \sigma^2 \gamma (1 - \gamma)^2 v = \frac{1}{2} \sigma^2 \gamma (1 - \gamma) (1 + \gamma^2) \neq 0$. This means there is no closed-form solution in the presence of consumption. Nevertheless, in the absence of consumption, we can find a solution; the result is provided next.

We conjecture our value function follows

$$J(x, v, t) = \frac{\zeta}{\gamma} h(t, v),$$

(4.27)

where $h(T, v) = 1$ for all $v$, which leads to the PDE

$$h_t + \left[ r - \delta - \frac{1}{2} \left( \rho \lambda_t + \sqrt{1 - \rho^2} \lambda_t (a \sqrt{V_t} + \frac{b}{\sqrt{V_t}})^2 \right) \right] h$$

$$+ \left[ \kappa (\theta - v) - \frac{1}{2} \sigma^2 \gamma (1 - \gamma) \right] h_v + \frac{1}{2} \sigma^2 \gamma h_{vv} = 0,$$

(4.28)
4.4 Robust consumption portfolio optimization under EUT

where

\[
\begin{align*}
\tilde{k} &= \kappa + \frac{\gamma}{\gamma - 1} \left( \rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2 \right) \sigma a, \\
\tilde{k} \theta &= \kappa \theta - \frac{\gamma}{\gamma - 1} \left( \rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2 \right) \sigma b.
\end{align*}
\] (4.29)

Proposition 4.3.2 (4/2 model in incomplete market, no consumption). Let \( \varepsilon_1 = 0 \) and

\[
\begin{align*}
\mu &= \frac{1}{2} \gamma \left( \rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2 \right) a^2, \\
v &= \frac{1}{2} \gamma \left( \rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2 \right) b^2.
\end{align*}
\] (4.30)

Assume the following four conditions

\[
\rho = 0, \quad \tilde{k} \theta \geq \frac{\sigma^2}{2}, \quad \mu > \frac{-\kappa^2}{2\sigma^2}, \quad v \geq \frac{(2\tilde{k} \theta - \sigma^2)^2}{8\sigma^2}.
\] (4.31)

Then, the solution of the HJB Equation (4.25) has representation (4.27) with

\[
h(v, t) = \exp \left\{ \left( r \gamma - \frac{\gamma}{\gamma - 1} (\bar{\lambda}_1 \rho + \bar{\lambda}_2 \sqrt{1 - \rho^2})^2 ab \right) (T - t) \right\} \times q(\tau, v; \alpha, \lambda, \mu, v).
\] (4.32)

with \( q(\tau, v; \alpha, \lambda, \mu, v) \) as per Equation (4.20) with associated \( m, D, \beta, \) and \( K. \)
Moreover, the optimal wealth exposure is given by

\[
(\pi)^* = \frac{\bar{\lambda}_2}{1 - \gamma}.
\] (4.33)

See the proof in Appendix C.2.

The previous result highlights limitations stemming from an extra, necessary condition (i.e., no leverage effect \( \rho = 0 \)) to achieve a closed-form solution, which coincides with the celebrated Merton’s solution.

4.4 Robust consumption portfolio optimization under EUT

Assume the investor is uncertain about the probability distribution for the reference model and consider a set of plausible, alternative models when making investment decisions. Specifically, the investor is uncertain about the distribution of \( Z_{t1} \) and \( Z_{t2}. \)

Let \( e := (e^1_t, e^2_t) \) be an \( \mathbb{R}^2 \)-valued \( \mathcal{F} \)-progressively measurable process and define the Radon-Nikodym derivative process by

\[
\xi_t = \frac{dP^e}{dP} |_{\mathcal{F}_t} = \exp \left\{ - \int^t_0 \left( \frac{(e^1_\tau)^2 + (e^2_\tau)^2}{2} d\tau + e^1_\tau dZ_{1\tau} + e^2_\tau dZ_{2\tau} \right) \right\}.
\] (4.34)
According to Girsanov’s theorem, the process
\[
\begin{bmatrix}
Z_{it} \\
\bar{Z}_{2t}
\end{bmatrix} = \left[ \int_{t}^{T} e^{\frac{r}{2}} d\tau \right] + \begin{bmatrix}
Z_{1t} \\
\bar{Z}_{2t}
\end{bmatrix},
\]
(4.35)
is a Wiener process under probability measure \( \mathbb{P}^e \). Let \( e \in [0, T] \) denote the set of all \( \mathcal{F}_t \)-progressively measurable processes such that the process (4.34) is a well-defined Radon-Nikodym derivative process. This formulation of incorporating the investor’s model uncertainty is actually allowing uncertainty on the drift of diffusion risk factors of the stock and its variance’s driver (i.e., \( Z_{2t} \) and \( Z_{1t} \), respectively).

The alternative model then follows:
\[
\begin{align*}
\frac{dS_t}{S_t} &= \left[ r + \rho \lambda_1(a \sqrt{V_t} + \frac{b}{V_T})^2 + \sqrt{1 - \rho^2} \lambda_2(a \sqrt{V_t} + \frac{b}{V_T})^2 - \rho(a \sqrt{V_t} + \frac{b}{V_T})e_t^v \\
&\quad - \sqrt{1 - \rho^2}(a \sqrt{V_t} + \frac{b}{V_T})e_t^c \right] dt + (a \sqrt{V_t} + \frac{b}{V_T})(\rho d\bar{Z}_{it} + \sqrt{1 - \rho^2} d\bar{Z}_{2t}),
\end{align*}
\]
\[
dv_t = (k(\theta - v_t) - \sigma \sqrt{V_t} e_t^v) dt + \sigma \sqrt{V_t} d\bar{Z}_{it},
\]
\[
\frac{dO_t}{O_t} = r dt + \frac{1}{O_t} \left[ m_1 \rho S_t + m_2 \frac{\sigma \sqrt{V_t}}{(a \sqrt{V_t} + \frac{b}{V_T})} \left( a \sqrt{V_t} + \frac{b}{V_T} \right) \left( \lambda_1(a \sqrt{V_t} + \frac{b}{V_T}) - e_t^c \right) dt + d\bar{Z}_{it} \right]
\]
\[
+ \frac{1}{O_t} \left[ m_3 \sqrt{1 - \rho^2 S_t} \left( a \sqrt{V_t} + \frac{b}{V_T} \right) \left( \lambda_2(a \sqrt{V_t} + \frac{b}{V_T}) - e_t^c \right) dt + d\bar{Z}_{2t} \right].
\]
(4.36)

The reward functional can be defined as in the previous section for a given probability measure \( \mathbb{P}^e \):
\[
w^v(x, v, t; \Theta, c) = \mathbb{E}^e \left[ e_{t(0)} \int_{t}^{T} e^{-\delta(t-\tau)} \eta^\gamma_{r} d\tau + e^{-\delta(T-t)} X^\gamma_{T} \right].
\]
(4.37)

In the presence of a preference for robustness, the investor’s objective is to minimize the penalty term and maximize his utility from intermediate consumption \( c_t \) and terminal wealth \( X_T^{\gamma} \):
\[
\hat{J}(x, v, t) = \sup_{\theta^e, \bar{\theta}^e, x^e \in \mathcal{U}} \inf_{c^e \in \mathcal{C}^{e, \gamma}} \left( w^e(x, v, t; \Theta, c) + \mathbb{E}^e \left[ \int_{t}^{T} \left( \frac{(e^c)^2}{2\Phi_1(x, v, \tau)} + \frac{(e^c)^2}{2\Phi_2(x, v, \tau)} \right) d\tau \right] \right),
\]
(4.38)

where \( \hat{J}(x, v, t) \) is the value function, and the last two terms serves as the penalty for deviating too far from the reference model. The space \( \mathcal{V} \) of \( \mathcal{F}_t \)-adapted processes \( e_t = (e^x, e^c) \in \mathbb{R}^2 \), is the set of perturbations; the space \( \mathcal{U} \) of admissible controls \( \{\Theta_t, c_t\}_{t \in [0, T]} \) (i.e., \( \Theta_t \in \mathbb{R}, c_t \in \mathbb{R}^+ \)) is the set of feedback admissible strategies. The perturbations \( e^c_t \) and \( e^x_t \) in the penalty term are scaled by \( \Phi_1 \) and \( \Phi_2 \), respectively. That is, the larger the values of \( \Phi_1 \) and \( \Phi_2 \), the smaller the penalties for deviating from the reference model, which implies that the investor is more uncertain about the model. Following Maenhout (2004), we assume
\[
\Phi_i = \frac{\phi_i}{\gamma \hat{J}(x, v, t)}, \quad i = 1, 2,
\]
(4.39)
where \( \phi_1 > 0 \) denotes the ambiguity-aversion parameters. In this specification, the optimal strategy would be independent of the current wealth level for a power utility investor, namely homothetic robustness in Maenhout (2004). Further, \( \phi_1 \) can be interpreted as ambiguity aversion regarding the volatility driver, while \( \phi_2 \) represents ambiguity about the stock process.

### 4.4.1 Complete market analysis

Let \( \pi_i^s \) be the fraction of wealth invested in the stock, \( \pi_i^o \) be the fraction of wealth invested in the option, and \( (1 - \pi_i^s - \pi_i^o) \) be the remaining portion of wealth invested in the money account, while \( c_t \) is consumption at time \( t \). The wealth \( X_t \) of the investor follows

\[
dX_t = X_t \left[ r + \Theta^v \lambda_1 (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) J_t^v - \Theta^s (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) e_t^v + \Theta^s \lambda_2 (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) e_t^s \right] dt
\]

\[
- c_t dt + X_t \left[ \Theta^v \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) J_t^v + \Theta^s \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) J_t^s \right]
\]

That is, if we can find wealth exposures \( \Theta^v \) and \( \Theta^s \) to the fundamental risk factors \( J_t^v \) and \( J_t^s \), the corresponding wealth weights \( \pi_i^s \) and \( \pi_i^o \) can also be obtained. The value function satisfies the HJBI (robust) equation:

\[
\sup_{\Theta^v, \Theta^s} \inf_{c_t, \pi^s, \pi^o} \left( u(c) - \delta \bar{J} + \bar{J}, \left[ X_t \left[ r + \Theta^v \lambda_1 (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) J_t^v - \Theta^s (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) e_t^s \right] dt \right. \right.
\]

\[
\left. - c_t dt \right] \left[ \Theta^v \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) J_t^v + \Theta^s \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) J_t^s \right]
\]

\[
\left. + \frac{1}{2} \sigma^2 X_t \left( \Theta^v \right)^2 + \left( \Theta^s \right)^2 \right] J_t^{xx} + \left( \kappa (\theta - v) - \sigma \sqrt{v} e_t^v \right) \right] J_t^v \]

\[
+ \frac{1}{2} \sigma^2 v J_t^{vv} + \sigma x (av + b) \Theta^v J_t^{xv} + \left( e_t^s \right)^2 + \left( e_t^s \right)^2 \right] \frac{2 \Phi_1}{2 \Phi_2} = 0,
\]

with boundary condition \( \bar{J}(x, v, T) = \frac{c_t}{\gamma} \). Similarly to Section 4.3.1, after solving the first order conditions, we conjecture a value function as follows:

\[
\bar{J}(x, v, t) = \frac{x^\gamma}{\gamma} \left( \bar{h}(t, v) \right)^{1-\gamma},
\]
Next, we present the main result of the section.

By setting \( \phi_1 = 0 \), we obtain

\[
(1 - \gamma) (\epsilon_1) \frac{1}{\gamma} \frac{\partial}{\partial \tau} + (1 - \gamma) \frac{\partial}{\partial \tau} \bar{h}_t + \left[ \frac{r - \frac{\delta}{\gamma}}{\gamma - 1 - \phi_1} - \frac{1}{2} \frac{\lambda_1^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2}{(a - \phi_1)} - \frac{1}{2} \frac{\lambda_2^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2}{(a - \phi_2)} \right] \gamma \frac{\partial}{\partial \tau} \bar{h}_t + \left[ \frac{\phi_1}{\gamma} - 1 - \gamma + \phi_1 \frac{\sigma \lambda_1 (av + b)}{\gamma - 1 - \phi_1} - \gamma (v - 1)(\frac{\phi_1}{\gamma} - 1) \sigma (av + b) \lambda_1 + \kappa (\theta - v) \right] \left( \frac{1}{\gamma - 1 - \phi_1} \right) \frac{\partial}{\partial \tau} \bar{h}_t = 0.
\]

Details of this calculation can be found in Appendix C.3. In order to find a solution we need to eliminate the term \( \frac{\partial h}{\partial \tau} \); this means:

\[
\frac{1}{2} (\gamma - \phi_1) \frac{\phi_1}{(\gamma - 1 - \phi_1)} - \frac{1}{2} \sigma^2 v \left( \frac{\gamma}{\gamma - 1} + \frac{\phi_1}{\gamma} \right) = 0,
\]

\[
\frac{(\gamma - \phi_1) (\phi_1) - 1}{(\gamma - 1 - \phi_1)} = \frac{\gamma}{1 - \gamma} + \frac{\phi_1}{\gamma},
\]

\[
\phi_1 = 0.
\]

By setting \( \phi_1 = 0 \), and rearranging terms we obtain

\[
(\epsilon_1) \frac{1}{\gamma} \frac{\partial}{\partial \tau} + \bar{h}_t + \left[ \frac{r - \frac{\delta}{\gamma}}{\gamma - 1 - \phi_1} - \frac{1}{2} \frac{\lambda_1^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2}{(a - \phi_1)} - \frac{1}{2} \frac{\lambda_2^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2}{(a - \phi_2)} \right] \gamma \frac{\partial}{\partial \tau} \bar{h}_t + \left[ \frac{\phi_1}{\gamma} - 1 - \gamma + \phi_1 \frac{\sigma \lambda_1 (av + b)}{\gamma - 1 - \phi_1} - \gamma (v - 1)(\frac{\phi_1}{\gamma} - 1) \sigma (av + b) \lambda_1 + \kappa (\theta - v) \right] \left( \frac{1}{\gamma - 1 - \phi_1} \right) \frac{\partial}{\partial \tau} \bar{h}_t = 0.
\]

where

\[
\bar{k} = \frac{\phi_1}{\gamma} - 1 - \gamma + \phi_1 \frac{\sigma \lambda_1 (av + b)}{\gamma - 1 - \phi_1} + \gamma (v - 1)(\frac{\phi_1}{\gamma} - 1) \sigma (av + b) \lambda_1 - \phi_1 \frac{\phi_1}{(\gamma - 1 - \phi_1)^2} + \kappa.
\]

Next, we present the main result of the section.

**Proposition 4.4.1 (4/2 model in complete market, robustness).** Let

\[
\mu = \frac{1}{2} \frac{\gamma}{1 - \gamma} \left( \frac{\lambda_1^2}{\gamma - 1 - \phi_1} + \frac{\lambda_2^2}{\gamma - 1 - \phi_2} \right) a^2,
\]

\[
\nu = \frac{1}{2} \frac{\gamma}{1 - \gamma} \left( \frac{\lambda_1^2}{\gamma - 1 - \phi_1} + \frac{\lambda_2^2}{\gamma - 1 - \phi_2} \right) b^2,
\]

(4.48)
and \( \phi_1 = 0 \) (condition \( 4.45 \)). Assume \( \tilde{k}, \tilde{\theta}, \mu, \text{and} \, \nu \) satisfy conditions \( 4.17 \). Then, the solution of the HJBI Equation \( 4.42 \) is \( \bar{h}(x, \nu, t) = \frac{\nu}{\gamma} \left( \bar{h}(t, \nu) \right)^{1-\nu} \), where \( \bar{h} \) solves the PDE in Equation \( 4.46 \) and admits the representation:

\[
\bar{h}(v, t) = (e_1)^{-1} \int_t^T g(v, \tau) d\tau + g(v, \tau), \tag{4.49}
\]

\[
g(v, \tau) = \exp \left\{ \frac{\nu}{1-\nu} \left[ r - \frac{\lambda_1^2 \nu b}{\gamma - 1 - \phi_1} - \frac{\lambda_2^2 \nu b}{\gamma - 1 - \phi_2} \right] (T - t) \right\} \times q(\tau, v, \alpha, \lambda, \mu, \nu), \tag{4.50}
\]

where \( \tau(t) = T - t \), and \( q(\tau, v, \alpha, \lambda, \mu, \nu) \) follows from Equation \( 4.20 \) with associated \( m, D, \beta, \text{and} \, K \).

Moreover, the optimal consumption-wealth ratio, and variance-stock exposures are given by

\[
\begin{align*}
\left( \frac{c}{x} \right)^* & = \bar{h}^{-1}(e_1)^{-\frac{\nu}{\gamma}}, \\
(\Theta^c)^* & = -\frac{\lambda_2}{\lambda_1}, \\
(\Theta^s)^* & = \frac{\lambda_2}{\gamma - 1 - \phi_2}.
\end{align*}
\]

The worst case measure is determined by

\[
\begin{align*}
(e^c)^* & = 0, \\
(e^s)^* & = -\frac{\phi_2 \lambda_2}{\gamma - 1 - \phi_2} \left( a \sqrt{\nu} + \frac{b}{\sqrt{\nu}} \right).
\end{align*}
\]

See the proof in Appendix C.3.

The previous result can be seen as a generalization of Proposition 4.3.1 by setting \( \phi_2 = 0 \). It should be noted that the closed-form solution does not support ambiguity-aversion or uncertainty on the variance driver (i.e., \( \phi_1 \) must be zero).

Important solutions can be produced in the absence of consumption. In this case, the candidate for the solution of the HJBI Equation \( 4.42 \) is \( \tilde{h}(x, \nu, t) = \frac{\nu}{\gamma} \bar{h}(t, \nu) \), where \( \bar{h} \) solves the PDE

\[
\begin{align*}
& \bar{h}_t + \left[ r - \frac{\lambda_1^2 (a \sqrt{\nu} + \frac{b}{\sqrt{\nu}})^2}{2 (\gamma - 1 - \phi_1)} - \frac{\lambda_2^2 (a \sqrt{\nu} + \frac{b}{\sqrt{\nu}})^2}{2 (\gamma - 1 - \phi_2)} \right] \gamma \bar{h} + \left[ \frac{(\phi_1 - \gamma) \sigma (av + b) \lambda_1}{(\gamma - 1 - \phi_1)} + \kappa (\theta - v) \right] \bar{h}_v + \frac{1}{2} \sigma^2 \nu \bar{h}_{vv} = 0, \\
& \bar{h} \big|_{v(t)} = \bar{h}(t, \nu),
\end{align*}
\]

with

\[
\begin{align*}
\tilde{k} & = \kappa - \frac{(\phi_1 - \gamma) \sigma b \lambda_1}{(\gamma - 1 - \phi_1)}, \\
\tilde{\theta} & = \kappa - \frac{(\phi_1 - \gamma) \sigma a \lambda_1}{(\gamma - 1 - \phi_1)}.
\end{align*}
\]

The main result is reflected in the next corollary.
Chapter 4. Optimal consumption and robust portfolio choice for the 3/2 and 4/2 SV models.

Corollary 4.4.2 (4/2 model in complete market, robustness, no consumption). Let

\[ \mu = \gamma \left( \frac{\lambda_1^2}{\gamma - 1 - \phi_1} + \frac{\lambda_2^2}{\gamma - 1 - \phi_2} \right) a^2, \]

\[ \nu = \gamma \left( \frac{\lambda_1^2}{\gamma - 1 - \phi_1} + \frac{\lambda_2^2}{\gamma - 1 - \phi_2} \right) b^2. \]  

(4.55)

Assume \( \epsilon_1 = 0 \), and

\[ \phi_1 = \frac{\gamma^2}{\gamma + 1}, \]  

\[ k \theta - \frac{(\phi_1 - \gamma) \sigma \beta \lambda_1}{(\gamma - 1 - \phi_1)} \geq \frac{\sigma^2}{2}, \]  

(4.56)

while \( \mu \) and \( \nu \) satisfy conditions (4.17). Then \( \bar{J}(x, \nu, t) = \frac{s^\gamma}{\gamma} \bar{h}(t, \nu) \), where \( \bar{h} \) has the representation:

\[ \bar{h}(\nu, t) = \exp \left\{ \left( r \gamma - \gamma a b \left( \frac{\lambda_1^2}{\gamma - 1 - \phi_1} + \frac{\lambda_2^2}{\gamma - 1 - \phi_2} \right) \right) (T - t) \right\} \times \bar{q}(\tau, \nu; \alpha, \lambda, \mu, \nu), \]  

(4.57)

where \( \tau(t) = T - t \), and \( \bar{q}(\tau, \nu; \alpha, \lambda, \mu, \nu) \) is the same as Equation (4.20) with associated \( m, D, \beta, \) and \( K \).

Moreover, the optimal variance-stock exposures are given by

\[ (\Theta^\nu)^* = \frac{\sigma \sqrt{\nu}}{a \sqrt{\nu} + \frac{b}{\sqrt{\nu}}} \frac{\bar{h}_v}{\bar{h}} - \lambda_1, \]  

\[ (\Theta^\delta)^* = \frac{-\lambda_2}{(\gamma - 1 - \phi_2)}. \]  

(4.58)

The worst case measure is determined by

\[ (e^\nu)^* = \sigma \sqrt{\nu} \frac{\bar{h}_v}{\bar{h}} \gamma - \lambda_1 \frac{\gamma^2}{\gamma + 1} (a \sqrt{\nu} + \frac{b}{\sqrt{\nu}}), \]  

\[ (e^\delta)^* = \frac{-\phi_2 \lambda_2}{\gamma - 1 - \phi_2} (a \sqrt{\nu} + \frac{b}{\sqrt{\nu}}). \]  

(4.59)

See proof in Appendix C.4.

In contrast to a solution in the presence of consumption (Proposition 4.4.1), here we can entertain non-zero ambiguity-aversion on variance (\( \phi_1 \)) and stock (\( \phi_2 \)), which provides a window into the impact of ambiguity-aversion in general.

4.4.2 Incomplete market

To provide a full analysis, and given the popularity of incomplete market solutions, here we address closed-form expressions in an incomplete market. Let \( \pi_t \) be the fraction of wealth
invested in the stock and \((1 - \pi_t)\) be the remaining portion of wealth invested in the money account, while \(c_t\) is the consumption at time \(t\). The wealth \(X_t\) of the investor follows:

\[
dX_t = X_t \left[ r + \pi_t \left( \rho \lambda_1 (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2 + \sqrt{1 - \rho^2} \lambda_2 (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2 - \rho (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) e_t^v \right) \right] dt - c_t dt + \pi_t X_t (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) (\rho dZ_t + \sqrt{1 - \rho^2} d\bar{Z}_t). \tag{4.60}
\]

The value function \(J(x, v, t)\) satisfies the HJBI equation:

\[
\sup_{\pi_t} \inf_{c^v, e^v} \left\{ u(c) - \delta J + J_t + \left[ x \left[ r + \pi_t \left( \rho \lambda_1 (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2 + \sqrt{1 - \rho^2} \lambda_2 (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2 - \rho (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) e_t^v \right) \right] \right] dt - c_t dt + \pi_t x (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) (\rho e_t + \sqrt{1 - \rho^2} e_t) dt \]
\[
+ \frac{1}{2} \sigma^2 v_t J_{vv} + \sigma x \pi \rho (av_t + b) J_v + \frac{(e_t^v)^2}{2\Phi_1} + \frac{(e_t^v)^2}{2\Phi_2} \right\} = 0, \tag{4.61}
\]

with boundary condition \(J(x, v, T) = \frac{x^2}{\gamma}\) and \(\Phi_i = \frac{\Phi_1}{\gamma}, i = 1, 2\). Similarly to the previous results, we conjecture our value function as follows:

\[
J(x, v, t) = \frac{x^2}{\gamma} \left( h(t, v) \right)^{1-\gamma}, \tag{4.62}
\]

where \(h(T, v) = 1\) for all \(v\). This conjecture leads to the following PDE for \(h\):

\[
(1 - \gamma)(\bar{e}_t)\frac{\partial h}{\partial t} + (1 - \gamma)h_t + \left[ r - \delta \frac{\partial}{\partial t} + \frac{1}{2} \left( \frac{\rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2}{\sqrt{\Phi_1}} \right)^2 \right] h_t + \left[ (1 - \gamma) (1 - \gamma) + \frac{\sigma \rho (1 - \gamma)}{\Phi_1 + \Phi_2 (1 - \rho^2) - (\gamma - 1)} \right] \left( \frac{\partial h}{\partial v} \right)_t + \left[ \frac{1}{2} \frac{\sigma v (1 - \gamma)(1 - \gamma) + (1 - \gamma) \frac{-\partial}{\partial t}}{\Phi_1 + \Phi_2 (1 - \rho^2) - (\gamma - 1)} \right] \frac{\partial^2 h}{\partial v^2} \tag{4.63}
\]

\[
+ \left[ \frac{1}{2} \sigma^2 v (1 - \gamma) \right] h_{vv} = 0.
\]

See details in Appendix C.5.

To produce solvable cases, we eliminate the term \(\frac{h_v}{h}\), dividing both sides simultaneously by
(1 − γ) leads to

\[
(\varepsilon_1)^{-\frac{1}{\gamma}} + h_t + \left[ r - \frac{\delta}{\gamma} + \frac{1}{2} \left( \frac{\rho \lambda_1 + \sqrt{1 - \rho^2 \lambda_2}}{(\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))} \right) \right] h_t \nonumber
\]

where

\[
\begin{align*}
\kappa_{\tilde{\theta}} &= \frac{\sigma \rho (\rho \lambda_1 + \sqrt{1 - \rho^2 \lambda_2}) b}{(\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))} (\gamma + \phi_1) + \kappa, \\
\tilde{\kappa} &= -\frac{\sigma \rho (\rho \lambda_1 + \sqrt{1 - \rho^2 \lambda_2}) a}{(\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))} (\gamma + \phi_1) + \kappa.
\end{align*}
\]

The main result is given next.

**Proposition 4.4.3 (4/2 model in incomplete market, robustness).** Let

\[
\begin{align*}
\mu &= \frac{1}{2} \frac{\gamma}{1 - \gamma} (\lambda_1 \rho + \lambda_2 \sqrt{1 - \rho^2})^2 a^2, \\
\nu &= \frac{1}{2} \frac{\gamma}{1 - \gamma} (\lambda_1 \rho + \lambda_2 \sqrt{1 - \rho^2})^2 b^2.
\end{align*}
\]

Assume

\[
(1 - \gamma) \gamma - \phi_1 = \frac{-\rho^2 (\gamma - \phi_1)^2 (1 - \gamma)}{\gamma (\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))},
\]

and \( \kappa, \tilde{\theta}, \mu, \) and \( \nu \) in Equation (4.66) satisfy conditions (4.17).

Then, a candidate solution of the HJBI equation (4.61) is \( J(x, v, t) = \frac{\mu}{\gamma} \left( h(t, v) \right)^{1-\gamma} \), where \( h \) admits the representation:

\[
h(v, t) = (\varepsilon_1)^{-\frac{1}{\gamma}} \int_t^T g(v, \bar{\tau}) d\bar{\tau} + g(v, \tau),
\]

\[
g(v, \tau) = \exp \left\{ \frac{\gamma}{1 - \gamma} \left( r - \frac{\delta}{\gamma} - \frac{\lambda_1 \rho b^2}{\gamma (1 - \phi_1)} - \frac{\lambda_2 \rho b^2}{\gamma (1 - \phi_2)} \right) (T - t) \right\} \times q(\tau, v; \alpha, \lambda, \mu, \nu),
\]

where the function \( q(\tau, v; \alpha, \lambda, \mu, \nu) \) is given by

\[
q(\tau, v; \alpha, \lambda, \mu, \nu) = \int_{\tau}^{\min(T, \tau + \Delta)} \int_{\tau}^{\min(T, \tau + \Delta)} \cdots \int_{\tau}^{\min(T, \tau + \Delta)} \left[ \frac{\gamma}{1 - \gamma} \left( r - \frac{\delta}{\gamma} - \frac{\lambda_1 \rho b^2}{\gamma (1 - \phi_1)} - \frac{\lambda_2 \rho b^2}{\gamma (1 - \phi_2)} \right) (T - t) \right] g(v, \tau) \times q(\tau, v; \alpha, \lambda, \mu, \nu),
\]

and the integration is performed over the appropriate regions.

\( \Delta \) is a small positive number chosen to ensure convergence of the integral.
with \( \tau(t) = T - t \), and \( q(\tau, v; \alpha, \lambda, \mu, \nu) \) follows from Equation (4.20) with associated \( m, D, \beta, \) and \( K \). Moreover, the optimal consumption-wealth ratio and stock allocation are given by

\[
\left( \frac{c}{\lambda} \right)^* = \frac{1}{h^{-1}(c) \gamma},
\]

\[
(\pi)^* = \frac{\left( \rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2 \right)}{\phi_1 \rho^2 + \phi_2 (1 - \rho^2) - (\gamma - 1)} + \frac{(1 - \frac{\mu}{\gamma}) \sigma \rho \sqrt{v}(1 - \gamma) h_v}{\phi_1 \rho^2 + \phi_2 (1 - \rho^2) - (\gamma - 1) \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) h_v}.
\]

(4.70)

The worst case measure is determined by

\[
(e^v)^* = \phi_1 [(\pi)^* \rho (a \sqrt{v} + \frac{b}{\sqrt{v}}) + \sigma \sqrt{v} 1 - \gamma h_v],
\]

\[
(e^x)^* = \phi_2 [(\pi)^* \sqrt{1 - \rho^2} (a \sqrt{v} + \frac{b}{\sqrt{v}})].
\]

(4.71)

See the proof in Appendix C.5.

Interestingly, here we manage to produce a closed-form solution in the presence of consumption, which was not possible in the non-robust analysis of section 4.3.2. Moreover, the robust incomplete market solution is not as simple as the non-robust Merton-like solution in Proposition 4.3.2 — there is a non-myopic term that does not go away as \( \rho \) goes to zero. This is an indication that in the presence of consumption, the non-robust solution shall also have had a non-myopic term.

In an incomplete market, in the absence of consumption, the solution is also simpler. Nonetheless we need to impose the condition\(^1\):

\[
\phi_1 = \frac{-(\gamma - \phi_1)^2 \rho^2}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)}.
\]

(4.72)

Then, \( J(x, v, t) = \frac{\pi}{\gamma} h(t, v) \) where \( h \) solves the PDE

\[
0 = h_t + \left[ \gamma - \frac{\rho \lambda_1 (a \sqrt{v} + \frac{b}{\sqrt{v}}) + \sqrt{1 - \rho^2} \lambda_2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right] h + \left[ \kappa (\theta - \nu) - (\gamma - \phi_1) \frac{(\rho \lambda_1 (a \sqrt{v} + \frac{b}{\sqrt{v}}) + \sqrt{1 - \rho^2} \lambda_2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2) \sigma \rho \sqrt{v}}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} h_v + \frac{1}{2} \sigma^2 v h_{vv} \right] \gamma (\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2) \right] h_v + \frac{1}{2} \sigma^2 v h_{vv},
\]

\[\Gamma(v, t) = \kappa - \nu \]

(4.73)

\(^1\)The condition leads to a feasible region of values for \( \phi_1 \) and \( \phi_2 \) given \( \rho \).
where
\[
\hat{\hat{\theta}} = \kappa \theta - (\gamma - \phi_1) \left( \rho \hat{\lambda}_1 + \sqrt{1 - \rho^2 \hat{\lambda}_2} \right) \frac{(\rho \hat{\lambda}_1 + \sqrt{1 - \rho^2 \hat{\lambda}_2})}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \sigma \rho b,
\]
\[
\hat{\kappa} = \kappa + (\gamma - \phi_1) \left( \rho \hat{\lambda}_1 + \sqrt{1 - \rho^2 \hat{\lambda}_2} \right) \frac{\rho \hat{\lambda}_1 + \sqrt{1 - \rho^2 \hat{\lambda}_2}}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \sigma \rho a.
\] (4.74)

If \( \hat{\kappa}, \hat{\theta}, \mu, \) and \( \nu \) satisfy conditions (4.17), then \( h \) admits a Feynman-Kac representation and can be solved as follows:
\[
h(v, t) = \exp \left\{ \left( r \gamma - \frac{(\tilde{\lambda}_1 \rho + \tilde{\lambda}_2 \sqrt{1 - \rho^2}) ah}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right)(T - t) \right\} \times q(t, v; \alpha, \lambda, \mu, \nu),
\] (4.75)
with parameters
\[
\mu = \frac{1}{2} \frac{(\tilde{\lambda}_1 \rho + \tilde{\lambda}_2 \sqrt{1 - \rho^2}) ah}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)},
\]
\[
\nu = \frac{1}{2} \frac{(\tilde{\lambda}_1 \rho + \tilde{\lambda}_2 \sqrt{1 - \rho^2}) bh}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)},
\] (4.76)
where \( q(T - t; v; \alpha, \lambda, \mu, \nu) \) comes from Equation (4.20) with associated \( m, D, \beta, \) and \( K \). The optimal wealth exposures are given by
\[
(\pi)^* = \frac{\sqrt{\nu} \left( \frac{\phi_1 - \gamma}{\gamma} \sigma \rho \right) h + \left( \rho \hat{\lambda}_1 \left( a \sqrt{\nu} + \frac{b}{\sqrt{\nu}} \right) + \sqrt{1 - \rho^2 \hat{\lambda}_2} (a \sqrt{\nu} + \frac{b}{\sqrt{\nu}}) \right) h}{(a \sqrt{\nu} + \frac{b}{\sqrt{\nu}}) \left( \gamma - 1 - \phi_1 \rho^2 - \phi_2 (1 - \rho^2) \right) h}.
\] (4.77)

The worst case measure is determined by
\[
(\varepsilon^*) = \phi_1 [ (\pi)^* \rho (a \sqrt{\nu} + \frac{b}{\sqrt{\nu}}) + \sigma \sqrt{\nu} \frac{1 - \gamma h_v}{h} ],
\]
\[
(\varepsilon^*) = \phi_2 [ (\pi)^* \sqrt{1 - \rho^2} (a \sqrt{\nu} + \frac{b}{\sqrt{\nu}}) ],
\] (4.78)

### 4.5 Numerical analysis

This section is divided in three subsections corresponding to the three most important contributions of the chapter. First, Section 4.5.1 presents the findings of closed-form solutions to a complete market with consumption (from Section 4.3.1). Second, Section 4.5.2 presents the solution to complete markets for ambiguity-averse investors (from Section 4.4.1). Third, Section 4.5.3 presents an incomplete market case with ambiguity-aversion and consumption (from Section 4.4.2).

Note that, we cannot use the estimation results of the “drift group” from Cheng and Escobar (2021) because of the new choice of MPR for 4/2 and 3/2 models. To accommodate to our
4.5. Numerical analysis

choice of MPR, we re-estimate the rate of market price of risk $\lambda$ for each model by fixing the excess return at $v_t = \theta$ (long-term value), in line with Cheng and Escobar (2021). Then, we follow the procedure of Cheng and Escobar (2021) and substitute $\lambda$ into the regression to update $\rho$ for each model. The estimation results and the other baseline parameters are presented in Tables 4.1 and 4.2, respectively. In this section, we set $\lambda_2 = 2$, and solve for $\lambda_1$ for each model according to the relationship $\lambda = \rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2$.

Table 4.1: Estimates among the various models

<table>
<thead>
<tr>
<th></th>
<th>4/2 Model</th>
<th>3/2 Model</th>
<th>Heston</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{k}$</td>
<td>7.3479</td>
<td>6.9884</td>
<td>14.6290</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>0.0328</td>
<td>0.0323</td>
<td>0.0315</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>0.6612</td>
<td>0.3760</td>
<td>0.5210</td>
</tr>
<tr>
<td>$\hat{a}$</td>
<td>0.9051</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\hat{b}$</td>
<td>0.0023</td>
<td>0.0268</td>
<td>0</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.7689</td>
<td>0.7910</td>
<td>-0.8129</td>
</tr>
<tr>
<td>$\bar{\lambda}$</td>
<td>3.0176</td>
<td>4.2973</td>
<td>2.8689</td>
</tr>
<tr>
<td>Theoretical leverage ($v_t = \theta$)</td>
<td>-0.7689</td>
<td>-0.7910</td>
<td>-0.8129</td>
</tr>
</tbody>
</table>

Table 4.2: Baseline parameters

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\delta$</th>
<th>$\gamma$</th>
<th>$v_0$</th>
<th>$t$</th>
<th>$T$</th>
<th>$\epsilon$</th>
<th>$v_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.02</td>
<td>-0.5</td>
<td>0.04</td>
<td>0</td>
<td>10</td>
<td>0.04</td>
<td></td>
</tr>
</tbody>
</table>

4.5.1 Complete market analysis with consumption

Figures 4.1, 4.2, and 4.3 present the optimal consumption-wealth ratio $c/x$ as a function of standard deviation (SD), investment horizon $T$, and risk-averse level $\gamma$, respectively. Intuitively, the optimal consumption-wealth ratio is related to the state of the economy. All models recommend an increase in consumption in a highly volatile economic state. In particular, the 4/2 model slightly recommends more consumption than the Heston model, while the 3/2 model suggests at least 20% more consumption. This behaviour of the 3/2 model may be explained by its excess return (i.e., $b^2/v_t$), which decreases with the increase of $v_t$. That is, the more volatile the market, the less excess return the 3/2 investor would get from investing in a risky asset. As a result, the investor would allocate his wealth into consumption to get higher utility.

On the other hand, both the Heston and the 4/2 model compensate the investor with higher excess return if the market becomes more risky. Hence, only a small portion of wealth is shifted from investing in risky assets into consumption. In general, the 3/2 model always implies the most wealth exposures, while the 4/2 model lies in between, closer to the conservative Heston model but with higher sensitivity to the changes in market conditions (SD), and risk-aversion level $\gamma$. 
Chapter 4. Optimal consumption and robust portfolio choice for the 3/2 and 4/2 SV models.

Figures 4.1, 4.2, and 4.3 present the plots of the optimal wealth exposure to the variance driver’s risk $\Theta^\nu$ as a function of SD, investment horizon $T$, and risk-aversion level $\gamma$, respectively. In contrast to the 3/2 model, the exposures to the variance driver’s risk under the Heston and the 4/2 model are insensitive to the changes in market conditions. That is, both the Heston and the 4/2 model suggest a constant level of total wealth exposure to variance risk. However, the 3/2 model disinvests the risk of the variance driver as the market gets into a highly volatile state, which can be understood as decreasing the holding on the asset associated with less excess return.

The positiveness of the wealth exposures among models may be explained by the correlation $\rho$ between the risk factors of the asset price and its variance driver for each model. Moreover, all three models recommend a constant level of wealth exposure in terms of investment horizon, as shown in Figure 4.5. Further, if the investor is less risk-averse, all three model suggest more aggressive wealth exposure in the absolute sense.

Figures 4.4, 4.5, and 4.6 present the plots of the optimal wealth exposure to the variance driver’s risk $\Theta^\nu$ as a function of SD, investment horizon $T$, and risk-aversion level $\gamma$, respectively. In contrast to the 3/2 model, the exposures to the variance driver’s risk under the Heston and the 4/2 model are insensitive to the changes in market conditions. That is, both the Heston and the 4/2 model suggest a constant level of total wealth exposure to variance risk. However, the 3/2 model disinvests the risk of the variance driver as the market gets into a highly volatile state, which can be understood as decreasing the holding on the asset associated with less excess return.

The positiveness of the wealth exposures among models may be explained by the correlation $\rho$ between the risk factors of the asset price and its variance driver for each model. Moreover, all three models recommend a constant level of wealth exposure in terms of investment horizon, as shown in Figure 4.5. Further, if the investor is less risk-averse, all three model suggest more aggressive wealth exposure in the absolute sense.

The plot of optimal wealth exposure to the stock’s risk $\Theta^S$ versus the risk-aversion level $\gamma$ is given in figure 4.7. As we expect, less risk-averse investors allocate more wealth to stocks.

The sensitivity analysis of parameters $a$, $b$ on the optimal consumption-wealth ratio $c/x$ and optimal wealth exposure $\Theta^\nu$ with the 4/2 model is explored in figures 4.8, 4.9, 4.10, and 4.11 respectively. Although the 3/2 model behaves differently from the Heston model from our previous observation, the consumption-wealth ratio trends seem dominated by $b$ (i.e., more sensitivity to $b$), while the wealth exposure $\Theta^\nu$ is dominated by the 1/2 component (i.e., more sensible to changes in $a$).
4.5. **Numerical analysis**

![Image](image1.png)

**Figure 4.7:** $\Theta^F$ vs. $\gamma$

![Image](image2.png)

**Figure 4.8:** $c/x$ vs. $a$

![Image](image3.png)

**Figure 4.9:** $c/x$ vs. $b$

![Image](image4.png)

**Figure 4.10:** $\Theta^v$ vs. $a$

![Image](image5.png)

**Figure 4.11:** $\Theta^v$ vs. $b$

### 4.5.2 Complete market analysis without consumption for ambiguity-averse investors

In this case, we have a constraint on the levels of ambiguity-aversion and risk-aversion allowed to produce closed-form solutions, see Equation (4.56), which is

$$
\phi_1 = \frac{\gamma^2}{\gamma + 1}.
$$

(4.79)

In this section, we continue using the baseline parameters, $\phi_1 = 0.5$ with $\gamma = -0.5$, and we further set $\phi_2 = 2$ in this section. The plots of optimal wealth exposures to the variance driver’s risk $\Theta^v$ versus SD and investment horizon $T$ are displayed in figures 4.12 and 4.13. It can be seen that all the three models are quite insensitive to changes in the state of volatility and investment horizon, whereas the $3/2$ model is apparently more aggressive than the Heston and the $4/2$ model by suggesting almost double the exposure to wealth.

![Image](image6.png)

**Figure 4.12:** $\Theta^v$ vs. SD

![Image](image7.png)

**Figure 4.13:** $\Theta^v$ vs. $T$
Chapter 4. Optimal consumption and robust portfolio choice for the 3/2 and 4/2 SV models.

The impact of the parameters $a$, $b$ on wealth exposure $\Theta^v$ with the 4/2 model can be found in figures 4.14 and 4.15, respectively. The marginal effect of the 1/2 component decreases dramatically when $a$ is greater than 0.5, while the 3/2 component $b$ suggests a slight increase in the exposure of wealth to the variance driver’s risk.

![Figure 4.14: $\Theta^v$ vs. $a$](image1)

![Figure 4.15: $\Theta^v$ vs. $b$](image2)

4.5.3 Incomplete market analysis with consumption for ambiguity-averse investors

This last analysis also requires a parametric constraint on the levels of ambiguity-aversion and risk-aversion; see Equation (4.67). The constraint is

$$\gamma - \phi_1(1 - \gamma) = \frac{-\rho^2(\gamma - \phi_1)^2(1 - \gamma)}{\gamma(\phi_1\rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))}. \quad (4.80)$$

To fulfill this condition, we set the risk-aversion level $\gamma = 0.5$, and the ambiguity level $\phi_1 = 3$, and then solve for $\phi_2 (= 3.5)$ accordingly.

The plots of the optimal consumption-wealth ratio $c/x$ and the optimal allocation $\pi$ versus a) SD and b) investment horizon $T$ are presented in figures 4.16, 4.17, and figures 4.18, 4.19, respectively. Generally speaking, by comparing figures 4.16 and 4.17 with figures 4.1 and 4.2, the investor who follows the 3/2 model benefits the most when allowed to hedge the variance and thus complete the market. Specifically, in a complete market, the 3/2 model leads to more consumption while in an incomplete market, the 3/2 model leads to the least consumption among models.

As Liu (2010), Maenhout (2006), and Pu and Zhang (2021) pointed out, a robust investor with risk-aversion level $\gamma$ and ambiguity-aversion level $\phi$ will follow the same strategy as that of a non-robust investor with risk-aversion level $\gamma + \phi$. Hence, the conservative behavior of the 3/2 model may be explained by its sensitivity to the risk-aversion level. In comparison, the Heston and 4/2 models provide a relatively robust and conservative consumption rate with 4/2 becoming closer to the Heston model in highly volatile markets. Figures 4.18 and 4.19 illustrate that the 4/2 model recommends a strategy that lies between the Heston and 3/2 models by being more aggressive than the Heston model but not as aggressive as the 3/2 model.
The sensitivity analyses of the optimal consumption-wealth ratio $c/x$ and optimal allocation $\pi$ to the parameters $a$ and $b$ are explored in figures 4.20, 4.21, 4.22, and 4.23 respectively. From figures 4.20 and 4.21, it is interesting to find that $b$ changing in $[0, 0.01]$ can affect the consumption-wealth ratio as much as that of $a$ changing in $[0, 2]$, thus confirming the larger impact of $b$ on consumption. Furthermore, the risky asset allocation decreases in both $a$ and $b$ with higher sensitivity to $a$. Comparing figures 4.8-4.9 and figures 4.20-4.21, the effects from $a$ and $b$ could be significantly different when accounting for market completeness.
Chapter 4. Optimal consumption and robust portfolio choice for the 3/2 and 4/2 SV models.

4.6 Conclusion

In this chapter, an optimal investment problem for a risk-averse investor under the 4/2 SV model and a 4/2-structured MPR is considered by combining with various elements of interest to scholars and practitioners. These elements include market completeness, terminal wealth with consumption, and ambiguity-aversion. By employing a corresponding derivative to complete the market, taking consumption into account, and allowing for different levels of uncertainty with respect to different risk factors, we orient our setting more closely to real world, which implies the importance of finding a closed-form solution. Although the non-affine nature of the 4/2 volatility and the 4/2-structured MPR is challenging, we found closed-form solutions for the case of a complete market with consumption, and for all the other interesting cases under certain conditions.

In the numerical part, we present and compare the portfolio strategies recommended by the 4/2, 3/2, and 1/2 models for an investor who either cares about consumption or concerns about misspecification of the model in a complete market, and we consider both consumption and ambiguity in an incomplete market using real-data parameters. The 4/2 and 1/2 models generally behave similarly in wealth exposures compared to that of 3/2 model. The 4/2 model behaves like an average by lying in-between the Heston and 3/2 models in consumption in a complete market, as well as in consumption and investment recommendation in an incomplete market.
Chapter 5

Multivariate 4/2 stochastic volatility model

5.1 Introduction

Multivariate stochastic covariance models are key in mathematical finance due to the widely confirmed presence of stochastic volatility and stochastic correlation in stock returns and other financial assets. Improving and creating new models is paramount to keep up with the evolving complexity of financial data. For relevancy and realism, new models shall be parsimonious, interpretable, and capable of delivering analytical solutions to important financial questions, from the price of a derivative to the optimal strategy in a portfolio.

In this chapter, we take advantage of a state-of-the-art stochastic volatility model, recently proposed by Grasselli (2017), to develop a multivariate 4/2 version with ample analytical properties, while easily interpretable parameters capture a variety of stylized facts to be described in the chapter. In particular, our model boasts stochastic volatility of the 4/2 type, stochastic correlation that avoids long periods of high or low values, leverage effect, and the rarely explored stochastic co-volatility movements (i.e., stochastic correlation among volatilities). Although we obtain a closed-form solution to characteristic functions under the pricing and historical measures, our application focuses on expected utility portfolio optimization. This is an area with fewer studies in the literature, and where interpretability of parameters and results is more challenging, which benefits our model.

Ball and Torous (2000) used a multivariate system of jump-diffusion processes to capture systematic risk, and they highlighted the importance of stochastic volatility and correlations in determining optimal portfolio choice. Ang and Bekaert (2002) solved a dynamic portfolio choice problem with a time-varying investment opportunity set using a regime-switching process. They captured volatility co-movements with two switching regimes, where the regimes were distinguished by the magnitudes of correlations and volatilities. Das and Uppal (2004) modeled co-volatility movements relying on introducing unpredictable Poisson shocks in asset returns and solving for optimal portfolio choice in a constant opportunity set. In 2007, Liu (2007) found analytical solutions for the largest family of solvable multivariate processes known today in the context of expected utility with consumption. These are the exponential affine and exponential quadratic families, and many important models can be derived from them. In particular, Buraschi et al. (2010) developed the intertemporal portfolio choice problem with the stochastic variance-covariance matrix specified as a Wishart diffusion process.
The Wishart process is arguably the richest stochastic covariance model in existence but a challenge in terms of interpretability. Other examples are Escobar et al. (2017), who solved portfolio allocation for the principal component stochastic volatility (PCSV) model in complete and incomplete markets. Compared to the Wishart process, the PCSV model and other orthogonally decomposable models, see De Col et al. (2013), Escobar-Anel et al. (2022), Escobar (2018), are more parsimonious and interpretable.

Instead of considering Heston-like stochastic components to build a multivariate structure, in this chapter we assume the newly proposed 4/2 stochastic volatility model. The state-of-the-art 4/2 model gained popularity for improving upon the limitations of the Heston and 3/2 stochastic volatility models, not only for pricing purposes but also in portfolio optimization. The model has been extended, in one dimension, with other stylized facts (for instance, jumps in Lin et al. (2017), and mean reversion in Escobar-Anel and Gong (2020)) within derivative pricing. Within the framework of portfolio optimization, Cheng and Escobar (2021) and Cheng and Escobar-Anel (2021) solved the expected utility problem for a robust portfolio choice, delivering optimal allocations dependant on market variance conditions. To the best of our knowledge, the only multivariate 4/2 models in the literature were proposed in Escobar-Anel and Gong (2020) and Cheng et al. (2019) in the context of derivative pricing.

The multivariate 4/2 model proposed in this chapter is not part of the family in Cheng et al. (2019), but it fits within the larger family proposed by Liu (2007). The model can capture stochastic volatility of a new type (i.e. a 4/2 two-factors, generalizing the 1/2 two-factors of Christoferstesen et al. (2009)), stochastic correlation among stocks driven by a convenient ratio of 4/2 processes, and stochastic correlation among variances (volatility co-movements) of a local type, as it is also driven by the same 4/2 processes. In terms of interpretation, we consider a common factor embedded in the idea of the capital asset pricing model (CAPM) to model systematic risk, but the 4/2 structure provides additional flexibility to control the correlation not only between stocks (via the parameter $\beta$) but also between stocks’ variances (via $b_3$). In this setting, we consider a portfolio optimization problem within the framework of EUT in an incomplete market.

The chapter is organized as follows. Section 5.2 describes the model, and the main statistics of interest (i.e., volatility and correlations among stocks and variances). Section 5.3 focuses on the properties of the model, namely changes of measure, and characteristic functions under historical and risk-neutral measures. The portfolio and solution are presented in Section 5.4. Finally, Section 5.5 presents a numerical analysis, which includes the impact of main statistics and parameters, in comparison with other popular models.

### 5.2 The model

Let us assume a financial market consisting of one risk-free asset and two risky assets (i.e., stocks), which can be traded continuously. Let all the stochastic processes introduced in this chapter be defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0,T]})$, where $\mathcal{F}_t$ is a right-continuous information filtration generated by standard Brownian motions (BM). The price process of the risk-free asset (money market) $M_t$ evolves according to

$$dM_t = M_{t-} rd_t, \ M_0 = 1,$$  \quad (5.1)
where the interest rate \( r \) is assumed to be constant. The price processes \( S_{it}, i = 1, 2 \), of the risky assets follow the structure

\[
dS_{1t} = S_{1t} \left[ \mu_1 dt + (\sqrt{\nu_{1t}} + \frac{b_1}{\sqrt{\nu_{1t}}})dW_{1t} \right], \quad S_{1}(0) = S_{1} > 0,
\]

\[
dS_{2t} = S_{2t} \left[ \mu_2 dt + \beta(\sqrt{\nu_{1t}} + \frac{b_2}{\sqrt{\nu_{2t}}})dW_{1t} + (\sqrt{\nu_{2t}} + \frac{b_2}{\sqrt{\nu_{2t}}})dW_{2t} \right], \quad S_{2}(0) = S_{2} > 0,
\]

\[
d\nu_{1t} = \kappa_1 (\nu_1 - \nu_{1t}) dt + \sigma_1 \sqrt{\nu_{1t}} dZ_{1t}, \quad \nu_{1}(0) = \nu_1 > 0,
\]

\[
d\nu_{2t} = \kappa_2 (\nu_2 - \nu_{2t}) dt + \sigma_2 \sqrt{\nu_{2t}} dZ_{2t}, \quad \nu_{2}(0) = \nu_2 > 0,
\]

where \( \nu_{1t}, i = 1, 2 \), are the variance drivers, Cox–Ingersoll–Ross (CIR) processes, with mean-reversion rate \( \kappa_i > 0 \), long-run mean \( \bar{\nu}_i > 0 \) and volatility of volatility \( \sigma_i > 0 \). The Feller conditions, i.e., \( 2\kappa_i\bar{\nu}_i \geq \sigma_i^2 \), for \( i = 1, 2 \) are imposed to keep the processes \( \nu_i \) strictly positive. The two standard BMs \( W_{it} \) and \( Z_{it} \) are correlated with parameter \( \rho_i \in (-1, 1) \); that is,

\[
d(W_{it}, Z_{it}) = \rho_i dt. \quad \text{Hence, we can write } dZ_{it} = \rho_i dW_{it} + \sqrt{1 - \rho_i^2} dW_{it}^Q, \quad \text{for independent Brownian motions } W_{it}^Q, i = 1, 2. \quad \text{Moreover, the parameters } b_1 \text{ and } b_2 \text{ are positive, while } b_3 \text{ can be any real number.}
\]

For the purpose of establishing both historical and risk-neutral measures for our model, we define the following market prices of equity risk:

\[
\begin{align*}
\lambda_1(v_{1t}) &= \lambda_1 \sqrt{\nu_{1t}}, \\
\lambda_2(v_{2t}) &= \lambda_2 \sqrt{\nu_{2t}},
\end{align*}
\]

where \( \lambda_1(v_{1t}) \) is the market price of risk for \( S_1 \) (i.e., with respect to \( W_{1t} \)), with \( \lambda_1 \) constant, while \( \lambda_2(v_{2t}) \) is the market price of idiosyncratic risk for \( S_2 \) (i.e., with respect to \( W_{2t} \)), with \( \lambda_2 \) constant. Here, \( \lambda_1 \) and \( \lambda_2 \) are controllers for the excess return of \( S_1 \) and \( S_2 \), respectively; thus, they should be positive (Ait-Sahalia and Kimmel, 2007). This setting of the market price of risk leads to the following drifts for the stocks:

\[
\begin{align*}
\mu_1 &= r + \lambda_1(v_{1t} + b_1) \\
\mu_2 &= r + \lambda_2(\nu_{1t} + b_3) + \lambda_2(v_{2t} + b_2),
\end{align*}
\]

where \( r \) is the risk-free rate of return. Similarly, the variance drivers \( \nu_{1t}, i = 1, 2 \) shall remain in the family of CIR processes under the risk-neutral measure. In particular, we will consider

\[
\begin{align*}
\lambda_1(v_{1t}) &= \lambda_1' \sqrt{\nu_{1t}}, \\
\lambda_2(v_{2t}) &= \lambda_2' \sqrt{\nu_{2t}},
\end{align*}
\]

where \( \lambda_1'(v_{1t}) \) is the market price of risk with respect to \( W_{1t}^Q \), with \( \lambda_1' \) constant, while \( \lambda_2'(v_{2t}) \) is the market price of risk with respect to \( W_{2t}^Q \), with \( \lambda_2' \) constant. This leads to the following CIR processes \( \nu_{it} \):

\[
\begin{align*}
d\nu_{1t} &= \left( \kappa_1 \nu_1 - \left( \kappa_1 + (\lambda_1 \rho_1 + \lambda_1' \sqrt{1 - \rho_1^2}) \sigma_1 \right) v_{1t} \right) dt + \sigma_1 \sqrt{\nu_{1t}} \left( \rho_1 dW_{1t}^Q + \sqrt{1 - \rho_1^2} dW_{1t}^{Q,1} \right), \\
d\nu_{2t} &= \left( \kappa_2 \nu_2 - \left( \kappa_2 + (\lambda_2 \rho_2 + \lambda_2' \sqrt{1 - \rho_2^2}) \sigma_2 \right) v_{2t} \right) dt + \sigma_2 \sqrt{\nu_{2t}} \left( \rho_2 dW_{2t}^Q + \sqrt{1 - \rho_2^2} dW_{2t}^{Q,1} \right).
\end{align*}
\]

(5.7)
The asset $S_1$ can be viewed as a market index, say the S&P 500. The risk from $S_1$ can be taken as systematic risk for individual stocks such as $S_2$. This is similar to the CAPM model, where $W_2$ would play the role of idiosyncratic risk. Moreover, as suggested by Christoffersen et al. (2009), a stochastic volatility model with two components is better at capturing the rich volatility structure implied from option prices (i.e., implied volatilities). This motivates the presence of two stochastic volatility drivers for $S_2$.

An important feature of our model is the stochastic volatility of the risky assets. To illustrate its structure clearly, let $\Sigma$, represent the instantaneous quadratic variation of $S_i$, $i = 1, 2$, then we have

\[
\begin{align*}
\text{d}\Sigma_1 &= d\left(\sqrt{\nu_{1t}} + \frac{b_1}{\sqrt{\nu_{1t}}}\right)^2 = (1 + b_1^2)\text{d}v_{1t}, \\
&= (1 + b_1^2)\left(\kappa_1 (\theta_1 - v_{1t}) \text{d}t + \sigma_1 \sqrt{\nu_{1t}} \text{d}Z_{1t}\right)
\end{align*}
\]

\[
\begin{align*}
\text{d}\Sigma_2 &= d\left(\beta^2(\sqrt{\nu_{1t}} + \frac{b_3}{\sqrt{\nu_{1t}}} + \sqrt{\nu_{2t}} + \frac{b_2}{\sqrt{\nu_{2t}}}\right)^2 \\
&= \beta^2(1 + b_1^2)\left(\kappa_1 (\theta_1 - v_{1t}) \text{d}t + \sigma_1 \sqrt{\nu_{1t}} \text{d}Z_{1t}\right) + (1 + b_2^2)\left(\kappa_2 (\theta_2 - v_{2t}) \text{d}t + \sigma_2 \sqrt{\nu_{2t}} \text{d}Z_{2t}\right).
\end{align*}
\]

(5.8)

Note that while $b_1, v_{1t}$ control the instantaneous variance of the first asset, we can use $b_2, v_{2t}$ to control the idiosyncratic variance of the second asset and therefore its instantaneous variance. This leaves $\beta$ and $b_3$ as free parameters, helpful to control other stylized facts of the model, as explained next. Moreover, the volatility structure implied by the model, $\Sigma^{1/2}dW$, is a combination of 1/2 and 3/2 processes, a natural generalization of the 4/2 model:

\[
\begin{align*}
\Sigma^{1/2} = \begin{pmatrix}
1 & 0 \\
\beta & 1
\end{pmatrix}
\begin{pmatrix}
\sqrt{\nu_{1t}} & 0 \\
0 & \sqrt{\nu_{2t}}
\end{pmatrix}
+ \begin{pmatrix}
0 & 1 \\
\beta b_3 & b_2
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{\nu_{1t}}} & 0 \\
0 & \frac{1}{\sqrt{\nu_{2t}}}
\end{pmatrix}.\end{align*}
\]

We now turn our attention to the stochastic correlation between the two risky assets. The formula is provided below:

\[
\begin{align*}
\rho_{stn} &= \frac{\langle \text{d}s_{1t}, \text{d}s_{2t} \rangle}{\sqrt{\langle \text{d}s_{1t}, \text{d}s_{1t} \rangle} \langle \text{d}s_{2t}, \text{d}s_{2t} \rangle} \\
&= \frac{(\sqrt{\nu_{1t}} + b_1 \sqrt{\nu_{1t}})\beta(\sqrt{\nu_{1t}} + b_1 \sqrt{\nu_{1t}})}{\sqrt{(\sqrt{\nu_{1t}} + b_1 \sqrt{\nu_{1t}})^2(\beta^2(\sqrt{\nu_{1t}} + b_1 \sqrt{\nu_{1t}})^2 + (\sqrt{\nu_{2t}} + b_2 \sqrt{\nu_{2t}})^2)}} \\
&= \frac{\beta}{\sqrt{\beta^2(\sqrt{\nu_{1t}} + b_1 \sqrt{\nu_{1t}})^2 + (\sqrt{\nu_{2t}} + b_2 \sqrt{\nu_{2t}})^2}} \\
&= \frac{\beta}{\sqrt{\beta^2 + (\sqrt{\nu_{1t}} + b_1 \sqrt{\nu_{1t}})^2}}. \\
\end{align*}
\]

(5.9)

\footnote{Our structural results can be extended to $n$ dimensions in many, easily interpretable ways, for instance with $dS_i = \text{diag}(S_i)\left(\mu_i dt + \text{Adiag} (\sqrt{\nu_i}) + B\text{diag}(\frac{1}{\sqrt{\nu_i}}) \text{d}W_i\right)$, where $A, B$ are $n \times n$ lower diagonal matrices. Each parametrization would lead to different conditions in the applications.}
where \( v_{1t}, v_{2t}, b_1, b_2 \) are positive, while \( \beta \) and \( b_3 \) are arbitrary. Note that \( \beta \) is the key parameter to capture this correlation, and this is how we should interpret it; this parameter goes all the way to CAPM models. If \( \beta < 0 (> 0) \), then the correlation between assets would be normally negative (positive). Nonetheless, if \( b_3 < 0 \), the \( \rho_{sto} \) can switch between positive and negative values, depending on the level of \( v_{1t} \). Moreover, if we set \( b_3 \neq b_1 \), the volatility of the common driver (systemic part) would not be identical for \( S_1 \) and \( S_2 \). These observations highlight the importance of \( b_3 \) in adding flexibility to the modeling of \( \rho_{sto} \).

Moreover, the correlation in our model is driven by a ratio of 4/2 processes, \( \frac{b_2}{b_1 + b_3} \). This is convenient as opposed to a ratio of 1/2 or 3/2. To see this, note that a 1/2 (Heston) fitting usually leads to a violation of the Feller condition. This means the variance is close to zero for long periods. In such case, the correlation would stay either very high or very low for long periods. A similar situation would happen with using a purely 3/2 model (the flip 1/2), the 3/2 would admit extreme paths for long periods, leading to very high/very low correlations. The correlation implied by the ratio of 4/2 avoids those pitfalls as the variance is bounded away from zero and, for low values of \( b \), would avoid sustained extreme values. This means the correlation would avoid long periods of low/high values. These are important differences with the stream of examples of multivariate processes in the literature constructed via independent one-dimensional stochastic volatility models, such as those by De Col et al. (2013) and Escobar et al. (2017). The observations highlight the flexibility and benefit of our model while still using a linear combination of independent one-dimensional processes.

Lastly, we feature another important stylized fact of financial data, called co-volatility movements (see Christodoulakis (2001), Escobar et al. (2017) and citations therein), which refer to the correlation between the movements of stocks’ volatilities. We denote it as \( \rho_{covo} \) and compute it next:

\[
\rho_{covo} = \text{Corr}(d\Sigma_1, d\Sigma_2) = \frac{\beta^2\sigma_1^2 v_{1t} + (b_1^2 + b_2^2)v_{1t}^2 + b_1^2b_3^2v_{1t}^3}{\sqrt{(1 + b_1^2 v_{1t})^2\sigma_1^2 v_{1t}} \times \left(\beta^2(1 + b_3^2 v_{1t})^2\sigma_1 v_{1t} + (1 + b_2^2 v_{1t})^2\sigma_2^2 v_{1t}\right)}
\]

\[
= \frac{\sigma_1 \sqrt{\nu_{1t}}}{\sqrt{\beta^2(1 + b_1^2 v_{1t})}}.
\]

(5.10)

This correlation is positive by construction, similar to empirical findings (see Escobar et al. (2017) and citations therein). We can control the level of this correlation via the parameter \( b_3 \). If \( b_3 = b_1 \), then the parameter \( \beta \) would have to control both correlations (\( \rho_{sto} \) and \( \rho_{covo} \)). However, the stochastic correlations between assets and the volatility co-movements are two separate stylized facts that practitioners must control and model separately. Hence, the degree of freedom brought by \( b_3 \) serves its key purpose here.

A closely relevant work is by Buraschi et al. (2010); the authors modelled the stochastic variance-covariance via a Wishart diffusion process. The importance of considering stochastic covariance in optimal asset allocation was thoroughly analyzed, and they found that the optimal hedging demand is significantly higher than that of constant correlations models. However, although the Wishart process captures most stylized facts of covariance structures, its parameters
Chapter 5. Multivariate 4/2 stochastic volatility model

suffer from limited interpretability. This is most evident when it comes to variance of variance
and co-volatility movements. These two concepts overlap at the level of the matrix \( Q \) in the
construction 
\[
\Sigma_2^2 = \left[ \Omega_Y + M \Sigma_2^2 + \Sigma_2^2 M \right] dt + \Sigma dB_t, \quad Q' dB_t \Sigma_t,
\]

hence obscuring their separate impact in applications. Our multivariate model offers a clear distinction between variance of variance (i.e., \( A_2^2 \)), and volatility co-movements (i.e., \( \rho_{\text{covo}} \)). Our analyses of their impact on optimal asset allocation will reveal their true importance.

5.3 Properties of the model

This section reports several properties of the proposed model. In particular, we obtain viable
changes of measure, the joint (and marginal) conditional c.f. for stocks, conditions for the
existence of the m.g.f., and the joint c.f. conditional on final variance.

First, we explore conditions that ensure the feasibility of the proposed changes of measure:

\[
d W_Q^i = \bar{\lambda}_i \sqrt{\sigma_i} dt + d W_i, \quad d W_Q^i, \quad d W_i = \bar{\lambda}_i \sqrt{\sigma_i} dt + d W_i,
\]

where \( \bar{\lambda}_i \) and \( \bar{\lambda}_i^+ \) are constants, with \( i = 1, 2 \).

**Proposition 5.3.1.** The change of measure is well defined under the following conditions:

\[
\max \{ |\bar{\lambda}_i|, |\bar{\lambda}_i^+| \} < \frac{\kappa_i}{\sigma_i}, \quad i = 1, 2, \tag{5.11}
\]

\[
\sigma_i^2 \leq 2 \kappa_i \sigma_l - 2 \max \{ |\kappa_1 \rho_1 b_1|, |\kappa_1 \rho_1 b_2| \}, \tag{5.12}
\]

\[
\sigma_i^2 \leq 2 \kappa_i \sigma_l - 2 |\kappa_1 \rho_1 b_2|, \tag{5.13}
\]

\[
\kappa_1 + \sigma_1 \rho_1 \beta > 0, \tag{5.14}
\]

\[
\kappa_2 + \sigma_2 \rho_2 > 0, \tag{5.15}
\]

\[
\kappa_i + \sigma_i \left( \rho_i \lambda_i + \sqrt{1 - \rho_i^2 \lambda_i^2} \right) > 0, \quad i = 1, 2. \tag{5.16}
\]

See Appendix D.1 for the proof.

Next, we provide a closed-form solution for the joint conditional generalized c.f. under the
risk-neutral measure \( Q \), defined as follows:

\[
\Psi(u_1, u_2; S_{1,0}, S_{2,0}, v_{10}, v_{20}) = \mathbb{E}_Q^Q \left[ \exp \left( u_1 \ln(S_1) + u_2 \ln(S_2) \right) \right]_{F_0},
\]

where \( u_1, u_2 \) are complex numbers. This is essential for pricing purposes. A similar result for
the c.f. under \( P \) is given in the appendix, Section D.2.2.

**Proposition 5.3.2.** Assume that \( (S_1, S_2) \) follow Equations (5.2) and (5.3). The joint conditional
Moreover, we define the following parameters that may facilitate the expressions afterwards:

\[
\begin{align*}
\Psi(u_1, u_2; S_{1,0}, S_{2,0}, v_{10}, v_{20}) &= \exp \left[ \left( u_1 \ln(S_{1,0}) + u_2 \ln(S_{2,0}) \right) + \left( u_1 + u_2 \right) \left( r - G_1 - \frac{\kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3 \kappa_1}{\sigma_1 \rho_1} + \frac{(u_1 + u_2)(1 - \rho_1^2)G_3}{\rho_1^2} \right) \right] \times 
\frac{(u_1 + u_2)}{\sigma_1 \rho_1} & \times \exp \left[ \left( \frac{G_3(u_1 + u_2)\log(v_{10})}{\sigma_1 \rho_1} \right) \times \phi_1(t, v_1; \alpha_1, \lambda_1, \mu_1, \nu_1) \times \phi_2(t, v_2; \alpha_2, \lambda_2, \mu_2, \nu_2) \right],
\end{align*}
\]

where

\[
\begin{align*}
G_0 &= \frac{u_1 + u_2 \beta^2}{u_1 + u_2}, \quad G_1 = \frac{u_1 b_1 + u_2 b_2 \beta^2}{u_1 + u_2}, \quad G_2 = \frac{u_1 b_1^2 + u_2 b_2 \beta^2}{u_1 + u_2}, \quad G_3 = \frac{u_1 b_1 + u_2 b_2 \beta}{u_1 + u_2}, \\
\alpha_1 &= -\frac{G_3(u_1 + u_2)}{\sigma_1 \rho_1}, \quad \lambda_1 = \frac{(u_1 + u_2)}{\sigma_1 \rho_1}, \\
\mu_1 &= -(u_1 + u_2) \left[ -1 - \frac{G_0}{\frac{G_3(u_1 + u_2)}{\sigma_1 \rho_1}} + \frac{\frac{(u_1 + u_2)(1 - \rho_1^2)G_3}{\rho_1^2}}{2} \right], \\
\nu_1 &= -(u_1 + u_2) \left[ -1 - \frac{G_2}{2} - \frac{G_3 \kappa_1}{2 \rho_1} + \frac{G_3 \sigma_1}{2 \rho_1} + \frac{(u_1 + u_2)(1 - \rho_1^2)G_3^2}{\rho_1^2} \right],
\end{align*}
\]

while

\[
\begin{align*}
\alpha_2 &= -\frac{u_2 b_2}{\sigma_2 \rho_2}, \quad \lambda_2 = -\frac{u_2}{\sigma_2 \rho_2}, \\
\mu_2 &= -u_2 \left[ -1 + \frac{\frac{\kappa_2 b_2}{\sigma_2 \rho_2} + \frac{1}{2} \frac{u_2 (1 - \rho_2^2)G_2}{\rho_2^2}}{2} \right], \\
\nu_2 &= -u_2 \left[ -1 + \frac{b_2^2}{2} - \frac{\kappa_2 \theta_2 b_2}{2 \sigma_2 \rho_2} + \frac{\sigma_2 b_2}{2 \rho_2} + \frac{1}{2} \frac{u_2 (1 - \rho_2^2)G_2}{\rho_2^2} \right].
\end{align*}
\]

Moreover, we define the following parameters that may facilitate the expressions afterwards:

\[
\begin{align*}
m_i &= \frac{1}{\sqrt{\sigma_i^2}} \sqrt{(2\kappa_i \theta_i - \sigma_i^2)^2 + 8\sigma_i^2 \nu_i}, \quad A_i = \kappa_i^2 + 2\mu_i \sigma_i^2, \\
\beta_i(t, v_i) &= \frac{2 \sqrt{A_i \nu_i}}{\sigma_i^2 \sinh \left( \frac{\sqrt{A_i}}{2} \right)}, \quad K_i(t) = \frac{1}{\sigma_i^2} \left( \sqrt{A_i} \coth \left( \frac{\sqrt{A_i} t}{2} \right) + \kappa_i \right), \quad (5.17)
\end{align*}
\]

Furthermore, when the complex numbers \((u_1, u_2)\) belong to the strip \(D_{0,\infty} = A_{0,\infty} + i\mathbb{R} \subset \mathbb{C} \) for
all $t \geq 0$, where the convergence set $\mathcal{A}_{0,\infty} \subset \mathbb{R}$ is given by

$$
\mathcal{A}_{0,\infty} = \{(u_1, u_2) \in \mathbb{R}^2 : A_i \geq 0 \text{ and the corresponding parameters } \mu_i, \nu_i, \alpha_i, \lambda_i \text{ for } i = 1, 2 \text{ satisfy conditions in (5.19) – (5.22)} \},
$$

with

$$
\mu_i > -\frac{\kappa^2_i}{2\sigma^2_i}, \quad \mu_i > -\frac{(2\kappa_i \theta_i - \sigma_i)^2}{8\sigma^2_i},
$$

(5.19)

$$
\nu_i \geq \frac{1}{2\sigma^2_i} (2\kappa_i \theta_i + \sigma^2_i + \sqrt{(2\kappa_i \theta_i - \sigma^2_i)^2 + 8\sigma^2_i \nu_i}),
$$

(5.20)

$$
\alpha_i < \frac{\sqrt{\kappa^2_i + 2\mu_i \sigma^2_i + \kappa_i}}{\sigma^2_i},
$$

(5.21)

$$
\lambda_i \geq -\frac{\sqrt{\kappa^2_i + 2\mu_i \sigma^2_i + \kappa_i}}{\sigma^2_i};
$$

(5.22)

or for $0 \leq t \leq t^*$ and $u_1, u_2 \in D_{0,t} = \mathcal{A}_{0,t} + i\mathbb{R}$ where

$$
\mathcal{A}_{0,t} = \{(u_1, u_2) \in \mathbb{R}^2 : 0 \leq A_i < -\frac{\sqrt{\kappa^2_i + 2\mu_i \sigma^2_i + \kappa_i}}{\sigma^2_i}, \text{ and parameters } \mu_i, \nu_i, \alpha_i \text{ for } i = 1, 2 \text{ satisfy conditions in (5.19) – (5.21)} \},
$$

(5.23)

and

$$
\tau^* = \inf \left\{ \frac{1}{\sqrt{A_1}} \log \left( 1 - \frac{2\sqrt{A_1}}{\kappa_i + \sigma^2_i \lambda_i + \sqrt{A_1}} \right), \frac{1}{\sqrt{A_2}} \log \left( 1 - \frac{2\sqrt{A_2}}{\kappa_i + \sigma^2_i \lambda_i + \sqrt{A_2}} \right) \right\},
$$

(5.24)

The functions $\phi_i$, for $i = 1, 2$, are well defined and given as

$$
\phi_i(t, v_i; \alpha_i, \lambda_i, \mu_i, \nu_i) = \left( \frac{\beta_i(t, v_i)}{2} \right)^{m_i+1} \frac{v_i^{\alpha_i}}{\sigma_i^{\nu_i}} K_i(t) \gamma \left( \frac{1}{2} + \frac{m_i}{2} + \frac{\kappa_i}{\sigma_i^2} \right) \times e^{\frac{1}{2} \left( \kappa_i^2 \theta_i r - \sqrt{\kappa_i^2 \nu_i \coth \left( \frac{\sigma_i v_i}{\sqrt{\kappa_i^2 \nu_i}} \right)} \right)} \Gamma \left( \frac{1}{2} + \frac{m_i}{2} + \frac{\kappa_i}{\sigma_i^2} \right) \Gamma (m_i + 1)
$$

$$
\times \frac{1}{4K_i(t)} \left( \frac{1}{2} + \frac{m_i}{2} + \frac{\kappa_i \theta_i}{\sigma_i^2}, m_i + 1, \beta_i(t, v_i)^2 \right),
$$

(5.25)

where $\Gamma(\cdot)$ and $\gamma_{1/2}$ denote the Gamma function and hypergeometric confluent function, respectively.

See Appendix D.2 for the complete proof. We may interchange log and ln throughout this chapter.
Proposition 5.3.3. The m.g.f. of the log-price conditional on \( v \), follows

\[
\Psi_v(u_1, u_2; S_{1,0}, S_{2,0}, v_1, v_2) = \mathbb{E}^Q \left[ \exp \left( u_1 \ln(S_1) + u_2 \ln(S_2) \right) \right]
\]

\[
= \exp \left( \left( u_1 \ln(S_{1,0}) + u_2 \ln(S_{2,0}) \right) + (u_1 + u_2) \left( r - G_1 - \frac{\kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3 \kappa_1}{\sigma_1 \rho_1} + \frac{(u_1 + u_2)(1 - \rho_1^2)}{\rho_1^2} \right) \right) \\
+ \frac{(u_1 + u_2)}{\sigma_1 \rho_1} (v_{1t} - v_{10}) + \frac{G_3 (u_1 + u_2)}{\sigma_1 \rho_1} (\log(v_{1t} - \log(v_{10}))) \times \phi_1(t, v_1; \mu_1, v_1) \\
\times \exp \left( u_2 \left( r - b_2 - \frac{\kappa_2 \theta_2}{\sigma_2 \rho_2} + \frac{b_2 \kappa_2}{\sigma_2 \rho_2} \right) + \frac{(1 - \rho_2^2)}{\rho_2^2} b_2 \right) + \frac{u_2}{\sigma_2 \rho_2} (v_{20} - v_{20}) + \frac{u_2 b_2}{\sigma_2 \rho_2} (\log(v_{2t}) - \log(v_{20})) \right) \\
\times \phi_2(t, v_2; \mu_2, v_2),
\]

\[
G_0 = \frac{u_1 + u_2 b_2^2}{u_1 + u_2}, \quad G_1 = \frac{u_1 b_1 + u_2 b_2 \beta_2^2}{u_1 + u_2}, \quad G_2 = \frac{u_1 b_1^2 + u_2 b_2^2 \beta_2^2}{u_1 + u_2}, \quad G_3 = \frac{u_1 b_1 + u_2 b_2 \beta_2}{u_1 + u_2},
\]

\[
\mu_1 = -(u_1 + u_2) \left[ -\frac{1}{2} G_0 + \frac{\kappa_1}{\sigma_1 \rho_1} + \frac{1}{2} \frac{(u_1 + u_2)(1 - \rho_1^2)}{\rho_1^2} \right],
\]

\[
\nu_1 = -(u_1 + u_2) \left[ -\frac{1}{2} G_2 - \frac{G_3 \kappa_1 \theta_1}{2 \sigma_1 \rho_1} + \frac{G_3 \sigma_1}{2 \rho_1} + \frac{1}{2} \frac{(u_1 + u_2)(1 - \rho_1^2)}{\rho_1^2} \right],
\]

while

\[
\mu_2 = -u_2 \left[ -\frac{1}{2} + \frac{b_2}{\sigma_2 \rho_2} + \frac{1}{2} \frac{u_2 (1 - \rho_2^2)}{\rho_2^2} \right],
\]

\[
\nu_2 = -u_2 \left[ -\frac{b_2^2}{2} - \frac{\kappa_2 \theta_2 b_2}{\sigma_2 \rho_2} + \frac{\sigma_2 b_2}{2 \rho_2} + \frac{1}{2} \frac{u_2 (1 - \rho_2^2)}{\rho_2^2} b_2^2 \right].
\]

Moreover, if we define \( A_i = \kappa_i^2 + 2 \mu_i \sigma_i^2 \) with

\[
\mu_i > -\frac{\kappa_i^2}{2 \sigma_i^2}, \quad \nu_i \geq -\frac{(2 \kappa_i \theta_i - \sigma_i)^2}{8 \sigma_i^2}, \quad (5.26)
\]
then the functions \( \phi_i \), for \( i = 1, 2 \), are well defined and given as

\[
\phi_i(t, v_i; \mu_i, v_i) = \mathbb{E}_t^0 \left[ \exp \left\{ -\mu_i \int_0^t v_i ds - v_i \int_0^t \frac{1}{v_i} \right\} \right] v_i \cdot \frac{\sqrt{A_i} \sinh \left( \frac{v_i}{2} \right)}{\kappa_i \sinh \left( \frac{\sqrt{A_i}}{2} \right)} \exp \left\{ \frac{v_{i0} + v_{it}}{\sigma^2_i} \left( \kappa_i \coth \frac{\sqrt{A_i}}{2} - \sqrt{A_i} \coth \frac{\sqrt{A_i}}{2} \right) \right\}
\]

\[
I_{\frac{d_i}{2}} \sqrt{2d_i (\sigma^2_i - \rho^2)}^{2 + 8d_i v_i} \left( \frac{2\sqrt{A_i \sinh \left( \frac{\sqrt{A_i}}{2} \right)}}{\sigma^2_i \sinh \left( \frac{\sqrt{A_i}}{2} \right)} \right),
\]

where \( I_n(z) \) is the modified Bessel function of the first kind.

See Appendix D.3 for the proof.

### 5.4 Portfolio problem formulation and solution

In this section, we consider an investor who aims at maximizing utility from terminal wealth at time \( T \) with CRRA risk preference, and allocates a proportion \( \pi_i \) of wealth to stock \( S_i \), \( i = 1, 2 \), while the rest goes to a bank account with constant interest rate \( r \). Furthermore, following the form of market price of risk described in the previous section (same as Cheng and Escobar (2021)), the wealth process for this investor under the historical measure evolves according to

\[
dX_t = X_t \left[ r + \pi_1 \lambda_1 (v_{1t} + b_1) + \pi_2 (\beta \lambda_1 (v_{1t} + b_1) + \lambda_2 (v_{2t} + b_2)) \right] dt + X_t \left[ \pi_1 \left( \sqrt{v_{1t}} + \frac{b_1}{\sqrt{v_{1t}}} \right) dW_1 + \pi_2 (\sqrt{v_{2t}} + \frac{b_2}{\sqrt{v_{2t}}}) dW_2 \right], \quad X(0) = x > 0,
\]

where \( x \) is the initial budget. Consider the following change of control

\[
\begin{cases}
\eta_1 = \pi_1 \left( \sqrt{v_{1t}} + \frac{b_1}{\sqrt{v_{1t}}} \right) + \pi_2 (\beta \lambda_1 (v_{1t} + b_1) + \lambda_2 (v_{2t} + b_2)) \\
\eta_2 = \pi_2 (\sqrt{v_{2t}} + \frac{b_2}{\sqrt{v_{2t}}})
\end{cases}
\]

then the wealth process follows

\[
dX_t = X_t \left[ r + \lambda_1 \sqrt{v_{1t}} \eta_1 + \lambda_2 \sqrt{v_{2t}} \eta_2 \right] dt + X_t \left[ \eta_1 dW_1 + \eta_2 dW_2 \right].
\]

The objective of the investor is to find an investment strategy that maximizes the expected terminal utility with utility function \( u(x) = \frac{x^{\gamma}}{\gamma} \), where \( \gamma < 1 \), and value function

\[
J(x, v, t) = \sup_{(\eta_1, \eta_2) \in \mathcal{U}} \mathbb{E}_{x,v,t}[u(X_T)].
\]

Here, \( v_t = (v_{1t}, v_{2t}) \), and \( \mathcal{U} \) denotes the space of admissible strategies (see Definition 3.2 in Cheng and Escobar (2021)). From the Dynamic Programming Principle, the HJB equation for
The optimal allocations are given as

\[
\pi_1^* = \left( \frac{v_1}{v_1 + b_1} \right) \frac{-\lambda_1 - \sigma_1 \rho_1 E(T-t)}{\gamma - 1} - \beta \left( \frac{v_1 + b_3}{v_2 + b_2} \right) \frac{\lambda_2 - \sigma_2 \rho_2 F(T-t)}{\gamma - 1},
\]

\[
\pi_2^* = \left( \frac{v_2}{v_2 + b_2} \right) \frac{-\lambda_2 - \sigma_2 \rho_2 F(T-t)}{\gamma - 1}.
\]
See Appendix D.4 for the complete proof.

In the multivariate 4/2 model, the optimal allocation of $S_2$ is the same as that in the univariate 4/2 model by Cheng and Escobar (2021). Moreover, there is a term that is proportional to the optimal allocation of $S_2$ in the weight of $S_1$, where the factor $\beta_{\mu \gamma}^2$ combines the effects of volatility co-movements and the correlation between assets. More importantly, this term in asset $S_1$ completely hedges away the risk from $W_1$ in asset $S_2$.

In the search for a verification result, the Proposition 5.4.2 next describes conditions for a real-valued and finite value function. The proof is provided in Appendix D.5.

**Proposition 5.4.2.** The function $J(x, v, t)$ is a well-defined solution to the HJB Equation (5.31) if

$$\frac{\gamma}{\gamma - 1} \tilde{\lambda}_i \sigma_i \left(2\rho_{ij} + \tilde{\lambda}_i \sigma_j\right) + \kappa_i^2 > 0, \quad i = 1, 2. \quad (5.38)$$

To ensure that the optimal control is the unique solution and that its associated value function solves the optimal problem, a verification result along the lines of Cheng and Escobar (2021) is provided next.\footnote{The proof, in the stream of Kraft et al. (2013) (Appendix C) is also entertained, which does not rely on a Lipschitz condition for the utility, hence permitting potential extensions to log and Epstein-Zin utilities. Conditions from both proofs are checked to be satisfied by the values of parameters in numerical analysis.}

**Theorem 5.4.3.** Consider a function $J(x, v, t) : [0, \infty) \times [0, \infty)^2 \times [0, T] \to \mathbb{R}$, such that

1) $J$ is real-valued, finite, once continuously differentiable in $t$ and twice continuously differentiable in $x$ and the vector $v$; and

2) $J$ satisfies Equation (5.31) and its terminal condition, with $J(x, v, t) = \frac{x^2}{2} h(v, t)$ for a positive function $h(v, t) = e^{D(T-t)+E(T-t)v_F+T(T-t)v^2};$

3) \[3.1\] The function $E(T-t)$ in Equation (5.34) must satisfy the following condition:

for $0 < \gamma < 1$, $-1 \leq \rho_1 < \frac{\gamma-1}{\gamma}$, and $\tilde{\lambda}_1 > \frac{\sigma_i^2 \left(1 - \frac{\gamma - 1}{\gamma} \rho_1 \right)^2 + 1 - \rho_1^2 \sigma_i^2 E(T)}{1 - \frac{\gamma - 1}{\gamma} \rho_1} > \frac{\sigma_i^2 \left(1 - \frac{\gamma - 1}{\gamma} \rho_1 \right)^2 + 1 - \rho_1^2 \sigma_i^2 E(T)}{1 - \frac{\gamma - 1}{\gamma} \rho_1},

$$-\frac{1}{2} \left[(1 - \frac{\gamma}{\gamma - 1} \rho_1)^2 + 1 - \rho_1^2 \sigma_i^2 E(T) - \frac{\tilde{\lambda}_1 \gamma}{\gamma - 1} (1 - \frac{\gamma}{\gamma - 1} \rho_1) \sigma_i E(T) + \left(\frac{\gamma}{\gamma - 1}\right)^2 \tilde{\lambda}_1^2 \right] \geq -\frac{\kappa_i^2}{2\sigma_i^2}. \quad (5.39)$$

3.2) While the function $F(T-t)$ in Equation (5.35) must satisfy the following condition:

for $0 < \gamma < 1$, $-1 < \rho_2 < \frac{\gamma - 1}{\gamma}$, and $\tilde{\lambda}_2 > \frac{\sigma_i^2 \left(1 - \frac{\gamma - 1}{\gamma} \rho_2 \right)^2 + 1 - \rho_2^2 \sigma_i^2 E(T)}{1 - \frac{\gamma - 1}{\gamma} \rho_2} > \frac{\sigma_i^2 \left(1 - \frac{\gamma - 1}{\gamma} \rho_2 \right)^2 + 1 - \rho_2^2 \sigma_i^2 E(T)}{1 - \frac{\gamma - 1}{\gamma} \rho_2},

$$-\frac{1}{2} \left[(1 - \frac{\gamma}{\gamma - 1} \rho_2)^2 + 1 - \rho_2^2 \sigma_i^2 F(T) - \frac{\tilde{\lambda}_2 \gamma}{\gamma - 1} (1 - \frac{\gamma}{\gamma - 1} \rho_2) \sigma_i F(T) + \left(\frac{\gamma}{\gamma - 1}\right)^2 \tilde{\lambda}_2^2 \right] \geq -\frac{\kappa_i^2}{2\sigma_i^2}. \quad (5.40)$$

4) Both functions $E(T-t)$ and $F(T-t)$ must satisfy the following condition as long as

i) $0 < \gamma < 1$, $-1 \leq \rho_1 < \frac{\gamma - 1}{\gamma}$, $0 < \tilde{\lambda}_1 < \frac{\sigma_i^2 \left(1 - \frac{\gamma - 1}{\gamma} \rho_1 \right)^2 + 1 - \rho_1^2 \sigma_i^2 E(T)}{1 - \frac{\gamma - 1}{\gamma} \rho_1}$ and $0 < \tilde{\lambda}_2 < \frac{\sigma_i^2 \left(1 - \frac{\gamma - 1}{\gamma} \rho_2 \right)^2 + 1 - \rho_2^2 \sigma_i^2 E(T)}{1 - \frac{\gamma - 1}{\gamma} \rho_2}$;
5.5 Numerical analysis

\( \text{ii)} \ 0 < \gamma < 1, \ \frac{\gamma - 1}{\gamma} \leq \rho_i < 0 \) and \( \bar{\lambda}_i > 0; \) or

\( \text{iii)} \ \gamma < 0 \) and \( \bar{\lambda}_i > 0: \)

\[
- \frac{1}{2} \frac{\gamma^2}{(\gamma - 1)^2} \bar{\lambda}_i^2 \geq - \frac{\kappa_i^2}{2\sigma_i^2}.
\]  

(5.41)

Then \( \eta_i^*, \ i = 1, 2 \) in Equation (5.36) denote the optimal control and \( J \) in Equation (5.32) is the corresponding value function. Hence, \( \pi_i^* \) in Equation (5.37) are the optimal allocations.

See Appendix D.6 for the complete proof.

5.5 Numerical analysis

In this section, we report the implication of our model for investors within EUT for a CRRA utility.

We first adapt the estimates of the one-dimensional 4/2 model by Cheng and Escobar (2021) to our two-dimensional setting. The estimation reported in the aforementioned paper, Chapter 2, was conducted based on S&P 500 and VIX data from January 2010 to the end of 2019. Here, we take key statistics of asset \( S_1 \), such as expected returns, expected variances, and expected volatility of variance, as benchmarks, adjusting parameters for asset \( S_2 \) such that similar values are ensured for the same key statistics of \( S_2 \). The parameters are presented in Table 5.1, and the key statistics are listed in Table 5.2.

More specifically, we first map the parameters for the variance driver \( v_{1_t} \) from the process of \( v_t \) in Cheng and Escobar (2021). To do this, let \( v_{1_t} = a^2 v_t \), where \( v_{1_t} \) is the first variance driver, while \( a \) and \( v_t \) are from Cheng and Escobar (2021). By Ito’s lemma, it can be shown that

\[
dv_{1_t} = \kappa (a^2 \theta - v_{1_t})dt + a \sigma \sqrt{v_{1_t}} dZ_1,
\]

(5.42)

which yields the values of the parameters in Equation (5.4): \( \kappa_1 = \kappa, \ \theta_1 = a^2 \theta, \) and \( \sigma_1 = a \sigma \). Then, we find the value of \( b_1 \) by setting the expected volatility to be the same:

\[
a \sqrt{\theta} + \frac{b}{\sqrt{\theta}} = \sqrt{\bar{\theta}} + \frac{b_1}{\sqrt{\bar{\theta}_1}},
\]

(5.43)

where \( v_t \) and \( v_{1_t} \) are targeted to their mean reversion levels \( \theta \) and \( \theta_1 \), respectively. Recall, parameters \( a \) and \( b \) are the proportions of the Heston component (i.e., \( a = 0.9051 \)) and the 3/2 component (i.e., \( b = 0.0023 \)). The mapping results of \( \kappa_1, \ \theta_1, \ \sigma_1, \) and \( b_1 \) are reported in Table 5.1. The parameters associated with \( S_2 \) are obtained by first setting the variance drivers to the level of their long-term mean reverting value, \( v_{1_t} = \bar{\theta}_i, \ i = 1, 2 \), and then ensuring that the key statistics in Table 5.2 are met.

Now, we can assess the impact of the raw new parameters, \( \beta \) (driver of \( \rho_{vol} \)), \( b_2 \) (driver of \( \sqrt{\sigma_2} \), volatility of \( S_2 \)) and \( b_3 \) (driver of \( \rho_{cov} \)), in the optimal allocations; see Figures 5.1, 5.2, and 5.3, respectively.
Chapter 5. Multivariate 4/2 stochastic volatility model

Table 5.1: Baseline parameters

<table>
<thead>
<tr>
<th></th>
<th>r</th>
<th>T</th>
<th>γ</th>
<th>κ₁</th>
<th>θ₁</th>
<th>σ₁</th>
<th>ρ₁</th>
<th>b₁</th>
<th>b₂</th>
<th>b₃</th>
<th>β</th>
<th>v₁ = θ₁</th>
<th>λ₁</th>
<th>λ₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>10</td>
<td>-1</td>
<td>7.3479</td>
<td>0.0269</td>
<td>0.5985</td>
<td>-0.7689</td>
<td>0.00208</td>
<td>0.0015</td>
<td>-0.0039</td>
<td>0.7139</td>
<td>0.0269</td>
<td>3</td>
<td>1.0857</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5.1: The impact of β on the optimal strategies. Panel (a) shows π₁ vs. β. Panel (b) shows π₂ vs. β.

Figure 5.2: The impact of b₂ on the optimal strategies. Panel (a) shows π₁ vs. b₂. Panel (b) shows π₂ vs. b₂.
Figure 5.3: The impact of $b_3$ on the optimal strategies. Panel (a) shows $\pi_1$ vs. $b_3$. Panel (b) shows $\pi_2$ vs. $b_3$.

The figures confirm the significant impact of $\beta$ and $b_3$ on allocations to the first asset and the influence of $b_2$ on allocations to both assets. The impact is consistent with findings by Buraschi et al. (2010). In their paper, the correlation between the asset ($\rho_{sto}$) and the volatility of assets ($\sqrt{\Sigma_1}, \sqrt{\Sigma_2}$) is related to the matrix $M$, while volatility co-movements ($\rho_{covo}$) are connected to the matrix $Q$ in the Wishart process. Specifically, the matrix $M$ drives the mean reversion of the variance-covariance matrix, and the matrix $Q$ determines the volatility of the variance-covariance matrix. That is, all the parameters in the matrix $M$ ($Q$) participate in explaining the key statistics $\sqrt{\Sigma_2}, \rho_{sto}$ ($\rho_{covo}$). This leads to difficulty not only in interpreting each parameter but also in extracting the pure effect (the combination of all interactions among parameters in the matrix) of the so-called statistic of interest on optimal allocation.

To extract the pure impacts of $\rho_{covo}$, $\rho_{sto}$ and $\sqrt{\Sigma_2}$ on the optimal allocations, we control the impacts of excess asset returns, covariances/variances, and volatility of variances on the optimal strategies by keeping them constant, as per Table 5.2. In particular, if $\rho_{covo}$ and $\rho_{sto}$ are not the examined parameters, then they are set to 0.5.

<table>
<thead>
<tr>
<th>Return ($\mu_i$)</th>
<th>$S_1$</th>
<th>$S_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1 = r + \lambda_1(v_1 + b_1)$</td>
<td>0.1068</td>
<td>$\mu_2 = 0.1$</td>
</tr>
<tr>
<td>Variance ($\Sigma_i$)</td>
<td>$\Sigma_1 = \left( \sqrt{v_1} + \frac{b_1}{\sqrt{v_1}} \right)^2$</td>
<td>0.0312</td>
</tr>
<tr>
<td>Volatility of $\Sigma_i$ ($A_i$)</td>
<td>$A_1 = (1 + b_1^2 v_1) r_1 \sqrt{v_1}$</td>
<td>0.0981</td>
</tr>
<tr>
<td>$\rho_{covo}$</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>$\rho_{sto}$</td>
<td>0.5</td>
<td></td>
</tr>
</tbody>
</table>

The statistics of interest, $\rho_{covo}$ and $\rho_{sto}$, are closely related to the parameters $b_2$, $b_3$, and $\beta$. Here, by controlling the variance of $S_2$ (i.e., $\Sigma_2$), we can work out the expressions for $b_2$, $b_3$, and $\beta$ in terms of $\rho_{covo}$, $\rho_{sto}$, and $\Sigma_2$. That is,
A complete market study may require difficult-to-price multi-asset derivatives, so it is left for future studies.
5.5 Numerical analysis

Figure 5.4: The impact of changes in $\rho_{covo}$ on the optimal strategies. Panel (a) shows $\pi_1$ vs. $\rho_{covo}$. Panel (b) shows $\pi_2$ vs. $\rho_{covo}$.

The significant influence of $\rho_{sto}$ on optimal allocations can be observed in Figure 5.5, where the maximal viable range is [0, 0.57] given that all the other parameters follow the baseline values. Not surprisingly there are important changes in allocation (e.g., threefold with changes in sign) for different values of $\rho_{sto}$. For comparison purposes, we also include Merton’s optimal allocation. For a fair comparison, we consider the Merton analog of our multivariate 4/2 model:

$$dS_{1t} = S_{1t} \left[ \mu_1^M dt + \sigma_1^M dW_{1t} \right],$$

$$dS_{2t} = S_{2t} \left[ \mu_2^M dt + \beta^M \sigma_1^M dW_{1t} + \sigma_2^M dW_{2t} \right],$$

where $\sigma_1^M = \sqrt{\bar{\lambda}_1 + \frac{b_1}{\sqrt{\rho_{sto}}}}$, $\mu_1^M = r + \bar{\lambda}_1 (\sigma_1^M)^2$, and $\mu_2^M = r + \bar{\lambda}_1 \beta^M (\sigma_1^M)^2 + \bar{\lambda}_2^M (\sigma_2^M)^2$, with $\bar{\lambda}_1$ the same as before while $\bar{\lambda}_2$ keeps adjusting to ensure constant $\mu_2^M$. Note that $b_3$ comes from the flexibility of the 4/2 model; hence, there is no $b_3$ in the Merton analog. The correlation between stocks would become

$$\rho_{sto}^M = \frac{\langle dS_{1t} \cdot dS_{2t} \rangle}{\sqrt{\langle dS_{1t}^2 \rangle} \cdot \sqrt{\langle dS_{2t}^2 \rangle}} = \frac{\beta^M \sigma_1^M}{\sqrt{\Sigma_2^M}},$$

where $\Sigma_2^M$ represents the variance of stock $S_2$; that is, $\Sigma_2^M = (\beta^M)^2(\sigma_1^M)^2 + (\sigma_2^M)^2$. With the same set values of $\rho_{sto}$, $\Sigma_2^M$, and $\mu_2^M$ as before, we can get

$$\beta^M = \frac{\rho_{sto} \sqrt{\Sigma_2^M}}{\sigma_1^M}, \quad \sigma_2^M = \sqrt{\Sigma_2^M - (\beta^M)^2(\sigma_1^M)^2}, \quad \bar{\lambda}_2^M = \frac{\mu_2^M - r - \bar{\lambda}_1 \beta^M (\sigma_1^M)^2}{(\sigma_2^M)^2}.$$

The optimal strategy for the Merton model is computed accordingly, such that

$$\pi_1^{Ms} = \frac{\bar{\lambda}_1}{1 - \gamma} - \beta^M \frac{\bar{\lambda}_2^M}{1 - \gamma}, \quad \pi_2^{Ms} = \frac{\bar{\lambda}_2^M}{1 - \gamma}.$$
It can be realized that both strategies recommended by the 4/2 model and the Merton model are highly sensitive to changes in correlations among assets. Specifically, as correlations among assets increase, the allocations in both assets shrink drastically in an incomplete market. We observe that both the 4/2 and Merton strategies offset the risk exposures of $S_2$ from the risk factor $W_1$ precisely, while the 4/2 strategy takes the risk from volatility co-movements into account and hence allocates less than the Merton strategy does in $S_1$. The 4/2 strategy does, however, allocate slightly more than the Merton strategy because of the hedging demand in $S_2$.

It should be noted that the 4/2 strategy would bring even more richness to the solution and differences to existing solutions as variance drivers $v_i$ change. This is because the optimal allocation depends explicitly on the volatility driver. In other words, the figures we have created assume that the volatility driver at the time of analysis is at its long-term value; if this were not the case, then we would expect larger differences to the Merton strategy and others.

Figure 5.5: The impact of the changes in $\rho_{sto}$ on the optimal strategies. Panel (a) shows $\pi_1$ vs. $\rho_{sto}$. Panel (b) shows $\pi_2$ vs. $\rho_{sto}$.

Figure 5.6 illustrates the impact of the volatility of $S_2$ on the optimal allocations, where we consider the range of $\sqrt{\Sigma_2}$ to be within $[0.2, 0.6]$.

The impact of the risk aversion level, $\gamma < 0$, on optimal allocations is illustrated in Figure 5.7, where the optimal allocations under the Merton model are provided as a benchmark.

## 5.6 Conclusion

In this chapter, we developed a multivariate 4/2 model that is flexible enough to capture, in a more interpretable way, stochastic correlations among assets and among variances (covolatility movements). We found closed-form expressions and conditions for well-defined changes of measure and c.f. and m.g.f. under risk-neutral and historical measures. We studied an expected utility portfolio optimization choice in an incomplete market setting. The optimal strategy implied by the multivariate 4/2 model was solved in closed form along with a verification theorem. The pure impacts of stochastic correlation among assets and among
5.6. Conclusion

Figure 5.6: The impact of volatility of $S_2$, i.e., $\sqrt{\Sigma_2}$, on the optimal strategies. Panel (a) shows $\pi_1$ vs. $\sqrt{\Sigma_2}$. Panel (b) shows $\pi_2$ vs. $\sqrt{\Sigma_2}$.

Figure 5.7: The impact of $\gamma$ on the optimal strategies. Panel (a) shows $\pi_1$ vs. $\gamma$, and Panel (b) shows $\pi_2$ vs. $\gamma$ along with the Merton strategy as a benchmark.
volatilities (co-volatility movements) on the optimal allocations were presented with the control of expected returns, variances, and volatility of variances. The numerical analysis indicates that although the new parameters have significant impact on optimality, this does not translate into a significant impact on co-volatility movements. Moreover, the importance of stochastic correlation and variances among stocks are confirmed.
Chapter 6
A class of portfolio optimization solvable problems

This chapter reveals the largest class of stochastic volatility processes solvable in closed form within expected utility theory for a hyperbolic absolute risk aversion investor. The risky-asset setting considers a framework outside the seminal work of Liu (2007), and highlights applications not yet studied in the literature. The work also demonstrates that analytical solutions for ambiguity-aversion analyses within the framework of Maenhout (2004) are feasible.

In this setting of continuous-time models with potentially incomplete markets, dynamic programming is one of the main approaches for tackling the implied optimal control problem associated with an expected utility theory (EUT) setting. Finding the optimal control (e.g., the optimal proportion allocated to the risky asset) involves solving a partial differential equation (PDE) that may not be tractable. Although numerical methods have advanced significantly in the past two decades, closed-form solutions are still desirable and convenient to gain a better understanding and interpretation of solutions.

This chapter has two objectives. First we aim to provide the most up-to-date review of solvable models within EUT for a hyperbolic absolute risk-averse (HARA) investor, many of which have not yet been implemented in the literature. Second, based on this collection of solvable cases, we use a simple change-of-control method to reveal a large family of fully solvable models, opening the door to far more complex and realistic configurations of diffusion and drift terms. For the sake of facilitating presentation and readability, we consider one risky asset with stochastic volatility in an incomplete market. Although extensions to multiple risky assets, complete markets, and other state variables are viable, they are more difficult to present in a granular form.

Liu (2007) made the most celebrated attempt at revealing a large family of solvable models in EUT, presenting the class of exponential-quadratic value functions and its originating multivariate models (quadratic returns and quadratic processes) for CRRA (constant relative risk aversion) utilities. The author detailed the functional forms of the drifts and diffusion terms of asset prices, state variables (stochastic volatility, or stochastic short rate, or predictors of stock returns), and the correlation between asset prices and state variables that allows for such...
exponential-quadratic form of the value function. Importantly, the paper was not concerned with detailing subclasses, leaving the door open to exploring members of such large family in terms of, for example, verification theorem, conditions for well-defined solutions, or further extensions in lower dimensions.

This chapter takes advantage of a one-dimensional setting (i.e. one risky asset, and one state variable) to perform a granular analysis of which specific stochastic volatility models would be solvable within HARA utilities. We go directly to the required PDE, showing that under certain conditions there are solvable cases outside the exponential-quadratic family. We then outline all these cases with the title “base cases”. We go beyond the base cases via a change of control. This technique originated in the seminal work of Liu and Pan (2003) dealing with financial derivatives, and it was later used widely, even for ambiguity-averse problems; see Escobar et al. (2015).

This chapter is organized as follows. Section 6.1 presents the general model for a stock process and its stochastic volatility. Section 6.2 formulates the optimal portfolio choice problem in an incomplete market and discusses the changing of control. Then, a list of specific solvable models (i.e., base cases) and their source paper are presented in Section 6.3.

### 6.1 The model

Let all the stochastic processes be defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0,T]})\), where \(\{\mathcal{F}_t\}_{t \in [0,T]}\) is a right-continuous information filtration generated by standard Brownian motions (BMs). Let us assume a general model with one risky asset and a state variable driving its diffusion and market price of risk terms:

\[
\frac{dS_t}{S_t} = \left[ r + \lambda(v_t)G(S_t, v_t, t) \right] dt + G(S_t, v_t, t)dW_t, \quad S(0) = S_0 > 0, \quad (6.1)
\]

\[
dv_t = m_1(v_t)dt + m_2(v_t)dZ_{1t}, \quad v(0) = v_0 > 0, \quad (6.2)
\]

where \(r\) is the risk-free interest rate; \(\lambda(v_t)\) is the market price of risk (MPR); \(G(S_t, v_t, t)\) represents the volatility of the risky asset; and \(m_1(v_t), m_2(v_t)\) are the drift and volatility of \(v_t\) respectively. The correlation between the underlying asset and the state variable is captured by \(W_t\) and \(Z_{1t}\) via the parameter \(\rho \in (-1, 1)\). For ease of representation, we write \(dW_t = \rho dZ_{1t} + \sqrt{1-\rho^2} dZ_{2t}\), where \(Z_2\) is another standard BM, independent of \(Z_1\). We assume all the coefficients of the SDEs above are progressively measurable with respect to the filtration \(\{\mathcal{F}_t\}_{t \in [0,T]}\).

**Assumption 6.1.1.**

1. To ensure uniqueness for the SDE (6.1), we assume (see Kraft (2005))

\[
\int_0^T \left[ |\lambda(v_t)|G(S_t, v_t, t) + G^2(S_t, v_t, t) \right] dt < \infty \quad a.s. \quad (6.3)
\]

2. For the SDE (6.2), we ensure existence of the solution adapted to the filtration \(\{\mathcal{F}_t\}_{t \in [0,T]}\) by checking the growth condition on its coefficients. That is,

\[
m_1^2(t, x_0) + m_2^2(t, x_0) \leq K^2(1 + x_0^2) \quad (6.4)
\]

for \(x_0 \in \mathbb{R}, |x_0| \leq N, 0 \leq t \leq T\) for arbitrary positive constants \(N\) and \(T\) for some \(K > 0\).
To ensure uniqueness in Equation (6.2), we need to employ the Yamada–Watanabe condition (Watanabe and Yamada, 1971); see Theorem 4 therein. This is, there exist real-valued, continuous, positive, and increasing functions $f(u)$ and $g(u)$ defined on $[0, C)$ for some $C > 0$, such that:

\[ |m_1(t, x_0) - m_1(t, y_0)| \leq g(|x_0 - y_0|), \]
\[ |m_2(t, x_0) - m_2(t, y_0)| \leq f(|x_0 - y_0|) \]

for all $x_0, y_0 \in \mathbb{R}$ such that $|x_0 - y_0| < C$; where $f(0) = g(0) = 0$, and $f^2(u)^{-1}$, $g(u)$ are concave, satisfying the relation,

\[ \int_{0^+} [f^2(u)u^{-1} + g(u)]^{-1} du = \infty, \] (6.6)

We consider a portfolio optimization problem in the framework of EUT with HARA utility; that is, $u(x) = \frac{1 - F}{x} F$ with $x > F$ and $\gamma \neq 0$. This is arguably the largest and most popular family of utilities among practitioners due to its flexibility in capturing risk preferences. The following are some interesting special cases of HARA utility, which can be adapted to our methodology:

- $\gamma = 2$ Quadratic utility\(^3\), $F = 0$, $\gamma < 1$ leads to Power utility (CRRA), and $\gamma \to 0$, $F = 0$ to Log utility.

An investor’s objective is to find an investment strategy that maximizes their utility from terminal wealth at time $T$. The portfolio consists of a risky asset, i.e., Equation (6.1), and a risk-free asset $M_t$ with dynamics

\[ dM_t = M_r dt, \] (6.7)

where $M_0 = 1$, and $r$ is the constant risk-free interest rate.

### 6.2 Problem formulation

For a HARA investor with a finite investment horizon, we denote $\pi_t$ as the proportion of wealth that the investor allocates to the stock, and the remaining portion of their wealth, $(1 - \pi_t)$, is kept in the risk-free bank account $M_t$. The goal of the investor is to find a strategy that maximizes their utility from terminal wealth; that is,

\[ J(x, v, t) = \sup_{\pi \in \mathcal{U}_t} \mathbb{E}_{x, v, t}[u(X_T)], \] (6.8)

where $J(x, v, t)$ is the value function.

Using a self-financing argument, the wealth $X_t$ of the investor follows the SDE:

\[ \frac{dX_t}{X_t} = \left[ r + \pi_t A(v_t) G(S_t, v_t, t) \right] dt + \pi_t G(S_t, v_t, t) dW_t. \] (6.9)

As before, we need conditions on $\pi_t$, such that the coefficients of the SDE above are progressively measurable with respect to the filtration $\mathcal{F}_t$, and the SDE (6.9) has a unique solution. These requirements lead to a set of admissible strategies (denoted by $\mathcal{U}_t$) as defined next:

\(^3\)This establishes a connection with the pre-commitment solution within mean-variance theory (MVT), see Zhou and Li (2000), Theorem 3.1.
**Definition 6.2.1.** $\pi$ is an admissible strategy if

1) $\pi$ is progressively measurable, and

2) For all $(x_0, v_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $t \in [0, T]$, the SDE (6.9) has a pathwise unique solution $\{X^x_t\}_{t \in [0, T]}$ under the risk-neutral measure $Q$, and

\[ \mathbb{E}_Q^{x_0, v_0, t} [u(X_t)] < \infty, \]

where $\mathbb{E}_Q^{x_0, v_0, t}[\cdot] = \mathbb{E}^{Q}[\cdot | X_t = x, v_t = v]$ denotes the conditional expectation.

Next, we consider a new control variable $\psi_t$, such that

\[ \psi_t = \pi_t G(S_t, v_t, t), \quad (6.10) \]

where $\psi_t$ satisfies

\[ \int_0^T (|\psi_t A(v_t)| + \psi_t^2) \, dt < \infty \text{ a.s.} \quad (6.11) \]

This last equation ensures that $\psi_t$ is an admissible control satisfying Definition 6.2.1, with the new set denoted by $\mathcal{U}$. We can write the portfolio optimization problem in terms of the new control as follows:

\[ J(x, v, t) = \sup_{\pi \in \mathcal{U}} \mathbb{E}_{x, v, t}[u(X_T)] = \sup_{\psi \in \mathcal{U}} \mathbb{E}_{x, v, t}[u(X_T)]. \quad (6.12) \]

The wealth process with the new control then follows the SDE:

\[ \frac{dX_t}{X_t} = \left[r + \psi_t A(v_t)\right] dt + \psi_t dW_t. \quad (6.13) \]

The wealth under $\psi_t$ looks simpler than in the original process with $\pi_t$. Moreover, if we solve the problem in terms of $\psi_t$ then we can produce the solution to the original problem. The rationale above means we should first find all cases solvable under the simpler wealth representation. That is, we should first describe all the functions $A(v_t), m_1(v_t), m_2(v_t)$ such that $\psi$ is solvable in closed form. These will be regarded as “base cases”.

**Remark**

1. This approach allows for solvability of arguably any diffusion term $G(S_t, v_t, t)$. That is, $G(\cdot)$ could be any well-defined function, finite and nonzero, supported by data. For instance, if $G(S_t, v_t, t) = S^\alpha_t \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right)$, where $0 \leq \alpha \leq 1$, then a local volatility 4/2 model would be targeted. Here, $G$ could even be a highly non-linear function from the realm of machine learning.

2. The approach implies the same value function and optimal wealth for infinitely many functional $G$’s. However, this is an illusion, as every choice of $G$ would lead to differently calibrated parameters; for example, the parameters in $v_t$ from the multiplicative model $G = S_t v_t$ would be different to those in $v_t$ if working with $G = v_t$. Thus, the value function and optimal wealth would be different due to the parameters.
6.3 Portfolio optimization and solvable “base cases”

As explained in the Section 6.2, the investor’s goal is to find a strategy that could maximize their terminal utility from terminal wealth. That is,

\[ J(x, v, t) = \sup_{\varphi \in \mathcal{U}} \mathbb{E}_{x,v,t}[u(X_T)]. \]  

(6.14)

Based on the principle of dynamic programming, the corresponding Hamilton-Jacobi-Bellman (HJB) equation satisfies

\[ \sup_{\psi} \left\{ J_t + \left[ r + \psi \lambda(v_t) \right] x J_x + \frac{1}{2} x^2 \psi^2 J_{xx} + m_1(v_t) J_v + \frac{1}{2} m_2^2(v_t) J_{vv} + x \psi p m_2(v_t) J_{xy} \right\} = 0, \]  

(6.15)

where \( J_t, J_x, J_{xx}, J_{vv}, J_{xy} \) are the first and second partial derivatives of the value function \( J \) with respect to \( t, x, v \). The above HJB equation also satisfies the boundary condition \( J(x, v, T) = \frac{\left( x - F e^{-r(T-t)} \right)^\gamma}{\gamma} h(v, t), \)  

Further, we assume a candidate value function of the form

\[ J(x, v, t) = \frac{(x - F e^{-r(T-t)})^\gamma}{\gamma} h(v, t), \]  

(6.16)

with terminal condition \( h(v, T) = 1 \). Using the first order condition in the maximization problem leads to a candidate optimal solution of \( \psi \) such that

\[ \psi^* = \frac{-x \rho m_2 J_{xy} - \lambda x J_x}{x^2 J_{xx}} = \frac{(x - F e^{-r(T-t)}) (-\rho m_2 h_v - \lambda h)}{x(y - 1)h}. \]  

(6.17)

Substituting \( \psi^* \) back and simplifying leads to a PDE in terms of the helper function \( h \):

\[ h_t + \left( r y - \frac{1}{2} \frac{\gamma}{y - 1} \lambda^2(v_t) \right) h_t + \left( m_1(v_t) - \frac{\gamma}{y - 1} \lambda(v_t) \rho m_2(v_t) \right) h_v + \frac{1}{2} m_2^2(v_t) h_{vv} - \frac{1}{2} \frac{\gamma}{y - 1} \rho^2 m_2^2(v_t) \frac{h^2}{h} = 0. \]  

(6.18)

Scholars have tackled the above PDE with the most general ansatz for \( h \) as an exponential-quadratic (see Liu (2007)). That is,

\[ h(v, t) = e^{A(t) + B(v) + \frac{1}{2} C(v)^2}. \]  

(6.19)

where \( \tau(T) = T - t \), and \( A(0) = B(0) = C(0) = 0 \) to ensure the terminal condition \( h(v, T) = 1 \) for all \( v \). Note that \( G(S_t, v_t, t) \) plays no role in the solvability of the PDE, while \( \lambda, m_1, m_2 \) are critical\(^4\).

Liu (2007) was the first to mention that a polynomial with an order higher than 2 cannot help in solving the PDE, but no proof was provided. Hence, in the next proposition, we conclude and prove the limitation on the order of the polynomial for the conjectured value function and the solvability of the portfolio optimization problem.

\(^4\)Although \( \rho \) could be made a function of \( v_t \), its bounded nature would create high nonlinearity in the PDE, jeopardizing any solvability.
Proposition 6.3.1. Consider a HARA investor, who aims at problem (6.12). Assume that the value function $J(x, v, t)$ is conjectured in the exponential family, such that

$$J(x, v, t) = \frac{(x - Fe^{-r(T-t)})^v}{\gamma} h(v, t) = \frac{(x - Fe^{-r(T-t)})^v}{\gamma} \exp\left\{ \sum_{k=0}^{n} A_k(\tau)v^k \right\}, \quad (6.20)$$

where $\tau(T) = T - t$, and $A_k(0) = 0$ for all $k = 0, 1, ..., n$. The ansatz cannot solve PDE (6.18) if the highest degree of the polynomial $n \geq 3$.

See proof in Appendix E.1.

The exponential-quadratic family for a CRRA setting ($F = 0, \gamma < 1$) was described by Liu (2007), leading to a quadratic diffusion process of the state variable and quadratic returns (see Definitions 1 and 2 therein). To facilitate the connection, we map the notation between the two papers for one risky asset and one state variable: $N = N_1 = 1, \eta = 0$ or 1; $X$ denotes the state variable, which is $v_t$ in our paper, and $P$ is the return of the risky asset, denoted $S$ in our paper. For the drift and diffusion process of the state variable $X$ (see Equations (9)-(11) in the referenced paper), and the drift and diffusion process of the risky asset $S$, (Equations (14)-(17)), we get:

1. $\eta = 1$ implies an OU process for $v_t$ (Table 1, column 2) with constant or linear MPR (Table 1, rows 2 and 4):

   $$\mu^X = k - KX = m_1, \quad \Sigma^X \Sigma^{X^T} = h_0 = m_2^2,$$
   $$\mu - r)(\Sigma\Sigma)^{-1}(\mu - r) = H_0 + 2H_1 \sqrt{H_0}X + H_1^2X^2 = \lambda^2,$$
   $$\Sigma^X \rho\Sigma^{-1}(\mu - r) = l_0 + l_1X = \rho m_2 \lambda. \quad (6.21)$$

2. $\eta = 0$, implies a process for $v_t$ slightly richer than a CIR (see Table 1, column 3 for the CIR, i.e. $h_0 = 0$) with constant and square root MPR (Table 1, rows 2 and 3)\textsuperscript{5}:

   $$\mu^X = k - KX = m_1, \quad \Sigma^X \Sigma^{X^T} = h_0 + h_1X = m_2^2,$$
   $$\mu - r)(\Sigma\Sigma)^{-1}(\mu - r) = c_0(h_0 + h_1X) = \lambda^2,$$
   $$\Sigma^X \rho\Sigma^{-1}(\mu - r) = c_1(h_0 + h_1X) = \rho m_2 \lambda. \quad (6.22)$$

Interestingly, our approach allows for richer risky asset diffusion terms, this is, our $G$ can be a function of the asset itself and time, more flexible than $\Sigma$ in Liu (2007). We also realize that Equation (9) in the referenced paper violates the Growth condition unless $K_2 = 0$ or $\eta = 0$, hence we have to limit the drift of the SDE for the state variable to the linear case. Several exponential-quadratic models have been thoroughly analyzed in the existing literature. For example, Kraft (2005) solved two base cases related to the Heston model (i.e. $G$ as a 1/2 with $v$ a CIR)\textsuperscript{6}, while Cheng and Escobar (2021) developed the solution for the 4/2 model of variance ($G$ as a 4/2).

\textsuperscript{5}other subcases appear if $\rho = 0$

\textsuperscript{6}Chacko and Viceira (2005) solved a case of an inverse CIR ($G$ as a 3/2) with infinite horizon hence outside our framework.
The PDE (6.18) can also be solved outside the family of exponential-affine structures, an example is available for $\rho = 0$ thanks to Lie symmetries for PDEs and the use of confluent hypergeometric functions (see Cheng and Escobar-Anel (2022)). This case fits into non-linear MPR within our context. For clarity, Table 6.1 summarizes a collection of solvable cases with the source of the main result, if no source is provided, it means the case has not yet been studied in the literature.

<table>
<thead>
<tr>
<th>MPR</th>
<th>Stochastic volatility $(v_t)$</th>
<th>OU process</th>
<th>CIR process</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>$G = v_t$</td>
<td>$G = C$: CRRA (Merton, 1969)</td>
<td>$G = \sqrt{v_t}$: CRRA (Kraft, 2005)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$G = \frac{v_t}{\sqrt{v_t}}$: CRRA (Merton, 1975)</td>
<td>$G = 1: MVT$ (Zhang, 2021a)</td>
</tr>
<tr>
<td>Linear in $\sqrt{v_t}$</td>
<td></td>
<td>$G = \frac{v_t}{\sqrt{v_t}}$: MVT (Zhang, 2021b)</td>
<td>$G = \frac{1}{\sqrt{v_t}}$: MVT (Zhang, 2021a)</td>
</tr>
<tr>
<td>Linear in $v_t$</td>
<td>$G = v_t$</td>
<td>$G = \frac{v_t}{\sqrt{v_t}}$: CRRA (Kraft, 2005)</td>
<td>$G = \frac{v_t}{\sqrt{v_t}}$: CRRA (Cheng and Escobar-Anel, 2021)</td>
</tr>
<tr>
<td>(exponential-quadratic)</td>
<td></td>
<td>$G = \frac{v_t}{\sqrt{v_t}}$: CRRA (Cheng and Escobar-Anel, 2022)</td>
<td>$G = \frac{1}{\sqrt{v_t}}$</td>
</tr>
</tbody>
</table>
Chapter 7

Conclusion

In this thesis, we study portfolio optimization problems using dynamic control theory for the 4/2 stochastic volatility model proposed by Grasselli (2017). We obtain the (robust) optimal strategies by solving the associated HJB(I) equations in both an incomplete and a complete market. Verification theorems are provided to ensure optimality. We perform sensitivity analysis based on parameters estimated from real-world data. The impact of important parameters is presented and the wealth-equivalent losses are addressed for typical suboptimal strategies. Moreover, consumption and a more preferable market price of risk are considered, and the solutions entail confluent hypergeometric functions due to their non-affine nature. Furthermore, a multivariate 4/2 stochastic model is proposed and constructed in the structure of a linear combination of independent 4/2 factors. We consider a portfolio optimization problem including two risky assets and derive conditional characteristic functions (c.f.) under both the real-world measure and risk-neutral measure, which can be used for risk management and options pricing. Lastly, the class of stochastic volatility processes solvable in closed form within expected utility theory for a HARA investor is explored.

With the contributions we have made in this thesis, there are still many limitations we realized from our results. Several extensions can be considered in future research but are not limited to:

- Numerically, our estimation approach depends on the minimum of variance process, which leads to biased estimators, moreover, a full multivariate estimation has not been performed. This limitation calls for improvement of the estimation method in future research.

- Theoretically, there is no consensus on what form of the market price of risk is the most appropriate one in the existing literature. Due to its relevance to the solvability and the analytical representation of the portfolio optimization problem, it is worthwhile to investigate this problem from solid empirical analysis.

- Explore portfolio solutions for the popular Epstein-Zin recursive utility investor.

- The multivariate model is worth investigating in at least two aspects: 1) a complete market, which can help to better extract the impact of co-volatility movements on optimal asset allocations; 2) an estimation for multiple assets.
Bibliography


Appendix A

Proofs for Chapter 2

A.1 Proof of conditions on change of measure

*Proof.* The first step is to ensure the change of measure is well-defined and for this we use
Novikov’s condition, i.e., generically for \( i = 1, 2 \)

\[
E \left[ \exp \left( \frac{1}{2} \int_0^T \lambda_i^2 (\sqrt{v_i})^2 ds \right) \right] = E \left[ \exp \left( \frac{\lambda_i^2}{2} \int_0^T v_i ds \right) \right] < \infty.
\]

From Cheng et al. (2019), in order for this expectation to exist, we need one condition:

\[
-\frac{\lambda_i^2}{2} > -\frac{\kappa^2}{2\sigma^2} \implies |\lambda_i| < \frac{\kappa}{\sigma}.
\] (A.1)

That is,

\[
\max \{|\lambda_1|, |\lambda_2|\} < \frac{\kappa}{\sigma}.
\]

The second step is to ensure the drift of the asset price is equal to the short rate under \( Q \), which is obviously satisfied here.

The third step ensures the discounted asset price process, \( \tilde{S}_t = e^{-rt}S_t \), is a true \( Q \)-martingale and not just a local \( Q \)-martingale, therefore it does not concern the change from \( P \) to \( Q \) but rather the martingale properties of the asset price under \( Q \) (see Grasselli (2017), section 2 for a similar situation). Recall,

\[
\frac{dS_t}{S_t} = r dt + \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \left( \rho dZ_{1t}^Q + \sqrt{1-\rho^2} dZ_{2t}^Q \right)
\]

\[
\frac{d\tilde{S}_t}{\tilde{S}_t} = \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \left( \rho dZ_{1t}^Q + \sqrt{1-\rho^2} dZ_{2t}^Q \right)
\]
Note, under $Q$, we have:

$$E^Q \left[ S_t \right]$$

$$= \widetilde{S}_0 E^Q \left\{ \exp \left\{ \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right) dW_s^Q - \frac{1}{2} \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right)^2 ds \right\} \right\}$$

$$= \widetilde{S}_0 E^Q \left\{ \exp \left\{ \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right) dZ_{ts}^Q + \sqrt{1 - \rho^2} \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right) dZ_{ts}^Q \right\} \right\}$$

$$\times \exp \left\{ -\frac{1}{2} \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right)^2 ds \right\}$$

$$= \widetilde{S}_0 E^Q \left\{ \exp \left\{ \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right) dZ_{ts}^Q - \frac{1}{2} \rho^2 \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right)^2 ds + \frac{1}{2} \rho^2 \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right)^2 ds \right\} \right\}$$

$$\times \exp \left\{ \sqrt{1 - \rho^2} \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right)^2 ds - \frac{1}{2} \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right)^2 ds \right\}$$

$$= \widetilde{S}_0 E^Q \left\{ \exp \left\{ \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right) dZ_{ts}^Q - \frac{1}{2} \rho^2 \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right)^2 ds \right\} \right\}$$

$$\times \exp \left\{ \sqrt{1 - \rho^2} \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right)^2 ds - \frac{1}{2} \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right)^2 ds \right\}$$

$$= \widetilde{S}_0 E^Q \left[ \xi_{1t} \xi_{2t} \right] = \widetilde{S}_0 E^Q [\xi_{1t}]$$

where we have used $dW_s^Q = \rho dZ_{ts}^Q + \sqrt{1 - \rho^2} dZ_{ts}^Q$, the independence of $Z_{ts}^Q$ and $Z_{ts}^Q$ (hence of $v_s$ and $Z_{ts}^Q$); and $\xi_{1t}, \xi_{2t}$ defined as follows:

$$\xi_{1t} = \exp \left\{ \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right) dZ_{ts}^Q - \frac{1}{2} \rho^2 \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right)^2 ds \right\}$$

$$\xi_{2t} = \exp \left\{ \sqrt{1 - \rho^2} \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right) dZ_{ts}^Q - \frac{1}{2} \rho^2 \int_0^t \left( a \sqrt{v_s} + \frac{b}{\sqrt{v_s}} \right)^2 ds \right\}$$

$\xi_{1t}$ is an exponential local martingale while $\xi_{2t}$ is an exponential martingale.

Hence, testing the martingale property for the discounted asset is equivalent to performing the Feller nonexplosion test for volatility using $\xi_{1t}$. Note $\xi_{1t}$ can be interpreted as a new change
of measure for the volatility process:

\[ dZ_{it}^{Q_1} = \rho \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) dt + dZ_{it}^0 \]

Hence we check the CIR process does not reach zero under \( Q_1 \):

\[ dv_t = \kappa (\theta - v_t) dt - \sigma \rho \sqrt{v_t} \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) dt + \sigma \sqrt{v_t} dZ_{it}^{Q_1} \]

\[ = (\kappa + \sigma \rho a) \left( \kappa - \sigma \rho b \right) \left( \frac{\kappa}{\kappa + \sigma \rho a} - v_t \right) dt + \sigma \sqrt{v_t} dZ_{it}^{Q_1} \]

This leads to the following conditions:

under \( Q_1 \):

\[ \sigma^2 \leq 2\kappa \theta - 2|\sigma \rho b|, \]
\[ \kappa + \sigma \rho a > 0 \]

under \( Q \):

\[ dv_t = \kappa (\theta - v_t) dt - \sigma \tilde{\lambda}_1 v_t dt + \sigma \sqrt{v_t} dZ_{it}^0 \]

\[ = \left( \kappa + \sigma \tilde{\lambda}_1 \right) \left( \frac{\kappa}{\kappa + \sigma \tilde{\lambda}_1} - v_t \right) dt + \sigma \sqrt{v_t} dZ_{it}^0 \]

\[ \sigma^2 \leq 2\kappa \theta \]
\[ \kappa + \sigma \tilde{\lambda}_1 > 0 \]

under \( P \):

\[ dv_t = \kappa (\theta - v_t) dt + \sigma \sqrt{v_t} dZ_t \]

\[ \sigma^2 \leq 2\kappa \theta . \]

These together lead to condition (2.12) and conditions (2.13), (2.14).

\[ \square \]

### A.2 Proof of optimal investment strategy

**Proof.** Proof of Proposition 2.3.1.

Separate out terms that involves \( \pi \) in Equation (2.16) and denote it as a function \( g(\pi) \):

\[ 0 = J_t + \kappa (\theta - v) J_v + \frac{1}{2} \sigma^2 v J_{vv} + \sup_{\pi} \left\{ x \left( r + \pi \tilde{\lambda}(av + b) \right) J_x \right. \]
\[ + \left. \frac{1}{2} \pi^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 J_{xx} + \pi x (av + b) \sigma \rho J_{xv} \right\} . \]
That is,

\[ 0 = J_t + \kappa(\theta - \nu)J_v + \frac{1}{2} \sigma^2 \nu J_{vv} + \sup_{\pi} \left( g(\pi) \right). \]

By the first order condition, we can find a candidate optimal investment strategy \( \pi \), i.e., \( \pi^* \), such that

\[ g'(\pi) = x \left( \dot{\lambda}(av + b) \right) J_x + x^2 \pi(a \sqrt{v} + b \sqrt{v})^2 J_{xx} + x(av + b) \sigma \rho J_{xy}. \]

Set \( g'(\pi) = 0 \) and solve for the candidate \( \pi^* \):

\[ \pi^* = -\frac{\sqrt{\nu} \sigma \rho J_{xy} - \dot{\lambda} \sqrt{v} J_x}{x(a \sqrt{v} + b \sqrt{v}) J_{xx}}. \quad (A.2) \]

Substituting \( \pi^* \) back into HJB equation, eliminating the “\( \sup \)”, cancelling, simplifying, and regrouping, it follows that

\[ 0 = J_t + \kappa(\theta - \nu)J_v + \frac{1}{2} \sigma^2 \nu J_{vv} + rxJ_x - \frac{\nu \sigma \rho \lambda}{J_{xx}} J_{xx} - \frac{1}{2} \frac{\nu \sigma^2 \rho^2}{J_{xx}} J_{xx} - \frac{1}{2} \frac{\dot{\lambda}^2 v}{J_{xx}} J_{xx}. \]

Assume the form of \( J(x, v, t) \) is given by \( J(x, v, t) = \frac{x^y}{v^z} h(t, v) \), where \( h(T, v) = 1 \), \( \forall v \). Thereby, it follows that

\[ J_t = \frac{x^y}{v^z} h_t, \quad J_v = \frac{x^y}{v^z} h_v, \quad J_x = x^{y-1} h, \quad J_{vv} = \frac{x^y}{v^z} h_{vv}, \quad J_{xx} = x^{y-1} h_v, \quad J_{xx} = (\gamma - 1)x^{y-2} h. \]

Moreover,

\[
\begin{align*}
\frac{J_{xx}^2}{J_{xx}} &= \frac{x^{2(y-1)}h^2}{(\gamma - 1)x^{y-2}h} = \frac{x^y h^2}{(\gamma - 1)x^{y-2}h}, \\
\frac{J_{xx} J_{xy}}{J_{xx}} &= \frac{x^{2(y-1)}h_v h_t}{(\gamma - 1)x^{y-2}h} = \frac{x^y h_v h_t}{(\gamma - 1)x^{y-2}h}, \\
\frac{J_{xx} J_{xy}}{J_{xx}} &= \frac{x^{2(y-1)}h_{vv}}{(\gamma - 1)x^{y-2}h} = \frac{x^y h_{vv}}{(\gamma - 1)x^{y-2}h}.
\end{align*}
\]

Substituting back into the PDE and simplifying by multiplying by the term \( \frac{x^y}{v^z} \) to both sides leads to

\[ 0 = h_t + \kappa(\theta - \nu)h_v + \frac{1}{2} \sigma^2 \nu h_{vv} + ryh - \nu \sigma \rho \lambda \frac{y h_v}{\gamma - 1} - \frac{1}{2} \nu \sigma^2 \rho^2 \frac{y h_{vv}}{(\gamma - 1) h} - \frac{1}{2} \frac{\dot{\lambda}^2 v}{\gamma - 1} \frac{y h}{\gamma - 1}. \quad (A.3) \]

Furthermore, assume that \( h(t, v) \) is of exponentially affine form, such that \( h(t, v) = \exp(A(\tau(t)) + B(\tau(t))v) \), with time horizon \( \tau(t) = T - t \) and therefore boundary conditions

\[ h(T, v) = 1 \quad \forall v \Rightarrow A(0) = A(\tau(T)) = 0, B(0) = B(\tau(T)) = 0. \quad (A.4) \]

This leads to

\[ h_t = (-A' - B'v)h, \quad h_v = Bh, \quad h_{vv} = B^2 h, \quad \frac{h_{v}^2}{h} = \frac{B^2 h^2}{h} = B^2 h. \]

Substituting again and simplifying leads to

\[ 0 = -A' - B'v + \kappa(\theta - \nu)B + \frac{1}{2} \sigma^2 v B^2 + ry - \nu \sigma \rho \lambda \frac{\gamma}{\gamma - 1} B - \frac{1}{2} \nu \sigma^2 \rho^2 \frac{\gamma}{\gamma - 1} B^2 - \frac{1}{2} \frac{\dot{\lambda}^2 v}{\gamma - 1}. \]
A.2. Proof of optimal investment strategy

Separating out $v$:

$$0 = -A' + \kappa \theta B + r \gamma + v \left[ -B' - \kappa B + \frac{1}{2} \sigma^2 B^2 - \sigma \rho \frac{\gamma}{\gamma-1} B - \frac{1}{2} \sigma^2 B^2 - \frac{1}{2} \frac{\gamma}{\gamma-1} B^2 - \frac{1}{2} \frac{\gamma^2}{\gamma-1} B^2 \right].$$

We end up with a term that is linear in $v$, but both "coefficients" are linear differential equations. Both of them have to be zero, such that

$$A_0 = \frac{\gamma}{\gamma-1} B + r,$$

$$B_0 = \frac{1}{2} \frac{\gamma}{\gamma-1} B^2 - \frac{1}{2} \frac{\gamma^2}{\gamma-1} B^2 \overset{\gamma_2}{\longrightarrow} \bar{\gamma}_2.$$ 

The Equation (A.5) is a so called Riccati equation with auxiliary parameters $k_i, i \in \{0, 1, 2\}$, which can be solved. Let $A(\tau), B(\tau)$ be two time dependent functions satisfying the equations

$$A'(\tau) = \kappa \theta B(\tau) + \gamma r,$$

$$B'(\tau) = \frac{1}{2} k_2 B(\tau)^2 - k_1 B(\tau) + \frac{1}{2} k_0,$$  \hspace{1cm} (A.6)

and the boundary conditions $A(0) = 0, B(0) = 0$ with constants $k_0, k_1, k_2$ satisfying $k_1^2 - k_0 k_2 > 0$. Define $k_3 := \sqrt{k_1^2 - k_0 k_2}.$ The right hand side of Equation (A.6) has roots

$$B_{1,2} = \frac{k_1 \pm \sqrt{k_1^2 - k_0 k_2}}{k_2} = \frac{k_1 \pm k_3}{k_2},$$

which is well defined due to the assumption made. Separating factors leads to

$$B'(\tau) = \frac{dB(\tau)}{d\tau} = \frac{k_2}{2} \left( B(\tau) - \frac{k_1 + k_3}{k_2} \right) \left( B(\tau) - \frac{k_1 - k_3}{k_2} \right).$$

Integrating yields

$$\int_0^{B(\tau)} \frac{1}{\left( B(\tau) - \frac{k_1 + k_3}{k_2} \right) \left( B(\tau) - \frac{k_1 - k_3}{k_2} \right)} dB = \int_0^{\tau} \frac{k_2}{2} dt$$

$$\ln \frac{B(\tau) - \frac{k_1 + k_3}{k_2}}{B(\tau) - \frac{k_1 - k_3}{k_2}} = \ln \frac{k_1 + k_3}{k_1 - k_3} = k_3 \tau$$

$$B(\tau) \left( 1 - \frac{k_1 + k_3}{k_1 - k_3} e^{k_3 t} \right) = \frac{k_1 + k_3}{k_2} e^{k_3 t},$$
where we implicitly assumed that $B(\tau) \neq \frac{k_1 + k_3}{k_2}$ and $B(\tau) \neq \frac{k_1 - k_3}{k_2}$, which will be shown later. We finally obtain

$$B(\tau) = \frac{(k_1 + k_3) \left(1 - e^{k_3 \tau}\right)}{k_2 \left(1 - \frac{k_1 + k_3}{k_1 - k_3} e^{k_3 \tau}\right)} = \frac{k_1^2 - k_3^2}{k_2 (k_1 - k_3 - (k_1 + k_3)e^{k_3 \tau})} \frac{(1 - e^{k_3 \tau})}{k_1 (1 - e^{k_3 \tau})} = \frac{e^{k_3 \tau} - 1}{k_1 - k_3 - (k_1 + k_3)e^{k_3 \tau} - k_1 + k_3} = \frac{k_0 (e^{k_3 \tau} - 1)}{2k_3 + (k_1 + k_3)(e^{k_3 \tau} - 1)}.$$ (A.7)

This leads to the following representation for $A(\tau)$,

$$A(\tau) = \gamma r \tau + \int_0^\tau \theta \kappa B(t) dt = \gamma r \tau + \theta \kappa \int_0^\tau \frac{k_0 (e^{k_3 \tau} - 1)}{2k_3 + (k_1 + k_3)(e^{k_3 \tau} - 1)} dt.$$

With $z(t) = 2k_3 + (k_1 + k_3)(e^{k_3 \tau} - 1)$, i.e., $t = \frac{1}{k_3} \ln\left(\frac{z - 2k_3 + k_1 + k_3}{k_1 - k_3}\right) = \frac{1}{k_3} \ln\left(\frac{z + k_1 - k_3}{k_1 + k_3}\right)$ and $dt = \frac{1}{k_3 (z + k_1 - k_3)} dz$, we obtain:

$$A(\tau) = \gamma r \tau + \theta \kappa \frac{k_0}{k_3} \left(\int_0^{z(\tau)} \frac{1}{z(k_1 + k_3)} dz - \int_0^{z(\tau)} \frac{1}{z(k_1 - k_3)} dz\right).$$

Note: $\frac{1}{z(z+a)} = \frac{A}{z} + \frac{D}{z+a} = \frac{C(a)}{z(z+a)} = \frac{C + Dz}{z(z+a)} \Leftrightarrow Ca = 1, C + D = 0$. Solving for C, D:

$$a = k_1 - k_3, C = -\frac{1}{k_1 - k_3}, D = -\frac{1}{k_1 - k_3}$$

$$A(\tau) = \gamma r \tau + \theta \kappa \frac{k_0}{k_3} \left(\int_0^{z(\tau)} \frac{1}{z(k_1 + k_3)} dz - \int_0^{z(\tau)} \frac{1}{z(k_1 - k_3)} dz + \int_0^{z(\tau)} \frac{1}{z + k_1 - k_3} dz\right)$$

$$= \gamma r \tau + \theta \kappa \frac{k_0}{k_3} \left[\ln z \bigg|_0^{z(\tau)} \frac{z(\tau)}{k_1 + k_3} - \ln z \bigg|_0^{z(\tau)} \frac{z(\tau)}{k_1 - k_3} + \ln z \bigg|_0^{z(\tau)} \frac{z(\tau)}{k_1 - k_3}\right]$$

with $k_0 = \frac{\theta k_0}{k_3} = \frac{k_1 - k_3}{k_3 k_2} = \frac{\theta k_1 - \theta k_3}{k_3 k_2}$, where $k_3 = k_1 - k_0 k_2$

$$A(\tau) = \gamma r \tau + \frac{\theta \kappa}{k_3 k_2} \left(-2k_3 \ln z \bigg|_0^{z(\tau)} + (k_1 + k_3) \ln(z + k_1 - k_3) \bigg|_0^{z(\tau)}\right)$$

Taking $2k_3$ out, we get

$$A(\tau) = \gamma r \tau + \frac{2 \theta \kappa}{k_2} \left(\frac{k_1 + k_3}{2k_3} \ln(z + k_1 - k_3) \bigg|_0^{z(\tau)} - \ln z \bigg|_0^{z(\tau)}\right)$$

Note $z(\tau) + k_1 - k_3 = (k_1 + k_3)(1 + e^{k_3 \tau} - 1) = (k_1 + k_3)(1 + e^{k_3 \tau} - 1) = (k_1 + k_3) e^{k_3 \tau}$

and $z(0) + k_1 - k_3 = 2k_3 - k_3 + k_1 = k_3 + k_1$

$$A(\tau) = \gamma r \tau + \frac{2 \theta \kappa}{k_2} \ln \left(\frac{2k_3 e^{k_3 \tau}}{2k_3 + (k_1 + k_3)(e^{k_3 \tau} - 1)}\right).$$
Then, we can obtain \( h(t, v) \), i.e., \( h(t, v) = e^{A(t,v)+B(t,v)r} \). Moreover, \( J(t, x, v) = \frac{\partial}{\partial t} h(t, v) \) and its partial derivatives can be obtained. Lastly, substitute these partial derivatives of \( J \) into the optimal strategy \( \pi^* \) from Equation (A.2), such that,

\[
\pi^* = -\frac{\sqrt{\sigma} \rho J_{xx} - \lambda \sqrt{J_s}}{x(a \sqrt{v} + \frac{b}{\sqrt{v}}) J_{xx}} = -\frac{\sqrt{\sigma} \rho B(T-t)}{(a \sqrt{v} + \frac{b}{\sqrt{v}})(1-\gamma)} + \frac{\lambda \sqrt{v}}{(a \sqrt{v} + \frac{b}{\sqrt{v}})(1-\gamma)}.
\]

\( \Box \)

### A.3 Proof of option price process

**Proof.** We have:

\[
\begin{align*}
\left[ \frac{dS_t}{dv_t} \right] &= \left[ S_t \left( r + (av_t + b)(\lambda_1 \rho + \lambda_2 \sqrt{1 - \rho^2}) \right) \right] dt \\
&+ \left[ \frac{S_t \rho (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}})}{\sigma \sqrt{v_t}} - S_t \sqrt{1 - \rho^2}(a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) \right] \left[ \frac{dZ_{1t}}{dZ_{2t}} \right],
\end{align*}
\]

where \( F_t = (S_t, v_t)^T \). Then the option price can be represented as \( O_t = m(F_t, t) \). On the one hand, according to Björk (2009), the option price should satisfy the following PDE,

\[
rm = m_t + rSm_s + (\kappa(\theta - v) - \sigma \sqrt{v} \lambda_1 \sqrt{v})m_v + \frac{1}{2} \text{trace} \left( \Sigma^F m_{f_j} \Sigma^F \right),
\]

where \( m_{f_j} \) is a matrix of mixed partial derivatives of the function \( m \). On the other hand, by applying Ito’s Lemma to the function \( m \), we can get

\[
dm = \left[ m_t + S_m \left( r + (av + b)(\lambda_1 \rho + \sqrt{1 - \rho^2} \lambda_2) \right) + m_v \kappa(\theta - v) + \frac{1}{2} \text{trace} \left( \Sigma^F m_{f_j} \Sigma^F \right) \right] dt + m_{f_j} \Sigma^F dZ_t,
\]

where \( dZ_t = \left[ \frac{dZ_{1t}}{dZ_{2t}} \right], \) and \( m_{f_j} = \left[ \frac{m_{s_j}}{m_{v_j}} \right] \) denotes the partial derivative of the function \( m \) with respect to \( S \) and \( v \) respectively. Now, we can combine the last two PDEs to obtain

\[
dm = \left[ rm + \sigma \lambda_1 v m_v + S_m (av + b)(\lambda_1 \rho + \sqrt{1 - \rho^2} \lambda_2) \right] dt \\
+ m_S \left( \rho (a \sqrt{v} + \frac{b}{\sqrt{v}})dZ_{1t} + \sqrt{1 - \rho^2}(a \sqrt{v} + \frac{b}{\sqrt{v}})dZ_{2t} \right) + m_v (\sigma \sqrt{v} dZ_{1t}).
\]

Rearranging and regrouping the equation by two risk factors, i.e., \( dZ_{1t} \) and \( dZ_{2t} \), such that

\[
dm = rm dt + \left[ \rho m_S (a \sqrt{v} + \frac{b}{\sqrt{v}}) + m_v \sigma \sqrt{v} \right] (\lambda_1 \sqrt{v} dt + dZ_{1t}) \\
+ \left[ m_S \sqrt{1 - \rho^2}(a \sqrt{v} + \frac{b}{\sqrt{v}}) \right] (\lambda_2 \sqrt{v} dt + dZ_{2t}).
\]
Thus, the dynamic of option price is given as

\[
d\frac{O_t}{O_t} = rdt + \frac{1}{O_t}\left[ mS\rho S_t + m_o \frac{\sigma \sqrt{v_t}}{\sqrt{v_t} + \frac{b}{\sqrt{v_t}}} \right] \left( a\sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \left( \lambda_1 \sqrt{v_t}dt + dZ_{1t} \right)
\]

\[
+ \frac{1}{O_t}\left[ mS\sqrt{1 - \rho^2} S_t \right] \left( a\sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \left( \lambda_2 \sqrt{v_t}dt + dZ_{2t} \right)
\]

Note: Here we factor out the term \( (a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}) \), which is the diffusion coefficient from stock process, together with the Brownian motions \( Z_{1t} \) and \( Z_{2t} \) that are involved in the stock process. Also, as the diffusion part of the stock process and stochastic volatility are of different forms, i.e., \( (a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}) \) vs. \( \sigma \sqrt{v_t} \), we cannot totally separate out \( v_t \) from \( K_t \). But this follows the same idea as Liu and Pan (2003) and Escobar et al. (2017).

\[\square\]

A.4 Proof of optimal investment strategy in complete market

Proof. Proof of Proposition 2.3.3.

The corresponding HJB equation that the value function should satisfy is given by

\[
0 = \bar{J}_t + \sup_{\eta \in \mathbb{R}^2} \left\{ \frac{1}{2} x^T \eta^T \bar{\bar{V}} \eta J_{xx} + x^T \left( r - \bar{\eta} \bar{\bar{V}} \bar{\lambda} \right) J_x + x^T \eta^T \bar{\bar{R}}^T \sigma \sqrt{v} J_{sv} + \frac{1}{2} \sigma^2 J_{vv} \right\},
\]

where \( \bar{R} = (1, 0) \) represents the correlation between the diffusion part of the stochastic volatility process (i.e., \( dZ_{1t} \)) and the diffusion part of the stock process (i.e., \( \rho dZ_{1t} + \sqrt{1 - \rho^2} dZ_{2t} \)). The function \( \bar{J} \) is the value function denoted in the case of complete market, such as

\[
\bar{J}(x, v, t) = \frac{x^T}{y} \bar{h}(t, v),
\]

with \( \bar{h}(T, v) = 1 \). Similarly to the value function in the incomplete market, \( \bar{J}_x, \bar{J}_v, \bar{J}_{xx}, \bar{J}_{sv}, \) and \( \bar{J}_{sv} \) are first and second partial derivatives of the function \( \bar{J} \), and it should satisfy the boundary condition \( \bar{J}(x, v, T) = \frac{x^T}{y} \). Plugging \( \bar{J}(x, v, t) \) into the HJB equation (2.23), we get

\[
0 = \bar{J}_t + \kappa(\theta - \nu)\bar{J}_v + \frac{1}{2} \sigma^2 \nu \bar{J}_{vv} + \sup_{\eta} \left\{ \bar{g}(\eta) \right\},
\]

where \( \bar{g}(\eta) = \frac{1}{2} x^T \eta^T \bar{\bar{V}} \eta J_{xx} + x^T \left( r + \eta^T \bar{\bar{V}} \bar{\lambda} \right) J_x + x^T \eta^T \bar{\bar{R}}^T \sigma \sqrt{v} J_{sv} \). With a first order condition on the function \( \bar{g}(\eta) \) and setting \( \bar{g}'(\eta) = 0 \), we get the candidate of optimal \( \eta^* \), this is:

\[
\eta^* = \left( \bar{V}^2 \right)^{-1} \left( -\bar{V}^T \sigma \sqrt{v} \bar{J}_{sv} / x \bar{J}_{xx} - \bar{\bar{V}} \bar{\lambda} / x \bar{J}_{xx} \right).
\]
Substituting the candidate for the optimal $\eta$ into the HJB and getting rid of the supremum leads to:

$$
0 = \ddot{J}_t + \frac{1}{2} \sigma^2 \nu \ddot{J}_v + \kappa (\theta - \nu) \dot{J}_v + \frac{1}{2} \left( \sigma^2 \nu \frac{\dddot{J}_v}{f_{xx}} + 2 \sigma \lambda_1 \nu \frac{\ddot{J}_v}{f_{xx}} + (\lambda_1^2 + \lambda_2^2) \nu \frac{\dot{J}_v}{f_{xx}} \right) + \nu x \ddot{J}_x - \sigma \lambda_1 \nu \frac{\ddot{J}_v}{f_{xx}} - (\lambda_1^2 + \lambda_2^2) \nu \frac{\dot{J}_v}{f_{xx}} - \sigma^2 \nu \frac{\dddot{J}_v}{f_{xx}} - \sigma \lambda_1 \nu \frac{\ddot{J}_v}{f_{xx}}.
$$

By assuming that

$$
\dot{J}(x, v, t) = \frac{\chi}{\gamma} \tilde{h}(t, v) = \frac{\chi}{\gamma} \exp \left( \bar{A}(T - t) + \bar{B}(T - t)v \right),
$$

with time horizon $\tau(t) = T - t$ and therefore boundary conditions

$$
\tilde{h}(T, v) = 1 \quad \forall v \Rightarrow \bar{A}(0) = \bar{A}(\tau(T)) = 0, \bar{B}(0) = \bar{B}(\tau(T)) = 0. \tag{A.8}
$$

Taking partial derivatives of $\dot{J}$, substituting into HJB and simplifying the equation by canceling out $\frac{\chi}{\gamma}$ leads to:

$$
0 = ry \tilde{h} + \dot{h}_t + \frac{1}{2} \sigma^2 \nu \ddot{h}_v + \kappa (\theta - \nu) \tilde{h}_v
$$

$$
- \frac{1}{2} \sigma^2 \nu \left( \frac{\gamma}{\gamma - 1} \tilde{h} \right) - \frac{1}{2} (\lambda_1^2 + \lambda_2^2) \nu \left( \frac{\gamma}{\gamma - 1} \tilde{h} \right) - \sigma \lambda_1 \nu \left( \frac{\gamma}{\gamma - 1} \tilde{h} \right).
$$

(A.9)

It is known that

$$
\ddot{h}_t = (-\bar{A}' - \bar{B}' \nu) \tilde{h}, \ \ddot{h}_v = \bar{B} \tilde{h}, \ \dddot{h}_v = \bar{B}^2 \tilde{h}, \ \frac{\dddot{h}_v}{\tilde{h}} = \frac{\bar{B}^2 \tilde{h}^2}{\tilde{h}} = \bar{B}^2 \tilde{h}.
$$

It follows canceling out $\tilde{h}$,

$$
0 = ry - \bar{A}' + \kappa \theta \bar{B} + \nu \left[ -\bar{B}' + \frac{1}{2} \sigma^2 \nu \bar{B}^2 - \kappa \bar{B} - \frac{1}{2} \sigma^2 \nu \left( \frac{\gamma}{\gamma - 1} \bar{B}^2 \right) - \frac{1}{2} (\lambda_1^2 + \lambda_2^2) \left( \frac{\gamma}{\gamma - 1} \right) - \sigma \lambda_1 \left( \frac{\gamma}{\gamma - 1} \bar{B} \right) \right].
$$

(A.10)

It turns out that this PDE is linear in $v$. We can separate $v$ out in a linear form, then we can get the functions $\bar{A}(\tau)$ and $\bar{B}(\tau)$ and solve for $\bar{h}$.

$$
0 = ry - \bar{A}' + \kappa \theta \bar{B} + \nu \left[ -\bar{B}' + \frac{1}{2} \sigma^2 \nu \bar{B}^2 - \kappa \bar{B} - \frac{1}{2} \sigma^2 \nu \left( \frac{\gamma}{\gamma - 1} \bar{B}^2 \right) - \frac{1}{2} (\lambda_1^2 + \lambda_2^2) \left( \frac{\gamma}{\gamma - 1} \right) - \sigma \lambda_1 \left( \frac{\gamma}{\gamma - 1} \bar{B} \right) \right].
$$

To satisfy the boundary condition, we need both terms of the linear function of $v$ to be zero,

$$
\bar{A}' = \kappa \theta \bar{B} + \gamma r, \tag{A.11}
$$

$$
\bar{B}' = \left( \frac{\sigma^2 - \gamma \sigma^2}{\gamma - 1} \right) \bar{B} - \left( \kappa + \lambda_1 \sigma \frac{\gamma}{\gamma - 1} \right) \bar{B} - \frac{1}{2} (\lambda_1^2 + \lambda_2^2) \frac{\gamma}{\gamma - 1} \bar{B} - \frac{1}{2} \sigma^2 \nu \left( \frac{\gamma}{\gamma - 1} \right) - \sigma \lambda_1 \left( \frac{\gamma}{\gamma - 1} \bar{B} \right).
$$

(A.12)
It can be seen that we get the same form of Riccati’s style PDE as that of incomplete market, and we can solve it similarly. Assume \( \bar{k}_3 = \sqrt{k_1^2 - k_0k_2} \), then we can solve for the functions \( \ddot{A}(\tau(t)) \) and \( \ddot{B}(\tau(t)) \), such that

\[
\ddot{A}(\tau(t)) = \gamma \tau t + \frac{2\theta k}{\bar{k}_2} \ln \left( \frac{2\bar{k}_3 e^{\frac{1}{2}k\tau} - \bar{k}_3}{2\bar{k}_3 + (\bar{k}_1 + \bar{k}_3) (e^{k\tau} - 1)} \right),
\]

(A.13)

\[
\ddot{B}(\tau(t)) = \frac{\bar{k}_0 (e^{k\tau} - 1)}{2\bar{k}_3 + (\bar{k}_1 + \bar{k}_3) (e^{k\tau} - 1)},
\]

(A.14)

leading to the optimal \( \eta \),

\[
\eta^* = \left[ \frac{1}{1 - \gamma} \left( \sigma \sqrt{\bar{B}(T - t)} + \bar{\lambda}_1 \sqrt{\bar{V}_t} \right) \right] = \left[ \frac{1}{1 - \gamma} \left( \frac{\bar{\lambda}_1 \sqrt{\bar{V}_t}}{\sqrt{\bar{V}_t}} \right) \right].
\]

It follows that the optimal allocation to stock and the option is \( \ddot{\pi}^* = (\ddot{G}^T)^{-1} \eta^* \). It can be computed explicitly.

\[
(G^T)^{-1} = \frac{1}{\rho L_1 - K_1 \sqrt{1 - \rho^2}} \left[ -\frac{L_1}{\sqrt{1 - \rho^2}} - K_1 \rho \right].
\]

That is,

\[
\ddot{\pi}^*_{S} = \frac{L_1 (\sigma \sqrt{\bar{B}(T - t)} + \bar{\lambda}_1 \sqrt{\bar{V}_t}) - K_1 \bar{\lambda}_2 \sqrt{\bar{V}_t}}{(1 - \gamma) (\rho L_1 - K_1 \sqrt{1 - \rho^2})},
\]

\[
\ddot{\pi}^*_{O} = \frac{-\sqrt{1 - \rho^2} (\sigma \sqrt{\bar{B}(T - t)} + \bar{\lambda}_1 \sqrt{\bar{V}_t}) + \rho \bar{\lambda}_2 \sqrt{\bar{V}_t}}{(1 - \gamma) (\rho L_1 - K_1 \sqrt{1 - \rho^2})},
\]

with \( L_1 = \frac{1}{\partial_t} \left( m_S \sqrt{1 - \rho^2} S_t \right) \) and \( K_1 = \frac{1}{\partial_t} \left( m_S \rho S_t + m_r \frac{\sigma \sqrt{\bar{V}_t}}{\sqrt{\bar{V}_t} + \frac{h}{\bar{V}_t}} \right) \), where \( m_S \) and \( m_r \) are partial derivatives of the option price function \( O_t = m(F_t, t) \). It can be calculated that \( (\rho L_1 - K_1 \sqrt{1 - \rho^2}) = -\sqrt{1 - \rho^2} \frac{1}{\partial_t} m_r \frac{\sigma \sqrt{\bar{V}_t}}{\sqrt{\bar{V}_t} + \frac{h}{\bar{V}_t}} \). After substituting and simplifying, we get the desired expressions. \( \Box \)

### A.5 Proof of technical conditions in incomplete market

**Proof.** Proof of Proposition 2.4.1.

To ensure that the solution is real-valued, we need to ensure the square roots are well-defined in the function \( B(T - t) \) (2.19). That is, \( k_1 > 0 \iff k_1^2 - k_0k_2 > 0 \).

\[
k_1^2 - k_0k_2 = \left( k - \bar{\lambda} \frac{\gamma \sigma \rho}{1 - \gamma} \right)^2 - \frac{\gamma}{1 - \gamma} \bar{\lambda}^2 \left( \sigma^2 + \frac{\gamma \sigma^2 \rho^2}{1 - \gamma} \right) = k^2 - 2k \bar{\lambda} \frac{\gamma}{1 - \gamma} \sigma \rho - \frac{\gamma}{1 - \gamma} \bar{\lambda}^2 \sigma^2.
\]
Thus, for all $\gamma < 1$, we need
\[
\kappa^2 - 2\kappa^3 \frac{\gamma}{1-\gamma} \sigma^2 - \frac{\gamma}{1-\gamma} \lambda^2 \sigma^2 > 0.
\]
Note that to ensure finiteness, if $\gamma < 0$, then $k_0 = \frac{\gamma}{1-\gamma} \lambda^2 < 0$, $\Leftrightarrow B(T-t) < 0$ and so is the $ln$ representation of $A(T-t)$. On the other hand, if $0 < \gamma < 1$, we will need $2k_3 + (k_1 + k_3)(e^{k_3^2 - 1}) \neq 0$, such that
\[
2k_3 + (k_1 + k_3)(e^{k_3^2 - 1}) \neq 0 \Leftrightarrow -(k_1 - k_3)
\]
\[
\left(1 - \frac{k_1 + k_3}{k_1 - k_3} e^{k_3^2 - 1}\right) \neq 0
\]
\[
\Leftrightarrow \frac{k_1 + k_3}{k_1 - k_3} > 1 \Leftrightarrow k_1 > k_3 = \sqrt{k_1^2 - k_0 k_2}
\]
\[
k_0 k_2 > 0 \Leftrightarrow \frac{\gamma}{1-\gamma} \lambda^2 \left(\sigma^2 + \frac{\gamma \sigma^2 \rho^2}{1-\gamma}\right) > 0 \Leftrightarrow \frac{\gamma}{1-\gamma} \left(\sigma^2 + \frac{\gamma \sigma^2 \rho^2}{1-\gamma}\right) \lambda^2 > 0,
\]
which is always satisfied. \hfill \Box

## A.6 Proof of technical conditions in complete market

**Proof.** Proof of Proposition 2.4.2

To ensure that the function is real-valued, we need to ensure the square roots are well-defined in the function $\tilde{B}(T-t)$ \eqref{3.5}. That is, $\tilde{k}_3 > 0 \Leftrightarrow \tilde{k}_3^2 - \tilde{k}_0 \tilde{k}_2 > 0$ to ensure the square roots are real numbers.

\[
\tilde{k}_3^2 - \tilde{k}_0 \tilde{k}_2 = \left(\kappa - \lambda_1 \sigma \frac{\gamma}{1-\gamma}\right)^2 - \frac{\gamma}{1-\gamma}(\lambda_1^2 + \lambda_2^2) \left(\sigma^2 + \frac{\gamma \sigma^2}{1-\gamma}\right)
\]
\[
= \kappa^2 - 2\kappa \lambda_1 \sigma \frac{\gamma}{1-\gamma} + \frac{\gamma^2}{(1-\gamma)^2} \lambda_1^2 \sigma^2 - \frac{\gamma}{1-\gamma}(\lambda_1^2 + \lambda_2^2) \sigma^2
\]
\[
= \kappa^2 + \frac{\gamma}{1-\gamma} \left(-2\kappa \lambda_1 \sigma + \frac{\gamma}{1-\gamma} \lambda_1^2 \sigma^2 - \frac{1}{1-\gamma}(\lambda_1^2 + \lambda_2^2) \sigma^2\right).
\]

For all $\gamma < 1$, we need
\[
\kappa^2 + \frac{\gamma}{1-\gamma} \left(-2\kappa \lambda_1 \sigma + \frac{\gamma}{1-\gamma} \lambda_1^2 \sigma^2 - \frac{1}{1-\gamma}(\lambda_1^2 + \lambda_2^2) \sigma^2\right) > 0.
\]

Note that to ensure finiteness, if $\gamma < 0$, then the argument of the exponential representation of $\tilde{J}(x, \nu, t)$ is negative because $\tilde{k}_0 = \frac{\gamma}{1-\gamma}(\lambda_1^2 + \lambda_2^2) < 0$, $\Leftrightarrow \tilde{B}(T-t) < 0$ and so is the $ln$ representation of $\tilde{A}(T-t)$. On the other hand, if $0 < \gamma < 1$, we will need $2\tilde{k}_3 + (\tilde{k}_1 + \tilde{k}_3)(e^{\tilde{k}_3^2 - 1}) \neq 0$, such that
\[
2\tilde{k}_3 + (\tilde{k}_1 + \tilde{k}_3)(e^{\tilde{k}_3^2 - 1}) \neq 0 \Leftrightarrow -(\tilde{k}_1 - \tilde{k}_3) \left(1 - \frac{\tilde{k}_1 + \tilde{k}_3}{\tilde{k}_1 - \tilde{k}_3} e^{\tilde{k}_3^2 - 1}\right) \neq 0
\]
\[
\Leftrightarrow \frac{\tilde{k}_1 + \tilde{k}_3}{\tilde{k}_1 - \tilde{k}_3} > 1 \Leftrightarrow \tilde{k}_1 > \tilde{k}_3 = \sqrt{\tilde{k}_1^2 - \tilde{k}_0 \tilde{k}_2}
\]
\[
\Leftrightarrow \tilde{k}_0 \tilde{k}_2 > 0 \Leftrightarrow \frac{\gamma}{(1-\gamma)^2}(\lambda_1^2 + \lambda_2^2) \sigma^2 > 0,
\]
which is always true. \hfill \Box
A.7 Proof of verification theorem, incomplete markets

Proof. Proof of Theorem 2.4.3.

Consider an arbitrary but fixed point \((\hat{x}, \hat{v}, \hat{t}) \in [0, T] \times [0, \infty) \times [0, \infty)\), and assume that the wealth of the investor at time \(\hat{t}\) is \(\hat{x}\), and \(v_t = \hat{v}\). For condition 1) from the first statement, the value function \(J\) is once continuously differentiable in \(t\), and twice differentiable in \(x\) and \(v\) by Proposition 2.4.1. For condition 2) from the first statement, the PDE is obviously satisfied by substituting the candidate optimal strategy \(\pi^*\), and the terminal condition follows by the definition of our help function \(h\). Furthermore, under the ansatz of value function \(J(x, v, t) = \frac{r}{\gamma} h(v, t)\), the dynamic of \(J\) can be determined, such that

\[
\begin{align*}
\frac{dJ}{dt} &= \frac{\partial J}{\partial x} dx + \frac{\partial J}{\partial v} dv + \frac{1}{2} \frac{\partial^2 J}{\partial x^2} (dx)^2 + \frac{1}{2} \frac{\partial^2 J}{\partial v^2} (dv)^2 + \frac{\partial J}{\partial x \partial v} (dx)(dv) \\
&= \left( \frac{X_t}{\gamma} h_t + \frac{X_t}{\gamma} h(r + \pi^* \lambda (av_t + b)) + \frac{X_t}{\gamma} \kappa (\theta - v_t) h_v + \frac{\gamma - 1}{2} X_t^2 (\pi^*)^2 (a \sqrt{v_t} + b \sqrt{v_t})^2 \\
&\quad + \frac{1}{2} \frac{X_t^2}{\gamma} h_v \sigma^2 v_t + \frac{2}{\gamma} X_t^2 h \pi^* \rho (a \sqrt{v_t} + b \sqrt{v_t}) \sigma \sqrt{v_t} \right) dt + \frac{X_t}{\gamma} h \pi^* (a \sqrt{v_t} + b \sqrt{v_t}) \rho dZ_t + \sqrt{1 - \rho^2} dZ_t. \\
\end{align*}
\]

At the same time, the optimal strategy \(\pi^*\) can be expressed as

\[
\pi^* = \frac{- \sqrt{v_t} \rho \frac{h_t}{h} - \lambda}{(a \sqrt{v_t} + b \sqrt{v_t})(\gamma - 1)}. 
\]

In addition, given the PDE (A.3) for \(h\), such that

\[
0 = h_t + \kappa (\theta - v) h_v + \frac{1}{2} \sigma^2 v h_{vv} + r y h - v \sigma \rho \lambda \frac{\gamma h_v}{\gamma - 1} - \frac{1}{2} v \sigma^2 \rho^2 \frac{\gamma h_v^2}{(\gamma - 1)h} - \frac{1}{2} \sigma^2 \frac{\gamma h}{\gamma - 1},
\]

the dynamic for \(J\) evolves as

\[
\frac{dJ}{J_t} = \frac{\gamma}{\gamma - 1} \left[ \left( \frac{\gamma - 1}{\gamma} - \rho^2 \right) \sigma B(T - t) - \rho \lambda \right] \sqrt{v_t} dZ_1 + \sqrt{1 - \rho^2} \frac{\gamma}{\gamma - 1} \left( - \sigma \rho B(T - t) - \lambda \right) \sqrt{v_t} dZ_2,
\]

By rewriting and substituting partial derivatives of the help function \(h\), it follows that

\[
\frac{dJ}{J_t} = \frac{\gamma}{\gamma - 1} \left[ \left( \frac{\gamma - 1}{\gamma} - \rho^2 \right) \sigma B(t) - \rho \lambda \right] \sqrt{v_t} (g_1(t)) dZ_1 + \sqrt{1 - \rho^2} \frac{\gamma}{\gamma - 1} \left( - \sigma \rho B(t) - \lambda \right) \sqrt{v_t} (g_2(t)) dZ_2,
\]

where \(B(T - t) = B(\tau(t)) = \frac{k_0 e^{b(\tau(t)-t)}}{\exp(k_1 e^{b(-\tau(t)-t)})} \) by Equation (2.19) with auxiliary parameters \(k_0, k_1, k_2, k_3\) defined in Proposition 2.3.1. By Ito’s lemma, the dynamics of \(f(J) = \log(J)\) can be derived,

\[
J_t = J_0 \cdot \exp \left( \frac{1}{2} \int_0^t \left( g_1^2(s) + g_2^2(s) \right) v_s ds + \int_0^t g_1(s) \sqrt{v_s} dZ_1 + \int_0^t g_2(s) \sqrt{v_s} dZ_2 \right) .
\]
According to 5.13 Corollary in Karatzas and Shreve (1998), if the above equation satisfies
\[
E \left[ \exp \left( \frac{1}{2} \int_0^t (g_1^2(s) + g_2^2(s))v_1(s)ds \right) \right] < \infty,
\]
then \( J_t \) is a martingale. Further, as \( v \) follows a CIR process, we can refer to Proposition 5.1 in Kraft (2005) which provides us a condition to ensure finiteness of such expectations, such that:
\[
\beta = \min_{t \in [0, T]} \left( -\frac{1}{2} (g_1^2(t) + g_2^2(t)) \right) \geq -\frac{k^2}{2\sigma^2},
\]
where
\[
-\frac{1}{2} (g_1^2(t) + g_2^2(t)) = -\frac{1}{2} \left[ (1 - \frac{\gamma (\gamma - 2)}{(y - 1)^2} \rho^2) \sigma^2 B(t) - t + 2 \frac{\gamma}{(y - 1)^2} \sigma B(t) \rho \bar{\lambda} + \left( \frac{\gamma}{y - 1} \right)^2 \bar{\lambda}^2 \right].
\]
Take the first order differentiation of \( B(T - t) \) with respect to \( t \), such that
\[
\left( -\frac{1}{2} (g_1^2(t) + g_2^2(t)) \right)' = (1 - \frac{\gamma (\gamma - 2)}{(y - 1)^2} \rho^2) \sigma^2 B'(T - t) B'(T - t) + \frac{\gamma}{(y - 1)^2} \sigma B'(T - t) \rho \bar{\lambda}.
\]
Then we have
\[
\begin{align*}
\begin{cases}
\text{If } 0 < \gamma < 1, & B'(T - t) < 0 \text{ in } t, B(0) = 0, \implies B(T - t) > 0; \\
\text{If } \gamma < 0, & B' (T - t) > 0 \text{ in } t, B(0) = 0, \implies B(T - t) < 0.
\end{cases}
\end{align*}
\]
As \( \bar{\lambda} > 0 \) and \( \rho < 0 \),
\[
\begin{align*}
\begin{cases}
\text{If } 0 < \gamma < 1 & \lambda > \left( \frac{\gamma}{y - 2} \right)^2 \rho^2 \sigma B(T - t), \implies \left( -\frac{1}{2} (g_1^2(t) + g_2^2(t)) \right) > 0, \text{ minimal at } t = 0 \\
& 0 < \lambda \leq \left( \frac{\gamma}{y - 2} \right)^2 \rho^2 \sigma B(T - t), \implies \left( -\frac{1}{2} (g_1^2(t) + g_2^2(t)) \right) < 0, \text{ minimal at } t = T \\
\text{If } \gamma < 0, & \lambda > \left( \frac{\gamma}{y - 2} \right)^2 \rho^2 \sigma B(T - t), \implies \left( -\frac{1}{2} (g_1^2(t) + g_2^2(t)) \right) < 0, \text{ minimal at } t = 0 \\
& 0 < \lambda \leq \left( \frac{\gamma}{y - 2} \right)^2 \rho^2 \sigma B(T - t), \implies \left( -\frac{1}{2} (g_1^2(t) + g_2^2(t)) \right) > 0, \text{ minimal at } t = T
\end{cases}
\end{align*}
\]
Note that when \( 0 < \gamma < 1 \), \( \frac{\gamma}{y - 2} \rho^2 > \left( \frac{\gamma}{y - 2} \right)^2 \rho^2 \sigma B(T - t) > 0 \) always holds. That is, \( \frac{\gamma}{y - 2} \rho^2 > \left( \frac{\gamma}{y - 2} \right)^2 \rho^2 \sigma B(T - t) > 0 \) with \( \rho < 0 \) and \( B(T - t) > 0 \). On the other hand, when \( \gamma < 0 \), \( \frac{\gamma}{y - 2} \rho^2 > 1 \) is always true. Thus, \( \rho^2 < \frac{\gamma}{y - 2} \rho^2 \sigma B(T - t) > 0 \) with \( \rho > 0 \) and \( B(T - t) < 0 \). On the other hand, when \( \gamma < 0 \), \( \frac{\gamma}{y - 2} \rho^2 > 1 \) is always true. Thus, \( \rho^2 < \frac{\gamma}{y - 2} \rho^2 \sigma B(T - t) > 0 \) with \( \rho < 0 \) and \( B(T - t) < 0 \).
To sum up, if \( 0 < \bar{\lambda} \leq \left( \frac{\gamma}{y - 2} \right)^2 \rho^2 \sigma B(T - t) \), then
\[
\beta = -\frac{1}{2} \left( 1 - \frac{\gamma (\gamma - 2)}{(y - 1)^2} \rho^2 \right) \sigma^2 B^2(T) + 2 \frac{\gamma}{(y - 1)^2} \sigma B(T) \rho \bar{\lambda} + \left( \frac{\gamma}{y - 1} \right)^2 \bar{\lambda}^2 \geq -\frac{k^2}{2\sigma^2}.
\]
If \( \bar{\lambda} > \left( \frac{\gamma}{y - 2} \right)^2 \rho^2 \sigma B(T - t) \), then
\[
\beta = -\frac{1}{2} \left[ (1 - \frac{\gamma (\gamma - 2)}{(y - 1)^2} \rho^2) \sigma^2 B^2(T) + 2 \frac{\gamma}{(y - 1)^2} \sigma B(T) \rho \bar{\lambda} + \left( \frac{\gamma}{y - 1} \right)^2 \bar{\lambda}^2 \right] \geq -\frac{k^2}{2\sigma^2}.
Now, it can be concluded that $J$ is a martingale. Further, as $J$ is a martingale, it follows that

$$
\mathbb{E}[U(X_T^\gamma)|\mathcal{F}_t] = \mathbb{E}[\frac{(X_T^\gamma)^\gamma}{\gamma}|\mathcal{F}_t] = \mathbb{E}[J(X_T^\gamma, v_T, T)|\mathcal{F}_t] = J(\bar{x}, \bar{v}, \bar{t}).
$$

For the second statement, let $\pi$ be an arbitrary admissible strategy and define the process $L_t$, $t \in [\bar{t}, T]$ by

$$
L_t := (X_{\bar{t}}^\pi)^{\gamma-1} X_t^\pi h(\pi, t).
$$

Given the dynamics of wealth $X_t$ in an incomplete market, and the dynamics of stochastic volatility $v_t$, apply Ito’s lemma to obtain the dynamics of $L_t$:

$$
dL_t = \frac{\partial L}{\partial X_t^\pi} dX_t^\pi + \frac{1}{2} \frac{\partial^2 L}{\partial (X_t^\pi)^2} (dX_t^\pi)^2 + \frac{\partial L}{\partial X_t^\pi} dh_t + \frac{1}{2} \frac{\partial^2 L}{\partial h_t^2} (dh_t)^2 + \frac{\partial^2 L}{\partial h_t \partial X_t^\pi} (dh_t)(dX_t^\pi) + \frac{\partial^2 L}{\partial h_t \partial X_t^\pi} (dL)(dX_t^\pi) + \frac{\partial^2 L}{\partial h_t \partial X_t^\pi} (dL)(dh_t).
$$

That is,

$$
dL_t = (X_{\bar{t}}^\pi)^{\gamma-1} X_t^\pi [(\gamma - 1) h(r + \pi_t \lambda a(v_t + b) + h(r + \pi_t \lambda a(v_t + b) + h_t \kappa (\theta - v_t) + h_t
$$

$$
+ \frac{1}{2} (\gamma - 1)(\gamma - 2) h(\pi_t)^2 (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2 + \frac{1}{2} h_t \sigma^2 v_t) + (\gamma - 1) \pi_t \lambda a(v_t + b) + h_t \kappa (\theta - v_t) + h_t
$$

$$
+ (\gamma - 1) h_t \pi_t \sigma (a v_t + b) \rho + h_t \pi_t \sigma (a v_t + b) \rho] dt + (X_{\bar{t}}^\pi)^{\gamma-1} X_t^\pi h_t \sigma \sqrt{v_t} dZ_t
$$

$$
+ (X_{\bar{t}}^\pi)^{\gamma-1} X_t^\pi h((\gamma - 1) \pi_t + \pi_t)(a \sqrt{v_t} + \frac{b}{\sqrt{v_t}})(\rho dZ_t + \sqrt{1 - \rho^2} dZ_{2t})
$$

$$
=: \mu^L dt + \Sigma^L_t dZ_t + \Sigma^L_{2t} dZ_{2t}.
$$

Thus, by rearranging terms in $\mu^L$, it leads to

$$
\mu^L = (X_{\bar{t}}^\pi)^{\gamma-1} X_t^\pi [r h + \gamma h \pi_t \lambda a(v_t + b) + h_t \kappa (\theta - v_t) + h_t + \frac{1}{2} (\gamma - 1) h(\pi_t)^2 (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2
$$

$$
+ \frac{1}{2} h_t \sigma^2 v_t + \gamma \pi_t \sigma (a v_t + b) \rho h_t + (\pi_t - \pi_t) (h_t \lambda a(v_t + b) + h_t (\gamma - 1) \pi_t (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2 + h_t \sigma (a v_t + b) \rho)].
$$

By substituting $\pi_t$ and cancelling out terms, it follows that $\mu^L = 0$ and the process of $L_t$ evolves as

$$
\frac{dL_t}{L_t} = \left( B \sigma \sqrt{v_t} + \rho [(\gamma - 1) \pi_t + \pi_t] (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) dZ_t + [(\gamma - 1) \pi_t + \pi_t] (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) \sqrt{1 - \rho^2} dZ_{2t}. \right)
$$

Thus, it follows that $L_t$ is a local martingale since all $X_t^\pi$, $X_t^\pi$, $\pi_t$, and $v_t$ are continuous functions. Furthermore, the help function $h(v, t) = e^{A(T-I)+B(T-I)v}$ is always positive, thus so is
the process $L_t$, which implies that it is a supermartingale (see the proof in Theorem 6.6 of Bain (2007)). Therefore, it can be derived that

\[
\mathbb{E}[U(X_T^x)|F_t] \leq \mathbb{E}[U(X_T^x)|F_t] + \mathbb{E}[U(X_T^x)(X_T^x - X_T^{x*})|F_t]
\]

(A.17)

\[
= \mathbb{E}[U(X_T^x)|F_t] + \mathbb{E}[L(T)|F_t] - \mathbb{E}[X_T^{x*}|F_t]
\]

(A.18)

\[
\leq \mathbb{E}[U(X_T^x)|F_t] + L(\tilde{t}) + \gamma \mathbb{E}[J(X_T^{x*}, v_T, T)|F_t]
\]

(A.19)

\[
= \mathbb{E}[U(X_T^x)|F_t] + (\tilde{x})^T \dot{h}(\tilde{v}, \tilde{t}) - \gamma J(\tilde{x}, \tilde{v}, \tilde{t})
\]

(A.20)

\[
= \mathbb{E}[U(X_T^x)|F_t] + (\tilde{x})^T \dot{h}(\tilde{v}, \tilde{t}) - (\tilde{x})^T h(\tilde{v}, \tilde{t}) = \mathbb{E}[U(X_T^x)|F_t].
\]

(A.21)

The first inequality is obtained by the concavity of utility function $U(x) = \frac{x^2}{2}$. The second equation takes apart the second term of the first inequality, and follows the definition of the process $L_t$ combined with $L_T := (X_T^{x*})^{T-1} X_T^{x*} h(v, T)$. The third inequality follows from the supermartingale property of $L_t$, such that $L_t \geq \mathbb{E}[L_T|F_t]$ and the definition of value function $J(x, v, t)$, such that:

\[
J(x, v, t) = \sup \mathbb{E}[U(X_T)|F_t] = \mathbb{E}[U(X_T^x)|F_t] = \mathbb{E}\left[\frac{(X_T^{x*})^T}{\gamma}\right].
\]

The fourth and fifth equation are obtained by the martingale property of the value function $J$ and substituting in $J(x, v, t) = \frac{x^2}{2} h(v, t)$. And the last equation obviously follows by cancelling out the last two terms. Now the proof shows that $J^* \leq J^{x*}$, thus $\pi^*$ is the optimal strategy and $J^{x*}$ is the value function that solves our original optimal problem.

\[\square\]

\section{A.8 Proof of verification theorem, complete markets}

\textbf{Proof.} Proof of Theorem 2.4.4.

For condition 1) in the first statement, the value function $\tilde{J}$ in a complete market satisfies them by Proposition 2.4.2. For condition 2) in the first statement, the value function $\tilde{J}$ in a complete market satisfies them by the HJB equation directly. Furthermore, under the ansatz of the value function $\tilde{J}(x, v, t) = \frac{x^2}{2} \tilde{h}(v, t)$, the dynamics of $\tilde{J}$ follow as

\[
d\tilde{J}_t = \frac{X_T^x}{\gamma} \left[ r \tilde{h}_t + \tilde{h}_t \kappa (\theta - v_t) + \frac{1}{2} \tilde{h}_{vv} \sigma^2 v_t + \gamma \tilde{h}(\eta_t^x)^T \tilde{V}_t \tilde{\lambda}_t + \frac{1}{2} \gamma (\gamma - 1) \tilde{h}(\eta_t^x)^T \tilde{V}_t^2 \eta_t^x \right]
\]

\[
+ \tilde{V}_t \tilde{h}(\eta_t^x)^T \sigma \sqrt{\tilde{V}_t} \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] dt + X_T^x \tilde{h}(\eta_t^x)^T \tilde{V}_t \left[ dZ_{1t} \right] + \frac{X_T^x}{\gamma} \tilde{h}_t \sigma \sqrt{\tilde{V}_t} dZ_{2t},
\]

with the candidate exposure as:

\[
\eta_t^x = (\tilde{V}_t)^{-1} \left( -\tilde{V}_t \tilde{R}^x \sigma \sqrt{\tilde{V}_t} \tilde{J}_{xx} - \tilde{V}_t \tilde{\lambda}_t \tilde{J}_x \tilde{J}_{xx} \right) = \left[ \begin{array}{ccc} \frac{\sigma \sqrt{\tilde{V}_t}}{(a \sqrt{\tilde{V}_t} + \frac{1}{\gamma})} & \frac{1}{\gamma} & -\frac{1}{\gamma-1} \left( a \sqrt{\tilde{V}_t} + \frac{1}{\gamma} \right) \\ \frac{1}{a \sqrt{\tilde{V}_t} + \frac{1}{\gamma}} & \frac{\tilde{h}_t}{\gamma} & -\frac{1}{\gamma-1} \left( a \sqrt{\tilde{V}_t} + \frac{1}{\gamma} \right) \\ -\frac{1}{\gamma-1} \left( a \sqrt{\tilde{V}_t} + \frac{1}{\gamma} \right) & -\frac{\lambda_t \sqrt{\tilde{V}_t}}{(a \sqrt{\tilde{V}_t} + \frac{1}{\gamma})} & \frac{\tilde{h}_t}{\gamma} \end{array} \right],
\]
where \( \bar{R} = (1, 0) \), \( \bar{V}_i = \begin{bmatrix} (a \sqrt{\bar{V}_i} + \frac{b}{\gamma \sqrt{\bar{V}_i}}) & 0 \\ 0 & (a \sqrt{\bar{V}_i} + \frac{b}{\gamma \sqrt{\bar{V}_i}}) \end{bmatrix} \), \( \lambda_i = \begin{bmatrix} \tilde{\lambda}_1 \sqrt{\bar{V}_i} \\ \tilde{\lambda}_2 \sqrt{\bar{V}_i} \end{bmatrix} \). Recall that the substitution of the ansatz \( \bar{J}(x, v, t) = \frac{v}{\gamma} \tilde{h}(v, t) \) leads to a PDE for \( \tilde{h} \):

\[
0 = ry\tilde{h} + \bar{h}_t + \frac{1}{2} \sigma^2 v \bar{h}_{vv} + \kappa(\theta - v)\bar{h}_v \tag{A.22}
\]

Substitute \( \eta_i \) into dynamic of \( \bar{J} \) and combine with Equation (A.22),

\[
d\bar{J}_i = \frac{X_i'}{\gamma} \left[ ry\tilde{h} + \bar{h}_t + \frac{1}{2} \sigma^2 v \bar{h}_{vv} - \frac{\gamma}{\gamma - 1} \bar{h}_t \lambda_1 v - \frac{\gamma}{\gamma - 1} \bar{h}(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2)v \right] \tag{A.23}
\]

by (A.24)

\[
- \frac{\gamma}{\gamma - 1} \bar{h}_t \lambda_1 v - \frac{\gamma}{\gamma - 1} \bar{h}_t \lambda_1 v \right] dt \tag{A.25}
\]

where

\[
\gamma \tilde{h}(\eta_i^T \bar{V}_i \lambda_i = \gamma \bar{h} \left[ \begin{bmatrix} \frac{-\sigma \sqrt{\bar{V}_i}}{\gamma} & \frac{1}{\gamma - 1} & \frac{-\tilde{\lambda}_1 \sqrt{\bar{V}_i}}{\gamma - 1} \\ 0 & \frac{-1}{\gamma - 1} & \frac{-1}{\gamma - 1} \end{bmatrix} \begin{bmatrix} \lambda_1 \sqrt{\bar{V}_i} \\ \lambda_2 \sqrt{\bar{V}_i} \end{bmatrix} \right] \right]
\]

\[
= - \frac{\gamma}{\gamma - 1} \bar{h}_t \bar{h}_t \lambda_1 v - \frac{\gamma}{\gamma - 1} \bar{h}(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2)v. \tag{A.23}
\]

\[
\frac{1}{\gamma (\gamma - 1)} \bar{h}(\eta_i^T \bar{V}_i \eta_i = \frac{1}{2} \frac{\gamma}{\gamma - 1} \bar{h} \left[ \begin{bmatrix} \frac{\gamma^2 v}{\gamma - 1} & \gamma \frac{\lambda_2^2 v}{\gamma - 1} \\ \frac{2 \sigma \lambda_1 v}{\gamma - 1} & \gamma \frac{\lambda_2^2 v}{\gamma - 1} \end{bmatrix} \begin{bmatrix} \lambda_1 \sqrt{\bar{V}_i} \\ \lambda_2 \sqrt{\bar{V}_i} \end{bmatrix} \right] \right]
\]

\[
= \frac{1}{2} \frac{\gamma}{\gamma - 1} \bar{h}_t \lambda_2^2 v + \frac{\gamma}{\gamma - 1} \bar{h}_t \lambda_1 v + \frac{\gamma}{\gamma - 1} \bar{h}(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2)v. \tag{A.24}
\]

\[
\gamma \tilde{h}_i(\eta_i^T \sigma \sqrt{\bar{V}_i} \bar{V}_i \begin{bmatrix} 1 \\ 0 \end{bmatrix} = - \frac{\gamma}{\gamma - 1} \bar{h}_t \lambda_1 v \tag{A.25}
\]
That is,
\[
\frac{d\hat{J}}{\hat{J}} = \frac{X_t^\gamma}{\gamma} \left[ r\hat{h} + \bar{h}_t + \frac{1}{2} \sigma^2 v_t \bar{h}_v + \kappa(\theta - v_t)\bar{h}_v \right]
\]
\[
- \frac{1}{2} \sigma^2 v_t \left( \frac{\gamma}{\gamma - 1} \bar{h}_t \right) - \frac{1}{2} \left( \lambda_1^2 + \lambda_2^2 \right) v_t \left( \frac{\gamma}{\gamma - 1} \bar{h}_v \right) - \sigma \lambda_1 v_t \left( \frac{\gamma}{\gamma - 1} \bar{h}_v \right) \right] dt
\]
\[
+ \frac{X_t^\gamma}{\gamma} \left[ \frac{1}{(\gamma - 1)} \sigma \sqrt{v_t} \bar{h}_v - \frac{\gamma}{\gamma - 1} \bar{h}_v \lambda_1 \sqrt{v_t} \right] dZ_{1t} - \frac{X_t^\gamma}{\gamma} \left[ \frac{\gamma}{\gamma - 1} \bar{h}_v \lambda_2 \sqrt{v_t} \right] dZ_{2t}
\]
\[
= \frac{X_t^\gamma}{\gamma} \left[ - \frac{1}{(\gamma - 1)} \sigma \sqrt{v_t} \bar{h}_v - \frac{\gamma}{\gamma - 1} \lambda_1 \sqrt{v_t} \right] dZ_{1t} - \frac{X_t^\gamma}{\gamma} \left[ \frac{\gamma}{\gamma - 1} \bar{h}_v \lambda_2 \sqrt{v_t} \right] dZ_{2t}.
\]
It is easy to tell that \(\hat{J}\) is a local martingale at this point. Moreover, we can rewrite the dynamics of \(d\hat{J}\), as
\[
\frac{d\hat{J}}{\hat{J}} = \left[ \frac{- \frac{1}{2} \sigma \bar{B}(T - t) - \frac{\gamma}{\gamma - 1} \lambda_1}{\bar{g}_i(t)} \sqrt{v_t} dZ_{1t} - \left[ \frac{\gamma}{\gamma - 1} \lambda_2 \right] \sqrt{v_t} dZ_{2t},
\]
where \(\bar{B}(T - t)\) is defined in Proposition 2.3.3, Equation (2.26). By Ito’s lemma,
\[
\hat{J} = J_0 \cdot \exp \left\{ - \frac{1}{2} \int_0^T \left[ \bar{g}_1^2(s) + \bar{g}_2^2(s) \right] v_t \right\} ds + \int_0^T \bar{g}_1(s) \sqrt{v_t} dZ_{1t} + \int_0^T \bar{g}_2(s) \sqrt{v_t} dZ_{2t} \}
\]
Similarly to the incomplete market case, we need:
\[
\beta = \min_{t \in [0, T]} \left\{ \frac{1}{2} \sigma \frac{\bar{B}^2(T - t) + 2\gamma \sigma \bar{B}(T - t) \lambda_1 + \gamma^2 (\lambda_1^2 + \lambda_2^2)}{\gamma} \right\} \geq \frac{\kappa^2}{2\sigma^2}.
\]
By taking first order differentiation of \(\bar{B}(T - t)\) with respect to \(t\), which is similar with that of the function \(\bar{B}(T - t)\) in incomplete market, such that
\[
\left( -\frac{1}{2} (\bar{g}_1^2(t) + \bar{g}_2^2(t)) \right)' = \frac{1}{(\gamma - 1)^2} \left[ \sigma^2 \bar{B}(T - t) \bar{B}'(T - t) + \gamma \sigma \lambda_1 \bar{B}'(T - t) \right].
\]
Then we have
\[
\begin{cases}
\text{If } 0 < \gamma < 1, \quad \bar{B}'(T - t) < 0 \text{ in } t, \quad \bar{B}(0) = 0, \quad \Rightarrow \quad \bar{B}(T - t) > 0; \\
\text{If } \gamma < 0, \quad \bar{B}'(T - t) > 0 \text{ in } t, \quad \bar{B}(0) = 0, \quad \Rightarrow \quad \bar{B}(T - t) < 0.
\end{cases}
\]
As \(\lambda_1 < 0\),
\[
\begin{cases}
\text{If } 0 < \gamma < 1, \quad \lambda_1 < \frac{\sigma \bar{B}(T - t)}{\gamma}, \quad \Rightarrow \quad \left( -\frac{1}{2} (\bar{g}_1^2(t) + \bar{g}_2^2(t)) \right)' > 0, \quad \text{minimal at } t=0 \\
\quad \quad \lambda_1 \geq \frac{\sigma \bar{B}(T - t)}{\gamma}, \quad \Rightarrow \quad \left( -\frac{1}{2} (\bar{g}_1^2(t) + \bar{g}_2^2(t)) \right)' < 0, \quad \text{minimal at } t=T \\
\text{If } \gamma < 0, \quad \left( -\frac{1}{2} (\bar{g}_1^2(t) + \bar{g}_2^2(t)) \right)' < 0, \quad \text{minimal at } t=T.
\end{cases}
\]
That is, if \(\lambda_1 < 0\) and \(\gamma < 0\) or \(0 > \lambda_1 \geq -\frac{\sigma \bar{B}(T - t)}{\gamma}\) and \(0 < \gamma < 1\),
\[
\beta = \frac{1}{2} \frac{\gamma^2}{(\gamma - 1)^2} (\lambda_1^2 + \lambda_2^2) \geq \frac{\kappa^2}{2\sigma^2}.
\]
If $\lambda_1 < -\frac{\sigma^2 T_j}{\gamma}$ and $0 < \gamma < 1$, then
\[
\beta = -\frac{1}{2} \frac{1}{(\gamma - 1)^2} \left[ \sigma^2 B^2(T) + 2\gamma \sigma B(T) \lambda_1 + \gamma^2 (\lambda_1^2 + \lambda_2^2) \right] \geq -\frac{\kappa^2}{2\gamma^2}.
\]

Now, it can be concluded that $\tilde{J}$ is a martingale. Then, it follows from the definition of martingale that
\[
\mathbb{E}[U(\tilde{X}^\eta_T) | X_t = x, v_i = v] = J(x, v, t).
\] (A.26)

For the second statement, let $\eta$ be an arbitrary admissible strategy and define the process $\tilde{L}_t = \{\tilde{X}^\eta_t\}_{t=0}^T \tilde{h}(v, t)$. Given the dynamics of wealth $X_t$ in complete market, i.e., Equation (2.22), and the dynamics of stochastic volatility $v_t$, applying Ito’s lemma to obtain the dynamics of $\tilde{L}_t$:
\[
d\tilde{L}_t = (\tilde{X}^\eta_t)^{\gamma-1} \tilde{X}^\eta_t \left[ (\gamma - 1) \tilde{h}(r + (\eta_i^T)^T \tilde{V}_i \bar{\lambda}_i) + \tilde{h}(r + \eta_i^T \tilde{V}_i \bar{\lambda}_i) + \tilde{h}(\theta - v_i) + \tilde{h}_i \right.
+ \frac{1}{2} (\gamma - 1)(\gamma - 2) \tilde{h}(\eta_i^T)^T \tilde{V}_i \eta_i^T + \frac{1}{2} \tilde{h}_v \sigma^2 v_i + (\gamma - 1) \tilde{h}(\eta_i^T)^T \tilde{V}_i (\eta_i^T)^T
+ (\gamma - 1) \tilde{h}(\eta_i^T)^T \sigma \sqrt{v_i} \tilde{V}_i \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] + \tilde{h}_v \eta_i^T \sigma \sqrt{v_i} \tilde{V}_i \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \right) dt + (\tilde{X}^\eta_t)^{\gamma-1} \tilde{X}^\eta_t \tilde{h}_v \sigma \sqrt{v_i} dZ_{1t}
+ (\tilde{X}^\eta_t)^{\gamma-1} \tilde{X}^\eta_t \tilde{h}(\gamma - 1)(\eta_i^T)^T + \eta_i^T)^T \tilde{V}_i \left[ \begin{array}{c} dZ_{1t} \\ dZ_{2t} \end{array} \right]
= \mu^T dt + \Sigma^\eta_1 dZ_{1t} + \Sigma^\eta_2 dZ_{2t}.
\] (A.27)

Rearranging terms, substituting $\eta_i^T$, simplifying and regrouping leads to:
\[
\mu^T = (\tilde{X}^\eta_t)^{\gamma-1} \tilde{X}^\eta_t \left[ \left( r \tilde{h} + \tilde{h}_i + \frac{1}{2} \sigma^2 v_i \tilde{h}_v + \kappa(\theta - v_i) \tilde{h}_v \right)
- \frac{1}{2} \sigma^2 v_i \left( \frac{\gamma}{\gamma - 1} \tilde{h}_v \right)
- \frac{1}{2} \left( \lambda_1^2 + \lambda_2^2 \right) v_i \left( \frac{\gamma}{\gamma - 1} \tilde{h}_v \right)
- \sigma \tilde{\lambda}_1 v_i \left( \frac{\gamma}{\gamma - 1} \tilde{h}_v \right)
+ (\eta_i^T - (\eta_i^T)^T) \begin{array}{c} 0 \\ 0 \end{array} \right] = 0,
\]
by Equation (A.22). Thus, it can be concluded that the process $\tilde{L}_t$ is a local martingale. As $\tilde{L}_t$ is a local martingale, it is a supermartingale resulting from its positivity (Bain, 2007). Moreover, $\tilde{J}^\eta \leq \bar{J}^\eta$ can be derived similarly to the incomplete market case, thus $\eta^*$ is the optimal strategy and $\bar{J}^\eta$ solves the original optimal problem can be concluded.

A.9 Proof of Corollary 2.5.3

Proof. For incomplete markets, substituting the suboptimal strategy into Equation (2.37) leads to
\[
0 = \frac{1}{\gamma}(-A^{(s)} - B^{(s)}v) + r + \frac{1}{1 - \gamma} (\bar{\lambda}^2 v) + \kappa(\theta - v) - B^{(s)}
- \frac{1}{2(1 - \gamma)} \bar{\lambda}^2 v + \frac{1}{2} \sigma^2 v \left( B^{(s)} \right)^2 + \frac{1}{1 - \gamma} v \sigma \rho B^{(s)}.
\]
Regrouping and multiplying by $\gamma$, it follows that in order to satisfy the boundary condition, both terms of the linear function of $v$ need to be zero,

$$A^{(s)} = \kappa \theta B^{(s)} + r \gamma,$$

$$B^{(s)} = \frac{1}{2} \sigma^2 (B^{(s)})^2 + \left( \frac{\gamma \lambda}{1 - \gamma} \sigma \rho - \kappa \right) B^{(s)} + \frac{\gamma \lambda^2}{2(1 - \gamma)}.$$

Thus, the function $A^{(s)}(\tau)$ and the function $B^{(s)}(\tau)$ are given by

$$A^{(s)}(\tau(t)) = r \gamma \tau + \frac{2 \theta \kappa}{k_2} \ln \left( \frac{2 k_3 e^{k_1 t + k_3 \tau}}{2 k_3 + (k_1 + k_3)(e^{k_1 \tau} - 1)} \right), \quad (A.28)$$

$$B^{(s)}(\tau(t)) = \frac{k_0 \left( e^{k_1 \tau} - 1 \right)}{2 k_3 + (k_1 + k_3)(e^{k_1 \tau} - 1)}, \quad (A.29)$$

with auxiliary parameters $k_0, k_1, k_2, k_3$ satisfying $k_1^2 - k_0 k_2 > 0$, such that

$$k_0 := \frac{\gamma \lambda^2}{1 - \gamma}, k_1 := \left( \kappa - \frac{\gamma \lambda}{1 - \gamma} \sigma \rho \right), k_2 := \sigma^2, k_3 := \sqrt{k_1^2 - k_0 k_2}. \quad (A.30)$$

With functions $A^{(s)}(\tau)$ and $B^{(s)}(\tau)$, the wealth-equivalent loss in the incomplete market under this suboptimal strategy can be determined.

\[ \square \]

**A.10 Proof of Corollary 2.5.4**

Proof. For complete markets, substituting the suboptimal strategy into Equation (2.39) leads to

$$0 = \frac{1}{\gamma} \left( \bar{A}^{(s)} - \bar{B}^{(s)} \right) v - \frac{1}{2(1 - \gamma)} \left( \bar{\lambda}_1^2 + \bar{\lambda}_2^2 \right) v$$

$$+ r + \left( \bar{\lambda}_1 + \bar{\lambda}_2 \right) \frac{1}{1 - \gamma} v - \frac{1}{1 - \gamma} \bar{\lambda}_1 \sigma \sqrt{\bar{B}^{(s)}} + \frac{1}{2} \sigma^2 v \left( \frac{1}{\gamma} \bar{B}^{(s)} \right)^2 + \frac{1}{2(1 - \gamma)} \left( \bar{\lambda}_1^2 + \bar{\lambda}_2^2 \right) + \kappa(\theta - v) \frac{1}{\gamma} \bar{B}^{(s)}.$$

Regrouping and multiplying by $\gamma$, it follows that in order to satisfy the boundary condition, both terms of the linear function of $v$ need to be zero,

$$\bar{A}^{(s)} = \kappa \theta \bar{B}^{(s)} + r \gamma,$$

$$\bar{B}^{(s)} = \frac{1}{2} \sigma^2 (\bar{B}^{(s)})^2 + \left( \frac{\gamma \bar{\lambda}_1 \sigma - \kappa}{1 - \gamma} \right) \bar{B}^{(s)} + \frac{\gamma \bar{\lambda}_1^2 + \bar{\lambda}_2^2}{2(1 - \gamma)}.$$

$$= \frac{1}{2} \frac{\sigma^2}{k_2} \left( \bar{B}^{(s)} \right)^2 - \left( \kappa - \frac{\gamma \bar{\lambda}_1 \sigma}{1 - \gamma} \right) \frac{\bar{B}^{(s)}}{k_1} + \frac{1}{2} \frac{\gamma \bar{\lambda}_1^2 + \bar{\lambda}_2^2}{k_0}.$$
Thus, the functions $\bar{A}^{(s)}(\tau)$ and $\bar{B}^{(s)}(\tau)$ are given by

$$
\bar{A}^{(s)}(\tau(t)) = \gamma\tau + \frac{2\theta\kappa}{k_2} \ln \left( \frac{2\bar{k}_3 e^{\frac{1}{k_3}}}{2\bar{k}_3 + (\bar{k}_1 + \bar{k}_3)(e^{\bar{k}_3\tau} - 1)} \right),
$$

$$
\bar{B}^{(s)}(\tau(t)) = \frac{\bar{k}_0 (e^{\bar{k}_3\tau} - 1)}{2\bar{k}_3 + (\bar{k}_1 + \bar{k}_3)(e^{\bar{k}_3\tau} - 1)},
$$

with auxiliary parameters $\bar{k}_0, \bar{k}_1, \bar{k}_2, \bar{k}_3$ satisfying $\bar{k}_1 - \bar{k}_0\bar{k}_2 > 0$, such that $\bar{k}_0 := \frac{\tau}{(1-\gamma)(\bar{\lambda}_1^2 + \bar{\lambda}_2^2)}$, $\bar{k}_1 := (\kappa - \frac{\tau}{1-\gamma} \bar{\lambda}_1\sigma)$, $\bar{k}_2 := \sigma^2$ and $\bar{k}_3 := \sqrt{\bar{k}_1^2 - \bar{k}_0\bar{k}_2}$. The wealth-equivalent loss in the complete market under this suboptimal strategy can be easily determined.

\[ \Box \]

## A.11 Proof of Corollary 2.5.5

**Proof.** Given that $\eta^{s(3)} = \left[ \begin{array}{c} \rho^{sS} \\ \sqrt{1-\rho^2} L_1 \end{array} \right] \left[ \begin{array}{c} \pi^{sS} \\ 0 \end{array} \right] = \left[ \begin{array}{c} \rho^{sS} \\ \sqrt{1-\rho^2}\pi^{sS} \end{array} \right]$, substituting $\eta^{s(3)}$ into (2.39), such that

$$
0 = \frac{1}{\gamma} (-\bar{A}' - \bar{B}'\nu) + \gamma - \frac{1}{2} \left[ \begin{array}{c} \pi^{sS} \\ (a \sqrt{\nu} + \frac{b}{\sqrt{\nu}}) \end{array} \right]^2 + (r + \rho\pi^{sS}(av + b) \bar{\lambda}_1 + \sqrt{1-\rho^2}\pi^{sS}(av + b) \bar{\lambda}_2) + \rho\pi^{sS}(av + b) \sigma \bar{B} + \frac{1}{2} \sigma^2v \frac{1}{\gamma} \bar{B}^2 + \kappa(\theta - \nu) \frac{1}{\gamma} \bar{B}.
$$

Substituting

$$
\pi^{sS} = \frac{\nu}{av + b} \frac{\lambda_2}{(1-\gamma)\sqrt{1-\rho^2}},
$$

it follows that

$$
0 = \frac{1}{\gamma} (-\bar{A}' - \bar{B}'\nu) - \frac{1}{2(1-\gamma)} \left[ \begin{array}{c} \lambda_2^2 \\ 1-\rho^2 \end{array} \right] \nu + r + \frac{\lambda_2(\rho \bar{\lambda}_1 + \sqrt{1-\rho^2}\bar{\lambda}_2)}{(1-\gamma)\sqrt{1-\rho^2}} \nu + \frac{\rho \lambda_2}{(1-\gamma)\sqrt{1-\rho^2}} (v\sigma \bar{B} + \frac{1}{2} \sigma^2 \nu \frac{1}{\gamma} \bar{B}^2 + \kappa(\theta - \nu) \frac{1}{\gamma} \bar{B}).
$$

Multiplying $\gamma$ and regrouping,

$$
0 = -\bar{A}' + r\gamma + \kappa\bar{B} + \nu \left\{ -\bar{B}' - \frac{\gamma}{2(1-\gamma)} \left[ \begin{array}{c} \lambda_2^2 \\ 1-\rho^2 \end{array} \right] \right\} + \frac{\lambda_2(\rho \bar{\lambda}_1 + \sqrt{1-\rho^2}\bar{\lambda}_2)}{(1-\gamma)\sqrt{1-\rho^2}} + \frac{1}{2} \sigma^2 B^2 - k\bar{B}).
$$
In order to satisfy the boundary condition, both terms of the linear function of \( v \) need to be zero,
\[
\hat{A}' = k\theta B + ry,
\]
\[
B' = \frac{1}{2} \frac{\sigma^2}{k_2} B^2 - \left( \kappa - \frac{\gamma}{1 - \gamma} \left( \frac{\rho_1 \lambda_2 \sigma}{\sqrt{1 - \rho^2}} \right) k_1 \right) B + \frac{1}{2} \left( \frac{\gamma}{1 - \rho^2} \kappa_0 \right) B \left[ \frac{2\lambda_2 \rho_1 + \sqrt{1 - \rho^2} \lambda_3}{\sqrt{1 - \rho^2}} \right].
\]

Thus, the function \( \hat{A}(\tau) \) and the function \( \hat{B}(\tau) \) are given by
\[
\hat{A}(\tau(t)) = \gamma rt + \frac{2\theta \kappa}{k_2} \ln \left( \frac{2\bar{k}_3 e^{\frac{1}{\gamma} k_2}}{2\bar{k}_3} \left( \frac{\bar{k}_1 + \bar{k}_3}{(\bar{k}_1 + \bar{k}_3)(e^{k_2} - 1)} \right) \right),
\]
\[
\hat{B}(\tau(t)) = \frac{\bar{k}_0}{2\kappa_3} \left( e^{k_2} - 1 \right),
\]
with auxiliary parameters \( \bar{k}_0, \bar{k}_1, \bar{k}_2, \bar{k}_3 \) satisfying \( \bar{k}_2 - \bar{k}_0 \bar{k}_2 > 0 \), such that \( \bar{k}_0 := \frac{\gamma}{1 - \gamma} \left( \frac{\lambda_2 \rho_1 + \sqrt{1 - \rho^2} \lambda_3}{\sqrt{1 - \rho^2}} \right) \),
\[
\bar{k}_1 := \kappa - \frac{\gamma}{1 - \gamma} \left( \frac{\rho_1 \lambda_2 \sigma}{\sqrt{1 - \rho^2}} \right), \bar{k}_2 := \sigma^2 \text{ and } \bar{k}_3 := \sqrt{\bar{k}_1^2 - \bar{k}_0 \bar{k}_2}.
\]
The wealth-equivalent loss in the complete market case without trading derivatives can be easily determined.

\section{A.12 Proof of Merton strategy}

\textit{Proof.} The main mathematical result of Grasselli (2017) is deriving a formula that gives a transform for the CIR process (i.e., the stochastic volatility \( v \) in our model), such that
\[
\phi(t, v_0; \alpha, \Lambda, \mu, v) = \mathbb{E} \left[ (v_T)^{-\alpha} e^{-\Lambda v_T - \mu \int_0^t v_s ds - \frac{\sigma^2}{2} \int_0^t v_s ds} \right]. \tag{A.31}
\]

On the other hand, recall the Feynman-Kac formula and consider the PDE:
\[
h_t + \bar{\mu} h_v + \frac{1}{2} \bar{\sigma}^2 h_{vv} - V h + f = 0,
\]
where \( v \in \mathbb{R}, t \in [0, T] \), and subject to the terminal condition \( h(v, T) = \phi(v) \). Additionally, \( \bar{\mu}, \bar{\sigma}, V, \phi, \) and \( f \) are known functions of \( v \) and \( t \). Then the Feynman-Kac formula gives the solution of \( h(v, t) \) in a conditional expectation form, such that
\[
h(v, t) = \mathbb{E} \left[ \int_t^T e^{-\int_t^r V(v_s, r)dr} f(v_r, r)dr + e^{-\int_t^T V(v_s, r)dr} \phi(v_T) \right]_{v_r = v},
\]
where $dv_t = \mu(v_t, s)ds + \sigma(v_t, s)dZ_t$, with initial condition $v_t = v$. By comparing these equations, the underlying functions can be matched accordingly, this leads to:

$$
\tilde{\mu}(v, t) = \kappa(\theta - v),
\tilde{\sigma}(v, t) = \sigma^2 v,
V(v, t) = -\left(r\gamma + \pi \gamma \bar{\lambda} (av + b) + \frac{1}{2} \pi^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 \gamma (\gamma - 1)\right),
$$

$$
= -\left(r\gamma + b\pi \gamma \bar{\lambda} + ab\pi^2 \gamma (\gamma - 1)\right)
+ \left(a \pi \gamma \bar{\lambda} + \frac{1}{2} \pi^2 a^2 \gamma (\gamma - 1)\right) v + \left(\frac{1}{2} \pi^2 b^2 \gamma (\gamma - 1)\right) \frac{1}{v},
$$

$$
f(v, t) = 0,
\phi(v_T) = h(v, T) = 1.
$$

Therefore, the solution of $h(v, t)$ can be obtained by the Feynman-Kac formula, such that

$$
h(v, t) = \mathbb{E}\left[e^{-\int_t^T \nu(v_s, \tau)d\tau} \phi(v_T)\right]_{v_t = v} \tag{A.32}
$$

$$
= \mathbb{E}\left[\exp\left(r\gamma + b\pi \gamma \bar{\lambda} + ab\pi^2 \gamma (\gamma - 1)\right)(T-t)
+ \left(a \pi \gamma \bar{\lambda} + \frac{1}{2} \pi^2 a^2 \gamma (\gamma - 1)\right) \int_t^T v_s d\tau + \left(\frac{1}{2} \pi^2 b^2 \gamma (\gamma - 1)\right) \int_t^T \frac{1}{v_s} d\tau\right].
$$

Now, we can match to Equation (A.31), leading to:

$$
\alpha = 0, \Lambda = 0, \mu = -\left(a \pi \gamma \bar{\lambda} + \frac{1}{2} \pi^2 a^2 \gamma (\gamma - 1)\right), \nu = -\left(\frac{1}{2} \pi^2 b^2 \gamma (\gamma - 1)\right).
$$

Then, we can use the formula from Grasselli (2017) (Theorem A.1.) to obtain the conditions that parameters $\alpha, \Lambda, \mu, \nu$ should satisfy to make sure Equation (A.32) is well-defined, and the fundamental solution of this expectation transform for the CIR process $dv_t = \kappa(\theta - v_t) + \sigma \sqrt{v_t}dZ_t$. Let $v_t$, $t \geq 0$ denote the solution of the CIR process with initial condition $v$ with $\kappa, \theta, \sigma < 0$ and $2\kappa \theta \geq \sigma^2$ (Feller condition). Consider $\alpha, \Lambda, \mu, \nu$, such that

$$
\mu > -\frac{\kappa^2}{2\sigma^2}, \nu \geq -\frac{(2\kappa \theta - \sigma^2)^2}{8\sigma^2},
\alpha = 0 < \frac{1}{2\sigma^2} \left(2\kappa \theta + \sigma^2 + \sqrt{(2\kappa \theta - \sigma^2)^2 + 8\sigma^2 \gamma}\right),
\Lambda = 0 \geq -\frac{\sqrt{\kappa^2 + 2\mu \sigma^2} + \kappa}{\sigma^2}.
$$
The following transform for the CIR process is well defined for all \( t \geq 0 \) and is given by

\[
\begin{align*}
    h(v, t) &= \exp \left\{ (r + b\pi\overline{\lambda} + ab\pi^2\gamma(\gamma - 1))(T - t) \right\} \phi(T - t, v; \Lambda, \mu, \nu) \\
          &= \exp \left\{ (r + b\pi\overline{\lambda} + ab\pi^2\gamma(\gamma - 1))(T - t) \right\} \\
          &\times \left( \frac{\beta(T - t, v)}{2} \right)^{m+1} \nu^{\frac{m}{2}} (K(T - t))^{-(\frac{1}{2} + \frac{m}{2} + \frac{\kappa}{2})} \\
          &\times e^{\frac{1}{\sigma^2}(\kappa^2(T-t)-\sqrt{\Lambda}\nu\coth(\frac{\sqrt{\Lambda}(T-t)}{2})+k\nu)} \frac{\Gamma(\frac{1}{2} + \frac{m}{2} + \frac{\kappa}{2})}{\Gamma(m + 1)} \\
          &\times \frac{\beta(T - t, v)^2}{4(K(T - t))},
\end{align*}
\]

with

\[
\begin{align*}
    m &= \frac{1}{\sigma^2} \sqrt{(2\kappa\theta - \sigma^2)^2 + 8\sigma^2\nu}, \quad A = \kappa^2 + 2\mu\sigma^2, \\
    \beta(T - t, v) &= \frac{2\sqrt{\Lambda}\nu}{\sigma^2 \sinh(\frac{\sqrt{\Lambda}(T-t)}{2})}, \quad K(T - t) = \frac{1}{\sigma^2} \left( \sqrt{\Lambda} \coth(\frac{\sqrt{\Lambda}(T-t)}{2}) + k \right),
\end{align*}
\]

and \( \Gamma, \, _1F_1 \) denote the Gamma and hypergeometric confluent function respectively. \( \square \)
Appendix B

Proofs for Chapter 3

B.1 Proofs

B.1.1 Proof of Proposition 3.3.1

Proof. Solving the infimum problem in (3.21) first, we obtain:

\[
\begin{align*}
\frac{c}{\theta_v} &= \Theta^v (a \sqrt{v} + \frac{b}{v}) x J_x + \sigma \sqrt{v} J_v \\
\frac{c}{\theta_v} &= \Theta^s (a \sqrt{v} + \frac{b}{v^2}) x J_x \\
\end{align*}
\]

\[\Rightarrow \begin{cases}
(e_v^*) &= \Phi_1 \left[ \Theta^v \left( a \sqrt{v} + \frac{b}{v^2} \right) x J_x + \sigma \sqrt{v} J_v \right] \\
(e_s^*) &= \Phi_2 \left[ \Theta^s \left( a \sqrt{v} + \frac{b}{v^2} \right) x J_x \right]
\end{cases}
\]  

(B.1)

Substituting the values of \( (e_v^*) \) and \( (e_s^*) \) from equation (B.1) into the robust HJB equation in equation (3.21), and cancelling and recombining terms leads to

\[
sup_{\theta^v, \theta^s} \left\{ J_i + x \left[ r + \Theta^v \lambda_1 (av + b) + \Theta^s \lambda_2 (av + b) \right] J_x \right. \\
+ \frac{1}{2} \Phi_1 \left[ (\Theta^v)^2 \right] x^2 (J_x)^2 + \frac{1}{2} \Phi_2 \left[ (\Theta^s)^2 \right] x^2 (J_x)^2 \\
- \frac{1}{2} \Phi_1 \left[ (\Theta^v)^2 \right] x^2 (J_x)^2 + \frac{1}{2} \Phi_2 \left[ (\Theta^s)^2 \right] x^2 (J_x)^2 \left\} = 0 \right.
\]

(B.2)

Solving the maximization problem leads to:

\[
\begin{align*}
\left\{ \begin{array}{l}
(\Theta^v)^* = \Phi_1 \left[ \Theta^v \left( a \sqrt{v} + \frac{b}{v^2} \right) x J_x - \lambda_1 (av + b) J_x - \lambda_2 (av + b) J_x \right] \\
(\Theta^s)^* = \frac{-x \lambda_3 (av + b) J_x}{x^2 (a \sqrt{v} + \frac{b}{v}) J_x - \Phi_1 \left[ a \sqrt{v} + \frac{b}{v^2} \right] x^2 (J_x)^2}
\end{array} \right.
\]

(B.3)
where $\Phi_1 = \frac{\phi_1}{\gamma}$, and $\Phi_2 = \frac{\phi_2}{\gamma}$ by following Maenhout (2004).

Moreover, we conjecture that our value function follows

$$J(x, v, t) = \frac{x^\gamma}{\gamma} \bar{h}(t, v) = \frac{x^\gamma}{\gamma} \exp \left\{ \bar{A}(T - t) + \bar{B}(T - t)v \right\},$$  \hspace{1cm} (B.4)

where $\bar{h}(T, v) = 1$ for all $v$, and $\tau(t) = T - t$. We consequently have the following partial derivatives for $J(x, v, t)$:

$$\bar{J}_t = \frac{x^\gamma}{\gamma} \bar{h}, \quad \bar{J}_e = \frac{x^\gamma}{\gamma} \bar{h}_v, \quad \bar{J}_s = x^{-1} \bar{h}, \quad \bar{J}_{vv} = \frac{x^\gamma}{\gamma} \bar{h}_{vv}, \quad \bar{J}_{sv} = x^{-1} \bar{h}_v, \quad \bar{J}_{sv} = (\gamma - 1)x^{-2} \bar{h}.$$

Furthermore, the partial derivatives of the function $\bar{h}(v, t)$ are given by

$$\bar{h}_t = (-A' - B'v)\bar{h}, \quad \bar{h}_v = B\bar{h}, \quad \bar{h}_{vv} = B^2\bar{h}, \quad \frac{\bar{h}_{tv}}{\bar{h}} = \frac{B^2\bar{h}^2}{\bar{h}} = B^2\bar{h},$$

with boundary conditions

$$\bar{h}(T, v) = 1 \forall v \Rightarrow \bar{A}(0) = \bar{A}(\tau(T)) = 0, \quad \bar{B}(0) = \bar{B}(\tau(T)) = 0.$$  \hspace{1cm} (B.5)

Substituting these partial derivatives into the optimal exposures ($\Theta^\gamma)^*$, ($\Theta^x)^*$ in equation (C.34), and into equation (C.30) to eliminate “sup” leads to

$$- \bar{A}' - B'v + ry + \frac{\phi_1 - \gamma}{(\gamma - 1 - \phi_1)} \sigma \bar{A}_1 v \bar{B} - \frac{\gamma \bar{A}_1^2}{(\gamma - 1 - \phi_2)} v - \frac{\gamma \bar{A}_2^2}{(\gamma - 1 - \phi_2)} v$$

$$+ \frac{1}{2} \frac{1}{\gamma (\gamma - 1 - \phi_1)^2} v B^2 (\gamma - 1) (\phi_1 - \gamma) \sigma \bar{A}_1 \bar{B} + \frac{1}{2} \frac{\bar{A}_1^2 (\gamma - 1)}{\gamma (\gamma - 1 - \phi_1)^2} v \bar{B} + \frac{1}{2} (\gamma - 1 - \phi_1)^2 v + \frac{1}{2} (\gamma - 1 - \phi_2)^2 v$$

$$+ \kappa \theta \bar{B} - \kappa v \bar{B} + \frac{1}{2} \sigma^2 v \bar{B}^2 + \frac{\phi_1 (\phi_1 - \gamma)^2}{(\gamma - 1 - \phi_1)^2} v \bar{B}^2 - \frac{\gamma \bar{A}_1 \sigma}{(\gamma - 1 - \phi_1)} v \bar{B}$$

$$- \frac{1}{2} \frac{\phi_1 (\phi_1 - \gamma)^2}{\gamma (\gamma - 1 - \phi_1)^2} v \bar{B}^2 + \frac{\phi_1 (\phi_1 - \gamma) \sigma \bar{A}_1}{(\gamma - 1 - \phi_1)^2} v \bar{B} - \frac{1}{2} \frac{\gamma \bar{A}_1^2}{(\gamma - 1 - \phi_1)^2} v - \frac{1}{2} \sigma^2 \frac{\bar{A}_1}{\gamma} \bar{B}^2$$

$$\bar{A}'(\tau) = ry + \kappa \theta \bar{B}(\tau) \quad \bar{B}'(\tau) = \frac{1}{2} \bar{B}^2(\tau) - \bar{B}(\tau) + \frac{1}{2} \bar{B}_0.$$  \hspace{1cm} (B.7)
where \( \bar{k}_0 = \frac{\gamma \lambda_1}{(\gamma - 1 - \phi_1)} - \frac{\gamma \lambda_2}{(\gamma - 1 - \phi_2)} \), \( \bar{k}_1 = \frac{(\phi_1 - \gamma) \bar{\lambda}_1}{(\gamma - 1 - \phi_1)} + \kappa \), \( \bar{k}_2 = \frac{(\phi_1 - \gamma) \bar{\lambda}_2}{(\gamma - 1 - \phi_1)} \), and \( \bar{k}_3 = \sqrt{\bar{k}_1^2 - \bar{k}_0 \bar{k}_2} \). Then, we can explicitly find the functions \( \bar{A}(\tau) \) and \( \bar{B}(\tau) \). The optimal exposures can be obtained:

\[
(\Theta^\gamma)^* = \frac{v}{av + b} \left( \frac{\phi_1 - \gamma}{\gamma - 1 - \phi_1} \bar{B}(\tau) - \frac{\bar{\lambda}_1}{\gamma - 1 - \phi_1} \right) \\
(\Theta^\delta)^* = \frac{v}{av + b} \left( \frac{-\bar{\lambda}_2}{\gamma - 1 - \phi_2} \right).
\]

Following from \( \begin{bmatrix} \Theta^\gamma \\ \Theta^\delta \end{bmatrix} = \bar{G}^T \begin{bmatrix} \bar{\eta}_1^\gamma \\ \bar{\eta}_1^\delta \end{bmatrix} \) in equation (3.17), where \( \bar{G} = \begin{bmatrix} \rho & \sqrt{1 - \rho^2} \\ \frac{1}{\rho L_1} & -K_1 \end{bmatrix} \), we have

\[
(\bar{G}^T)^{-1} = \frac{1}{\rho L_1 - K_1 \sqrt{1 - \rho^2}} \begin{bmatrix} L_1 & -K_1 \\ -\sqrt{1 - \rho^2} & \rho \end{bmatrix}.
\]

The worst case measure is determined by

\[
(\epsilon^\gamma)^* = \phi_1 \left( \frac{-\sigma}{\gamma (\gamma - 1 - \phi_1)} \bar{B}(\tau) - \frac{\bar{\lambda}_1}{\gamma - 1 - \phi_1} \right) \sqrt{\nu} \\
(\epsilon^\delta)^* = \phi_2 \left( \frac{-\bar{\lambda}_2}{\gamma - 1 - \phi_2} \right) \sqrt{\nu}.
\]

**B.1.2 Proof of Proposition 3.3.2**

**Proof.** For this condition, we show the optimal Radon-Nikodym derivative of \( \mathbb{P}'^\gamma \) with respect to \( \mathbb{P} \) in the complete market, such that

\[
\xi_t^\gamma = \frac{d\mathbb{P}'^\gamma}{d\mathbb{P}} | F_t = \exp \left\{ - \int_0^t \left( \frac{(\epsilon^\gamma_t)^2 + (\epsilon^\delta_t)^2}{2} d\tau + \epsilon^\gamma_t dZ_{1\tau} + \epsilon^\delta_t dZ_{2\tau} \right) \right\}
\]

is a \( \mathbb{P} \)-martingale to ensure a well-defined \( \mathbb{P}'^\gamma \). We thus consider sufficient conditions based on Novikov’s equation as follows:

\[
\mathbb{E}^\mathbb{P} \left[ \exp \left( \int_0^T \frac{(\epsilon^\gamma_t)^2 + (\epsilon^\delta_t)^2}{2} dt \right) \right] < \infty,
\]

where the optimal perturbations are given in equation (3.28):

\[
(\epsilon^\gamma)^* = \phi_1 \left( \frac{-\sigma}{\gamma (\gamma - 1 - \phi_1)} B(T - t) - \frac{\bar{\lambda}_1}{\gamma - 1 - \phi_1} \right) \sqrt{\nu} \\
(\epsilon^\delta)^* = \frac{-\phi_2 \bar{\lambda}_2}{\gamma - 1 - \phi_2} \sqrt{\nu}.
\]
We then consider the process $\xi_t$, defined as

$$\xi_t = (e^{t^*})^2 + (e^{s^*})^2$$

$$\xi_t = \phi_1^2\left(\frac{-\sigma}{\gamma(y - 1 - \phi_1)} \tilde{B}(T - t) - \frac{\lambda_1}{(y - 1 - \phi_1)}\right)^2 v_t + \phi_2^2\left(\frac{-\lambda_2}{(y - 1 - \phi_2)}\right)^2 v_t$$

$$\xi_t = \left[\phi_1^2\left(\frac{\sigma^2}{\gamma^2(y - 1 - \phi_1)^2} \tilde{B}^2(T - t) + \frac{2\sigma \lambda_1}{\gamma(y - 1 - \phi_1)^2} \tilde{B}(T - t) + \left(\frac{\phi_1^2 \lambda_1^2}{(y - 1 - \phi_1)^2} + \frac{\phi_2^2 \lambda_2^2}{(y - 1 - \phi_2)^2}\right)\right] v_t.\right.$$  

(B.13)

Since this process is a product of a time-dependent term and the volatility driver $v_t$, we then rewrite it as

$$\xi_t = \tilde{K}(T - t)v_t. \quad \text{(B.14)}$$

Hence, Novikov’s condition for the Radon-Nikodym derivative becomes

$$\mathbb{E}^p\left[\exp\left\{\int_0^T \frac{\xi_t^2}{2} dt\right\}\right] = \mathbb{E}^p\left[\exp\left\{\frac{1}{2} \int_0^T \tilde{K}(T - t)v_t dt\right\}\right]$$

$$< \mathbb{E}^p\left[\exp\left\{\frac{1}{2} \sup_{0 < t < T} \tilde{K}(T - t)v_t dt\right\}\right] = \mathbb{E}^p\left[\exp\left\{\frac{1}{2} \tilde{K} \int_0^T v_t dt\right\}\right].$$

(B.15)

where we denote $\tilde{K} := \sup_{0 < t < T} [\tilde{K}(T - t)]$, so $\tilde{K}$ is independent of $t$, and it can be removed from the integral. Furthermore, by Proposition 5.1 according to Kraft (2005), if condition

$$\tilde{K} \leq \frac{\kappa^2}{\sigma^2} \quad \text{(B.16)}$$

is satisfied, then we have

$$\mathbb{E}^p\left[\exp\left\{\int_0^T \frac{\xi_t^2}{2} dt\right\}\right] < \mathbb{E}^p\left[\exp\left\{\frac{1}{2} \tilde{K} \int_0^T v_t dt\right\}\right] < \infty. \quad \text{(B.17)}$$

To ensure a real-valued solution, we must ensure that the square roots are well-defined in function $\tilde{B}(T - t)$ in equation (3.23). That is,

$$\tilde{k}_3 > 0 \iff \tilde{k}_1^2 - \tilde{k}_0 \tilde{k}_2 > 0$$

$$\iff \left(\frac{(\phi_1 - \gamma \eta_1)}{(y - 1 - \phi_1)} + k\right)^2 + \left(\frac{\gamma \lambda_1^2}{(y - 1 - \phi_2)} + \frac{\gamma \lambda_2^2}{(y - 1 - \phi_1)}\right) > 0. \quad \text{(B.18)}$$

Note that when $0 < \gamma < 1$, the above condition is satisfied automatically. In other words, the condition $\tilde{k}_1^2 - \tilde{k}_0 \tilde{k}_2 > 0$ is required to ensure real-valued square roots if we have $\gamma < 0$.

We must further ensure the finiteness of function $\tilde{f}(x, v, t)$, so that the exponential function $\tilde{h}(v, t)$ does not go to infinity. Equivalently, the function $\tilde{B}(T - t)$ in equation (3.23) must remain finite to lead to a well-defined function $\tilde{A}(T - t)$. On the one hand, if $\gamma < 0$, then the argument of the exponential in $\tilde{f}$ is negative because $\tilde{k}_0 < 0$ yields a negative $\tilde{B}(T - t)$ and the
logarithm representation of $\tilde{A}(T-t)$. Thus, the finiteness condition always holds for $\gamma < 0$. On the other hand, when $0 < \gamma < 1$, then function $B(T-t)$ is finite if

$$2\tilde{k}_3 + (\tilde{k}_1 + \tilde{k}_3) (e^{k_3} - 1) \neq 0.$$  \hspace{1cm} (B.19)

It then follows that

$$- (\tilde{k}_1 - \tilde{k}_3) \left(1 - \frac{(\tilde{k}_1 + \tilde{k}_3) e^{k_3}}{(\tilde{k}_1 - \tilde{k}_3)}\right) \neq 0 \iff \frac{\tilde{k}_1 + \tilde{k}_3}{\tilde{k}_1 - \tilde{k}_3} > 1$$

\hspace{1cm} (B.20)

$$\iff \tilde{k}_1 > \tilde{k}_3 = \sqrt{\tilde{k}_1^2 - \tilde{k}_2} \iff \tilde{k}_0 \tilde{k}_2 > 0$$

That is,

$$\left(- \frac{\gamma \lambda_1^2}{(\gamma - 1 - \phi_1)} - \frac{\gamma \lambda_2^2}{(\gamma - 1 - \phi_2)}\right) \left(- \frac{(\phi_1 - \gamma)\sigma^2}{\gamma(\gamma - 1 - \phi_1)}\right) > 0.$$  \hspace{1cm} (B.21)

$\Box$

### B.1.3 Proof of Proposition 3.3.3

**Proof.** We follow along the lines of Pu and Zhang (2021)’s proposition 3.5 and their application to the Heston (1/2) model in theorem 4.2.

Let $\tilde{J} \in C^{2,1}((0, \infty) \times \mathbb{R} \times [0, T])$ be a solution to the HJBI equation

$$\sup_{\Theta^\gamma, \Theta^\sigma} \inf_{\epsilon^x, \epsilon^v} \left( \tilde{J}_t + x \left[ r + \Theta^\gamma \tilde{a}(av + b) - \Theta^\gamma \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) \epsilon^x_\gamma + \Theta^\sigma \tilde{a}(av + b) - \Theta^\sigma \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) \epsilon^v_\sigma \right] \tilde{J}_t + \frac{1}{2} \epsilon^x_\gamma \left[ (\Theta^\gamma)^2 + (\Theta^\sigma)^2 \right] \tilde{J}_{xx} + \left[ \kappa (\theta - v) - \sigma \sqrt{v} \epsilon^v_\sigma \right] \tilde{J}_v + \frac{1}{2} \epsilon^v_\sigma \left[ \sigma^2 + \frac{(\epsilon^v)^2_1}{2\Theta^\sigma} + \frac{(\epsilon^v)^2_2}{2\Theta^\sigma} \right] = 0,$$

$$\tilde{J}(x, v, T) = \frac{\chi^\gamma}{\gamma}.$$

The monotonicity condition, i.e. condition (iii) in Pu and Zhang (2021)’s proposition 3.5, holds for power utility, $u(X_T) = (x^{2\gamma})^\gamma$, where $\gamma < 1$ and $\neq 0$.

Next we validate conditions (iv) and (v) in Pu and Zhang (2021)’s proposition 3.5:

(iv) The local martingale

$$\int_0^T \frac{\partial \tilde{J}}{\partial u}(u_t, t)^T \Lambda(u_t, t) d\tilde{Z}_t,$$

is a true martingale for any $(\Theta^\gamma, \Theta^\sigma) \in \mathcal{U}_t$ and $\epsilon \in \epsilon_t$, where $u_t = (X_t, v_t)^T$ stands for the state variables, $\tilde{Z}_t = (\tilde{Z}_{1t}, \tilde{Z}_{2t})^T$, and

$$\Lambda = \begin{pmatrix} \Theta^\gamma x \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) & \Theta^\sigma x \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) \\ \sigma \sqrt{v} & 0 \end{pmatrix};$$

(v) For any $(\Theta^\gamma, \Theta^\sigma) \in \mathcal{U}_t$ and $\epsilon \in \epsilon_t$, we have $\mathbb{E} \left[ \int_0^T |\tilde{J}(u_t, t) - I| dt \right] < \infty$. 

The proof is divided into three steps, in step 1 we show the optimal control is admissible, then we validate condition (iv) in step 2, and finally we validate condition (v) in step 3.

**Step 1:** Here we check that the optimal wealth exposure \( \Theta^* = (\Theta^t)^*, (\Theta^x)^* \) in equation (3.25) is admissible. The proof is similar to that of Proposition C.6 in Kraft et al. (2013).

Step (1i): Change of measure. From the wealth process in equation (3.16) and \( v_t \) in equation (3.14), \( X_0^{\gamma^t} \) can be computed:

\[
X_0^{\gamma^t} = x^{\gamma^t} \exp \left\{ q \int_0^t \left[ r + \Theta^t \bar{\lambda}_1 (av_t + b) - \Theta^t \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) e^t + \Theta^x \bar{\lambda}_2 (av_t + b) - \Theta^x \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) e^x \right. \\
- \frac{1}{2} \left( \Theta^t \right)^2 + \left( \Theta^x \right)^2 \right\} \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right)^2 \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \exp \left[ \int_0^t \Theta^t \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) dZ_t + \int_0^t \Theta^x \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) d\tilde{Z}_t \right].
\]

It follows that

\[
X_0^{\gamma^t} = x^{\gamma^t} \exp \left\{ q \int_0^t \left[ r + \left( \Theta^t \bar{\lambda}_1 + \Theta^x \bar{\lambda}_2 - \Theta^t \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) e^t \right. \\
- \Theta^x \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) e^x \right] (av_t + b) \right\} \right. \right\} \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right)^2 \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \exp \left[ \int_0^t \Theta^t \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) dZ_t + \int_0^t \Theta^x \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) d\tilde{Z}_t \right].
\]

hence for a sufficiently large \( K \),

\[
X_0^{\gamma^t} \leq K \exp \left\{ - q \int_0^t \left[ \Psi_s ds \right] Z_t \right\}
\]

where

\[
Z_t := Z_1 Z_2 = \exp \left\{ - \frac{1}{2} q^2 \gamma^2 \int_0^t \left( \Theta^t \right)^2 \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right)^2 ds + q \int_0^t \Theta^t \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) dZ_t \right\}
\]

\[
\times \exp \left\{ - \frac{1}{2} q^2 \gamma^2 \int_0^t \left( \Theta^x \right)^2 \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right)^2 ds + q \int_0^t \Theta^x \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) d\tilde{Z}_t \right\}
\]

\[
\Psi_s := \frac{1 - q \gamma}{2} \left( \Theta^t \right)^2 + \left( \Theta^x \right)^2 \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right)^2 \left( \Theta^t \bar{\lambda}_1 + \Theta^x \bar{\lambda}_2 - \Theta^t \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) e^t \right. \\
- \Theta^x \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) e^x \right] (av_t + b).
\]

Recall that

\[
|\Theta^t| = \left| \frac{1}{\gamma - 1 - \phi_1} \frac{1}{\gamma - 1 - \phi_1} B(T - t) - \frac{\bar{\lambda}_1}{\gamma - 1 - \phi_1} B(T) - \frac{\bar{\lambda}_1}{\gamma - 1 - \phi_1} : K^t,
\]

\[
|\Theta^x| = \left| - \frac{\bar{\lambda}_2}{\gamma - 1 - \phi_2} \right| \leq \left| \frac{\bar{\lambda}_2}{\gamma - 1 - \phi_2} \right| : K^x.
\]
\[
\frac{e^y}{\sqrt{v}} := |e_1| = \phi_1 \left| \frac{-\sigma \bar{b} (T - t)}{\gamma (\gamma - 1 - \phi_1)} - \bar{\lambda}_1 \right| \leq \phi_2 \left| \frac{-\sigma \bar{b} (T)}{\gamma (\gamma - 1 - \phi_1)} - \bar{\lambda}_1 \right| := K^e,
\]

\[
\frac{e^S}{\sqrt{v}} := |e_2| = \phi_2 \left| \frac{-\bar{\lambda}_2}{\gamma - 1 - \phi_2} \right| \leq \phi_3 \left| \frac{-\bar{\lambda}_2}{\gamma - 1 - \phi_2} \right| := K^s,
\]

where \( \phi_1, \phi_2 > 0, \bar{\lambda}_1 < 0, \bar{\lambda}_2 \) can be positive or negative, and

\[
\begin{cases}
\gamma < 0, \frac{\partial \bar{b}}{\partial t} < 0 \text{ with } \bar{b}(0) = 0 \implies \bar{b}(T - t) < 0, \\
0 < \gamma < 1 \left\{ \begin{array}{l}
\phi_1 > \gamma, \frac{\partial \bar{b}}{\partial t} > 0 \text{ with } \bar{b}(0) = 0 \implies \bar{b}(T - t) > 0, \\
\phi_1 < \gamma, \frac{\partial \bar{b}}{\partial t} < 0 \text{ with } \bar{b}(0) = 0 \implies \bar{b}(T - t) < 0.
\end{array} \right.
\]

Then, with \(|\Theta_1| \leq K^e, |\Theta_2| \leq K^s\) for all \( t \in [0, T] \), we have that \( Z_t \) satisfies Novikov’s condition

\[
E \left[ \exp \left( \frac{1}{2} q^2 \gamma^2 \int_0^T (\Theta^e)^2 + (\Theta^s)^2 \left( a \sqrt{v_i} + \frac{b}{\sqrt{v_i}} \right)^2 dt \right) \right] = E \left[ \exp \left( \frac{1}{2} q^2 \gamma^2 \int_0^T (\Theta_1)^2 + (\Theta_2)^2 v_i dt \right) \right] < \infty
\]

for some \( q > 2 \) and

\[
\frac{1}{2} q^2 \gamma^2 \left( (K^e)^2 + (K^s)^2 \right) \leq \frac{\kappa^2}{2 \sigma^2}.
\]

This follows by Lemma C.2 in Kraft et al. (2013), and it is recorded in assumption (H4). Further simplifying the above conditions, we get

\[
4 \gamma^2 \sigma^2 \left( (K^e)^2 + (K^s)^2 \right) < \kappa^2,
\]

leading to \( Z_t \) being a martingale.

Then, under the equivalent measure \( \bar{\mathbb{P}} \) given by \( \frac{d\bar{\mathbb{P}}}{d\mathbb{P}} = Z_t \), the process \( Z_{u|} = \bar{Z}_{u|} - q \gamma \int_0^u \Theta^e \left( a \sqrt{v_i} + \frac{b}{\sqrt{v_i}} \right) ds \) is a standard Wiener process. With \( \bar{\mathbb{E}} \) denoting the expectation operator under measure \( \bar{\mathbb{P}} \), we have

\[
\bar{\mathbb{E}} \left[ X_t \gamma \right] \leq K \bar{\mathbb{E}} \left[ \exp \left( - q \gamma \int_0^t \Psi^e ds \right) \right].
\]

Step (ii): Dynamic of \( v_i \), under \( \bar{\mathbb{P}} \) follows

\[
dv_i = \left[ \kappa (\theta - v_i) - \sigma \sqrt{v_i} e^i \right] dt + \sigma \sqrt{v_i} \left( q \gamma \Theta^e \left( a \sqrt{v_i} + \frac{b}{\sqrt{v_i}} \right) ds + dZ_{u|} \right) = \left[ \kappa \theta + \kappa q \gamma \sigma \left( \frac{v_i}{a v_i + b} \Theta^e \right) dt + \left( \kappa - q \gamma \sigma \left( \frac{v_i}{a v_i + b} \Theta^e \right) v_i \right) dt + \sigma \sqrt{v_i} dZ_{u|} \right] \]

\[
= \left[ \kappa \theta - \left( \kappa - q \gamma \sigma \Theta^e \right) v_i \right] dt + \sigma \sqrt{v_i} dZ_{u|},
\]
Laplace transform of the integrated squared-root process, with assumption (H3) ensures that from equation (B.39). Now, by equation (B.40), where the second inequality follows from equation (B.41), and the third inequality follows equation (3.34), (3.35), (3.36), and (3.37). From equation (B.36), we have where defined in equation (B.27) is given as Step (1iii): We next prove the inequality where and is a CIR process for all for all , and 2 sufficiently small. Thus, according to the comparison result of Yamada and Watanabe, Theorem 43.1 in Rogers and Williams (2000), we have and is a CIR process where is given in (3.38).

Step (1iii): We next prove the inequality for all Recall the process defined in equation (B.27) is given as where and can be negative or positive according to empirical finding. The inequality follows equation (3.34), (3.35), (3.36), and (3.37). From equation (B.36), we have

\[
\mathbb{E}\left[X_t^\gamma\right] \leq K \mathbb{E}\left[\exp\left\{-q\gamma \int_0^t \Psi_s ds\right\}\right] \leq K \mathbb{E}\left[\exp\left\{-q\gamma \int_0^t \bar{\Psi}_v ds\right\}\right] \leq K \mathbb{E}\left[\exp\left\{-q\gamma \int_0^{t'} \bar{\nu}_v ds\right\}\right],
\]

(B.42)

where the second inequality follows from equation (B.41), and the third inequality follows from equation (B.39). Now, by equation (B.40), is a CIR process under the measure , and assumption (H3) ensures that

\[-q\gamma \bar{\nu} \leq \frac{\kappa^2}{2\sigma^2},\]

(B.43)

with 2 sufficiently small. Hence, by the Pitman-Yor lemma (Pitman and Yor (1982)) for the Laplace transform of the integrated squared-root process,

\[
\sup_{t \in [0,T]} K \mathbb{E}\left[\exp\left\{-q\gamma \bar{\Psi} \int_0^{t'} \bar{\nu}_v ds\right\}\right] \leq K < \infty.
\]

(B.44)
That is,
\[ \mathbb{E}\left[ X^2_t \right] \leq K. \] (B.45)

**Step 2:** Verification of martingale condition. To verify the local martingale \( \int_0^T \frac{\partial \tilde{I}}{\partial u}(u_t, t)^T \Lambda(u_t, t) d\tilde{Z}_t \) is a true martingale, it is sufficient to show the integration condition, which is
\[ \mathbb{E} \left[ \int_0^T \left\| \frac{\partial \tilde{I}}{\partial u}(u_t, t)^T \Lambda(u_t, t) \right\|^2 ds \right] < \infty. \] (B.46)

Let \((\Theta^\gamma, \Theta^\gamma) \in \mathcal{A}_t\) be admissible wealth exposures. Consider \( \mathbb{E} \left[ \int_0^T \left( F_{1t}^2 + F_{2t}^2 \right) dt \right] \), where
\[
F_{1t} := \tilde{I}_t(X_t, v_t, t) \Theta^\gamma X_t \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) + \tilde{I}_t(X_t, v_t, t) \sigma \sqrt{v_t} \\
= X_t^\gamma \tilde{h} \Theta^\gamma \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) + X_t^\gamma \tilde{h} \sigma \sqrt{v_t} = X_t^\gamma \tilde{h} \left( \Theta_1 + \frac{B}{\gamma} \right) \sqrt{v_t},
\]
\[
F_{2t} := \tilde{I}_t(X_t, v_t, t) \Theta^\gamma X_t \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \\
= X_t^\gamma \tilde{h} \Theta^\gamma \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) = X_t^\gamma \tilde{h} \Theta_2 \sqrt{v_t},
\]

Then it follows that
\[
F_{1t}^2 + F_{2t}^2 = \left( X_t^\gamma \tilde{h} \right)^2 \left[ \left( \Theta_1 \right)^2 + \left( \Theta_2 \right)^2 \right] + \frac{B^2}{\gamma^2} \sigma^2 + 2 \Theta_1 \frac{B}{\gamma} \sigma \sqrt{v_t},
\] (B.48)

where \( \Theta_1, \Theta_2, \) and \( \tilde{B}(\cdot) \) are uniformly bounded, and \( \tilde{h} = \exp \left\{ \tilde{A} + \tilde{B} v_t \right\} \). In addition, if \( \gamma \leq 0 \) or \( 0 < \phi_1 \leq \gamma < 1 \) then, by equation (B.32) we obtain:
\[ \| \tilde{h} \| \leq \exp \left\{ \| A \| \right\} \] (B.49)

where \( \| A \| := \max_{s,t \in [0,T]} |\tilde{A}(s,t)| < \infty \). Hence, in order to show \( \mathbb{E} \left[ \int_0^T \left( F_{1t}^2 + F_{2t}^2 \right) dt \right] < \infty \), it suffices to show that
\[ \mathbb{E} \left[ \int_0^T G_t^2 dt \right] = \int_0^T \mathbb{E} \left[ G_t^2 \right] dt < \infty, \] (B.50)

where \( G_t := X_t^\gamma \tilde{h} \sqrt{v_t} \). Again, by Pitman and Yor (1982), we have \( \sup_{t \in [0,T]} \mathbb{E} \left[ v_t^p \right] < \infty \) for any \( p > 1 \). By Hölder’s inequality, it is sufficient to show \( \int_0^T \mathbb{E} \left[ |H_t|^{pq} \right] dt < \infty \) for some \( q > 2 \), where
\[ H_t := X_t^\gamma \tilde{h}. \] (B.51)
The upper bound for $|\tilde{h}|$ implies that $|H_t| \leq X_T^q e^{\|A\|}$, $t \in [0, T]$.

Recall the admissible condition in equation (B.45), $\mathbb{E}[X_{t}^{\text{up}}] \leq K$, if $q - 2$ is sufficient small. It then leads to

$$\mathbb{E}[|H_t|^q] \leq K^\frac{q}{2} e^{\|A\|}$, for all $t \in [0, T]$, some $K > 0$. (B.52)

Consequently,

$$\int_0^T \mathbb{E}[|H_t|^\frac{q}{2}] \, dt \leq \int_0^T K^\frac{q}{2} e^{\|A\|} \, dt = K^\frac{q}{2} e^{\|A\|} T < \infty. \quad (B.53)$$

The proof of the integration condition is completed, so is the martingale condition.

**Step 3:** Verification of the condition $\mathbb{E}[\int_0^T |\tilde{J}(u, t) - I_t| \, dt] < \infty$ for $(\Theta; e) \in \mathcal{A}$. We firstly show that

$$\mathbb{E}[\int_0^T |\tilde{J}(X_t, v, t)| \, dt] < \infty. \quad (B.54)$$

In fact,

$$\mathbb{E}[\int_0^T |\tilde{J}(X_t, v, t)| \, dt] = \mathbb{E}[\int_0^T \frac{X_t^q}{\gamma} \tilde{h}(v, t) \, dt] = \int_0^T \mathbb{E}[\frac{X_t^q}{\gamma} \tilde{h}(v, t)] \, dt \quad (B.55)$$

By the expressions for the wealth in Equation (3.16) and volatility process in Equation (3.14), we can compute $X_T$, such that

$$X_T = \exp\left\{ \int_0^T \left[ r + \Theta^v \tilde{\lambda}_1(a v_t + b) - \Theta^\gamma \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) e^\gamma_t + \Theta^A \tilde{\lambda}_2(a v_t + b) - \Theta^A \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) e^A_t \right. \right. \right.$$

$$\left. \left. \left. - \frac{1}{2} \left( (\Theta^v)^2 + (\Theta^\gamma)^2 \right) \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right)^2 \right] \, dt \right. \right. \right. \right. \right.$$  

$$\left. \left. + \left[ \int_0^T \Theta^v \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) dZ_{1_t} + \int_0^T \Theta^\gamma \left( a \sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) dZ_{2_t} \right] \right\} \geq 0. \quad (B.56)$$

From step 2, we know that $\int_0^T \mathbb{E}[X_t^q \tilde{h}] \, dt < \infty$, thus,

$$\mathbb{E}[\int_0^T |\tilde{J}(X_t, v, t)| \, dt] = \int_0^T \mathbb{E}[|\frac{X_t^q}{\gamma} \tilde{h}(v, t)|] \, dt < \infty. \quad (B.57)$$

Next, we prove $\mathbb{E}[\int_0^T |I_t| \, dt] < \infty$, where

$$I_t = \mathbb{E}[\frac{X_t^q}{\gamma} + \int_0^T \left( \frac{(e^\gamma_t)^2}{2\Phi_1(X, v, t)} + \frac{(e^A_t)^2}{2\Phi_2(X, v, t)} \right) \, dt]. \quad (B.58)$$

for all $(\Theta; e) \in \mathcal{A}$, $t \in [0, T]$. It is easy to observe that it is sufficient to verify that

$$\mathbb{E}[\int_0^T \left( \frac{(e^\gamma_t)^2}{2\Phi_1(X, v, t)} + \frac{(e^A_t)^2}{2\Phi_2(X, v, t)} \right) \, dt] < \infty. \quad (B.59)$$
with \( \Phi_t = \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \). We then obtain

\[
\mathbb{E}\left[ \int_t^T \left( \frac{(e^x)^2}{2\Phi_1(X,v,t)} + \frac{(e^x)^2}{2\Phi_2(X,v,t)} \right) dt \right] = \int_t^T \mathbb{E}\left[ \left( \frac{(e^x)^2}{2\Phi_1(X,v,t)} + \frac{(e^x)^2}{2\Phi_2(X,v,t)} \right) \right] dt \leq \int_t^T \frac{1}{2} \mathbb{E}\left[ \left( X^T \dot{h} \right) \left( \phi_1 g_1^2 + \phi_2 g_2^2 \right) \right] dt \leq K \int_t^T \mathbb{E}[v] dt = K\theta(T-t) < \infty,
\]

(B.60)

where \( \phi_1, \phi_2 > 0; g_1 = \frac{c_1}{\phi_1} \) and \( g_2 = \frac{c_2}{\phi_2} \) are bounded because \( e_1 \) and \( e_2 \) are bounded, i.e., (3.36), (3.37). The proof of \( \mathbb{E}[\int_0^T |\tilde{J}(u,t) - I|dt] < \infty \) is completed by verifying \( \mathbb{E}[\int_0^T |\tilde{J}(u,t)|dt] < \infty \) and \( \mathbb{E}[\int_0^T |I|dt] < \infty \), respectively.

\[ \Box \]

### B.2 Extra Proofs

#### B.2.1 Proof of the form of the option price process

**Proof.** We have:

\[
\begin{align*}
\left[ \frac{dS_t}{dv_t} \right] &= \left[ S_t \left( r + (av_t + b)(\bar{\lambda}_1 \rho + \bar{\lambda}_2 \sqrt{1 - \rho^2}) \right) \right] dt \\
&= \left[ \frac{S_t \rho(a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) + \sqrt{1 - \rho^2}(a \sqrt{v_t} + \frac{b}{\sqrt{v_t}})}{\sigma \sqrt{v_t}} \right] \left[ dS_t \sigma \sqrt{v_t} \right] \left[ dZ_{v_t} \right],
\end{align*}
\]

where \( F_t = (S_t, v_t)^T \). Then the option price can be represented as \( O_t = m(F_t, t) \). On the one hand, according to Björk (2009), the option price should satisfy the following PDE,

\[
rm = m_t + rS m_S + \left( \kappa(\theta - v) - \sigma \sqrt{v} \bar{\lambda}_1 \sqrt{\bar{\lambda}} \right) m_v + \frac{1}{2} \text{trace} \left( \Sigma^{T \Sigma \Sigma^T} f f \right),
\]

where \( m_{ff} \) is the matrix of mixed partial derivatives of the function \( m \). On the other hand, by applying Ito’s Lemma to the function \( m \), we can get

\[
dm = \left[ m_t + S m_S \left( r + (av + b)(\bar{\lambda}_1 \rho + \sqrt{1 - \rho^2} \bar{\lambda}_2) \right) + m_v \kappa(\theta - v) + \frac{1}{2} \text{trace} \left( \Sigma^{T \Sigma \Sigma^T} f f \right) \right] dt + m_T \text{d}Z_t,
\]

where \( dZ_t = \left[ dZ_{v_t}, dZ_{s_t} \right] \) and \( m_t = \left[ m_s \right. \left. m_v \right] \) denotes the partial derivatives of the function \( m \) with respect to \( S \) and \( v \) respectively. Now, we can combine the last two PDEs to obtain

\[
dm = \left[ rm + \sigma \bar{\lambda}_1 v m_v + S m_S (av + b)(\bar{\lambda}_1 \rho + \sqrt{1 - \rho^2} \bar{\lambda}_2) \right] dt \\
+ m_S \left( \rho(a \sqrt{v} + \frac{b}{\sqrt{v}}) dZ_{v_t} + \sqrt{1 - \rho^2}(a \sqrt{v} + \frac{b}{\sqrt{v}}) dZ_{s_t} \right) + m_v \left( \sigma \sqrt{v} dZ_{v_t} \right).
\]
Rearranging and regrouping the equation by the two risk factors, i.e., $dZ_{1t}$ and $dZ_{2t}$, we obtain
\[
dm = rm dt + \left[ \rho m_S \frac{a \sqrt{v}}{\sqrt{\nu}} + m, \sigma \sqrt{\nu} \right] (\bar{\lambda}_1 \sqrt{\nu} dt + dZ_{1t}) \\
+ \left[ m_S \sqrt{1 - \rho^2} \frac{a \sqrt{v}}{\sqrt{\nu}} \right] (\bar{\lambda}_2 \sqrt{\nu} dt + dZ_{2t}).
\]

Thus, the dynamics of the option price is given as
\[
\frac{dO_t}{O_t} = r dt + \underbrace{\frac{1}{O_t} \left( m_S p S_t + m_v \left( \frac{\sigma}{a \sqrt{\nu} + \frac{b}{\sqrt{\nu}}} \right) \right) \left( a \sqrt{\nu} + \frac{b}{\sqrt{\nu}} \right) \left( \bar{\lambda}_1 \sqrt{\nu} dt + dZ_{1t} \right)}_{K_t} \\
+ \underbrace{\frac{1}{O_t} \left( m_S \sqrt{1 - \rho^2} S_t \right) \left( a \sqrt{\nu} + \frac{b}{\sqrt{\nu}} \right) \left( \bar{\lambda}_2 \sqrt{\nu} dt + dZ_{2t} \right)}_{L_t}.
\]

Note: Here we factor out the term \( \left( a \sqrt{\nu} + \frac{b}{\sqrt{\nu}} \right) \), which is the diffusion coefficient from the stock process, together with the Brownian motions $dZ_{1t}$ and $dZ_{2t}$ that appear in the stock price process. Also, as the diffusion parts of the stock process and stochastic volatility are of different forms, i.e., \( a \sqrt{\nu} + \frac{b}{\sqrt{\nu}} \) vs. \( \sigma \sqrt{\nu} \), we cannot totally separate out $v_t$ from $K_t$. But this follows the same idea as Liu and Pan (2003) and Escobar et al. (2017). \( \Box \)

### B.2.2 Proof of Proposition 3.4.1

**Proof.** Solving the infimum problem of equation (3.41) first, we obtain:

\[
\begin{align*}
\frac{e^v}{\theta_1} &= \Theta^{r} \left( a \sqrt{\nu} + \frac{b}{\sqrt{\nu}} \right) x_{f_1}^{(s)} + \sigma \sqrt{\nu} \bar{x}_{f_1}^{(s)} \\
\frac{e^v}{\theta_2} &= \Theta^{\bar{s}} \left( a \sqrt{\nu} + \frac{b}{\sqrt{\nu}} \right) x_{f_2}^{(s)} 
\end{align*}
\Rightarrow \begin{cases} (e^v)^* = \Phi_1 \Theta^{r} \left( a \sqrt{\nu} + \frac{b}{\sqrt{\nu}} \right) x_{f_1}^{(s)} + \sigma \sqrt{\nu} \bar{x}_{f_1}^{(s)} \\
(e^\bar{s})^* = \Phi_2 \Theta^{\bar{s}} \left( a \sqrt{\nu} + \frac{b}{\sqrt{\nu}} \right) x_{f_2}^{(s)} \end{cases}.
\]

(B.61)
Substituting the values of \((e^r)^*\) and \((e^s)^*\) from equation (B.61) into the robust HJB equation, i.e., equation (3.41), we obtain the following equation that the function \(\bar{J}^{(s)}\) has to satisfy:

\[
\bar{J}_t^{(s)} + \left[ r + \Theta^y \lambda_1 (av + b) - \Theta^x \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) \Phi_1 \left[ \Theta \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x \bar{J}_t^{(s)} + \sigma \sqrt{v} \bar{J}_v^{(s)} \right] + \right. \\
+ \Theta^x \bar{\lambda}_2 (av + b) - \Theta^y \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) \Phi_2 \left[ \Theta \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x \bar{J}_v^{(s)} \right] \\
+ \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \left[ (\Theta^y)^2 + (\Theta^x)^2 \right] \bar{J}_{st}^{(s)} + \left[ \kappa (\theta - v) - \sigma \sqrt{v} \Phi_1 \left[ \Theta \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x \bar{J}_v^{(s)} + \sigma \sqrt{v} \bar{J}_v^{(s)} \right] \right] \bar{J}_v^{(s)} \\
+ \frac{1}{2} \sigma^2 v \bar{J}_{vv}^{(s)} + \sigma x (av + b) \Theta^y \bar{J}_{sv}^{(s)} + \frac{(\Phi_1 \left[ \Theta \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x \bar{J}_v^{(s)} + \sigma \sqrt{v} \bar{J}_v^{(s)} \right] )^2}{2 \Phi_1} + \frac{(\Phi_2 \left[ \Theta \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x \bar{J}_v^{(s)} + \sigma \sqrt{v} \bar{J}_v^{(s)} \right] )^2}{2 \Phi_2} \\
= 0
\]

\(\text{(B.62)}\)

Cancelling and recombining terms, it shows that solving next equation is equivalent to solving the problem in equation (3.41):

\[
\bar{J}_t^{(s)} + \left[ r + \Theta^y \lambda_1 (av + b) + \Theta^y \bar{\lambda}_2 (av + b) \right] \bar{J}_v^{(s)} \\
+ \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \left[ (\Theta^y)^2 + (\Theta^x)^2 \right] \bar{J}_{st}^{(s)} + \left[ \kappa (\theta - v) \bar{J}_v^{(s)} + \frac{1}{2} \sigma^2 v \bar{J}_{vv}^{(s)} + \sigma x (av + b) \Theta^y \bar{J}_{sv}^{(s)} \right. \\
- \frac{1}{2} \frac{\phi_1}{\gamma \bar{J}_v^{(s)}} \left[ (\Theta^y)^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 x^2 (\bar{J}_v^{(s)})^2 + \sigma^2 v (\bar{J}_v^{(s)})^2 + 2 \Theta^y (av + b) \sigma x \bar{J}_v^{(s)} \right] \\
- \frac{1}{2} \frac{\phi_2}{\gamma \bar{J}_v^{(s)}} \left[ (\Theta^x)^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 x^2 (\bar{J}_v^{(s)})^2 \right] = 0.
\]

\(\text{(B.63)}\)

The wealth-equivalent loss is defined as

\[
\bar{J} \left( x(1 - \bar{L}^{(s)}), v, t \right) = \bar{J}^{(s)}(x, v, t),
\]

\(\text{(B.64)}\)

where \(\bar{J}^{(s)}(x, v, t)\) is the value function of an investor that allocates his wealth following a suboptimal rule, and \(\bar{L}^{(s)}(v, t)\) is the proportion of wealth loss incurred by adopting a suboptimal strategy as a function of wealth \(x\) and instantaneous volatility driver \(v\). Moreover, if the form of the value function is assumed to be

\[
\bar{J}^{(s)}(x, v, t) = \frac{\gamma}{\bar{J}^{(s)}(v, t)},
\]

\(\text{(B.65)}\)
with boundary condition \( \tilde{h}^{(s)}(v, T) = 1 \), then substituting the partial derivatives of \( \tilde{f}^{(s)}(x, v, t) \), we obtain the following equation that is enough to solve the problem in (3.41):

\[
\begin{align*}
\tilde{h}_t^{(s)} + \left[ r + \Theta^v \bar{\lambda}_1 (av + b) + \Theta^s \bar{\lambda}_2 (av + b) \right] \gamma \tilde{h}^{(s)} \\
+ \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \left[ (\Theta^v)^2 + (\Theta^s)^2 \right] \gamma (\gamma - 1) \tilde{h}^{(s)} + \kappa (\theta - v) \tilde{h}^{(s)} + \frac{1}{2} \sigma^2 v \tilde{h}^{(s)} + \sigma (av + b) \Theta^v \gamma \tilde{h}^{(s)} \\
- \frac{\bar{\phi}_1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \gamma \tilde{h}^{(s)} + \sigma^2 v \frac{1}{\gamma} \tilde{h}^{(s)} + 2 \Theta^v (av + b) \sigma \tilde{h}^{(s)} \\
- \frac{\bar{\phi}_2}{2} \left( \Theta^s \right)^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \gamma \tilde{h}^{(s)} = 0.
\end{align*}
\]  

(B.66)

Furthermore, if we also assume that the value function of the suboptimal strategy \( \Pi^{(s)} \) admits an exponential-affine form such as

\[
\tilde{f}^{(s)}(x, v, t) = \frac{x^\gamma}{\gamma} \tilde{h}^{(s)}(v, t) = \frac{x^\gamma}{\gamma} \exp \left\{ \tilde{A}^{(s)}(T - t) + \tilde{B}^{(s)}(T - t) v \right\},
\]  

(B.67)

with \( \tau(t) = T - t \), and boundary condition \( \tilde{A}^{(s)}(\tau(T)) = 0 \) and \( \tilde{B}^{(s)}(\tau(T)) = 0 \). Directly, the equation (3.42) can be rewritten by substituting the partial derivatives of the function \( \tilde{h}^{(s)} \) as follows:

\[
\begin{align*}
- (\tilde{A}^{(s)})' - (\tilde{B}^{(s)})' v + ry + \left[ \Theta^v \bar{\lambda}_1 (av + b) + \Theta^s \bar{\lambda}_2 (av + b) \right] \gamma \\
+ \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \left[ (\Theta^v)^2 + (\Theta^s)^2 \right] \gamma (\gamma - 1) \tilde{h}^{(s)} + \frac{1}{2} \sigma^2 v \tilde{h}^{(s)} + \sigma (av + b) \Theta^v \gamma \tilde{B}^{(s)} \\
- \frac{\bar{\phi}_1}{2} \left( \Theta^v \right)^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \gamma + \sigma^2 v \frac{1}{\gamma} \tilde{B}^{(s)} + 2 \Theta^v (av + b) \sigma \tilde{B}^{(s)} \\
- \frac{\bar{\phi}_2}{2} \left( \Theta^s \right)^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \gamma = 0.
\end{align*}
\]  

(B.68)

Then the wealth-equivalent loss function \( \bar{L}^{(s)}(v, t) \) admits the expression

\[
\bar{L}^{(s)}(v, t) = 1 - \left( \frac{\tilde{h}^{(s)}(v, t)}{\tilde{h}(v, t)} \right)^{\frac{1}{\gamma}},
\]  

(B.69)

\[
= 1 - \exp \left\{ \frac{1}{\gamma} \left[ \tilde{A}^{(s)}(\tau) - \tilde{A}(\tau) + \left( \tilde{B}^{(s)}(\tau) - \tilde{B}(\tau) \right) v \right] \right\},
\]

where \( \tilde{A}(\tau) \) and \( \tilde{B}(\tau) \) are time-dependent functions that appear in the value function \( \tilde{f}(x, v, t) \) under the optimal strategy, i.e., equation (3.23), while \( \tilde{A}^{(s)}(\tau) \) and \( \tilde{B}^{(s)}(\tau) \) are functions that characterize the value function \( \tilde{f}^{(s)}(x, v, t) \) under a suboptimal strategy. Further, the worst case measure from equation (B.61) is given by

\[
\begin{align*}
\{ (e^y)^{(s)} & = \bar{\phi}_1 \left( \Theta^v \right)^{(s)} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) + \sigma \sqrt{v} \frac{1}{\gamma} \tilde{B}^{(s)}(\tau) \\
(e^s)^{(s)} & = \bar{\phi}_2 \left( \Theta^s \right)^{(s)} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right),
\end{align*}
\]  

(B.70)
where \((\Theta^s)^{(i)}\) and \((\Theta^y)^{(i)}\) correspond to a suboptimal strategy \(\Pi^{(i)}\). Also, \(\overline{\phi}_1 = \phi_1\), and \(\overline{\phi}_2 = \phi_2\) whenever the suboptimal strategy does not have any assumption about the ambiguity parameters.

\section{B.2.3 Proof of Corollary 3.4.2}

\textbf{Proof.} By substituting the suboptimal strategy \(\Pi^{(i)}\) in equation (3.45) into equation (B.68), it follows

\begin{align}
- (\overline{A}^{(s)})' - (\overline{B}^{(y)})' v + r\gamma + \left[ \frac{-\sigma v \overline{B}(\tau) - \overline{\lambda}_1 v}{(\gamma - 1)} + \frac{-\overline{\lambda}_2 v}{(\gamma - 1)} \right] \gamma \\
+ \frac{1}{2} \left[ \frac{-\sigma v \sqrt{\overline{B}(\tau)} - \overline{\lambda}_1 \sqrt{v}}{(\gamma - 1)} + \frac{-\overline{\lambda}_2 \sqrt{v}}{(\gamma - 1)} \right] \gamma \\
+ \frac{k(\theta - v)\overline{B}^{(s)}}{2} + \frac{1}{2} \sigma^2 v (\overline{B}^{(s)})^2 + \frac{-\sigma v \overline{B}(\tau) - \overline{\lambda}_1 v}{(\gamma - 1)} \gamma \overline{B}^{(s)} \\
- \frac{\overline{\phi}_1}{2} \left[ \frac{-\sigma v \sqrt{\overline{B}(\tau) - \overline{\lambda}_1 \sqrt{v}}}{(\gamma - 1)^2} \gamma + \sigma^2 v \frac{1}{\gamma} (\overline{B}^{(s)})^2 + 2 \frac{-\sigma v \overline{B}(\tau) - \overline{\lambda}_1 v}{(\gamma - 1)} \gamma \right] \\
- \frac{\overline{\phi}_2}{2} \left[ \frac{-\overline{\lambda}_2 \sqrt{v}}{(\gamma - 1)^2} \gamma \right] = 0,
\end{align}

where \(\overline{A}^{(s)}\) and \(\overline{B}^{(s)}\) are functions associated with the value function \(\overline{J}^{(s)}\) under the suboptimal strategy \(\Pi^{(s)}\). Rearrange terms leads to

\begin{align}
- (\overline{A}^{(s)})' + \kappa \theta \overline{B}^{(s)} + r\gamma \\
+ \left[ - (\overline{B}^{(s)})' + \left[ \frac{-\sigma \overline{B}(\tau) - \overline{\lambda}_1}{(\gamma - 1)} \lambda_1 + \frac{-\overline{\lambda}_2}{(\gamma - 1)} \lambda_2 \right] \gamma + \frac{1}{2} \left[ \frac{-\sigma \overline{B}(\tau) - \overline{\lambda}_1}{(\gamma - 1)} \lambda_1 + \frac{-\overline{\lambda}_2}{(\gamma - 1)} \lambda_2 \right] \gamma \\
- \frac{k(\theta - v)\overline{B}^{(s)}}{2} + \frac{1}{2} \sigma^2 v (\overline{B}^{(s)})^2 + \frac{-\sigma \overline{B}(\tau) - \overline{\lambda}_1}{(\gamma - 1)} \gamma \overline{B}^{(s)} \\
- \frac{\overline{\phi}_1}{2} \left[ \frac{-\sigma \overline{B}(\tau) - \overline{\lambda}_1}{(\gamma - 1)^2} \gamma + \sigma^2 v \frac{1}{\gamma} (\overline{B}^{(s)})^2 + 2 \frac{-\sigma \overline{B}(\tau) - \overline{\lambda}_1}{(\gamma - 1)} \gamma \right] \\
- \frac{\overline{\phi}_2}{2} \left[ \frac{-\overline{\lambda}_2 \sqrt{v}}{(\gamma - 1)^2} \gamma \right] = 0.
\end{align}

We obtain an equation that is linear in \(v\), but both “coefficients” satisfy linear differential equations. In order to satisfy the boundary condition on the functions \(\overline{A}^{(s)}\) and \(\overline{B}^{(s)}\), both of the “coefficients” have to be zero, such that:

\begin{align}
(\overline{A}^{(s)})'(\tau) &= r\gamma + \kappa \theta \overline{B}^{(s)}(\tau) \\
(\overline{B}^{(s)})'(\tau) &= \frac{1}{2} k_2 (\overline{\overline{B}^{(s)}}(\tau)) - \frac{1}{2} k_4 \overline{B}^{(s)}(\tau) + \frac{1}{2} k_0 
\end{align}

\section{B.2.4 Numerical Results
where
\[
\tilde{\ell}_0^{(s)} = 2 \left[ \frac{(-\sigma \hat{B}(\tau) - \tilde{\lambda}_1)}{(\gamma - 1)} - \frac{1}{(\gamma - 1)} \right] + \frac{\tilde{\lambda}_2}{(\gamma - 1)} y + \left[ \frac{(-\sigma \hat{B}(\tau) - \tilde{\lambda}_1)^2}{(\gamma - 1)} + \frac{(-\tilde{\lambda}_2)^2}{(\gamma - 1)} \right] y
\]
\[
- \phi_1 \left[ \frac{(-\sigma \hat{B}(\tau) - \tilde{\lambda}_1)^2}{(\gamma - 1)^2} y \right] - \phi_2 \left[ \frac{(-\tilde{\lambda}_2)^2}{(\gamma - 1)^2} y \right]
\]
\[
\tilde{\ell}_1^{(s)} = \kappa - \sigma \left[ \frac{(-\sigma \hat{B}(\tau) - \tilde{\lambda}_1)}{(\gamma - 1)} \right] \left( y - \phi_1 \right)
\]
\[
\tilde{\ell}_2^{(s)} = \sigma^2 (1 - \frac{\phi_1}{\gamma})
\]
\[
\tilde{\ell}_3^{(s)} = \sqrt{\left( \tilde{\ell}_1^{(s)} \right)^2 - \frac{\tilde{\ell}_0^{(s)} \tilde{\ell}_2^{(s)}}{\kappa}}
\]

and the function \( \hat{B}(\tau) \) follows equation (3.23) that characterized the value function \( \hat{J}(x, v, t) \) under the optimal strategy in the complete market. By solving the Riccati equation, we have
\[
\tilde{B}^{(s)}(\tau) = \frac{\tilde{\ell}_0^{(s)} \left( e^{\tilde{\ell}_0^{(s)} \tau} - 1 \right)}{2 \tilde{\ell}_3^{(s)} \left( \tilde{\ell}_1^{(s)} + \tilde{\ell}_2^{(s)} \right) \left( e^{\tilde{\ell}_1^{(s)} \tau} - 1 \right)}
\]
\[
\tilde{A}^{(s)}(\tau) = \gamma r \tau + \frac{2 \theta k}{k_2^{(s)}} \ln \left( \frac{2 \tilde{\ell}_3^{(s)} e^{\tilde{\ell}_0^{(s)} \tau}}{2 \tilde{\ell}_3^{(s)} + \left( \tilde{\ell}_1^{(s)} + \tilde{\ell}_2^{(s)} \right) \left( e^{\tilde{\ell}_1^{(s)} \tau} - 1 \right)} \right),
\]

with auxiliary parameters \( \tilde{\ell}_0^{(s)} \), \( \tilde{\ell}_1^{(s)} \), \( \tilde{\ell}_2^{(s)} \), and \( \tilde{\ell}_3^{(s)} \) above. Further, knowing the functions \( \tilde{A}^{(s)}(\tau), \tilde{B}^{(s)}(\tau) \), as well as \( \hat{A}(\tau), \hat{B}(\tau) \) from optimal strategy, i.e., equation (3.23), the wealth-equivalent loss from ignoring model uncertainty can be quantified, such that
\[
\tilde{L}^{(s)}(v, t) = 1 - \exp \left\{ \frac{1}{\gamma} \left[ \left( \tilde{A}^{(s)}(\tau) - \hat{A}(\tau) \right) + \left( \tilde{B}^{(s)}(\tau) - \hat{B}(\tau) \right) v \right] \right\}
\]
(B.76)

Further, the worst case measure under the suboptimal strategy \( \Pi^{(s)} \) follows
\[
\left\{ \begin{array}{l}
(\epsilon^r)^{(s)} = \phi_1 [ (\Theta^r)^{(s)} (a \sqrt{v} + \frac{k}{\sqrt{\gamma}}) + \sigma \sqrt{\frac{1}{\gamma}} \tilde{B}^{(s)}(\tau) ] \\
(\epsilon^v)^{(s)} = \phi_2 [ (\Theta^v)^{(s)} (a \sqrt{v} + \frac{k}{\sqrt{\gamma}}) ] \\
\end{array} \right.
\]
(B.77)

where \( (\Theta^r)^{(s)} \) and \( (\Theta^v)^{(s)} \) correspond to a suboptimal strategy \( \Pi^{(s)} \).
B.2.4 Proof of Corollary 3.4.3

Proof. Substituting the suboptimal strategy $\Pi^{(s)}$ into equation (B.68) leads to

$$
- (\bar{A}^{(s)})' - (\bar{B}^{(s)})' v + ry + \left[ - \frac{\rho \bar{\lambda}_2}{(\gamma - 1 - \phi_2)^2(1 - \rho^2)} \bar{\lambda}_1 - \frac{\bar{\lambda}_2}{\gamma - 1 - \phi_2} \right] v
$$

$$
+ \frac{1}{2} \gamma \left[ \frac{\rho \bar{\lambda}_2}{(\gamma - 1 - \phi_2)^2(1 - \rho^2)} + \frac{\bar{\lambda}_2}{(\gamma - 1 - \phi_2)^2} \right] \gamma (\gamma - 1) + \kappa(\theta - v)(\bar{B}^{(s)})^2 + \frac{1}{2} \sigma^2 v(\bar{B}^{(s)})^2
$$

$$
- \sigma v \frac{\rho \bar{\lambda}_2}{(\gamma - 1 - \phi_2)^2(1 - \rho^2)} \bar{B}^{(s)}
$$

$$
- \frac{\bar{\lambda}_2}{2} \frac{\gamma^2}{(\gamma - 1 - \phi_2)^2(1 - \rho^2)} \gamma v + \sigma^2 \frac{1}{2} \gamma (\gamma - 1) + \kappa(\theta - v)(\bar{B}^{(s)})^2 - 2 \frac{\rho \bar{\lambda}_2}{(\gamma - 1 - \phi_2)^2(1 - \rho^2)} \sigma \bar{B}^{(s)}
$$

$$
- \frac{\bar{\lambda}_2}{2} \frac{\gamma^2}{(\gamma - 1 - \phi_2)^2(1 - \rho^2)} \gamma v = 0.
$$

(B.78)

where $\bar{A}^{(s)}$ and $\bar{B}^{(s)}$ are functions associated with the value function $\bar{v}^{(s)}$ under the suboptimal strategy $\Pi^{(s)}$. Now rearrange terms by separating terms with $v$, such as

$$
- (\bar{A}^{(s)})' + \kappa \theta \bar{B}^{(s)} + ry
$$

$$
+ \left[ - (\bar{B}^{(s)})' - \frac{\rho \gamma \bar{\lambda}_1 \bar{\lambda}_2}{(\gamma - 1 - \phi_2)^2(1 - \rho^2)} - \frac{\gamma \bar{\lambda}_2^2}{\gamma - 1 - \phi_2} + \frac{1}{2} \gamma (\gamma - 1) \left[ \frac{\rho \bar{\lambda}_2}{(\gamma - 1 - \phi_2)^2(1 - \rho^2)} + \frac{\bar{\lambda}_2^2}{(\gamma - 1 - \phi_2)^2} \right] \right]
$$

$$
- \kappa \bar{B}^{(s)} + \frac{1}{2} \sigma^2 (\bar{B}^{(s)})^2 - \sigma \frac{\rho \bar{\lambda}_2}{(\gamma - 1 - \phi_2)^2(1 - \rho^2)} \bar{B}^{(s)}
$$

$$
- \frac{\rho \bar{\lambda}_2}{2} \frac{\gamma^2}{(\gamma - 1 - \phi_2)^2(1 - \rho^2)} + \frac{\sigma^2}{2} (\bar{B}^{(s)})^2 - 2 \frac{\rho \bar{\lambda}_2}{(\gamma - 1 - \phi_2)^2(1 - \rho^2)} \sigma \bar{B}^{(s)}
$$

$$
- \frac{\gamma \bar{\lambda}_2^2}{2} \frac{\gamma^2}{(\gamma - 1 - \phi_2)^2(1 - \rho^2)} v = 0.
$$

(B.79)

We end up with an equation that is linear in $v$, and both “coefficients” are linear differential equations. In particular, both of the “coefficients” of function $\bar{A}^{(s)}$ and $\bar{B}^{(s)}$ have to be zero to fulfill the boundary condition, such that:

$$
\begin{align*}
(\bar{A}^{(s)})'(\tau) &= ry + \kappa \theta \bar{B}^{(s)}(\tau) \\
(\bar{B}^{(s)})'(\tau) &= \frac{1}{2} \frac{\gamma^2}{\bar{\lambda}_2^2} (\bar{B}^{(s)}(\tau))^2 - \frac{\bar{\lambda}_2}{2} \bar{B}^{(s)}(\tau) + \frac{1}{2} \bar{\lambda}_0^{(s)}
\end{align*}
$$

(B.80)
B.2. Extra Proofs

where

\[ k_0^{(s_2)} = -2 \left[ \frac{\rho \gamma \lambda_1 \lambda_2}{(\gamma - 1 - \phi_2) \sqrt{1 - \rho^2}} + \frac{\gamma \lambda_2^2}{\gamma - 1 - \phi_2} \right] + \gamma (\gamma - 1) \left[ \frac{\rho^2 \lambda_2^2}{(\gamma - 1 - \phi_2)^2 (1 - \rho^2)} + \frac{\lambda_2^2}{(\gamma - 1 - \phi_2)^2} \right] \]

- \phi_1 \left[ \frac{\gamma \rho^2 \lambda_2^2}{(\gamma - 1 - \phi_2)^2 (1 - \rho^2)} \right] - \phi_2 \left[ \frac{\gamma \lambda_2^2}{(\gamma - 1 - \phi_2)^2} \right]

\[ k_1^{(s_2)} = \kappa + (\gamma - \phi_1) \frac{\sigma \rho \lambda_2}{(\gamma - 1 - \phi_2) \sqrt{1 - \rho^2}} \]

\[ k_2^{(s_2)} = \sigma^2 (1 - \frac{\phi_1}{\gamma}) \]

\[ k_3^{(s_2)} = \sqrt{(k_1^{(s_2)})^2 - k_0^{(s_2)} k_2^{(s_2)}}. \]  

(B.81)

and function $\bar{B}(\tau)$ is given in equation (3.23) that characterized value function $\bar{J}(x, v, t)$ under the optimal strategy in the incomplete market. Thereby, the wealth-equivalent loss can be found according to equation (3.44):

\[ \bar{L}^{(s_2)}(v, t) = 1 - \exp \left\{ \frac{1}{\gamma} \left[ (\bar{A}^{(s_2)}(\tau) - \bar{A}(\tau)) + (\bar{B}^{(s_2)}(\tau) - \bar{B}(\tau)) v \right] \right\}. \]  

(B.82)

where $\bar{A}(\tau)$ and $\bar{B}(\tau)$ characterize value function $\bar{J}(x, v, t)$ under the optimal strategy in the complete market, i.e., equation (3.23). Further, the worst case measure under suboptimal strategy $\Pi^{(s_2)}$ follows

\[ \begin{cases} 
(e^{v})^{(s_2)} = \phi_1 \left[ (\Theta^v)^{(s_2)} (a \sqrt{\nu} + \frac{b}{\sqrt{\nu}}) + \sigma \sqrt{\nu} \gamma \bar{B}^{(s_2)}(\tau) \right] \\
(e^{v})^{(s_2)} = \phi_2 \left[ (\Theta^S)^{(s_2)} (a \sqrt{\nu} + \frac{b}{\sqrt{\nu}}) \right]
\end{cases} \]  

(B.83)

where $(\Theta^v)^{(s_2)}$ and $(\Theta^S)^{(s_2)}$ are corresponding to a suboptimal strategy $\Pi^{(s_2)}$.  

\[ \Box \]
B.2.5 Proof of Proposition 3.4.4

Proof. Substituting the suboptimal strategy \( \Pi^{(s)} \) into equation (3.42) leads to

\[
\tilde{h}_i^{(s)} + \left[ r + \left( \frac{\phi_i - 1}{\gamma} \sigma \tilde{B}(\tau) \bar{\lambda}_1 \right) \lambda_1 (a v + b) + \left( \frac{-\bar{\lambda}_2}{(\gamma - 1 - \phi_2) a} \right) \lambda_2 (a v + b) \right] \gamma \tilde{h}_i^{(s)} \\
+ \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) \left( (\frac{\phi_i - 1}{\gamma} \sigma \tilde{B}(\tau) \bar{\lambda}_1)^2 + \left( \frac{-\bar{\lambda}_2}{(\gamma - 1 - \phi_2) a} \right)^2 \right) \gamma (\gamma - 1) \tilde{h}_i^{(s)} + \kappa (\theta - v) \tilde{h}_i^{(s)} + \frac{1}{2} \sigma^2 v \tilde{h}_i^{(s)} \\
+ \sigma (a v + b) \left( \frac{\phi_i - 1}{(\gamma - 1 - \phi_2) a} \right) \gamma \tilde{h}_i^{(s)} \\
- \frac{\phi_1}{2} \left( \frac{\phi_i - 1}{(\gamma - 1 - \phi_2) a} \right)^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \gamma \tilde{h}_i^{(s)} + \sigma^2 v \frac{1}{\gamma} \tilde{h}_i^{(s)} + 2 \left( \frac{\phi_i - 1}{(\gamma - 1 - \phi_2) a} \right) (a v + b) \sigma \tilde{h}_i^{(s)} \\
= 0.
\] (B.84)

Further, regroup terms in the last equation by various partial derivatives of function \( \tilde{h}_i^{(s)} \), such as

\[
\tilde{h}_i^{(s)} = \left\{ - r y - \left( \frac{\phi_i - 1}{\gamma} \sigma \tilde{B}(\tau) \bar{\lambda}_1 \right) \gamma \lambda_1 (a v + b) + \left( \frac{-\bar{\lambda}_2}{(\gamma - 1 - \phi_2) a} \right) \gamma \lambda_2 (a v + b) \\
- \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \left( (\frac{\phi_i - 1}{\gamma} \sigma \tilde{B}(\tau) \bar{\lambda}_1)^2 + \left( \frac{-\bar{\lambda}_2}{(\gamma - 1 - \phi_2) a} \right)^2 \right) \gamma (\gamma - 1) \\
+ \frac{\phi_1}{2} \left( \frac{\phi_i - 1}{(\gamma - 1 - \phi_2) a} \right)^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \gamma + \frac{\phi_2}{2} \left( \frac{-\bar{\lambda}_2}{(\gamma - 1 - \phi_2) a} \right)^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \gamma \right\} \tilde{h}_i^{(s)} \\
+ \left[ \kappa (\theta - v) + \sigma (a v + b) \left( \frac{\phi_i - 1}{(\gamma - 1 - \phi_2) a} \right) \gamma \phi_1 \left( \frac{\phi_i - 1}{(\gamma - 1 - \phi_2) a} \right) (a v + b) \sigma \right] \tilde{h}_i^{(s)} \\
+ \frac{1}{2} \sigma^2 v \tilde{h}_i^{(s)} - \frac{\phi_1}{2} \sigma^2 v \frac{1}{\gamma} \tilde{h}_i^{(s)} + \frac{1}{2} \tilde{h}_i^{(s)} = 0,
\] (B.85)

with terminal condition \( \tilde{h}_i^{(s)}(v, T) = 1 \). That is,

\[
\tilde{h}_i^{(s)} = \left\{ - r y - \left[ \frac{\phi_i - 1}{(\gamma - 1 - \phi_2) a} \right] \gamma \lambda_1 (a v + b) + \left( \frac{-\bar{\lambda}_2}{(\gamma - 1 - \phi_2) a} \right) \gamma \lambda_2 (a v + b) \\
+ \left[ \frac{\phi_1 - (\gamma - 1)}{2} \left( \frac{\phi_i - 1}{(\gamma - 1 - \phi_2) a} \right)^2 \gamma \right. \\
\left. + \frac{\phi_2 - (\gamma - 1)}{2} \left( \frac{-\bar{\lambda}_2}{(\gamma - 1 - \phi_2) a} \right)^2 \gamma \right] \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \gamma \right\} \tilde{h}_i^{(s)} \\
+ \left[ \kappa (\theta - v) + (\gamma - \phi_1) \sigma (a v + b) \right] \tilde{h}_i^{(s)} + \frac{1}{2} \sigma^2 v \tilde{h}_i^{(s)} - \frac{\phi_1}{2} \sigma^2 v \frac{1}{\gamma} \tilde{h}_i^{(s)} + \frac{1}{2} \tilde{h}_i^{(s)} = 0.
\] (B.86)
Assume that \( \phi_1 = 0 \), by Feynman-Kac formula, the solution can be found as a conditional expectation:

\[
\tilde{h}^{(0)}(v, t) = \mathbb{E}_0 \left\{ e^{\int_v^T \left( -\frac{\lambda_2}{2(y - 1) a} \gamma_2 \right) \psi_2 + \frac{\phi_2 - (y - 1)}{2(y - 1 - \phi_2) a} \gamma V_2 \right\} \cdot \int_v^T \left( \frac{1}{\sqrt{v_r}} \right) d\tau \],
\]

Moreover, if this suboptimal strategy is either \( \lambda_1 = 0 \) or the auxiliary parameters in the function \( \tilde{B}(\tau) \) satisfying \( \tilde{k}_3 = \sqrt{k_1^2 - k_0 k_2} = 0 \), then an explicit expression for function \( \tilde{h}^{(0)}(v, t) \) that satisfies the value function \( \tilde{F}^{(0)}(x, v, t) \) under this suboptimal strategy can be found. Hence, the time-dependent function \( \tilde{B}(\tau) \) will vanish (being myopic). The last equation can then be simplified, such as

\[
\tilde{h}^{(0)}(v, t) = \mathbb{E}_0 \left\{ e^{\int_v^T \left( -\frac{\lambda_2}{2(y - 1) a} \gamma_2 \right) \psi_2 + \frac{\phi_2 - (y - 1)}{2(y - 1 - \phi_2) a} \gamma V_2 \right\} \cdot \int_v^T \left( \frac{1}{\sqrt{v_r}} \right) d\tau \],
\]

That is,

\[
\tilde{h}^{(0)}(v, t) = e^{\gamma(T - t)} \times \mathbb{E}_0 \left\{ e^{\int_v^T \left( \frac{1}{2} \left( \frac{\lambda_2^2 \gamma}{2(y - 1) - \phi_2} \right) \right) \psi_2 + \frac{\phi_2 - (y - 1)}{2(y - 1 - \phi_2) a^2} \gamma V_2 \right\} \cdot \int_v^T \left( \frac{1}{\sqrt{v_r}} \right) d\tau \],
\]

Comparing to Grasselli (2017)’s Theorem A.1, we are dealing with a special case of the expectation with

\[
\alpha = 0, \quad \lambda = 0, \quad \mu = \frac{1}{2} \frac{\lambda_2^2 \gamma}{2(y - 1) - \phi_2}, \quad \nu = \frac{1}{2} \frac{b^2 \lambda_2^2 \gamma}{(y - 1 - \phi_2) a^2}.
\]

Following the theorem above-mentioned, the expectation in the above equation can be expressed explicitly for all \( \tau \geq 0 \) where \( \tau(t) = T - t \):

\[
q(\tau, v; \alpha, \lambda, \mu, \nu) = \mathbb{E}_0 \left\{ e^{\int_v^T \left( \frac{1}{2} \frac{\lambda_2^2 \gamma}{2(y - 1) - \phi_2} \right) \psi_2 + \frac{\phi_2 - (y - 1)}{2(y - 1 - \phi_2) a^2} \gamma V_2 \right\} \cdot \int_v^T \left( \frac{1}{\sqrt{v_r}} \right) d\tau \] = \left( \frac{\beta(\tau, v)}{2} \right)^{m+1} v^{\frac{\nu}{\alpha}} (1 + K(\tau))^{-\frac{\gamma}{2} + \frac{\gamma}{2} - \nu} \times e^{\frac{1}{2} \left( e^{\theta \tau - \sqrt{D(\tau) \coth \left( \frac{2\nu}{\alpha} + \frac{2\nu}{\alpha} \right)}} \right) ^{\frac{1}{2} + \frac{\gamma}{2} - \nu}} \times F_1 \left( \frac{1}{2} + m - \alpha + \frac{\nu}{\alpha}, m + 1, \frac{\beta(\tau, v)^2}{4(\lambda + K(\tau))} \right),
\]

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with
\[ m = \frac{1}{\sigma^2} \sqrt{(2\kappa\theta - \sigma^2)^2 + 8\sigma^2\nu}, \]
\[ D = \kappa^2 + 2\mu\sigma^2, \]
\[ \beta(\tau, \nu) = \frac{2 \sqrt{D\nu}}{\sigma^2 \sinh\left(\frac{\sqrt{D}\tau}{2}\right)}, \] (B.92)
\[ K(\tau) = \frac{1}{\sigma^2} \left( \sqrt{D} \coth\left( \frac{\sqrt{D}\tau}{2} \right) + \kappa \right), \]

if the following conditions on the parameters \( \alpha = 0, \lambda = 0, \mu = -\frac{\xi}{2}, \) and \( \nu = \frac{\nu^2}{2(\gamma - 1) - 2\xi} \) can be satisfied:
\[ \mu > \frac{-\kappa^2}{2\sigma^2}, \]
\[ \nu \geq \frac{(2\kappa\theta - \sigma^2)^2}{8\sigma^2}, \] (B.93)
\[ \alpha < \frac{1}{2\sigma^2} \left( 2\kappa\theta + \sigma^2 \sqrt{(2\kappa\theta - \sigma^2)^2 + 8\sigma^2\nu} \right), \]
\[ \lambda \geq \frac{-\kappa^2 + 2\mu\sigma^2 + \kappa}{\sigma^2}. \]

Moreover, the function \( \bar{h}^{(x)}(v, t) \) can be expressed explicitly as
\[
\bar{h}^{(x)}(v, t) = e^{\gamma(T-t)} q(\tau, v; \alpha, \lambda, \mu, \nu)
\]
\[ = \left( \frac{\beta(\tau, \nu)}{2} \right)^{m+1} v^{-\frac{m}{2}} \left( \lambda + K(\tau) \right)^{-\left(\frac{1}{2} + \frac{\kappa^2}{2} \right)} \]
\[ \times e^{\frac{1}{2} \left( \sqrt{D} \coth\left( \frac{\sqrt{D} \tau}{2} \right) \right)} \frac{\Gamma\left( \frac{1}{2} + \frac{m + 1}{2} \right)}{\Gamma\left( \frac{1}{2} + \frac{m + 1}{2} \right)} \]
\[ \times {}_1 F_1 \left( \frac{1}{2} + \frac{m}{2} - \alpha + \frac{\kappa \theta}{\sigma^2}, m + 1, \frac{-\beta(\tau, \nu)^2}{4(\lambda + K(\tau))} \right), \] (B.94)

with parameters \( m, D, \beta(\tau, \nu), \) and \( K(\tau) \) as presented in equation (3.57) and conditions on the parameters \( \alpha, \lambda, \mu, \) and \( \nu \) as shown in equation (3.55). Knowing the function \( \bar{h}^{(x)}(v, t) \), we can find the wealth-equivalent loss following equation (3.44), such that
\[ \bar{L}^{(x)}(v, t) = 1 - \left( \frac{\bar{h}^{(x)}(v, t)}{\bar{h}(v, t)} \right)^{\frac{1}{\gamma}}, \] (B.95)

where the function \( \bar{h}^{(x)}(v, t) \) characterizes the suboptimal strategy of following the Heston strategy, ignoring the uncertainty of stochastic volatility, and also being myopic, while the function \( \bar{h}(v, t) \) characterizes the optimal value function in the complete market. \( \square \)
B.2.6 Proof of Proposition 3.4.5

Proof. Substituting the suboptimal strategy $\Pi^{(s)}$ into equation (3.42) leads to

$$
\bar{h}_v^{(s)} + \left[ r + \frac{-\bar{\lambda}_M}{(\gamma - 1 - \phi_2)} \bar{\lambda}_2 (av + b) \right] \gamma \bar{h}_v^{(s)} \\
+ \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \left( \frac{-\bar{\lambda}_M}{(\gamma - 1 - \phi_2)} \right)^2 \gamma (\gamma - 1) \bar{h}_v^{(s)} + \kappa (\theta - v) \bar{h}_v^{(s)} + \frac{1}{2} \sigma^2 \bar{v} \bar{h}_v^{(s)} \\
- \frac{\phi_1}{2} \left[ \sigma^2 \frac{1}{\gamma} \left( \frac{\bar{h}_v^{(s)}}{\bar{h}_v^{(s)}} \right) \right] - \frac{\phi_2}{2} \left[ \frac{-\bar{\lambda}_M}{(\gamma - 1 - \phi_2)} \right]^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \gamma \bar{h}_v^{(s)} = 0.
$$

(B.96)

Regroup terms in the last equation by various partial derivatives of the function $\bar{h}_v^{(s)}$, such as

$$
\bar{h}_v^{(s)} + \left[ r + \frac{-\bar{\lambda}_M}{(\gamma - 1 - \phi_2)} \bar{\lambda}_2 (av + b) + \frac{\gamma - 1 - \phi_2}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \left( \frac{-\bar{\lambda}_M}{(\gamma - 1 - \phi_2)} \right)^2 \gamma \bar{h}_v^{(s)} \\
+ \kappa (\theta - v) \bar{h}_v^{(s)} + \frac{1}{2} \sigma^2 \bar{v} \bar{h}_v^{(s)} - \frac{\phi_1}{2} \left[ \sigma^2 \frac{1}{\gamma} \left( \frac{\bar{h}_v^{(s)}}{\bar{h}_v^{(s)}} \right) \right] = 0.
$$

(B.97)

Since there is no stochastic volatility in Merton’s model, there should be no ambiguity about the volatility driver, i.e., $\phi_1 = 0$. Then, canceling and recombining terms leads to

$$
\bar{h}_v^{(s)} + \left[ -r + \frac{\bar{\lambda}_M}{(\gamma - 1 - \phi_2)} \bar{\lambda}_2 (av + b) - \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \frac{\bar{\lambda}_M^2}{(\gamma - 1 - \phi_2)} \gamma \bar{h}_v^{(s)} \\
+ \kappa (\theta - v) \bar{h}_v^{(s)} + \frac{1}{2} \sigma^2 \bar{v} \bar{h}_v^{(s)} = 0,
$$

(B.98)

with terminal condition $\bar{h}_v^{(s)}(v, T) = 1$. Further, by the Feynman-Kac formula, the solution can be found as a conditional expectation:

$$
\bar{h}_v^{(s)}(v, t) = \mathbb{E}^Q \left[ \exp \left\{ - \int_t^T \left\{ -ry + \frac{\bar{\lambda}_M}{(\gamma - 1 - \phi_2)} \bar{\lambda}_2 y (av + b) - \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \frac{\bar{\lambda}_M^2}{(\gamma - 1 - \phi_2)} \gamma d\tau \right\} \right] h(v, T) \mid v_i \right].
$$

(B.99)

That is,

$$
\bar{h}_v^{(s)}(v, t) = \mathbb{E}^Q \left[ \exp \left\{ - \int_t^T \left\{ -ry + \frac{ab \bar{\lambda}_M \bar{\lambda}_2 \gamma}{(\gamma - 1 - \phi_2)} - \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \frac{\bar{\lambda}_M^2 \gamma}{(\gamma - 1 - \phi_2)} \right\} \right] \right] \cdot \mathbb{E}^Q \left[ \exp \left\{ - \int_t^T - \frac{\bar{\lambda}_M \bar{\lambda}_2 y a}{(\gamma - 1 - \phi_2)} - \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \frac{\bar{\lambda}_M^2 \gamma}{(\gamma - 1 - \phi_2)} \right\} \right] v_i d\tau - \left[ \int_t^T \frac{1}{v_i} d\tau \right] \mid v_i \right].
$$

(B.100)

Apply Grasselli (2017)’s Theorem A.1 to the above expectation with

$$
\alpha = 0, \quad \lambda = 0, \quad \mu = \frac{\bar{\lambda}_M \bar{\lambda}_2 \gamma a}{(\gamma - 1 - \phi_2)} + \frac{1}{2} \left( a \right)^2 \frac{\bar{\lambda}_M^2 \gamma}{(\gamma - 1 - \phi_2)}, \quad \nu = \frac{1}{2} b^2 \frac{\bar{\lambda}_M^2 \gamma}{(\gamma - 1 - \phi_2)}.
$$

(B.101)
If these parameters, i.e., equation (B.101), can satisfy conditions

\[
\begin{align*}
\mu &> -\frac{k^2}{2\sigma^2}, \\
\nu &> -\frac{(2k\theta - \sigma^2)^2}{8\sigma^2}, \\
\alpha &< \frac{1}{2\sigma^2}(2k\theta + \sigma^2 \sqrt{(2k\theta - \sigma^2)^2 + 8\sigma^2\nu}), \\
\lambda &> -\frac{\sqrt{k^2 + 2\beta^2} + \kappa}{\sigma^2}, \\
\end{align*}
\]  
(B.102)

then the expectation can admit an explicit expression, such that

\[
q(\tau, v; \alpha, \lambda, \mu, \nu) = \mathbb{E}\left[ \exp \left\{ -\frac{\lambda_M \lambda_2^\gamma a}{(y - 1 - \phi_2)} - \frac{1}{2} d^2 \frac{\lambda_M \gamma}{(y - 1 - \phi_2)} \int_t^T v_t d\tau - \frac{1}{2} b^2 \frac{\lambda_M \gamma}{(y - 1 - \phi_2)} \int_t^T \frac{1}{v_t} d\tau \right\} | v_t \right]
\]

\[
= \left( \frac{\beta(\tau, v)}{2} \right)^{m+1} v^{-\frac{\alpha}{\sigma^2}} (\lambda + K(\tau))^{-\left(\frac{1}{2} + \frac{m}{2} + \frac{\theta}{\sigma^2}\right)} \times e^{\frac{1}{\sigma^2} \left( \frac{1}{\sigma^2} - \frac{\theta}{\sigma^2} \right) v^2} \times \Gamma \left( \frac{1}{2} + \frac{m}{2} + \alpha + \frac{\theta}{\sigma^2} \right) \Gamma \left( \frac{1}{2} + \frac{m}{2} + \alpha + \frac{\theta}{\sigma^2} \right) \left( T - t \right)
\]

\[
\times F_1 \left( \frac{1}{2} + \frac{m}{2} + \alpha + \frac{\theta}{\sigma^2}, m + 1, \frac{\beta(\tau, v)^2}{4(\lambda + K(\tau))} \right) \right),
\]  
(B.103)

with parameters \( m, D, \beta(\tau, v), \text{and} \ K(\tau) \) follows in equation (3.64) with \( \alpha, \lambda, \mu, \text{and} \ \nu \) presented in equation (B.101). Consequently, the explicit expression of the expectation leads us to find a close-form solution for the function \( \bar{h}^{(v)}(v, t) \) under Merton’s strategy,

\[
\bar{h}^{(v)}(v, t) = \exp \left\{ \left[ r y + \frac{ab \lambda_M^2 \gamma - b \lambda_M \lambda_2 \gamma}{(y - 1 - \phi_2)} \right](T - t) \right\}
\]

\[
\times \mathbb{E}\left[ \exp \left\{ -\frac{\lambda_M \lambda_2^\gamma a}{(y - 1 - \phi_2)} - \frac{1}{2} d^2 \frac{\lambda_M \gamma}{(y - 1 - \phi_2)} \int_t^T v_t d\tau - \frac{1}{2} b^2 \frac{\lambda_M \gamma}{(y - 1 - \phi_2)} \int_t^T \frac{1}{v_t} d\tau \right\} | v_t \right]
\]

\[
= \left( \frac{\beta(\tau, v)}{2} \right)^{m+1} v^{-\frac{\alpha}{\sigma^2}} (\lambda + K(\tau))^{-\left(\frac{1}{2} + \frac{m}{2} + \frac{\theta}{\sigma^2}\right)} \times \exp \left( \frac{1}{\sigma^2} \left( \frac{1}{\sigma^2} - \frac{\theta}{\sigma^2} \right) v^2 \right) \times \Gamma \left( \frac{1}{2} + \frac{m}{2} + \alpha + \frac{\theta}{\sigma^2} \right) \Gamma \left( \frac{1}{2} + \frac{m}{2} + \alpha + \frac{\theta}{\sigma^2} \right) \left( T - t \right)
\]

\[
\times F_1 \left( \frac{1}{2} + \frac{m}{2} + \alpha + \frac{\theta}{\sigma^2}, m + 1, \frac{\beta(\tau, v)^2}{4(\lambda + K(\tau))} \right) \right),
\]  
(B.104)

with parameters \( m, D, \beta(\tau, v), \text{and} \ K(\tau) \) as presented in equation (3.64) and parameters \( \alpha, \lambda, \mu, \text{and} \ \nu \) as shown in equation (B.101). Finally, the wealth-equivalent loss under Merton’s strategy can be determined by the following equation (3.44), such that

\[
\bar{L}^{(v)}(v, t) = 1 - \left( \frac{\bar{h}^{(v)}(v, t)}{\bar{h}(v, t)} \right)^{\frac{1}{2}},
\]  
(B.105)
where the function $\tilde{h}(v, t)$ is given in equation (3.63), which characterizes the suboptimal strategy of following Merton’s strategy, while the function $h(v, t)$ characterizes the optimal value function in the complete market.

\[ \square \]

### B.2.7 Proof of detection-error probability

**Proof.** Define the conditional characteristic functions for the logarithm of the Radon-Nikodym derivatives for the reference model $\xi_{1, t}$ and the alternative model $\xi_{2, t}$, such as

\[
\begin{align*}
    f_1(\omega, t, N) &= \mathbb{E}^P\left[e^{\rho \xi_{1, t}N} | F^v_t \right] = \mathbb{E}^P\left[\Xi_{1,N}^{\omega} | F^v_t \right] \tag{B.106} \\
    f_2(\omega, t, N) &= \mathbb{E}^P\left[e^{\rho \xi_{2, t}N} | F^v_t \right] = \mathbb{E}^P\left[\Xi_{1,N}^{\omega} | F^v_t \right] \tag{B.107}
\end{align*}
\]

where $i = \sqrt{-1}$, $\omega$ is the transform variable, and $\xi_{1,t} = \ln \Xi_{1,t}$ is used. Note that the second line of the characteristic function $f_2$ performed the change of measure from the measure $P$ to the measure $\mathbb{P}$ by multiplying by the Radon-Nikodym derivative $\Xi_{1,t} = e^{\xi_{1,t}}$ to facilitate the procedure afterwards.

By the Feynman-Kac theorem, the conditional characteristic functions $f_1$ and $f_2$ can satisfy the PDE, known as the Kolmogorov backward equation, such that

\[
\begin{align*}
    \frac{df_1}{dt} + \kappa(\theta - v) \frac{df_1}{dv} + \frac{1}{2} \mathbb{E}^2_{1,t} \left( (e^v)'^2 + (e^\bar{v})'^2 \right) \frac{d^2 f_1}{dv^2} + \frac{1}{2} \sigma^2 \mathbb{E}^2_{1,t} \frac{d^2 f_1}{dv^2} - \Xi_{1,t} e^v \sigma \sqrt{\Xi_{1,t}} \frac{d^2 f_1}{d\Xi_{1,t}dv} &= \tag{B.108} \\
    \frac{df_2}{dt} + \kappa(\theta - v) \frac{df_2}{dv} + \frac{1}{2} \mathbb{E}^2_{1,t} \left( (e^v)'^2 + (e^\bar{v})'^2 \right) \frac{d^2 f_2}{dv^2} + \frac{1}{2} \sigma^2 \mathbb{E}^2_{1,t} \frac{d^2 f_2}{dv^2} - \Xi_{1,t} e^v \sigma \sqrt{\Xi_{1,t}} \frac{d^2 f_2}{d\Xi_{1,t}dv} &= \tag{B.109}
\end{align*}
\]

with corresponding terminal conditions $f_1(\omega, N, N) = \Xi_{1,N}^{\omega}$ and $f_2(\omega, N, N) = \Xi_{1,N}^{\omega+1}$, respectively. Furthermore, we try to find a solution for the PDE (B.108) by assuming that it follows the exponential-affine form, such that

\[
f_1(\omega, t, N) = \Xi_{1,N}^{\omega} \exp (C(t)v + D(t)), \tag{B.110}
\]

with terminal conditions $C(N) = D(N) = 0$. The partial derivatives of $f_1$ then follow

\[
\begin{align*}
    \frac{df_1}{dt} &= f_1(C'v + D'), \quad \frac{df_1}{dv} = f_1C, \quad \frac{d^2 f_1}{dv^2} = f_1C^2, \tag{B.111} \\
    \frac{d^2 f_1}{d\Xi_{1,t}dv} &= i\omega(\omega - 1)\Xi_{1,t}^{\omega-2} \exp (Cv + D), \tag{B.112} \\
    \frac{d^2 f_1}{d\Xi_{1,t}dv} &= i\omega C\Xi_{1,t}^{\omega-1} \exp (Cv + D). \tag{B.113}
\end{align*}
\]

Substituting these partial derivatives into PDE (B.108), we get

\[
\begin{align*}
    f_1(C'v + D') + \kappa(\theta - v)f_1C + \frac{1}{2} \mathbb{E}^2_{1,t} \left( (e^v)'^2 + (e^\bar{v})'^2 \right) i\omega(\omega - 1)\Xi_{1,t}^{\omega-2} \exp (Cv + D) \\
    + \frac{1}{2} \sigma^2 v f_1C^2 - \Xi_{1,t} e^v \sigma \sqrt{\Xi_{1,t}} C\Xi_{1,t}^{\omega-1} \exp (Cv + D) = 0,
\end{align*}
\]
which can be simplified by dividing \( f_1 \) simultaneously

\[
(C'v + D') + \kappa(\theta - v)C + \frac{1}{2} \left((e')^2 + (e')^2\right) i\omega(i\omega - 1) + \frac{1}{2} \sigma^2 vC^2 - e_i^i\sigma \sqrt{v}\omega C = 0.
\]

Regrouping terms yields

\[
D' + \kappa\theta C + \left(C' - \kappa C + \frac{1}{2} \left(q_i^i + q_i^i\right) i\omega(i\omega - 1) + \frac{1}{2} \sigma^2 C^2 - q_i^i\sigma i\omega C \right)v = 0,
\]

where \( q_i^i := e_i^i/\sqrt{v} \) and \( q_i^i := e_i^i/\sqrt{v} \) so that \( q_i \) and \( q_i^i \) are time-dependent. By the boundary condition, we end up with a system of nonlinear ODEs with complex-valued coefficients, such that

\[
\begin{aligned}
D' + \kappa\theta C &= 0 \\
C' - \kappa C + \frac{1}{2} \left(q_i^i + q_i^i\right) i\omega(i\omega - 1) + \frac{1}{2} \sigma^2 C^2 - q_i^i\sigma i\omega C &= 0,
\end{aligned}
\]

subject to \( C(N) = D(N) = 0 \). Note that the time-dependent nature of the coefficients in the above ODE is due to the presence of the time-dependent function \( B(T - t) \), which is related to the optimal portfolio allocation and appears in the perturbation \( e_i^i \) (and so is \( q_i^i \). The time-dependency of the coefficients makes it difficult to find a closed-form solution to the system of ODEs above. Nevertheless, if we assume the function \( B(T - t) \) to be a fixed constant within the time interval \([0, N]\), then a closed-form solution to the system of ODEs above can be obtained.

Likewise, we conjecture a solution for the PDE (B.109) by assuming that it follows the exponential-affine form, such that

\[
f_2(\omega, t, N) = \Xi_1^{i\omega+1} \exp(E(t)v + F(t)), \quad (B.115)
\]

with terminal conditions \( E(N) = F(N) = 0 \). The partial derivatives of \( f_2 \) then follow

\[
\begin{aligned}
\frac{df_2}{dt} &= f_2(E'v + F'), \\
\frac{df_2}{dv} &= f_2E, \\
\frac{d^2f_2}{dv^2} &= f_2E^2,
\end{aligned}
\]

\[
\begin{aligned}
\frac{d^2f_2}{d\Xi_{1,i}^2} &= (i\omega + 1)\Xi_{1,i}^{i\omega} \exp(Ev + F), \\
\frac{d^2f_2}{d\Xi_{1,i} dv} &= (i\omega + 1)\Xi_{1,i}^{i\omega} \exp(Ev + F).
\end{aligned}
\]

Substituting these partial derivatives into the PDE (B.109), we get a system of ODEs subject to the terminal condition \( E(N) = F(N) = 0 \), such that

\[
\begin{aligned}
F' + \kappa\theta E &= 0 \\
E' - \kappa E + \frac{1}{2} \left(q_i^i + q_i^i\right) i\omega(i\omega + 1) + \frac{1}{2} \sigma^2 E^2 - q_i^i\sigma(i\omega + 1)E &= 0,
\end{aligned}
\]

where \( q_i := e_i^i/\sqrt{v} \) and \( q_i^i := e_i^i/\sqrt{v} \) so that \( q_i \) and \( q_i^i \) are time-dependent.
Table B.1: Comparison of risk premium parameters

<table>
<thead>
<tr>
<th>Model</th>
<th>Escobar et al. (2015)</th>
<th>current paper</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{dS_t}{S_t} = \ldots dt + \ldots dW^S_t$</td>
<td>$\frac{dS_t}{S_t} = \ldots dt + \ldots (\rho dZ_1 + \sqrt{1 - \rho^2} dV)$</td>
<td>$\frac{dS_t}{S_t} = \ldots dt + \ldots (\rho dZ_1 + \sqrt{1 - \rho^2} dV)$</td>
</tr>
<tr>
<td>$dv_t = \ldots dt + \ldots (\rho dW^S_t + \sqrt{1 - \rho^2} dV_t)$</td>
<td>$dv_t = \ldots dt + \ldots dZ_1$</td>
<td>$dv_t = \ldots dt + \ldots dZ_1$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.4</td>
<td>-0.7689</td>
</tr>
<tr>
<td>premium for diffusive price risk</td>
<td>$\lambda_1 = 4$</td>
<td>$\lambda = 2.9428$</td>
</tr>
<tr>
<td>premium for volatility risk</td>
<td>$\rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2 = (-0.4) \cdot 4 + \sqrt{1 - 0.4^2} \cdot (-6) = -2.8294$ (by setting $\lambda_2 = 1.2$)</td>
<td>$\lambda_1 = 4$, $\lambda_2 = 1.2$, $\lambda = 2.9428$</td>
</tr>
</tbody>
</table>

Table B.2: Parameter values

| 4/2 model | $\hat{k}$ | 7.3479 | 0.0328 | 0.6612 | 0.9051 | 0.0023 | -0.7689 | 2.9428 |
| Heston model | 14.6290 | 0.0315 | 0.5210 | 1 | 0 | -0.8129 | 3.007 |
| base-case parameters | 5 | 0.13 | 0.25 | 1 | 0 | -0.4 | 4 |

Moreover, based on the results from Maenhout (2006), the detection-error probability for a finite length of $N$ can be obtained via the conditional characteristic function, such that

$$\epsilon_N(\phi_1, \phi_2) = \frac{1}{2} - \frac{1}{2\pi} \int_0^\infty \left( \text{Re} \left[ f_2(\omega, 0, N) \right] - \text{Re} \left[ f_1(\omega, 0, N) \right] \right) d\omega, \quad (B.120)$$

where $f_1(\omega, 0, N) = \exp(C(0)v_0 + D(0))$ and $f_2(\omega, 0, N) = \exp(E(0)v_0 + F(0))$ are accessible.

B.3 Supplementary Figures and Tables

Optimal exposures/allocations in complete/incomplete markets versus $\phi_1$ and $\phi_2$ when tomorrow’s variance $z_t$ is greater or smaller than today’s variance $z_0$ as a follow-up to Chapter 2. As there is not much difference in the pattern, we only present the case when $z_0 = z_t$ in the main text.

In order to be more comparable to the existing literature, we present the base-case parameters (Liu and Pan, 2003; Escobar et al., 2015) in Table B.2 and show (1) the losses from ignoring ambiguity (Figures B.3a and B.3b), (2) the losses from Heston strategy (Figures B.5a and B.5b), respectively.
Figure B.1: Optimal exposures $\Theta^v$ in complete market as function of ambiguity aversion parameters $\phi_1$ and $\phi_2$ when tomorrow’s variance $z_t$ is greater or smaller than today’s variance $z_0$. Panel (a) indicates optimal $\Theta^v$ vs. $\phi_1$ in complete market. Panel (b) indicates optimal $\Theta^v$ vs. $\phi_1$ in complete market. Panel (c) indicates optimal $\Theta^v$ vs. $\phi_2$ in complete market. Panel (d) indicates optimal $\Theta^v$ vs. $\phi_2$ in complete market.
B.3. Supplementary Figures and Tables

Figure B.2: Optimal exposures $\Theta_s$ in complete market as function of ambiguity aversion parameter $\phi_2$ when tomorrow’s variance $z_t$ is greater or smaller than today’s variance $z_0$. Panel (a) exhibits optimal $\Theta_s$ vs. $\phi_2$ in complete market. Panel (b) exhibits optimal $\Theta_s$ vs. $\phi_2$ in complete market.

Figure B.3: Wealth-equivalent loss (WEL) from ignoring model uncertainty as function of ambiguity aversion parameters $\phi_1$ and $\phi_2$ using JBF parameters. Panel (a) shows WEL vs. $\phi_1$ using JBF parameters. Panel (b) shows WEL vs. $\phi_2$ using JBF parameters.
Figure B.4: Upper bound of penalty term as function of ambiguity aversion parameters $\phi_1$ and $\phi_2$. Panel (a) demonstrates penalty term vs. $\phi_1$. Panel (b) demonstrates penalty term vs. $\phi_2$.

Figure B.5: Approximated wealth-equivalent loss (WEL) from following Heston strategy as function of ambiguity aversion parameter $\phi_1$ and $\phi_2$. Panel (a) reports approximated EWL from Heston strategy vs. $\phi_1$, JBF parameters. Panel (b) reports approximated EWL from Heston strategy vs. $\phi_2$, JBF parameters.
B.4 Supplementary Sections

B.4.1 The incomplete market case

Let \( \pi \) be the fraction of wealth invested in the stock, and \( (1 - \pi) \) be the remaining portion of wealth invested in the money account. The wealth \( X_i \) of the investor follows

\[
\frac{dX_i}{X_i} = \pi \frac{dS_i}{S_i} + (1 - \pi)rdt
\]

\[
= \left[ r + \pi \left( \lambda(\alpha v_i + b) - \rho(a \sqrt{v_i} + \frac{b}{\sqrt{v_i}})e^v_i - \sqrt{1 - \rho^2}(a \sqrt{v_i} + \frac{b}{\sqrt{v_i}})e^s_i \right) \right] dt
\]

\[
+ \pi(a \sqrt{v_i} + \frac{b}{\sqrt{v_i}})(\rho dZ_i + \sqrt{1 - \rho^2}d\tilde{Z}_i),
\]

where \( \lambda = \rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2. \)

The value function, i.e., \( J(x, v, t) \), satisfies the robust HJB equation:

\[
\sup_{\pi} \inf_{\mathcal{E}} \left\{ J_x + x \left[ r + \pi \left( \lambda(\alpha v + b) - \rho(a \sqrt{v} + \frac{b}{\sqrt{v}})e^v_i - \sqrt{1 - \rho^2}(a \sqrt{v} + \frac{b}{\sqrt{v}})e^s_i \right) \right] \right\}
\]

\[
= \sup_{\pi} \inf_{\mathcal{E}} \left\{ J_x + x \left[ r + \pi \left( \lambda(\alpha v + b) - \rho(a \sqrt{v} + \frac{b}{\sqrt{v}})e^v_i - \sqrt{1 - \rho^2}(a \sqrt{v} + \frac{b}{\sqrt{v}})e^s_i \right) \right] \right\} J_x
\]

\[
+ \frac{1}{2} \sigma^2 v J_{xx} + \sigma x \rho(\alpha v + b) J_{xv} + \left( \frac{e^v}{2} + \frac{e^s}{2} \right) = 0.
\]

with boundary condition \( J(x, v, T) = \frac{x^\tau}{\gamma} \).

**Proposition B.4.1.** The solution to (B.122) is given by

\[
J(x, v, t) = \frac{x^\gamma}{\gamma} \exp \left\{ A(T - t) + B(T - t)v \right\},
\]

where \( \tau(t) = T - t, A(\tau) \) and \( B(\tau) \) are given by

\[
A(\tau(t)) = \gamma \tau(t) + \frac{2\theta_k}{k_2} \ln \left( \frac{2k_3 e^{k_3 \tau(t)}}{2k_3 + k_1 + k_3 (e^{k_3 \tau(t)} - 1)} \right),
\]

\[
B(\tau(t)) = \frac{k_0 (e^{k_3 \tau(t)} - 1)}{2k_3 + (k_1 + k_3) (e^{k_3 \tau(t)} - 1)},
\]

with auxiliary parameters

\[
k_0 = \frac{-\lambda^2 \gamma}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)},
\]

\[
k_1 = \kappa - \frac{\lambda (\phi_1 - \gamma) \sigma \rho}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)},
\]

\[
k_2 = -\frac{(\phi_1 - \gamma)^2 \sigma^2 \rho^2}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} + \sigma^2 - \frac{1}{\gamma} \phi_1 \sigma^2, k_3 = \sqrt{k_1^2 - k_0 k_2}.
\]
The optimal strategy is given as
\[
\pi^* = \frac{\nu}{av + b} \left( \frac{\phi_1 \gamma v \sigma B(t(t)) - \lambda}{(y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right).
\] (B.126)

The worst case measure is determined by:
\[
(e^\xi)^* = \phi_1 \left[ \frac{\rho \left( \frac{\phi_1 \gamma v \sigma B(t(t)) - \lambda}{(y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} + \frac{\sigma B(t(t))}{\gamma} \right)}{\sqrt{v}} \right] \quad (B.127)
\]
\[
(e^\xi)^* = \phi_2 \left[ \frac{\sqrt{1 - \rho^2} \left( \frac{\phi_1 \gamma v \sigma B(t(t)) - \lambda}{(y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right)}{\sqrt{v}} \right].
\]

The proof follows similarly to the complete market case.

The optimal strategy in the incomplete market is given in equation (B.126). Similar to what we observe in the complete market, the optimal allocation is related to the factor \( \frac{\nu}{av + b} \). Also, it is consists of a myopic component and an inter-temporal hedging component. In particular, the ambiguity averse parameters \( \phi_1 \) and \( \phi_2 \) have impacts on both myopic and hedging terms simultaneously. The worst case probability measure \( (e^\xi)^* \) and \( (e^\xi)^* \) in equation (B.127) are proportional to \( \sqrt{v} \). Different from what we observed in the complete market case, \( (e^\xi)^* \) and \( (e^\xi)^* \) depend on both ambiguity parameters \( \phi_1 \) and \( \phi_2 \). For this reason, the impacts from \( \phi_1 \) or \( \phi_2 \) are nontrivial. Also, it can be noticed that in the incomplete market case, both \( (e^\xi)^* \) and \( (e^\xi)^* \) are time-dependent.

Next, we provide conditions on the optimal change of measure \( \xi^\pi \) determined by the requirement that the perturbations of the volatility driver \( (e^\xi)^* \) and stock \( (e^\xi)^* \) be well-defined, and for the solution \( J(x, v, t) \) to be real-valued and finite in an incomplete market.

**Proposition B.4.2.** For the optimal Radon-Nikodym \( \xi^\pi \) in the incomplete market to be a well-defined density, parameters should satisfy the condition:
\[
\sup_{0 < t < T} K(T - t) \leq \frac{\kappa^2}{\sigma^2} \quad (B.128)
\]

where \( K(T - t) = \phi_1^2 \left( \frac{\sigma^2}{\gamma(y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right) \left( \frac{\phi_1 \gamma v \sigma B(T(t)) - \lambda}{(y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right) \right)^2 \) with function \( B(T - t) \) from equation (B.124).

The proof is similar to the complete market case.

**Proposition B.4.3.** Assume that all the above parametric conditions are satisfied, the function \( J(x, v, t) \) in (B.123) is a well-defined solution to the robust HJB equation in (B.122) if the parameters satisfy the following technical conditions: If \( 0 < \gamma < 1 \), then the following conditions are needed to ensure the value function \( J(x, v, t) \) in (B.123) in the incomplete market to be real-valued and finite:
\[
\begin{align*}
(k - \frac{\lambda (\phi_1 - \gamma)}{(y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)})^2 &> 0, \\
(\frac{\phi_1 \gamma^2 v \sigma^2}{\gamma (y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)})^2 &+ \left( \frac{\phi_1 \gamma^2 v \sigma^2}{\gamma (y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right)^2 &> 0, \\
(\frac{\lambda \gamma}{(y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)})^2 &+ \left( \frac{\phi_1 \gamma^2 v \sigma^2}{\gamma (y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right)^2 &> 0.
\end{align*}
\] (B.129)

(B.130)
The proof is similar to the complete market case.

### B.4.2 Losses from Suboptimal strategies in an incomplete market

In this section, we analyze some suboptimal strategies with regard to the wealth-equivalent utility loss in an incomplete market. Specifically, we consider the suboptimal allocations including ignoring model uncertainty and ignoring parameters \( a \) or \( b \).

The indirect utility function of an investor that follows a suboptimal strategy is denoted by \( J^{(s)} \), such that

\[
J^{(s)}(x, v, t) = \inf_{(c, l) \in \mathcal{L}(x, t)} \left\{ w(c, l) + \mathbb{E}_{t}^{P} \left[ \int_{t}^{T} \left( \frac{(e^{1})^{2}}{2\Phi_{1}(\tau, X_{\tau}, v_{\tau})} + \frac{(e^{2})^{2}}{2\Phi_{2}(\tau, X_{\tau}, v_{\tau})} \right) d\tau \right] \right\},
\]

where \( J^{(s)}(x, v, t) \) is the value function and the suboptimal \( \Pi^{(s)} \) is in the space of admissible strategies \( \mathcal{U} \). In addition, the penalty terms that are scaled by the \( \Phi_{i} \)'s are assumed to be

\[
\Phi_{i} = \frac{\bar{\phi}_{i}}{\gamma J^{(s)}(x, v, t)}, \quad i = 1, 2,
\]

with ambiguity aversion parameters \( \bar{\phi}_{i} > 0 \). For suboptimal strategies that make different assumptions on ambiguity compared to the optimal strategy, the parameters \( \bar{\phi}_{i}, i = 1, 2 \) are different from the parameters \( \phi_{i}, i = 1, 2 \), which are used in the optimal strategy in equation (B.126). By definition, the relationship between the value function in a complete market \( J(x, v, t) \) and \( J^{(s)}(x, v, t) \) is \( J \succeq J^{(s)} \) with equality occurring when the investment allocation is optimal.

The value function, i.e., \( J^{(s)}(x, v, t) \), satisfies the robust HJB equation:

\[
\inf_{c, l} \left\{ J^{(s)}_{t} + \left( \hat{\lambda} \left( av + b \right) - \rho \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) e_{t} + \sqrt{1 - \rho^{2}} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) e_{t}^{S} \right) J^{(s)}_{x} \right\}
+ \frac{1}{2} \sqrt{a \sqrt{v} + \frac{b}{\sqrt{v}}} \pi^{2} J^{(s)}_{xx} + \left[ \kappa (\theta - v) - \sigma \sqrt{v} e_{t}^{S} \right] J^{(s)}_{v} + \frac{1}{2} \sigma^{2} v J^{(s)}_{vv} + \sigma x \pi (av + b) J^{(s)}_{xv} + \frac{(e^{v})^{2}}{2\Phi_{1}} + \frac{(e^{S})^{2}}{2\Phi_{2}} = 0,
\]

with boundary condition \( J^{(s)}(x, v, T) = \frac{v}{T} \). Here \( J^{(s)}_{t}, J^{(s)}_{x}, J^{(s)}_{v}, J^{(s)}_{xx}, J^{(s)}_{xv}, \) and \( J^{(s)}_{vv} \) are the first and second partial derivatives of the function \( J^{(s)} \) with respect to \( t, x, \) and \( v \).

**Proposition B.4.4.** Assume that \( \Pi^{(s)} \) is a function of \((t, v)\); the solution to the PDE (B.133) is then given by

\[
J^{(s)}(x, v, t) = \frac{x^{\gamma}}{\gamma} h^{(s)}(v, t),
\]
where the function $h^{(s)}$ satisfies the PDE:

$$h_t^{(s)} + \left[ r + \pi \lambda (av + b) \right] y h^{(s)} + \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 \gamma (y - 1) h^{(s)} + \kappa (\theta - v) h_v^{(s)} + \frac{1}{2} \sigma^2 v h_{vv}^{(s)} + \sigma \rho (av + b) y h_v^{(s)}$$

$$- \frac{\phi_1}{2} \pi^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) \rho^2 \gamma h^{(s)} + \sigma^2 v \left( \frac{1}{y} h_{\gamma}^{(s)} + 2 \pi (av + b) \rho y h_v^{(s)} \right)$$

$$- \frac{\phi_2}{2} \pi^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) (1 - \rho^2) y h^{(s)} = 0.$$  \hspace{1cm} (B.134)

with boundary condition $h^{(s)}(v, T) = 1$. Furthermore, the wealth-equivalent utility loss $L$ in the incomplete market is

$$L^{(s)}(v, t) = 1 - \left( \frac{h^{(s)}(v, t)}{h(v, t)} \right)^{\frac{1}{2}}.  \hspace{1cm} (B.135)$$

In particular, if $h^{(s)}$ follows an exponential-affine form, such as $h^{(s)}(v, t) = \exp(A^{(s)}(\tau) + B^{(s)}(\tau)v)$, where $\tau(t) = T - t$, then the wealth-equivalent utility loss $L$ in the complete market is given by

$$L^{(s)}(v, t) = 1 - \exp \left( (A^{(s)}(\tau) - A(\tau)) + (B^{(s)}(\tau) - B(\tau)) v \right) \frac{1}{\gamma}.  \hspace{1cm} (B.136)$$

The worst case measure is given by

$$\begin{cases}
(e^{V})^{(s)} = \phi_1 \pi^{(s)} \rho \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) + \sigma \sqrt{v} \tilde{B}^{(s)}(\tau) \\
(e^{S})^{(s)} = \phi_2 \pi^{(s)} \sqrt{1 - \rho^2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right),
\end{cases}  \hspace{1cm} (B.137)$$

where $\pi^{(s)}$ is a suboptimal strategy. Also, $\tilde{\phi}_1 = \phi_1$, and $\tilde{\phi}_2 = \phi_2$ whenever the suboptimal strategy does not have any assumption about the ambiguity parameters.

See Appendix B.4.6 for the complete proof.

Figure B.6: Detection-error probabilities as function of ambiguity aversion parameters $\phi_1$ and $\phi_2$ in an incomplete market. Panel (a) shows DEP in an incomplete market with $\phi_1 = \phi_2$. Panel (b) shows DEP in an incomplete market as $\phi_2$ changes. Panel (c) shows DEP in an incomplete market as $\phi_1$ changes.
Figure B.7: Adjustments $e^v/\sqrt{v}$ and $e^S/\sqrt{v}$ to parameters $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$, respectively, caused by ambiguity aversion in an incomplete market. Panel (a) presents $e^v/\sqrt{v}$ vs. $\phi_1$ in an incomplete market. Panel (b) presents $e^S/\sqrt{v}$ vs. $\phi_2$ in an incomplete market.

Different from the complete market case, where $e^v/\sqrt{v}$ and $e^S/\sqrt{v}$ are only affected by their corresponding uncertainty parameters $\phi_1$ and $\phi_2$, Figure B.7a and Figure B.7b indicate that $e^v/\sqrt{v}$ and $e^S/\sqrt{v}$ are both decreasing in $\phi_1$ and increasing in $\phi_2$ in the incomplete market case.

With regard to the optimal allocation in the stock in an incomplete market, the stock allocation is influenced by both uncertainty parameters $\phi_1$ and $\phi_2$ with a similar trend. Besides, as a follow-up of Cheng and Escobar (2021), the optimal wealth exposures/allocations in complete/incomplete market for tomorrow’s variance greater than or less than today’s variance are provided in Figures B.1a, B.1b, B.1c, B.1d, B.2a, and B.2b for a complete market, and in Figures B.9a, B.9b, B.9c, B.9d for an incomplete market. Overall, tomorrow’s variance does not have a significant impact on the optimal exposures/allocations.

**B.4.3 Proof of Proposition B.4.1**

*Proof.* Solving the infimum problem first, we obtain:

$$
\begin{align*}
\frac{e^v}{\delta_1} &= \pi \rho \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x + \sigma \sqrt{v} J_v, \\
\frac{e^S}{\delta_1} &= \pi \sqrt{1 - \rho^2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x
\end{align*}
\Rightarrow
\begin{align*}
(e^v)^* &= \Phi_1 \left[ \pi \rho \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x + \sigma \sqrt{v} J_v \right], \\
(e^S)^* &= \Phi_2 \left[ \pi \sqrt{1 - \rho^2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x \right].
\end{align*}
$$

(B.138)

Substituting the values of $(e^v)^*$ and $(e^S)^*$ from equation (B.138) into the robust HJB equation,
i.e., equation (B.122), we obtain the following equation that function $J$ has to satisfy:

$$
\sup_{\pi} \left\{ J_x + x \left[ r + \pi \lambda (av + b) \right] J_x - \pi \rho (a \sqrt{v} + \frac{b}{\sqrt{v}}) \Phi_1 \left[ \pi \rho \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x + \sigma \sqrt{v} J_x \right] J_x \\
- x \pi \sqrt{1 - \rho^2 (a \sqrt{v} + \frac{b}{\sqrt{v}}) \Phi_2 \left[ \pi \sqrt{1 - \rho^2 (a \sqrt{v} + \frac{b}{\sqrt{v}}) x J_x + \sigma \sqrt{v} J_x \right] J_x \\
+ \frac{1}{2} x^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 J_{xx} + \kappa (\theta - v) J_x - \sigma \sqrt{v} \Phi_1 \left[ \pi \rho \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x + \sigma \sqrt{v} J_x \right] J_x \\
+ \frac{1}{2} \sigma^2 v J_{vv} + \sigma x \pi \rho (av + b) J_{xv} + \frac{\Phi_1 \left[ \pi \rho \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x + \sigma \sqrt{v} J_x \right] J_x + \frac{\Phi_2 \left[ \pi \sqrt{1 - \rho^2 (a \sqrt{v} + \frac{b}{\sqrt{v}}) x J_x + \sigma \sqrt{v} J_x \right] J_x \right)^2}{2 \Phi_1} \\
+ \Phi_2 \left[ \pi \sqrt{1 - \rho^2 (a \sqrt{v} + \frac{b}{\sqrt{v}}) x J_x + \sigma \sqrt{v} J_x \right] J_x \right]\right\} = 0.
$$

(B.139)

Cancelling and recombing terms,

$$
\sup_{\pi} \left\{ J_x + r x J_x + \pi \lambda (av + b) x J_x + \frac{1}{2} x^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 J_{xx} + \kappa (\theta - v) J_x + \frac{1}{2} \sigma^2 v J_{vv} \\
+ \sigma x \pi \rho (av + b) J_{xv} - \Phi_1 \left[ \pi^2 \rho^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 \pi^2 x^2 (J_x)^2 + 2 \pi \rho \sigma (av + b) x J_x J_v + \sigma^2 v (J_x)^2 \right] \\
- \frac{1}{2} \left[ \pi^2 (1 - \rho^2) (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 \pi^2 (J_x)^2 \right]\right\} = 0.
$$

(B.140)
Figure B.9: Optimal allocation to stock as a function of the ambiguity aversion parameters $\phi_1$ and $\phi_2$ in an incomplete market when tomorrow’s variance is greater or smaller than today’s variance. Panel (a) displays optimal $\pi$ vs. $\phi_1$ in an incomplete market. Panel (b) displays optimal $\pi$ vs. $\phi_1$ in an incomplete market. Panel (c) displays optimal $\pi$ vs. $\phi_2$ in an incomplete market. Panel (d) displays optimal $\pi$ vs. $\phi_2$ in an incomplete market.
Solving the maximization problem:

\[ 0 = \lambda (av + b) x J_x + x^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 J_{xx} \pi + \sigma x p (av + b) J_{xv} \]

\[ - \Phi_1 \rho^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 x (J_x)^2 \pi - \Phi_1 \rho \sigma (av + b) x J_x J_v - \Phi_2 (1 - \rho^2) (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 x (J_v)^2 \pi. \]  

That is,

\[ (\pi)^* = \frac{\Phi_1 \rho \sigma (av + b) x J_x J_v - (\lambda (av + b)) x J_x - \sigma x p (av + b) J_{xv}}{x^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 J_{xx} - \Phi_1 \rho^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 x (J_x)^2 - \Phi_2 (1 - \rho^2) (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 x (J_v)^2}, \]

(B.141)

where \( \Phi_1 = \frac{\phi_1}{\gamma} \), and \( \Phi_2 = \frac{\phi_2}{\gamma} \) by following Maenhout (2004).

Moreover, we conjecture that the value function has the following form

\[ J(x, v, t) = \frac{x^y}{\gamma} h(t, v), \]  

(B.143)

where \( h(T, v) = 1 \) for all \( v \). Thereby, we have the following partial derivatives for \( J(x, v, t) \):

\[ J_x = \frac{x^y}{\gamma} h_1, \quad J_v = \frac{x^y}{\gamma} h_v, \quad J_{xx} = x^{y-1} h, \quad J_{vv} = \frac{x^y}{\gamma} h_{vv}, \quad J_{xv} = x^{y-1} h_v, \quad J_{xx} = (y - 1) x^{y-2} h. \]

Substitute these partial derivatives into the optimal allocation \( \pi \) from equation (C.75), and into equation (B.140) to eliminate “sup”:

\[ (\pi)^* = \frac{(av + b) \left( \frac{\phi_1 - \gamma}{\gamma} \sigma p h_v - \lambda \hat{h} \right)}{\left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 (y - 1) h - \phi_1 \rho \sigma^2 \hat{p}^2 - \phi_2 h(1 - \rho^2)} \]  

(B.144)

\[ h_v + r \gamma h + \frac{\lambda \gamma v \left( \frac{\phi_1 - \gamma}{\gamma} \sigma p h_v - \lambda \hat{h} \right)}{(y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} + \frac{1}{2} \gamma (y - 1) v \left( \frac{\phi_1 - \gamma}{\gamma} \sigma p h_v - \lambda \hat{h} \right)^2 \]

\[ + \frac{1}{2} \sigma^2 \gamma h_v v + \gamma \sigma p h_v \frac{v \left( \frac{\phi_1 - \gamma}{\gamma} \sigma p h_v - \lambda \hat{h} \right)}{h (y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \]

\[ - \frac{\phi_1}{2 h} \left( \rho^2 \gamma \frac{v \left( \frac{\phi_1 - \gamma}{\gamma} \sigma p h_v - \lambda \hat{h} \right)^2}{(y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} + 2 \rho \sigma h_v \frac{v \left( \frac{\phi_1 - \gamma}{\gamma} \sigma p h_v - \lambda \hat{h} \right)}{(y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} + \frac{\sigma^2 v}{\gamma} h_v^2 \right) \]

\[ - \frac{\phi_2}{2 h} \left( 1 - \rho^2 \gamma \frac{v \left( \frac{\phi_1 - \gamma}{\gamma} \sigma p h_v - \lambda \hat{h} \right)^2}{(y - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right) = 0 \]  

(B.145)
We further conjecture that the function $h(v, t)$ has the form

$$h(v, t) = \exp \left\{ A(T - t) + B(T - t)v \right\}, \quad (B.146)$$

with boundary condition

$$h(T, v) = 1 \forall v. \quad (B.147)$$

The partial derivatives of the function $h(v, t)$ are given by

$$h_t = (-A' - B'v)h, \quad h_v = Bh, \quad h_{vv} = B^2h, \quad \frac{h^2}{h} = \frac{B^2h^2}{h} = B^2h.$$  

By substituting the partial derivatives of $h(v, t)$ into the robust HJB equation above, we can further simplify:

$$- A' - B'v + r\gamma + \frac{1}{2} \gamma \lambda^2v - \frac{1}{2} \frac{(\phi_1 - \gamma)^2}{\gamma} \sigma^2 \rho^2vB^2 - \sigma \rho (\gamma - \phi_1)v \lambda B \quad \left[ (\gamma - 1) - \phi_1 \rho^2 - \phi_2(1 - \rho^2) \right]$$

$$+ \kappa (\theta - v)B + \frac{1}{2} \sigma^2vB^2 - \frac{1}{2\gamma} \phi_1 \sigma^2vB^2 = 0 \quad (B.148)$$

Rewriting the last equation into a linear PDE in $v$ leads to

$$- A' + r\gamma + \kappa \theta B$$

$$\left\{ - B' - \frac{1}{2} \gamma \lambda^2 + \frac{1}{2} \frac{(\phi_1 - \gamma)^2}{\gamma} \sigma^2 \rho^2B^2 - \sigma \rho (\gamma - \phi_1)\lambda B \quad \left[ (\gamma - 1) - \phi_1 \rho^2 - \phi_2(1 - \rho^2) \right] \right.$$  

$$- \kappa B + \frac{1}{2} \sigma^2B^2 - \frac{1}{2\gamma} \phi_1 \sigma^2B^2 \right\} v = 0 \quad (B.149)$$

In order to fulfill the boundary condition of $h(v, T)$, both of the “coefficients” have to be zero:

$$A' = \kappa \theta B + \gamma r$$

$$B' = \frac{1}{2} \left\{ \frac{(\phi_1 - \gamma)^2}{\gamma} \sigma^2 \rho^2 \right\}_{k_2} + \sigma^2 - \frac{1}{\gamma} \phi_1 \sigma^2 \right\}_{k_1} B^2 - \kappa - \frac{\lambda (\phi_1 - \gamma) \sigma \rho}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2(1 - \rho^2)} \right\}_{k_1} B$$

$$+ \frac{1}{2} \left\{ \frac{-\lambda^2}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2(1 - \rho^2)} \right\}_{k_0}$$  

$$(B.150)$$
The worst case measure is determined by

\[
\begin{aligned}
A'(\tau) &= \kappa \theta B(\tau) + \gamma r \\
B'(\tau) &= \frac{1}{2} k_2 B(\tau)^2 - k_1 B(\tau) + \frac{1}{2} k_0
\end{aligned}
\]  

(B.151)

with boundary conditions \( h(T, v) = 1 \ \forall \ v \) and \( \tau(t) = T - t \), we have

\[
A(0) = A(\tau(T)) = 0, B(0) = B(\tau(T)) = 0
\]

(B.152)

with constants \( k_0, k_1, k_2 \) satisfying \( k_1^2 - k_0 k_2 > 0 \). The problem is solvable with expressions for \( A, B \) as follows:

\[
A(\tau(t)) = \gamma r t + \frac{2\theta k}{k_2} \ln \left( \frac{2 k_3 e^{k_1 + k_3 \tau}}{2 k_3 + (k_1 + k_3) (e^{k_3 \tau} - 1)} \right),
\]

(B.153)

\[
B(\tau(t)) = \frac{k_0 (e^{\lambda t} - 1)}{2 k_3 + (k_1 + k_3) (e^{k_3 \tau} - 1)},
\]

with parameters

\[
\begin{align*}
k_0 &= \frac{-\bar{\lambda}^2 \gamma}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \\
k_1 &= \kappa - \frac{\bar{\lambda} (\phi_1 - \gamma) \sigma \rho}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \\
k_2 &= -\frac{(\phi_1 - \gamma) \sigma^2 \rho^2}{\gamma (\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} + \sigma^2 \left[ 1 - \frac{1}{\gamma} \phi_1 \sigma \rho \right] \\
k_3 &= \sqrt{k_1^2 - k_0 k_2}.
\end{align*}
\]

(B.154)

This leads to the explicit form of the optimal strategy

\[
\pi^* = \frac{\nu}{av + b} \left( \frac{\phi_1 - \gamma \sigma \rho B(\tau(t)) - \bar{\lambda}}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right).
\]

(B.155)

The worst case measure is determined by

\[
\begin{align*}
(e^\gamma)^* &= \phi_1 \left[ \frac{\rho \left( \frac{\phi_1 - \gamma \sigma \rho B(\tau(t)) - \bar{\lambda}}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right) + \frac{1}{\gamma} \sigma \rho B(\tau(t))}{\sqrt{v}} \right] \\
(e^\delta)^* &= \phi_2 \left[ \sqrt{1 - \rho^2} \left( \frac{\phi_1 - \gamma \sigma \rho B(\tau(t)) - \bar{\lambda}}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right) \right] \sqrt{v}.
\end{align*}
\]

(B.156)
B.4. Supplementary Sections

B.4.4 Proof of Proposition B.4.2

Proof. For the first condition, we show that the optimal Radon-Nikodym derivative of $\mathbb{P}^\ast$ with respect to $\mathbb{P}$ in the incomplete market, such that

$$
\xi_t^\ast = \mathbb{E} \left[ \frac{d\mathbb{P}^\ast}{d\mathbb{P}} | \mathcal{F}_t \right] = \exp \left\{ - \int_0^t \left( \frac{(e_t^v)^2 + (e_t^S)^2}{2} d\tau + e_t^v dZ_{1 \tau} + e_t^S dZ_{2 \tau} \right) \right\} 
$$

is a $\mathbb{P}$-martingale to ensure a well-defined $\mathbb{P}^\ast$. Thus, we consider sufficient conditions based on Novikov’s equation as follows:

$$
\mathbb{E}\mathbb{P} \left[ \exp \left\{ \int_0^T \frac{(e_t^v)^2 + (e_t^S)^2}{2} dt \right\} \right] < \infty, 
$$

where the optimal perturbations are given in equation (B.127):

$$
(e_v)^\ast = \phi_1 \left( \frac{\rho \left( \frac{\phi - \gamma}{\gamma} \sigma p B(T - t) - \lambda \right)}{\gamma (1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right) + \frac{\sigma B(T - t)}{\gamma} \sqrt{\nu} 
$$

$$
(e_S)^\ast = \phi_2 \left( \frac{\sqrt{1 - \rho^2} \left( \frac{\phi - \gamma}{\gamma} \sigma p B(T - t) - \lambda \right)}{\gamma (1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right) \sqrt{\nu}.
$$

We then consider the process $\xi_t$ defined as

$$
\xi_t = (e_t^v)^2 + (e_t^S)^2 = \phi_1^2 \left( \frac{\rho \left( \frac{\phi - \gamma}{\gamma} \sigma p B(T - t) - \lambda \right)}{\gamma (1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right) + \frac{\sigma B(T - t)}{\gamma} \sqrt{\nu} \right)^2 v_t + \phi_2^2 \left( \frac{\sqrt{1 - \rho^2} \left( \frac{\phi - \gamma}{\gamma} \sigma p B(T - t) - \lambda \right)}{\gamma (1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right)^2 v_t 
$$

$$
= \phi_1^2 \left( \frac{\sigma^2 \rho}{\gamma^2} B^2(T - t) + \frac{\sigma \rho}{\gamma} \left( \frac{\phi \nu - \gamma}{\gamma} \sigma p B(T - t) - \lambda \right) B(T - t) \right) v_t \right)^2 
$$

$$
+ \phi_2^2 \left( \frac{\sigma^2 \rho^2 + \phi_2 (1 - \rho^2)}{\gamma (1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right)^2 v_t. 
$$

Since this process is a product of a time-dependent term and the volatility driver $v_t$, we then rewrite it as

$$
\xi_t = K(T - t) v_t. 
$$
Hence, Novikov’s condition for the Radon-Nikodym derivative becomes
\[
\mathbb{E}^P \left[ \exp \left\{ \int_0^T \frac{\xi_t^2}{2} dt \right\} \right] = \mathbb{E}^P \left[ \exp \left\{ \frac{1}{2} \int_0^T K(T - t)v_t dt \right\} \right] < \mathbb{E}^P \left[ \exp \left\{ \frac{1}{2} \int_0^T \sup_{0 < t < T} \{K(T - t)v_t dt\} \right\} \right] = \mathbb{E}^P \left[ \exp \left\{ \frac{1}{2} K \int_0^T v_t dt \right\} \right],
\]
where we denote \( K := \sup_{0 < t < T} \{K(T - t)\} \), so \( \bar{K} \) is independent of \( t \) and it can be taken out from the integral. Further, by proposition 5.1 in Kraft (2005), if the condition
\[
K \leq \frac{k^2}{\sigma^2}
\]
is satisfied, then we have
\[
\mathbb{E}^P \left[ \exp \left\{ \int_0^T \frac{\xi_t^2}{2} dt \right\} \right] < \mathbb{E}^P \left[ \exp \left\{ \frac{1}{2} \bar{K} \int_0^T v_t dt \right\} \right] < \infty.
\]
\(\blacksquare\)

### B.4.5 Proof of Proposition B.4.3

**Proof.** To ensure that the function is real-valued, we need to make sure the square roots are well-defined in the function \( B(T - t) \) in equation (B.124). That is,
\[
k_3 > 0 \iff k_1^2 - k_0k_2 > 0
\]
\[
\iff \left( \kappa - \frac{\lambda(\phi_1 - \gamma)\sigma^2}{(\gamma - 1) - \phi_1\rho^2 - \phi_2(1 - \rho^2)} \right)^2
\]
\[
+ \left( \frac{-\lambda^2\gamma}{(\gamma - 1) - \phi_1\rho^2 - \phi_2(1 - \rho^2)} \right) - \frac{(\phi_1 - \gamma)^2\sigma^2\rho^2}{(\gamma - 1) - \phi_1\rho^2 - \phi_2(1 - \rho^2)} + \left( 1 - \frac{\phi_1}{\gamma} \right) > 0.
\]

Note that when \( 0 < \gamma < 1 \), the above condition is satisfied if \( \gamma > \phi_1 \). Otherwise, the condition \( k_1^2 - k_0k_2 > 0 \) is required to ensure real-valued square roots.

We further need to ensure finiteness of the function \( J(x, v, t) \), so that the exponential function \( h(v, t) \) does not go to infinity. Equivalently, we need the function \( B(T - t) \) in equation (B.124) to remain finite, so it leads to a well-defined function \( A(T - t) \). On the one hand, if \( \gamma < 0 \), the argument of the exponential in \( J \) is negative because \( k_0 < 0 \) gives negative \( B(T - t) \)
and due to the logarithm representation of $A(T-t)$. Thus, the finiteness condition always holds for $\gamma < 0$. On the other hand, when $0 < \gamma < 1$, the function $B(T-t)$ is finite if

$$2k_3 + (k_1 + k_3) (e^{k_3} - 1) \neq 0. \quad (B.166)$$

It then follows that

$$-(k_1 - k_3) \left( 1 - \frac{(k_1 + k_3)}{(k_1 - k_3)} e^{k_3} \right) \neq 0 \quad \iff \quad \frac{k_1 + k_3}{k_1 - k_3} > 1 \quad (B.167)$$

That is,

$$\left( \frac{-3^{2} \gamma}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right) \left( \frac{(\phi_1 - \gamma) \sigma^2 \rho^2}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} + (1 - \phi_1) \sigma^2 \right) > 0. \quad (B.168)$$

**B.4.6 Proof of Proposition B.4.4**

*Proof.* Solving the infimum problem first, we obtain:

$$\begin{align*}
\left\{ \begin{array}{l}
\frac{d' \mathcal{J}^e}{d_1} = \pi \rho \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x + \sigma \sqrt{v} J_v \\
\frac{d' \mathcal{J}^e}{d_2} = \pi \sqrt{1 - \rho^2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x
\end{array} \right. \\
\implies \left\{ \begin{array}{l}
(e^e)^* = \Phi_1 \left[ \pi \rho \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x + \sigma \sqrt{v} J_v \right] \\
(e^e)^* = \Phi_2 \left[ \pi \sqrt{1 - \rho^2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x \right].
\end{array} \right.
\end{align*} \quad (B.169)$$

Substituting the values of $(e^e)^*$ and $(e^e)^*$ from equation (B.169) into the robust HJB equation, i.e., equation (B.133), we obtain the following equation that function $J^{(e)}$ has to satisfy:

$$\begin{align*}
J_t^{(e)} + x \left[ r + \pi \lambda (a v + b) \right] J_x^{(e)} - x \pi \rho (a \sqrt{v} + \frac{b}{\sqrt{v}}) \Phi_1 \left[ \pi \rho \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x^{(e)} + \sigma \sqrt{v} J_v^{(e)} \right] J_x^{(e)} & \\
- x \pi \sqrt{1 - \rho^2} (a \sqrt{v} + \frac{b}{\sqrt{v}}) \Phi_2 \left[ \pi \sqrt{1 - \rho^2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x^{(e)} \right] J_x^{(e)} & \\
+ \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 J_{xx}^{(e)} + \kappa (\theta - v) J_{t}^{(e)} - \sigma \sqrt{v} \Phi_1 \left[ \pi \rho \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x^{(e)} + \sigma \sqrt{v} J_v^{(e)} \right] J_x^{(e)} & \\
+ \frac{1}{2} \sigma^2 v J_{vv}^{(e)} + \sigma x \pi \rho (a v + b) J_{xv}^{(e)} + \Phi_1^2 \left[ \pi \rho \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x^{(e)} + \sigma \sqrt{v} J_v^{(e)} \right]^2 & \\
+ \frac{1}{2} \Phi_2^2 \left[ \pi \sqrt{1 - \rho^2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x J_x^{(e)} \right]^2 & = 0
\end{align*} \quad (B.170)$$
Cancelling and recombining terms, we have that solving next equation is equivalent to solving the problem in equation (B.133):

\[
J_t^{(s)} + x \left[ r + \pi \hat{\lambda} (av + b) \right] J_v^{(s)} \\
+ \frac{1}{2} x^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 J_{xx}^{(s)} + \kappa (\theta - v) J_v^{(s)} + \frac{1}{2} \sigma^2 v J_{vv}^{(s)} + \sigma x \pi p (av + b) J_{xv}^{(s)} \\
- \frac{1}{2} \frac{\partial}{\partial J} \left[ \pi^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \rho^2 x^2 (J_x^{(s)})^2 + \sigma^2 v (J_v^{(s)})^2 + 2 \pi (av + b) x \sigma p x J_x^{(s)} J_v^{(s)} \right] \\
- \frac{1}{2} \frac{\partial}{\partial J} \left[ \pi^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 (1 - \rho^2) x^2 (J_v^{(s)})^2 \right] = 0.
\]

(B.171)

The wealth-equivalent loss is defined as

\[
J \left( x (1 - L^{(s)}), v, t \right) = J^{(s)} (x, v, t),
\]

(B.172)

where \( J^{(s)} (x, v, t) \) is the value function of an investor that allocates his wealth following a suboptimal rule in an incomplete market, and \( L^{(s)} (v, t) \) is the proportion of wealth loss incurred by adopting a suboptimal strategy as a function of wealth \( x \) and the instantaneous volatility driver \( v \). Moreover, if the form of the value function is assumed to be

\[
J^{(s)} (x, v, t) = \frac{x^\gamma}{\gamma} h^{(s)} (v, t),
\]

(B.173)

with boundary condition \( h^{(s)} (v, T) = 1 \), then substituting the partial derivatives of \( J^{(s)} (x, v, t) \), we obtain the following equation that is enough to solve the problem in (B.133):

\[
h_t^{(s)} + \left[ r + \pi \hat{\lambda} (av + b) \right] \gamma h^{(s)} \\
+ \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 \gamma (y - 1) h^{(s)} + \kappa (\theta - v) h_v^{(s)} + \frac{1}{2} \sigma^2 v h_{vv}^{(s)} + \sigma x \pi p (av + b) \gamma h_{xv}^{(s)} \\
- \frac{\partial}{\partial J} \left[ \pi^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \rho^2 \gamma (y - 1) h^{(s)} + \sigma^2 v \frac{1}{\gamma} \frac{(h_v^{(s)})^2}{h^{(s)}} + 2 \pi (av + b) x \sigma p h_{xv}^{(s)} \right] \\
- \frac{\partial}{\partial J} \left[ \pi^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 (1 - \rho^2) \gamma h^{(s)} \right] = 0.
\]

(B.174)

Furthermore, if we also assume that the value function of the suboptimal strategy \( \pi^{(s)} \) admits an exponential-affine form such as

\[
J^{(s)} (x, v, t) = \frac{x^\gamma}{\gamma} h^{(s)} (v, t) = \frac{x^\gamma}{\gamma} \exp \left\{ A^{(s)} (T - t) + B^{(s)} (T - t) v \right\},
\]

(B.175)

with \( \tau (t) = T - t \), and boundary condition \( A^{(s)} (\tau (T)) = 0 \) and \( B^{(s)} (\tau (T)) = 0 \), directly, the equation (B.134) can be rewritten by substituting the partial derivatives of function \( h^{(s)} \) as
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follows

\[- (A^{(s)})' - (B^{(s)})'v + r\gamma + \pi \tilde{\lambda}(av + b)\gamma
\]
\[+ \frac{1}{2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 \gamma (\gamma - 1) + k(\theta - v)B^{(s)} + \frac{1}{2} \sigma^2 v (B^{(s)})^2 + \sigma\pi\rho (av + b)\gamma B^{(s)}\]
\[- \frac{\tilde{\varphi}_1}{2} \left[ \pi^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \rho^2 \gamma + \sigma^2 v \left( B^{(s)} \right)^2 + 2\pi (av + b)\sigma\rho B^{(s)} \right] - \frac{\tilde{\varphi}_2}{2} \left[ \pi^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 (1 - \rho^2) \gamma \right] = 0. \]
(B.176)

Then the wealth-equivalent loss function \(L^{(s)}\) admits the expression

\[L^{(s)}(v, t) = 1 - \left( \frac{h^{(s)}(v, t)}{\bar{h}(v, t)} \right)^{\frac{1}{\gamma}}, \]
\[= 1 - \exp \left\{ \frac{1}{\gamma} \left[ (A^{(s)}(\tau) - A(\tau)) + (B^{(s)}(\tau) - B(\tau)) \right] \right\}, \]
(B.177)

where \(A(\tau)\) and \(B(\tau)\) are time-dependent functions that feature the value function \(J(x, v, t)\) under the optimal strategy, i.e., equation (B.124), while \(A^{(s)}(\tau)\) and \(B^{(s)}(\tau)\) are functions that characterize the value function \(J^{(s)}(x, v, t)\) under a suboptimal strategy. Further, the worst case measure from equation (B.169) is given by

\[\left\{ \begin{array}{l}
(e^y)^{(s)} = \tilde{\varphi}_1 \left[ \pi^*(\rho \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) + \sigma \sqrt{\gamma} \tilde{B}^{(s)}(\tau) \right]
\vspace{0.5cm}
(e^\xi)^{(s)} = \tilde{\varphi}_2 \left[ \pi^*(\sqrt{1 - \rho^2} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) \right],
\end{array} \right. \]
(B.178)

where \(\pi^{(s)}\) is a suboptimal strategy. Also, \(\tilde{\varphi}_1 = \phi_1\), and \(\tilde{\varphi}_2 = \phi_2\) whenever the suboptimal strategy does not have any assumption about the ambiguity parameters.
Appendix C

Proofs for Chapter 4

C.1 Proof of Proposition 4.3.1 (Complete Market, No Robustness, Consumption)

Proof. Solving the maximization problem for intermediate consumption:

\[
0 = u'(c) - \bar{J}_x = \epsilon_1 c^{\gamma-1} - \bar{J}_x
\]

That is,

\[
c^* = (\bar{J}_x)^{\frac{1}{\gamma-1}}(\epsilon_1)^{-\frac{1}{\gamma-1}},
\]

where the risk averse parameter \( \gamma < 1 \).

Solving the maximization problem for wealth exposures:

\[
\begin{cases}
0 &= x\lambda_1(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2\bar{J}_x + x^2\left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}\right)^2\Theta^\nu \bar{J}_{xx} + \sigma x(\nu + b)\bar{J}_{xv} \\
0 &= x\lambda_2(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2\bar{J}_x + x^2\left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}\right)^2\Theta^\nu \bar{J}_{xx}
\end{cases}
\]

That is,

\[
\begin{align*}
(\Omega^\nu)^* &= -\frac{-x\lambda_1(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2\bar{J}_x - \sigma x(\nu + b)\bar{J}_{xv}}{x^2(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2\bar{J}_{xx}} \\
(\Theta^\nu)^* &= -\frac{-x\lambda_2(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2\bar{J}_x}{x^2(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})^2\bar{J}_{xx}} = \frac{-x\lambda_2\bar{J}_x}{x\bar{J}_{xx}}.
\end{align*}
\]

Under the conjecture of the value function in (4.13):

\[
\bar{J}(x, v, t) = \frac{x^\nu}{\gamma} \left(h(t, v)\right)^{1-\gamma},
\]
We aim at an expected value representation of (4.15) with \( \tilde{h} \) where

\[
\frac{1 - \gamma}{\gamma} x^\gamma h^{-\gamma} \tilde{h},
\]

\[
\frac{\gamma}{\gamma} (1 - \gamma) \tilde{h}^{-\gamma} \tilde{h},
\]

\[
x^\gamma h^{1-\gamma},
\]

\[
-(1 - \gamma) x^\gamma h^{-\gamma} \tilde{h}^2 + \frac{\gamma}{\gamma} (1 - \gamma) \tilde{h}^{-\gamma} \tilde{h},
\]

\[
(1 - \gamma) x^\gamma h^{-\gamma} \tilde{h},
\]

\[
(\gamma - 1) x^\gamma h^{-\gamma} \tilde{h}.
\]

Substituting these partial derivatives into the candidates of optimal consumption wealth ratio from Equation (C.2), wealth exposures in Equation (C.4), and the PDE (4.12), we have

\[
(1 - \gamma)(\epsilon_t) - \frac{1}{\gamma} + (1 - \gamma) \tilde{h}_t + \left[ r - \frac{\delta}{\gamma} - \frac{1}{2} \frac{\lambda_1^2(a \sqrt{v} + b \sqrt{v})^2}{\gamma - 1} - \frac{1}{2} \frac{\lambda_2^2(a \sqrt{v} + b \sqrt{v})^2}{\gamma - 1} \right] \gamma \tilde{h}
\]

\[
+ \left[ \frac{- \gamma (1 - \gamma) \sigma \lambda_1 (av + b)}{\gamma - 1} + \gamma (1 - \gamma) \sigma (av + b) \lambda_1 + \kappa (\theta - v) \right] (1 - \gamma) \tilde{h}_v + \left[ \frac{1}{2} \gamma^2 (1 - \gamma) \right] \tilde{h}_{vv} = 0.
\]

(C.6)

It can be seen that there is no nonlinear term in the PDE, thereby, no parameter condition is needed to find an explicit solution for \( \tilde{h} \).

For clarity we divide both sides of the equation by \((1 - \gamma)\) so that the coefficient of \( \tilde{h}_t \) is 1:

\[
(\epsilon_t) - \frac{1}{\gamma} + \tilde{h}_t + V(v, t) \tilde{h} + \Gamma(v, t) \tilde{h}_v + \frac{1}{2} \gamma^2 \tilde{h}_{vv} = 0,
\]

(C.7)

where \( V(v, t) \) and \( \Gamma(v, t) \) are defined as follow:

\[
V(v, t) = \left[ r - \frac{\delta}{\gamma} - \frac{1}{2} \frac{\lambda_1^2(a \sqrt{v} + b \sqrt{v})^2}{\gamma - 1} - \frac{1}{2} \frac{\lambda_2^2(a \sqrt{v} + b \sqrt{v})^2}{\gamma - 1} \right] \frac{\gamma}{1 - \gamma}
\]

\[
= \frac{\lambda_1^2 + \lambda_2^2}{2(\gamma - 1)^2} (a \sqrt{v} + b \sqrt{v})^2 + \frac{\delta - \gamma r}{\gamma - 1}
\]

\[
= \bar{a} v + \bar{b} \frac{1}{v} + \bar{c}
\]

\[
\Gamma(v, t) = \left[ \frac{- \gamma (1 - \gamma) \sigma \lambda_1 (av + b)}{\gamma - 1} + \gamma (1 - \gamma) \sigma (av + b) \lambda_1 + \kappa (\theta - v) \right]
\]

\[
= \bar{k} \bar{\theta} - \bar{k} v
\]

with \( \bar{a} = a^2 \frac{(\lambda_1^2 + \lambda_2^2)}{2(\gamma - 1)^2}, \quad \bar{b} = b^2 \frac{(\lambda_1^2 + \lambda_2^2)}{2(\gamma - 1)^2}, \quad \bar{c} = 2ab \frac{(\lambda_1^2 + \lambda_2^2)}{2(\gamma - 1)^2} + \frac{\delta - \gamma r}{\gamma - 1} \) and \( \bar{k} \) and \( \bar{\theta} \) can be found in Equation (4.15).

We aim at an expected value representation of \( \tilde{h} \) where \( v \) stands for a convenient stochastic
process. This is an application of the Feynman-Kac formula, therefore the coefficients in (4.14) must satisfy the conditions of Theorem 1 and Lemmas 2 and 3 in Heath and Schweizer (2000).

In the notation of Heath and Schweizer (2000) we have: \( X = v, D = (0, \infty), b(t, v) = \Gamma(v, t), \Sigma(t, v) = \sigma \sqrt{v}, c(t, v) = V(t, v), g(t, v) = (e_1)^{-\frac{1}{2\gamma}} \), \( \bar{h}(t, x) = 1 \), \( u(t, v) = \bar{h}(t, v), u(T, v) = \bar{h}(T, v) = 1 \) and \( a(t, x) = \sigma^2 v \). The process \( v_t \) should follow the SDE: \( dv_t = \Gamma(v, t) dt + \sigma \sqrt{v} dZ_{1,t} \).

Using the same arguments as in their section 2.1 (an application on the Heston model), we can conclude that \( h \) admits the Feynman-Kac representation:

\[
\bar{h}(v, t) = \mathbb{E}^Q \left[ \int_t^T \exp \left\{ \int_t^\tau V(v_\tau, \tau) d\tau \right\} \left( (e_1)^{-\frac{1}{2\gamma}} \right) d\bar{\tau} + \exp \left\{ \int_t^T V(v_\tau, \tau) d\tau \right\} h(v, T) | v_t \right].
\]

Moreover, we have \( \exp \left\{ \int_t^T V(v_\tau, \tau) d\tau \right\} \geq 0 \) hence we can apply Tonelli's Theorem to exchange integral and expectation on the first term:

\[
\bar{h}(v, t) = \int_t^T \mathbb{E}^Q \left[ \left( (e_1)^{-\frac{1}{2\gamma}} \right) \exp \left\{ \int_t^\tau V(v_\tau, \tau) d\tau \right\} | v_t \right] d\bar{\tau} + \mathbb{E}^Q \left[ \exp \left\{ \int_t^T V(v_\tau, \tau) d\tau \right\} | v_t \right].
\]

Here, \( \tau(t) = T - t \), and \( g(v, \tau) \) can be rewritten as

\[
g(v, \tau) = \mathbb{E}^Q \left[ \exp \left\{ \int_t^T V(v_\tau, \tau) d\tau \right\} | v_t \right]
= \exp \left\{ \frac{\gamma}{1 - \gamma} \left( r - \frac{\delta}{\gamma} - \frac{\lambda^2_{1} ab}{\gamma - 1} - \frac{\lambda^2_{2} ab}{\gamma - 1} \right) (T - t) \right\} \times \mathbb{E}^Q \left[ \exp \left\{ - \mu \int_t^T v_\tau d\tau - \nu \int_t^T \frac{1}{v_\tau} d\tau \right\} | v_t \right].
\]

with parameters

\[
\alpha = 0,
\lambda = 0,
\mu = \frac{1}{2} \frac{\gamma}{1 - \gamma} \left( \frac{\lambda^2_{1} a}{\gamma - 1} + \frac{\lambda^2_{2} b}{\gamma - 1} \right) a^2,
\nu = \frac{1}{2} \frac{\gamma}{1 - \gamma} \left( \frac{\lambda^2_{1} a}{\gamma - 1} + \frac{\lambda^2_{2} b}{\gamma - 1} \right) b^2.
\]

Note that the conditional expectation in \( g(v, t) \) is taken under probability measure \( \mathbb{Q} \) such that \( v_t \) has drift \( \Gamma(v, t) \) in Equation (4.14) instead of \( \kappa (\theta - v) \). The Feller condition is assumed to be
satisfied by the new drift, hence we have:

\[
\Gamma(v,t) = \frac{-1 - \gamma + \gamma(\gamma - 1)^2}{\gamma - 1} \sigma(b\bar{v} + \kappa\theta + v)
\]

Furthermore, the function \( q(v,t) \) of (C.10) can be solved explicitly by Grasselli (2017)’s result for all \( \tau \geq 0 \) where \( \tau = T - t \):

\[
q(\tau, v; \alpha, \lambda, \mu, v) = \mathbb{E}^0\left[ \exp\left( -\mu \int_t^T v \, d\tau - v \int_t^T \frac{1}{v} \, d\tau \right) \mid v_t \right]
\]

\[
= \left( \frac{\beta(\tau,v)}{2} \right)^{\frac{m+1}{v}} (\lambda + K(\tau))^{-\frac{m}{2} + \alpha + \frac{\kappa \theta}{\sigma^2}}
\]

\[
\times e^{\frac{1}{\sigma^2}(\kappa \theta - m \alpha + \frac{\kappa \theta}{\sigma^2})} \left( \frac{1}{2} + \frac{m}{2} - \alpha + \frac{\kappa \theta}{\sigma^2}, m + 1, \frac{\beta(\tau,v)}{4(\lambda + K(\tau))} \right),
\]

with

\[
m = \frac{1}{\sigma^2} \sqrt{(2\kappa \theta - \sigma^2)^2 + 8\sigma^2 v},
\]

\[
D = \kappa^2 + 2\mu\sigma^2,
\]

\[
\beta(\tau,v) = \frac{2\sqrt{Dv}}{\sigma^2 \sinh\left( \frac{\sqrt{D} \tau}{2} \right)},
\]

\[
K(\tau) = \frac{1}{\sigma^2} \left( \sqrt{D} \coth\left( \frac{\sqrt{D} \tau}{2} \right) + \kappa \right),
\]
Further, if $\alpha, \lambda, \mu,$ and $\nu$ in Equation (C.11) satisfy following conditions,

\[
\mu > \frac{-\bar{k}^2}{2\sigma^2}, \\
\nu \geq -\frac{(2\bar{k}^2 - \sigma^2)^2}{8\sigma^2}, \\
\alpha < \frac{1}{2\sigma^2} \left( 2\bar{k}^2 + \sigma^2 \sqrt{2(2\bar{k}^2 - \sigma^2)^2 + 8\sigma^2\nu} \right), \\
\lambda \geq \frac{\sqrt{k^2 + 2\mu\sigma^2 + \bar{k}}}{\sigma^2}.
\]  

(C.15)

Moreover, the optimal wealth exposures and consumption-wealth ratio are given by

\[
\begin{pmatrix}
\dfrac{\xi}{\Theta} \\
\Phi
\end{pmatrix}^* = \begin{pmatrix}
\bar{h}^{-1}(\epsilon) \\
(\gamma-1) & (\gamma-1) \\
\end{pmatrix} = \begin{pmatrix}
\frac{\sigma \nu \bar{v}}{(a \sqrt{\nu} + \frac{b}{\sqrt{\nu}})} & \frac{h}{\nu} - \frac{\lambda}{\gamma-1} \\
\frac{\sigma \nu \bar{v}}{(a \sqrt{\nu} + \frac{b}{\sqrt{\nu}})} & \frac{h}{\nu} - \frac{\lambda}{\gamma-1} \\
\end{pmatrix}. 
\]  

(C.16)

\[\square\]

### C.2 Proof of Proposition 4.3.2 (Incomplete Market, no Robustness, no Consumption)

Proof. Solve for $\pi^*$,

\[
xJ_x\left[ \rho \lambda_1(a \sqrt{\nu} + \frac{b}{\sqrt{\nu}})^2 + \sqrt{1 - \rho^2 \lambda_2(a \sqrt{\nu} + \frac{b}{\sqrt{\nu}})^2} \right] + \nu x^2 \left( a \sqrt{\nu} + \frac{b}{\sqrt{\nu}} \right)^2 \pi J_{xx} \\
+ \sigma x \nu (av + b) J_{yv} = 0.
\]

(C.17)

It follows that

\[
\pi^* = \frac{xJ_x\left[ \rho \lambda_1(a \sqrt{\nu} + \frac{b}{\sqrt{\nu}})^2 + \sqrt{1 - \rho^2 \lambda_2(a \sqrt{\nu} + \frac{b}{\sqrt{\nu}})^2} \right] + \sigma x \nu (av + b) J_{yv}}{-\nu x^2 \left( a \sqrt{\nu} + \frac{b}{\sqrt{\nu}} \right)^2 J_{xx}}
\]

(C.18)

In the case of no intermediate consumption, we conjecture our value function as follows

\[J(x, v, t) = \frac{x^\gamma}{\gamma} h(t, v),\]

(C.19)

where $h(T, v) = 1$ for all $v$. Thereby, we have following partial derivatives for $J(x, v, t)$:

\[J_x = \frac{x^\gamma}{\gamma} h_x, \quad J_y = \frac{x^\gamma}{\gamma} h_v, \quad J_{xx} = x^{\gamma-1} h, \quad J_{yy} = \frac{x^\gamma}{\gamma} h_{vv}, \quad J_{xy} = x^{\gamma-1} h_v, \quad J_{xx} = (\gamma - 1)x^{\gamma-2} h.\]
C.2. **Proof of Proposition 4.3.2 (Incomplete Market, no Robustness, no Consumption)**

Substituting these partial derivatives into the optimal allocation \( \pi^* \) from Equation (C.18),

\[
(\pi)^* = \frac{x^\gamma \left[ \rho \lambda_1 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 + \sqrt{1-\rho^2} \lambda_2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 \right] + \sigma_p (av + b)x^\gamma h_v}{ -(\gamma - 1) \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 x^\gamma h_v} - \frac{\rho \lambda_1 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 + \sqrt{1-\rho^2} \lambda_2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2}{ -(\gamma - 1) \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2} .
\]

(C.20)

And then substituting the partial derivatives and the optimal allocation above into Equation (4.25) to eliminate “sup” and further simplify leads to

\[
h_t + \left[ ry - \frac{\gamma}{2} \left( \rho \lambda_1 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 + \sqrt{1-\rho^2} \lambda_2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 \right) \right] h + \left[ -\frac{\gamma}{2} \sigma_p^2 v \right] h_v^2 + \left[ \kappa(\theta - v) - \gamma \frac{\rho \lambda_1 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2 + \sqrt{1-\rho^2} \lambda_2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2}{(\gamma - 1)} \sigma_p \sqrt{v} \right] h_v + \frac{1}{2} \sigma^2 v h_{vv} = 0.
\]

(C.21)

It can be seen that if we have

\[
\rho = 0,
\]

(C.22)

the PDE above becomes linear:

\[
h_t + V(v, t)h + \Gamma(v, t)h_v + \frac{1}{2} \sigma^2 v h_{vv} = 0,
\]

(C.23)

where \( V(v, t) \) and \( \Gamma(v, t) \) are defined in Equation (4.28). Further, the coefficients satisfy the conditions of Heath and Schweizer (2000) as per the previous proposition, and \( h \) admits the Feynman-Kac representation:

\[
h(v, t) = \mathbb{E}[0] \left[ \exp \left\{ \int_t^T V(v, \tau) d\tau \right\} h(v, T) \mid v \right] \\
= \mathbb{E}[0] \left[ \exp \left\{ \left( \int_T^T \frac{\gamma}{\gamma - 1}(\rho \lambda_1 \psi + \lambda_2 \sqrt{1-\rho^2}) a\psi \right)(T-t) \right\} \times \mathbb{E}[0] \left[ \exp \left\{ -\mu \int_T^T v \psi d\tau - \psi \int_T^T \frac{1}{v} d\tau \right\} \mid v \right] \right]_{\psi = q(T-t, v \psi, \lambda_1 \psi, \lambda_2, \mu)}
\]

(C.24)

with parameters

\[
\alpha = 0, \\
\lambda = 0, \\
\mu = \frac{1}{2} \frac{\gamma}{\gamma - 1}(\rho \lambda_1 \psi + \sqrt{1-\rho^2} \lambda_2) a^2, \\
\psi = \frac{1}{2} \frac{\gamma}{\gamma - 1}(\rho \lambda_1 \psi + \sqrt{1-\rho^2} \lambda_2)^2 b^2.
\]

(C.25)
Note that the conditional expectation is taken under the probability measure $Q$ such that $v_t$ has drift $\Gamma(v, t)$ in Equation (C.23) instead of $\kappa(\theta - v)$. The Feller condition is assumed to be satisfied by the new drift:

$$\begin{align*}
\Gamma(v, t) &= \kappa(\theta - v) - \gamma \frac{\left(\rho \lambda_1 (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) + \sqrt{1 - \rho^2 \lambda_2 (a \sqrt{v} + \frac{b}{\sqrt{v}})}\right)}{(\gamma - 1)} \sigma \rho \sqrt{v} \\
&= \kappa(\theta - v) - \gamma \frac{\left(\rho \lambda_1 + \sqrt{1 - \rho^2 \lambda_2}\right) \sigma \rho (av + b)}{\gamma - 1} \\
&= \left(\kappa \theta - \gamma \frac{\left(\rho \lambda_1 + \sqrt{1 - \rho^2 \lambda_2}\right) \sigma \rho \beta}{\gamma - 1}\right) - \left(\kappa + \gamma \frac{\left(\rho \lambda_1 + \sqrt{1 - \rho^2 \lambda_2}\right) \sigma \rho}{\gamma - 1}\right) v \\
&\implies \kappa \theta - \gamma \frac{\left(\rho \lambda_1 + \sqrt{1 - \rho^2 \lambda_2}\right) \sigma \rho \beta}{\gamma - 1} \geq \frac{\sigma^2}{2}
\end{align*}$$

(C.26)

Further, if $\alpha$, $\lambda$, $\mu$, and $\nu$ in Equation (C.25) satisfy the conditions in Equation (C.15), then Equation of (4.32) can be solved explicitly by Grasselli (2017)'s result like Equation (4.20) with associated $m$, $D$, $\beta$, and $K$ like Equation (4.21). Note that the last two conditions for $\alpha = \lambda = 0$ are satisfied directly. Thus, the dependence of the function $q(\cdot)$ on $\alpha$ and $\lambda$ can be omitted.

Moreover, the optimal strategy with $\rho = 0$ is given by

$$(\pi)^* = \frac{-\sqrt{v} \sigma \rho h_v - \left(\rho \lambda_1 + \sqrt{1 - \rho^2 \lambda_2}\right) \left(a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}\right) h}{(a \sqrt{v} + \frac{b}{\sqrt{v}})(\gamma - 1) h}$$

$$= \frac{-\lambda_2 (a \sqrt{v_t} + \frac{b}{\sqrt{v_t}}) h}{(a \sqrt{v} + \frac{b}{\sqrt{v}})(\gamma - 1) h}$$

$$= \frac{-\lambda_2}{\gamma - 1}$$

(C.27)

\[\hfill \square\]

C.3 Proof of Proposition 4.4.1 (Complete Market, Robustness, Consumption)

Proof. Solving the minimization problem in (4.42) first, we obtain:

$$\begin{align*}
\left\{ \frac{c}{\phi_1^v} = \Theta^v \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x \bar{J}_x + \sigma \sqrt{v} \bar{J}_v \right. \\
\left. \frac{c}{\phi_2} = \Theta^\delta \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x \bar{J}_x \right\} \implies \left\{ \left( e^v \right)^* = \Phi_1 \left[ \Theta^v \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x \bar{J}_x + \sigma \sqrt{v} \bar{J}_v \right] \\
\left( e^\delta \right)^* = \Phi_2 \left[ \Theta^\delta \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) x \bar{J}_x \right] \right. \\
\left. \right\} \quad (C.28)
\end{align*}$$
C.3. Proof of Proposition 4.4.1 (Complete Market, Robustness, Consumption)

Substituting the values of \((c^*)^*\) and \((\xi^*)^*\) from Equation (C.28) into the robust HJB equation, i.e., Equation (4.42), we obtain the following equation that the function \(\bar{J}\) has to satisfy:

\[
\sup_{\theta^v, \theta^r, c} \left\{ u(c) - \delta \bar{J} + \bar{J}_x + \left[ x r + \Theta^r \lambda_1(a \sqrt{\bar{V}} + b \sqrt{V})^2 - \Theta^r \left( a \sqrt{\bar{V}} + b \sqrt{V} \right) \Phi_1 \left[ \Theta^r \left( a \sqrt{\bar{V}} + b \sqrt{V} \right) x \bar{J}_x + \sigma \sqrt{\bar{V}} \bar{J}_v \right] \right. \\
+ \Theta^r \lambda_2(a \sqrt{\bar{V}} + b \sqrt{V})^2 \left. - \Theta^r \left( a \sqrt{\bar{V}} + b \sqrt{V} \right) \Phi_2 \left[ \Theta^r \left( a \sqrt{\bar{V}} + b \sqrt{V} \right) x \bar{J}_x + \sigma \sqrt{\bar{V}} \bar{J}_v \right] \right] \right\} \\
+ \frac{1}{2} x^2 \left( a \sqrt{\bar{V}} + b \sqrt{V} \right) \left[ (\Theta^r)^2 + (\Theta^r)^2 \right] \bar{J}_{xx} \right. \\
+ \left. \left[ \kappa (\theta - v) - \sigma \sqrt{\bar{V}} \Phi_1 \left[ \Theta^r \left( a \sqrt{\bar{V}} + b \sqrt{V} \right) x \bar{J}_x + \sigma \sqrt{\bar{V}} \bar{J}_v \right] \right] \bar{J}_x \right) \\
+ \left. \frac{1}{2} \sigma^2 \nu \bar{J}_{vv} + \sigma x (av + b) \Theta^r \bar{J}_{vx} + \frac{\left( \Phi_1 \left[ \Theta^r \left( a \sqrt{\bar{V}} + b \sqrt{V} \right) x \bar{J}_x + \sigma \sqrt{\bar{V}} \bar{J}_v \right] \right)^2}{2 \Phi_1} + \frac{\left( \Phi_2 \left[ \Theta^r \left( a \sqrt{\bar{V}} + b \sqrt{V} \right) x \bar{J}_x + \sigma \sqrt{\bar{V}} \bar{J}_v \right] \right)^2}{2 \Phi_2} \right) \\
= 0
\]

(C.29)

Canceling and recombining terms:

\[
\sup_{\theta^v, \theta^r, c} \left\{ u(c) - \delta \bar{J} + \bar{J}_x + x \left[ r + \Theta^r \lambda_1(a \sqrt{\bar{V}} + b \sqrt{V})^2 + \Theta^r \lambda_2(a \sqrt{\bar{V}} + b \sqrt{V})^2 \right] \bar{J}_x - c \bar{J}_x \right) \\
+ \frac{1}{2} x^2 \left( a \sqrt{\bar{V}} + b \sqrt{V} \right) \left[ (\Theta^r)^2 + (\Theta^r)^2 \right] \bar{J}_{xx} + \kappa (\theta - v) \bar{J}_x + \frac{1}{2} \sigma^2 \nu \bar{J}_{vv} + \sigma x (av + b) \Theta^r \bar{J}_{vx} \\
- \frac{1}{2} \Phi_1 \left[ \Theta^r \left( a \sqrt{\bar{V}} + b \sqrt{V} \right) x \bar{J}_x + \sigma \sqrt{\bar{V}} \bar{J}_v \right] \bar{J}_{xx} + \sigma^2 \nu \bar{J}_{xx} \right) + 2 \Theta^r (av + b) \sigma x \bar{J}_x \bar{J}_v \right) \\
- \frac{1}{2} \Phi_2 \left[ \Theta^r \left( a \sqrt{\bar{V}} + b \sqrt{V} \right) x \bar{J}_x + \sigma \sqrt{\bar{V}} \bar{J}_v \right] \bar{J}_{xx} \right) \right) \right] = 0
\]

(C.30)

Solving the maximization problem for intermediate consumption:

\[
0 = u'(c) - \bar{J}_x \\
= \epsilon_1 c^{\gamma - 1} - \bar{J}_x
\]

(C.31)

That is,

\[
c^* = (\bar{J}_x)^{\frac{1}{\gamma}}(\epsilon_1)^{-\frac{1}{\gamma}},
\]

where the risk aversion parameter satisfies \(\gamma < 1\).

Solving the maximization problem for wealth exposures:

\[
\begin{align*}
0 &= x \lambda_1(a \sqrt{\bar{V}} + b \sqrt{V})^2 \bar{J}_x + x^2 \left( a \sqrt{\bar{V}} + b \sqrt{V} \right)^2 \Theta^r \bar{J}_{xx} + \sigma x (av + b) \bar{J}_{vx} - \Phi_1 \Theta^r \left( a \sqrt{\bar{V}} + b \sqrt{V} \right)^2 \bar{J}_x \\
- \Phi_1 (av + b) \sigma x \bar{J}_x \bar{J}_v \\
0 &= x \lambda_2(a \sqrt{\bar{V}} + b \sqrt{V})^2 \bar{J}_x + x^2 \left( a \sqrt{\bar{V}} + b \sqrt{V} \right)^2 \Theta^r \bar{J}_{xx} - \Phi_2 \Theta^r \left( a \sqrt{\bar{V}} + b \sqrt{V} \right)^2 \bar{J}_x \end{align*}
\]

(C.33)
That is,
\[
\begin{align*}
(\Theta^r)^* & = \frac{\Phi_1 (av + b)(x_p - x_l(\alpha \sqrt{\gamma} + \frac{\alpha}{\gamma})^2 J_l - \sigma \varepsilon J_w)}{x^2 (a \sqrt{\gamma} + \frac{\alpha}{\gamma}) J_l - \Phi_1 (av + b)^2 J_w} \\
(\Theta^v)^* & = \frac{-\Phi_2 (av + b)(x_p - x_l(\alpha \sqrt{\gamma} + \frac{\alpha}{\gamma})^2 J_l - \sigma \varepsilon J_w)}{x^2 (a \sqrt{\gamma} + \frac{\alpha}{\gamma}) J_l - \Phi_2 (av + b)^2 J_w}
\end{align*}
\]
(C.34)

where \( \Phi_1 = \frac{\Phi_1}{\gamma^2} \), and \( \Phi_2 = \frac{\Phi_2}{\gamma^2} \) by following Maenhout (2004).

We conjecture the following representation of the value function:
\[
\tilde{J}(x, v, t) = \frac{x^\gamma}{\gamma} (\tilde{h}(t, v))^{1-\gamma},
\]
(C.35)
where \( \tilde{h}(T, v) = 1 \) for all \( v \). Thereby, we have following partial derivatives for \( \tilde{J}(x, v, t) \):
\[
\begin{align*}
\tilde{J}_t & = \frac{1 - \gamma}{\gamma} x^\gamma \tilde{h}^{1-\gamma} \tilde{h} \\
\tilde{J}_v & = \frac{x^\gamma}{\gamma} (1 - \gamma) \tilde{h}^{1-\gamma} \tilde{h} \\
\tilde{J}_x & = x^{\gamma-1} \tilde{h}^{1-\gamma} \\
\tilde{J}_{vv} & = -(1 - \gamma) x^\gamma \tilde{h}^{\gamma-1} \tilde{h}^2 + \frac{x^\gamma}{\gamma} (1 - \gamma) \tilde{h}^{1-\gamma} \tilde{h} \\
\tilde{J}_{vx} & = (1 - \gamma) x^{\gamma-1} \tilde{h}^{1-\gamma} \tilde{h} \\
\tilde{J}_{xx} & = (\gamma - 1) x^{\gamma-2} \tilde{h}^{1-\gamma} 
\end{align*}
\]

Substituting these partial derivatives into the candidates for the optimal consumption wealth ratio from Equation (C.32) and wealth exposures in Equation (C.34), we have
\[
\begin{align*}
\left( \frac{\varepsilon}{\xi} \right)^* & = \tilde{h}^{-1} (\varepsilon_1)^{-1} \\
(\Theta^r)^* & = \frac{(1-\gamma)(\frac{\gamma}{\gamma} - 1)\varepsilon \tilde{h} - 3(\varepsilon \sqrt{\gamma} + \frac{\varepsilon}{\gamma})}{\gamma (\gamma - 1 - \phi)} \\
(\Theta^v)^* & = -\frac{\gamma - 1 - \phi}{(\gamma - 1 - \phi)^2}
\end{align*}
\]
(C.36)

Next, we substitute the above expressions for \( c^* \), \( (\Theta^r)^* \), and \( (\Theta^v)^* \) into Equation (C.30) to
C.3. Proof of Proposition 4.4.1 (Complete Market, Robustness, Consumption)

eliminate “sup”:

\[ \frac{x^y h^{-\gamma}(\varepsilon_i)}{x^y h^{-\gamma} + 1 - \frac{\gamma}{y} x^y h^{-\gamma} h_i} \]

\[ + x^y \left[ r h^{-\gamma} + \frac{(1 - \gamma)(\frac{\phi_1}{\gamma} - 1)\sigma \bar{h}_1 (av + b)}{\gamma - 1 - \phi_1} \bar{h}_v h^{-\gamma} - \frac{\bar{h}_1^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2}{\gamma - 1 - \phi_1} \bar{h}_v h^{-\gamma} - \frac{\bar{h}_1^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2}{\gamma - 1 - \phi_2} \bar{h}_v h^{-\gamma} \right] \]

\[ - x^y \bar{h}_v (\varepsilon_i) \frac{1}{\gamma} + \frac{1}{2} (\gamma - 1) x^y \left[ (1 - \gamma)^2 \frac{\phi_1}{\gamma} (1 - 1)^2 \sigma^2 v (1 - \gamma)x^y \bar{h}_v^{-\gamma} h_v^2 + \frac{1}{2} \sigma^2 v \frac{1}{\gamma} (1 - \gamma) \bar{h}_v^{-\gamma} h_v \right] \]

\[ + 2(1 - \gamma)(\frac{\phi_1}{\gamma} - 1)\sigma (av + b) \bar{h}_v h^{-\gamma} h_v + \frac{\bar{h}_1^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2}{(\gamma - 1 - \phi_2)^2} \bar{h}_v h^{-\gamma} \]

\[ + \kappa(\theta - v) \frac{x^y}{\gamma} (1 - \gamma) \bar{h}_v^{-\gamma} h_v - \frac{1}{2} \sigma^2 v (1 - \gamma) x^y \bar{h}_v^{-\gamma} h_v + \frac{1}{2} \sigma^2 v \frac{x^y}{\gamma} (1 - \gamma) \bar{h}_v^{-\gamma} h_v \]

\[ + \sigma (1 - \gamma) x^y \left[ \frac{(1 - \gamma)(\frac{\phi_1}{\gamma} - 1)\sigma v}{\gamma - 1 - \phi_1} h_v^{-\gamma} h_v^2 - \frac{\bar{h}_1 (av + b)}{\gamma - 1 - \phi_1} \bar{h}_v h^{-\gamma} h_v \right] \]

\[ - \frac{\phi_1}{2} x^y \left[ \frac{(1 - \gamma)^2 (\frac{\phi_1}{\gamma} - 1)^2 \sigma^2 v}{(\gamma - 1 - \phi_1)^2} h_v^{-\gamma} h_v^2 + \frac{\bar{h}_1^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2}{(\gamma - 1 - \phi_1)^2} h_v^{-\gamma} h_v \right] \]

\[ + \sigma^2 v \frac{x^y}{\gamma^2} (1 - \gamma)^2 h_v^{-\gamma} h_v^2 + 2\sigma \frac{1 - \gamma}{\gamma} x^y \left[ \frac{(1 - \gamma)(\frac{\phi_1}{\gamma} - 1)\sigma v}{\gamma - 1 - \phi_1} h_v^{-\gamma} h_v^2 - \frac{\bar{h}_1 (av + b)}{\gamma - 1 - \phi_1} \bar{h}_v h^{-\gamma} h_v \right] \]

\[ - \frac{\phi_2}{2} \frac{\bar{h}_1^2 (a \sqrt{v} + \frac{b}{\sqrt{v}})^2}{(\gamma - 1 - \phi_2)^2} x^y h_v^{-\gamma} = 0. \]

(C.37)
Dividing by term $\frac{\delta}{\gamma} \tilde{h}^{-\gamma}$ leads to

\begin{equation}
(e_1)^{-\frac{1}{\gamma} - \delta \tilde{h}} + (1 - \gamma)\tilde{h}_t + \gamma [(1 - \gamma)\left(\frac{\phi_1}{\gamma} - 1\right)\sigma \lambda_1 (av + b)\tilde{h}_v - \frac{\lambda_1^2 (a \sqrt{v} + \frac{h}{\sqrt{v}})^2}{\gamma - 1 - \phi_1} \tilde{h} - \frac{\lambda_2^2 (a \sqrt{v} + \frac{h}{\sqrt{v}})^2}{\gamma - 1 - \phi_2} \tilde{h}]
- (1 - \gamma)\left(\frac{\phi_1}{\gamma} - 1\right)\sigma \lambda_1 (av + b)\tilde{h}_v - \frac{\lambda_2^2 (a \sqrt{v} + \frac{h}{\sqrt{v}})^2}{(\gamma - 1 - \phi_2)^2} \tilde{h}
= 0.
\end{equation}

(C.38)

Regrouping the above equation, we have

\begin{equation}
(1 - \gamma)(e_1)^{-\frac{1}{\gamma}} + (1 - \gamma)\tilde{h}_t + \gamma [(1 - \gamma)\left(\frac{\phi_1}{\gamma} - 1\right)\sigma \lambda_1 (av + b)\tilde{h}_v - \frac{\lambda_1^2 (a \sqrt{v} + \frac{h}{\sqrt{v}})^2}{\gamma - 1 - \phi_1} \frac{h}{\gamma} + \frac{\lambda_2^2 (a \sqrt{v} + \frac{h}{\sqrt{v}})^2}{(\gamma - 1 - \phi_2)^2} \frac{h}{\gamma}]
+ \frac{\phi_1 \lambda_1^2 (a \sqrt{v} + \frac{h}{\sqrt{v}})^2}{2 (\gamma - 1 - \phi_1)^2} \frac{h}{\gamma} + \frac{\phi_2 \lambda_2^2 (a \sqrt{v} + \frac{h}{\sqrt{v}})^2}{2 (\gamma - 1 - \phi_2)^2} \frac{h}{\gamma}
+ \gamma [(1 - \gamma)\left(\frac{\phi_1}{\gamma} - 1\right)\sigma \lambda_1 (av + b)\tilde{h}_v - \frac{\lambda_1^2 (a \sqrt{v} + \frac{h}{\sqrt{v}})^2}{\gamma - 1 - \phi_1} \frac{h}{\gamma} + \frac{\lambda_2^2 (a \sqrt{v} + \frac{h}{\sqrt{v}})^2}{(\gamma - 1 - \phi_2)^2} \frac{h}{\gamma}]
- \gamma [(1 - \gamma)\left(\frac{\phi_1}{\gamma} - 1\right)\sigma \lambda_1 (av + b)\tilde{h}_v - \frac{\lambda_1^2 (a \sqrt{v} + \frac{h}{\sqrt{v}})^2}{\gamma - 1 - \phi_1} \frac{h}{\gamma} + \frac{\lambda_2^2 (a \sqrt{v} + \frac{h}{\sqrt{v}})^2}{(\gamma - 1 - \phi_2)^2} \frac{h}{\gamma}]
= 0.
\end{equation}

(C.39)
C.3. Proof of Proposition 4.4.1 (Complete Market, Robustness, Consumption)

Simplifying leads to:

\[(1 - \gamma)(\epsilon_1)^{-\frac{1}{\gamma t}} + (1 - \gamma)\bar{h}_t + \left[ r - \frac{\delta}{\gamma} - \frac{\lambda_1^2(a \sqrt{v} + \frac{\rho}{\sqrt{v}})^2}{2} - \frac{\lambda_2^2(a \sqrt{v} + \frac{\rho}{\sqrt{v}})^2}{2} \right] \gamma \bar{h} + \left[ \frac{\phi_1}{\gamma} - 1 - \gamma + \phi_1 \right] \sigma(\bar{v} + b)\lambda_1 + \kappa(\theta - v) \]

\[+ \phi_1 \left( \frac{\phi_1}{\gamma} - 1 \right) \sigma(\bar{v} + b)\lambda_1 \gamma - (\gamma - 1)(\phi_1 - 1)\sigma(\bar{v} + b)\lambda_1 \gamma + \kappa(\theta - v) \]

\[+ \frac{1}{2}(\gamma - \phi_1) \frac{(\phi_1 - 1)\sigma^2\lambda_1}{(y_1 - \phi_1)^2} \left[ (1 - \gamma)\bar{h}_v + \left[ \frac{1}{2}\sigma^2\gamma(1 - \gamma) \right] \bar{h}_v \right] = 0. \tag{C.40} \]

In order to find a solution we need to eliminate the term \((\bar{h}_v)^2\), this means:

\[\frac{1}{2}(\gamma - \phi_1) \frac{(\phi_1 - 1)\sigma^2\lambda_1}{(y_1 - \phi_1)^2} \left[ \frac{\gamma}{1 - \gamma} + \frac{\phi_1}{\gamma} \right] = 0, \tag{C.41} \]

Then we have a linear PDE,

\[\left( \epsilon_1 \right)^{-\frac{1}{\gamma t}} + \bar{h}_t + \left[ r - \frac{\delta}{\gamma} - \frac{\lambda_1^2(a \sqrt{v} + \frac{\rho}{\sqrt{v}})^2}{2} - \frac{\lambda_2^2(a \sqrt{v} + \frac{\rho}{\sqrt{v}})^2}{2} \right] \gamma \bar{h} + \left[ \frac{\phi_1}{\gamma} - 1 - \gamma + \phi_1 \right] \sigma(\bar{v} + b)\lambda_1 + \kappa(\theta - v) + \phi_1 \left( \frac{\phi_1}{\gamma} - 1 \right) \sigma(\bar{v} + b)\lambda_1 \]

\[\right) \bar{h}_v = 0. \tag{C.42} \]

Further, in order to apply the Feynman-Kac formula, we divide both sides of the equation by \((1 - \gamma)\) so that the coefficient of \(\bar{h}_t\) is 1:

\[\left( \epsilon_1 \right)^{-\frac{1}{\gamma t}} + \bar{h}_t + V(v, t)\bar{h} + \Gamma(v, t)\bar{h}_v + \frac{1}{2}\sigma^2\gamma\bar{h}_v = 0. \tag{C.43} \]

This is an application of the Feynman-Kac formula, therefore the coefficients in (4.46) must satisfy the conditions of Theorem 1 and Lemmas 2 and 3 in Heath and Schweizer (2000).

In the notation of Heath and Schweizer (2000) we have: \(X = v\), \(D = (0, \infty)\), \(b(t, v) = \Gamma(v, t)\), \(\Sigma(t, v) = \sigma \sqrt{v}, c(t, v) = V(t, v)\), \(g(t, v) = (\epsilon_1)^{-\frac{1}{\gamma t}}\), \(\hat{h}(t, x) = \hat{h}(t, v)\), \(u(t, v) = \hat{h}(t, v), u(T, v) = \hat{h}(T, v) = 1\) and \(a(t, x) = \sigma^2\gamma\). The process \(v\) should follow the SDE: \(dv_t = \Gamma(v, t)dt + \sigma \sqrt{v}dZ_{1,t}\).
Using the same arguments as in their section 2.1 (an application to the Heston model), we can conclude that $\hat{h}$ admits the Feynman-Kac representation:

\[
\hat{h}(v, t) = \mathbb{E}^Q \left[ \int_t^T \exp \left\{ \int_t^\tau V(v, \tau) \, d\tau \right\} \left( (\epsilon_1)^{-1} \right) d\tau + \exp \left\{ \int_t^T V(v, \tau) \, d\tau \right\} h(v, T) | v_t \right].
\]

Moreover, given that $V(v, t): \mathbb{R}^+ \times [0, T] \rightarrow \mathbb{R}$ is a measurable function, and $\exp \left\{ \int_t^T V(v, \tau) \, d\tau \right\} \geq 0$ then we can apply Tonelli’s Theorem to the first term:

\[
\hat{h}(v, t) = \int_t^T \mathbb{E}^Q \left[ \left( (\epsilon_1)^{-1} \right) \exp \left\{ \int_t^\tau V(v, \tau) \, d\tau \right\} \right] d\tau + \mathbb{E}^Q \left[ \exp \left\{ \int_t^T V(v, \tau) \, d\tau \right\} \right] v_t.
\]

(C.45)

Here, $\tau(t) = T - t$ and $g(v, \tau)$ can be rewritten as

\[
g(v, \tau) = \mathbb{E}^Q \left[ \exp \left\{ \int_t^T V(v, \tau) \, d\tau \right\} \right] v_t
\]

\[
= \exp \left\{ \frac{\gamma}{1 - \gamma} (r - \frac{\lambda_0}{\gamma - 1 - \phi_1} - \frac{\lambda_0}{\gamma - 1 - \phi_2}) (T - t) \right\} \times \mathbb{E}^Q \left[ \exp \left\{ -\mu \int_t^T v_t \, d\tau - \nu \int_t^T \frac{1}{v_t} \, d\tau \right\} \right] v_t.
\]

(C.46)

with parameters

\[
\alpha = 0,
\lambda = 0,
\mu = \frac{1}{2} \frac{\gamma}{1 - \gamma} \left( \frac{\lambda_0^2}{\gamma - 1 - \phi_1} + \frac{\lambda_0^2}{\gamma - 1 - \phi_2} \right) \sigma^2,
\nu = \frac{1}{2} \frac{\gamma}{1 - \gamma} \left( \frac{\lambda_0^2}{\gamma - 1 - \phi_1} + \frac{\lambda_0^2}{\gamma - 1 - \phi_2} \right) \beta^2.
\]

(C.47)

Note that the conditional expectation in $g(v, t)$ is taken under the probability measure $\mathbb{Q}$ such that $v_t$ has drift $\Gamma(v, t)$ in Equation (4.46) instead of $\kappa(\theta - v)$. The Feller condition is assumed.
C.4 Proof of Corollary 4.4.2 (Complete Market, Robustness, No Consumption)

Proof. In the case of no intermediate consumption, we conjecture our value function follows

$$\bar{J}(x, v, t) = \frac{x^\gamma}{\gamma} \bar{h}(t, v),$$

(C.51)

where $\bar{h}(T, v) = 1$ for all $v$. Thereby, we have following partial derivatives for $\bar{J}(x, v, t)$:

$$\bar{J}_x = \frac{x^\gamma}{\gamma} \bar{h}_t, \quad \bar{J}_v = \frac{x^\gamma}{\gamma} \bar{h}_t, \quad \bar{J}_s = x^{\gamma-1} \bar{h}, \quad \bar{J}_y = \frac{x^\gamma}{\gamma} \bar{h}, \quad \bar{J}_vy = x^{\gamma-1} \bar{h}, \quad \bar{J}_sx = (\gamma - 1)x^{\gamma-2} \bar{h}.$$
Substituting these partial derivatives into the optimal exposures \((\Theta')^*, (\Theta^v)^*\) from Equation (C.34), and then into Equation (C.30) to eliminate “\(sup\)’’:

\[
\begin{align*}
(\Theta')^* &= \left(\frac{\l_1 - 1}{\gamma - 1 - \phi_1}\right) \frac{\sigma \sqrt{h_v} - \l_1 (a \sqrt{v} + \frac{b}{\sqrt{v}}) \bar{h}}{(a \sqrt{v} + \frac{b}{\sqrt{v}})}, \\
(\Theta^v)^* &= \left(\frac{-\l_2}{\gamma - 1 - \phi_2}\right)
\end{align*}
\]  
(C.52)

\[
\frac{x^\gamma}{\gamma} \frac{\partial}{\partial h_v} + \left[ r + \left( \frac{\Theta}{\gamma - 1 - \phi_1} \right) \frac{\sigma \sqrt{h_v} - \l_1 (a \sqrt{v} + \frac{b}{\sqrt{v}}) \bar{h}}{(a \sqrt{v} + \frac{b}{\sqrt{v}})} \right] \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \\
+ \left( \frac{-\l_2}{\gamma - 1 - \phi_2} \right) \frac{\Theta (a \sqrt{v} + \frac{b}{\sqrt{v}})^2}{(a \sqrt{v} + \frac{b}{\sqrt{v}})} \frac{1}{\gamma} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \\
+ \frac{1}{2} \frac{x^\gamma}{\gamma} \frac{\partial^2}{\partial h_v^2} + \frac{1}{2} \frac{\sigma^2 \sqrt{v} \frac{x^\gamma}{\gamma} \frac{h_v}{v} + \sigma x (av + b) \left( \frac{\Theta}{\gamma - 1 - \phi_1} \right) \frac{\sigma \sqrt{h_v} - \l_1 (a \sqrt{v} + \frac{b}{\sqrt{v}}) \bar{h}}{(a \sqrt{v} + \frac{b}{\sqrt{v}})} \right) \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \\
+ \frac{1}{2} \Phi_1 \left( \frac{\Theta}{\gamma - 1 - \phi_1} \right) \frac{\sigma \sqrt{h_v} - \l_1 (a \sqrt{v} + \frac{b}{\sqrt{v}}) \bar{h}}{(a \sqrt{v} + \frac{b}{\sqrt{v}})} \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) \left( \frac{x^\gamma}{\gamma} \frac{\partial}{\partial h_v} \right)^2 \\
+ \frac{1}{2} \Phi_2 \left( \frac{\Theta}{\gamma - 1 - \phi_2} \right) \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right) \left( \frac{x^\gamma}{\gamma} \frac{\partial}{\partial h_v} \right)^2 = 0.
\]  
(C.53)
Simplifying the equation above and substituting $\Phi_1 = \frac{\phi_1}{\gamma + h}$ and $\Phi_2 = \frac{\phi_2}{\gamma + h}$:

\[
\frac{x^y}{\gamma} \tilde{h}_t + \left[ r + \left( \frac{\phi_1}{\gamma} - 1 \right) \sigma \sqrt{\tilde{h}_t} - \bar{\lambda}_1(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}}) \tilde{h}_t \right] + \frac{\bar{\lambda}_2(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2}{(\gamma - 1 - \phi_1) \tilde{h}_t} + \left( \frac{-\bar{\lambda}_2}{(\gamma - 1 - \phi_2) \tilde{h}_t} \right)^2 \left( \gamma - 1 \right) x^y \tilde{h}
\]
\[
+ \frac{1}{2} \left( a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}} \right) ^2 \left[ \left( \frac{\phi_1}{\gamma} - 1 \right) \sigma \sqrt{\tilde{h}_t} - \bar{\lambda}_1(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}}) \tilde{h}_t \right] ^2 - \frac{\bar{\lambda}_2(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2}{(\gamma - 1 - \phi_1) \tilde{h}_t} + \left( \frac{-\bar{\lambda}_2}{(\gamma - 1 - \phi_2) \tilde{h}_t} \right)^2 \left( \gamma - 1 \right) x^y \tilde{h}
\]
\[
+ \kappa(\theta - v) \frac{x^y}{\gamma} \tilde{h}_v + \frac{1}{2} \sigma^2 v \frac{x^y}{\gamma} \tilde{h}_v v + \sigma (a v + b) \left( \frac{\phi_1}{\gamma} - 1 \right) \sigma \sqrt{\tilde{h}_t} - \bar{\lambda}_1(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}}) \tilde{h}_t \right] ^2 - \frac{\bar{\lambda}_2(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2}{(\gamma - 1 - \phi_1) \tilde{h}_t} + \left( \frac{-\bar{\lambda}_2}{(\gamma - 1 - \phi_2) \tilde{h}_t} \right)^2 \left( \gamma - 1 \right) x^y \tilde{h}
\]
\[
+ \frac{1}{2} \left( a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}} \right) ^2 \left( \frac{\phi_1}{\gamma} - 1 \right) \sigma \sqrt{\tilde{h}_t} - \bar{\lambda}_1(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}}) \tilde{h}_t \right] ^2 - \frac{\bar{\lambda}_2(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2}{(\gamma - 1 - \phi_1) \tilde{h}_t} + \left( \frac{-\bar{\lambda}_2}{(\gamma - 1 - \phi_2) \tilde{h}_t} \right)^2 \left( \gamma - 1 \right) x^y \tilde{h}
\]
\[
= 0.
\]

(C.54)

Furthermore,

\[
\frac{x^y}{\gamma} \tilde{h}_t + \left[ r + \left( \frac{\phi_1}{\gamma} - 1 \right) \sigma \sqrt{\tilde{h}_t} - \bar{\lambda}_1(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}}) \tilde{h}_t \right] + \frac{\bar{\lambda}_2(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2}{(\gamma - 1 - \phi_1) \tilde{h}_t} - \frac{\bar{\lambda}_2(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2}{(\gamma - 1 - \phi_2) \tilde{h}_t} \left( \gamma - 1 \right) x^y \tilde{h}
\]
\[
+ \frac{1}{2} \left( a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}} \right) ^2 \left[ \left( \frac{\phi_1}{\gamma} - 1 \right) \sigma \sqrt{\tilde{h}_t} - \bar{\lambda}_1(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}}) \tilde{h}_t \right] ^2 - \frac{\bar{\lambda}_2(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2}{(\gamma - 1 - \phi_1) \tilde{h}_t} + \left( \frac{-\bar{\lambda}_2}{(\gamma - 1 - \phi_2) \tilde{h}_t} \right)^2 \left( \gamma - 1 \right) x^y \tilde{h}
\]
\[
+ \kappa(\theta - v) \frac{x^y}{\gamma} \tilde{h}_v + \frac{1}{2} \sigma^2 v \frac{x^y}{\gamma} \tilde{h}_v v + \sigma (a v + b) \left( \frac{\phi_1}{\gamma} - 1 \right) \sigma \sqrt{\tilde{h}_t} - \bar{\lambda}_1(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}}) \tilde{h}_t \right] ^2 - \frac{\bar{\lambda}_2(a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2}{(\gamma - 1 - \phi_1) \tilde{h}_t} + \left( \frac{-\bar{\lambda}_2}{(\gamma - 1 - \phi_2) \tilde{h}_t} \right)^2 \left( \gamma - 1 \right) x^y \tilde{h}
\]
\[
= 0.
\]

(C.55)
Open the squared terms,

\[
x^\gamma \ddot{h}_v + \left[ r + \left( \frac{\phi_i}{\gamma} - 1 \right) \sigma \ddot{\lambda}_1 (av + b) \ddot{h}_v - \frac{\ddot{\lambda}_1^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2}{(\gamma - 1 - \phi_1) \ddot{h}} - \frac{\ddot{\lambda}_2^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2}{(\gamma - 1 - \phi_2) \ddot{h}} \right] x^\gamma \ddot{h}
\]

\[
+ \frac{1}{2} \left[ \frac{\phi_i}{\gamma} - 1 \right]^2 \sigma^2 v^2 \ddot{h}_v^2 - 2 \left( \frac{\phi_i}{\gamma} - 1 \right) \sigma \ddot{\lambda}_1 (av + b) \ddot{h}_v \ddot{h} + \ddot{\lambda}_1^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2 \ddot{h}_v \ddot{h} + \ddot{\lambda}_2^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2 \ddot{h}_v \ddot{h} \right] (\gamma - 1) x^\gamma \ddot{h}
\]

\[
+ \kappa(\theta - v) x^\gamma \ddot{h}_v + \frac{1}{2} \sigma^2 v x^\gamma \ddot{h}_v + \sigma \sqrt{\gamma} \left( \frac{\phi_i}{\gamma} - 1 \right) \sigma \sqrt{\gamma} \ddot{h}_v - \ddot{\lambda}_1 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}}) \ddot{h}_v \right] x^\gamma \ddot{h}_v
\]

\[
- \frac{1}{2} \phi_1 \left[ \frac{\phi_i}{\gamma} - 1 \right]^2 \sigma^2 v^2 \ddot{h}_v^2 - 2 \left( \frac{\phi_i}{\gamma} - 1 \right) \sigma \ddot{\lambda}_1 (av + b) \ddot{h}_v \ddot{h} + \ddot{\lambda}_1^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2 \ddot{h}_v \ddot{h} + \ddot{\lambda}_2^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2 \ddot{h}_v \ddot{h} \right] (\gamma - 1) \ddot{h}_v
\]

\[
- \frac{1}{2} \phi_2 \left[ \frac{\phi_i}{\gamma} - 1 \right]^2 \sigma^2 v^2 \ddot{h}_v^2 - 2 \left( \frac{\phi_i}{\gamma} - 1 \right) \sigma \ddot{\lambda}_1 (av + b) \ddot{h}_v \ddot{h} + \ddot{\lambda}_1^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2 \ddot{h}_v \ddot{h} + \ddot{\lambda}_2^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2 \ddot{h}_v \ddot{h} \right] (\gamma - 1) \ddot{h}_v
\]

Moreover, we divide each term by \( \frac{\gamma}{\gamma} \):

\[
\ddot{h}_v + \left[ r + \left( \frac{\phi_i}{\gamma} - 1 \right) \sigma \ddot{\lambda}_1 (av + b) \ddot{h}_v - \frac{\ddot{\lambda}_1^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2}{(\gamma - 1 - \phi_1) \ddot{h}} - \frac{\ddot{\lambda}_2^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2}{(\gamma - 1 - \phi_2) \ddot{h}} \right] \gamma \ddot{h}
\]

\[
+ \frac{1}{2} \left[ \frac{\phi_i}{\gamma} - 1 \right]^2 \sigma^2 v^2 \ddot{h}_v^2 - 2 \left( \frac{\phi_i}{\gamma} - 1 \right) \sigma \ddot{\lambda}_1 (av + b) \ddot{h}_v \ddot{h} + \ddot{\lambda}_1^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2 \ddot{h}_v \ddot{h} + \ddot{\lambda}_2^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2 \ddot{h}_v \ddot{h} \right] (\gamma - 1) \gamma \ddot{h}
\]

\[
+ \kappa(\theta - v) \ddot{h}_v + \frac{1}{2} \sigma^2 v \ddot{h}_v + \sigma \sqrt{\gamma} \left( \frac{\phi_i}{\gamma} - 1 \right) \sigma \sqrt{\gamma} \ddot{h}_v - \ddot{\lambda}_1 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}}) \ddot{h}_v \right] \gamma \ddot{h}_v
\]

\[
- \frac{1}{2} \phi_1 \left[ \frac{\phi_i}{\gamma} - 1 \right]^2 \sigma^2 v^2 \ddot{h}_v^2 - 2 \left( \frac{\phi_i}{\gamma} - 1 \right) \sigma \ddot{\lambda}_1 (av + b) \ddot{h}_v \ddot{h} + \ddot{\lambda}_1^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2 \ddot{h}_v \ddot{h} + \ddot{\lambda}_2^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2 \ddot{h}_v \ddot{h} \right] \gamma \ddot{h}_v
\]

\[
+ \frac{1}{2} \phi_2 \left[ \frac{\phi_i}{\gamma} - 1 \right]^2 \sigma^2 v^2 \ddot{h}_v^2 - 2 \left( \frac{\phi_i}{\gamma} - 1 \right) \sigma \ddot{\lambda}_1 (av + b) \ddot{h}_v \ddot{h} + \ddot{\lambda}_1^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2 \ddot{h}_v \ddot{h} + \ddot{\lambda}_2^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2 \ddot{h}_v \ddot{h} \right] \gamma \ddot{h}_v
\]

\[
- \frac{1}{2} \phi_2 \left[ \frac{\phi_i}{\gamma} - 1 \right]^2 \sigma^2 v^2 \ddot{h}_v^2 - 2 \left( \frac{\phi_i}{\gamma} - 1 \right) \sigma \ddot{\lambda}_1 (av + b) \ddot{h}_v \ddot{h} + \ddot{\lambda}_1^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2 \ddot{h}_v \ddot{h} + \ddot{\lambda}_2^2 (a \sqrt{\gamma} + \frac{b}{\sqrt{\gamma}})^2 \ddot{h}_v \ddot{h} \right] \gamma \ddot{h}_v = 0.
\]

(C.56)
Regrouping in the above equation leads to
\[
\bar{h}_t + \left[ r + \frac{(\phi_1 - 1) \sigma (av + b) \lambda_1}{(y - 1 - \phi_1)} \bar{h}_t - \frac{\lambda_1^2 (a \sqrt{v} + \frac{b \sqrt{v}}{v})}{(y - 1 - \phi_1)} - \frac{\lambda_2^2 (a \sqrt{v} + \frac{b \sqrt{v}}{v})^2}{(y - 1 - \phi_2)} \right] \gamma \bar{h}
\]
\[
+ \frac{1}{2} \left( \frac{\phi_1 - 1}{(y - 1 - \phi_1)^2} \sigma^2 v \bar{h}^2 \right) - \frac{2 (\phi_1 - 1) \sigma (av + b) \lambda_1}{(y - 1 - \phi_1)^2} \bar{h}_t + \frac{\lambda_1^2 (a \sqrt{v} + \frac{b \sqrt{v}}{v})}{(y - 1 - \phi_1)^2} \bar{h} + \frac{\lambda_2^2 (a \sqrt{v} + \frac{b \sqrt{v}}{v})^2}{(y - 1 - \phi_2)^2} \bar{h} \right] (y - 1) \gamma 
\]
\[
+ \kappa (\theta - v) \bar{h}_t + \frac{1}{2} \sigma^2 \bar{v} \bar{h}_v + \frac{(\phi_1 - 1) \sigma^2 v \bar{h}^2}{(y - 1 - \phi_1)} \gamma - \frac{\lambda_1 \sigma (av + b)}{(y - 1 - \phi_1)^2} \bar{h}_t \gamma 
\]
\[
- \frac{1}{2} \phi_1 \gamma \frac{(\phi_1 - 1)^2 \sigma^2 v \bar{h}^2}{(y - 1 - \phi_1)^2} - \frac{2 (\phi_1 - 1) \sigma (av + b) \lambda_1}{(y - 1 - \phi_1)^2} \bar{h}_t + \frac{\lambda_1^2 (a \sqrt{v} + \frac{b \sqrt{v}}{v})}{(y - 1 - \phi_1)^2} \bar{h} + \frac{\lambda_2^2 (a \sqrt{v} + \frac{b \sqrt{v}}{v})^2}{(y - 1 - \phi_2)^2} \bar{h} \right] (y - 1) \gamma 
\]
\[
+ \frac{2 (\phi_1 - 1) \sigma^2 v \bar{h}^2}{(y - 1 - \phi_1)^2} \gamma - \frac{2 \lambda_1 \sigma (av + b)}{(y - 1 - \phi_1)^2} \bar{h}_t \gamma 
\]
\[
- \frac{1}{2} \phi_2 \gamma \frac{(\phi_1 - 1)^2 \sigma^2 v}{(y - 1 - \phi_1)^2} \bar{h}^2 = 0.
\]
\[\text{(C.58)}\]

Regrouping in the above equation leads to
\[
\bar{h}_t + \left[ r + \frac{\lambda_1^2 (a \sqrt{v} + \frac{b \sqrt{v}}{v})}{(y - 1 - \phi_1)} - \frac{\lambda_1^2 (a \sqrt{v} + \frac{b \sqrt{v}}{v})}{(y - 1 - \phi_2)} + \frac{1}{2} \frac{\lambda_1^2 (a \sqrt{v} + \frac{b \sqrt{v}}{v})^2}{(y - 1 - \phi_1)^2} (y - 1) \right] \gamma \bar{h}
\]
\[
+ \frac{1}{2} \frac{\lambda_1^2 (a \sqrt{v} + \frac{b \sqrt{v}}{v})^2}{(y - 1 - \phi_2)^2} (y - 1) - \frac{1}{2} \phi_1 \gamma \frac{(\phi_1 - 1) \sigma (av + b) \lambda_1}{(y - 1 - \phi_1)^2} \gamma (y - 1) \gamma + \kappa (\theta - v) \bar{h}_t + \frac{\lambda_1 \sigma (av + b)}{(y - 1 - \phi_1)^2} \gamma 
\]
\[
+ \phi_1 \gamma \frac{(\phi_1 - 1) \sigma (av + b) \lambda_1}{(y - 1 - \phi_1)^2} \gamma + \frac{1}{2} \phi_1 \gamma \frac{\lambda_1 \sigma (av + b)}{(y - 1 - \phi_1)^2} \gamma \bar{h}_t + \frac{1}{2} \sigma^2 v \bar{h}_v 
\]
\[
+ \frac{1}{2} \frac{(\phi_1 - 1)^2 \sigma^2 v}{(y - 1 - \phi_1)^2} (y - 1) \gamma + \frac{(\phi_1 - 1) \sigma^2 v}{(y - 1 - \phi_1)^2} \gamma - \frac{1}{2} \phi_1 \gamma \frac{(\phi_1 - 1)^2 \sigma^2 v}{(y - 1 - \phi_1)^2} \gamma 
\]
\[
- \frac{1}{2} \phi_1 \gamma \frac{(\phi_1 - 1)^2 \sigma^2 v}{(y - 1 - \phi_1)^2} \bar{h}^2 = 0.
\]
\[\text{(C.59)}\]

Simplifying leads to,
\[
\bar{h}_t + \left[ r - \frac{1}{2} \frac{\lambda_1^2 (a \sqrt{v} + \frac{b \sqrt{v}}{v})^2}{(y - 1 - \phi_1)} - \frac{1}{2} \frac{\lambda_1^2 (a \sqrt{v} + \frac{b \sqrt{v}}{v})^2}{(y - 1 - \phi_2)} \gamma \bar{h}
\]
\[
+ \left[ \frac{(\phi_1 - 1) \sigma (av + b) \lambda_1}{(y - 1 - \phi_1)^2} + \kappa (\theta - v) \right] \gamma \bar{h}_t + \frac{1}{2} \sigma^2 v \bar{h}_v 
\]
\[
+ \left[ - \frac{1}{2} \frac{(\phi_1 - 1) \sigma^2 v}{(y - 1 - \phi_1)^2} - \frac{1}{2} \phi_1 \frac{\sigma^2 v}{(y - 1 - \phi_1)^2} \bar{h}^2 = 0.
\]
\[\text{(C.60)}\]

In order to eliminate the non-linear term, we need
\[
\phi_1 = \frac{\gamma^2}{\gamma + 1}.
\]
\[\text{(C.61)}\]
Thereby, we have a linear PDE
\[ \ddot{h}_t + V(v, t)\dot{h} + \Gamma(v, t)\dot{h}_t + \frac{1}{2}\sigma^2 v\ddot{h}_{vv} = 0. \] (C.62)

Further, the coefficients of the above PDE satisfy the conditions of Heath and Schweizer (2000) as per the previous proposition, so that \( \dot{h} \) admits the Feynman-Kac representation:
\[
\begin{aligned}
\dot{h}(v, t) &= \mathbb{E}^0\left[ \exp\left( \int_t^T V(v, \tau) d\tau \right) h(v, T) \mid v_t \right] \\
&= \exp\left( \left( r - \frac{\lambda_1^2 ab}{\gamma - 1 - \phi_1} - \frac{\lambda_2^2 ab}{\gamma - 1 - \phi_2} \right)(T - t) \right) \times \mathbb{E}^0\left[ \exp\left( -\mu \int_t^T v_t d\tau - \nu \int_t^T \frac{1}{v_t} d\tau \right) \mid v_t \right],
\end{aligned}
\]
\[\hat{\mathbb{Q}}(T; v_t, \alpha, \lambda, \mu, v)\] (C.63)

with parameters
\[
\begin{aligned}
\alpha &= 0, \\
\lambda &= 0, \\
\mu &= \frac{1}{2} \gamma \left( \frac{\lambda_1^2}{\gamma - 1 - \phi_1} + \frac{\lambda_2^2}{\gamma - 1 - \phi_2} \right) a^2, \\
\nu &= \frac{1}{2} \gamma \left( \frac{\lambda_1^2}{\gamma - 1 - \phi_1} + \frac{\lambda_2^2}{\gamma - 1 - \phi_2} \right) b^2.
\end{aligned}
\] (C.64)

Note that the conditional expectation is taken under the probability measure \( \hat{\mathbb{Q}} \) such that \( v_t \) has drift \( \Gamma(v, t) \). The Feller condition is assumed to be satisfied by the new drift:
\[
\Gamma(v, t) = \frac{(\phi_1 - \gamma) \sigma (av + b) \lambda_1}{(\gamma - 1 - \phi_1)} + \kappa(\theta - v)
\]
\[
= \left( \kappa \theta - \frac{(\phi_1 - \gamma) \sigma b \lambda_1}{(\gamma - 1 - \phi_1)} - \kappa \frac{(\phi_1 - \gamma) \sigma a \lambda_1}{(\gamma - 1 - \phi_1)} \right) v
\]
\[
\Rightarrow \kappa \theta - \frac{(\phi_1 - \gamma) \sigma b \lambda_1}{(\gamma - 1 - \phi_1)} \geq \frac{\sigma^2}{2}.
\] (C.65)

Further, if \( \alpha, \lambda, \mu, \) and \( \nu \) satisfy conditions (C.15), \( \dot{h}(v, t) \) can be solved explicitly by Grasselli (2017)'s result like Equation (4.20) with associated \( m, D, \beta, \) and \( K \) like Equation (4.21). Note that the last two conditions for \( \alpha = \lambda = 0 \) are satisfied directly. Thus, the dependence of the function \( g(\cdot) \) on \( \alpha \) and \( \lambda \) can be omitted. Moreover, the optimal wealth exposures with \( \phi_1 = \frac{\lambda_2}{\gamma + 1} \) are given by
\[
\left\{ \begin{array}{l}
(\Theta^\lambda)_\alpha^* = \frac{(\phi_1 - 1) \sigma b \sqrt{\lambda_1 (a \sqrt{\lambda_2} + \frac{h}{\sqrt{\lambda_2}})} h}{(\gamma - 1 - \phi_1) \sigma b \sqrt{\lambda_1 (a \sqrt{\lambda_2} + \frac{h}{\sqrt{\lambda_2}})}} = \frac{\sigma b \sqrt{\lambda_1 (a \sqrt{\lambda_2} + \frac{h}{\sqrt{\lambda_2}})} h}{(\gamma - 1 - \phi_1) \sigma b \sqrt{\lambda_1 (a \sqrt{\lambda_2} + \frac{h}{\sqrt{\lambda_2}})}}
\\
(\Theta^\beta)_\lambda^* = \frac{\sigma \sqrt{\lambda_2} \frac{h}{\sqrt{\lambda_2}}}{(\gamma - 1 - \phi_2) h} - \lambda_1
\end{array} \right.
\] (C.66)
The worst case measure is determined by
\[
\begin{align*}
(e^\gamma)^* &= \phi_1\left[\sigma \sqrt{\frac{b}{\tilde{V}_{\gamma+1}}} - \tilde{\lambda}_1(a \sqrt{\tilde{V}} + \frac{b}{\tilde{V}})\right] \\
&= \sigma \sqrt{\frac{b}{\tilde{V}}} \gamma - \tilde{\lambda}_1 \frac{\gamma}{\gamma+1}(a \sqrt{\tilde{V}} + \frac{b}{\tilde{V}}) \\
(e^\delta)^* &= \frac{-\phi_{2,1}}{\gamma-1-\phi_2}(a \sqrt{\tilde{V}} + \frac{b}{\tilde{V}}),
\end{align*}
\]
(C.67)

C.5 Proof of Proposition 4.4.3 (Incomplete Market, Robustness, Consumption)

**Proof.** Following the similar procedure as before, we solve an infimization problem in terms of $e^\gamma$ and $e^\delta$ first, and then substitute back into the the robust HJB equation (4.61) to solve the maximization problem in terms of the optimal allocation $\pi$.

Solving the infimization problem leads to
\[
\begin{align*}
\left(\frac{e^\gamma}{\phi_1}\right) &= \pi \rho \left( a \sqrt{\tilde{V}} + \frac{b}{\sqrt{\tilde{V}}} \right) xJ_x + \sigma \sqrt{\tilde{V}} J_v \\
\left(\frac{e^\delta}{\phi_2}\right) &= \pi \sqrt{1 - \rho^2} \left( a \sqrt{\tilde{V}} + \frac{b}{\sqrt{\tilde{V}}} \right) xJ_x & \implies (e^\gamma)^* &= \Phi_1 \left[ \pi \rho \left( a \sqrt{\tilde{V}} + \frac{b}{\sqrt{\tilde{V}}} \right) xJ_x + \sigma \sqrt{\tilde{V}} J_v \right] \\
&= \Phi_1 \left[ \pi \sqrt{1 - \rho^2} \left( a \sqrt{\tilde{V}} + \frac{b}{\sqrt{\tilde{V}}} \right) xJ_x \right].
\end{align*}
\]
(C.68)

The HJB equation then follows
\[
\begin{align*}
&\sup_{\pi,c} \left\{ u(c) - \delta J + J_t + (\chi + \frac{1}{2}\rho \tilde{\lambda}_1(a \sqrt{\tilde{V}} + \frac{b}{\sqrt{\tilde{V}}})^2 + \sqrt{1 - \rho^2} \tilde{\lambda}_2(a \sqrt{\tilde{V}} + \frac{b}{\sqrt{\tilde{V}}})^2 \right. \\
&\quad - \rho(a \sqrt{\tilde{V}} + \frac{b}{\sqrt{\tilde{V}}}) \Phi_1 \left[ \pi \rho \left( a \sqrt{\tilde{V}} + \frac{b}{\sqrt{\tilde{V}}} \right) xJ_x + \sigma \sqrt{\tilde{V}} J_v \right] \\
&\quad - \sqrt{1 - \rho^2}(a \sqrt{\tilde{V}} + \frac{b}{\sqrt{\tilde{V}}}) \Phi_2 \left[ \pi \sqrt{1 - \rho^2} \left( a \sqrt{\tilde{V}} + \frac{b}{\sqrt{\tilde{V}}} \right) xJ_x \right] - c \right) J_x + \frac{1}{2} \sigma^2 \left( a \sqrt{\tilde{V}} + \frac{b}{\sqrt{\tilde{V}}} \right)^2 \pi^2 J_{xx} \\
&\quad + \left[ \kappa(\theta - \varphi) - \sigma \sqrt{\tilde{V}} \Phi_1 \left[ \pi \rho \left( a \sqrt{\tilde{V}} + \frac{b}{\sqrt{\tilde{V}}} \right) xJ_x + \sigma \sqrt{\tilde{V}} J_v \right] \right] J_v \\
&\quad + \frac{1}{2} \sigma^2 J_{vv} + \sigma \chi \rho (\alpha + b) J_{xx} + \rho \Phi_1 \left[ \pi \rho \left( a \sqrt{\tilde{V}} + \frac{b}{\sqrt{\tilde{V}}} \right) xJ_x + \sigma \sqrt{\tilde{V}} J_v \right] \\
&= 0.
\end{align*}
\]
(C.69)
Simplifying leads to

\[
\sup_{\pi,c}\left\{ u(c) - \delta J + J_x + \left( x + \pi \left[ \rho \tilde{l}_1(a \sqrt{v}) + \frac{b}{\sqrt{v}} \right]^2 + \sqrt{1 - \rho^2} \tilde{l}_2(a \sqrt{v}) + \frac{b}{\sqrt{v}} \right]^2 \right\} - c \right) J_x \\
- \Phi_1 \left[ \pi^2 \rho^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 J_x^2 \right] + \frac{1}{2} \pi^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 J_x^2 \\
- \Phi_2 \left[ \pi^2 (1 - \rho^2) \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 J_x^2 \right] + \frac{1}{2} \pi^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 J_x^2 \\
+ \kappa(\theta - v)J_v - \Phi_1 \left[ \pi \rho \sigma (av + b) J_x J_v + \sigma^2 v J_v^2 \right] \\
+ \frac{1}{2} \sigma^2 v J_v + \sigma x \pi \rho (av + b) J_v + \\
\Phi_2 \left[ \pi^2 (1 - \rho^2) \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 J_x^2 \right] \\
= 0. \\
\] (C.70)

That is,

\[
\sup_{\pi,c}\left\{ u(c) - \delta J + J_x + \left( x + \pi \left[ \rho \tilde{l}_1(a \sqrt{v}) + \frac{b}{\sqrt{v}} \right]^2 + \sqrt{1 - \rho^2} \tilde{l}_2(a \sqrt{v}) + \frac{b}{\sqrt{v}} \right]^2 \right\} - c \right) J_x \\
+ \frac{1}{2} \pi^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 J_x^2 + \kappa(\theta - v)J_v + \frac{1}{2} \sigma^2 v J_v + \pi x \pi \rho (av + b) J_v \\
- \Phi_1 \left[ \pi^2 \rho^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 J_x^2 \right] + \frac{2}{2} \pi^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 J_x^2 \\
- \Phi_2 \left[ \pi^2 (1 - \rho^2) \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi^2 J_x^2 \right] \\
= 0. \\
\] (C.71)

Solve for \( c^* \),

\[
0 = u'(c) - J_x \\
= \epsilon_1 \epsilon^{-1} - J_x \\
\] (C.72)

It follows that

\[
c^* = (J_x)^{\frac{1}{\gamma_1}} (\epsilon_1)^{-\frac{1}{\gamma_1}}. \\
\] (C.73)
C.5. Proof of Proposition 4.4.3 (Incomplete Market, Robustness, Consumption)

Solve for \( \pi^* \),

\[
x J_t \left[ \rho \lambda_1 \left( a \sqrt{v} + \frac{b}{\sqrt{v}_t} \right)^2 + \sqrt{1 - \rho^2} \lambda_2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}_t} \right)^2 \right] + x^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 \pi J_{xx}
\]

\[+ \sigma x \rho (av + b) J_{xv} - \Phi_1 \left[ \pi \rho^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 x^2 J_{xv}^2 + \rho \sigma (av + b) x J_x J_v \right]
\]

\[= 0.
\]

(C.74)

It follows that

\[
\pi^* = \frac{x J_t \left[ \rho \lambda_1 \left( a \sqrt{v} + \frac{b}{\sqrt{v}_t} \right)^2 + \sqrt{1 - \rho^2} \lambda_2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}_t} \right)^2 \right] + \sigma x \rho (av + b) J_{xv} - \Phi_1 \rho^2 (av + b) J_{xv}}{\Phi_1 \rho^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 x^2 J_{xv}^2 + \Phi_2 (1 - \rho^2) \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 x^2 J_{xv}^2 - x^2 \left( a \sqrt{v} + \frac{b}{\sqrt{v}} \right)^2 J_{xx}}.
\]

(C.75)

In the case of considering both intermediate consumption and terminal wealth, we conjecture our value function has the following form

\[
J(x, v, t) = \frac{x^y}{\gamma} \left( h(t, v) \right)^{1-\gamma},
\]

(C.76)

where \( h(T, v) = 1 \) for all \( v \). Thereby, we have following partial derivatives for \( J(x, v, t) \):

\[
J_t = \frac{1 - \gamma}{\gamma} x^y h^{1-\gamma} h_t,
\]

\[
J_v = \frac{x^y}{\gamma} (1 - \gamma) h^{-\gamma} h_v,
\]

\[
J_x = x^{\gamma - 1} h^{1-\gamma},
\]

\[
J_{vv} = -(1 - \gamma) x^{\gamma - 1} h^{-\gamma} h_{vv} + \frac{x^y}{\gamma} (1 - \gamma) h^{-\gamma} h_{vv},
\]

\[
J_{xv} = (1 - \gamma) x^{\gamma - 1} h^{-\gamma} h_v,
\]

\[
J_{xx} = (\gamma - 1) x^{\gamma - 2} h^{1-\gamma}.
\]

Substituting these partial derivatives into the candidates for the optimal consumption in Equation (C.73) and optimal allocation in Equation (C.75) with \( \Phi_1 = \frac{\delta_1}{\gamma} = \frac{\delta_1}{\gamma h^{1-\gamma}} \) as in Maenhout.
Substituting the partial derivatives and the optimal allocation in Equation above into Equation (2004), we have

\[
\begin{align*}
\frac{c^*}{\pi^*} &= xh^{-1}(e_1)^{-\frac{1}{\gamma^*}} \\
&= xh^{-1} \left[ \rho \bar{\lambda}_1(a \sqrt{\bar{\nu}} + \frac{b}{\gamma^*})^2 + \sqrt{1 - \rho^2} \bar{\lambda}_2(a \sqrt{\bar{\nu}} + \frac{b}{\gamma^*})^2 \right] + c_p(a + b)(1 - \gamma) \chi_h^{-\gamma} h_c - \frac{\lambda_1}{\sqrt{1 - \rho^2}} \rho (a + b) \frac{b}{\gamma^*} x_h^{1 - \gamma} h_c \\
&= \frac{xh^{-1} \left[ \rho \bar{\lambda}_1(a \sqrt{\bar{\nu}} + \frac{b}{\gamma^*})^2 + \sqrt{1 - \rho^2} \bar{\lambda}_2(a \sqrt{\bar{\nu}} + \frac{b}{\gamma^*})^2 \right] + c_p(a + b)(1 - \gamma) \chi_h^{-\gamma} h_c - \frac{\lambda_1}{\sqrt{1 - \rho^2}} \rho (a + b) \frac{b}{\gamma^*} x_h^{1 - \gamma} h_c}{\phi \mu^2 \left[ \rho \bar{\lambda}_1(a \sqrt{\bar{\nu}} + \frac{b}{\gamma^*})^2 + \sqrt{1 - \rho^2} \bar{\lambda}_2(a \sqrt{\bar{\nu}} + \frac{b}{\gamma^*})^2 \right] + c_p(a + b)(1 - \gamma) \chi_h^{-\gamma} h_c - \frac{\lambda_1}{\sqrt{1 - \rho^2}} \rho (a + b) \frac{b}{\gamma^*} x_h^{1 - \gamma} h_c} \\
&= \frac{xh^{-1} \left( \rho \bar{\lambda}_1 + \sqrt{1 - \rho^2} \bar{\lambda}_2 \right) \left[ a \sqrt{\bar{\nu}} + \frac{b}{\gamma^*} \right]^2 + (1 - \frac{\lambda_1}{\sqrt{1 - \rho^2}} \rho (a + b)(1 - \gamma) \chi_h^{-\gamma} h_c}{\phi \mu^2 \left( \rho \bar{\lambda}_1 + \sqrt{1 - \rho^2} \bar{\lambda}_2 \right) \left[ a \sqrt{\bar{\nu}} + \frac{b}{\gamma^*} \right]^2 + (1 - \frac{\lambda_1}{\sqrt{1 - \rho^2}} \rho (a + b)(1 - \gamma) \chi_h^{-\gamma} h_c} \\
&= \frac{xh^{-1} \left( \rho \bar{\lambda}_1 + \sqrt{1 - \rho^2} \bar{\lambda}_2 \right) \left[ a \sqrt{\bar{\nu}} + \frac{b}{\gamma^*} \right]^2 + (1 - \frac{\lambda_1}{\sqrt{1 - \rho^2}} \rho (a + b)(1 - \gamma) \chi_h^{-\gamma} h_c}{\phi \mu^2 \left( \rho \bar{\lambda}_1 + \sqrt{1 - \rho^2} \bar{\lambda}_2 \right) \left[ a \sqrt{\bar{\nu}} + \frac{b}{\gamma^*} \right]^2 + (1 - \frac{\lambda_1}{\sqrt{1 - \rho^2}} \rho (a + b)(1 - \gamma) \chi_h^{-\gamma} h_c} \\
\end{align*}
\]

(C.77)

Substituting the partial derivatives and the optimal allocation in Equation above into Equation

\[\text{Equation (2004)}\]
(C.71) to eliminate “sup”, and simplifying the equation lead to

\[
(1 - \gamma)(\varepsilon_1)\frac{r + (1 - \gamma)h_t + ry}{\sup_p (C.78)}
\]

\[
+ \frac{(\rho \hat{\lambda}_1 + \sqrt{1 - \rho^2 \hat{\lambda}_2})^2(a \sqrt{v} + \frac{h_v}{\sqrt{v}})^2}{(\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))} yh
\]

\[
+ \frac{(1 - \frac{\phi_1}{\gamma})^{(1 - \gamma)}(1 - \gamma)\sigma \rho (av + b)(\rho \hat{\lambda}_1 + \sqrt{1 - \rho^2 \hat{\lambda}_2})}{\sup_p (C.78)} yh_v
\]

\[
+ \frac{1}{2} \frac{\gamma(\theta - \nu)(1 - \gamma)h_v + \frac{1}{2} \sigma^2 v(1 - \gamma)\gamma h_v^2}{(C.78)}
\]

\[
+ \frac{\sigma \rho (1 - \gamma)(\rho \hat{\lambda}_1 + \sqrt{1 - \rho^2 \hat{\lambda}_2})(av + b)}{\sup_p (C.78)} yh_v + \frac{\sigma^2 \rho^2 (1 - \frac{\phi_1}{\gamma})(1 - \gamma)^2 v}{\sup_p (C.78)} yh
\]

\[
- \frac{\phi_1}{2} \frac{\rho^2(\rho \hat{\lambda}_1 + \sqrt{1 - \rho^2 \hat{\lambda}_2})^2(a \sqrt{v} + \frac{h_v}{\sqrt{v}})^2}{(\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))^2} yh
\]

\[
+ 2\sigma^2 \rho^2 \frac{(1 - \frac{\phi_1}{\gamma})(1 - \gamma)\gamma h_v^2}{\sup_p (C.78)} + \frac{\sigma \rho (\rho \hat{\lambda}_1 + \sqrt{1 - \rho^2 \hat{\lambda}_2})(av + b)}{\sup_p (C.78)} yh_v
\]

\[
- \frac{\phi_2}{2} \frac{(1 - \rho^2)(\rho \hat{\lambda}_1 + \sqrt{1 - \rho^2 \hat{\lambda}_2})^2(a \sqrt{v} + \frac{h_v}{\sqrt{v}})^2}{(\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))^2} yh
\]

\[
+ \frac{(1 - \rho^2)(1 - \gamma)^2 \sigma^2 \rho^2 v}{\sup_p (C.78)} yh
\]

\[
= 0.
\]

(C.78)
Regrouping terms leads to

\[
(1 - \gamma)(e_i - \frac{1}{\delta^2}) + (1 - \gamma)h_i
\]

\[
+ \left[ r - \frac{\delta}{\gamma} + \frac{(\rho\lambda_1 + \sqrt{1 - \rho^2\lambda_2})^2(a\sqrt{\nu} + \frac{b}{\sqrt{\nu}})^2}{\phi_1\rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1)} + \frac{1}{2}(\gamma - 1) \frac{(\rho\lambda_1 + \sqrt{1 - \rho^2\lambda_2})^2(a\sqrt{\nu} + \frac{b}{\sqrt{\nu}})^2}{(\phi_1\rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))^2}
\]

\[
- \frac{1}{2}(\phi_1\rho^2 + \phi_2(1 - \rho^2))(\rho\lambda_1 + \sqrt{1 - \rho^2\lambda_2})^2(a\sqrt{\nu} + \frac{b}{\sqrt{\nu}})^2
\]

\[
\cdot \left( \frac{1}{\phi_1\rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1)} \right)^2 \gamma h
\]

\[
+ \left[ (1 - \frac{\phi_1}{\gamma})(1 - \gamma)\sigma_\rho(\nu + b)(\rho\lambda_1 + \sqrt{1 - \rho^2\lambda_2}) - \gamma \frac{1}{\phi_1\rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1)} \gamma (1 - \gamma)
\]

\[
- \phi_1\sigma_\rho \frac{(\rho\lambda_1 + \sqrt{1 - \rho^2\lambda_2})(\nu + b)(1 - \frac{\phi_1}{\gamma})(1 - \gamma)\sigma_\rho}{\phi_1\rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1)} h
\]

\[
+ \left[ \frac{1}{2} \frac{(1 - \frac{\phi_1}{\gamma})^2(1 - \gamma)^2\sigma^2\rho^2\nu}{(\phi_1\rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))^2} \gamma (\gamma - 1) + \frac{1}{2}\sigma^2\nu(1 - \gamma)\gamma
\]

\[
+ \frac{1}{2}\sigma^2\rho^2(1 - \frac{\phi_1}{\gamma})(1 - \gamma)^2\nu
\]

\[
\cdot \left( \frac{1}{\phi_1\rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1)} \right)^2 \gamma
\]

\[
- \phi_1\sigma^2\rho^2 \frac{(1 - \frac{\phi_1}{\gamma})(1 - \gamma)\nu}{(\phi_1\rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))} (1 - \gamma) - \frac{\phi_1}{2}\sigma^2\nu(1 - \gamma)^2 \left[ \frac{h_i^2}{h} \right]
\]

\[
+ \left[ \frac{1}{2}\sigma^2\nu(1 - \gamma) \right] h_{iv} = 0.
\]
That is,

\[(1 - \gamma)(\varepsilon_1)^{-\frac{1}{\gamma}} + (1 - \gamma)h_t + \left[ r - \delta + \frac{1}{\gamma} \frac{(\rho \lambda_1 + \sqrt{1 - \rho^2 \lambda_2})^2(a \sqrt{V} + \frac{b}{\sqrt{V}})^2}{(\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))} \right] \gamma h_t + \left[ \kappa(\theta - \nu)(1 - \gamma) + \frac{\sigma \rho (1 - \gamma)(\rho \lambda_1 + \sqrt{1 - \rho^2 \lambda_2})(av + b)}{(\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))} (\gamma + \phi_1) \right] h_v \]

\[+ \frac{1}{2} \sigma^2 (1 - \gamma)(\gamma - \phi_1(1 - \gamma)) + \frac{1}{2} (\gamma - \phi_1) \frac{\sigma^2 \rho^2 (1 - \phi_2)(1 - \gamma)^2 \nu}{(\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))} \frac{h^2_v}{h} \]

\[+ \frac{1}{2} \sigma^2 (1 - \gamma) h_v = 0. \tag{C.80} \]

We can rewrite it as:

\[(\varepsilon_1)^{-\frac{1}{\gamma}} + h_t + \left[ r - \frac{\delta}{\gamma} + \frac{1}{2} \frac{(\rho \lambda_1 + \sqrt{1 - \rho^2 \lambda_2})^2(a \sqrt{V} + \frac{b}{\sqrt{V}})^2}{(\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))} \right] \frac{\gamma h_t}{1 - \gamma} + \left[ \kappa(\theta - \nu) + \frac{\sigma \rho (\rho \lambda_1 + \sqrt{1 - \rho^2 \lambda_2})(av + b)}{\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1)} (\gamma + \phi_1) \right] h_v + \frac{1}{2} \sigma^2 \gamma h_v = 0. \tag{C.81} \]

where we have eliminated the non-linear term by enforcing:

\[\gamma - \phi_1(1 - \gamma) = \frac{-\rho^2 (\gamma - \phi_1)^2 (1 - \gamma)}{\gamma (\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1))}. \tag{C.82} \]

Then we have a linear PDE

\[(\varepsilon_1)^{-\frac{1}{\gamma}} + h_t + V(v, t) h_t + \Gamma(v, t) h_v + \frac{1}{2} \sigma^2 h_v = 0. \tag{C.83} \]

Further, the coefficients in (4.64) satisfy the condition of Heath and Schweizer (2000) as per previous proposition, then \( h \) admits the Feynman-Kac representation:

\[ h(v, t) = \mathbb{E}^D \left[ \int_t^T \exp \left( \int_t^\tau V(v, \tau) d\tau \right) \left( \varepsilon_1 \right)^{-\frac{1}{\gamma}} d\tau + \exp \left( \int_t^T V(v, \tau) d\tau \right) h(v, T) | v_t \right]. \tag{C.84} \]

Moreover, given that \( V(v, t): \mathbb{R}^+ \times [0, T] \rightarrow \mathbb{R} \) is a measurable function, such that \( \exp \left( \int_t^T V(v, \tau) d\tau \right) \geq 0 \) we can then apply Fubini’s theorem, in particular the version of Tonelli’s Theorem, to the
first term:

\[
h(v, t) = \int_t^T \mathbb{E}^Q \left[ \left( (\epsilon_1)^{-\frac{1}{\gamma}} \right) \exp \left\{ \int_t^\tau V(v, \tau) d\tau \right\} \mid v_i \right] d\tau + \mathbb{E}^Q \left[ \exp \left\{ \int_t^\tau V(v, \tau) d\tau \right\} \mid v_i \right]
\]

\[
= (\epsilon_1)^{-\frac{1}{\gamma}} \int_t^T \mathbb{E}^Q \left[ \exp \left\{ \int_t^\tau V(v, \tau) d\tau \right\} \mid v_i \right] d\tau + \mathbb{E}^Q \left[ \exp \left\{ \int_t^\tau V(v, \tau) d\tau \right\} \mid v_i \right].
\]

(C.85)

Here, \( \tau(t) = T - t \) and \( g(v, \tau) \) can be rewritten as

\[
g(v, \tau) = \mathbb{E}^Q \left[ \exp \left\{ \int_t^\tau V(v, \tau) d\tau \right\} \mid v_i \right]
\]

\[
= \exp \left\{ \frac{\gamma}{1 - \gamma} \left[ \frac{\lambda_1 \rho + \lambda_2 \sqrt{1 - \rho^2}}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right] (T - t) \right\} \times \mathbb{E}^Q \left[ \exp \left\{ -\mu \int_t^\tau v_\tau d\tau - v \int_t^\tau \frac{1}{\tau} d\tau \right\} \mid v_i \right],
\]

(C.86)

with parameters

\[
\alpha = 0,
\]

\[
\lambda = 0,
\]

\[
\mu = \frac{1}{2} \frac{\gamma}{1 - \gamma} \left[ \frac{(\lambda_1 \rho + \lambda_2 \sqrt{1 - \rho^2})^2}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right] a^2,
\]

(C.87)

\[
\nu = \frac{1}{2} \frac{\gamma}{1 - \gamma} \left[ \frac{(\lambda_1 \rho + \lambda_2 \sqrt{1 - \rho^2})^2}{(\gamma - 1) - \phi_1 \rho^2 - \phi_2 (1 - \rho^2)} \right] b^2.
\]

Note that the conditional expectation in \( g(v, t) \) is taken under the probability measure \( \mathbb{Q} \) such that \( v_i \) has drift \( \Gamma(v, t) \) instead of \( \kappa(\theta - v) \). The Feller condition is assumed to be satisfied by the new drift:

\[
\Gamma(v, t) = \kappa(\theta - v) + \frac{\sigma \rho (\rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2)(av + b)}{\phi_1 \rho^2 + \phi_2 (1 - \rho^2) - (\gamma - 1)} (\gamma + \phi_1)
\]

\[
= \left( \frac{\sigma \rho (\rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2)b}{\phi_1 \rho^2 + \phi_2 (1 - \rho^2) - (\gamma - 1)} (\gamma + \phi_1) + \kappa \theta \right) - \left( \frac{\sigma \rho (\rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2)a}{\phi_1 \rho^2 + \phi_2 (1 - \rho^2) - (\gamma - 1)} (\gamma + \phi_1) + \kappa \nu \right)
\]

\[
\Rightarrow \frac{\sigma \rho (\rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2)b}{\phi_1 \rho^2 + \phi_2 (1 - \rho^2) - (\gamma - 1)} (\gamma + \phi_1) + \kappa \theta \geq \frac{\sigma^2}{2}.
\]

(C.88)

Further, if \( \alpha, \lambda, \mu, \) and \( \nu \) in Equation (C.87) satisfy the conditions (C.15), \( h \) can be solved explicitly by Grasselli (2017)'s result like Equation (4.20) with associated \( m, D, \beta, \) and \( K \) like
Equation (4.21). Note that the last two conditions for $\alpha = \lambda = 0$ are satisfied directly. Thus, the dependence of the function $q(\cdot)$ on $\alpha$ and $\lambda$ can be omitted. The optimal portfolio and consumption-wealth ratio are given by

$$
\begin{align*}
\left( \xi \right)^* &= h^{-1}(\epsilon_1)^{-1} \\
\left( \pi \right)^* &= \frac{h}{\pi(1+\bar{\rho}^2\bar{\lambda}_2)(\pi\sqrt{\nu} + \frac{b}{\sqrt{\nu}})^2 + (\pi\alpha + b)(1-\gamma)h,} \\
&= \frac{\phi(\pi^2 + \phi_1^2(1-\gamma) - (\pi\alpha + b)(1-\gamma)h}{(\phi_1^2 + \phi_2^2(1-\gamma))\sqrt{\nu} + \frac{b}{\sqrt{\nu}} h}}. 
\end{align*}
(C.89)
$$

The worst case measure is determined by

$$
\begin{align*}
\left( e^v \right)^* &= \phi^v[(\pi)^*\rho(\alpha \sqrt{\nu} + \frac{b}{\sqrt{\nu}}) + \sigma \sqrt{\nu} \frac{h}{1-\gamma} ] \\
\left( e^s \right)^* &= \phi^s[(\pi)^* \sqrt{1-\rho^2}(\alpha \sqrt{\nu} + \frac{b}{\sqrt{\nu}})].
\end{align*}
(C.90)
$$

\[ \Box \]

C.6 Extra plots for Section 4.5.1

The plots of the optimal consumption-wealth ratio $c/x$ versus the premiums of market price of risk $\bar{\lambda}_1, \bar{\lambda}_2$, and the relative importance ratio $\epsilon_1$ are presented in figures C.1, C.2, and C.3 respectively.

![Figure C.1: $\xi$ vs. $\bar{\lambda}_1$](image1)
![Figure C.2: $\xi$ vs. $\bar{\lambda}_2$](image2)
![Figure C.3: $\xi$ vs. $\epsilon_1$](image3)

The plots of the optimal wealth exposures $\Theta^v, \Theta^s$ versus the premiums of market price of risk $\bar{\lambda}_1, \bar{\lambda}_2$ are presented in figures C.4, C.5, and C.6 respectively.

![Figure C.4: $\Theta^v$ vs. $\bar{\lambda}_1$](image4)
![Figure C.5: $\Theta^v$ vs. $\bar{\lambda}_2$](image5)
![Figure C.6: $\Theta^s$ vs. $\bar{\lambda}_2$](image6)
C.7 Extra plots for Section 4.5.2

The plots of optimal wealth exposures $\Theta^r$, $\Theta^f$ versus the premiums of market price of risk $\bar{\lambda}_1$, and $\bar{\lambda}_2$ are presented in figures C.7, C.8, and C.9 respectively.

![Figure C.7: $\Theta^r$ vs. $\bar{\lambda}_1$](image)

![Figure C.8: $\Theta^r$ vs. $\bar{\lambda}_2$](image)

![Figure C.9: $\Theta^f$ vs. $\bar{\lambda}_2$](image)

C.8 Extra plots for Section 4.5.3

The plots of the optimal consumption-wealth ratio $c/x$ versus the premiums of market price of risk $\bar{\lambda}_1$, $\bar{\lambda}_2$, and the relative importance ratio $\epsilon_1$ are presented in figures C.10, C.11, and C.12 respectively.

![Figure C.10: $\xi_x$ vs. $\bar{\lambda}_1$](image)

![Figure C.11: $\xi_x$ vs. $\bar{\lambda}_2$](image)

![Figure C.12: $\xi_x$ vs. $\epsilon_1$](image)

The plots of the optimal allocation $\pi$ versus the premiums of market price of risk $\bar{\lambda}_1$, and $\bar{\lambda}_2$ are presented in figures C.13, and C.14 respectively.

![Figure C.13: $\pi$ vs. $\bar{\lambda}_1$](image)

![Figure C.14: $\pi$ vs. $\bar{\lambda}_2$](image)
Appendix D

Proofs for Chapter 5

D.1 Proof of changes of measure

Proof. Proof of Proposition 5.3.1.

The first step is similar to Cheng and Escobar (2021), where we ensure the change of measure is well-defined. For this, we use Novikov’s condition, i.e., generically

$$
\mathbb{E}\left[ \exp\left( \frac{1}{2} \int_0^T \lambda^2 \left( \sqrt{v_i} \right)^2 ds \right) \right] = \mathbb{E}\left[ \exp\left( \frac{\lambda^2}{2} \int_0^T v_i ds \right) \right] < \infty.
$$

From Cheng et al. (2019), in order for this expectation to exist, we need one condition:

$$
-\frac{\lambda^2}{2} > -\frac{\kappa^2}{2\sigma^2} \implies |\lambda| < \frac{\kappa}{\sigma}. \quad (D.1)
$$

That is, we need

$$
\max \{ |\bar{\lambda}_i|, |\bar{\lambda}_i^{\dagger} | \} < \frac{\kappa_i}{\sigma_i}, i = 1, 2,
$$

where $\bar{\lambda}_i > 0$ (Aït-Sahalia and Kimmel, 2007; Escobar et al., 2015), and Chernov and Ghysels (2000); Bakshi and Kapadia (2003); Escobar et al. (2015) find that volatility risk is negatively priced, i.e., $\bar{\lambda}_i^{\dagger} < 0$.

The second step is to ensure the drift of the asset price is equal to the short rate under $Q$, which is obviously satisfied here.

The third step ensures the discounted asset price process, $\bar{S}_i = e^{-\tau} S_i, i = 1, 2$, is a true $Q$-martingale and not just a local $Q$-martingale. See similar discussions in Grasselli (2017); Cheng and Escobar (2021). Here, we check the martingale properties of the asset prices for $S_{1t}$ and $S_{2t}$ under $Q$ respectively. Recall that for asset $S_1$,

$$
\frac{dS_{1t}}{S_{1t}} = rdt + \left( \sqrt{v_{1t}} + \frac{b_1}{\sqrt{v_{1t}}} \right) dW_{1t}^Q, \quad S_{1}(0) = S_1,
$$

$$
\frac{dS_{2t}}{S_{2t}} = \left( \sqrt{v_{2t}} + \frac{b_1}{\sqrt{v_{2t}}} \right) dW_{2t}^Q, \quad S_{1}(0) = \bar{S}_1.
$$
Then, under $Q$ we have

$$
E^Q[S_{1t}] = \mathcal{S}_1 E^Q \left\{ \int_0^t \left( \sqrt{v_{1s}} + \frac{b_1}{\sqrt{v_{1s}}} \right) dW^Q_{1s} - \frac{1}{2} \int_0^t \left( \sqrt{v_{1s}} + \frac{b_1}{\sqrt{v_{1s}}} \right)^2 ds \right\}
$$

where we define the exponential local martingale process $\xi_{1t}$ via

$$
\xi_{1t} = \exp \left\{ \int_0^t \left( \sqrt{v_{1s}} + \frac{b_1}{\sqrt{v_{1s}}} \right) dZ^Q_{1s} - \frac{1}{2} \rho_1^2 \int_0^t \left( \sqrt{v_{1s}} + \frac{b_1}{\sqrt{v_{1s}}} \right)^2 ds \right\}.
$$

Testing the martingale property for the discounted asset is equivalent to performing the Feller nonexplosion test for volatility using $\xi_{1t}$. Note $\xi_{1t}$ can be interpreted as a new change of measure for the volatility process:

$$
dZ^Q_{1t} = \rho_1 \left( \sqrt{v_{1t}} + \frac{b_1}{\sqrt{v_{1t}}} \right) dt + dZ^Q_{1t}.
$$

Hence we check the CIR process does not reach zero under $Q1$:

\[
\Rightarrow \quad d\gamma_{1t} = \kappa_1(\gamma_1 - \gamma_{1t}) dt - \sigma_1 \gamma_{1t} \left( \sqrt{\gamma_{1t}} + \frac{b_1}{\sqrt{\gamma_{1t}}} \right) dt + \sigma_1 \sqrt{\gamma_{1t}} dZ^Q_{1t}
\]

\[
= (\kappa_1 + \sigma_1 b_1) \left( \frac{\kappa_1 \gamma_1 - \sigma_1 \gamma_{1t} b_1}{\kappa_1 + \sigma_1 b_1} - \gamma_{1t} \right) dt + \sigma_1 \sqrt{\gamma_{1t}} dZ^Q_{1t}.
\]

This leads to the following conditions:

under $Q1$:

$$
\begin{align*}
\sigma_1^2 &\leq 2 \kappa_1 b_1 - \frac{1}{2} |\sigma_1 b_1|, \\
\kappa_1 + \sigma_1 b_1 &> 0
\end{align*}
$$

under $Q$:

\[
d\gamma_{1t} = \kappa_1(\gamma_1 - \gamma_{1t}) dt - \sigma_1 \left( \rho_1 \lambda_{1t} + \sqrt{1 - \rho_1^2 \lambda_{1t}^2} \right) \gamma_{1t} dt + \sigma_1 \sqrt{\gamma_{1t}} \left( \rho_1 dW^Q_{1t} + \sqrt{1 - \rho_1^2} dW^Q_{1t} \right)
\]

\[
= \left( \kappa_1 + \sigma_1 \left( \rho_1 \lambda_{1t} + \sqrt{1 - \rho_1^2 \lambda_{1t}^2} \right) \right) \left( \frac{\kappa_1 \gamma_{1t}}{\kappa_1 + \sigma_1 \left( \rho_1 \lambda_{1t} + \sqrt{1 - \rho_1^2 \lambda_{1t}^2} \right)} - \gamma_{1t} \right) dt \\
+ \sigma_1 \sqrt{\gamma_{1t}} \left( \rho_1 dW^Q_{1t} + \sqrt{1 - \rho_1^2} dW^Q_{1t} \right)
\]
where we define the exponential local martingale process

\[ \sigma_t^2 \leq 2 \kappa_1 \theta_1 \]

\[ \kappa_1 + \sigma_1 \left( \rho_1 \lambda_1 + \sqrt{1 - \rho_1^2 \lambda_1^2} \right) > 0 \]

under \( P \):

\[ dv_{1t} = \kappa_1(\theta_1 - v_{1t})dt + \sigma_1 \sqrt{v_{1t}}dZ_{1t} \]

\[ = \kappa_1(\theta_1 - v_{1t})dt + \sigma_1 \sqrt{v_{1t}}(\rho_1 dW_{1t} + \sqrt{1 - \rho_1^2 dW_{1t}^2}) \]

\[ \sigma_t^2 \leq 2 \kappa_1 \theta_1. \]

Recall that for the asset \( S_2 \),

\[ \frac{dS_{2t}}{S_2} = \rho_2 \beta \left( \sqrt{v_{2t}} + \frac{b_2}{\sqrt{v_{2t}}} \right) dW_{2t}^Q + \left( \sqrt{v_{2t}} + \frac{b_2}{\sqrt{v_{2t}}} \right) dW_{2t}^Q, \quad S_2(0) = S_2, \]

\[ \frac{dS_{2t}}{S_2} = \beta \left( \sqrt{v_{2t}} + \frac{b_2}{\sqrt{v_{2t}}} \right) dW_{2t}^Q + \left( \sqrt{v_{2t}} + \frac{b_2}{\sqrt{v_{2t}}} \right) dW_{2t}^Q, \quad \bar{S}_2(0) = \bar{S}_2. \]

Then, under \( Q \) we have

\[ E^Q \left[ \bar{S}_{2t} \right] \]

\[ = \bar{S}_2 E^Q \left[ \exp \left\{ \int_0^t \alpha \left( \sqrt{v_{1s}} + \frac{b_3}{\sqrt{v_{1s}}} \right) dW_{1s}^Q - \frac{1}{2} \beta^2 \int_0^t \left( \sqrt{v_{1s}} + \frac{b_3}{\sqrt{v_{1s}}} \right)^2 ds \right\} \right] \]

\[ \times \exp \left\{ \int_0^t \left( \sqrt{v_{2s}} + \frac{b_2}{\sqrt{v_{2s}}} \right) dW_{2s}^Q - \frac{1}{2} \int_0^t \left( \sqrt{v_{2s}} + \frac{b_2}{\sqrt{v_{2s}}} \right)^2 ds \right\} \]

\[ = \bar{S}_2 E^Q \left[ \exp \left\{ \int_0^t \left( \sqrt{v_{1s}} + \frac{b_3}{\sqrt{v_{1s}}} \right) dZ_{1s}^Q - \frac{1}{2} \rho_1^2 \beta^2 \int_0^t \left( \sqrt{v_{1s}} + \frac{b_3}{\sqrt{v_{1s}}} \right)^2 ds \right\} \right] \]

\[ \times \exp \left\{ \int_0^t \left( \sqrt{v_{2s}} + \frac{b_2}{\sqrt{v_{2s}}} \right) dZ_{2s}^Q - \frac{1}{2} \rho_2^2 \int_0^t \left( \sqrt{v_{2s}} + \frac{b_2}{\sqrt{v_{2s}}} \right)^2 ds \right\} \]

\[ = \bar{S}_2 E^Q \left[ \xi_{1t} \xi_{2t} \right] = \bar{S}_2 E^Q \left[ \xi_{1t} \right] E^Q \left[ \xi_{2t} \right], \]

where we define the exponential local martingale process \( \xi_{it} \) via

\[ \xi_{1t} = \exp \left\{ \int_0^t \left( \sqrt{v_{1s}} + \frac{b_3}{\sqrt{v_{1s}}} \right) dZ_{1s}^Q - \frac{1}{2} \rho_1^2 \beta^2 \int_0^t \left( \sqrt{v_{1s}} + \frac{b_3}{\sqrt{v_{1s}}} \right)^2 ds \right\} \]

\[ \xi_{2t} = \exp \left\{ \int_0^t \left( \sqrt{v_{2s}} + \frac{b_2}{\sqrt{v_{2s}}} \right) dZ_{2s}^Q - \frac{1}{2} \rho_2^2 \int_0^t \left( \sqrt{v_{2s}} + \frac{b_2}{\sqrt{v_{2s}}} \right)^2 ds \right\}. \]
Testing the martingale property for the discounted asset is equivalent to performing the Feller nonexplosion test for volatility using $\xi_a$. Note $\xi_a$ can be interpreted as a new change of measure for the volatility processes:

$$dZ_{1t}^0 = \rho_1 \beta \left( \frac{b_3}{\sqrt{v_{1t}}} \right) dt + dZ_{1t}^0$$
$$dZ_{2t}^0 = \rho_2 \left( \frac{b_3}{\sqrt{v_{2t}}} \right) dt + dZ_{2t}^0.$$  

Hence we check the CIR processes do not reach zero under $Q_1$:

$$\implies dv_{1t} = \kappa_1 (\theta_1 - v_{1t}) dt - \sigma_1 \rho_1 \beta \sqrt{v_{1t}} \left( \frac{b_3}{\sqrt{v_{1t}}} \right) dt + \sigma_1 \sqrt{v_{1t}} dZ_{1t}^{01}$$
$$= (\kappa_1 + \sigma_1 \rho_1 \beta) \left( \frac{\kappa_1 \theta_1 - \sigma_1 \rho_1 \beta b_3}{(\kappa_1 + \sigma_1 \rho_1 \beta)} - v_{1t} \right) dt + \sigma_1 \sqrt{v_{1t}} dZ_{1t}^{01}.$$  

$$\implies dv_{2t} = \kappa_2 (\theta_2 - v_{2t}) dt - \sigma_2 \rho_2 \sqrt{v_{2t}} \left( \frac{b_3}{\sqrt{v_{2t}}} \right) dt + \sigma_2 \sqrt{v_{2t}} dZ_{2t}^{01}$$
$$= (\kappa_2 + \sigma_2 \rho_2) \left( \frac{\kappa_2 \theta_2 - \sigma_2 \rho_2 b_3}{(\kappa_2 + \sigma_2 \rho_2)} - v_{2t} \right) dt + \sigma_2 \sqrt{v_{2t}} dZ_{2t}^{01}.$$  

This leads to the following conditions:

**under $Q_1$:**

$$\sigma_1^2 \leq 2\kappa_1 \theta_1 - 2 |\sigma_1 \rho_1 \beta b_3|,$$
$$\sigma_2^2 \leq 2\kappa_2 \theta_2 - 2 |\sigma_2 \rho_2 b_2|,$$
$$\kappa_1 + \sigma_1 \rho_1 \beta > 0,$$
$$\kappa_2 + \sigma_2 \rho_2 > 0$$

**under $Q$:**

$$dv_{1t} = \kappa_1 (\theta_1 - v_{1t}) dt - \sigma_1 \left( \rho_1 \bar{\lambda}_1 + \sqrt{1 - \rho_1^2 \bar{\lambda}_1^2} \right) v_{1t} dt + \sigma_1 \sqrt{v_{1t}} \left( \rho_1 dW_{1t}^Q + \sqrt{1 - \rho_1^2} dW_{1t}^{QL} \right)$$
$$= \left( \kappa_1 + \sigma_1 \left( \rho_1 \bar{\lambda}_1 + \sqrt{1 - \rho_1^2 \bar{\lambda}_1^2} \right) \right) \left( \frac{\kappa_1 \theta_1}{\kappa_1 + \sigma_1 \left( \rho_1 \bar{\lambda}_1 + \sqrt{1 - \rho_1^2 \bar{\lambda}_1^2} \right)} - v_{1t} \right) dt$$
$$+ \sigma_1 \sqrt{v_{1t}} \left( \rho_1 dW_{1t}^Q + \sqrt{1 - \rho_1^2} dW_{1t}^{QL} \right)$$

$$dv_{2t} = \kappa_2 (\theta_2 - v_{2t}) dt - \sigma_2 \left( \rho_2 \bar{\lambda}_2 + \sqrt{1 - \rho_2^2 \bar{\lambda}_2^2} \right) v_{2t} dt + \sigma_2 \sqrt{v_{2t}} \left( \rho_2 dW_{2t}^Q + \sqrt{1 - \rho_2^2} dW_{2t}^{QL} \right)$$
$$= \left( \kappa_2 + \sigma_2 \left( \rho_2 \bar{\lambda}_2 + \sqrt{1 - \rho_2^2 \bar{\lambda}_2^2} \right) \right) \left( \frac{\kappa_2 \theta_2}{\kappa_2 + \sigma_2 \left( \rho_2 \bar{\lambda}_2 + \sqrt{1 - \rho_2^2 \bar{\lambda}_2^2} \right)} - v_{2t} \right) dt$$
$$+ \sigma_2 \sqrt{v_{2t}} \left( \rho_2 dW_{2t}^Q + \sqrt{1 - \rho_2^2} dW_{2t}^{QL} \right)$$
D.2. Proof of joint conditional c.f.

Proof of Proposition 5.3.2.

D.2.1 Under $Q$:

Consider assets $S_1$ and $S_2$ under the risk-neutral measure $Q$, such that

$$
\frac{dS_{1t}}{S_{1t}} = r dt + \sigma_1 \sqrt{v_{1t}} dW_{1t}, \quad S_{1}(0) = S_1 > 0,
$$

$$
\frac{dS_{2t}}{S_{2t}} = r dt + \beta(\sqrt{v_{1t}} + \frac{b_1}{\sqrt{v_{1t}}})dW_{1t} + (\sqrt{v_{2t}} + \frac{b_2}{\sqrt{v_{2t}}})dW_{2t}, \quad S_{2}(0) = S_2 > 0.
$$

Define the following $Y_i$, where $i = 1, 2, 3$:

$$
\begin{align*}
    dY_{1t} &= \left[ r - \frac{1}{2} \left( \sqrt{v_{1t}} + \frac{b_1}{\sqrt{v_{1t}}} \right)^2 \right] dt + \left( \sqrt{v_{1t}} + \frac{b_1}{\sqrt{v_{1t}}} \right) dW_{1t}, \\
    dY_{2t} &= \left[ r - \frac{1}{2} \left( \sqrt{v_{2t}} + \frac{b_2}{\sqrt{v_{2t}}} \right)^2 \right] dt + \left( \sqrt{v_{2t}} + \frac{b_2}{\sqrt{v_{2t}}} \right) dW_{2t}, \\
    dY_{3t} &= \left[ r - \frac{1}{2} \beta^2 \left( \sqrt{v_{1t}} + \frac{b_3}{\sqrt{v_{1t}}} \right)^2 \right] dt + \beta(\sqrt{v_{1t}} + \frac{b_3}{\sqrt{v_{1t}}})dW_{1t}.
\end{align*}
$$

It can be seen that $Y_{1t}$ and $Y_{3t}$ are about BM $W_{1t}$, while $Y_{2t}$ is about BM $W_{2t}$. Thus, $Y_{2t}$ is independent of $Y_{1t}$ and $Y_{3t}$ because of the independence between $W_{1t}$ and $W_{2t}$.
The joint conditional c.f. follows

\[
\mathbb{E}
\left[
\exp\left(u_1 \ln(S_1) + u_2 \ln(S_2)\right) \bigg| F_t
\right]
\]

\[
= \mathbb{E}
\left[
\exp\left(u_1 Y_{1,t} + u_2 (Y_{2,t} + Y_{3,t} - r t)\right) \bigg| F_t
\right]
\]

\[
= \mathbb{E}^{-r t u_2}
\left[
\exp\left(u_1 Y_{1,t}\right) \cdot \exp\left(u_2 (Y_{2,t} + Y_{3,t})\right) \bigg| F_t
\right]
\]

\[
= \mathbb{E}^{-r t u_2}
\left[
\exp\left(u_1 Y_{1,t} + u_2 Y_{3,t}\right) \bigg| F_t
\right]
\cdot \mathbb{E}
\left[
\exp\left(u_2 Y_{2,t}\right) \bigg| F_t
\right].
\]

(D.2)

The last equality is based on the fact that \(Y_{2,t}\) is independent of \(Y_{1,t}\) and \(Y_{3,t}\). Note that the second expectation closely followed Grasselli (2017), Proposition 3.1. Now, we focus on the first expectation.

We firstly express \(u_1 Y_{1,t} + u_2 Y_{3,t}\), such that

\[
u_1 Y_{1,t} + u_2 Y_{3,t}
= u_1
\left[Y_{1,0} + \int_0^t \left[ r - \frac{1}{2}(\sqrt{\nu_1} + \frac{b_1}{\sqrt{\nu_1}})^2 \right] ds + \int_0^t (\sqrt{\nu_1} + \frac{b_1}{\sqrt{\nu_1}}) dW_{1,s}\right]
\]

\[
+ u_2 \left[Y_{3,0} + \int_0^t \left[ r - \frac{1}{2}\beta^2(\sqrt{\nu_1} + \frac{b_3}{\sqrt{\nu_1}})^2 \right] ds + \beta \int_0^t (\sqrt{\nu_1} + \frac{b_3}{\sqrt{\nu_1}}) dW_{1,s}\right]
\]

\[
= \left(u_1 Y_{1,0} + u_2 Y_{3,0}\right) + \int_0^t (u_1 + u_2) \left[ r - \frac{1}{2}(G_0 v_1 + 2G_1 + \frac{G_2}{v_1}) \right] ds + \int_0^t (u_1 + u_2) (\sqrt{\nu_1} + \frac{G_3}{\sqrt{\nu_1}}) dW_{1,s},
\]

(D.3)

where

\[
G_0 = \frac{u_1 + u_2 \beta^2}{u_1 + u_2}, \quad G_1 = \frac{u_1 b_1 + u_2 b_3 \beta^2}{u_1 + u_2}, \quad G_2 = \frac{u_1 b_1^2 + u_2 b_3^2 \beta^2}{u_1 + u_2}, \quad G_3 = \frac{u_1 b_1 + u_2 b_3 \beta}{u_1 + u_2}.
\]
Substituting into the first expectation, we have
\[
\mathbb{E}\left[ \exp \left\{ u_1 Y_{1,t} + u_2 Y_{3,t} \right\} | \mathcal{F}_t \right] = \mathbb{E}\left[ \mathbb{E}\left[ \exp \left\{ u_1 Y_{1,t} + u_2 Y_{3,t} \right\} | \mathcal{G}_t \right] | \mathcal{F}_t \right] \\
= \exp \left\{ (u_1 Y_{1,0} + u_2 Y_{3,0}) + (u_1 + u_2)(r - G_1) t \right\} \\
\times \mathbb{E}\left[ \mathbb{E}\left[ \exp \left\{ \int_0^t (u_1 + u_2) \left[ - \frac{1}{2} (G_0 v_1 + \frac{G_2}{v_1}) \right] ds + \int_0^t (u_1 + u_2) (\sqrt{v_1} + \frac{G_3}{\sqrt{v_1}}) dW_{1,s} \right] | \mathcal{G}_t \right] | \mathcal{F}_t \right] \\
= \exp \left\{ (u_1 Y_{1,0} + u_2 Y_{3,0}) + (u_1 + u_2)(r - G_1) t \right\} \\
\times \mathbb{E}\left[ \mathbb{E}\left[ \exp \left\{ - \frac{1}{2} (u_1 + u_2) \int_0^t (G_0 v_1 + \frac{G_2}{v_1}) ds \right\} \mathbb{E}\left[ \exp \left\{ \int_0^t (u_1 + u_2) (\sqrt{v_1} + \frac{G_3}{\sqrt{v_1}}) dW_{1,s} \right\} | \mathcal{G}_t \right] | \mathcal{F}_t \right] \\
= \exp \left\{ (u_1 Y_{1,0} + u_2 Y_{3,0}) + (u_1 + u_2)(r - G_1) t \right\} \\
\times \mathbb{E}\left[ \mathbb{E}\left[ \exp \left\{ - \frac{1}{2} (u_1 + u_2) G_0 \int_0^t v_1 ds - \frac{1}{2} G_2 (u_1 + u_2) \int_0^t \frac{1}{v_1} ds \right\} \mathbb{E}\left[ \exp \left\{ \int_0^t (u_1 + u_2) (\sqrt{v_1} + \frac{G_3}{\sqrt{v_1}}) dW_{1,s} \right\} | \mathcal{G}_t \right] | \mathcal{F}_t \right], \\
\]
where the first equality is based on the tower property, and \( \mathcal{G}_t \) is a filtration that contains \( \mathcal{F}_t \) and information about \( v_{1,t} \). The second equality simply substitutes the expression for \( u_1 Y_{1,t} + u_2 Y_{3,t} \) obtained before. The third equality follows from taking out terms that do not depend on \( \mathcal{G}_t \) from the inner expectation. The last two equalities simplify and reorganize terms.

Now, recall that
\[
\int_0^t dv_{1,s} = \int_0^t \kappa_1 (\theta_1 - v_{1,s}) ds + \int_0^t \sigma_1 \sqrt{v_{1,s}} dZ_{1,s},
\]
where \( dZ_{1,t} = \rho_1 dW_{1,t} + \sqrt{1 - \rho_1^2} dW_{1,t}^L \). Thus, we have
\[
\int_0^t \sqrt{v_{1,s}} dW_{1,s} = \frac{1}{\sigma_1 \rho_1} \left[ (v_{1,t} - v_{1,0}) - \int_0^t \kappa_1 (\theta_1 - v_{1,s}) ds - \sqrt{1 - \rho_1^2} \int_0^t \sigma_1 \sqrt{v_{1,s}} dW_{1,s}^L \right].
\tag{D.4}
\]

Also, we have
\[
\int_0^t d \log(v_{1,s}) = \int_0^t \frac{\sigma_1}{\sqrt{v_{1,s}}} dZ_{1,s} + \int_0^t \left( \frac{\kappa \theta_1}{v_{1,s}} - \kappa \right) ds - \int_0^t \frac{\sigma_1^2}{2 v_{1,s}} ds,
\]
where \(dZ_{1t} = \rho_1 dW_{1t} + \sqrt{1 - \rho_1^2} dW_{1t}^\perp\). Thereby,

\[
\int_0^t \frac{1}{\sqrt{v_{1s}}} dW_{1s} = \frac{1}{\sigma_1 \rho_1} \left[ (\log(v_{1t}) - \log(v_{10})) - \int_0^t \left( \frac{\kappa_1 \theta_1}{v_{1s}} - \kappa_1 \right) ds + \int_0^t \frac{\sigma_1^2}{2v_{1s}} ds - \sqrt{1 - \rho_1^2} \int_0^t \frac{\sigma_1}{\sqrt{v_{1s}}} dW_{1s}^\perp \right].
\]

(D.5)

Substituting the expressions for \(\int_0^t \sqrt{v_{1s}} dW_{1s}\) and \(\int_0^t \frac{1}{\sqrt{v_{1s}}} dW_{1s}\) into the inner expectation,

\[
\mathbb{E} \left[ \exp \left\{ u_1 Y_{1,t} + u_2 Y_{3,t} \right\} \bigg| F_t \right]
\]

\[=
\exp \left\{ \left( u_1 Y_{1,0} + u_2 Y_{3,0} \right) + (u_1 + u_2) \left( r - G_1 \right) t \right]\]

\[\times \mathbb{E} \left[ \exp \left\{ -\frac{1}{2} \left( u_1 + u_2 \right) G_0 \int_0^t v_{1s} dt - \frac{1}{2} G_2 (u_1 + u_2) \int_0^t \frac{1}{v_{1s}} ds + \frac{(u_1 + u_2)}{\sigma_1 \rho_1} \left[ (v_{1t} - v_{10}) - \int_0^t \kappa_1 (\theta_1 - v_{1s}) ds \right] 
\right]
\[+ \frac{G_3 (u_1 + u_2)}{\sigma_1 \rho_1} \left[ (\log(v_{1t}) - \log(v_{10})) - \int_0^t \left( \frac{\kappa_1 \theta_1}{v_{1s}} - \kappa_1 \right) ds + \int_0^t \frac{\sigma_1^2}{2v_{1s}} ds \right] \right] \]

\[\times \mathbb{E} \left[ \exp \left\{ -\frac{(u_1 + u_2)}{\rho_1} \int_0^t (\sqrt{v_{1s}} + \frac{G_3}{\sqrt{v_{1s}}}) dW_{1s}^\perp \right\} \bigg| F_t \right] \]

Given the information/path of \(v_t\) for \(t \geq 0\), the random variable inside this expectation is a lognormal with \(\mu = 0\) and \(\sigma = -\frac{(u_1 + u_2)}{\rho_1} \int_0^t (\sqrt{v_{1s}} + \frac{G_3}{\sqrt{v_{1s}}}) dW_{1s}^\perp\). Thus, the inner expectation is the mean of this lognormal, such that

\[
\mathbb{E} \left[ \exp \left\{ -\frac{(u_1 + u_2)}{\rho_1} \int_0^t (\sqrt{v_{1s}} + \frac{G_3}{\sqrt{v_{1s}}}) dW_{1s}^\perp \right\} \bigg| F_t \right]
\]

\[=
\exp \left\{ \mu + \frac{1}{2} \sigma^2 \right\}
\]

\[=
\exp \left\{ 0 + \frac{1}{2} \frac{(u_1 + u_2)^2 (1 - \rho_1^2)}{\rho_1^2} \int_0^t (v_{1s} + 2G_3 + \frac{G_3^2}{v_{1s}}) ds \right\}. \]
Substituting in the above expression and regrouping terms,

\[
\mathbb{E}\left[ \exp \left\{ u_1 Y_{1,t} + u_2 Y_{3,t} \right\} \bigg| \mathcal{F}_t \right] \\
= \exp \left\{ (u_1 Y_{1,0} + u_2 Y_{3,0}) + (u_1 + u_2) \left( r - G_1 - \frac{\kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3 \kappa_1}{\sigma_1 \rho_1} + \frac{(u_1 + u_2)(1 - \rho_1^2)}{\rho_1^2} \right) \right\} \\
- \frac{(u_1 + u_2)}{\sigma_1 \rho_1} v_{10} - \frac{G_3 (u_1 + u_2)}{\sigma_1 \rho_1} \log(v_{10}) \bigg] \\
\times \mathbb{E}\left[ \exp \left\{ (u_1 + u_2) \left[ - \frac{1}{2} G_0 + \frac{\kappa_1}{\sigma_1 \rho_1} + \frac{1}{2} \frac{(u_1 + u_2)(1 - \rho_1^2)}{\rho_1^2} \right] \left( r - G_1 - \frac{\kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3 \kappa_1}{\sigma_1 \rho_1} + \frac{(u_1 + u_2)(1 - \rho_1^2)}{\rho_1^2} \right) \right\} \bigg| \mathcal{F}_t \right] \\
+ \frac{(u_1 + u_2)}{\sigma_1 \rho_1} v_{1l} - \frac{G_3 (u_1 + u_2)}{\sigma_1 \rho_1} \log(v_{1l}) \bigg] \times \phi_1,
\] (D.6)

where

\[
\phi_1(t, v_1; \alpha_1, \lambda_1, \mu_1, v_1) = \mathbb{E}\left[ v_{1l}^{\alpha_1} \cdot \exp \left\{ - \lambda_1 v_{1l} - \mu_1 \int_0^t v_{1s} ds - v_1 \int_0^t \frac{1}{v_{1s}} \right\} \bigg| \mathcal{F}_t \right],
\] (D.7)

with parameters

\[
\alpha_1 = -\frac{G_3 (u_1 + u_2)}{\sigma_1 \rho_1},
\]
\[
\lambda_1 = -\frac{(u_1 + u_2)}{\sigma_1 \rho_1},
\]
\[
\mu_1 = -(u_1 + u_2) \left[ - \frac{1}{2} G_0 + \frac{\kappa_1}{\sigma_1 \rho_1} + \frac{1}{2} \frac{(u_1 + u_2)(1 - \rho_1^2)}{\rho_1^2} \right],
\]
\[
\nu_1 = -(u_1 + u_2) \left[ - \frac{1}{2} G_2 - \frac{G_3 \kappa_1}{\sigma_1 \rho_1} + \frac{G_3 \kappa_1}{\sigma_1 \rho_1} + \frac{1}{2} \frac{(u_1 + u_2)(1 - \rho_1^2)}{\rho_1^2} \right].
\]

By Grasselli (2017) Theorem A.1, we know the analytical representation of \( \phi_1(t, v_1) \) with parameters \( \alpha_1, \lambda_1, \mu_1, \nu_1 \) above.
Now, back to Equation (D.2). If we express the second expectation similarly, we have

\[
\mathbb{E}\left[ \exp\left\{ u_1 \ln(S_1) + u_2 \ln(S_2) \right\} \bigg| \mathcal{F}_t \right] 
= e^{\tau u_2} \mathbb{E}\left[ \exp\left\{ u_1 Y_{1,t} + u_2 Y_{3,t} \right\} \bigg| \mathcal{F}_t \right] \cdot \mathbb{E}\left[ \exp\left\{ u_2 Y_{2,t} \right\} \bigg| \mathcal{F}_t \right] 
= e^{\tau u_2} \times \exp\left( u_1 Y_{1,0} + u_2 Y_{3,0} \right) + (u_1 + u_2) \left( r - G_1 - \frac{\kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3 \kappa_1}{\sigma_1 \rho_1} + \frac{(u_1 + u_2)(1 - \rho_2^2) G_3}{\rho_1^2} \right) t 
\times \phi_1(t, v_1; \alpha_1, \lambda_1, \mu_1, v_1) 
\times \phi_2(t, v_2; \alpha_2, \lambda_2, \mu_2, v_2) 
= \exp\left( u_1 \ln(S_{1,0}) + u_2 \ln(S_{2,0}) \right) + (u_1 + u_2) \left( r - G_1 - \frac{\kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3 \kappa_1}{\sigma_1 \rho_1} + \frac{(u_1 + u_2)(1 - \rho_2^2) G_3}{\rho_1^2} \right) t 
\times \phi_1(t, v_1; \alpha_1, \lambda_1, \mu_1, v_1) 
\times \phi_2(t, v_2; \alpha_2, \lambda_2, \mu_2, v_2),
\]

where

\[
\phi_2(t, v_2; \alpha_2, \lambda_2, \mu_2, v_2) = \mathbb{E}\left[ v_2^{-u_2} \cdot \exp\left\{ -\lambda_2 v_2 t - \mu_2 \int_0^t v_2 ds - v_2 \int_0^t \frac{1}{v_2} ds \right\} \bigg| \mathcal{F}_t \right], \quad (D.8)
\]

with parameters

\[
\alpha_2 = -\frac{u_2 b_2}{\sigma_2 \rho_2}, \\
\lambda_2 = -\frac{u_2}{\sigma_2 \rho_2}, \\
\mu_2 = -u_2 \left[ -\frac{1}{2} + \frac{\kappa_2}{\sigma_2 \rho_2} + \frac{1}{2} \frac{u_2(1 - \rho_2^2)}{\rho_2^2} \right], \\
\nu_2 = -u_2 \left[ -\frac{b_2^2}{2} - \frac{\kappa_2 \theta_2}{\sigma_2 \rho_2} + \frac{\sigma_2 b_2}{2 \rho_2} + \frac{1}{2} \frac{u_2(1 - \rho_2^2)}{\rho_2^2} b_2^2 \right].
\]
Moreover, if the parameters $\alpha_i, \lambda_i, \mu_i,$ and $v_i,$ for $i = 1, 2,$ satisfy the following conditions

$$
\mu_i > -\frac{\kappa_i^2}{2\sigma_i^2},
$$

$$
v_i \geq -\frac{(2\kappa_i\theta_i - \sigma_i)^2}{8\sigma_i^2},
$$

$$
\alpha_i < \frac{1}{2\sigma_i^2}(2\kappa_i\theta_i + \sigma_i^2 + \sqrt{(2\kappa_i\theta_i - \sigma_i^2)^2 + 8\sigma_i^2v_i}),
$$

$$
\lambda_i \geq -\frac{\sqrt{\kappa_i^2 + 2\mu_i\sigma_i^2 + \kappa_i^2}}{\sigma_i^2},
$$

then the expectations in Equations (D.7) and (D.8) are well defined for all $t \geq 0$ and are given by

$$
\phi_i(t, v_i; \alpha_i, \lambda_i, \mu_i, v_i) = \left(\frac{\beta_i(t, v_i)}{2}\right)^{m_i - \frac{\kappa_i^2}{\sigma_i^2}}K_i(t)\Gamma\left(\frac{3}{2} + \frac{m_i}{2} + \frac{\kappa_i\theta_i}{\sigma_i^2}, m_i + 1, \frac{\beta_i(t, v_i)^2}{4K_i(t)}\right)
$$

$$
\times_1 F_1\left(\frac{1}{2} + \frac{m_i}{2} + \frac{\kappa_i\theta_i}{\sigma_i^2}, m_i + 1, \frac{\beta_i(t, v_i)^2}{4K_i(t)}\right),
$$

with

$$
m_i = \frac{1}{\sigma_i^2}\sqrt{(2\kappa_i\theta_i - \sigma_i^2)^2 + 8\sigma_i^2v_i}, A_i = \kappa_i^2 + 2\mu_i\sigma_i^2,
$$

$$
\beta_i(t, v_i) = \frac{2}{\sigma_i^2}\sqrt{A_i}v_i, K_i(t) = \frac{1}{\sigma_i^2}\left(\sqrt{A_i}\coth\left(\frac{\kappa_i\theta_i}{2}\right) + \kappa_i\right).
$$

**D.2.2 Under $P$:**

Next, we consider assets $S_1$ and $S_2$ under the real world measure $P,$ such that

$$
\frac{dS_{1t}}{S_{1t}} = \mu^1 dt + \left(\sqrt{v_{1t}} + \frac{b_1}{\sqrt{v_{1t}}}\right)dW_{1t}, S_{1}(0) = S_1 > 0,
$$

$$
\frac{dS_{2t}}{S_{2t}} = \mu^2 dt + \beta\left(\sqrt{v_{1t}} + \frac{b_3}{\sqrt{v_{1t}}}\right)dW_{1t} + \left(\sqrt{v_{2t}} + \frac{b_2}{\sqrt{v_{2t}}}\right)dW_{2t}, S_{2}(0) = S_2 > 0,
$$

Define the following $Y_{it},$ where $i = 1, 2, 3:

$$
dY_{1t} = \left[\mu_1 - \frac{1}{2}\left(\sqrt{v_{1t}} + \frac{b_1}{\sqrt{v_{1t}}}\right)^2\right]dt + \left(\sqrt{v_{1t}} + \frac{b_1}{\sqrt{v_{1t}}}\right)dW_{1t},
$$

$$
dY_{2t} = \left[\mu_2 - \frac{1}{2}\left(\sqrt{v_{2t}} + \frac{b_2}{\sqrt{v_{2t}}}\right)^2\right]dt + \left(\sqrt{v_{2t}} + \frac{b_2}{\sqrt{v_{2t}}}\right)dW_{2t},
$$

$$
dY_{3t} = \left[\mu_3 - \frac{1}{2}\beta^2\left(\sqrt{v_{1t}} + \frac{b_3}{\sqrt{v_{1t}}}\right)^2\right]dt + \beta\left(\sqrt{v_{1t}} + \frac{b_3}{\sqrt{v_{1t}}}\right)dW_{1t}.
It can be seen that \( Y_{1,t} \) and \( Y_{3,t} \) are measurable functions of the Brownian motions \( W_{1t} \), while \( Y_{2,t} \) is measurable functions of the Brownian motion \( W_{2t} \). Thus, \( Y_{2,t} \) is independent of \( Y_{1,t} \) and \( Y_{3,t} \) because of the independence between \( W_{1t} \) and \( W_{2t} \).

Further, let \( \lambda_1(v_1) \) represent the market price of risk from \( W_1 \) with rate \( \bar{\lambda}_1 \), while \( \lambda_2(v_2) \) represents the market price of risk from \( W_2 \) with rate \( \bar{\lambda}_2 \). To keep the form of the market price of risk general, we consider

\[
\begin{align*}
\mu_1 &= r + \bar{\lambda}_1(B_1v_1 + C_1 + \frac{D_1}{v_1}) \\
\mu_2 &= r + \bar{\lambda}_2(B_2v_2 + C_2 + \frac{D_2}{v_2}) \\
\mu_3 &= r + \bar{\lambda}_3(B_3v_1 + C_3 + \frac{D_3}{v_1}),
\end{align*}
\]

where \( B_i, C_i, \) and \( D_i, i = 1, 2, 3, \) are constants.

Thus, it can easily see that

\[
\begin{align*}
\mu^{S1} &= \mu_1 \\
\mu^{S2} &= \mu_2 + \mu_3 - r.
\end{align*}
\]

The joint conditional c.f. follows

\[
\begin{align*}
\mathbb{E} \left[ \exp \left( u_1 \ln(S_1) + u_2 \ln(S_2) \right) \right| \mathcal{F}_t] &= \mathbb{E} \left[ \exp \left( u_1 Y_{1,t} + u_2 (Y_{2,t} + Y_{3,t} - rt) \right) \right| \mathcal{F}_t] \\
&= \mathbb{E} \left[ \exp \left( u_1 Y_{1,t} \right) \cdot \exp \left( u_2 (Y_{2,t} + Y_{3,t} - rt) \right) \right| \mathcal{F}_t] \\
&= e^{-u_2rt} \mathbb{E} \left[ \exp \left( u_1 Y_{1,t} + u_2 Y_{3,t} \right) \right| \mathcal{F}_t] \cdot \mathbb{E} \left[ \exp \left( u_2 Y_{2,t} \right) \right| \mathcal{F}_t] \\
&= e^{-u_2rt} \mathbb{E} \left[ \exp \left( u_1 Y_{1,t} + u_2 Y_{3,t} \right) \right| \mathcal{F}_t] \cdot \mathbb{E} \left[ \exp \left( u_2 Y_{2,t} \right) \right| \mathcal{F}_t].
\end{align*}
\]

The last equality is based on the fact that \( Y_{2,t} \) is independent of \( Y_{1,t} \) and \( Y_{3,t} \).
We first express the inside of the first expectation, such that

\[ u_1 Y_{1,t} + u_2 Y_{3,t} \]

\[ = u_1 \left[ Y_{1,0} + \int_0^t \left( u_1 - \frac{1}{2} \left( \sqrt{v_1} + \frac{b_1}{\sqrt{v_1}} \right)^2 \right) ds + \int_0^t \left( \sqrt{v_1} + \frac{b_1}{\sqrt{v_1}} \right) dW_{1,s} \right] 
+ u_2 \left[ Y_{3,0} + \int_0^t \left( u_2 - \frac{1}{2} \beta \left( \sqrt{v_1} + \frac{b_3}{\sqrt{v_1}} \right)^2 \right) ds + \beta \int_0^t \left( \sqrt{v_1} + \frac{b_3}{\sqrt{v_1}} \right) dW_{1,s} \right] 
= \left( u_1 Y_{1,0} + u_2 Y_{3,0} \right) + \int_0^t (u_1 + u_2) \left[ r + \frac{u_1}{u_1 + u_2} \lambda_1 (B) v_1 + C + \frac{D_1}{v_1} \right] + \frac{u_2}{u_1 + u_2} \lambda_1 (B) v_1 + C + \frac{D_3}{v_1} ds 
- \frac{1}{2} \left( G_0' v_1 + 2 G_1' + \frac{G_2'}{v_1} \right) ds + \int_0^t (u_1 + u_2) \left( \sqrt{v_1} + \frac{G_3'}{v_1} \right) dW_{1,s} 
= \left( u_1 Y_{1,0} + u_2 Y_{3,0} \right) + \int_0^t (u_1 + u_2) \left[ r + \lambda_1 (B) v_1 + C + \frac{D_1}{v_1} \right] ds 
+ \int_0^t (u_1 + u_2) \left( \sqrt{v_1} + \frac{G_3'}{v_1} \right) dW_{1,s}, 
\]

where

\[ B = \frac{u_1 B_1 + u_2 B_2}{u_1 + u_2}, \quad C = \frac{u_1 C_1 + u_2 C_2}{u_1 + u_2}, \quad D = \frac{u_1 D_1 + u_2 D_2}{u_1 + u_2}, \]

\[ G_0' = \frac{u_1 + u_2 b_2^2}{u_1 + u_2}, \quad G_1' = \frac{u_1 b_1 + u_2 b_3 b_2^2}{u_1 + u_2}, \quad G_2' = \frac{u_1 b_1^2 + u_2 b_3^2 b_2^2}{u_1 + u_2}, \quad G_3' = \frac{u_1 b_1 + u_2 b_3 b_2^2}{u_1 + u_2}. \]

We can further simplify the expression above

\[ u_1 Y_{1,t} + u_2 Y_{3,t} \]

\[ = \left( u_1 Y_{1,0} + u_2 Y_{3,0} \right) + \int_0^t (u_1 + u_2) \left[ r - \frac{1}{2} \left( G_0 v_1 + 2 G_1 + \frac{G_2}{v_1} \right) \right] ds 
+ \int_0^t (u_1 + u_2) \left( \sqrt{v_1} + \frac{G_3}{v_1} \right) dW_{1,s}, 
\]

where

\[ G_0 = G'_0 - 2 \lambda_1 B, \quad G_1 = G'_1 - 2 \lambda_1 C, \quad G_2 = G'_2 - 2 \lambda_1 D, \quad G_3 = G'_3. \]

Now, we arrive at an expression that is similar to Equation (D.3). Thus, we can directly apply the result of the first expectation, i.e., \( E \left[ \exp \left( u_1 Y_{1,t} + u_2 Y_{3,t} \right) \right] \) in Equation (D.6), under the risk-neutral \( Q \) with parameters \( G_0, G_1, G_2, \) and \( G_3 \) specified above and the corresponding
\(\phi()\) like Equation (D.7):

\[
\mathbb{E}\left[ \exp \left\{ u_1 Y_{1,t} + u_2 Y_{3,t} \right\} \mid \mathcal{F}_t \right] \\
= \mathbb{E}\left[ \exp \left\{ \left( u_1 Y_{1,0} + u_2 Y_{3,0} \right) + \int_0^t \left( u_1 + u_2 \right) \left[ r - \frac{1}{2} \left( G_0 v_1 + 2 G_1 + \frac{G_2}{v_1} \right) \right] ds \right. \right. \\
+ \left. \left. \int_0^t \left( u_1 + u_2 \right) \left( \sqrt{v_1} + \frac{G_3}{\sqrt{v_1}} \right) dW_{1,s} \right] \mid \mathcal{F}_t \right] \\
= \exp \left\{ \left( u_1 Y_{1,0} + u_2 Y_{3,0} \right) + \left( u_1 + u_2 \right) \left( r - G_1 \right) t \right\} \\
\times \mathbb{E}\left[ \exp \left\{ - \frac{1}{2} \left( u_1 + u_2 \right) G_0 \int_0^t v_{1,s} ds - \frac{1}{2} G_2 \left( u_1 + u_2 \right) \int_0^t \frac{1}{v_{1,s}} ds \right. \right. \\
+ \left. \frac{G_3 \left( u_1 + u_2 \right)}{\sigma_1 \rho_1} \left[ \left( \log(v_{1,t}) - \log(v_{1,0}) \right) - \int_0^t \left( \frac{\kappa_1 \theta_1}{v_{1,s}} - \kappa_1 \right) ds + \int_0^t \frac{\sigma_1^2}{2v_{1,s}} ds \right] \right\} \\
\times \mathbb{E}\left[ \exp \left\{ - \frac{\left( u_1 + u_2 \right)}{\rho_1} \int_0^t \left( \sqrt{v_{1,s}} + \frac{G_3}{\sqrt{v_{1,s}}} \right) dW_{1,s} \right] \mid \mathcal{F}_t \right] \\
= \exp \left\{ \left( u_1 Y_{1,0} + u_2 Y_{3,0} \right) + \left( u_1 + u_2 \right) \left( r - G_1 \right) t \right\} \\
\times \mathbb{E}\left[ \exp \left\{ - \frac{1}{2} \left( u_1 + u_2 \right) G_0 \int_0^t v_{1,s} ds - \frac{1}{2} G_2 \left( u_1 + u_2 \right) \int_0^t \frac{1}{v_{1,s}} ds \right. \right. \\
+ \left. \frac{G_3 \left( u_1 + u_2 \right)}{\sigma_1 \rho_1} \left[ \left( \log(v_{1,t}) - \log(v_{1,0}) \right) - \int_0^t \left( \frac{\kappa_1 \theta_1}{v_{1,s}} - \kappa_1 \right) ds + \int_0^t \frac{\sigma_1^2}{2v_{1,s}} ds \right] \right\} \\
\times \exp \left\{ 0 + \frac{1}{2} \left( u_1 + u_2 \right)^2 \left( 1 - \rho_1^2 \right) \int_0^t \left( v_{1,s} + 2 G_3 + \frac{G_3^2}{v_{1,s}} \right) ds \right\} \left[ \mathcal{F}_t \right] \\
= \exp \left\{ \left( u_1 Y_{1,0} + u_2 Y_{3,0} \right) + \left( u_1 + u_2 \right) \left( r - G_1 - \frac{\kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3 \kappa_1}{\sigma_1 \rho_1} + \frac{\left( u_1 + u_2 \right) \left( 1 - \rho_2^2 \right) G_3}{\rho_1^2} \right) t \right. \\
- \left. \frac{\left( u_1 + u_2 \right)}{\sigma_1 \rho_1} - \frac{G_3 \left( u_1 + u_2 \right)}{\sigma_1 \rho_1} \log(v_{1,0}) \right\} \times \phi_1, \\
\text{where} \\
\phi_1(t, v_{1}; \alpha_1, \lambda_1, \mu_1, v_{1}) = \mathbb{E}\left[ v_{1,0}^{x_{1,0}} \cdot \exp \left\{ - \lambda_1 v_{1,s} - \mu_1 \int_0^t v_{1,s} ds - \lambda_1 \int_0^t 1 \right\} \mid \mathcal{F}_t \right], \quad (D.11)
D.2. Proof of joint conditional c.f.

with parameters

\[ \begin{align*}
\alpha_1 &= -\frac{G_3(u_1 + u_2)}{\sigma_1 \rho_1} = -\frac{G_3'(u_1 + u_2)}{2 \rho_1} \\
\lambda_1 &= \frac{(u_1 + u_2)}{\sigma_1 \rho_1} \\
\mu_1 &= -(u_1 + u_2) \left[ -\frac{1}{2} G_0 + \frac{\kappa_1}{\sigma_1 \rho_1} + \frac{1}{2} \frac{(u_1 + u_2)(1 - \rho_1^2)}{\rho_1^2} \right] \\
\nu_1 &= -(u_1 + u_2) \left[ -\frac{1}{2} (G_0 - 2 \lambda_1 B) + \frac{\kappa_1}{\sigma_1 \rho_1} + \frac{1}{2} \frac{(u_1 + u_2)(1 - \rho_1^2)}{\rho_1^2} \right] \\
\nu_1 &= -(u_1 + u_2) \left[ -\frac{1}{2} (G_2 - 2 \lambda_1 D) - \frac{G_3 \kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3' \sigma_1}{2 \rho_1} + \frac{1}{2} \frac{(u_1 + u_2)(1 - \rho_1^2)G_3'^2}{\rho_1^2} \right].
\end{align*} \]

Note that \( \mathcal{G}_t \) in the second equality is a filtration that contains \( \mathcal{F}_t \) and the information/path about \( \nu_{1t} \), and we only leave the terms that depend on \( \mathcal{G}_t \) in the inner expectation. The third equation comes from substituting the process of \( \int_0^t \sqrt{v_{1s}} dW_{1s} \) and \( \int_0^t \frac{1}{\sqrt{v_{1s}}} dW_{1s} \) from Equations (D.4) and (D.5) into the inner expectation. Given the information of \( \nu_{1t} \), the inner expectation is a lognormal with \( \mu = 0 \), and \( \sigma = \frac{-(u_1 + u_2) \sqrt{1 - \rho_1^2}}{\rho_1} \int_0^t \left( \sqrt{v_{1s}} + \frac{G_3}{\sqrt{v_{1s}}} \right) dW_{1s} \). Thus, the fourth equation takes the expectation of a lognormal. The last equation regroups terms.

For the second expectation, we have

\[ \mathbb{E} \left[ \exp \left\{ u_2 Y_{2,1} \right\} \right] \]

\[ = \mathbb{E} \left[ \exp \left\{ u_2 \int_0^t \left[ \mu_2 - \frac{1}{2} \left( \sqrt{v_2} + \frac{b_2}{\sqrt{v_2}} \right)^2 \right] ds + \int_0^t \left( \sqrt{v_2} + \frac{b_2}{\sqrt{v_2}} \right) dW_{2s} \right\} \right] \]

\[ = \mathbb{E} \left[ \exp \left\{ u_2 \int_0^t \left[ r + \lambda_2 (B v_2 + C_2 + \frac{D_2}{\sqrt{v_2}}) \right] ds + \int_0^t \left( \sqrt{v_2} + \frac{b_2}{\sqrt{v_2}} \right) dW_{2s} \right\} \right] \]

\[ = \mathbb{E} \left[ \exp \left\{ u_2 \int_0^t \left[ r + \lambda_2 (B v_2 + C_2 + \frac{D_2}{\sqrt{v_2}}) \right] ds + \int_0^t \left( \sqrt{v_2} + \frac{b_2}{\sqrt{v_2}} \right) dW_{2s} \right\} \right] \]

\[ = \exp \left\{ u_2 Y_{2,0} \right\} \mathbb{E} \left[ \exp \left\{ u_2 \int_0^t \left[ r - \frac{1}{2} (\bar{G}_0 v_2 + \bar{G}_1 + \bar{G}_2 \frac{v_2}{v_2}) \right] ds + \int_0^t \left( \sqrt{v_2} + \frac{G_3}{\sqrt{v_2}} \right) dW_{2s} \right\} \right] \]

where

\[ \bar{G}_0 = 1 - 2 \lambda_2 B_2, \quad \bar{G}_1 = 2b_2 - 2 \lambda_2 C_2, \quad \bar{G}_2 = b_2^2 - \lambda_2 D_2, \quad \bar{G}_3 = b_2. \]

Moreover, substituting the processes of \( \int_0^t \sqrt{v_{2s}} dW_{2s} \) and \( \int_0^t \frac{1}{\sqrt{v_{2s}}} dW_{2s} \) in a manner similar
to that in Equations (D.4) and (D.5), we have

\[
\mathbb{E}\left[ \exp\left\{ u_2 Y_{2,t} \right\} \Big| \mathcal{F}_t \right] = \exp\left\{ u_2 Y_{2,0} + u_2 \left( r - \tilde{G}_1 - \frac{\kappa_2 \theta}{\sigma_2 \rho_2} + \frac{\tilde{G}_3 \kappa_2}{\sigma_2 \rho_2} + \frac{1 - \rho_2^2}{\rho_2} \bar{G}_3 \right) t - \frac{u_2}{\sigma_2 \rho_2} v_{20} - \frac{u_2 \bar{G}_3}{\sigma_2 \rho_2} \log(v_{20}) \right\}
\]

\[
\times \phi_2(t, v_2; \alpha_2, \lambda_2, \mu_2, v_2),
\]

where

\[
\phi_2(t, v_2; \alpha_2, \lambda_2, \mu_2, v_2) = \mathbb{E}\left[ v_2^{-\alpha_2} \cdot \exp\left\{ -\lambda_2 v_{2t} - \mu_2 \int_0^t v_{2s} ds - v_2 \int_0^t \frac{1}{v_{2s}} \right\} \Big| \mathcal{F}_t \right], \tag{D.12}
\]

with parameters

\[
\alpha_2 = -\frac{u_2 \bar{G}_3}{\sigma_2 \rho_2} = -\frac{u_2 b_2}{\sigma_2 \rho_2}
\]

\[
\lambda_2 = -\frac{u_2}{\sigma_2 \rho_2}
\]

\[
\mu_2 = -u_2 \left[ -\frac{1}{2} \tilde{G}_0 + \frac{\kappa_2}{\sigma_2 \rho_2} + \frac{1}{2} \frac{u_2 (1 - \rho_2^2)}{\rho_2} \right]
\]

\[
= -u_2 \left[ -\frac{1}{2} (1 - 2 \lambda_2 B_2) + \frac{\kappa_2}{\sigma_2 \rho_2} + \frac{1}{2} \frac{u_2 (1 - \rho_2^2)}{\rho_2} \right]
\]

\[
v_2 = -u_2 \left[ -\frac{\tilde{G}_2}{2} - \frac{\kappa_2 \theta_2 \bar{G}_3}{\sigma_2 \rho_2} + \frac{\sigma_2 \bar{G}_3}{2 \rho_2} + \frac{1}{2} \frac{u_2 (1 - \rho_2^2) \bar{G}_3}{\rho_2} \right]
\]

\[
= -u_2 \left[ -\frac{(b_2^2 - \lambda_2 D_2)^2}{2} - \frac{\kappa_2 \theta_2 b_2}{\sigma_2 \rho_2} + \frac{\sigma_2 b_2}{2 \rho_2} + \frac{1}{2} \frac{u_2 (1 - \rho_2^2) b_2}{\rho_2} \right].
\]
The joint conditional c.f. follows

\[ \mathbb{E} \left[ \exp \left\{ u_1 \ln(S_1) + u_2 \ln(S_2) \right\} \bigg| \mathcal{F}_t \right] \]

\[ = e^{u_2 \bar{G}_2} \cdot \mathbb{E} \left[ \exp \left\{ u_1 Y_{1,t} + u_2 Y_{3,t} \right\} \bigg| \mathcal{F}_t \right] \cdot \mathbb{E} \left[ \exp \left\{ u_2 Y_{2,t} \right\} \bigg| \mathcal{F}_t \right] \]

\[ = e^{u_2 \bar{G}_2} \cdot \mathbb{E} \left[ \exp \left\{ (u_1 Y_{1,0} + u_2 Y_{3,0}) + (u_1 + u_2)(r - G_1 - \frac{\kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3 \kappa_1}{\sigma_1 \rho_1} + \frac{(u_1 + u_2)(1 - \rho^2)G_3}{\rho_1^2})t \right\} - \frac{(u_1 + u_2)}{\sigma_1 \rho_1} \log(v_{10}) \right] \times \phi_1 \]

\[ \times \exp \left\{ u_2 Y_{2,0} + u_2 \left( \frac{\kappa_2 \theta_2}{\sigma_2 \rho_2} + \frac{G_3 \kappa_2}{\sigma_2 \rho_2} + \frac{(1 - \rho_2^2)\bar{G}_3}{\rho_2^2} \right) - \frac{u_2}{\sigma_2 \rho_2} \log(v_{20}) \right\} \times \phi_2(t, v_{3,0}; \alpha_2, \lambda_2, \mu_2, v_{2,0}) \]

\[ = \exp \left\{ (u_1 \ln(S_1) + u_2 \ln(S_2)) + (u_1 + u_2)(r - G_1 - \frac{\kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3 \kappa_1}{\sigma_1 \rho_1} + \frac{(u_1 + u_2)(1 - \rho^2)G_3}{\rho_1^2})t \right\} - \frac{(u_1 + u_2)}{\sigma_1 \rho_1} \log(v_{10}) \right] \times \phi_1 \]

\[ \times \exp \left\{ u_2 \left( \frac{\kappa_2 \theta_2}{\sigma_2 \rho_2} + \frac{G_3 \kappa_2}{\sigma_2 \rho_2} + \frac{(1 - \rho_2^2)\bar{G}_3}{\rho_2^2} \right) - \frac{u_2}{\sigma_2 \rho_2} \log(v_{20}) \right\} \times \phi_2(t, v_{3,0}; \alpha_2, \lambda_2, \mu_2, v_{2,0}) \]

where the last equality used the relation \( u_1 \ln(S_1) + u_2 \ln(S_2) = u_1 Y_{1,0} + u_2 (Y_{3,0} + Y_{2,0} - rt) \). The functions \( \phi_1(\cdot) \) and \( \phi_2(\cdot) \) are defined in Equations (D.11) and (D.12), respectively.

Now, given the closed form solution of joint conditional c.f. under \( P \) and the form of the assets’ return assumed in Equation (D.9), we can conclude that the 1/2, 3/2, or 4/2-like MPRs are solvable.

\[ \square \]

D.3 Proof of joint c.f. for exact simulation

Proof. Proof of Proposition 5.3.3.

Using similar arguments as we drive the joint c.f. in Proposition 5.3.2, we get the first
expectation follows

\[
\mathbb{E} \left[ \exp \left\{ u_1 Y_{1,t} + u_2 Y_{3,t} \right\} \right] \\
= \exp \left\{ u_1 Y_{1,0} + u_2 Y_{3,0} \right\} + (u_1 + u_2) \left( r - G_1 - \frac{\kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3 \kappa_1}{\sigma_1 \rho_1} + \frac{(u_1 + u_2)(1 - \rho^2)G_3}{\rho_i^2} \right) t \\
+ \frac{(u_1 + u_2)}{\sigma_1 \rho_1} (v_{1t} - v_{10}) + \frac{G_3 (u_1 + u_2)}{\sigma_1 \rho_1} (\log(v_{1t}) - \log(v_{10})) \\
\times \mathbb{E} \left[ \exp \left\{ (u_1 + u_2) \left[ - \frac{1}{2} G_0 + \frac{\kappa_1}{\sigma_1 \rho_1} + \frac{1}{2} \left( \frac{u_1 + u_2}{\rho_i^2} \right) \right] \int_0^t v_{1s} ds \right\} \right] (D.13) \\
+ (u_1 + u_2) \left[ - \frac{1}{2} G_2 - \frac{G_3 \kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3 \sigma_1^2}{2 \sigma_1 \rho_1} + \frac{1}{2} \left( \frac{u_1 + u_2}{\rho_i^2} \right) \right] \int_0^t v_{1s} ds \right\} \right] (D.13) \\
\times \mathbb{E} \left[ \exp \left\{ (u_1 Y_{1,0} + u_2 Y_{3,0}) \right\} + (u_1 + u_2) \left( r - G_1 - \frac{\kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3 \kappa_1}{\sigma_1 \rho_1} + \frac{(u_1 + u_2)(1 - \rho^2)G_3}{\rho_i^2} \right) t \\
+ \frac{(u_1 + u_2)}{\sigma_1 \rho_1} (v_{1t} - v_{10}) + \frac{G_3 (u_1 + u_2)}{\sigma_1 \rho_1} (\log(v_{1t}) - \log(v_{10})) \right\} \times \phi_1, \\
\text{where}
\]

\[
\phi_1(t, v_1; \alpha_1, \lambda_1, \mu_1, v_1) = \mathbb{E} \left[ \exp \left\{ - \mu_1 \int_0^t v_{1s} ds - v_1 \int_0^t \frac{1}{v_{1s}} \right\} \right] (D.14) \\
\]

with parameters

\[
\mu_1 = -(u_1 + u_2) \left[ - \frac{1}{2} G_0 + \frac{\kappa_1}{\sigma_1 \rho_1} + \frac{1}{2} \left( \frac{u_1 + u_2}{\rho_i^2} \right) \right] \\

v_1 = -(u_1 + u_2) \left[ - \frac{1}{2} G_2 - \frac{G_3 \kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3 \sigma_1^2}{2 \sigma_1 \rho_1} + \frac{1}{2} \left( \frac{u_1 + u_2}{\rho_i^2} \right) G_3^2 \right].
\]
Therefore, the joint c.f. conditional on \(\nu_t\) is given by

\[
\mathbb{E} \left[ \exp \left\{ u_1 \ln(S_{1,t}) + u_2 \ln(S_{2,t}) \right\} \mid \nu_t \right]
\]

\[
= e^{-rtu} \mathbb{E} \left[ \exp \left\{ u_1 Y_{1,t} + u_2 Y_{2,t} \right\} \right] \cdot \mathbb{E} \left[ \exp \left\{ u_2 Y_{2,t} \right\} \right] 
\]

\[
= e^{-rtu} \times \exp \left( u_1 Y_{1,0} + u_2 Y_{3,0} \right) + (u_1 + u_2) \left( r - G_1 - \frac{\kappa_1 \theta_1}{\sigma_1 \rho_1} + \frac{G_3 \kappa_1}{\sigma_1 \rho_1} + \frac{(u_1 + u_2)(1 - \rho^2)G_3}{\rho_1^2} \right) \]

\[
+ \left( \frac{u_1 + u_2}{\sigma_1 \rho_1} \right) (v_{1t} - v_{10}) + \frac{G_3(u_1 + u_2)}{\sigma_1 \rho_1} \left( \log(v_{1t} - \log(v_{10})) \right) \times \phi(t, \nu_t; \nu_1^0, \nu_1^1) \]

\[
\times \exp \left( u_2 Y_{2,0} + u_2 \left( r - b_2 - \frac{\kappa_2 \theta_2}{\sigma_2 \rho_2} + \frac{b_2 \kappa_2}{\sigma_2 \rho_2} + \frac{(1 - \rho_2^2)}{\rho_2^2}b_2 \right) \right) + \frac{u_2}{\sigma_2 \rho_2} (v_{2t} - v_{20}) + \frac{u_2 b_2}{\sigma_2 \rho_2} \left( \log(v_{2t}) - \log(v_{20}) \right) \]

\[
\times \phi(t, \nu_t; \mu_2, \nu_2) \]

where

\[
\phi(t, \nu_t; \mu_2, \nu_2) = \mathbb{E} \left[ \exp \left\{ -\mu_2 \int_0^t v_{2s} \, ds - \nu_2 \int_0^t \frac{1}{v_{2s}} \right\} \right]. \quad (D.15)
\]

with parameters

\[
\mu_2 = -u_2 \left[ -\frac{1}{2} + \frac{\kappa_2}{\sigma_2 \rho_2} + \frac{u_2 (1 - \rho_2^2)}{2 \rho_2^2} \right] 
\]

\[
v_2 = -u_2 \left[ \frac{b_2^2}{2} - \frac{\kappa_2 \theta_2 b_2}{\sigma_2 \rho_2} + \frac{\sigma_2 b_2}{2 \rho_2^2} + \frac{u_2 (1 - \rho_2^2)}{2 \rho_2^2} b_2^2 \right]. 
\]

Moreover, if for \(i = 1, 2\), we define

\[
A_i = \kappa_i^2 + 2 \mu_i \sigma_i^2, \quad (D.16)
\]

with

\[
\mu_i > -\frac{\kappa_i^2}{2 \sigma_i^2}, \quad (D.17)
\]

\[
v_i \geq \frac{(2 \kappa_i \theta_i - \sigma_i)^2}{8 \sigma_i^2}, \quad (D.18)
\]
then the functions \( \phi_i \), for \( i = 1, 2 \), are well defined and given as

\[
\phi_i(t, v_i; \mu_i, v_i) = \mathbb{E}\left[ \exp \left\{ -\mu_i \int_0^t v_i ds - v_i \int_0^t \frac{1}{v_i} \right\} \left| v_i \right| \right]
\]

\[
= \frac{\sqrt{A_i} \sinh \left( \frac{\kappa t}{2} \right)}{\kappa_i \sinh \left( \frac{\kappa t}{2} \right)} \exp \left\{ \frac{\kappa t}{\sigma_i^2} \right\} \exp \left\{ \frac{\kappa t}{\sigma_i^2} \right\} \frac{\sinh \left( \frac{\kappa t}{2} \right)}{\sinh \left( \frac{\kappa t}{2} \right)} \right\}
\]

\[
\frac{I_{\alpha}(t, \sigma_i^2)}{\alpha \pi} \frac{2 \sqrt{A_i \sigma_i^2 \gamma_v} \left( \frac{2 \sqrt{A_i \sigma_i^2 \gamma_v}}{\sigma_i^2 \sinh \left( \frac{\kappa t}{2} \right)} \right)}{I_{\alpha}(t, \sigma_i^2) \left( \frac{2 \sqrt{A_i \sigma_i^2 \gamma_v}}{\sigma_i^2 \sinh \left( \frac{\kappa t}{2} \right)} \right)}
\]

where \( I_n(z) \) is the modified Bessel function of the first kind (Grasselli, 2017).

\[\square\]

### D.4 Proof of optimal strategies

**Proof.** Proof of Proposition 5.4.1.

Recall Equation (5.31), which is equivalent to

\[
0 = J_i + \kappa_1 (\theta_1 - v_1) J_{v_1} + \kappa_2 (\theta_2 - v_2) + \frac{1}{2} \sigma_1^2 v_1 J_{v_1 v_1} + \frac{1}{2} \sigma_2^2 v_2 J_{v_2 v_2} + \sup_{\eta_1, \eta_2} \left\{ g(\eta_1, \eta_2) \right\},
\]

where

\[
g(\eta_1, \eta_2) = x \left[ r + \lambda_1 \sqrt{v_1} \eta_1 + \lambda_2 \sqrt{v_2} \eta_2 \right] J_x + x \left[ \sigma_1 \sqrt{v_1} J_{v_1} + \sigma_2 \sqrt{v_2} J_{v_2} \right].
\]

The first order condition of \( g(\eta_1, \eta_2) \) with respect to \( \eta_1 \) and \( \eta_2 \) are given as

\[
\begin{cases}
g'(\eta_1) = x \lambda_1 \sqrt{v_1} J_x + x^2 \eta_1 J_{x x} + x \sigma_1 \sqrt{v_1} J_{v_1 v_1} \\
g'(\eta_2) = x \lambda_2 \sqrt{v_2} J_x + x^2 \eta_2 J_{x x} + x \sigma_2 \sqrt{v_2} J_{v_2 v_2}.
\end{cases}
\]

Setting \( g'(\eta_1) = 0 \) and \( g'(\eta_2) = 0 \) and solving for the candidate \( \eta_1^* \) and \( \eta_2^* \), we get

\[
\begin{cases}
\eta_1^* = -\frac{x \lambda_1 \sqrt{v_1} J_x}{x^2 J_{x x}} \\
\eta_2^* = -\frac{x \lambda_2 \sqrt{v_2} J_x}{x^2 J_{x x}}.
\end{cases}
\] (D.19)

Assume the form of \( J(x, v, t) = J(x, v_1, v_2, t) \) is given by \( J(x, v_1, v_2, t) = \frac{x^\gamma}{\gamma} h(t, v_1, v_2) \), where \( h(T, v_i) = 1, \forall v_1, v_2 \). Thereby, it follows that

\[
J_i = \frac{x^\gamma}{\gamma} h_i, J_{v_i} = \frac{x^\gamma}{\gamma} h_{v_i}, J_x = x^{\gamma - 1} h, J_{v_1 v_1} = \frac{x^\gamma}{\gamma} h_{v_1 v_1}, J_{v_2 v_2} = x^{\gamma - 1} h_{v_2}, J_{x x} = (\gamma - 1)x^{\gamma - 2} h,
\]
for \( i = 1, 2 \). Substituting the optimal candidates \( \eta^*_i, \eta^*_i \) and the partial derivatives of \( J \) back into the PDE and simplifying by multiplying the term \( \frac{v}{\gamma} \) on both sides leads to

\[
0 = h_i + ryh - \frac{\gamma}{\gamma - 1} h_i \lambda_i^2 v_i - \frac{\gamma}{\gamma - 1} h_i \Lambda_i \sigma_1 v_i \rho_1 - \frac{\gamma}{\gamma - 1} h_i \lambda_i^2 v_i - \frac{\gamma}{\gamma - 1} h_i \Lambda_i \sigma_2 v_2 \rho_2 \\
+ \kappa_1(\theta_1 - v_i)h_v + \kappa_2(\theta_2 - v_2)h_v + \frac{1}{2} \left[ \lambda_i^2 v_i \frac{\gamma}{\gamma - 1} h_i + 2 \sigma_1 \lambda_i v_i \rho_1 \frac{\gamma}{\gamma - 1} h_i + \sigma_1^2 v_i \rho_1^2 \frac{\gamma}{\gamma - 1} h_i \right]

\]

\[
+ \frac{\lambda_i^2 v_i}{\gamma - 1} h_i + 2 \frac{\sigma_2 \lambda_i v_i \rho_2}{\gamma - 1} h_i + \frac{\sigma_2^2 v_i \rho_2^2}{\gamma - 1} h_i \\
- \lambda_i \lambda_i v_i \rho_1 \frac{\gamma}{\gamma - 1} h_i - \sigma_1^2 v_i \rho_1^2 \frac{\gamma}{\gamma - 1} h_i - \lambda_i \lambda_i v_i \rho_2 \frac{\gamma}{\gamma - 1} h_i - \sigma_2^2 v_i \rho_2^2 \frac{\gamma}{\gamma - 1} h_i.
\]

(D.20)

Furthermore, assume that \( h(t, v_1, v_2) \) is of exponentially affine form, such that \( h(t, v) = \exp\left\{ D(\tau(t)) + E(\tau(t))v_1 + F(\tau(t))v_2 \right\} \), with time horizon \( \tau(t) = T - t \) and therefore boundary conditions

\[
h(T, v_1, v_2) = 1 \quad \forall \, v_1, v_2 \Rightarrow D(0) = D(\tau(T)) = 0, E(0) = E(\tau(T)) = 0, F(0) = F(\tau(T)) = 0.
\]

This leads to

\[
h_i = (-D' - E'v_i - F'v_2)h_j, \quad h_i = Eh, \quad h_2 = Fh, \quad v_i, v_2 = E^2h, \quad h_v = F^2h, \quad \frac{h^2_i}{h} = E^2h, \quad \frac{h^2_2}{h} = F^2h.
\]

Substituting again and simplifying leads to

\[
(-D' - E'v_i - F'v_2) + ry - \frac{\gamma}{\gamma - 1} \lambda_i^2 v_i - \frac{\gamma}{\gamma - 1} E \lambda_i \sigma_1 v_i \rho_1 - \frac{\gamma}{\gamma - 1} \lambda_i^2 v_i - \frac{\gamma}{\gamma - 1} E \lambda_i \sigma_2 v_2 \rho_2 \\
+ \kappa_1(\theta_1 - v_i)E + \kappa_2(\theta_2 - v_2)F + \frac{1}{2} \left[ \lambda_i^2 v_i \frac{\gamma}{\gamma - 1} E + 2 \sigma_1 \lambda_i v_i \rho_1 \frac{\gamma}{\gamma - 1} E + \sigma_1^2 v_i \rho_1^2 \frac{\gamma}{\gamma - 1} E \right]

\]

\[
+ \left[ \lambda_i^2 v_i \gamma + 2 \sigma_2 \lambda_i v_i \rho_2 \gamma + \sigma_2^2 v_i \rho_2^2 \gamma \right] + \frac{1}{2} \sigma_1^2 v_i E^2 \\
+ \frac{1}{2} \sigma_2^2 v_i F^2 - \frac{\lambda_i \lambda_i v_i \rho_1 \gamma}{\gamma - 1} E - \sigma_1^2 v_i \rho_1^2 \frac{\gamma}{\gamma - 1} E - \frac{\lambda_i \lambda_i v_i \rho_2 \gamma}{\gamma - 1} E - \sigma_2^2 v_i \rho_2^2 \frac{\gamma}{\gamma - 1} F^2 = 0.
\]

Separating out \( v_1 \) and \( v_2 \) and simplifying, we have

\[
- D' + ry + \kappa_1 \theta_1 E + \kappa_2 \theta_2 F \\
+ \left[ - E' - \frac{1}{2} \gamma \lambda_i^2 - \frac{\gamma}{\gamma - 1} \lambda_i \sigma_1 \rho_1 E - \kappa_1 E + \frac{1}{2} \sigma_1^2 E^2 - \frac{1}{2} \sigma_1^2 \rho_1^2 \frac{\gamma}{\gamma - 1} E^2 \right] v_1
\]

\[
+ \left[ - F' - \frac{1}{2} \gamma \lambda_i^2 - \frac{\gamma}{\gamma - 1} \lambda_i \sigma_2 \rho_2 F - \kappa_2 F + \frac{1}{2} \sigma_2^2 F^2 - \frac{1}{2} \sigma_2^2 \rho_2^2 \frac{\gamma}{\gamma - 1} F^2 \right] v_2 = 0.
\]

We end up with a term that is linear in \( v_1 \) and \( v_2 \), but all the “coefficients” satisfy linear differential equations. All of them have to be zero, such that

\[
\begin{align*}
D' &= ry + \kappa_1 \theta_1 E + \kappa_2 \theta_2 F, \\
E' &= \frac{1}{2} k^E E^2 - k^E F + \frac{1}{2} k^E_0, \\
F' &= \frac{1}{2} k^F E^2 - k^F F + \frac{1}{2} k^F_0,
\end{align*}
\]
Moreover, the optimal allocations, $\pi_1^*$ and $\pi_2^*$, can be obtained via $\eta_1^*$ and $\eta_2^*$

$$\pi_1^* = \frac{\sqrt{v_1} + \frac{b_1}{\sqrt{v_1}}}{\sqrt{v_1} + \frac{b_1}{\sqrt{v_1}}} \eta_1^* - \beta \left( \sqrt{v_1} + \frac{b_1}{\sqrt{v_1}} \right) \eta_2^*$$

$$= \eta_1^* - \eta_2^* - \frac{\beta \eta_2^*}{\sqrt{v_1} + \frac{b_1}{\sqrt{v_1}}}$$

$$= \frac{v_1}{v_1 + b_1} \left( \sqrt{v_1} + \frac{b_1}{\sqrt{v_1}} - \gamma \right) - \beta \frac{v_1 + b_1}{v_1 + b_1 + \gamma} \left( \sqrt{v_1} + \frac{b_1}{\sqrt{v_1}} - \gamma \right)$$

$$= \frac{v_2}{v_2 + b_2} \left( \sqrt{v_2} + \frac{b_2}{\sqrt{v_2}} - \gamma \right) - \beta \frac{v_1 + b_1}{v_1 + b_1 + \gamma} \left( \sqrt{v_1} + \frac{b_1}{\sqrt{v_1}} - \gamma \right)$$

$$= \frac{\eta_2^*}{\sqrt{v_2} + \frac{b_2}{\sqrt{v_2}}}$$

$$= \frac{v_2}{v_2 + b_2} \left( \sqrt{v_2} + \frac{b_2}{\sqrt{v_2}} - \gamma \right) - \beta \frac{v_1 + b_1}{v_1 + b_1 + \gamma} \left( \sqrt{v_1} + \frac{b_1}{\sqrt{v_1}} - \gamma \right)$$

$$= \frac{\sqrt{v_2} + \frac{b_2}{\sqrt{v_2}}}{v_2 + b_2} \left( \sqrt{v_2} + \frac{b_2}{\sqrt{v_2}} - \gamma \right) - \beta \frac{v_1 + b_1}{v_1 + b_1 + \gamma} \left( \sqrt{v_1} + \frac{b_1}{\sqrt{v_1}} - \gamma \right)$$

$$= \frac{v_2}{v_2 + b_2} \left( \sqrt{v_2} + \frac{b_2}{\sqrt{v_2}} - \gamma \right) - \beta \frac{v_1 + b_1}{v_1 + b_1 + \gamma} \left( \sqrt{v_1} + \frac{b_1}{\sqrt{v_1}} - \gamma \right)$$

$$\square$$
D.5 Proof of technical conditions

Proof. To ensure that the value function is real-valued, we need to ensure the square roots are
well-defined in functions $E(T-t)$ and $F(T-t)$ in Equations (5.34), (5.35). That is,
\[ k_0^F > 0 \iff (k_1^F)^2 - k_0^F k_2^F > 0, \]
\[ k_3^F > 0 \iff (k_4^F)^2 - k_0^F k_2^F > 0. \]

That is, the condition for $E(T-t)$ follows:
\[
(k_1^F)^2 - k_0^F k_2^F = \left( \frac{\gamma}{\gamma-1} \lambda_1 \sigma_1 \rho_1 + \kappa_1 \right)^2 - \left(-\frac{\gamma}{\gamma-1} \lambda_1^2 \cdot \sigma_1^2(1-\rho_1^2 \frac{\gamma}{\gamma-1}) \right) \\
= \frac{\gamma^2}{(\gamma-1)^2} \lambda_1^2 \sigma_1^2 \rho_1^2 + 2 \frac{\gamma}{\gamma-1} \lambda_1 \sigma_1 \rho_1 \kappa_1 + \kappa_1^2 + \frac{\gamma}{\gamma-1} \lambda_1^2 \sigma_1^2(1-\rho_1^2 \frac{\gamma}{\gamma-1}) \\
= 2 \frac{\gamma}{\gamma-1} \lambda_1 \sigma_1 \rho_1 \kappa_1 + \kappa_1^2 + \frac{\gamma}{\gamma-1} \lambda_1^2 \sigma_1^2 \\
= \frac{\gamma}{\gamma-1} \lambda_1 \sigma_1 \left(2 \rho_1 \kappa_1 + \lambda_1 \sigma_1 \right) + \kappa_1^2.
\]

Thus, we need $\frac{\gamma}{\gamma-1} \lambda_1 \sigma_1 \left(2 \rho_1 \kappa_1 + \lambda_1 \sigma_1 \right) + \kappa_1^2 > 0$ for all $\gamma < 1$. Similarly for the function $F(T-t)$, we need $(k_1^F)^2 - k_0^F k_2^F > 0$.

Note that to ensure finiteness, if $\gamma < 0$,
\[ k_0^E = -\frac{\gamma}{\gamma-1} \lambda_1^2 < 0 \iff E(T-t) < 0, \]
\[ k_0^F = -\frac{\gamma}{\gamma-1} \lambda_2^2 < 0 \iff F(T-t) < 0, \]

and so are the two \emph{ln} representations of the function $D(T-t)$ in Equation (5.33).

If $0 < \gamma < 1$, we need $2k_3^F + (k_1^F + k_5^F)(e^{k_5^F} - 1) \neq 0$, such that
\[
2k_3^F + (k_1^F + k_5^F)(e^{k_5^F} - 1) \neq 0 \\
\iff -(k_1^F - k_3^F) \left(1 - \frac{k_1^F + k_5^F}{k_1^F - k_3^F} e^{k_5^F} \right) \neq 0 \\
\iff \frac{k_1^F + k_5^F}{k_1^F - k_3^F} > 1 \iff k_1^F > k_3^F = \sqrt{(k_4^F)^2 - k_0^F k_2^F} \\
\iff k_0^F k_5^F > 0 \iff \left(-\frac{\gamma}{\gamma-1} \lambda_2^2 \right) \left(\sigma_1^2(1-\rho_1^2 \frac{\gamma}{\gamma-1}) \right) > 0,
\]
which is always satisfied. Similarly for the function $F(T-t)$:
\[
2k_3^F + (k_1^F + k_5^F)(e^{k_5^F} - 1) \neq 0 \\
\iff k_0^F k_5^F > 0 \iff \left(-\frac{\gamma}{\gamma-1} \lambda_2^2 \right) \left(\sigma_2^2(1-\rho_2^2 \frac{\gamma}{\gamma-1}) \right) > 0,
\]
which is always true.
Proof of the verification theorem 5.4.3

Proof. Consider an arbitrary but fixed point \((\bar{x}, \bar{v}_1, \bar{v}_2, i) \in [0, T] \times [0, \infty) \times [0, \infty) \times [0, \infty)\), and assume that the wealth of the investor at time \(\bar{t}\) is \(\bar{x}\), and \(v_i = \bar{v}_i, i = 1, 2\). For condition 1), the value function \(J\) is once continuously differentiable in \(t\), and twice differentiable in \(x\) and \(v_i\) by Proposition 5.4.2. For condition 2), the PDE is obviously satisfied by substituting the candidate optimal control \(\eta^*_i\), while the terminal condition follows by the definition of our helper function \(h\).

Furthermore, under the ansatz for the value function \(J(x, v, t) = \frac{v^2}{2} h(v, t), \ v = (v_1, v_2)\), the dynamics of \(J\) can be derived as

\[
dJ_t = \frac{\partial J}{\partial t} dt + \frac{\partial J}{\partial x} dx + \frac{\partial J}{\partial v_1} dv_1 + \frac{\partial J}{\partial v_2} dv_2 + \frac{1}{2} \frac{\partial^2 J}{\partial x^2} (dx)^2 + \frac{1}{2} \frac{\partial^2 J}{\partial v_1^2} (dv_1)^2 + \frac{1}{2} \frac{\partial^2 J}{\partial v_2^2} (dv_2)^2 + \frac{\partial^2 J}{\partial x \partial v_1} (dx)(dv_1)
\]

\[
+ \frac{\partial^2 J}{\partial x \partial v_2} (dx)(dv_2) + \frac{\partial^2 J}{\partial v_1 \partial v_2} (dv_1)(dv_2)
\]

\[
= \left[ \frac{X_1}{\gamma} h_t + \frac{X_1}{\gamma} h_r + \eta_1 \frac{\lambda_1}{\sqrt{v_1}} \sqrt{v_1} + \eta_2 \frac{\lambda_2}{\sqrt{v_2}} \sqrt{v_2} \right] + \frac{X_1}{\gamma} h_{v_1} + \frac{X_1}{\gamma} h_{v_2} + \eta_2 h_{v_2} + \frac{X_1}{\gamma} h_{v_1} + \frac{X_1}{\gamma} h_{v_2} + \frac{X_1}{\gamma} h_{v_1} + \frac{X_1}{\gamma} h_{v_2}
\]

\[
+ \gamma - 1 \frac{X_1}{\gamma} h((\eta_1)^2 + (\eta_2)^2) + \frac{1}{2} \frac{X_1}{\gamma} h_{v_1} v_1 \sigma_1 v_1 + \frac{1}{2} \frac{X_1}{\gamma} h_{v_2} v_2 \sigma_2 v_2 + \frac{X_1}{\gamma} h_{v_1} \eta_1 \rho_1 \sigma_1 + \frac{X_1}{\gamma} h_{v_2} \eta_2 \rho_2 \sigma_2 + \frac{X_1}{\gamma} h_{v_1} \eta_2 \rho_2 \sigma_2 + \frac{X_1}{\gamma} h_{v_2} \eta_1 \rho_1 \sigma_1
\]

\[
+ X_1 h(\eta_1^* dW_t + \eta_2^* dW_t) + \frac{X_1}{\gamma} h_{v_1} \sigma_1 \sqrt{v_1} dZ_{t_1} + \frac{X_1}{\gamma} h_{v_2} \sigma_2 \sqrt{v_2} dZ_{t_2}.
\]

At the same time, the optimal strategy \(\eta^*_i\) for \(i = 1, 2\) can be expressed as

\[
\eta^*_i = \frac{-\tilde{\lambda}_i}{(\gamma - 1)} v_i - \sigma_i \sqrt{v_i} \rho \frac{h_v}{h_k}.
\]

In addition, we use the PDE (D.20) for \(h\), hence the dynamics for \(J\) evolves as follows:

\[
dJ_t = \frac{X_1}{\gamma} \left[ \frac{\gamma}{\gamma - 1} (-\tilde{\lambda}_1 h \sqrt{v_1} - \sqrt{v_1} \sigma_1 \rho_1 \sqrt{h_{v_1}}) dW_t + \frac{\gamma}{\gamma - 1} (-\tilde{\lambda}_2 h \sqrt{v_2} - \sqrt{v_2} \sigma_2 \rho_2 \sqrt{h_{v_2}}) dW_t + \sigma_1 h_{v_1} \rho_1 dW_{v_1} + \sqrt{1 - \rho_2^2} dW_{v_2} + \sigma_2 h_{v_2} \rho_2 dW_{v_2} + \sqrt{1 - \rho_1^2} dW_{v_1} + \sqrt{1 - \rho_2^2} dW_{v_2} \right]
\]

By rewriting and substituting partial derivatives of the helper function \(h\), it follows that

\[
\frac{dJ_t}{J_t} = \left( -\frac{\tilde{\lambda}_1}{\gamma - 1} + (1 - \frac{1}{\gamma - 1} \rho_1) \sigma_1 E(T - t) \sqrt{v_1} dW_{v_1} + \left( -\frac{\tilde{\lambda}_2}{\gamma - 1} + (1 - \frac{1}{\gamma - 1} \rho_2) \sigma_2 F(T - t) \sqrt{v_2} dW_{v_2} \right) + \sigma_1 \sqrt{1 - \rho_2^2} E(T - t) \sqrt{v_1} dW_{v_1} + \left( \sigma_2 \sqrt{1 - \rho_1^2} F(T - t) \sqrt{v_2} dW_{v_2} \right) \right.
\]

\[
+ \left( \sigma_1 \sqrt{1 - \rho_2^2} E(T - t) \sqrt{v_1} dW_{v_1} + \left( \sigma_2 \sqrt{1 - \rho_1^2} F(T - t) \sqrt{v_2} dW_{v_2} \right) \right)
\]

where \(E(T - t) = E(t) = \frac{k(t') - 1}{2k(t') + (k(t') + k(t') - 1)}\), and \(F(T - t) = F(t) = \frac{k(t') - 1}{2k(t') + (k(t') + k(t') - 1)}\), as per Equations (5.34) and (5.35) with auxiliary parameters defined in Proposition 5.4.1.
By Ito’s lemma, the dynamics of \( f(J) = \log(J) \) can be derived,

\[
J_t = J_0 \cdot \exp \left\{ -\frac{1}{2} \int_0^t \left( (g^2_1(s) + g^2_3(s))v_1 + (g^2_2(s) + g^2_3(s))v_2 \right) ds + \int_0^t g_1(s) \sqrt{v_1} dW_{1s} + \int_0^t g_2(s) \sqrt{v_2} dW_{2s} \\
+ \int_0^t g_3(s) \sqrt{v_1} dW_{1s}^\perp + \int_0^t g_4(s) \sqrt{v_2} dW_{2s}^\perp \right\}
\]

According to 5.13 Corollary in Karatzas and Shreve (1998), if the above equation satisfies

\[
\mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^t \left( (g^2_1(s) + g^2_3(s))v_1 + (g^2_2(s) + g^2_3(s))v_2 \right) ds \right\} \right] < \infty,
\]

then \( J_t \) is a martingale. Since \( v_1 \) and \( v_2 \) are independent, we have

\[
\mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^t \left( (g^2_1(s) + g^2_3(s))v_1 + \frac{1}{2} \int_0^t (g^2_2(s) + g^2_3(s))v_2 ds \right) \right\} \\
= \mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^t (g^2_1(s) + g^2_3(s))v_1 ds \right\} \right] \times \mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^t (g^2_2(s) + g^2_3(s))v_2 ds \right\} \right].
\]

That is, we can impose:

\[
\mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^t (g^2_1(s) + g^2_3(s))v_1 ds \right\} \right] < \infty, \quad \text{and} \quad \mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^t (g^2_2(s) + g^2_3(s))v_2 ds \right\} \right] < \infty.
\]

Further, as \( v_{it}, i = 1, 2 \), follow CIR processes, we can refer to Proposition 5.1 in Kraft (2005) which provides us a condition to ensure finiteness of such expectations. For the first expectation in Equation (D.22), related to \( v_{1t} \), it requires that

\[
\beta_1 = \min_{\gamma \in (0, 1]} \left( -\frac{1}{2} (g^2_1(t) + g^2_3(t)) \right) \geq -\frac{\kappa^2_1}{2\sigma^2_3},
\]

where

\[
-\frac{1}{2} (g^2_1(t) + g^2_3(t)) = -\frac{1}{2} \left( \frac{(1-\gamma)(1-\rho^2_1)}{\gamma-1} + 1-\rho^2_1 \right) \sigma^2_1 E(T-t) - 2 \frac{\lambda_1}{\gamma-1} \sigma_1 (1-\gamma-\rho_1) E(T-t) + \frac{\gamma}{\gamma-1} \lambda^2_1.
\]

Taking the first order differentiation of \( E(T-t) \) with respect to \( t \), leads to:

\[
\left( -\frac{1}{2} (g^2_1(t) + g^2_3(t)) \right)' = \left( \frac{(1-\gamma)(1-\rho^2_1)}{\gamma-1} + 1-\rho^2_1 \right) \sigma^2_1 E(T-t) E'(T-t) - 2 \frac{\lambda_1}{\gamma-1} \sigma_1 (1-\gamma-\rho_1) E'(T-t).
\]

Then we have

\[
\begin{cases}
\text{If } 0 < \gamma < 1, & E'(T-t) < 0 \text{ in } t, \\ E(0) = 0, & \implies E(T-t) > 0;
\end{cases}
\]

\[
\begin{cases}
\text{If } \gamma < 0, & E'(T-t) > 0 \text{ in } t, \\ E(0) = 0, & \implies E(T-t) < 0.
\end{cases}
\]
Given that \( \bar{\lambda}_1 > 0 \) and \(-1 < \rho_1 < 0\),

\[
\begin{cases}
(1 - \frac{\gamma}{\gamma - 1} \rho_1) < 0 & \Rightarrow -1 \leq \rho_1 < \frac{\gamma - 1}{\gamma} \\
0 < \gamma < 1 & \Rightarrow 0 < \bar{\lambda}_1 < \frac{\sigma_1(1 - \frac{\gamma}{\gamma - 1} \rho_1)^2 + 1 - \rho_1^2}{1 - \frac{\gamma}{\gamma - 1} \rho_1} E(T - \bar{\lambda}_1) \\
(1 - \frac{\gamma}{\gamma - 1} \rho_1) > 0 & \Rightarrow \frac{\gamma - 1}{\gamma} \leq \rho_1 < 0, \text{ and } \bar{\lambda}_1 > 0 > \frac{\sigma_1(1 - \frac{\gamma}{\gamma - 1} \rho_1)^2 + 1 - \rho_1^2}{1 - \frac{\gamma}{\gamma - 1} \rho_1} E(T - \bar{\lambda}_1)
\end{cases}
\]

If \( \gamma < 0 \), given that \( \bar{\lambda}_1 > 0 \), \( \Rightarrow \left( -\frac{1}{2} (g_1^2(t) + g_2^2(t)) \right) < 0 \), minimal at \( t=T \)

Note that when \( (1 - \frac{\gamma}{\gamma - 1} \rho_1) > 0 \), if \( 0 < \bar{\lambda}_1 < \frac{\sigma_1(1 - \frac{\gamma}{\gamma - 1} \rho_1)^2 + 1 - \rho_1^2}{1 - \frac{\gamma}{\gamma - 1} \rho_1} E(T - \bar{\lambda}_1) \), then \( \left( -\frac{1}{2} (g_1^2(t) + g_2^2(t)) \right) > 0 \). However, the upper bound of \( \bar{\lambda}_1 \) is a negative number, which violates \( \bar{\lambda}_1 > 0 \) and thus not viable.

To sum up, if either i) \( 0 < \gamma < 1 \), \(-1 \leq \rho_1 < \frac{\gamma - 1}{\gamma} \), and \( 0 < \bar{\lambda}_1 < \frac{\sigma_1(1 - \frac{\gamma}{\gamma - 1} \rho_1)^2 + 1 - \rho_1^2}{1 - \frac{\gamma}{\gamma - 1} \rho_1} E(T - \bar{\lambda}_1) \); or ii) \( 0 < \gamma < 1 \), \( \frac{\gamma - 1}{\gamma} \leq \rho_1 < 0 \), and \( \bar{\lambda}_1 > 0 \); or iii) \( \gamma < 0 \) and \( \bar{\lambda}_1 > 0 \), then

\[
\beta_1 = -\frac{1}{2} \frac{\gamma^2}{(\gamma - 1)^2} \bar{\lambda}_1^2 \geq -\frac{\kappa_1^2}{2\sigma_1^2}.
\]

On the other hand, if \( 0 < \gamma < 1 \), \(-1 \leq \rho_1 < \frac{\gamma - 1}{\gamma} \), and \( \bar{\lambda}_1 > \frac{\sigma_1(1 - \frac{\gamma}{\gamma - 1} \rho_1)^2 + 1 - \rho_1^2}{1 - \frac{\gamma}{\gamma - 1} \rho_1} E(T - \bar{\lambda}_1) \), then

\[
\beta_1 = -\frac{1}{2} \left[ (1 - \frac{\gamma}{\gamma - 1} \rho_1)^2 + 1 - \rho_1^2 \right] \sigma_1^2 E(T) - 2 \frac{\bar{\lambda}_1 \gamma}{\gamma - 1} (1 - \frac{\gamma}{\gamma - 1} \rho_1) \sigma_1 E(T) + \frac{\gamma}{\gamma - 1} \bar{\lambda}_1^2 \geq -\frac{\kappa_1^2}{2\sigma_1^2}.
\]

For the second expectation Equation (D.22), related to \( v_{2r} \), a similar argument applies. It requires that

\[
\beta_2 = \min_{t \in [0,T]} \left( -\frac{1}{2} (g_1^2(t) + g_2^2(t)) \right) \geq -\frac{\kappa_2^2}{2\sigma_2^2},
\]

where

\[
-\frac{1}{2} (g_1^2(t) + g_2^2(t)) = -\frac{1}{2} \left[ (1 - \frac{\gamma}{\gamma - 1} \rho_2)^2 + 1 - \rho_2^2 \right] \sigma_2^2 F^2(T - t) - 2 \frac{\bar{\lambda}_2 \gamma}{\gamma - 1} \sigma_2 (1 - \frac{\gamma}{\gamma - 1} \rho_2) F(T - t) + \frac{\gamma}{\gamma - 1} \bar{\lambda}_2^2.
\]

Similarly, taking the first order differentiation of \( F(T - t) \) with respect to \( t \),

\[
\left( -\frac{1}{2} (g_1^2(t) + g_2^2(t)) \right)' = \left( (1 - \frac{\gamma}{\gamma - 1} \rho_2)^2 + 1 - \rho_2^2 \right) \sigma_2^2 F(T - t) F'(T - t) - \frac{\bar{\lambda}_2 \gamma}{\gamma - 1} \sigma_2 (1 - \frac{\gamma}{\gamma - 1} \rho_2) F'(T - t).
\]

Then we have

\[
\begin{cases}
\text{If } 0 < \gamma < 1, \ F'(T - t) < 0 \text{ in } t, \ F(0) = 0, \Rightarrow F(T - t) > 0; \\
\text{If } \gamma < 0, \ F'(T - t) > 0 \text{ in } t, \ F(0) = 0, \Rightarrow F(T - t) < 0.
\end{cases}
\]
Given that $\lambda_2 > 0$ and $-1 < \rho_2 < 0$, it can be summarized that if either (i) $0 < \gamma < 1$, $-1 \leq \rho_2 < \frac{\gamma-1}{\gamma}$, and $0 < \lambda_2 < \frac{\sigma_2^2(1-\rho_2^2)\gamma^2+\rho_2^2F(T)\gamma^2}{\gamma^2-1}$; (ii) $0 < \gamma < 1$, $\frac{\gamma-1}{\gamma} \leq \rho_2 < 0$, and $\lambda_2 > 0$; or (iii) $\gamma < 0$ and $\lambda_2 > 0$, then

$$\beta_2 = -\frac{1}{2} \frac{\gamma^2}{(\gamma-1)^2} \lambda_2^2 \geq -\frac{\kappa_2^2}{2\sigma_2^2}.$$ 

If $0 < \gamma < 1$, $-1 \leq \rho_2 < \frac{\gamma-1}{\gamma}$, and $\lambda_2 > \frac{\sigma_2^2(1-\rho_2^2)\gamma^2+\rho_2^2F(T)\gamma^2}{\gamma^2-1}$, then

$$\beta_2 = -\frac{1}{2} \left( (1-\frac{\gamma}{\gamma-1}\rho_2)^2 + 1 - \rho_2^2 \right) \sigma_2^2 F(T) - 2 - \frac{\lambda_2 \gamma}{\gamma-1} \frac{1}{\gamma-1} \gamma - 1 \rho_2 \sigma_2 F(T) + \left( \frac{\gamma}{\gamma-1} \right)^2 \lambda_2^2 \right] \geq -\frac{\kappa_2^2}{2\sigma_2^2}.$$ 

Now, it can be concluded that $J$ is a martingale. Further, as $J$ is a martingale, it follows that

$$\mathbb{E}[U(X_T^n)\mid F_I] = \mathbb{E}[\frac{(X_T^n)^\gamma}{\gamma} \mid F_I] = \mathbb{E}[J(X_T^n, v_{1T}, v_{2T}, T)\mid F_I] = J(\tilde{x}, \tilde{v}, \tilde{v}_2, \tilde{t}).$$

For the statement of optimality, let $\eta$ be an arbitrary admissible strategy and define the process $L_\pi, t \in [\bar{t}, T]$ by

$$L_\pi := (X_t^n)^{-1} X_t^n h(v_1, v_2, t).$$

Given the dynamics of wealth $X_t$, and the dynamics of stochastic volatility $v_1, v_2$, applying Ito’s lemma to obtain the dynamics of $L_\pi$:

$$dL_\pi = \frac{\partial L}{\partial X^n_\pi} dX_\pi^n + \frac{1}{2} \frac{\partial^2 L}{\partial (X^n_\pi)^2} (dX_\pi^n)^2 + \frac{\partial L}{\partial \eta} d\eta + \frac{1}{2} \frac{\partial^2 L}{\partial \eta^2} (d\eta)^2 + \frac{\partial^2 L}{\partial \eta \partial X^n_\pi} (dh)(dX_\pi^n) + \frac{\partial^2 L}{\partial d\eta^2} (dh)(d\eta).$$

That is,

$$dL_\pi = (X_t^n)^{-1} X_t^n \left[ (\gamma - 1) h(r + \lambda_1 \sqrt{v_1} \eta_{1t}^n + \lambda_2 \sqrt{v_2} \eta_{2t}^n) + h(r + \lambda_1 \sqrt{v_1} \eta_{1t} + \lambda_2 \sqrt{v_2} \eta_{2t}) ight. \\
+ h_{v_1} \lambda_1 (\theta_1 - v_{1t}) + h_{v_2} \lambda_2 (\theta_2 - v_{2t}) + \frac{1}{2} h_{v_1} \sigma_2^2 v_{1t} + \frac{1}{2} h_{v_2} \sigma_2^2 v_{2t} + h_{\eta_1} \lambda_1 \eta_{1t} + \frac{1}{2} (\gamma - 1)(\gamma - 2) h((\eta_{1t}^n)^2 + (\eta_{2t}^n)^2) + (\gamma - 1) h(\eta_{1t} \eta_{1t} + \eta_{2t} \eta_{2t}) \\
+ \left( (\gamma - 1)(\eta_{1t} \sigma_1 \eta_1 \sqrt{v_1} \eta_{1t} + \eta_{2t} \sigma_2 \eta_2 \sqrt{v_2} \eta_{2t}) + (\eta_{1t} \sigma_1 \eta_1 \sqrt{v_1} \eta_{1t} + \eta_{2t} \sigma_2 \eta_2 \sqrt{v_2} \eta_{2t}) \right) dt \\
+ (X_t^n)^{-1} X_t^n \left( h_{v_1} \sigma_1 \sqrt{v_1} dZ_{1t} + h_{v_2} \sigma_2 \sqrt{v_2} dZ_{2t} \right) \\
+ (X_t^n)^{-1} X_t^n \left( (\eta_{1t} \sigma_1 \eta_1 dW_{1t} + \eta_{2t} \sigma_2 \eta_2 dW_{2t}) + (X_t^n)^{-1} X_t^n \eta_{1t} \eta_1 dW_{1t} + \eta_{2t} \eta_2 dW_{2t} \right) \\
=: \mu^t dt + \Sigma_{1t} dW_{1t} + \Sigma_{2t} dW_{2t} + \Sigma_{1t} dW_{1t} + \Sigma_{2t} dW_{2t},$$

(D.24)
where \( dZ_{it} = \rho_i dW_{it} + \sqrt{1 - \rho_i^2} dW_{it}^i, \) \( i = 1, 2. \) Rearranging terms in \( \mu^L \) leads to

\[
\mu^L = (X^*_{it})^{-1} X^*_{it} \left[ ryh + \gamma h(\bar{\lambda}_1 \sqrt{v_1} \eta_{1t} + \bar{\lambda}_2 \sqrt{v_2} \eta_{2t}) + h_{1t} \eta_{1t} + h_{2t} \eta_{2t} + h_{v1} \sigma_1 \sqrt{v_1} + h_{v2} \sigma_2 \sqrt{v_2} \right] + \frac{1}{2} h_{v1v1} \sigma_1^2 v_{1t} + \frac{1}{2} h_{v2v2} \sigma_2^2 v_{2t} + \frac{1}{2} (\gamma - 1) h((\eta_{1t})^2 + (\eta_{2t})^2) + \gamma (\eta_{1t} \sigma_1 \sqrt{v_1} + \eta_{2t} \sigma_2 \sqrt{v_2}) + (\eta_{1t} - \eta_{2t}) \left( \bar{\lambda}_1 \sqrt{v_1} + h(\gamma - 1) \eta_{1t} + h_{v1} \sigma_1 \sqrt{v_1} \right) + (\eta_{2t} - \eta_{2t}) \left( \bar{\lambda}_2 \sqrt{v_2} + h(\gamma - 1) \eta_{2t} + h_{v2} \sigma_2 \sqrt{v_2} \right).
\]

By substituting \( \eta_{it}^* \) of Equation (D.21), cancelling out terms and Equation (D.20), it follows that \( \mu^L = 0 \) and the process of \( L_t \) evolves as

\[
dL_t = \left( \frac{E_{\eta_{1t}} \sqrt{v_1} \rho_1 + (\gamma - 1) \eta_{1t} + \eta_{1t}}{\Sigma^2_{1t}} \right) dW_{1t} + \left( \frac{E_{\eta_{2t}} \sqrt{v_2} \rho_2 + (\gamma - 1) \eta_{2t} + \eta_{2t}}{\Sigma^2_{2t}} \right) dW_{2t} + \sqrt{1 - \rho_1^2} E_{\eta_{1t}} \sqrt{v_1} dW_{1t}^i + \sqrt{1 - \rho_2^2} E_{\eta_{2t}} \sqrt{v_2} dW_{2t}^i.
\]

Thus, it follows that \( L_t \) is a local martingale since all \( X^*_{it}, X^*_{it}, \eta_{it}, \) and \( v_{it} \) are continuous functions. Furthermore, the helper function \( h(v_1, v_2, t) = e^{D(T - t) + E(T - t)v_1 + F(T - t)v_2} \) is always positive, thus so is the process \( L_t, \) which implies that it is a supermartingale (see the proof in Theorem 6.6 in Bain (2007)). The next step follows similarly as Cheng and Escobar (2021), provided here for completeness.

\[
\mathbb{E}[U(X^*_{it})|\mathcal{F}_t] \leq \mathbb{E}[U(X^*_{it})|\mathcal{F}_t] + \mathbb{E}[U(X^*_{it})(X^*_{it} - X^*_{it})|\mathcal{F}_t] = \mathbb{E}[U(X^*_{it})|\mathcal{F}_t] + \mathbb{E}[L(T)|\mathcal{F}_t] - \mathbb{E}[(X^*_{it})^\gamma|\mathcal{F}_t] \leq \mathbb{E}[U(X^*_{it})|\mathcal{F}_t] + L(T) - \gamma \mathbb{E}[J(X^*_{it}, v_{1T}, v_{2T}, T)|\mathcal{F}_t] = \mathbb{E}[U(X^*_{it})|\mathcal{F}_t] + \gamma \mathbb{E}[J(x, v_{1t}, v_{2t}, \tilde{t}) - \gamma J(x, \tilde{v}_1, \tilde{v}_2, \tilde{t})] = \mathbb{E}[U(X^*_{it})|\mathcal{F}_t].
\]

The first inequality is obtained by the concavity of the utility function \( U(x) = \frac{h}{x}. \) The second equation takes apart the second term of the first inequality, and follows the definition of the process \( L_t, \) combined with \( L_T := (X^*_{it})^{-1} X^*_{it} h(v_1, v_2, T). \) The third inequality follows from the supermartingale property of \( L_t, \) such that \( L_t \geq \mathbb{E}[L_T|\mathcal{F}_t] \) and the definition of the value function \( J(x, v_{1t}, v_{2t}, \tilde{t}), \) such that:

\[
J(x, v_{1t}, v_{2t}, \tilde{t}) = \sup \mathbb{E}[U(X_T)|\mathcal{F}_t] = \mathbb{E}[U(X^*_{it})|\mathcal{F}_t] = \mathbb{E}[\frac{(X^*_{it})^\gamma}{\gamma}|\mathcal{F}_t].
\]

The fourth and fifth equation are obtained by the martingale property of the value function \( J \) and substituting in \( J(x, v_{1t}, v_{2t}, \tilde{t}) = \frac{h}{x} h(v_1, v_2, \tilde{t}). \) And the last equation obviously follows by cancelling out the last two terms. Now the proof shows that \( J^u \leq J^*, \) thus \( \pi^* \) is the optimal control, \( \pi^* \) is the optimal allocation, and \( J^u \) is the value function that solves our optimal problem.

\[\square\]
Appendix E

Proofs for Chapter 6

E.1 Proof of exponential-polynomial

Proof. We will first consider the constant, linear and quadratic cases, then we show the cubic and via induction that higher order won’t work. Note, we will also highlight the implications in terms of $A(v_t), m_1(v_t)$ and $m_2(v_t)$.

Exponential-constant:

$$h(v, t) = \exp\{A(T - t)\}, \quad \text{(E.1)}$$

where $\tau(t) = T - t$ and $A(0) = 0$. Substituting the partial derivatives of $h$ and simplifying leads to

$$-A' + \left( r\gamma - \frac{1}{2}\frac{\gamma}{\gamma - 1} A^2(v) \right) = 0, \quad \text{(E.2)}$$

which simply implies a constant market price of risk ($\lambda$) and any feasible $m_1, m_2$.

Exponential-linear:

$$h(v, t) = \exp\{A(T - t) + B(T - t)v\}, \quad \text{(E.3)}$$

where $\tau(t) = T - t$, and $A(0) = B(0) = 0$. Substituting leads to

$$-A' - B' + \left( r\gamma - \frac{1}{2}\frac{\gamma}{\gamma - 1} A^2(v) \right) + \left( m_1(v) - \frac{\gamma}{\gamma - 1}\lambda m_2(v) \right) B + \frac{1}{2} \left( m_2^2(v) - \frac{\gamma}{\gamma - 1}\rho^2 m_2^2(v) \right)' B^2 = 0. \quad \text{(E.4)}$$

For the above PDE to be linear in $v$, the following functions have to be either constant or linear in $v$:

$$m_1(\cdot), \quad \lambda(\cdot)m_2(\cdot), \quad m_2^2(\cdot), \quad A^2(\cdot) \quad \text{(E.5)}$$

This means, $m_1$ constant or linear, with $\lambda$ and $m_2$ square-root.

Exponential-quadratic:

$$h(v, t) = \exp\{A(T - t) + B(T - t)v + \frac{1}{2} C(T - t)v^2\}, \quad \text{(E.6)}$$
where \( \tau(t) = T - t, \) and \( A(0) = B(0) = C(0) = 0. \) We obtain:

\[
- A' - B'v - \frac{1}{2} C'v^2 + \left( r \gamma - \frac{1}{2} \frac{\gamma}{\gamma - 1} A^2(v) \right) + \left( m_1(v) - \frac{\gamma}{\gamma - 1} \lambda(v) \rho m_2(v) \right) (B + Cv) \\
+ \frac{1}{2} m_2^2(v) \left( B^2 + 2BCv + C^2v^2 + C \right) - \frac{1}{2} \frac{\gamma}{\gamma - 1} \rho^2 m_2^2(v) \left( B^2 + 2BCv + C^2v^2 \right) = 0.
\]

(E.7)

For the above PDE to be solvable, the power of the state variable \( v \) can only be 0, 1, and 2, this means:

\[
\begin{align*}
& m_1(v), \quad \lambda(v)m_2(v) \text{ linear in } v, \\
& \lambda^2(v) \text{ quadratic in } v, \\
& m_2^2(v) \text{ constant.}
\end{align*}
\]

This is, market price of risk linear in \( v \) or constant, while \( v \) follows an Ornstein-Uhlenbeck (OU) process.

**Exponential-cubic:** Assume:

\[
h(v, t) = \exp \left( A(T - t) + B(T - t)v + \frac{1}{2} C(T - t)v^2 + \frac{1}{3} D(T - t)v^3 \right),
\]

(E.9)

where \( \tau(t) = T - t, \) and \( A(0) = B(0) = C(0) = D(0) = 0. \) The partial derivatives of \( h \) are

\[
\begin{align*}
& h_t = h \left( -A' - B'v - \frac{1}{2} C'v^2 - \frac{1}{3} D'v^3 \right) \\
& h_v = h \left( B + Cv + Dv^2 \right) \\
& h_{vv} = h \left( C + 2Dv \right) + h \left( B + Cv + Dv^2 \right)^2 \\
& \quad = h \left( C + B^2 + (2D + 2BC)v + (C^2 + 2BD)v^2 + 2CDv^3 + D^2v^4 \right) \\
& \frac{h^2}{h} = \frac{h^2 \left( B + Cv + Dv^2 \right)^2}{h} \\
& \quad = h \left( B^2 + 2BCv + (C^2 + 2BD)v^2 + 2CDv^3 + D^2v^4 \right).
\end{align*}
\]

(E.10)

By substituting the partial derivatives of \( h \) and simplifying leads to

\[
- A' - B'v - \frac{1}{2} C'v^2 - \frac{1}{3} D'v^3 + \left( r \gamma - \frac{1}{2} \frac{\gamma}{\gamma - 1} A^2(v) \right) + \left( m_1(v) - \frac{\gamma}{\gamma - 1} \lambda(v) \rho m_2(v) \right) (B + Cv + Dv^2) \\
+ \frac{1}{2} m_2^2(v) \left( C + B^2 + (2D + 2BC)v + (C^2 + 2BD)v^2 + 2CDv^3 + D^2v^4 \right) \\
- \frac{1}{2} \frac{\gamma}{\gamma - 1} \rho^2 m_2^2(v) \left( B^2 + 2BCv + (C^2 + 2BD)v^2 + 2CDv^3 + D^2v^4 \right) = 0.
\]

(E.11)

Note that there are terms involving \( v^4. \) For the above PDE to be solvable, we have to cancel the terms with \( v^4, \) this is:

\[
1 = \frac{\gamma}{\gamma - 1} \rho^2.
\]

(E.12)
However, this condition is not feasible for $\gamma < 1$. Therefore, the conjecture of an exponential-cubic form of $h$ cannot solve the optimal problem because the terms involving $v_{1}^{4}$ in the PDE cannot be matched. The same problem can be inferred for higher polynomials.

**Exponential-$n^{th}$**: If $h(v, t)$ follows

$$h(v, t) = \exp \left( A_0(\tau) + \sum_{k=1}^{n} \frac{1}{k} A_k(\tau)v^k \right),$$

where $\tau(T) = T - t$, and $A_k(0) = 0$ for all $k = 0, 1, \ldots, n$. The partial derivatives of $h$ are

$$h_{\tau} = h \left( -A'_0 - \sum_{k=1}^{n} \frac{1}{k} A_k(\tau)v^k \right),$$

$$h_{v_k} = h \left( A_1 + \sum_{k=2}^{n} A_k(\tau)v^{k-1} \right),$$

$$h_{v_{k+1}} = h \left( A_2 + \sum_{k=3}^{n} (k-1)A_k(\tau)v^{k-2} \right) + h \left( A_1 + \sum_{k=2}^{n} A_k(\tau)v^{k-1} \right)^2,$$

$$\frac{h_{v_k}}{h} = \frac{h^2 \left( A_1 + \sum_{k=2}^{n} A_k(\tau)v^{k-1} \right)}{h} = h \left( A_1 + \sum_{k=2}^{n} A_k(\tau)v^{k-1} \right)^2. \tag{E.14}$$

By substituting the partial derivatives of $h$ into Equation (6.18), it leads to

$$\left( -A'_0 - \sum_{k=1}^{n} \frac{1}{k} A_k(\tau)v^k \right) + \left( r\gamma - \frac{1}{2} \gamma - 1 \right) \lambda v^2(\tau) + \left( m_1(v) - \frac{\gamma}{\gamma - 1} \lambda \right) \left( A_1 + \sum_{k=2}^{n} A_k(\tau)v^{k-1} \right)$$

$$+ \frac{1}{2} \lambda m_2(v) \left( A_2 + \sum_{k=3}^{n} (k-1)A_k(\tau)v^{k-2} \right) + \left( A_1 + \sum_{k=2}^{n} A_k(\tau)v^{k-1} \right)^2 \right]$$

$$- \frac{1}{2} \gamma \frac{\gamma}{\gamma - 1} \rho^2 m_2^2(v) \left( A_1 + \sum_{k=2}^{n} A_k(\tau)v^{k-1} \right)^2 = 0, \tag{E.15}$$

This involves $v^{n+1}, \ldots, v^{2n-2}$, for $n \geq 3$. These are terms that cannot be matched by the conjectured helper function unless $1 = \gamma \rho^2$. However, this condition is not viable for $\gamma < 1$. That is, for $n \geq 3$, the exponential-polynomial form of the function $h$ cannot help in solving the optimal portfolio problem.

### E.2 Multidimensional portfolio optimization in the family of exponential-polynomial

A multidimensional setting implies multiple risky-assets and multiple state variables, this is very complex to describe in details. The main difficulties are the lack of a list of viable multidimensional processes for state variables (in our notation, vector $m_1$, and matrix $m_2$), and the
flexibility in the dependence structure of state variables and risky-assets (denoted $\rho$). On the other hand, the only information needed from the risky assets is the market price of risk ($\lambda$). Therefore, we can only provide a few insights into the multidimensional world, the comments have been added to the paper as an appendix.

The idea of a change of control can be easily extended to a multidimensional setting with beneficial implications. Assume there are $M$ risky assets following the SDE:

$$dS_t = D[S_t][r + G(S_t, V_t, t)\lambda(V_t)]dt + D[S_t]G(S_t, V_t, t)dw_t,$$

(E.16)

where $D[S_t]$ stands for diagonal form of vector $S$, $r = r1^M$ with $1^M = [1, ..., 1]^T \in \mathbb{R}^M$, $w_t$ is a standard $M$-dimensional Brownian motion, $\lambda(\cdot)$ is a $M \times 1$ vector of functions on the $N$-dimensional state variable vector $V_t$, $G$ is a $P$-a.s. invertible $M \times M$ matrix, both functions of $V_t$, and $w_t$ is a $N$-dimensional standard Brownian motion. We assume the correlation matrix between $dw_t$ and $dZ_t$ is $\rho(V_t)dt$, where $\rho(V_t)dt$ is a $N \times M$ matrix function of $V_t$.

Denote $\pi_t$ as the $M$-dimensional vector of the portfolio weights of risky assets, the wealth of the investor then follows

$$dX_t = [r + \pi_t G(S_t, V_t, t)\lambda(V_t)]dt + \pi_t G(S_t, V_t, t)dw_t.$$

(E.18)

Our setting above has important similarities and differences with Liu (2007), for clarity the mapping between the notations is as follows: $S_t = P_t$, $X_t = V_t$, $\mu(X_t) = r + G\lambda(V_t)$, $\Sigma = G$, $\mu^X = m_1(V_t)$, $\Sigma^X = m_2(V_t)$, and wealth $W_t = X_t$. In particular,

$$\mu^X = m_1(V_t), \quad \Sigma^X = m_2(V_t)m_2(V_t)^T,$$

(E.19)

Equations (9)-(11) in Liu (2007) impose specific conditions on $\mu^X$ and $\Sigma^X\Sigma^{X^T}$ for solvability. In reality, Equation (9) violates the Growth condition unless $\zeta_2 = 0$ or $\eta = 0$, hence we limit the SDE for state variable to the linear case.

The terms in Equations (13)-(17) in Liu (2007) would read as follow in our notation,

$$r = \delta_0, \quad (\mu - r)^T(\Sigma \Sigma^T)^{-1}(\mu - r) = \lambda(V_t)\lambda(V_t)^T,$$

$$(\Sigma^X \rho \Sigma^{-1})(\mu - r) = \lambda(V_t)\rho(V_t)m_2(V_t),$$

$$\Sigma^X \rho^T \Sigma^{X^T} - \Sigma^X \Sigma^{X^T} = m_2(V_t)\rho(V_t)\rho(V_t)^T - m_2(V_t)m_2(V_t)^T.$$

(E.20)

The flexibility of Liu’s Equation (13) does not apply to our paper since we do not consider stochastic interest rate. On the other hand, we relaxed $\Sigma(X)$ in our setting by allowing it to depend on time and stock prices, i.e. $\Sigma = G(S_t, V_t, t)$. We can now impose the quadratic conditions of Liu’s Equations (14)-(17) to our problem, which together with a convenient ansatz to control for the HARA (i.e. $J(x, v, t) = (x - F \alpha r^{-\alpha})^\gamma h(v, t)$) leads to the same PDE for $h(v, t)$ as in Liu’s Equation (8).\footnote{There are non-affine solutions available by allowing, for instance, $\lambda(V_t) = V_t^{-1}$ (non-quadratic). This example follows from the one-dimensional case reported in Cheng and Escobar-Anel (2022).}
In the search for the "base cases" in multidimensions, we follow similarly to the one risky asset. We consider a new control vector \( \psi_t \), such that,

\[
\psi_t^T = \pi_t^T \mathbf{G}(S_t, V_t, t),
\]

(E.21)

where \( \psi_t \) is a \( M \times 1 \) vector, which also satisfies admissibility conditions. The wealth process with the new control then follows the simpler SDE:

\[
\frac{dX_t}{X_t} = (r + \psi_t^T \lambda(V_t))dt + \psi_t^T dW_t.
\]

(E.22)

This highlights that the important terms for solvability are: the market price of risk, \( \lambda \), the drift and quadratic variation terms of the state variables, \( m_1, m_2, m_1^2 \), and the correlation of state variable and stocks, \( \rho \) via \( \lambda pm_2 \).

### E.3 Robust portfolio optimization in the family of exponential-polynomial

We follow a robust portfolio choice problem for an ambiguity-averse investor along the lines of the work of Maenhout (2004), Escobar et al. (2015) among others. The conditions implied by the solvability of the exponential-polynomial family conjecture are the same as that of a non-ambiguity-averse problem. In other words, the solvability of the models is not affected in a robust portfolio analysis.

The dynamics of the asset price and a state variable follow Equations (6.1) and (6.2). The investor is uncertain about the distribution of the randomness from \( Z_{1t} \) and \( Z_{2t} \).

Let \( e := (e^1_t, e^2_t) \) be an \( \mathbb{R}^2 \)-valued, \( \mathcal{F}_t \)-progressively measurable process and define the Radon-Nikodym derivative process by

\[
\xi_t = \frac{d\mathbb{P}^e}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp \left\{ -\int_0^t \left( \frac{(e^1_t)^2 + (e^2_t)^2}{2} d\tau + e^1_t dZ_{1t} + e^2_t dZ_{2t} \right) \right\}.
\]

(E.23)

According to Girsanov's theorem, the process

\[
\begin{bmatrix}
Z_{1t} \\
Z_{2t}
\end{bmatrix} = \begin{bmatrix}
\int_0^t e^1_t d\tau \\
\int_0^t e^2_t d\tau
\end{bmatrix} + \begin{bmatrix}
Z_{1t} \\
Z_{2t}
\end{bmatrix},
\]

(E.24)

is a Wiener process under the probability measure \( \mathbb{P}^e \). Let \( \varepsilon[0, T] \) denote the set of all \( \mathcal{F}_t \)-progressively measurable processes such that the process (E.23) is a well-defined Radon-Nikodym derivative process. This formulation of incorporating an investor’s model uncertainty actually allows for uncertainty in the drift of diffusion risk factors of the stock and its variance driver (i.e., \( Z_{2t} \) and \( Z_{1t} \), respectively).

The alternative model then follows

\[
\frac{dS_t}{S_t} = \left[ r + \lambda(v_t) G(S_t, v_t, t) - \left( \rho e^v_t + \sqrt{1 - \rho^2} e^\delta_t \right) G(S_t, v_t, t) \right] dt + G(S_t, v_t, t) \left( \rho d\bar{Z}_{1t} + \sqrt{1 - \rho^2} d\bar{Z}_{2t} \right),
\]

\[
dv_t = [m_1(v_t) - e^v m_2(v_t)] dt + m_2(v_t) d\bar{Z}_{1t}.
\]

(E.25)
Note that the alternative model setup allows for different levels of ambiguity in the risky asset and the state variable that drives its diffusion and market price of risk.

Let \( \bar{\pi} \) be the fraction of wealth invested in the risky asset, and the rest of the wealth \((1 - \bar{\pi}) \) is invested in the money account \( M_t \). The wealth \( X_t \) then follows

\[
\frac{dX_t}{X_t} = \bar{\pi} \frac{dS_t}{S_t} + (1 - \bar{\pi})rdt
\]

\[
= \left[ r + \bar{\pi} \left( \lambda(v_t)G(S_t, v_t, t) - G(S_t, v_t, t) \left( \rho \psi_e^2 + \sqrt{1 - \rho^2} \psi_e^2 \right) \right) \right] dt + \bar{\pi}G(S_t, v_t, t) \left( pdZ_{1t} + \sqrt{1 - \rho^2}dZ_{2t} \right),
\]

(E.26)

Consider a change of control such that

\[
\psi_t = \bar{\pi}G(S_t, v_t, t),
\]

(E.27)

where \( \psi_t \) belongs to the space of admissible strategies. The wealth process can be rewritten as

\[
\frac{dX_t}{X_t} = \left[ r + \psi_t \left( \lambda(v_t) - \left( \rho \psi_e^2 + \sqrt{1 - \rho^2} \psi_e^2 \right) \right) \right] dt + \psi_t \left( pdZ_{1t} + \sqrt{1 - \rho^2}dZ_{2t} \right).
\]

(E.28)

The objective of the ambiguity-averse agent with HARA utility is to maximize the expected utility from his terminal wealth \( X_T \). Define the reward function realized when choosing an alternative model specified by \( e \) as

\[
w^e(x, v; \psi) = \frac{1}{Y} \mathbb{E}_{\mathcal{F}_{t,x,v}} [(X_T - F)^Y],
\]

(E.29)

and the indirect utility function (value function) is defined as

\[
J(x, v, t) = \sup_{\psi \in U} \inf_{e \in \mathcal{E} \cap [t, T]} \left\{ w^e(x, v; \psi) + \mathbb{E}_{t} \left[ \int_{t}^{T} \frac{(e^\tau)^2}{2\Phi_1(\tau, X_{\tau}, v_{\tau})} + \frac{(e^\tau)^2}{2\Phi_2(\tau, X_{\tau}, v_{\tau})} d\tau \right] \right\},
\]

(E.30)

where \( \mathcal{U} \) denotes the space of admissible strategies defined in Definition 6.2.1.

By the dynamic programming approach, the value function, i.e., \( J(x, v, t) \), satisfies the robust HJB equation:

\[
sup_{\psi} \inf_{e \in \mathcal{E}^t,e^t} \left\{ J_t + x \left[ r + \psi \left( \lambda(v_t) - \left( \rho \psi_e^2 + \sqrt{1 - \rho^2} \psi_e^2 \right) \right) \right] J_t + \frac{1}{2} x^2 \psi^2 J_{xx} \right.
\]

\[
+ \left[ m_1(v_t) - m_2(v_t) \psi_e^2 \right] J_t + \frac{1}{2} m_2^2(v_t) J_{vw} + \psi \rho m_2(v_t) J_{v\psi} + \frac{\psi^2}{2\Phi_1} + \frac{(e^\tau)^2}{2\Phi_2} = 0,
\]

(E.31)

with boundary condition \( J(x, v, T) = \frac{(x-F)^Y}{Y} \), and \( \Phi_i = \frac{\phi_i}{\beta} \), \( i = 1, 2 \). Then by assuming the value function in the form of \( J(x, v, t) = \frac{(x-F \exp^{-r(T-t)})^Y}{Y} h(v, t) \), solving the minimization and maximization problem, substituting the partial derivatives of \( J \), and simplifying the PDE, we
obtain
\[
\begin{align*}
& h_t + \left( ry + \frac{1}{2} \gamma \left( \frac{\lambda^2(v_t)}{\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1)} \right) \right) h \\
& + \left( m_1(v_t) + \frac{1}{2} \gamma \left( \frac{2 \lambda(v_t) \rho \left( 1 - \frac{\phi_2}{\gamma} \right)}{\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1)} \right) m_2(v_t) \right) h_t + \frac{1}{2} m_2^2(v_t) h_{vv} \\
& + \left( - \frac{1}{2} \frac{\phi_1}{\gamma} + \frac{1}{2} \gamma \left( \frac{\rho^2 \left( 1 - \frac{\phi_2}{\gamma} \right)^2}{\phi_1 \rho^2 + \phi_2(1 - \rho^2) - (\gamma - 1)} \right) m_2^2(v_t) \right) \frac{h_t^2}{h} = 0.
\end{align*}
\]  
\text{(E.32)}

This PDE is structurally the same as equation (6.18) hence leading to the same family of solutions.
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