Wideband and Relativistic Superradiance in Astrophysics

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Abstract

In the quantum phenomenon of superradiance (SR) a population of inverted particles evolves, through its interaction with the quantized vacuum radiation field, into a highly entangled state capable of generating much greater radiative emission than predicted by the independent spontaneous decay of its constituent particles. The phenomenon has recently been applied to transient astrophysical processes but has thus far been restricted to particles sharing a common velocity. This thesis researches the effects of astrophysical velocity distributions upon SR, which are distinct from conventional regimes of the quantum optics literature in that they may possess extremely wide bandwidths, turbulent statistical properties, or highly relativistic mean velocities. An important result of this thesis is the derivation of two novel algorithms for simulating widely Doppler broadened SR, each offering improved numerical complexity scaling over conventional methods. In the first, a Fourier domain representation is derived for the velocity dependent partial differential equations describing a population inversion interacting with the radiation field; this representation generalises an existing quasi-steady state maser model to the transient SR regime. In the second, the electric field is represented by a collection of fields, each representing photon creation or annihilation on resonance with a particular velocity channel; the symmetry of this representation leads to a numerically advantageous algorithm for many velocity broadened systems. I apply this latter algorithm to investigate the effects of pumping mechanisms and velocity distribution statistics upon transient SR processes in widely broadened astrophysical media. I demonstrate that the orientation of the pumping mechanism as well as turbulent properties of the velocity distribution critically affect transient SR structure in a widely Doppler broadened sample. The final project of this thesis develops a relativistic model of SR built upon canonical quantization of a covariant Lagrangian for the matter-radiation interaction. I apply the diagrammatic method alongside numerical techniques to compute the particle state reduced density operator’s time evolution from the relativistic two-particle SR Hamiltonian, and make quantitative conclusions regarding the effect of relativistic velocity coherence upon SR intensity measurements in the observer’s frame.

Keywords: Superradiance, Radiation Dynamics, Radiation Mechanisms, Quantum Optics, Entanglement, Numerical Methods, Special Relativity, Klein-Gordon
Summary for Lay Audience

Even in a pitch dark vacuum, the physics of the very small describes so-called quantum fluctuations of light. A particle possessing energy internal to itself can interact with these fluctuations and release its energy as a single unit of light known as a photon.

Sometimes, particles within a gas interact with quantum fluctuations of light independently. Other times, a collection of particles interacts with fluctuations in an exotic fashion whose whole cannot be understood as a sum of its parts. When the language of mathematical physics carefully tells the story of such a group, it cannot be said that any single member emits a photon; rather, the group itself collectively emits a photon. During this process the particles become tangled up together and lose their identities. This entangled group interacts strongly with quantum fluctuations of light and emits an intense stream of photons in a phenomenon known as superradiance.

Superradiance can describe some astrophysical observations, but such environments often possess wide spreads in their particles’ velocities. Velocities shift particles’ natural frequencies (similar to a passing train’s pitch modulation) such that particles of different velocities talk to different photons, reducing their entanglement. This thesis analyses the effect of a wide velocity spread upon superradiance.

Certain mathematical objects describing a superradiant sample rotate at extremely fast rates. Computer simulation is simplified by spinning one’s perspective, analogous to a strobe light bringing a spinning fan to a standstill. When multiple velocities are present, however, objects rotate at different rates and simulation is complicated by the lack of a shared “strobe rate” to simultaneously simplify all particles’ mathematical descriptions. This thesis derives two novel simulation algorithms: one which transforms our approach to the spinning mathematical objects, and another which juggles a collection of strobes. Both methods dramatically improve simulation speed and are used to investigate superradiance in astrophysics.

Astrophysical sources can travel near the speed of light, which requires that superradiance be reformulated consistently with Einstein’s special theory of relativity. This thesis derives a model of superradiance capturing relativistic time dilation and the relativistic transformation of light, and calculates consequences for observations.
Co-Authorship

This thesis comprises three papers. Chapter 2 is already published in the Monthly Notices of the Royal Astronomical Society (MNRAS), Chapter 3 is being submitted imminently (July 2022) to MNRAS, and Chapter 4 has been submitted to Physical Review A and is currently undergoing the review process. Publication details are provided in the opening of each chapter. All three papers were co-authored with Dr. Martin Houde, who supervised the corresponding projects.

All project ideas were established in collaboration with Dr. Houde, who provided feedback on results; who recommended theoretical investigations to undertake, analytical derivations to attempt, and numerical simulations to perform; and who provided meaningful conversations regarding astrophysical context or appropriate physical regimes of interest. Dr. Houde additionally assisted with multiple proofreading iterations for all five chapters of the thesis.

Dr. Fereshteh Rajabi reviewed and provided helpful feedback on all three papers. Dr. Rajabi’s suggestions included potential content additions, conceptual rephrasing recommendations, re-mapping of emphasis upon topics most relevant to researchers in the field, and structural re-arrangements. Dr. Rajabi also provided verbal discussions which assisted C. Wyenberg’s theoretical understanding during composition.

Dr. Boy Lankhaar provided important insight into the Menegozzi & Lamb algorithm referenced heavily in Chapter 2 as well as into maser theory and numerical modelling. Section 2.1, Paragraphs VI and VII were written in their first drafts by Dr. Lankhaar before undergoing revisions in collaboration with C. Wyenberg.

Mr. Mohammed Chamma and Ms. Aishwarya Kumar provided helpful proofreading of all three papers (Chapters 2–4) as well as extensive conversations during research group meetings regarding the projects undertaken and the interpretation of nuanced results. Both Mr. Chamma and Ms. Kumar discovered typing errors and made phrasing recommendations upon review of draft manuscripts, as well as provided feedback regarding legibility and structure.

The authorship allocation and ordering of all three papers (published or in submission) reflect the degree of contribution provided by those persons listed above. Dr. Houde, as the supervisor, is listed last.
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I wish to thank Dr. Martin Houde for taking a chance on me and committing to my academic and professional success. I understood that I was an unconventional and unfamiliar student to him when I returned to academia from industry, and I will always be immensely grateful to him for having given me this opportunity to fulfill a life’s dream. Dr. Houde turned out to be the kindest, most patient, and most confidence-inspiring supervisor I could ever have hoped for. It has truly been an honour and a joy to work with him.

I also wish to thank my parents, who supported me immeasurably throughout my education. My dad employed me on his construction crew during every summer and winter academic vacation, teaching me the value of hard work, patience, and gratitude for the opportunity of education. I keenly recall when theoretical physics became one of his pop-science reading fascinations during slower construction seasons, and my love for the field is in no small way owed to the mindset of wonder and curiosity which he instilled in me at a very young age.

I am also grateful for my mom’s model of tenacity and kindness. Regarding the former quality, I trace my keen academic focus back to her example set as a hardworking and disciplined teacher during my elementary years. Regarding the latter, I have always been impressed by the kindness and respect which she showed her immigrant adult English as a second language students arriving in Canada from war-torn countries. My mom taught me to always be kind to those who have not had the opportunity of an education, to appreciate that all people are equally brilliant, and to never take pride in academic accomplishments afforded to me by the privilege of a stable Canadian environment.

Both of my parents have also been incredibly encouraging and proud of me for my academic pursuits, never suggesting I pursue a financially lucrative job, but rather stressing that I strive for intellectual fulfillment, kindness, and servitude to others in my career. Furthermore, they have always been immensely kind to me, and I appreciate that my academic success is owed to their loving and supportive upbringing.

I am grateful to my brother Jay and to his wife Dani; to Jay for his lifelong friendship, but also, in an academic respect, for listening to my lengthy physics monologues whenever possible. Theoretical physics is a fascinating field, but it can be a lonely one—and even the
most beautiful concept is nothing if not shared with someone. Jay constantly listens to me with patience and enthusiasm, and I would never have pursued this field if not for the thrill of sharing it with him through undergrad, grad school, and onward. I am grateful to my encouraging and patient sister-in-law, Dani, who accepted me into their home on many occasions during my doctoral studies and who always fully supports my (admittedly quite excessive) time spent discussing physics with Jay.

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I acknowledge that my doctoral studies at Western University have been undertaken on the traditional lands of the Anishinaabek (Ah-nish-in-a-bek), Haudenosaunee (Ho-den-no-show-nee), Lūnaapēewak (Len-ahpay-wuk) and Chonnonton (Chun-ongk-ton) Nations, on lands connected with the London Township and Sombra Treaties of 1796 and the Dish with One Spoon Covenant Wampum.

With this, I respect the longstanding relationships that Indigenous Nations have to this land, as they are the original caretakers. I acknowledge historical and ongoing injustices that Indigenous Peoples (First Nations, Métis and Inuit) endure in Canada, and we at Western University accept responsibility as a public institution to contribute toward revealing and correcting miseducation as well as renewing respectful relationships with Indigenous communities through our teaching, research and community service.
Dedication

To my parents, who taught me curiosity and instilled in me a love for learning;

To Jay and Dani, who patiently endure my lengthy physics monologues;

and

With special thanks to Charlie, Lily, and Thea.
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List of Abbreviations, Symbols, and Nomenclature

CTD: Conventional Time Domain

IF: Integral Fourier

LMI: Local Mode Interaction

LMI\textsuperscript{IF}: Local Mode Interaction (in the IF algorithm)

LMI\textsuperscript{SF}: Local Mode Interaction (in the SF algorithm)

MB: Maxwell-Bloch

ML: Menegozzi & Lamb

QED: Quantum Electrodynamics

QO: Quantum Optics

SF: Supplementary Fields

SM: Statistical Mechanic(al/s)

SR: Superradiance

SVEA: Slowly-Varying Envelope Approximation

TML: Transient Menegozzi & Lamb

WDB: Widely Doppler Broadened
Chapter 1

Introduction

1.1 Motivation

The interaction between radiation and matter is arguably the most important process in all of physics. From the historical development of the theory of the atom, to all experiments in atomic physics, to our discovery of quantum mechanics, to the introduction of quantum field theory through the necessity of a quantum electrodynamic theory of the atom, and to the creation of innumerable modern technologies—the interaction between radiation and matter is ubiquitous. The process is also theoretically described to an incredible degree of accuracy; for example, a quantum electrodynamic calculation of the anomalous magnetic dipole moment agrees with experiment (as of this writing) to within better than one part in a billion [13].

The radiation-matter interaction is as rich in its applications as it is mathematically complex. The aforementioned calculation involves a sophisticated machinery drawing from many branches of advanced mathematics, it invokes deep arguments about the nature of physical reality, and it demands substantial computational resources to execute. Owing to its complexity, it is not surprising that the process requires a variety of models across differing fields of research. Depending upon the dominant feature or the degree of accuracy required, researchers may apply partial differential equations (the Maxwell-Bloch equations in a polarised medium); the quantum harmonic oscillator creation and annihilation formalism; fully relativistic loop diagrams (again, the anomalous magnetic dipole moment example); statistical mechanics (radiative processes); or even simple geometrical and custom differential models (radiative transfer).
In light of this diversity, it is not surprising that some of the interaction’s associated phenomena or modelling techniques may be well understood by one field of research but entirely overlooked by another. Furthermore, cross-pollination in this regard may be entirely absent, as the domains of differing research areas simply may not overlap.

It is cause for excitement when a feature of the interaction between radiation and matter which is simple, fundamental, and well-understood within one area of physics suddenly finds application within another. Such is the case, we believe, with the recent application of a quantum optical coherent radiative emission process to the field of astrophysics. The phenomenon is known to the quantum optics community as superradiance (SR) and is characterised by enhanced radiative emission intensity. SR is a correlation effect wherein the highly entangled quantum state of an ensemble of invertible particles couples strongly to the quantized radiation field. The resulting emission intensity amplitude and transient features are quite distinct from those expected from a classical statistical analysis of independent spontaneous emission from the ensemble’s constituent particles.

Dr. Fereshteh Rajabi, Dr. Martin Houde, and others have recently applied the theory of SR to the interstellar medium. Rajabi and Houde first demonstrated SR’s applicability to recurrent maser flares in the interstellar medium, where numerical models successfully fit high intensity, varying time-scales, and time ordering of spectral lines in data from G107.298+5.639. In 2018, Houde, Mathews, and Rajabi continued to propose SR as a mechanism for Fast Radio Bursts, suggesting their origin in maser harboring regions and demonstrating consistency with SR’s temporal features and ringing effects. Shortly thereafter, Houde, Rajabi, et al. then demonstrated a mechanism by which a Fast Radio Burst SR event could be triggered, considering a pulsar initiating an SR burst while also being amplified by the SR system.

The application of SR to astrophysics is exciting for multiple reasons. From a utilitarian perspective, the quantitative success of the initial work by Rajabi, Houde, et al. establishes SR as a useful tool in astrophysics research—and we expect a novel research tool to immediately present new avenues of research. The application of SR to astrophysics also has fascinating theoretical implications: it may potentially shift thought processes around familiar problems or it may pose questions never before considered. A quantum coherence effect witnessed on a scale spanning 100 to 1000 astronomical units and involving the entanglement of an estimated
10^{30–10^{32}} molecules [24] should capture the imagination of any physicist. The rich and unique features of QM coherence have rarely (if ever) been observed on scales larger than a few kilometers; indeed, outside of highly theoretical black hole thought experiments, we are unaware of any such large extended phenomenon which cannot be accurately modelled by the classical extension of local QM models.

1.1.1 Purpose of the thesis

This thesis will advance the theory of SR for its application to realistic non-relativistic and relativistic Doppler broadened astrophysical sources. Sections 1.2–1.5 of this chapter lay the groundwork for making these statements precise. We begin by building a foundational understanding of the theory and history of SR in Section 1.2, followed in Section 1.3 by a survey of research methodologies and necessary approximations made in numerical simulations of SR which are crucial to the work of this thesis. We describe conditions necessary for establishing SR in Section 1.4, which are important when determining the phenomenon’s candidate astrophysical environments. After a summary of SR features in Section 1.5, we will be equipped to properly formulate the plan for the thesis in Section 1.6.

1.2 Foundations

We present now a brief introduction to the theory and history of SR. We do not intend to complicate the discussion at this point with excessive mathematical formalism, but rather to highlight the phenomenon’s profoundly quantum mechanical nature. We will therefore open with an analogy from spin-$\frac{1}{2}$ neutrons, which we find both elegant and historically important as having been first presented in the introduction of Robert Dicke’s seminal paper [9] and having guided his development of the coherency theory therein. We will thereafter proceed to compute some basic results (with only as little mathematical rigor as needed) from a very simple SR system: the two-level, four-atom small sample. This example highlights principal features of the theory and provides physical intuition and context for the work of this thesis. Finally, we conclude this foundational section with a survey of historical theoretical and experimental developments in SR.
1.2.1 Dicke’s seminal work

The theoretical foundations for SR were laid in the seminal paper by Dicke [9] which predicted key features of the phenomenon in both small and large samples of atoms. Dicke’s paper is particularly elegant, presenting arguments rooted in symmetry, exploiting an isomorphism to the theory of angular momentum from elementary quantum mechanics, and invoking quantum electrodynamics in only its most simple form. The concepts presented by Dicke have remained pedagogically important to SR research, even as theoretical models have evolved to increasing degrees of complexity.

Motivation: Dicke’s angular momentum analogy

As a motivating introduction to the profound simplicity and importance of quantum entanglement in SR, we present the fascinating thought experiment from the opening discussion of Dicke [9]. Dicke demonstrated that quantum superposition effects are fundamental to the theory of radiating ensembles via an analogy with the angular momentum transitions of neutrons in a magnetic field. Considered in isolation, a single spin-$\frac{1}{2}$ neutron’s doublet degeneracy is lifted by the presence of a magnetic field and it may decay from the spin-up state to the spin-down state with the accompanying emission of a photon [9].

Suppose that we now introduce a second neutron in the spin-down state next to the first neutron in the initial spin-up state; both neutrons are exposed to a magnetic field, but they are neither interacting nor overlapping in their wavefunctions (so no fermionic exchange symmetrisation of the quantum state is required). We might naively expect the presence of the second neutron to have no effect upon the first (they are not interacting, after all), and predict for the first neutron the usual transition rate and its inevitable decay to the spin-down state. In fact, this classical expectation is entirely wrong. The initial state $|\uparrow\downarrow\rangle$ must be identified as the superposition of the symmetric and antisymmetric states

$$|\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}} |s\rangle + \frac{1}{\sqrt{2}} |a\rangle,$$

(1.1)
where
\[ |s\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right) \quad \text{and} \quad |a\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \] are, respectively, members of the triplet total angular momentum collection of states (total angular momentum number \( s = 1 \) including the three states \(|\uparrow\uparrow\rangle, |s\rangle, \text{and} |\downarrow\downarrow\rangle\)) and of the singlet state (\( s = 0 \), of which \(|a\rangle\) is the only member). Since the magnetic interaction Hamiltonian does not couple states of differing total angular momentum, a transition from the singlet state to a triplet state (example, \(|a\rangle \rightarrow |\downarrow\downarrow\rangle\)) is forbidden. The antisymmetric part of the \(|\uparrow\downarrow\rangle\) state will therefore not decay to the doubly spin-down state. We come to the important realisation that when repeated measurements are made on many systems prepared in the \(|\uparrow\downarrow\rangle\) state, only half will ever radiate, while the other half will become trapped in the antisymmetric state. This result is entirely at odds with our common intuition of two independent neutrons.

Although at this point serving as only a motivating analogy, the problem of angular momentum transitions can, in fact, be brought into mathematical correspondence with the quantum theory of radiative transitions within an ensemble of two-level particles. Dicke proceeded to develop the theory of SR with the tools of the angular momentum raising and lowering operators as they manifest themselves in the correspondence to the two-level particle energy state space. He made quantitative predictions of photon emission rates in small samples (defined as possessing particle separations \( l \) much shorter than the wavelength of emission \( \lambda \)), and continued to present coherence as the fundamental feature of SR. Finally, Dicke concluded with generalisations of SR to large samples (\( l \gg \lambda \)) and presented the common photon interaction as the mediator of coherence between distantly separated radiating particles.

**The Dicke model: a four-atom system**

In order to establish a tangible physical intuition for SR, we perform in this section the detailed emission rate calculations for four two-level particles contained in a small volume of linear dimension \( l \) much less than the wavelength of dipole emission \( \lambda \) (\( l \ll \lambda \)). We assume that the particles interact with the vacuum radiation field but are far enough separated so as not to interact with each other through their Van der Waals dipole coupling. The physical exchange symmetry of the problem imposes an exchange symmetry upon the Hamiltonian; that is, it
must be invariant under permutation of particles. We set as our initial condition the exchange-
symmetric fully excited state

$$|\Psi(t = 0)\rangle = |e(1)\rangle \otimes |e(2)\rangle \otimes |e(3)\rangle \otimes |e(4)\rangle \equiv |S_4\rangle.$$  

(1.3)

We shall herein abbreviate all states as, for example, $|eeee\rangle$ (with the four particles identified
by their positions from left to right and the tensor products suppressed).

Because both the initial condition and the Hamiltonian are exchange symmetric, the system
is constrained to evolve within the symmetric subspace $S$ of the full system Hilbert space:

$$|\Psi(t)\rangle \in S = \text{Span}\left\{|S_0\rangle, |S_1\rangle, |S_2\rangle, |S_3\rangle, |S_4\rangle\right\} \subset \mathcal{H}$$

(1.4)

where the subscript denotes the number of excited atoms in each symmetric state, explicitly:

$$|S_4\rangle = |eeee\rangle,$$

(1.5)

$$|S_3\rangle = \frac{1}{2} (|eeeg\rangle + |eege\rangle + |egee\rangle + |geee\rangle),$$

(1.6)

$$|S_2\rangle = \frac{1}{\sqrt{6}} (|eegg\rangle + |eegg\rangle + |egge\rangle + |geeg\rangle + |gege\rangle + |ggee\rangle),$$

(1.7)

$$|S_1\rangle = \frac{1}{2} (|eggg\rangle + |gegg\rangle + |ggee\rangle + |ggge\rangle), \text{ and}$$

(1.8)

$$|S_0\rangle = |gggg\rangle.$$  

(1.9)

We will build the dynamics of these states from a generalisation of the evolution of a single
particle, which we briefly describe now. Were we to consider only the Hamiltonian of the parti-
cle energy levels, an atom initially prepared in the excited state $|e\rangle$ would remain $|e\rangle$ indefinitely.
However, the particle exists within the vacuum quantized radiation field, whose Hamiltonian
contains an interaction term between the excited state and the radiation mode on resonance
with the particle’s energy level separation. The combined $|\text{Atom}\rangle \otimes |\text{Radiation}\rangle$ state evolves
with mixing between the $|e\rangle$ and $|g\rangle$ states, and one may make a measurement of whether a
radiation mode (or photon) has been created within a given time period. The full computation
yields the Dirac formula, which states that the probability $W_1$ of one atom radiating per unit
time is proportional to the expectation value of the raising and lowering operators:

\[ W_1 = \Gamma \langle \hat{D}_1^+ \hat{D}_1^- \rangle, \quad (1.10) \]

where the “1” denotes a single particle, \( \Gamma \) is the single-particle decay rate of the transition, and

\[ \hat{D}_1^+ = |e\rangle \langle g|, \quad (1.11) \]

\[ \hat{D}_1^- = |g\rangle \langle e| = (\hat{D}_1^+)^\dagger. \quad (1.12) \]

At the level of rigor of this introduction, we naturally expect equation (1.10) to generalise to our four-particle system as:

\[ W_{4\text{atoms}} = \Gamma \langle \hat{D}_4^+ \hat{D}_4^- \rangle, \quad (1.13) \]

where the four-particle raising and lowering operators are the totally symmetric raising and lowering operators:

\[ \hat{D}_4^+ = \hat{D}_1^+(1) + \hat{D}_1^+(2) + \hat{D}_1^+(3) + \hat{D}_1^+(4) \quad (1.14) \]

and \( \hat{D}_4^- = (\hat{D}_4^+)^\dagger \). The numerical values in parentheses denote the particular particle being raised or lowered. Note that it is implied that a term such as, for example, \( \hat{D}_1^+(1) \) abbreviates the full expression \( \hat{D}_1^+(1) \otimes \hat{I}(2) \otimes \hat{I}(3) \otimes \hat{I}(4) \). In what follows we shall generally drop the subscript “4” and all operators will be implied to act on our four-particle system unless explicitly stated otherwise.

We may use expression 1.13 to compute the probability per unit time of decaying from any of the \( |S_j\rangle \). For example,

\[ \Gamma \langle \hat{D}^+ \hat{D}^- \rangle_{S_4} = \Gamma \langle eeee \mid \sum_j \hat{D}_1^+(j) \left( \sum_k \hat{D}_1^-(k) \right) eeee \rangle \]

\[ = \Gamma \langle eeee \mid \sum_{jk} \hat{D}_1^+(j) \hat{D}_1^-(k) eeee \rangle. \quad (1.15) \]

In this particular calculation the only non-vanishing terms will be, significantly, only the \( j = k \) terms such that

\[ W_4 = \Gamma \langle \hat{D}^+ \hat{D}^- \rangle_{S_4} = 4\Gamma, \quad (1.17) \]
which is just the expected decay rate for four particles (four times that for one particle).

The situation differs dramatically when we consider decay from the other states, for example \(|S_2\).

\[
\Gamma \langle \hat{D}^+ \hat{D}^- \rangle_{S_2} = \Gamma \langle S_2 \rangle \sum_{j,k} \hat{D}^+(j) \hat{D}^-(k) |S_2\rangle \\
= \frac{\Gamma}{6} (\langle eegg \rangle + \langle egeg \rangle + \langle egge \rangle + \langle geeg \rangle + \langle gege \rangle + \langle ggee \rangle) \\
\times \sum_{j,k} \hat{D}^+(j) \hat{D}^-(k) (|eegg\rangle + |egeg\rangle + |egge\rangle + |geeg\rangle + |gege\rangle + |ggee\rangle).
\]

In evaluating equation (1.19) we must count how many ways a ket on the right may be de-excited in its \(k\)th atom, excited in its \(j\)th atom, and thereafter project non-vanishingly onto a bra on the left. By symmetry we can count how many ways this may be done on the first ket \(|eegg\rangle\) and then multiply by six. There are two ways it can be transformed into itself; one way into each of \(\langle egeg \rangle, \langle egeg \rangle, \langle egge \rangle,\) and \(\langle gege \rangle;\) and no ways into \(\langle ggee \rangle\). This gives a total of \(6 \times 6 = 36\) ways which, when multiplied by the factor of \(\Gamma/6\) out front, yields \(W_2 = 6\Gamma\).

We are thus led to the following important intuitive realisation regarding SR: that the emission rate is enhanced by the level correlation between particles, as we witness here in our attempts to de-excite \(k\)th particles and excite \(j\)th particles, and then project onto another state. We are in essence counting the correlation between \(k\)th and \(j\)th particles. For \(N = 4\) the result is not especially dramatic; however, by counting considerations, one can see that the emission rate of the \(S_{N/2}\) state is in fact \(W_{N/2} = \frac{N}{2} \left( \frac{N}{2} + 1 \right) \Gamma\), which scales not as \(N\) but as \(N^2\) for large \(N\).

Continuing with our four-particle example, by the same procedure we may compute \(W_3 = 6\Gamma\) and \(W_1 = 4\Gamma\). Obviously, \(W_0 = 0\). We may use these emission rates to model the population evolution of each symmetric level. The decay rate of the population \(P_j\) of the \(j\)th level is proportional to its own spontaneous decay rate plus the rate of entry from decay of the level
1.2. Foundations

above it; i.e.,

\[
\frac{dP_4}{dt} = -W_4 P_4 \quad (1.20)
\]
\[
\frac{dP_3}{dt} = -W_3 P_3 + W_4 P_4 \quad (1.21)
\]
\[
\frac{dP_2}{dt} = -W_2 P_2 + W_3 P_3 \quad (1.22)
\]
\[
\frac{dP_1}{dt} = -W_1 P_1 + W_2 P_2 \quad (1.23)
\]
\[
\frac{dP_0}{dt} = W_1 P_1. \quad (1.24)
\]

Substituting our values for the \(W_j\) yields a set of five coupled differential equations in the five unknown level populations. We assume the initial conditions \(P_4(0) = 1\) and \(P_{j\neq4}(0) = 0\) and solve to obtain the exact solution:

\[
P_4(t) = e^{-4\Gamma t} \quad (1.25)
\]
\[
P_3(t) = 2e^{-4\Gamma t} - 2e^{-6\Gamma t} \quad (1.26)
\]
\[
P_2(t) = 6e^{-4\Gamma t} - 6(1 + 2\Gamma t) e^{-6\Gamma t} \quad (1.27)
\]
\[
P_1(t) = 9(-3 + 4\Gamma t) e^{-4\Gamma t} + 9(3 + 2\Gamma t) e^{-6\Gamma t} \quad (1.28)
\]
\[
P_0(t) = 9(3 - 4\Gamma t) e^{-4\Gamma t} - 4(7 + 6\Gamma t) e^{-6\Gamma t} + 1. \quad (1.29)
\]

The total emission rate \(I(t)\) is (negative) the sum of the decay rates, or (negative) the sum of all the first terms on the right-hand side of equations (1.20)–(1.23):

\[
I(t) = W_4 P_4 + W_3 P_3 + W_2 P_2 + W_1 P_1 \quad (1.30)
\]
\[
= 4\Gamma P_4 + 6\Gamma P_3 + 6\Gamma P_2 + 4\Gamma P_1. \quad (1.31)
\]

The population level transients and the total emission rate are shown in Figure 1.1, which demonstrates four typical features of the SR cascade:

1. Initial intensity matching the case of independent atoms,

2. A short time delay to peak intensity,
3. Peak intensity exceeding that of independent atoms, and

4. A subsequent rapid decrease in emission falling off faster than that of independent atoms.

![Population Level Transients for the Dicke States of the Four-Atom Small Sample](image1.jpg)

![Four-Atom Emission Intensity Transients](image2.jpg)

Figure 1.1: Top: Plots of population level transients for the five Dicke states of a four-atom small sample. Bottom: Plot of total emission rate from all Dicke states; independent atoms expectation and superradiant results shown.

In order to build further intuition into the emission dynamics it is illustrative to consider a cascade diagram from the fully excited state to the fully ground state, as depicted in Figure 1.2. The four features enumerated previously can be loosely associated, respectively, with the following four observations of Figure 1.2:
1. The initial emission rate matches the independent radiators result as the initial state possesses no correlation between atoms, and follows only the four independent radiative avenues (arrows) for emission,

2. The delayed build-up to peak emission is related to the delay in cascading to the maximally correlated mid-level Dicke state $S_2$,

3. The high peak intensity is related to the large multitude of parallel decay avenues (arrows) between the Dicke states $S_3$, $S_2$, and $S_1$; and

4. The rapid falloff is due to total energy conservation in conjunction with item #3.

Figure 1.2: Qualitative visualisation of the emission cascade down Dicke states of the four-atom small sample.

### 1.2.2 History and experiment

Dicke’s original work [9] was followed by a large amount of theoretical research. In 1969, Agarwal [1] introduced a time-evolution equation and imposed important approximations related to particle-radiation correlations. The Italian physicist F.T. Arecchi first described the phenomenon as a cooperative enhancement effect in 1970 [2], and R. Bonifacio et. al [7, 4, 5, 6] analysed the damping of the cooperative effect as a function of sample size and other parameters. R. Friedberg and S.R. Hartmann published a number of papers describing realistic modifiers to SR such as inhomogeneous broadening and spatial variations in atomic positions.
Many other papers in the 1970s established important results for dephasing processes, time scale conditions necessary to establish SR, and propagation effects ([20]; cf. also references [2]–[51] of Gross and Haroche [20]).

The experimental observation of SR was made possible with the advancement of dye-pulsed laser systems and was first observed in an HF gas by Skribanowitz et al in 1973 [38, 3]. Figure 1.3 depicts the experiment schematically. A pulsed laser beam first established a population inversion by pumping on resonance between the lowest energy state \( |f\rangle \) and an excited state \( |e\rangle \). The excited molecules demonstrated SR on the \( |e\rangle \to |g\rangle \) transition, which possessed a dipole transition in the infrared; i.e., far removed from the pumping frequency.

![Figure 1.3: Typical pumping schematic for the experimental observation of SR. Adapted from Gross and Haroche [20].](image)

Both the HF experiments of Skribanowitz, Herman, MacGillivray, and Feld [38] and subsequent CH\(_3\)F experiments [37] demonstrated SR in the far-infrared via optical pumping by CO\(_2\) lasers. From 1976 to 1978, SR was observed in the near infrared from Na, Tl, Cs, Li, and Rb [18, 14, 42, 21]; in 1979, in the visible from Sr and Eu [8]; and also in 1979, in the submillimetre from Cs and Na [19].
1.2.3 Coherence and superradiance

The fundamental requirement for SR emission is the build-up of a large degree of correlation (or coherence) between different particles. We have thus far highlighted the quantum mechanical nature of SR, but it should be clarified that—although QM evolution and entanglement are responsible for establishing phase coherence between atoms—it is the coherence itself which yields the important $N^2$ intensity scaling with population size $N$. This coherence can be understood differently in different contexts: in the small sample, the indiscernibility of particles imposes a permutation symmetry which in turn dictates evolution into the symmetric so-called “Dicke states”. The Dicke states possess high degrees of correlation, thereby generating strong global dipole moments which, in turn, increase the emission intensity [20]. Alternatively, in the large sample limit ($l \gg \lambda$), coherence is mathematically described by phase correlation in the polarisation established via the mediating radiation field. We stress the fundamental importance of coherence at the outset of our discussion of SR applications and theory, as the preservation of coherence guides the development of SR models, numerical approximations, and conditions for the realisation of SR [44].

1.2.4 Superradiant transitions and notation

The SR phenomenon is not defined by the physical nature of the transition (be it electric dipole, magnetic dipole, or other) and may be observed in any ensemble of particles for which each member is capable of radiative emission in isolation. For simplicity, the evolution equations of the following sections (equations (1.45)–(1.47), for example) use the notation of an electric dipole transition (as is often found in the literature [20]), and refer only to electric field and electric polarisation envelope operators. Alternatively, the 21 cm Hydrogen line is, for example, a magnetic dipole transition and would therefore lead to modified expressions involving magnetic field and magnetisation envelope operators [33]. The generality of transition type is implied throughout this thesis, and revisions to the magnetic transition type would be a straightforward exercise having no effect upon the important results to follow.
1.3 Theoretical Models

A full quantum mechanical description of an SR system is usually—apart from the simplest examples—intractably large. The density operator of an $N$-particle system, for example, evolves in a $2^{2N}$-dimensional Hilbert space [3]. It is therefore necessary to articulate a given SR problem in the language and model appropriate to the physical quantities of interest. Additionally, numerical approximation decisions must carefully preserve the important quantum coherence features. In this section we present the dominant pictures of SR, we identify new directions for theoretical approaches to SR, and we discuss important approximation methods and their resulting models’ domains of validity.

1.3.1 Quantum mechanical pictures of superradiance

SR research is primarily conducted within either the Schrödinger or the Heisenberg pictures of quantum mechanics. Although theoretically equivalent, each method offers a unique perspective on the phenomenon and offers its own advantages or disadvantages across differing regimes. We present here a high-level summary of these two dominant research pictures, as well as an introduction to the diagrammatic representation of the Schrödinger picture time-evolution operator, a framework that will be fundamental to Chapter 4 of this thesis.

The Schrödinger picture

Because the Schrödinger picture of quantum mechanics follows the time-evolution of the state of the system, it is naturally useful for describing the physical particle+radiation state of an SR system. With a system prepared in an excited state and the background electromagnetic field in its initial vacuum ground state, Schrödinger’s equation describes the evolution of the system state within the $\mathcal{H}_{\text{par}} \times \mathcal{H}_{\text{rad}}$ product space of the particle and radiation Hilbert spaces (in all mathematical expressions we abbreviate “particle” and “radiation” as “par” and “rad”, respectively) via

$$|\Psi(t = 0)\rangle \rightarrow |\Psi(t)\rangle \in \mathcal{H}_{\text{par}} \times \mathcal{H}_{\text{rad}}$$

(1.32)
through the total particle+radiation Hamiltonian:

\[ i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H}_{\text{tot}} |\Psi(t)\rangle = (\hat{H}_{\text{par}} + \hat{H}_{\text{rad}} + \hat{V}_{\text{int}}) |\Psi(t)\rangle \]

(noting, of course, that $|\Psi(t)\rangle$ is not in general separable; i.e. $|\Psi(t)\rangle \neq |\psi(t)\rangle_{\text{par}} \otimes |\phi(t)\rangle_{\text{rad}}$ for $|\psi(t)\rangle_{\text{par}} \in \mathcal{H}_{\text{par}}$ and $|\phi(t)\rangle_{\text{rad}} \in \mathcal{H}_{\text{rad}}$). The interaction between the particles and the radiation field is described by the $\hat{V}_{\text{int}}$ term.

In principle, equation (1.33) enables us to follow the time-evolution of every variable describing the state of the entire system; however, we are often concerned with only the particle sub-state. To that end, let us first elevate the full system state to a density operator as

\[ \hat{\rho}_\Psi(t) = |\Psi(t)\rangle \langle \Psi(t)| \]

from which the reduced density operator of the particle subspace is computed via a partial trace over the radiation field,

\[ \hat{\rho}_{\text{par}}(t) \equiv \text{Tr}_{\text{rad}} \left[ \hat{\rho}_\Psi(t) \right] \]

After quantizing the electromagnetic field, introducing a dipole interaction between the particle and the field, computing the evolution of the full system density operator, and tracing out the radiation field, numerous authors [3, 20, 15] derive the important SR Schrödinger picture evolution equation, also known as the SR master equation,

\[ \frac{d\hat{\rho}_{\text{at}}}{dt} = \frac{1}{i\hbar} \left[ \sum_j \hat{H}_j, \hat{\rho}_{\text{at}} \right] - \frac{c d^2}{16 \epsilon_0 \pi^3 \hbar} \int_0^{+\infty} d\tau \int_0^{+\infty} dk k^3 \int d\Omega (\epsilon \cdot \epsilon_0)^2 \sum_{ij} e^{ik(r_j-r_i)} \
\times \left\{ \hat{D}_j e^{-i\hat{H}_j \tau / \hbar} \hat{D}_j e^{i\hat{H}_j \tau / \hbar} \hat{\rho}_{\text{at}} - e^{-i\hat{H}_j \tau / \hbar} \hat{D}_j e^{i\hat{H}_j \tau / \hbar} \hat{\rho}_{\text{at}} \hat{D}_j + \text{adjoint of the integral} \right\} \]
quasi-spin operators $\hat{D}_j$ acting in each particle’s energy state sub-space.

It is not our objective to analyse the SR master equation here, but only to demonstrate that such an evolution equation exists and that its projection upon the isolated particle state energy eigenvectors provides insight into important physical processes. As noted in Section 1.3, however, a density operator quantum state description becomes quickly impractical (scaling in dimension as $2^{2N}$ for $N$ atoms). This description is therefore primarily a theoretical tool, where SR features can be related to such total state derived metrics as dipole correlation, coherence and dephasing measures, and exchange symmetries. However, for practical numerical calculations, research involving a large number of particles typically turns to the Heisenberg picture of quantum mechanics.

**The Heisenberg picture**

The Heisenberg picture of quantum mechanics follows the time-evolution of the quantum observables and is therefore useful for describing the limited number of experimentally important properties while suppressing the excessive details of the internal degrees of freedom within the SR system. For example, researchers may be concerned with a net sample polarisation, a total emission intensity, or a total population inversion. Although the evolution of such quantities is necessarily determined by the entire (complicated) system, internal complexities can be reduced by careful approximations which preserve the important underlying QM features.

At a high-level, the Heisenberg model of SR is developed as follows. Macroscopic operators representing polarisation and population inversion are coarsely centered upon each particle, while the quantized radiation field operators retain their continuous parametrisation over the sample volume. Under an appropriate Hamiltonian the Heisenberg equations of motion yield the following self-consistent system of partial differential equations for the radiation field mode operators $\hat{E}^{\pm}$, the polarisation operators $\hat{P}^{\pm}$, and the population inversion operator $\hat{N}$.
Theoretical Models

[3]:

\[ \frac{\partial \hat{N}}{\partial t} = \frac{i}{\hbar} \hat{E}^- \cdot [\hat{P}^+ - \hat{P}^-] + \frac{i}{\hbar} \left[ \hat{P}^+ - \hat{P}^- \right] \cdot \hat{E}^+ \]  
(1.37)

\[ \frac{\partial \hat{P}^+}{\partial t} = i \omega_0 \hat{P}^+ + 2i \frac{d^2}{\hbar} \epsilon_a \left\{ \epsilon_a \cdot \left( \hat{E}^- \hat{N} + \hat{N} \hat{E}^+ \right) \right\} \]  
(1.38)

\[ \frac{\partial^2 \hat{E}^+(r,t)}{\partial t^2} - c^2 \nabla \cdot \nabla \hat{E}^+(r,t) = -\frac{1}{\epsilon_0} \frac{\partial^2 \hat{P}^-(r,t)}{\partial t^2} \]  
(1.39)

where \( \epsilon_a \) is a unit vector in the direction of the dipole orientation.

These equations look formally like the classical Maxwell-Bloch equations for the evolution of an ensemble of dipoles coupled to its radiation field but, importantly, involve in this case the non-commuting operators which give the system its QM character [20]. We will herein refer to equations (1.37)–(1.39) as the Maxwell-Bloch (MB) equations, understanding always that we are working with fully quantum mechanical operators (and not semi-classical quantities).

The quantum mechanical character of equations (1.37)–(1.39) is essential to the development of SR. Consider the sudden inversion of the sample at \( t = 0 \) as established by the initial conditions \( N(t = 0) = N_0 \) and \( P^+(t = 0) = E^+(t = 0) = 0 \). It is straightforward to see, from inspection, that a classical interpretation of the quantities appearing in (1.37)–(1.39) will not respond to the initial population inversion and will remain forever in its initial state.

Actually, a fully quantum mechanical calculation finds [20] that the MB equations may be evolved as describing classical quantities, so long as they are prescribed initial conditions consistent with initial quantum mechanical fluctuations in the polarisation and the quantized vacuum radiation field. Such initial conditions are established through the Schrödinger picture time-evolution of the expectation values of the population inversion, the polarisation, and the electric field. The system very quickly develops a non-zero polarisation, such that the MB equations (1.37)–(1.39) should be understood to describe an ensemble of semi-classical trajectories commencing from a statistical distribution of non-vanishing initial polarisation configurations. This result is discussed in more mathematical detail in Chapter 3.
The diagrammatic method

The Schrödinger picture time-evolution of a quantum mechanical system described by a Hamiltonian of the form
\[ \hat{H} = \hat{H}_0 + \hat{V}_{\text{int}}, \]
where \( \hat{V}_{\text{int}} \) is an interaction perturbation of a well understood Hamiltonian \( \hat{H}_0 \), may be represented by an infinite series of diagrams. We defer the detailed derivation of this diagrammatic representation to Chapter 4, where a Schrödinger picture analysis shows that the evolution of the system may be understood in terms of elementary “stories” involving discrete transition events between eigenstates of the unperturbed \( \hat{H}_0 \).

For example, the diagram of Figure 1.4 tells the story in which two initially excited particles denoted by the combined quantum state \( |ee\rangle \) at \( t = t_0 \) first emit a photon of mode \( k \) and transition into the symmetric state \( |s\rangle \) at \( t = t_1 \), then emit a second photon of mode \( k' \) and transition into the doubly ground state at \( t = t_2 \), and finally reabsorb the photon of mode \( k' \) at time \( t = t_3 \) and exist in the symmetric state \( |s\rangle \) thereafter. The full system time-evolution is determined by an integration over all possible intermediate times \( t_n \) and a sum over all possible stories consistent with the transition events described by the interaction \( \hat{V}_{\text{int}} \) (of which Figure 1.4 is only one example). These concepts will be fully formalised in Chapter 4.

![Diagram](image.png)

Figure 1.4: One example story contributing to the full diagrammatic representation of the time-evolution of an initially doubly-excited state.

The diagrammatic representation is particularly efficient when analysing the time-evolution of the density operator \( \rho(t) \). From a standard result of the literature [3], it turns out that \( \rho(t) \) contains contributions similar to the diagram of Figure 1.4 evolving forward in time, as well as contributions equivalent to a formal time-reversal of such figures; moreover, photon emis-
tion/absorption elements in the forward time contributions always emerge concurrently with absorption/emission elements (respectively) in the reverse time contributions. The diagrammatic representation of $\rho(t)$ is thus comprised of a series of diagrams of the form depicted in Figure 1.5; i.e., possessing a forward time contribution (top, denoted with arrows from left to right), and a reverse time contribution (bottom, denoted with arrows from the right to left), as well as photon lines conjoining the two.

![Figure 1.5](image_url)

Figure 1.5: One example contribution to the full diagrammatic representation of the density operator time-evolution.

Diagrams such as that depicted in Figure 1.5 are in fact precise mathematical statements, as each component of a diagram corresponds to a specific function (a quantum mechanical propagator) and connections between components (the vertices at the discrete temporal times) correspond to convolutions between adjacent functions. Unfortunately, even the simple diagram of Figure 1.5 is associated with a lengthy mathematical calculation involving four convolutions.

The mathematics are considerably simplified when evaluating the particle state reduced density operator $\rho_{\text{par}}(t)$ defined as

$$\rho_{\text{par}}(t) = \text{Tr}_{\text{rad}} \left[ \rho(t) \right].$$

(1.41)

By a standard result of the literature, the trace operation causes the photon emission/absorption and absorption/emission events in the forward- and reverse-time contributions to occur simul-
taneously; i.e., for example, the trace over the radiation field of Figure 1.5 yields $t_1 = t'_1$, which is depicted pictorially as a vertical photon. Additionally, the convolution structure between propagators corresponding to individual segments of the diagram transforms to a convolution between propagators corresponding to separate regions between vertical photon lines. We generalise the vertical photon result to a relativistic Hamiltonian in Chapter 4, where we demonstrate that each vertical photon introduces an overall Lorentz factor multiplier corresponding to the (potentially relativistic) centre of mass velocity of the SR system.

The diagrammatic method becomes especially useful when analysing SR emission from particles travelling at different velocities. Consider, for example, the diagram of Figure 1.6 representing the time-evolution of the $\rho_{gg}(t)$ reduced density operator matrix element. We may colloquially describe the process as follows. First, at $t = t_1$ the forward-time part (top) emits a photon from the first particle which is absorbed in the reverse-time part (bottom) by the second particle. Next, at $t = t_2$ the forward-time part emits a photon from the second particle which is absorbed in the reverse-time part by the first particle. For this particular diagram, each absorption and emission event occurs between distinct particles. We expect that such a process will be strongest when the particles’ velocities are perfectly matched; when their velocities differ, however, a photon emitted/absorbed by one particle will be Doppler offset relative to the natural frequency of the other. The diagrammatic method thus gives us a natural means to visualise the effect of velocity mismatches upon SR.

Figure 1.6: One example contribution to the diagrammatic representation of the particle state reduced density operator time-evolution.

The qualitative description above is made mathematically rigorous in Chapter 4, where the diagrammatic method is in fact used to construct a precise metric for evaluating the depen-
dency of SR upon the velocity coherence of (potentially relativistic) particles. It is found that the careful tracking of particle velocity has the unfortunate side effect of breaking the aforementioned convolution structure, and we develop an algorithm which replaces conventional convolution-based methods. Chapter 4 closes with an application of the model to the simple case of two relativistic particles demonstrating SR.

1.3.2 Approximations

Both the SR master equation and the MB operator evolution equations are fully quantum mechanical descriptions of SR, and both capture the important quantum fluctuation behaviour not described by semi-classical models. There is, however, a distinguishing feature between the two pictures: the MB equations are exact (at least, for the relatively simple systems and physics which they model) while the SR master equation is derived under the Born-Markov approximation (see below). As we have already repeatedly stressed, exact descriptions of SR systems are immensely complex and approximations must be performed to make the MB equations numerically tractable [20].

In fact, the approximation made in the derivation of the SR master equation is rooted in a similar physical assumption as that used to simplify the MB equations [20]. For this reason, we will briefly present first the SR master equation’s so-called Born-Markov approximation, which provides intuition into important physical concepts of particle and field correlations. We will then move on to describe in greater detail the slowly-varying envelope approximation (SVEA) of the MB equations, with which this thesis will work throughout Chapters 2 and 3.

The Born-Markov approximation to the superradiance master equation

In the succinct narrative of the SR master equation derivation presented in Section 1.3.1, it was discussed that the particle observables are isolated from the full system via a partial trace over the radiation field. In fact, that partial trace must be performed on an integral over the entire history of the system, as described by the integro-differential exact solution to the von
Neumann equation [20],

$$\frac{d\hat{\rho}_{\text{par}}}{dt} = -\frac{1}{\hbar^2} \text{Tr}_{\text{rad}} \left\{ \int_0^\infty d\tau \left[ \hat{\mathcal{V}}(t), \left[ \hat{\mathcal{V}}(t - \tau), \hat{\rho}_\Psi(t - \tau) \right] \right] \right\}, \quad (1.42)$$

where $\hat{\rho}_{\text{par}}$ denotes the particle density operator, $\hat{\mathcal{V}}$ the particle-field interaction operator, and $\hat{\rho}_\Psi$ the full system density operator. The tildes denote operators expressed in the interaction picture defined by the unitary transformation $\hat{U} = \exp \left[ -i \left( \hat{H}_{\text{par}} + \hat{H}_{\text{rad}} \right) t / \hbar \right]$. At a high level, the interaction picture unitary transformation allows one to work with deviations of the full time evolution from that time evolution which would be exhibited by an unperturbed system lacking the interaction Hamiltonian term.

The evolution of $\hat{\rho}_{\text{par}}$ at a given moment thus depends upon the state of the system at all earlier times. If we consider the feasibility of a numerical solution to equation (1.42), we notice that its integrand is completely revised over its entire interval upon each advancement of $t$ (i.e., a numerical solution at time $t$ cannot make use of a value of the integral computed at the prior discrete time $t - dt$ and retained in memory). The Born-Markov approximation makes two assumptions to suppress the infinite memory of equation (1.42); namely,

1. As noted in Section 1.2.3, the essential property of SR is the development of correlation between particles; however, the Born approximation neglects the accumulation of correlation between the particles and the field and, in fact, successfully preserves SR features.

2. The Markov approximation makes a similarly careful suppression of non-essential correlations by considering the particle-field correlation time as negligible.

Taken together, these two approximations remove the earlier memory of the particle density operator in the integrand and yield the SR master equation (see equation (1.36)), which relates the density operator's present time derivative to only its present value (albeit weighted against integrands oscillating over all time). Such a differential equation can be solved via typical Runge-Kutta methods.
The slowly-varying envelope approximation to the Maxwell-Bloch equations

As discussed in the opening of this section, no approximation is made during a derivation of the MB equations (1.37)–(1.39). They must therefore possess—in some underlying structure—the temporal memory exposed by the von Neumann equation (1.42) during the SR master equation’s derivation (prior to the Born-Markov approximation). In fact, this memory is present in the second-order propagation effects underlying the third MB equation (1.39). The numerical complexity of the problem therefore exposes itself differently here: including all correlations suppressed by the Born-Markov approximation in the Schrödinger picture would amount, in the Heisenberg picture, to evolving the set of coupled MB equations on an excessively fine temporal step size compared to the characteristic evolution time of the relevant physical properties [20].

Research with the MB equations therefore often makes the Slowly-Varying Envelope Approximation (SVEA), which factors out the fast oscillations of the field to work instead with slowly varying envelope operators \( \hat{E}(z, t) \) and \( \hat{P}(z, t) \) according to the definitions:

\[
\hat{E}^{\pm \text{rad}} = \hat{E}(z, t)e^{\pm i(\omega_0 t - k_0 z)} \epsilon_a \\
\hat{P}^{\pm}(r, t) = \hat{P}(z, t)e^{\pm i(\omega_0 t - k_0 z)} \epsilon_a.
\]

Upon substitution into the MB equations (1.37)–(1.39), it is conventional to eliminate the fast-rotating terms \( \mathcal{E}^+ \mathcal{P}^- \) and \( \mathcal{E}^- \mathcal{P}^+ \) (the so-called “rotating wave approximation”) and to recognise that \( \partial / \partial z \ll \omega_0 / c \) and \( \partial / \partial \tau \ll \omega_0 \) when acting on the envelope functions. We obtain that

\[
\frac{\partial \hat{N}}{\partial t} = \frac{i}{\hbar} \left[ \hat{P}^+ \hat{E}^+ - \hat{E}^- \hat{P}^- \right] \\
\frac{\partial \hat{P}^+}{\partial t} = \frac{2i d^2}{\hbar \epsilon} \hat{E}^- \hat{N} \\
\left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) \hat{E}^+ = \frac{i\omega_0}{2\epsilon_0 c} \hat{P}^-.
\]

If we work in retarded time \( \tau = t - z/c \), equations (1.45) and (1.46) retain their forms while equation (1.47) becomes

\[
\frac{\partial \hat{E}^+}{\partial z} = \frac{i\omega_0}{2\epsilon_0 c} \hat{P}^-.
\]
Equations (1.45)–(1.47) or their retarded time versions are referred to as the MB equations under the SVEA and form the basis of Chapters 2 and 3; we will herein refer to them as simply the MB equations, although we will revise them to account for velocity dependence in Section 1.3.3 and augment them with terms capturing realistic relaxation and decoherence effects in Section 1.4.2.

### 1.3.3 Velocity dependence

This thesis studies the effects of wide inhomogeneous broadening and relativistic particle velocities upon SR emission. In this section we introduce velocity dependence in both the Heisenberg (MB equations) and Schrödinger pictures of SR. Studies of the velocity dependent MB equations exist in the literature for narrow velocity extents [20, 3, 27, 30]; that is, for velocity bandwidths on the order of the SR transient bandwidth. This thesis, on the other hand, studies SR emission from a velocity distribution of total bandwidth many orders of magnitude wider than the SR transient bandwidth, as is typical of an astrophysical environment. The recovery of the velocity dependent MB equations below is a standard result that serves as the foundation for the wide velocity bandwidth work of Chapters 2 and 3.

The MB equations are generalised below to a velocity distribution by a straightforward analysis of the effect of a Galilean frame transformation on equations (1.37)–(1.39). The MB equations are not, however, capable of modelling the initial quantum mechanical fluctuations of SR nor the emission of SR radiation from a highly relativistic source. We will demonstrate in Chapter 4 that the Schrödinger picture diagrammatic method is appropriate to such tasks, but the complicated mathematics of that later work are outside of the scope of this introduction. Instead, we introduce here the general role of velocity dependence in the Schrödinger picture of quantum mechanics and discuss how these initial results shall influence our later work in Chapter 4.

**Velocity dependence in the Maxwell-Bloch equations**

Suppose that a single particle is modelled by equations (1.37)–(1.39) in its rest frame, which travels with one-dimensional velocity $u$ relative to some observer frame. Consider a trans-
formation to the observer frame. The operator acting on the electric field on the left side of equation (1.39) is a covariant expression; i.e., even for a fully relativistic transformation it is Lorentz invariant, and it is therefore also invariant to first order in $u/c$ under a non-relativistic Galilean transformation. Consider a Galilean transformation to the observer frame coordinates defined as

$$z_{\text{obs}} = z + ut \quad \text{and} \quad t_{\text{obs}} = t. \quad (1.49)$$

We now define our slowly-varying polarisation envelope in terms of observer coordinates; i.e.,

$$\hat{P}^\pm(\mathbf{r}, t) = \hat{P}(z, t_{\text{obs}}) e^{\pm i(\omega_0 t_{\text{obs}} - k_0 z)} e_a, \quad (1.50)$$

such that the left side of equation (1.38) evaluates to

$$\frac{\partial \hat{P}^+}{\partial t} = \left( \frac{\partial}{\partial t_{\text{obs}}} + u \frac{\partial}{\partial z} \right) \left[ \hat{P}(z, t_{\text{obs}}) e^{\pm i(\omega_0 t_{\text{obs}} - k_0 z)} e_a \right] \quad (1.51)$$

$$= \left[ \left( \frac{\partial}{\partial t_{\text{obs}}} + u \frac{\partial}{\partial z} \right) \pm i \omega_0 \mp uk_0 \right] \hat{P}(z, t_{\text{obs}}) e^{\pm i(\omega_0 t_{\text{obs}} - k_0 z)} e_a. \quad (1.52)$$

Noting that $k_0 = \omega_0/c$ and recognising again that $\partial/\partial z \ll \omega_0/c$, we have that

$$\frac{\partial \hat{P}^+}{\partial t} \approx \left\{ \frac{\partial}{\partial t_{\text{obs}}} \pm i \omega_0 \left( 1 - \frac{u}{c} \right) \right\} \hat{P}(z, t_{\text{obs}}) e^{\pm i(\omega_0 t_{\text{obs}} - k_0 z)} e_a, \quad (1.53)$$

such that its envelope-factored equivalent equation (1.46) reads in the observer frame,

$$\frac{\partial \hat{P}^+}{\partial t_{\text{obs}}} = i \omega_0 \frac{u}{c} \hat{P}^+ + \frac{2id^2}{\hbar} \hat{E}^\perp \hat{N}. \quad (1.54)$$

The first term on the right hand side of equation (1.54) describes the familiar non-relativistic Doppler shift.

Consider now a distribution of particles travelling at various velocities as denoted by velocity channel subscripts $v$ in the population inversion and polarisation; i.e.,

$$\hat{N} \to \hat{N}_v \quad \text{and} \quad \hat{P}^+ \to \hat{P}^+_v. \quad (1.55)$$

We expect the inversion or polarisation of a given velocity channel to be affected only by its
own polarisation’s or inversion’s interaction with the field, such that equations (1.45) and (1.54) become simply

\[ \frac{\partial \hat{N}_v}{\partial \tau} = \frac{i}{\hbar} \left[ \hat{P}_v \hat{E}^+ - \hat{E}^- \hat{P}_v \right] \]

\[ \frac{\partial \hat{P}_v^+}{\partial \tau} = i \omega_0 v \hat{P}_v^+ + \frac{2i d^2}{\hbar} \hat{E}^- \hat{N}_v, \]

where we have dropped the “obs” subscript and now established the observer frame as our preferred frame possessing retarded coordinates \((\tau, z)\).

The electric field is generated by all polarisation velocity channels. If \( F_v \) denotes the velocity distribution, then equation (1.48) becomes

\[ \frac{\partial \hat{E}^+}{\partial z} = \frac{i \omega_0}{2 \epsilon_0 c} \int dv F_v \hat{P}_v^- \]

Equations (1.56)–(1.58) are referred to as the velocity dependent MB equations throughout this thesis and they form the foundation of Chapters 2 and 3.

**Velocity dependence in the Schrödinger picture of quantum mechanics**

The most simple system able to demonstrate SR is a two-particle, two-level sample. The Schrödinger picture of quantum mechanics completely describes the quantum evolution of such a sample and its accompanying quantized radiation field and is, therefore, best suited to our analysis of relativistic two-particle SR in Chapter 4. We approach the problem in that chapter via the diagrammatic method, and defer deriving the detailed model of relativistic velocity-dependent SR to then; however, it is helpful at this point to make some introductory statements regarding the manifestation of velocity dependence in the Schrödinger picture of quantum mechanics quite generally.

Simple coordinate transformations—such as exploited in the previous section—are not appropriate to the Schrödinger picture. The issue lies in the fact that, although the wavefunction of a system is a complex-valued function of the space and time coordinates of a given reference frame, it is not strictly speaking a field on the coordinate manifold; rather, it is a mathematical tool that results from canonical quantization of a system in that particular reference frame. To
clarify this point, consider the position-basis wavefunction of a one-dimensional free particle in a momentum eigenstate $|p\rangle$:

$$\langle x, t | \Psi \rangle = \Psi(x, t) = e^{i\left(px-tp^2/(2m)\right)/\hbar}.$$  \hspace{1cm} (1.59)

Suppose this wavefunction were a complex-valued scalar field tied to space and time. If this were the case, then under a Galilean transformation to a new frame moving with velocity $u$ relative to the original frame; i.e.,

$$(x', t') = (x - ut, t) \quad \text{or} \quad (x, t) = (x' + ut', t')$$  \hspace{1cm} (1.60)

the wavefunction would transform to

$$\Psi'(x', t') = e^{i\left[p(x'+ut')-tp^2/(2m)\right]/\hbar}$$  \hspace{1cm} (1.61)

which certainly cannot be manipulated into the wavefunction of a free particle of momentum $p - mu$ (as it should).

The wavefunction is not a physical, scalar field tied to space and time. It does not transform consistently with a change in reference frame coordinates; rather, it is a mathematical tool for making calculations in a given frame which happens to take the form of a complex-valued function that depends upon the spatial and temporal coordinates of that particular frame.

If coordinate transformation alone cannot describe the wavefunction of a particle in a new frame, then how does it transform? The answer comes from a more detailed study of the canonical quantization process. The wavefunction is transformed between two frames by the following steps.

1. Determine the effect of frame transformation upon the classical Hamiltonian and its canonically conjugate variables, and

2. Relate eigenstates of the canonically quantized operators between the two frames accordingly.

For example, the earlier Galilean transformation of a free particle causes the canonical mo-
momentum variable $p$ of the classical Hamiltonian to map to the canonical momentum variable $p - mu$ in the new frame; thus, we should map the state

$$|p\rangle \rightarrow |p'\rangle = |p - mu\rangle$$ (1.62)

which has wavefunction

$$\langle x', t' | p - mu \rangle = e^{i \left[ (p - mu) x' - t' (p - mu)^2 / (2m) \right] / \hbar}.$$

(1.63)

The above discussion implies that we should not introduce velocity dependence to an SR sample by a simple Galilean transformation of Schrödinger’s equation; rather, we introduce velocity dependence in the quantum mechanical state description. This concept is further complicated by the role of locality in SR. Small sample SR, for example, is defined by particles localised to a volume of characteristic length much smaller than the wavelength of emission. Strictly speaking, such a state cannot be an eigenstate of the particles’ centre of mass momenta; however, the scales of interest to us are not sufficient to cause concern. Our (and the literature’s) applications of SR typically involve emission wavelengths many orders of magnitude longer than the position uncertainty associated with the required precision in the particles’ centre of mass momenta.\(^1\) It is therefore conventional to treat the centre of mass positions classically.

The problem becomes nuanced, however, if we wish to analyse the effect of velocity offsets between particles. In such a case the momentum of each particle plays a central role and thus cannot be brushed under the rug. A fully formal approach to the problem would describe each particle’s centre of mass coordinate quantum state as a superposition of momentum eigenstates, centred upon the particle’s assigned momentum and of momentum spread consistent with the particle’s position localisation within one emission wavelength. This approach becomes quickly complex as it involves the time-evolution of a superposition of many states, each of which couples to many other states through the interaction potential.

In Chapter 4 we take the approach of treating the centre of mass positions and momenta of the particles classically (i.e., simultaneously well-defined), but tracking the momenta of states

\(^1\)That is, the particles must have such sufficiently well-defined and similar momenta that their photons are not Doppler offset from one another, thereby enabling the particles to evolve into an entangled state.
1.4 Establishing Superradiance

A system must possess certain physical characteristics and mechanisms in order to exhibit SR. It must be able to preserve velocity coherence across its radiating members; it must possess a pumping mechanism capable of generating a population inversion; and the onset of an SR pulse must be identifiable with a triggering event. In this section we detail these requirements as well as their implications for the work of this thesis.

Before these requirements may be discussed, however, the defining distinction between superradiance and the maser action must be clarified. Because these two phenomena share similar requirements, and because transition to the SR mode is described as a departure from the masing mode, it is critical that this distinction be properly understood at the outset.

1.4.1 Superradiance versus the maser action

Superradiance is, of course, a cooperative emission process whose radiative features cannot be explained by the sum of its individual parts, independently; however, it should be distinguished from another (related) collective process: that of amplification by stimulated emission. We compare SR to microwave amplified stimulated emission of radiation without a resonant cavity; i.e., the mirror-less maser, or amplification by spontaneous emission (ASE).

Indeed, SR systems and masers possess a number of similarities:

1. Both require a pumping source and a population inversion,

2. Both may be modelled with the MBEs under the SVEA,

3. Both require a strong degree of velocity coherence, and

4. Both are emission phenomena whose collective intensities exceed the sums of their parts.

Despite the fact that both SR and masing are described by the MBEs and their polarisation and inversion density operators (cf. item #2), SR and masing occur in very separate regions of
the MBEs’ parameter space [12]. In a maser, the decay rates of all members involved is much shorter than the evolution time scale of the polarisation and population inversion envelopes. The total radiative power of the system is therefore enhanced as it operates in the steady-state regime [16], wherein the pumped inverted population is continually decaying but also amplifying emission at some effective gain. Acting as a classical statistical ensemble of uncorrelated radiators, the intensity of a saturated maser thus scales linearly with the population size \( N \) of the system, albeit more intensely than simply a multiple \( N \) of the individual spontaneous decay rate.

Conversely, in a superradiant system—considered as another limiting case of the MBEs—the polarisation and inversion density operators evolve much faster than the time scale of incoherent decay. The SR system may therefore establish a large degree of coherence across its members, such that its polarisation envelope scales with the population size \( N \), providing a positive feedback mechanism to enhance each member’s decay. The SR system is therefore not characterized by a radiative steady-state, but instead by a swift cascade from the fully inverted state to the ground state: an emission process enhanced by the system’s coherent polarisation and thus scaling in intensity as \( N^2 \) [24].

### 1.4.2 Conditions for superradiance

Whether or not a system can exhibit superradiance is determined by its ability to establish coherence across its radiating members. The build-up to coherence occurs over a characteristic SR time delay \( \tau_D \). In the small sample Dicke model \((l \ll \lambda)\), coherence increases as the system cascades from the initial fully excited \(|ee...e\rangle\) state (possessing zero atom-atom correlation) through the symmetric Dicke states to the maximally coherent half de-excited symmetric state \( \sum_{\text{all exchange permutations}} |e_1 e_2 ... e_{N/2} g_{N/2+1} g_{N/2+2} ... g_N \rangle \). In this case, coherence is established over \( \tau_D \approx T_R \ln (N) \), where the characteristic duration of the SR pulse is \( T_R = 1/(\Gamma N) \) for \( \Gamma \) the characteristic spontaneous emission rate of a single atom. In the large sample model, the characteristic time depends upon the geometry. For a ‘pencil-shaped’ extended system with
total population $N$, length $L$, and number density $n$, the time delay is approximately [3, 35]

$$\tau_D \simeq \frac{T_R}{4} \left| \ln \left( \frac{1}{\pi \sqrt{N}} \right) \right|^2,$$

(1.64)

where

$$T_R \simeq \frac{8\pi}{3nL^2}\Gamma^{-1}$$

(1.65)

is the characteristic duration of the SR pulse, which may vary over many orders of magnitude.

In the present work we are interested in astrophysical processes for which $T_R$ is on the order of $10^8 \text{ s}$ as in, for example, those transient processes modelled in Chapter 2 (initial population inversion $n_0 = 1.5 \times 10^{-12} \text{ cm}^{-3}$, length $L = 2 \times 10^{15} \text{ cm}$, wavelength $\lambda = 4.48 \text{ cm}$, and spontaneous emission rate $\Gamma = 1.72 \times 10^{-9} \text{ s}^{-1}$) for which $T_R = 8.1 \times 10^4 \text{ s}$. These timescales are many orders of magnitude longer than typical laboratory demonstrations of SR; for example, the room-temperature HF gas observations of SR by Skribanowitz, Herman, MacGillivray, and Feld [38] possessed $T_R \sim 100 \text{ ns}$.

There are, however, numerous effects which can destroy coherence. In order for a system to exhibit SR features, these disturbances must evolve over time scales larger than $\tau_D$. We refer to such processes in opposition to coherence as dephasing and relaxation processes. Dephasing processes may be present in the ideal models themselves—as in, for example, the dipole-dipole Van der Waals dephasing described below—or may be caused by realistic physical phenomena which must be introduced ad-hoc into the models, such as thermal effects (destroying velocity coherence), collisional broadening, or inhomogeneous environments. If $T_1 = T_2 \equiv T'$ for relaxation and dephasing timescales $T_1$ and $T_2$, then a necessary condition for establishing SR is that $\tau_D < T'$ [35]. In the case of our pencil sample—cf. equations (1.64) and (1.65)—this establishes a critical column density ($nL_{\text{crit}}$) given by [27, 12]

$$(nL)_{\text{crit}} \approx \frac{2\pi}{3\lambda^2} \Gamma^{-1} \left| \ln \left( \frac{1}{\pi \sqrt{N}} \right) \right|^2.$$

(1.66)

**Phenomenological relaxation and dephasing terms**

The MBEs are phenomenologically augmented with terms capturing realistic dephasing and relaxation effects in the ISM, as well as the presence of a continuous pumping source [31].
Adapting the work of Mandel [28] to the electric dipole example, we introduce constant inversion/polarisation pumping sources $\Lambda^{(N/P)}$, as well as a population inversion decay time $T_1$ and a natural dephasing time $T_2$. Our first two equations (1.45) and (1.46) become:

\[
\frac{\partial \hat{N}}{\partial t} = \frac{i}{\hbar} [\hat{P}^+ \hat{E}^+ - \hat{E}^- \hat{P}^-] - \frac{1}{T_1} (\hat{N} - N_{eq}) + \Lambda^{(N)} \tag{1.67}
\]

\[
\frac{\partial \hat{P}^+}{\partial t} = \frac{2i d^2}{\hbar} \hat{E}^- \hat{N} - \frac{1}{T_2} \hat{P}^+ + \Lambda^{(P)}, \tag{1.68}
\]

where $N_{eq}$ is the equilibrium inversion density in the absence of the field interaction.

**A dephasing example: the Van der Waals interaction** As a physical example of a dephasing process described by the $T_2$ term of equation (1.68), consider the effect of inhomogeneous Van der Waals coupling between atoms in an otherwise exchange-symmetric three-atom system (a reduced version of the four-atom system of section 1.2.1). Such inhomogeneity could be introduced, for example, by unequal spacing between atomic sites, in which the $r^{-3}$ interaction would vary across differing pairs.

![Asymmetrical geometry inducing Van der Waals dephasing](image)

**Figure 1.7:** Asymmetrical geometry inducing Van der Waals dephasing. Adapted from Gross and Haroche [20].

We follow in this example the theory of Gross and Haroche [20]. Consider an isosceles triangle geometry in which the Van der Waals interaction between the 1st and 2nd atoms differs
1.4. Establishing Superradiance

in strength from that between 1 and 3, and between 2 and 3. The geometry is depicted in Figure
1.7. The geometrical asymmetry introduces off-diagonal perturbative coupling operators which
are not invariant by permutation of atoms 1 and 3 or 2 and 3 [20]:

$$\hat{\Omega} = \Omega_{12} \left( |eeg\rangle\langle ege| + |ege\rangle\langle eeg| \right).$$

(1.69)

The state $|S_2\rangle = (|eeg\rangle + |ege\rangle + |gee\rangle)/\sqrt{3}$ is a member of the \{\|eeg\>, |ege\>, |gee\>\} subspace. In the absence of the perturbation we would compute its decay rate via the expectation value of
the product $\hat{D}^+ \hat{D}^-$ (cf. Section 1.2.1); however, we suppose that the introduction of $\hat{\Omega}$ induces
a transformation of $|S_2\rangle$ faster than the characteristic decay time, thus evolving as:

$$|S_2(\tau)\rangle = \frac{1}{\sqrt{3}} \left( e^{-i\Omega_{12}\tau} |eeg\rangle + e^{-i\Omega_{12}\tau} |ege\rangle + |gee\rangle \right).$$

(1.70)

The non-vanishing terms of the expectation value $\langle \hat{D}^+ \hat{D}^- \rangle_{S_2}$ are:

$$\frac{e^{i\Omega_{12}\tau}}{\sqrt{3}} \langle ege| \hat{D}_1^+ \hat{D}_2^+ \frac{1}{\sqrt{3}} |gee\rangle = \frac{e^{i\Omega_{12}\tau}}{3},$$

$$\frac{e^{i\Omega_{12}\tau}}{\sqrt{3}} \langle eeg| \hat{D}_2^+ \hat{D}_3^+ \frac{e^{-i\Omega_{12}\tau}}{\sqrt{3}} |ege\rangle = \frac{1}{3}, \text{ and}$$

$$\frac{1}{\sqrt{3}} \langle gee| \hat{D}_3^+ \hat{D}_1^+ \frac{e^{-i\Omega_{12}\tau}}{\sqrt{3}} |eeg\rangle = \frac{e^{-i\Omega_{12}\tau}}{3}.$$

(1.71)

The correlation factors between atoms—represented by the right-hand terms of equations 1.71—are therefore dephased and will no longer sum coherently. As the decay rate of the level population is proportional to the expectation value $\langle \hat{D}^+ \hat{D}^- \rangle$, the coherent emission enhancement may be destroyed if $\Omega_{12}$ is sufficiently large.

1.4.3 Inversion and triggering

In order to initiate the superradiant emission cascade, a system must possess a population
inversion and be made to exceed the critical column density of equation (1.66). The inverted
state is not one of thermodynamic equilibrium, and therefore demands an external pumping
source. Lasers typically serve this purpose in the laboratory (as in, for example, the 1973
experiment by Skribanowitz, Herman, MacGillivray, and Feld [38] described in Section 1.2.2),
but to propose SR in regions of the ISM, research must identify an astronomical-scale pumping
source as well as a transient mechanism by which a system may cross the critical column
density threshold.

**Feasibility of inversion and triggering in the ISM—CH$_3$OH/H$_2$O line ordering**

We highlight the work of Rajabi et al. in analysing methanol and water spectral line flares from
G107.298+5.639 [35], which demonstrates the feasibility of identifying astrophysical population
inversions and triggering mechanisms. G107.298+5.639 is an intermediate-mass young
stellar object demonstrating methanol 6.7-GHz and water 22-GHz maser activity. The need
for a pumping source and population inversion led Rajabi et al. to search for signs of SR in
published data from this maser-harbouring region of the ISM [43, 40, 39]. There, they found
periodic intensity flares from both transition lines occurring, curiously, with consistent order-
ing. Such ordering is difficult to capture with a typical maser model; however, it is a natural
feature of the superradiant mode acting upon two transitions: the time delay of (for example)
equation (1.64) is set by the particular transition, and therefore establishes an ordering to the
intensity flares of different molecular species experiencing a common triggering event. Al-
though the common triggering event is not specifically identified in their work, Rajabi et al.
highlight an example of a periodically varying infrared luminosity in a similar young proto-
stellar system. Such infrared intensity is identified as a candidate pumping source and—due
to its periodicity—a candidate mechanism for periodically elevating the population inversion
above the critical column density threshold.

**Turbulence in the interstellar medium**

There exist other potential mechanisms for crossing the critical column density threshold of
equation (1.66) than simply an increase in pump rate. The dephasing time-scale $T_2$ could
fluctuate during the natural hydrodynamic evolution of the gas, and thereby change the critical
threshold and initiate SR.

The existence of significant turbulence in the ISM was first proposed by von Weizsäcker in
1951 [41] and identified through velocity dispersions thereafter by Hoerner [23]. Turbulence is
very difficult to detect as observations are, of course, integrations along lines of sight through
gas density distributions; therefore, sophisticated statistical inversion methods analysing tem-
poral and spatial variability of emissions must be applied. Turbulence models were not anal-
ysed in more detail until the recent few decades, when researchers began to study the power-law
of the spatial power spectrum of the HI emission [17] and absorption [10], of the CO and $^{13}$CO
line emission [22], and of the dust thermal emission, finding power law indices consistent with
Kolmogorov turbulence [11, 41, 23].

A local sub-sample of a wide velocity distribution, possessing a population inversion den-
sity $nL < (nL)_{\text{crit}}$, could transition to a regime capable of SR emission if turbulence in its hy-
drodynamic environment were to cause its column density to exceed $(nL)_{\text{crit}}$; alternatively, tur-
bulence could modify the velocity distribution statistics and the spectral interactions between
SR transient processes generated from neighbouring bandwidths of a widely inhomogeneously
broadened system. In this thesis we will investigate the effect of velocity distribution noise
upon SR temporal structure and determine statistical criteria for a highly saturated turbulent
system to generate transient SR features.

### 1.5 Features of Superradiance

Having incidentally noted some of superradiance’s important features during our introduction
to the phenomenon, we now summarize, expand upon, and quantify those features; specifically
those which are most important to astrophysical processes.

#### 1.5.1 Intensity

The $N^2$ intensity scaling with population size $N$ of a superradiant sample has been well de-
scribed in a qualitative sense, but should now be quantified and augmented with realistic ef-

facts. Robert Dicke did, in fact, extend his analysis to a large population within a small sample
($l \ll \lambda$) to predict a maximum intensity [9, 20]

$$I_{\text{SR}} = \frac{N^2 \hbar \omega \Gamma}{2} \frac{1}{\cosh^2 \left[ N \Gamma \left( \tau - \tau_D \right)/2 \right]}.$$  (1.72)
where $\omega$ is the angular frequency of the transition, $\Gamma$ the single-atom/molecule spontaneous emission rate, and $\tau_D = \ln(N)/(\Gamma N)$ the characteristic superradiant delay time.

### 1.5.2 Ringing

In a large scale system, the intensity of equation (1.72) is in fact modulated by a *ringing* propagation effect. In the cylindrical large-sample model, for example, multiple consecutive superradiant bursts with exponentially decaying peak intensities are generated through photon re-absorption and re-emission along the longitudinal discharge length [31]. A numerical solution of the Maxwell-Bloch equations captures this effect as depicted in Figure 1.8 below.

![Normalized Intensity vs. Time](image)

Figure 1.8: Ringing effect in a large-sample SR system. Reproduced from Rajabi [31] with permission.

### 1.5.3 Delays, the Arecchi-Courtens condition, and swept coherence

The onset of superradiance after an initial inversion pulse is delayed by a characteristic time $\tau_D$ associated with the buildup to coherence across the sample. The delay time $\tau_D$ is determined by the atoms’ or molecules’ natural spontaneous decay rate $\Gamma$, by the participating population size $N$, and by the geometry of the sample. In the ideal small sample ($l \ll \lambda$), for example, $\tau_D = \ln(N)/(\Gamma N)$ [9, 20].
The superradiant pulse possesses a characteristic duration width $T_R$ which—in conjunction with $\tau_D$—is critical to establishing SR against competing dephasing and relaxation effects. The characteristic temporal width decreases with sample size; in the ideal small sample, it simply scales inversely with $N$: $T_{R, \text{small}} = \tau_{sp}/N$, where $\tau_{sp}$ is the spontaneous emission time of a single atom or molecule. As a more realistic example, in the extended cylindrical model of length $L$, column density $n$, and emission wavelength $\lambda$, $T_R$ is given by equation (1.65) as stated in Section 1.4.2.

The two characteristic times $\tau_D$ and $T_R$ compete against the relaxation and dephasing times $T_1$ and $T_2$ of equations (1.67) and (1.68) which oppose the establishment of SR. Additionally, a third characteristic time should be considered when describing superradiance across a large extended sample: the speed-of-light time of propagation $t_c = L/c$ across a sample possessing length $L$ must be less than the characteristic width $T_R$, or else separate regions of the sample will emit photons independently before global coherence may be established through the mediating radiation field. The condition $t_c < T_R$ is known as the Arecchi-Courtens condition [2], and a sample violating it should be modeled as multiple independent superradiant samples.

The Arecchi-Courtens condition applies to a uniform inversion; however, if the sample is inverted by a swept excitation [20, 4, 29], the retarded time of excitation can match the retarded time of radiation field coupling between atoms, effectively rendering the inversion globally simultaneous in the retarded time coordinate along the longitudinal direction of discharge. Such SR systems can, therefore, theoretically possess infinite longitudinal coherence, but would still be limited in the number of participating transverse atoms by the constraint that coherence can only be established within a sub-volume of Fresnel number on the order of (or less than) unity.

1.6 Outline of the Thesis

This thesis advances the theory of velocity dependent SR to widely Doppler broadened and relativistic media. A widely Doppler broadened velocity distribution is one for which the Doppler shift across its total velocity extent is much greater than the bandwidth of the transient emission process. As discussed in detail in the introductions of Chapters 2–4, these two regimes are of particular interest to astrophysics; however, the models, algorithms, and results derived
in this thesis are relevant to any systems lying in these regimes.

Conventional numerical simulation of a widely Doppler broadened distribution in the time domain scales in numerical complexity as $O(n^2)$ with the number of velocity channels $n$. Chapter 2 therefore begins with the derivation of a novel $O(n)$-complex algorithm for simulating the velocity dependent MB equations. The algorithm is a generalisation of the methods of Megen-gozi and Lamb [30] to the transient SR regime. Chapter 2 also serves as a theoretical and numerical study of the relationship between the quasi-steady state maser regime (as originally modelled in [30]) and the non-linear transient SR regime.

The thesis continues in Chapter 3 to compute results in the widely Doppler broadened regime. We investigate how SR transient structure is modified as distributions are constructed of successively wider total velocity extents, and in so doing demonstrate the emergence of extremely wide bandwidth polarisation phase correlation. We study the response of a widely Doppler broadened system to transverse and swept inversion mechanisms, and we demonstrate that the Arecchi-Courtens condition limits polarisation phase correlation in the transverse case. We also investigate the effect of stochastic features within a wide velocity distribution, and we develop for this purpose a second novel $O(n)$-complex algorithm which is particularly efficient for simulating such systems. We argue that certain statistical characteristics can limit the formation of wide polarisation phase correlation for even a swept inversion process, and we demonstrate numerically that such statistical arrangements retain temporal SR structure in the widely Doppler broadened limit.

Finally, we close in Chapter 4 with a study of SR from a highly relativistic source. We construct a novel Schrödinger picture model of a small number of particles with which we investigate velocity coherence requirements for observing SR decay rate enhancement in the observer frame. Our system is admittedly quite simplistic, but our model and methods are foundational for further research into the observation of SR from a highly relativistic source.

1.7 Closing Remarks

Superradiance is a remarkable expression of the beautiful features of foundational QM.

It is, on the one hand, quite difficult to find any physical phenomenon which is not at least
an *indirect* validation of QM. From the structure of the periodic table, to the stimulated emission of a laser, to the electrical conductivity of metals, much of experimental physics—and, moreover, much of our daily experiences—are at some level verifications of Schrödinger’s equation. However, the truly fascinating and counter-intuitive aspects of QM are often assumed to be relegated to complex laboratory or thought experiments. The violation of Bell’s inequalities, the strange correlation of hugely separated entangled atoms, or adherence to the Heisenberg inequality are QM features not typically witnessed at the macroscopic level.

Astrophysical SR is a very direct expression of QM entanglement on a much larger scale. As demonstrated in the opening description of the small sample limit [9], the enhanced intensity of SR emission is a consequence of a large degree of entanglement in the system. The quadratic intensity scaling with population size is, therefore, direct macroscopic evidence of a classically counter-intuitive quantum state. An SR ensemble truly acts as a single quantum unit, and SR emission is evidence of the irreducible state of the system. In an SR process it is necessary to recognise the entire system response as a cooperative dynamic radiation process which cannot be reconciled with concepts of rate-balancing equations or of atoms or molecules radiating or absorbing in separate, discrete events.

In light of its intrinsically non-classical behaviour, the promising application of SR to astrophysical data by Rajabi, Houde, et al. [24, 25, 26, 32, 33, 34, 35, 36] is extremely exciting. Moreover, the novelty of the SR picture sparks one’s imagination and suggests a plethora of new research directions. In addition to the concrete research steps outlined in the previous section, this thesis offers unique insights into familiar processes (e.g., a departure from the conventional radiative transfer picture in maser hosting regions), discusses foundational physical concepts (e.g., the relationship between a quantum mechanical measurement process and a relativistic frame transformation), and employs a variety of mathematical and numerical methods (from partial differential equations to diagrammatic techniques). The richness and novelty of the SR picture offers a gestalt shift in perspective and presents a multitude of new and exciting thought avenues. We believe that the application of SR to astrophysics is in a very promising infancy.
Chapter 1 Bibliography


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Chapter 2

A Fourier Domain Algorithm for the Accelerated Simulation of Widely Doppler Broadened Superradiance

A version of this chapter has been published in the Monthly Notices of the Royal Astronomical Society as


2.1 Introduction

The theory of quantum electrodynamics (QED) describes the emission of a photon from an excited atom or molecule\(^1\) through its interaction with the quantized radiation field. More generally, a large collection of molecules interacting with their common radiation field can produce complex spontaneous radiative phenomena. In the well-known process of microwave amplification by stimulated emission of radiation, for example, the presence of a photon in the radiation field as generated from one molecule enhances the emission rate from adjacent molecules. In an ideal system, in which stimulated emission occurs between molecules of

\(^{1}\)For brevity, we shall herein refer to “molecules” only; however, all discussion and results to follow apply equally well to either atoms or molecules.
similar velocities and without dephasing interactions, stimulated photons possess frequencies very near those of the stimulating photons. Such processes lead to a high degree of coherence within the radiation field. Conversely, a realistic amplification by stimulated emission process is usually only weakly coherent, in that broad velocity distributions as well as collisional and other dephasing processes together determine the quasi-steady state radiation field, population inversion level, and polarisation profiles [26]. In astrophysics, microwave amplification by stimulated emission of radiation (maser) processes have been observed in regions containing sufficiently velocity-coherent gases of molecules [9, 15].

There also exists a cooperative coherent spontaneous emission process related to (but distinct from) amplification by stimulated emission, known as R. H. Dicke’s superradiance (SR) and first described in Dicke [6]. In the SR process a sample of excited molecules characterised by slow relaxation and dephasing time scales evolves, through interaction with the common radiation field, into a quantum state possessing a high degree of entanglement between the constituent molecules’ individual excitation states. These entangled states couple strongly to the radiation field and produce enhanced emission rates as compared to those rates predicted for independently radiating molecules. Recent work [28, 29, 17, 18, 30, 26, 25] has demonstrated that SR is a strong candidate for describing transient astrophysical processes demonstrating sharp flux rises in maser-harbouring regions, which possess conditions similar to those prerequisite to the development of SR. In Rajabi and Houde [29], for example, it was argued that the relative delay (∼20 days) and differing duration (∼7 days and ∼20 days) of similarly periodic (∼34.4 days) methanol 6.7-GHz and water 22-GHz flares (respectively), observed in the intermediate-mass young stellar object G107.298+5.639 [34], is difficult to explain by the quasi-steady state dynamics of a maser model with periodic pumping source. Conversely, an SR numerical model naturally reproduced the distinct temporal timescales between both flares, while being triggered by a single common periodic population inversion source.

The recent applications of SR to astrophysics [28, 29, 17, 18, 30, 26, 25] have been restricted to slices of velocity coherent (on resonance) populations, where the relevant Maxwell-Bloch (MB) equations describe a gas of molecules travelling at only a single shared velocity. It is the objective of the present work to extend modelling of SR to realistic velocity distributions. Such an extension is ultimately motivated by our desire to eventually study coherence in
observational data.

The emergence of coherence in the transfer of radiation through stimulated emission processes has a long history in the theory of astrophysical maser propagation [9, 15]. It is commonly assumed that only processes of absorption and stimulated emission contribute to the propagation of maser radiation [13, 9, 15]. Such modelling is justified in the quasi-steady state limit [26], under the assumptions of (i) incoherent radiation, where the radiation field frequency modes fulfill Gaussian statistics and are uncorrelated [20], and (ii) the steady state of the molecular populations [20, 13, 15]. Proper modelling of the amplification of radiation in a population inverted medium has revealed that these assumptions are warranted for unsaturated masers [7, 8], where the dephasing timescale is much shorter than the timescale of stimulated emission processes.

However, already at low degrees of maser saturation it has been shown that coherence emerges in both the radiation field and between the populations [23, 7, 8], rendering the quasi-steady state limit invalid [15]. Attempts have been made to partially account for the coherence properties of radiation [12, 11], but their utility is limited to low degrees of saturation. To properly account for the coherence properties in the maser process, one needs to solve the full MB equations across a velocity distribution [15, 23, 31], as was done in Menegozzi and Lamb [23] and previously in quantum optics studies of SR [31, 21, 3]. In light of the efficiency and effectiveness of the methods of Menegozzi and Lamb [23] in describing the emergence of coherence in the maser regime of the MB equations, we investigate in this paper the application of their methods to the high field transient SR regime.

The paper is organised as follows. In Section 2.2 we introduce methods for simulating collective and cooperative emission processes across velocity distributions and we derive the time domain envelope factorisation of the MB equations for modelling a one-dimensional quasi-steady state maser process and a transient SR process. After discussing the computational complexity of the time domain representation, the Fourier space representation of Menegozzi and Lamb [23] is introduced (herein referred to as the ML representation and its solution method as the ML algorithm). We take the opportunity at this point to generalise an approximation method of Menegozzi and Lamb [23] to the so-called local mode interaction (LMI) approximation, which offers higher degrees of fidelity where regions of higher field strength demand
and which translates in a straightforward manner to the transient work of later sections. The ML method’s advantageous computational scaling complexity and drawbacks compared to the time domain method are discussed.

In Section 2.3 we describe the fundamental distinctions between maser and SR processes which we expect to complicate the application of the ML algorithm to SR. We discuss the validity of a periodic temporal Fourier series representation of the spatial propagation of spectral noise in a quasi-steady state maser process (as conducted by Menegozzi and Lamb [23]) and we discuss the challenges faced by such a representation when simulating a transient SR process.

In Section 2.4 we investigate the performance of the ML algorithm in the transition from the unsaturated maser to the saturated maser quasi-steady states, as well as in the high field transient SR regime. We simulate a one-dimensional sample configured to demonstrate all such processes at different positions along its length. We make two evaluations of the ML algorithm with this system. First, we investigate the performance of the LMI approximation in the transition from regions characterised by weak field (unsaturated) masers to regions characterised by strong field (saturated) masers. Second, after making a minor revision which enables us to enforce temporal initial conditions, we investigate the ML algorithm’s ability to model transient SR processes within the sample.

Upon demonstrating the inaccuracy of the transient application of the ML algorithm, we proceed to the central work of this paper. In Section 2.5 we construct a manifestly unique Fourier representation of the MB equations which generalises the ML algorithm. This representation is capable of modelling high field strength transient SR processes, may be executed with varying degrees of approximation fidelity, and retains the conventional ML algorithm’s improved computational complexity scaling over the time domain method. We demonstrate the successful simulation of all SR regions of the system investigated in the prior Section 2.4, and we characterise approximation fidelity requirements for future simulations of transient SR processes with our new algorithm.

Appendix A provides a rigorous justification for the LMI approximation from perturbation theory and appendix B expresses the novel Fourier representation central to this paper in a format more naturally suited to numerical simulation; namely, in its real and imaginary parts.
2.2 Collective and Cooperative Emission Processes Across Velocity Distributions and the Menegozzi and Lamb Method

2.2.1 Modelling maser and superradiant processes across velocity distributions

There are three common methods for modelling the maser action across wide incoherent velocity distributions: first, by a theory of rate-balanced excitation and de-excitation of velocity sub-populations, with accompanying equations of radiative transfer [9]; second, by the master equation describing the evolution of the quantum mechanical density operator [14, 23, 15]; and third, by the Heisenberg equations describing the time evolution of expectation values of the population inversion, polarisation, and field operators within the Heisenberg picture of QED [16, 27]. The latter two methods lead to the velocity dependent MB equations. All three methods must model, in some manner, the relatively weak coherence of the maser action within narrow velocity slices of the global incoherent velocity distribution. Compared to the intensity that would be generated by a fully coherent population sharing a single velocity, the total intensity generated by the incoherent distribution is reduced by the independence of these velocity slices.

Maser rate-balancing algorithms treat the radiators as statistically independent, and therefore do not generalise to SR modelling; conversely, the MB equations—being derived from the fully quantum mechanical density operator master equation or Heisenberg equations—continue to describe cooperative coherent emission in the transient SR regime. The MB equations are the starting point of Menegozzi and Lamb [23] and of our present work. A derivation of the MB equations as a valid representation of transient SR processes (under reasonable approximations) can be found in the literature [2, 21, 16, 1, 3, 27]. We turn now to discuss the MB equations over a velocity distribution, with the objective of constructing a numerically efficient algorithm for solving them in the transient SR regime.


2.2.2 The Maxwell-Bloch equations and the slowly-varying envelope approximation

Derivations of the MB equations start from the two-level (with ground state \( |g\rangle \) and excited state \( |e\rangle \)) model of a molecule possessing an electric or magnetic transition matrix element. The population inversion density \( N(\mathbf{r}) \) of the sample is defined as a coarse-grained function over the sample volume, valued at the \( k^{\text{th}} \) molecular site with the expectation value of the \( k^{\text{th}} \) molecule’s population inversion operator \( |e_k\rangle \langle e_k| - |g_k\rangle \langle g_k| \). The polarisation \( \mathbf{P}(\mathbf{r}) \) is a coarse-grained vector field over the sample volume, valued at each site with the expectation value of the molecular dipole operator weighted by the local population density.\(^2\) The quantum mechanical Heisenberg equations determine the self-consistent evolution of the population inversion density, the polarisation or magnetisation, and the field amplitude through the MB equations \([2, 21, 16, 3, 27, 28]\).

For a one-dimensional sample extended along the \( z \) axis with all dipole moments, the media polarisation, and the field polarisation oriented along a fixed orientation perpendicular to the \( z \) axis, the MB equations across a velocity distribution with an electric dipole transition are \([16]\)

\[
\left[ \frac{\partial}{\partial t} + v \frac{\partial}{\partial z} \right] N_v = \frac{i}{\hbar} (E^+ + E^-) (P^+_v - P^-_v) \tag{2.1}
\]

\[
\left[ \frac{\partial}{\partial t} + v \frac{\partial}{\partial z} \right] P^+_v = i \omega_0 P^+_v + 2i \frac{d^2}{\hbar} (E^+ + E^-) N_v \tag{2.2}
\]

\[
\left[ \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial z^2} \right] E^+ = -\frac{1}{\epsilon_0} \int \text{d}v F(v) \frac{\partial^2 P^-_v}{\partial t^2} \tag{2.3}
\]

where \( N_v \) is half the population inversion for those molecules travelling with velocity \( v \), \( P^+_v \) are the forward (+) and reverse (−) rotating parts\(^3\) of the polarisation for those molecules travelling with velocity \( v \), and \( E^\pm \) are the forward and reverse rotating parts of the electric field. We note that in the quantum mechanical limit from which the derivation of the MB equations starts, the \( P^+ \) (\( P^- \)) correspond to molecular raising (lowering) operators and the \( E^+ \) (\( E^- \)) to photon annihilation (creation) operators. All quantities depend upon only position \( z \) and time \( t \). The

\(^2\)In the case of the electric dipole transition in the two-level basis, the dipole operator of the \( k^{\text{th}} \) molecule is \( \mathbf{d} \langle |e_k\rangle \langle g_k| + |g_k\rangle \langle e_k| \rangle \) for a molecule with dipole moment \( \mathbf{d} \).

\(^3\)If \( f(z, t) = \int_{-\infty}^{\infty} f(z, \omega) e^{i\omega t} d\omega \), then \( f^+(z, t) \) are defined as \( f^-(z, t) = \int_0^{+\infty} f(z, \omega) e^{i\omega t} d\omega \) and \( f^+(z, t) = \int_{-\infty}^{0} f(z, \omega) e^{i\omega t} d\omega \).
angular frequency of emission is $\omega_0$ in the rest frame, the molecular dipole moment is $d$, and $F(v)$ is defined such that the fraction of molecules of velocity between $v$ and $v + dv$ is $F(v)dv$ (where $\int F(v)dv = 1$).

We now make a change of variables to the retarded time $\tau = t - z/c$ and factor the polarisations and electric field with envelope functions\(^4\) as

\begin{align*}
P_\pm^v (z, \tau) &= \bar{\mathcal{P}}_\pm^v (z, \tau) e^{\pm i\omega(1 + v/c)\tau} \\
E_\pm^v (z, \tau) &= \mathcal{E}_\pm^v (z, \tau) e^{\mp i\omega\tau}.
\end{align*}

If we neglect the fast-rotating terms $E^+P^-_v$ and $E^-P^+_v$ (the so-called “rotating wave approximation”) in equation (2.1) and recognise that $\partial/\partial z \ll \omega_0/c$ and $\partial/\partial \tau \ll \omega_0$ when acting on the envelope functions, we arrive at the so-called slowly-varying envelope approximation (SVEA) of the MB equations\[^5\].

\begin{align*}
\frac{\partial N_v}{\partial \tau} &= i\frac{\hbar}{\omega_0} (\bar{\mathcal{P}}_v^+ \mathcal{E}^+ e^{i\omega_v z/c} - \bar{\mathcal{P}}_v^- \mathcal{E}^- e^{-i\omega_v z/c}) - \frac{N_v}{T_1} + \Lambda^{(N)} \tag{2.6} \\
\frac{\partial \bar{\mathcal{P}}_v^+}{\partial \tau} &= i\frac{2d^2}{\hbar} \mathcal{E}^- N_v e^{-i\omega_v z/c} - \frac{\bar{\mathcal{P}}_v^+}{T_2} + \Lambda^{(P)} \tag{2.7} \\
\frac{\partial \mathcal{E}_v^+}{\partial z} &= i\frac{\omega_0}{2\varepsilon_0 c} \int dv (1 + v/c) F(v) \bar{\mathcal{P}}_v^- e^{-i\omega_v z/c}, \tag{2.8}
\end{align*}

where we have introduced population inversion and polarisation pumping sources $\Lambda^{(N)}(\tau)$ and $\Lambda^{(P)}(\tau)$, respectively, as well as non-coherent relaxation and dephasing time scales $T_1$ and $T_2$, respectively. The factor $(1 + v/c)$ in (2.8) is retained only for the discussion of the following paragraph, but is replaced in all practical computations by $1 + v/c \approx 1$. We make two observations on equations (2.6)–(2.8).

First, if the velocity distribution $F(v)$ is a purely coherent one at some velocity $v_0$—that is, if $F(v) = \delta(v - v_0)$—then our choice of Doppler shifted polarisation envelope frequency causes the system of equations (2.6)–(2.8) to appear exactly as a coherent system with $\omega_0 \rightarrow \omega'_0 = \omega_0(1 + v_0/c)$, if only we redefine our field factorisation of equation (2.5) by the Doppler shifted

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\(^4\)The “bar” on $\bar{\mathcal{P}}_v$ distinguishes our polarisation envelopes from the literature, in that we factor by Doppler shifted frequencies on a per-channel basis.

\(^5\)Our form of the SVEA of the MB equations with a velocity distribution differs slightly from that of Gross and Haroche [16] or Andreev [1] due to our Doppler shifting of the polarisation envelopes.
frequency $\omega_0'$ (upon doing so all manifest velocity reference vanishes and all occurrences of $\omega_0$ become $\omega_0'$). Therefore, although equations (2.6)–(2.8) are not manifestly symmetric across velocity channels, the channel dependent factors of $\nu$ are merely artifacts of our choice of reference electric field envelope frequency $\omega_0$. Each velocity slice sees a physically equivalent system centred upon its own Doppler shifted natural frequency of oscillation. When generalising to a wide velocity distribution $F(\nu) \neq \delta(\nu - \nu_0)$, a given velocity slice should couple most strongly to those Fourier components of the field neighbouring its natural Doppler shifted frequency. This physical argument will motivate the local mode interaction approximation in Section 2.2.5.

Second, because the physics should be symmetric across velocities, we decay and pump the polarisation on resonance with a velocity channel’s Doppler shifted frequency; i.e., no velocity dependent exponential multiplies $\bar{P}_\nu^{\pm}/T_2$ nor $\Lambda^{(P)}$ in equation (2.7), despite $\bar{P}_\nu$ being Doppler shifted relative to $P_\nu^\pm$ of equation (2.2). Had we originally introduced pumping and decay terms to equation (2.2), we would have accidentally neglected this physical symmetry across velocities.

2.2.3 Spontaneous emission and the initial Bloch angle prescription

We stated in Section 2.2.1, without proof, a basic tenet of this paper: that the MB equations accurately model SR transient processes. A precise derivation of this result can be found in the literature [2, 21, 16, 3], but it is necessary to describe here a feature of the derivation relevant to the modelling of transient processes with the MB equations.

It is apparent from equations (2.1)–(2.3) that the MB equations alone will not model even the simplest spontaneous emission process. Starting from an initially inverted population possessing no polarisation, and in the absence of an electric field, we expect a sample to spontaneously emit photons and eventually generate a non-zero electric field and non-zero polarisation. Instead, upon inspection of the SVEA MB equations, we see that the null right side of equation (2.7) will never allow the sample to acquire a polarisation (nor an electric field).

In order to model any spontaneous emission processes, the polarisation initial conditions of the semi-classical MB equations must be prescribed by a purely quantum mechanical analysis.
In Gross and Haroche [16] it is shown that interaction of the inverted molecules with fluctuations of the quantized radiation field leads, very quickly, to a classical ensemble of non-zero polarisation configurations. The continued evolution of the system is then described by the collection of trajectories of the MB equation determined by this ensemble of initial conditions. Transients of population inversion, polarisation, and fields are computed from expectation values averaged over these simulated trajectories.

In fact, the averaging operation yields negligible modifications to our results. We will be concerned only with the degree of polarisation built up in the interaction of the initial population inversion with fluctuations of the quantized vacuum radiation field. This polarisation value is that for which the conventional Bloch angle \( \theta_B \) (defined via \( \tan[\theta_B(z, \tau)] = |P(z, \tau)| / [N(z, \tau) d] \)) has tipped to \( \theta_{B,0} = 2/\sqrt{N_{\text{mol}}} \) [24, 16], where \( N_{\text{mol}} \) is the number of molecules in the sample.\(^6\) For a sample of a large number of molecules with initial population inversion \( N_0 \), this prescribes an initial polarisation \( P_0 \) according to \( P_0/d = N_0 \tan(\theta_{B,0}) \approx N_02/\sqrt{N_{\text{mol}}} \).

### 2.2.4 Computational complexity of the Maxwell-Bloch equations in the time domain

Although equations (2.6)–(2.8) analytically remove the stiffest\(^7\) temporal propagation term \( i\omega_0 P^v_\nu \) from equation (2.2), a lower degree of stiffness remains present within the exponentials \( \exp[\pm i\omega_0 (v/c) \tau] \) (had we not Doppler shifted our polarisation envelopes, this stiffness would have emerged in a term of the form \( i\omega_0 (v/c) P^v_\nu \)).

The order of total numerical complexity to a Runge-Kutta solution of equations (2.6)–(2.8) is degraded by the increasing stiffness of these exponential terms with increasing width of the velocity distribution. As \( F(v) \) widens, these exponentials oscillate at higher frequencies and demand finer time stepping to avoid aliasing of their cycles. No analytical factorisation can remove this stiffness, which forces a time domain algorithm to be \( O(N^2) \) complex in the

---

\(^6\)Actually, \( N_{\text{mol}} \) should be replaced here by the number of interacting molecules. This is problematic, as the number of interacting molecules within a velocity distribution is not well-defined at this point in our analysis. We discuss this point further in Section 2.6.3 on future research.

\(^7\)The term “stiff” has various usages in the literature. A “stiff” term in this paper is any derivative generating term (any term on the right side of our differential equations as written) which places finer step size demands (relative to adjacent generating terms) upon the numerical algorithm.
number $N$ of velocity channels simulated. Doubling the velocity width, for example, demands both that twice as many channels be simulated and that each be simulated with twice as fine a time step in a Runge-Kutta propagation of equations (2.6) and (2.7).

### 2.2.5 The Menegozzi & Lamb Method

We desire to reduce the order of numerical complexity in simulating our system by turning to physical arguments. To this end, we review in this section the one-dimensional maser simulation algorithm developed by Menegozzi and Lamb [23] within a temporal Fourier series representation of the MB equations.

This representation will introduce two numerical advantages. First, it will allow a simulation to crop the spectrum of each velocity channel’s population inversion and polarisation transients to those spectral components lying within a limited neighbourhood of the channel’s natural Doppler shifted frequency. Second, it will enable the assertion of what we refer to here as the LMI approximation, which suppresses the algebraic coupling of inversion and polarisation velocity channels to electric field modes sufficiently far removed from their natural frequencies. This approximation is presented in only a limiting case in Menegozzi and Lamb [23], but is naturally generalised in the present work.

Importantly, although the LMI approximation will remove the formal direct mathematical coupling between distant frequency modes, it will not necessarily remove the possibility of physical coupling and correlation between distant modes through indirect, transitive\(^8\) means. In Section 2.5.2 we will investigate the accuracy of this approximation in the SR domain; it will then become the task of future research, operating within these LMI approximation fidelity constraints, to quantify the degree of transitive coupling and correlation between SR processes across a broad velocity distribution. The algorithm we develop in Section 2.5 will enable this future research.

In the remainder of this Section 2.2.5 the notation is our own but the theory loosely follows

\(^8\)We colloquially describe “transitive” coupling by the following example. Suppose that velocity channels $A$, $B$, $C$ would naturally radiate field modes of frequencies $\omega_A < \omega_B < \omega_C$, and that our LMI approximation is only so sufficiently wide as to mathematically couple $A \Rightarrow \{\omega_A, \omega_B\}; B \Rightarrow \{\omega_A, \omega_B, \omega_C\}; C \Rightarrow \{\omega_B, \omega_C\}$ (“$\Rightarrow$” denotes “couples to”). Although channel $A$ does not mathematically couple to field mode $\omega_C$, we recognise that channel $A$ may indirectly physically couple to $\omega_C$ via the transitive coupling $A \Rightarrow \omega_B \Rightarrow B \Rightarrow \omega_C$.\]
2.2. Introducing Menegozzi and Lamb [23].

The Menegozzi & Lamb representation of the Maxwell-Bloch equations

For a simulation of duration \(T\), the population inversion, the polarisation envelopes, the pumping sources, and the field envelopes are expanded in Fourier series of mode separation \(d\omega = 2\pi/T\). Additionally, the velocity distribution is partitioned with a granularity \(dv\) of the equivalent Doppler shift \(d\omega\) between adjacent channels; namely, \(dv = c d\omega/\omega_0\). If \(p\) denotes the integer multiple of \(dv\) identifying a velocity channel then, for example, \(N_{p,m}\) denotes the \(m\)th frequency mode of the population inversion of a velocity slice centred at velocity \(v = pdv\). Explicitly, the expansions read as

\[
N_p = \sum_m N_{p,m}(z) e^{i m d\omega \tau} \tag{2.9}
\]
\[
\tilde{\Phi}_p^\pm = \sum_m \tilde{\Phi}_{p,m}^\pm (z) e^{\pm i m d\omega \tau} \tag{2.10}
\]
\[
\tilde{\mathcal{E}}^\pm = \sum_m \tilde{\mathcal{E}}_{m}^\pm (z) e^{\mp i m d\omega \tau} \tag{2.11}
\]
\[
\Lambda^{(N/P)} = \sum_m \Lambda_m^{(N/P)} e^{i m d\omega \tau}. \tag{2.12}
\]

Upon substitution into the MB equations and some changes of summation orders and variables (the details of which are omitted here), the first two MB equations (2.6) and (2.7) translate to the algebraic Fourier mode relations

\[
(i m d\omega) N_{p,m} = i \hbar \sum_m \left( \bar{\mathcal{E}}_{p,m}^+ \mathcal{E}_{p+m-m}^- - \bar{\mathcal{E}}_{p,m}^- \mathcal{E}_{p+m-m}^+ \right) - \frac{N_{p,m}}{T_1} + \Lambda_m^{(N)} \tag{2.13}
\]
\[
(i m d\omega) \tilde{\Phi}_{p,m}^+ = \frac{i 2 d^2}{\hbar} \sum_m \left( \bar{\mathcal{E}}_{p,m}^- \mathcal{E}_{p+m-m}^- \right) - \frac{\tilde{\Phi}_{p,m}^+}{T_2} + \Lambda_m^{(P)}. \tag{2.14}
\]

Equations (2.13) and (2.14) are mathematically equivalent to equations (2.30) and (2.29), respectively, of Menegozzi and Lamb [23], despite differing notation and algebraic rearrangement. The form here will prove advantageous for our work generalising the algorithm to the transient domain in Section 2.5 and for our introduction of the LMI approximation. The third
MB equation (2.8) reads

\[
\frac{dE^\pm_m}{dz} = i\frac{d\omega}{2\epsilon_0} \sum_p F_p \bar{P}_{p,m-p}^\pm,
\]

where \( F_p = F(pdv) \). We refer to equations (2.13)–(2.15) as the Menegozzi & Lamb (ML) representation of the MB equations.

**Solution method, spectral limiting, and the local mode interaction approximation**

Solving the ML equations is a straightforward numerical task. Starting from the Fourier representation \( E^\pm_m \) of a given incident electric field time dependence \( E(z = 0, \tau) \) at the start of the sample, equations (2.13) and (2.14) form a linear system, which is solved for the population inversion and polarisation modes at \( z = 0 \); next, the polarisation modes are used to propagate the electric field modes forward one step in \( z \) via equation (2.15). These two steps loop along the entire length of the sample. In a practical numerical scheme, a fourth-order Runge-Kutta abstraction of the \( z \)-stepping is employed.

In their native form, equations (2.13) and (2.14) are computationally expensive in the number of velocity channels \( N \): for each channel added to the system, the resulting expansion of the electric field spectrum which enters on the right side of equations (2.13) and (2.14) implies expansion of the range of \( m \) in \( \bar{P}_{p,m}^\pm \). Increasing the number of velocity channels thus increases the size of the linear system of unknown modes that must be solved for every velocity channel. The operation of solving a linear system of equations is \( \sim O(N^3) \) complex,\(^9\) so that the total algorithm is \( \sim O(N^4) \) complex.

The order of complexity of the ML algorithm is dramatically reduced by limiting the range of the spectral mode index of all velocity channels’ population inversions and polarisations (the index \( m \) of \( \bar{P}_{p,m}^\pm \)). We herein refer to this approximation as spectral limiting. If \( m \) is limited to a fixed-size (independent of the number of velocity channels introduced) neighbourhood of 0, the size of the linear system of equations for each channel does not grow with \( N \); as \( F(v) \) widens, numerical operations therefore increase proportional only to the number of velocity channels needing to be solved. Simulating the ML equations under spectral limiting is thus \( O(N) \) complex.

\(^9\)This complexity can be reduced to \( \sim O(N^{2.5}) \) by employing an efficient linear system solver.
An additional numerical approximation introduced in Appendix C of Menegozzi and Lamb [23] is to couple the population inversion and polarisation modes of a particular velocity channel to interact only with that field mode corresponding to its natural Doppler shifted resonance. In their work, Menegozzi and Lamb [23] eliminate reference to the polarisation modes and achieve said approximation by recognising the dominant terms in the remaining system of equations for the unknown population inversion modes. In our present form, and with $\tilde{P}_v$ factored about its natural Doppler shifted frequency, such an approximation is achieved by truncating to $\tilde{m} = 0$. This choice of term may appear at first glance ambiguous; however, a rigorous justification for it may be found in Appendix A.

The formulation of the approximation in Menegozzi and Lamb [23] corresponds, in our present form, to summation over the trivial set $\tilde{m} \in \{0\}$. Such an approximation is sufficiently accurate for the unsaturated maser domain but, as Menegozzi & Lamb rightly argue, becomes inaccurate in regions of high field strength. Our formulation suggests a natural generalisation of the approximation which permits its assertion to varying degrees of fidelity. We extend the $\tilde{m}$ summation over a finite neighbourhood of 0. This is the mathematical expression of the LMI approximation introduced colloquially near the end of Section 2.2.2. We will investigate the performance of the LMI approximation within regions of increasing field strength in Section 2.4.3. Notice that decreasing LMI approximation fidelity (narrowing the range of the $\tilde{m}$ summation) improves the sparsity of the linear system of equations (2.13) and (2.14), and therefore offers a further reduction in numerical operations. The simulations throughout this work implement an LMI approximation of fixed fidelity across all $z$ positions. Such fidelity could, in theory, be made to vary as a function of $z$, and we discuss this possible generalisation in Section 2.6.3 on future work.

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\[10\] We refer the reader to the paragraphs immediately preceding and following equations (C11) and (C12) of Menegozzi and Lamb [23].
2.3 Transient Superradiance Processes Versus Quasi-Steady State Maser Processes

In this section we differentiate between the transient nature of an SR process and the quasi-steady state nature of a maser process. For a comprehensive comparison of the two processes, see Rajabi and Houde [26].

2.3.1 Superradiance as a transient process

Superradiance is fundamentally a transient phenomenon, involving a series of distinct events and the evolution from an initial energetic inverted population level to a dramatically altered final inversion level [26]. This transient system evolution is perhaps best qualitatively understood in the Schrödinger picture of the QED of a collection of \( n \) molecules.

We imagine a system initially prepared with all molecules excited and the quantized radiation field in the vacuum state,

\[
|\Psi\rangle_{\text{initial}} = |e_1 e_2 \ldots e_n\rangle \otimes |0\rangle_{\text{rad}},
\]

(2.16)

where \( |e_1 e_2 \ldots e_n\rangle \) denotes a tensor product of molecular states which may each assume either the excited state \( |e\rangle \) or the ground state \( |g\rangle \), and where \( |0\rangle_{\text{rad}} \) denotes the vacuum state of the radiation field.

It should be understood that the instantaneous inversion of the entire sample described by equation (2.16) is an idealisation intended, for the purpose of this section, to capture the salient features of SR. In a realistic astrophysical environment we could imagine a system starting for \( t < t_{\text{init}} \) in a highly inverted state accurately modelled by maser theory. The initial preparation of equation (2.16) would be realised by a pumping flare of finite duration shorter than that of the SR transient process, but of sufficient amplitude to raise the inverted population column density above the SR threshold. In Rajabi, Houde, Bartkiewicz, Olech, Szymczak, and Wolak [30], for example, the 6.7 GHz methanol emission is believed to be pumped by infrared radiation from dust [32] demonstrating outbursts [5, 35] which are proposed to initiate the onset of SR.

We assume that the relevant photon wavelengths are much larger than the size of a molecule,
and we neglect the Hilbert space associated with the motions of the molecules’ centres of masses. We also assume that the molecules are sufficiently separated so as not to require wave-function symmetrisation.\(^{11}\) The total Hamiltonian \(H_{\text{tot}}\) of the system is the sum of the internal Hamiltonian \(H_{\text{mol}}\) of two-level molecular excitations, of the radiation Hamiltonian \(H_{\text{rad}}\), and of the field-molecule interaction Hamiltonian \(H_{\text{int}}\). Tensor products of free molecular excitation states and radiation field Fock states, such as that of equation (2.16), are eigenstates of \(H_{0} \equiv H_{\text{mol}} + H_{\text{rad}}\), but not of \(H_{\text{tot}} = H_{0} + H_{\text{int}}\). Treating \(H_{\text{int}}\) as a perturbation to \(H_{0}\) yields transition amplitudes between eigenstates of \(H_{0}\).

In his seminal paper Dicke [6] describes first the small sample limit, defined such that molecules are separated by a distance much less than their spontaneous emission wavelength \(\lambda\) but much greater than any intermolecular interaction length. Although the molecules are fundamentally distinguishable in the small sample limit, they are not distinguishable via observation of an emitted photon. Thus, if spontaneous emission is described by the transition to some singular de-excited molecular state \(|\phi\rangle_{\text{mol}}\) with an accompanying photon of mode \(p\),

\[
|e_1 e_2 \ldots e_n\rangle_{\text{mol}} \otimes |0\rangle_{\text{rad}} \rightarrow |\phi\rangle_{\text{mol}} \otimes |1_p\rangle_{\text{rad}},
\]  

then the final molecular state \(|\phi\rangle_{\text{mol}}\) must be indeterminate in the identity of the ground state molecule. The precise calculations of Dicke [6] show that the most probable molecular state is the symmetric superposition of all possible configurations having one molecule in the ground state; i.e., \(|\phi\rangle_{\text{mol}} = \left(1/\sqrt{n}\right) \sum_k |e_1 e_2 \ldots g_k \ldots e_n\rangle \equiv |s(1)\rangle\), where we define \(|s(k)\rangle\) as the symmetric superposition of all states possessing \(k\) molecules in the ground state. Note that \(|s(k)\rangle\) is an entangled state for \(0 < k < n\). By energy conservation, the energy of the emitted photon matches the loss in molecular excitation energy \(\hbar \omega_0\).

The transient SR process is the cumulative effect of the cascade down the totally symmetric excitation states \(|s(k)\rangle\) with accompanying photon emission as depicted in Figure 2.1, where emission rates vary with \(k\). The full perturbation calculation finds that the halfway state \(|s(n/2)\rangle\)^{12} couples most strongly to the radiation field, so that emission is maximised after some

---

\(^{11}\)Incidentally, as we describe later in this section, the calculation of Dicke [6] yields transitions through totally symmetric states only. This result is a consequence of the permutation symmetry of the interaction Hamiltonian, and not a consequence of spin-statistic imposed symmetrisation.

\(^{12}\)for even \(n\); \(|s(n/2 \pm 1/2)\rangle\) for odd \(n\)
\[ |s(0)\rangle = |e_1 e_2 \ldots e_n\rangle \]
\[ \Downarrow \sim \sim \hbar \omega_0 \]
\[ |s(1)\rangle = \frac{1}{\sqrt{n}} (|g_1 e_2 \ldots e_n\rangle + |e_1 g_2 \ldots e_n\rangle + \cdots + |e_1 e_2 \ldots g_n\rangle) \]
\[ \Downarrow \sim \sim \hbar \omega_0 \]
\[ \cdots \]
\[ \Downarrow \sim \sim \hbar \omega_0 \]
\[ |s(n)\rangle = |g_1 g_2 \ldots g_n\rangle \]

Figure 2.1: Schematic of the cascade down the ladder of symmetric excitation states with accompanying photon emission.

delay \( \tau_D \) required to reach this state. The cascade may be modelled by a Markovian traversal through the symmetric states with a Lindblad operator describing photon loss to the environment. Analysis of transitions between symmetric states provides a radiation intensity transient, where the intensity is derived from the expectation value of the transition rate as a function of time, as averaged over many repetitions of the full stochastic cascade \([6, 27, 33]\).

The SR process is thus an inherently transient one, characterised by discrete events and phases: first, a pumping event initiates the inversion of the system at some time \( \tau_0 \); second, the system evolves over a time \( \tau_D \) to the maximal emission state \( |s(n/2)\rangle \); and third, the emission eventually concludes when the system has evolved into the final fully ground state \( |s(n)\rangle = |g_1 g_2 \ldots g_n\rangle \).

This qualitative evolution carries over, with some modification, to the more complicated case of an extended sample with molecules distributed over distances much greater than \( \lambda \). The evolution of such a system is described by equations (2.6)–(2.8). We can demonstrate the transient features of SR in the simplest case of a one-dimensional extended sample without a velocity distribution (at resonance) and without relaxation or dephasing effects,\(^{14}\) where the so-called sine-Gordon equation describes the evolution of the Bloch angle \( \theta_B \) according to \([16,\ldots]\).

---

\(^{13}\)Averaging is justified in astrophysical observations by the fact that the unresolved source is composed of many statistically independent SR cylinders, each representing a single realisation of the SR cascade experiment.

\(^{14}\)If the relaxation time \( T_1 \) matches the dephasing time \( T_2 \), an analytical solution exists which generalises equation (2.18); see Rajabi and Houde [26].
2.3. Transient Superradiance Processes Versus Quasi-Steady State Maser Processes

\[
\begin{align*}
\frac{d^2 \theta_B}{dq^2} + \frac{1}{q} \frac{d\theta_B}{dq} = \sin (\theta_B),
\end{align*}
\]  

(2.18)

where \( q \) is the dimensionless parameter \( q = 2 \sqrt{z \tau / LT_R} \) for a sample of length \( L \). The characteristic timescale \( T_R \) is determined by the sample’s length, by the inverse of the molecules’ Einstein coefficient of isolated spontaneous emission rate \( \tau_{sp} \), by the spontaneous emission wavelength \( \lambda \), and by the sample’s inverted population density \( n \) as

\[
T_R = \tau_{sp} \frac{8\pi}{3n\lambda^2 L}.
\]  

(2.19)

It is important to note that both the relaxation and dephasing timescales \( T_1 \) and \( T_2 \) of the sample must approximately exceed the characteristic timescale \( T_R \), if the system is to reach the highly entangled maximal emission state and thus demonstrate SR.

A plot of an SR transient generated by the sine-Gordon equation is depicted in Figure 2.2, which clearly displays the transient SR phases of a buildup to the maximal emission state and a decay to the fully ground state. Note that in the extended one-dimensional sample, the initial emission can partially re-invert the sample downstream and lead to subsequent emissions, thereby producing the ringing effect visible in Section C of Figure 2.2.

2.3.2 The maser action as a quasi-steady state process

In contrast to SR, the maser action is effectively modelled as a quasi-steady state process involving a large number of concurrent, rate-balanced emission and stimulation events [10, 9, 15, 26]. These events are connected across molecular sites through the seed photon: a photon of mode \( p \) emitted at a molecular site \( j \) can enhance the probability of emission into the same mode \( p \) at another molecular site \( k \) through a process such as, for example,

\[
|e_k\rangle_{\text{mol}} \otimes |1_p\rangle_{\text{rad}} \rightarrow |g_k\rangle_{\text{mol}} \otimes |2_p\rangle_{\text{rad}}.
\]  

(2.20)
Figure 2.2: Normalised radiation intensity transient at the end-fire \( z = L \) of a one-dimensional sample of 200 molecules, inverted at \( \tau = \tau_0 \) and exhibiting SR. Computed from the sine-Gordon limit of the MB equations. Section A: Buildup to the maximal emission state \( |s(n/2)\rangle \). Section B: Decay to the fully ground state \( |s(n)\rangle \). Section C: Re-inversion downstream and emission ringing.

Although the maser process physically couples distinct molecular sites \( j \) and \( k \), the evolution between emission and stimulation events follows classical statistics. That is, emission of photon \( p \) is unambiguously associated with the transition of some single molecule \( j \) from the excited to the ground state; then, under the assumption that molecule \( j \) is in the ground state and that the radiation field possesses photon \( p \), the probability of stimulation of a photon \( p \) at another site \( k \) may be computed; if emission occurs, molecule \( k \) definitively transitions from \( |e_k\rangle \) to \( |g_k\rangle \).\(^{15}\)

This analysis of a large number of transition processes each into states well-defined in the identity of the emitting molecule is in contrast to the cascade process of SR. The SR photon emission discussion following equation (2.17), for example, described transition into a single-photon state which was an entangled superposition of the excitation states of different molecular sites; i.e., into the state \( |s(1)\rangle \) that was indeterminate in the identity of the emitting molecule.

\(^{15}\)We present these single-photon descriptions for the purpose of providing fundamental physical insight only. In typical numerical work, however, maser simulations operate on density matrices at exceedingly larger scales than those of single-photon processes. Most numerical maser models do not explicitly simulate the individual molecular events described here.
molecule. In the maser case, the relaxation and dephasing timescales are much shorter than the time-scale for the evolution of the system. For a more detailed discussion of the role of relaxation, dephasing, and SR characteristic timescales in differentiating the maser and SR regimes, see Rajabi and Houde [26].

Whereas the SR cascade unfolds as a transient process, the maser action may be successfully modelled as a quasi-steady state process; for a comprehensive summary of the theory of astrophysical masers, see Elitzur [9] and Gray [15]. The maser analysis in Menegozzi and Lamb [23], for example, investigates continuous emission in the quasi-steady state limit, where the quantity of interest to the simulation is the field spectrum profile along the sample’s length. Such a profile results from the complicated inter-operation of pumping, decay, stimulated emission, and dephasing processes within a velocity distribution of molecules. Despite describing a quasi-steady state, the field spectrum certainly contains a rich ensemble of various off-resonance frequencies, and therefore varies significantly in time over the Fourier expansion period $T$ (even within the rotating envelope picture). The assumption of Menegozzi and Lamb [23] is that the artificiality of such a representation’s indefinite periodicity (in integer multiples of $T$) does not detract from its ability to describe quasi-steady state features within a single simulation period $T$, such as deviations from a Gaussian white noise spectral distribution along the sample’s length.

It should be noted that although the maser action is a quasi-steady state process, it can demonstrate transient behaviour in the following limited sense. In the maser regime, calculation of the characteristic timescale $T_R$ yields a value much greater than either $T_1$ or $T_2$. As a result, the system tracks in lock-step with any transient behaviour of the inversion pump, the polarisation pump, or the incident electric field. A transient maser process is simply an immediate quasi-steady state response to variations in pump levels or incident field strengths. For a more thorough discussion see Rajabi and Houde [26], where the maser domain is formally identified with those processes for which

$$\frac{\partial N}{\partial \tau} \ll \frac{N}{T_1} \quad \text{and} \quad \frac{\partial P^+}{\partial \tau} \ll \frac{P^+}{T_2}.$$  \hspace{1cm} (2.21)

Conversely, in an SR process $T_R \ll T_1, T_2$ so that the total transient response of the system
possesses a finite memory. The inversion, polarisation, and electric field do not track in lock-step with the pump or incident field sources, and the system state at any given time depends upon its own history. When we distinguish SR as a transient process versus the maser action as a quasi-steady state process, we do so in this nuanced manner. An SR transient response to variations in pump or incident field sources is a complex dynamic process which may exhibit dramatically different timescales than those presented by the sources. It is formally identified in Rajabi and Houde [26] with those processes for which
\[ \frac{\partial N}{\partial \tau} \gg \frac{N}{T_1} \text{ and } \frac{\partial P^+}{\partial \tau} \gg \frac{P^+}{T_2}. \] (2.22)

### 2.3.3 Fourier representations of quasi-steady state versus transient processes

The ML representation of the MB equations, albeit advantageous for numerical complexity purposes, introduces two complications. First, being an expansion in periodic basis functions, it forces periodicity upon the solutions. Second, it removes the ability to impose temporal initial conditions upon the inversion and polarisation. As discussed in our closing paragraph of the previous section, neither of these complications hinders the simulation of a quasi-steady state process, where periodicity is a reasonable approximation and where initial conditions are irrelevant. The ML algorithm is therefore naturally suited to the analysis of a quasi-steady state maser, including investigations of Gaussian white noise propagation down a sample’s length or of radiation coherence (to which the algorithm was indeed applied in Menegozzi and Lamb [23]).

As discussed in Section 2.3.1, SR is a transient process with distinct initial and final configurations. Such a process is fundamentally non-periodic, and quantities of interest (including total radiated energy, peak intensity, and process time scales) are strongly dependent upon initial conditions. We therefore expect difficulties to arise when applying the ML algorithm to SR processes, and we demonstrate shortly in Section 2.4.4 the inability of the ML algorithm to converge to the correct Fourier representation of non-periodic transient SR solutions to the MB equations.
These limitations for modelling transients may appear, upon first consideration, insurmountable by any Fourier series representation of the MB equations: any such representation being (by construction) periodic in the simulation duration $T$, and thus ill-suited to describing the evolution between significantly different initial and final configurations. Indeed, any Fourier series expansion of such a process will introduce ringing artifacts when inverted back to the time domain; however, important physical quantities of interest (such as total radiated energy, characteristic timescales, etc.) can be accurately described by a proper Fourier series expansion, if only the algorithm used is able to converge to it.

2.4 Performance of the Menegozzi & Lamb Algorithm

We investigate in this section the performance of the ML algorithm in the transition from modelling quasi-steady state maser processes at unsaturated field strengths, to maser processes at saturated field strengths, to transient SR processes. For this purpose we simulate a single one-dimensional sample which contains regions characterised by all the aforementioned processes. The experiment is detailed in Section 2.4.1 and a reference correct solution is computed in Section 2.4.2 from the time domain representation of the MB equations (2.6)–(2.8). The performance of the ML algorithm in computing the quasi-steady state of the system within regions of increasing field strength under varying degrees of LMI approximation fidelity is evaluated in Section 2.4.3. After slight algebraic revision, the performance of the ML algorithm in simulating transient responses is evaluated in Section 2.4.4, where we demonstrate the inability of the ML algorithm to converge at all to correct transients within regions demonstrating high field strength, transient SR processes.

2.4.1 Experiment description

A sample is initially prepared at $\tau = 0$ in the fully inverted state, but with a molecular column density sufficient to initiate an SR transient only near the end of the sample ($z = L$) when a constant coherent incident electric field $E(z = 0, \tau) = E_0 = 1 \times 10^{-16} \text{ V/m}$ is applied at the start of the sample\(^{16}\) (the column density refers to the number of inverted molecules per unit area.

\(^{16}\)An incident field is not essential as the SR process can be initiated by a sufficient column density alone.
projected along the sample’s full length; for a detailed discussion of the critical column density threshold necessary for SR, see Rajabi and Houde [26]). The sample possesses a velocity distribution of narrow extent; specifically, 21 velocity channels are simulated, separated by the fundamental velocity differential \( dv = (2\pi/T) (c/\omega_0) \) established by the simulation duration \( T \).

We simulate the cylindrical sample of methanol molecules (\( \omega_0 = 2\pi \times 6.7 \text{ GHz}, d = 0.7 \text{ D} \)) described in Rajabi and Houde [26] over a duration \( T = 10^8 \text{ s} \), having length \( L = 2 \times 10^{15} \text{ cm} \), radius \( w = 5.4 \times 10^7 \text{ cm} \), population inversion relaxation time constant \( T_1 = 1.64 \times 10^7 \text{ s} \), and polarisation dephasing time constant \( T_2 = 1.55 \times 10^6 \text{ s} \). Despite such time scales being orders of magnitude longer than those typically used in maser models of star formation regions, there is in theory no physical barrier to their occurrence. In Rajabi, Houde, Bartkiewicz, Olech, Szymczak, and Wolak [30] it is shown that \( T_2 = 1.55 \times 10^6 \text{ s} \) corresponds to a gas density of approximately \( 10^5 \text{ cm}^{-3} \). Rather than precluding such a large value of \( T_2 \) on account of an assumption of high gas density, the very natural fit of the SR model in Rajabi, Houde, Bartkiewicz, Olech, Szymczak, and Wolak [30] under such a value of \( T_2 \) may be considered evidence of low gas density in the star formation region which it models.

Our simulation differs from that of Rajabi and Houde [26] in our initial population inversion of \( N_0 = 1.5 \times 10^{-12} \text{ cm}^{-3} \) at \( \tau = 0 \) (cf. \( N_0 = 3.3 \times 10^{-12} \text{ cm}^{-3} \) in Rajabi and Houde [26]) and in our non-vanishing incident \( E_0 \). We apply a constant restoring population inversion pump equal to the relaxation rate; i.e., \( \Lambda^{(N)}(\tau) = N_0/(2T_1) \). The velocity distribution is uniform, so that \( F(v) = 1/\Delta v \) for the total velocity width \( \Delta v = 21dv = 9.4 \times 10^{-9} \text{ m/s} \). The molecular density for the present \( \Delta v \) corresponds to an inverted molecular density on the order of \( 0.1 \text{ cm}^{-3} \) for a realistic velocity distribution of \( \Delta v \approx 1 \text{ km/s} \). Note that the cylindrical dimensions correspond to a Fresnel number \( \pi w^2/L\lambda \) of unity.

### 2.4.2 Reference time domain solution

Reference intensity transients \( I(z, \tau) = c\epsilon_0 |E(z, \tau)|^2/2 \) at each of six positions along the length of the sample, normalised to the incident intensity \( I_0 = c\epsilon_0 |E_0|^2/2 \) at the start of the sample,

\[ ^{17} \text{This difference being in addition to the key distinction that the present work models a (non-trivial) velocity distribution.} \]

\[ ^{18} \text{Recall that } N_r \text{ is defined as half the population inversion, hence the factor of } 1/2 \text{ in the restoring pump.} \]
are readily computed from the time domain representation of the MB equations (2.6)–(2.8). Our solution is a generalisation of the technique of Mathews [22], Houde, Rajabi, Gaensler, Mathews, and Tranchant [18], Rajabi, Houde, Bartkiewicz, Olech, Szymczak, and Wolak [30], and Rajabi and Houde [26] to a distribution of velocity channels. To commence the simulation, the inverted populations and polarisations of all velocity channels are set to their initial conditions at \( \tau = 0 \) (recalling the polarisation initial condition prescription of Section 2.2.3), and the electric field throughout the sample is initialised via a fourth-order Runge-Kutta \( z \)-propagation of equation (2.8) from \( z = 0 \) to \( z = L \). The electric field is then used to perform a fourth-order Runge-Kutta time advancement (\( \tau \rightarrow \tau + d\tau \)) of all population inversion and polarisation velocity channels, the electric field is re-propagated along \( z \), and the process repeats until \( \tau = T \). The results are shown in Figure 2.3.

The time domain algorithm is a perfectly valid one and is, in fact, more computationally efficient than either the ML algorithm or the integral Fourier (IF) algorithm (to be introduced in Section 2.5) for simulating our present narrow velocity distribution. Although this velocity extent is physically trivial, it spans a numerically non-trivial multiple (21) of the fundamental angular frequency differential \( 2\pi/T \) determined by the duration of the simulation. This experiment is thus a meaningful investigation of numerical accuracy; it is the goal of the IF algorithm of Section 2.5 to enable, in future research, the efficient simulation of physically non-trivial velocity distributions which are otherwise intractable with the time domain representation of the MB equations.

### 2.4.3 The Menegozzi & Lamb algorithm in the quasi-steady state domain

We first evaluate the ML algorithm with varying degrees of LMI fidelity against the quasi-steady state regime located on the right side of Figure 2.3. We solve the ML equations (2.13)–(2.15) via the procedure described in Section 2.2.5 and with summations limited to \( \tilde{m} \in [-N_{\text{int}}, +N_{\text{int}}] \) for varying values of the local mode interaction distance \( N_{\text{int}} \).

The ML algorithm accurately converges to the steady state intensity profile; however, the LMI approximation fidelity requirements increase along the length of the sample as the system
Figure 2.3: Intensity transients (normalized to the incident intensity $I_0$ at the start of the sample) at varying positions along a one-dimensional sample possessing a velocity distribution, inverted at $\tau = 0$, and exhibiting SR at its end-fire ($z = L$). Computed from the time domain representation of the MB equations. Note the logarithmic scale for the intensity.
enters the saturated maser regions. Note that the saturated maser region can be identified as $z \gtrsim 0.6 L$, where the logarithm of the steady state intensity begins to deviate from constant-step increases when scanning vertically up the right side of Figure 2.3. We summarise these findings by presenting the ML simulation of equations (2.13)–(2.15) for our prototypical experiment in Figure 2.4, computed with the LMI approximation at five degrees of fidelity, and in each case with spectral limiting (the range limitation on the mode index $m$ of each velocity channel) enforced to $m \in [-50, +50]$.

The ML simulation converges to correct quasi-steady state intensities in the unsaturated maser domain across all degrees of LMI fidelity, but degrades with reduced LMI fidelity (moving to the right in Figure 2.4) at greater $z$ positions (moving up in Figure 2.4). This widened LMI mode coupling requirement at higher maser saturation suggests that an algorithm for simulating SR transients in the Fourier representation within regions of high field strength will also demand increased LMI approximation fidelity.

2.4.4 The Menegozzi & Lamb algorithm in the transient domain

Although not naturally suited to modelling transient processes in its raw form of equations (2.13)–(2.15), the ML representation may be slightly revised to investigate the transient region on the left side of Figure 2.3.

We begin this revision by noticing that the $m = 0$ cases of equations (2.13) and (2.14) are, in fact, assertions of periodicity in the inversion and polarisation: when $m = 0$, the right sides of equations (2.13) and (2.14) represent the zeroth modes of the Fourier series expansions of the right sides of equations (2.6) and (2.7), which are the time derivatives of the inversion and polarisation. Generally speaking, the zeroth Fourier mode of a function is computed by integrating the function over the expansion domain $T$; thus, the $m = 0$ equations are statements that the integral of the time derivatives of the inversion and polarisation must vanish; i.e., that they must be periodic in $T$.

This is a redundant statement to the assumption that the inversion and polarisation be represented, in the first place, by Fourier series expansions in the simulation duration $T$. We
Figure 2.4: Quasi-steady state intensities (normalized to the incident intensity $I_0$ at the start of the sample) from the ML simulation of the system of Figure 2.3, with LMI approximation interaction truncated to 10, 8, 6, 4, and 2 neighbouring modes. The fidelity requirements increase in the high field region $z > 0.6L$, where the error in intensity becomes highly sensitive to reductions in $N_{int}$. For reference, the leftmost column ($N_{int} = 10$) is effectively coincident with the true quasi-steady state values (cf. the right side of Figure 2.3).
therefore drop the $m = 0$ cases of equations (2.13) and (2.14), and replace them with statements imposing our initial conditions upon the system in the ML representation; namely,

\[
N_p(z, \tau = 0) \equiv N_{p,0}(z) \Rightarrow \sum_m N_{p,m}(z) = N_{p,0}(z) \tag{2.23}
\]

\[
\bar{\mathcal{P}}_p^\pm(z, \tau = 0) \equiv \bar{\mathcal{P}}_{p,0}^\pm(z) \Rightarrow \sum_m \mathcal{P}_{p,m}^\pm(z) = \mathcal{P}_{p,0}^\pm(z). \tag{2.24}
\]

We refer to equations (2.13) and (2.14), absolved of the $m = 0$ case and augmented with the initial condition equations (2.23) and (2.24), as the transient Menegozzi & Lamb (TML) algorithm. The transients produced by the TML algorithm for our prototypical experiment are shown in Figure 2.5. Simulation in this case is executed with high LMI fidelity ($N_{\text{int}} = 30$) and generous spectral limiting; i.e., $m \in [-50, +50]$.

The TML simulation demonstrates Gibbs ringing phenomena [4] in positions advanced along the length of the sample, where SR processes yield substantial differences between the temporal initial and final configurations of the inversion and polarisation over the simulation duration $T$. Such ringing is present in any Fourier representation of a function which differs in value at the endpoints of the expansion interval [4]; however, the overall form of the TML response is sensitive to this ringing and (problematically) renders this algorithm incapable of accurately modelling transient SR processes.

The TML simulation appears reasonably capable of modelling the transient approach to the unsaturated maser steady states for $z \lesssim 0.6L$, but begins to fail beyond $z \approx 0.6L$ where the Gibbs ringing amplitude nears the magnitude of the imposed temporal initial conditions. For $z = 0.8L$ the peak amplitude near $\tau = 1.6 \times 10^7$ s is incorrect by more than a factor of 2 and the solution varies erratically in amplitude for all $\tau$ (note the logarithmic vertical axis of Figure 2.5). For $z \gtrsim 0.85$ the algorithm is completely unstable. Although the solutions at all $z$ correctly realise the temporal initial conditions enforced by equations (2.23) and (2.24), asserting those conditions within a time interval demonstrating strong Gibbs ringing is a physically meaningless exercise.

We turn now in Section 2.5 to derive a new Fourier representation of the MB equations which yields meaningful transients immune to the temporal boundary Gibbs ringing phenomenon.
Figure 2.5: Intensity transients generated by a TML simulation of the experiment, with LMI approximation truncated to 30 modes. Note that transients are recovered in the time domain by performing an inverse FFT on the modes computed via the TML algorithm. Top: transients shown for $0.70L \leq z \leq 0.80L$, linear vertical axis. Bottom: transients shown for $0.0L \leq z \leq 0.6L$, logarithmic vertical axis. Gibbs ringing phenomena are visible in the logarithmic plots near the temporal boundaries.
2.5 The Integral Fourier Representation of the Maxwell-Bloch Equations

In this section we derive a new Fourier representation of the MB equations that is manifestly distinct from the ML algorithm. This new representation is essentially the Fourier series of the integral form of the MB equations, and is therefore referred to as the integral Fourier (IF) representation. Our IF representation naturally facilitates the enforcement of initial conditions, yields solutions converging to the optimal Fourier series representation of the correct time domain solution, contains a translation of the LMI approximation, and achieves $O(N)$ complexity in the number of velocity channels $N$.

2.5.1 The Integral Fourier representation of a general first-order temporal propagation equation

We first derive the integral Fourier representation of a generic first-order temporal propagation problem, expressed as

$$\frac{dF}{dt} = G[F(t), H(t)], \quad (2.25)$$

where we seek to find the temporal propagation of $F$ from some specified initial conditions, given some generating expression $G$ (which may potentially involve $F$) and (importantly) working exclusively within Fourier series expansions of the unknown quantities. Let $G$ be expanded as

$$G(t) = \sum m G_m e^{im\omega t}, \quad (2.26)$$

where $d\omega = 2\pi/T$ for a simulation duration $T$, and where the $G_m$ may potentially involve the coefficients $F_m$ and $H_m$ in the Fourier series expansions of the solution $F$ and the forcing function $H$,

$$F(t) = \sum m F_m e^{im\omega t}, \quad (2.27)$$

$$H(t) = \sum m H_m e^{im\omega t}. \quad (2.28)$$
The usual Fourier representation of equation (2.25) appropriate to steady state modelling is obtained by equating mode expansion coefficients upon insertion of equations (2.26)–(2.28) into equation (2.25). Motivated by our desire to introduce initial conditions into the Fourier representation, we instead first integrate both sides of equation (2.25); the left side introduces $F(t = 0)$ as desired, while the right side we may analytically integrate in its Fourier representation. In equations, we have

\[ F(t) - F(0) = \int_0^t dt' \sum_m G_m e^{im\omega t'} \]  

(2.29)

\[ = \int_0^t dt' G_0 + \sum_{m \neq 0} \int_0^t dt' G_m e^{im\omega t'} \]  

(2.30)

\[ = G_0 t + \sum_{m \neq 0} \frac{G_m}{im\omega} (e^{im\omega t} - 1). \]  

(2.31)

Let us now express the function $t$ in a Fourier series as $t = \sum_m T_m e^{im\omega t}$ (we will explicitly compute $T_m$ in a moment). We express $F$ on the left side in its Fourier series to obtain

\[ \sum_m F_m e^{im\omega t} - F(0) = G_0 \sum_m T_m e^{im\omega t} + \sum_{m \neq 0} \frac{G_m}{im\omega} (e^{im\omega t} - 1), \]  

(2.32)

or

\[ F_0 - F(0) + \sum_{m \neq 0} F_m e^{im\omega t} = G_0 T_0 - \sum_{m \neq 0} \frac{G_m}{im\omega} + \sum_{m \neq 0} \left( G_0 T_m + \frac{G_m}{im\omega} \right) e^{im\omega t}. \]  

(2.33)

Equating mode coefficients of equal frequencies generates the algebraic relations

\[ F_0 = F(0) + G_0 T_0 - \sum_{m \neq 0} \frac{1}{im\omega} G_m \]  

(2.34)

\[ F_m = G_0 T_m + \frac{1}{im\omega} G_m \]  

for $m \neq 0$.  

(2.35)

We compute $T_m$ by the usual Fourier expansion coefficient calculation,

\[ T_m = \frac{1}{T} \int_0^T t e^{-im\omega t} dt = \begin{cases} \frac{\pi}{d\omega} & m = 0 \\ \frac{i}{md\omega} & m \neq 0. \end{cases} \]  

(2.36)
2.5. The Integral Fourier Representation of the Maxwell-Bloch Equations

Equations (2.34) and (2.35) now simplify to

\[ \mathbb{F}_0 = F(0) + \sum_m T_m \mathbb{G}_m \]  
\[ \mathbb{F}_m = T_m (\mathbb{G}_0 - \mathbb{G}_m) \text{ for } m \neq 0. \]  

We refer to equations (2.37) and (2.38) as the integral Fourier (IF) representation of the first-order temporal propagation equation (2.25). Note that if the expression \( \mathcal{G} \) of equation (2.25) contains \( F \), then the \( \mathbb{G}_m \) contain the \( \mathbb{F}_m \) on the right sides of equations (2.37) and (2.38), so that they must be solved as a linear system in the unknowns \( \mathbb{F}_m \).

2.5.2 The Integral Fourier representation of the Maxwell-Bloch equations

We seek now to cast the MB equations into the IF representation. Recognising the left sides of equations (2.13) and (2.14) as derivative operators acting on the inversion and polarisation, we can immediately infer from equations (2.13) and (2.14) that if

\[ \frac{\partial N_p}{\partial \tau} = \sum_m e^{im\omega \tau} \mathbb{G}_m^{(N)} \]  
\[ \frac{\partial \mathbb{P}^+}{\partial \tau} = \sum_m e^{im\omega \tau} \mathbb{G}_m^{(P)}, \]  

then

\[ \mathbb{G}_m^{(N)} = \frac{i}{\hbar} \sum_m (\mathbb{F}_{p,m}^+ \mathbb{E}_{m-m+p}^+ - \mathbb{F}_{p,m}^- \mathbb{E}_{m+m+p}^-) - \frac{N_{p,m}}{T_1} + \mathbb{I}^{(N)}_m \]  
\[ \mathbb{G}_m^{(P)} = \frac{2i d^2}{\hbar} \sum_m (N_{p,m} \mathbb{E}_{m-m+p}^-) - \frac{\mathbb{F}_{p,m}^+}{T_2} + \mathbb{L}^{(P)}_m. \]

Comparing equations (2.39)–(2.42) to the generic form of equation (2.26), we can immediately apply our IF representation formulae of equations (2.37) and (2.38) to obtain the IF
representation of the MB equations,

\[
\begin{align*}
N_{p,0} &= N_p(0) + \sum_m T_m \left[ \frac{i}{\hbar} \sum_{\bar{m}} \left( \tilde{F}_{p,\bar{m}}^+ B_{\bar{m}-m+p}^+ - \tilde{F}_{p,\bar{m}}^- B_{\bar{m}-m+p}^- \right) - \frac{N_{p,m}}{T_1} + \mathbb{I}^{(N)}_m \right] \\
N_{p,m \neq 0} &= T_m \left( \frac{i}{\hbar} \sum_{\bar{m}} \left[ \tilde{F}_{p,\bar{m}}^+ \left( B_{\bar{m}+p}^+ - B_{\bar{m}-m+p}^+ \right) - \tilde{F}_{p,\bar{m}}^- \left( B_{\bar{m}+p}^- - B_{\bar{m}-m+p}^- \right) \right] \\
&\quad + \frac{1}{T_1} \left( N_{p,m} - N_{p,0} \right) + \left( \mathbb{I}^{(N)}_0 - \mathbb{I}^{(N)}_m \right) \right) \\
\mathbb{E}_{p,0}^+ &= \mathbb{E}_{p}^+ (0) + \sum_m T_m \left[ \frac{2id^2}{\hbar} \sum_{\bar{m}} \left( N_{p,\bar{m}} B_{\bar{m}-m+p}^- \right) - \frac{\mathbb{E}_{p,m}}{T_2} + \mathbb{I}^{(P)}_m \right] \\
\mathbb{E}_{p,m \neq 0}^+ &= T_m \left[ \frac{2id^2}{\hbar} \sum_{\bar{m}} N_{p,\bar{m}} \left( B_{\bar{m}+p}^- - B_{\bar{m}-m+p}^- \right) + \frac{1}{T_2} \left( \mathbb{E}_{p,m} - \mathbb{E}_{p,0}^+ \right) + \left( \mathbb{I}^{(P)}_0 - \mathbb{I}^{(P)}_m \right) \right].
\end{align*}
\]  

At a given \( z \) position with known electric field modes \( B_{\bar{m}}^\pm \), the above system of equations can be solved for the inversion and polarisation Fourier modes. Note that the electric field modes are propagated forward starting from \( z = 0 \) via equation (2.15).

**The local mode interaction approximation and numerical considerations**

The LMI approximation translates naturally to the IF representation under our Doppler shifted envelope factorisation of the polarisations. Equations (2.43)–(2.46) are formulated such that the LMI approximation is realised by restricting all occurrences of \( \bar{m} \) to \( \bar{m} \in [-N_{\text{int}}, +N_{\text{int}}] \) for a desired mode interaction truncation distance \( N_{\text{int}} \).

The spectral limiting of the \( m \) index in each of \( N_{p,m} \) and \( \mathbb{E}_{p,m}^\pm \) achieves \( O(N) \) complexity in the number of velocity channels \( N \); however, an additional efficiency is gained by exchanging summation orders in equations (2.43) and (2.45) and introducing the array

\[
\Xi_{a}^\pm = \sum_m T_m B_{a+m}^\pm,
\]  

(2.47)
such that equations (2.43) and (2.45) become

\[ \mathbb{N}_{p,0} = N_p (0) + \frac{i}{\hbar} \sum_m (\Xi^+_{m+p,\hbar} - \Xi^-_{m+p,\hbar}) + \sum_m T_m \left( \frac{\mathbb{L}_m(N) - \mathbb{N}_{p,m}}{T_1} \right) \]  

(2.48)

\[ \tilde{\mathcal{P}}_{p,0}^+ = \tilde{\mathcal{P}}_{p}^+(0) + \frac{2i d^2}{\hbar} \sum_m \Xi^-_{p-m}\mathbb{N}_{p,\hbar} + \sum_m T_m \left( \frac{\mathbb{L}_m^{(P)} - \tilde{\mathcal{P}}_{p,m}^+}{T_2} \right) \]  

(2.49)

At a given \( z \) position, the \( \Xi^\pm_a \) need only be calculated once, and may then be re-used to generate each linear system to be solved for each velocity channel of mode \( p \). Note that because the inversion is real, its modes are related via \( \mathbb{N}_{p,m} = \mathbb{N}_{p,-m}^* \), which reduces both the number of unknowns and the extents of inversion mode summations. After applying such simplifications, the real and imaginary representation of the IF system of equations is provided in Appendix B.

It is helpful to define and clarify the ranges of all indices in equations (2.48) and (2.49), as well as in the definition of \( \Xi^\pm_a \). We allow the velocity channel index \( p \) to vary above and below the on-resonance \( p = 0 \) central channel by the side channel distance \( N_{\text{sch}} \), so that \( p \in [-N_{\text{sch}}, +N_{\text{sch}}] \) for a total velocity channel count of \( 2N_{\text{sch}} + 1 \). The mode index \( m \) of the \( p \)th channel’s inversion (or polarisation) expansion coefficients \( \mathbb{N}_{p,m} (\tilde{\mathcal{P}}^\pm_{p,m}) \) varies above and below the \( m = 0 \) central mode by the side mode distance \( N_{\text{sm}} \) reflecting the degree of spectral limiting applied, so that \( m \in [-N_{\text{sm}}, +N_{\text{sm}}] \) for a total inversion (polarisation) mode count of \( 2N_{\text{sm}} + 1 \). The electric field mode index \( m \) of \( \Xi^\pm_m \) is judiciously limited to \( m \in [-N_{\text{sch}} + N_{\text{sm}}, +N_{\text{sch}} + N_{\text{sm}}] \) for a total electric field mode count of \( 2(N_{\text{sch}} + N_{\text{sm}}) + 1 \). All occurrences of the index \( \hat{m} \) vary over the interaction distance \( N_{\text{int}} \) such that \( \hat{m} \in [-N_{\text{int}}, +N_{\text{int}}] \). As a result of the above chosen ranges, the index \( a \) of the \( \Xi^\pm_a \) array extends over \( a \in [-N_{\text{sch}}, +N_{\text{sch}} + N_{\text{int}}] \).

### 2.5.3 Simulation

The result of simulating the same system of Figure 2.3 now with the IF algorithm is shown in Figure 2.6, where the inversion and polarisation are spectrally limited to \( N_{\text{sm}} = 50 \) and the LMI approximation is truncated to \( N_{\text{int}} = 30 \) neighbouring modes.

In contrast to the ML simulation of Figure 2.5, the IF simulation provides sustained accu-
Figure 2.6: Intensity transients generated by an IF simulation of the experiment, with LMI approximation interaction truncated to 30 modes. Top: transients shown for $0.7L \leq z \leq 0.8L$, linear vertical axis. Bottom: transients shown for $0.0L \leq z \leq 0.6L$, logarithmic vertical axis. Reference time domain simulations are superimposed in dotted lines.
2.5. The Integral Fourier Representation of the Maxwell-Bloch Equations

Figure 2.7: Intensity transients generated by an IF simulation of the experiment, with LMI approximation interaction truncated to 30 modes. Transients shown for $0.7L \leq z \leq L$. Reference time domain simulations are superimposed in dotted lines.

Despite the Gibbs ringing near the temporal boundaries of Figures 2.6 and 2.7, the total transient shape appears insensitive to such artifacts. The Gibbs phenomenon is also visible in the population inversion transients; we show in Figure 2.8 the transients of the population inversion of the central velocity channel at various positions along the length of the sample. The accuracy of the IF algorithm is again verified in Figure 2.8 where, importantly, the total response is unaffected by the aggressive Gibbs ringing at the boundaries. For completeness, we show in Figure 2.9 the imaginary part of the polarisation of the central velocity channel at various positions along the length of the sample (in this particular simulation the real part of the central velocity channel’s polarisation is negligible).
Figure 2.8: Population inversion transients generated by an IF simulation of the experiment, with LMI approximation interaction truncated to 30 modes. Reference time domain simulations are superimposed in dotted lines. Note that the molecular density used for the $\Delta v$ of this simulation corresponds to an inverted molecular density on the order of $0.1 \text{ cm}^{-3}$ for a realistic velocity distribution of $\Delta v \approx 1 \text{ km/s}$.

Figure 2.9: Imaginary part of polarisation transients generated by an IF simulation of the experiment, with LMI approximation interaction truncated to 30 modes. Reference time domain simulations are superimposed in dotted lines.
2.5.4 Local mode interaction fidelity requirements

We turn now to quantify the effect of reducing the LMI fidelity upon the accuracy of the SR transients generated. In Figure 2.10 we plot the intensity transients for varying LMI truncation extents at two positions along the length of the sample. At $z = 0.6L$ the system does not yet demonstrate SR; i.e., there is no loss of population inversion until $z \approx 0.7L$, as can be seen in Figure 2.8. Conversely, at $z = L$ the system generates a strong SR pulse with significant and fast variations in the population inversion, as well as a peak SR intensity greatly exceeding the steady state value.

We observe that the peak intensity at $z = 0.6L$ remains very accurate (it tracks well with the $N_{\text{int}} = 30$ case) down to an LMI truncation extent as low as $N_{\text{int}} \approx 10$, and reasonably accurate down to $N_{\text{int}} \approx 5$. Conversely, the SR transient at $z = L$ acquires a moderate error already at $N_{\text{int}} = 20$, a significant error at $N_{\text{int}} = 10$, and becomes completely unstable at $N_{\text{int}} = 5$. These observations suggest that the LMI fidelity requirements increase as the system enters the transient SR domain.

Let us compare the bandwidth of the SR pulse in the bottom plot of Figure 2.10 to the LMI truncation extent. The pulse has a duration on the order of $\sim T/20$ (note that $T = 1 \times 10^8$ s, despite the reduced plot viewing widths), or a bandwidth of $\Delta \omega \approx 20 (2\pi/T)$. The natural angular frequencies of adjacent velocity channels are separated by an amount $2\pi/T$, according to our prescription of a channel separation $d\nu = (c/\omega_0) 2\pi/T$. The bandwidth of the SR pulse thus covers the bandwidth of approximately 20 velocity channels, which is the value of $N_{\text{int}}$ at which the approximated transient begins to depart from the true transient. This suggests that the LMI velocity channel truncation extent, interpreted according to the equivalent Doppler shift across the interacting channels, must exceed the bandwidth of the transient generated.
Figure 2.10: IF simulation intensity transients at $z = 0.6L$ (top) and $z = L$ (bottom) for varying degrees of LMI approximation fidelity. The $N_{\text{int}} = 30$ plots (solid lines) may serve as reference true transients, being effectively coincident with the (correct) transients generated by a time domain simulation. Significant deviations begin to emerge below $N_{\text{int}} \approx 5$ at $z = 0.6L$ (low field strength), and below $N_{\text{int}} \approx 20$ at $z = L$ (high field strength).
2.6 Discussion

2.6.1 Conclusions

The ML algorithm was originally developed by Menegozzi and Lamb [23] (and extended by Dinh-V-Trung [7] and Dinh-V-Trung [8]) to investigate noise propagation and formation of coherence in a quasi-steady state maser process. Motivated by the advantageous $O(N)$ complexity (in the velocity channel count $N$) achieved by the ML algorithm over the $O(N^2)$ complexity of a time domain simulation of the MB equations with a velocity distribution, we have investigated the application of the ML algorithm to the modelling of SR transient processes.

The ML algorithm accurately describes the quasi-steady state maser regime. After a minor revision to its algebraic mode relations (a replacement of the $m = 0$ cases with initial condition assertions), it also describes weak field transients (see $z \lesssim 0.6L$ in Figure 2.5); however, we have demonstrated it to be unacceptably sensitive to Gibbs phenomena in the case of strong field SR transients ($z \gtrsim 0.7L$ in Figure 2.5). The transient ML algorithm does not accurately converge to the true transient if the amplitude of the Gibbs ringing exceeds the magnitude of the initial conditions asserted, and it is therefore unable to describe SR transient processes where the loss in population inversion caused by the SR cascade causes the temporal initial and final configurations to differ substantially.

We have developed a manifestly unique Fourier representation of the MB equations which lends itself naturally to the assertion of initial conditions and which accurately models all SR transient processes described by the MB equations. The IF algorithm is robustly insensitive to Gibbs phenomena, yielding total transients which accurately replicate key SR features such as peak intensity delay time and peak intensity amplitude. Most importantly, the spectral limiting and LMI approximations of the ML algorithm translate naturally into the IF algorithm, so that the latter is also $O(N)$ complex in the number of velocity channels $N$.

We have observed that the fidelity requirements of the LMI approximation made in either the ML representation or the IF representation of the MB equations increase as a system approaches high field strength regimes. We suggest that the fidelity of the LMI approximation used when simulating SR transients in the IF representation be such that the natural frequency extent of the spread in velocity channel interactions exceeds the bandwidth of the transient
response.

2.6.2 Limitations of the Integral Fourier Method

In a footnote to Section 2.2.3, it was emphasised that although the initial Bloch tipping angle is prescribed by the number of interacting molecules $N_{\text{mol int}}$, such number is not known \textit{a priori} when simulating across a wide velocity distribution. In fact, $N_{\text{mol int}}$ could more accurately be estimated from simulations enabled by this paper, which would provide an estimate of the local number of coherent interacting neighbouring molecules within a subset of the full global velocity distribution. Alternatively, the purely quantum mechanical arguments of Gross and Haroche [16] which inform the initial Bloch angle prescription could be re-examined in the context of a velocity distribution, with the objective of obtaining a revised expression for $N_{\text{mol int}}$. In most astrophysical situations, however, other radiative processes are of sufficient intensity to render the initial tipping angle irrelevant. In Rajabi and Houde [26], for example, the background radiation of the interstellar medium is shown to dominate over any initial radiation field resulting from the non-zero initial tipping angle.

On a numerical efficiency note, it should be pointed out that although the IF algorithm achieves $O(N)$ complexity in the number of velocity channels $N$ simulated (compared to $O(N^2)$ for the time domain method), it does not overtake the time domain method until approximately $N > 100$. The simplicity of the time domain method, owing in part to its lack of matrix inversion operations, therefore makes it the superior choice of algorithm for simulating nearly coherent velocity distributions.

2.6.3 Future Work

This paper is a proof of operation of the IF algorithm, and all simulations were performed over a velocity extent demanding a numerical simulation duration as brief as $\sim 40$ seconds when solving in the time domain on a modern (2020 quad core) CPU over a $500 \times 500$-point $(n_\tau \times n_z)$ grid. These velocity extents are numerically non-trivial and serve as valid demonstrations of proper convergence to the reference time domain solution, but at present remain physically trivial ($\sim 10^{-6}$ cm/s for the samples simulated in Section 2.5). This paper enables future re-
2.6. Discussion

search to apply the $O(N)$ complex IF algorithm to physically realistic velocity distributions for the study of SR processes in astrophysical gases.

The objective of future simulations will be to characterise the degree of coherence expected of transient astrophysical SR events generated from wide velocity distributions of molecules. It is the hope that the coherence characteristics predicted by these simulations will constitute a metric by which to ascertain the presence of transient SR events within observational data.

Simulations of SR processes across wide velocity distributions will also provide realistic corrections to physical requirements for a system to demonstrate SR, as well as to characteristic features of SR transients. Such parameters include the critical column density threshold required to initiate an SR transient event, the time delay $\tau_D$ to peak intensity of the SR pulse, and the characteristic time scale $T_R$ for the dissipation of energy from the system.

Additionally, the IF simulation enables investigations of noise propagation along the length of a sample during a transient SR process. In the original work of Menegozzi and Lamb [23], an incident electric field possessing a broad and decoherent spectrum of Gaussian white noise was set incident upon the $z = 0$ face of a one-dimensional maser sample, and the coherence of the emerging radiation at the end-fire $z = L$ was evaluated. Similarly, the IF simulation enables future research to investigate the relationship between the statistics of incident radiation noise at $z = 0$ and the coherence of emerging radiation at $z = L$ when the relevant collective emission process occurring within the sample is that of a transient SR event.

Finally, a possible generalisation of the IF algorithm introduced briefly near the end of Section 2.2.5 warrants elaboration. The simulations of this paper were executed with a fixed LMI approximation fidelity for all $z$; however, the velocity channel interaction distance $N_{\text{int}}$ could, in theory, be made to vary as a function of $z$ (or more precisely, as a function of the local degree of saturation). The present code fixes a global $N_{\text{int}}$ according to the highest degree of saturation occurring at $z = L$. Conversely, for $z$ moderately less than $L$, lower degrees of saturation reduce the LMI approximation fidelity requirements. A reduction in $N_{\text{int}}$ at such positions would reduce the numerical complexity of solving the coupled system of equations (2.43)–(2.46) within the majority of the sample, and could feasibly improve computation speed by an order of magnitude.
**Data Availability Statement**

The data pipeline is made available at:

https://github.com/cwyenber/MandL-to-Superradiance

and maintained by C.M.W. The figures in this paper were prepared using the matplotlib package [19].
Appendix A

Formal Justification of the LMI Truncation Range in the Integral Fourier Algorithm

We formally justify the centering of the LMI approximation about \( \bar{m} = 0 \) on the summation index \( \bar{m} \) of equations (2.13) and (2.14) via comparison to a perturbative solution in the electric field strength. Consider first a rearrangement of equations (2.13) and (2.14) into the form

\[
\bar{N}_{p,m} = \left( imd \omega + \frac{1}{T_1} \right)^{-1} \left[ \frac{i}{\hbar} \sum_{m} \left( \bar{P}^+_{p,m} \bar{E}^+_{p+m} - \bar{P}^-_{p,m} \bar{E}^-_{p+m} \right) + \mathbb{L}^{(N)} \right] \\
\bar{\bar{P}}^+_{p,m} = \left( imd \omega + \frac{1}{T_2} \right)^{-1} \left[ \frac{i2d^2}{\hbar} \sum_{m} \left( \bar{N}_{p,m} \bar{E}^-_{p+m} \right) + \mathbb{L}^{(P)} \right].
\]

(A.1)  (A.2)

Let us define vectors of inversion modes and polarisation modes for the \( p^{\text{th}} \) velocity channel as

\[
\begin{align*}
\vec{N} &= \begin{pmatrix}
\vdots \\
N_{p,-1} \\
N_{p,0} \\
N_{p,1} \\
\vdots
\end{pmatrix} ; \\
\vec{\bar{P}} &= \begin{pmatrix}
\vdots \\
\bar{P}^+_{p,-1} \\
\bar{P}^+_{p,0} \\
\bar{P}^+_{p,1} \\
\vdots
\end{pmatrix} ; \\
\vec{\bar{\bar{P}}} &= \begin{pmatrix}
\bar{N} \\
\bar{\bar{P}}^+
\end{pmatrix}.
\end{align*}
\]

(A.3)
so that equations (A.1) and (A.2) may be expressed as

\[ \vec{x} = \vec{M}_E \vec{x} + \vec{b} \]  

(A.4)

where

\[
\vec{M}_E = \begin{bmatrix}
0 & \frac{\vec{E}}{(im\omega + \frac{1}{T_1})} & -\frac{\vec{E}^*}{(im\omega + \frac{1}{T_1})} \\
2d^2 \frac{\vec{E}^*}{(im\omega + \frac{1}{T_2})} & 0 & 0 \\
-2d^2 \frac{\vec{E}}{(-im\omega + \frac{1}{T_2})} & 0 & 0
\end{bmatrix}
\]  

(A.5)

for submatrices \( \vec{E} \) and \( \vec{E}' \) having elements

\[
\left[ \vec{E} \right]_{m,\bar{m}} = \mathbb{E}^{+}_{p+\bar{m}-m} \quad \text{and} \quad \left[ \vec{E}' \right]_{m,\bar{m}} = \left[ \vec{E}^* \right]_{-m,\bar{m}},
\]

(A.6)

and where \( * \) denotes the complex conjugate and \( \dagger \) the adjoint. In deriving the submatrix in the third row and first column of equation (A.5), we took the complex conjugate of equation (A.2), recognised that \( N^*_p,\bar{m} = N_{p,-\bar{m}}, \) and made a change of summation variable \( \bar{m} \rightarrow -\bar{m}. \) The vector \( \vec{b} \) has upper elements

\[
\left[ \vec{b} \right]_{m,\text{upper}} = \mathbb{J}^{(N)}_m \left( im\omega + \frac{1}{T_1} \right)^{-1},
\]

(A.7)

middle elements

\[
\left[ \vec{b} \right]_{m,\text{middle}} = \mathbb{J}^{(P)}_m \left( im\omega + \frac{1}{T_2} \right)^{-1},
\]

(A.8)

and lower elements

\[
\left[ \vec{b} \right]_{m,\text{lower}} = \mathbb{J}^{(P)*}_m \left( -im\omega + \frac{1}{T_2} \right)^{-1}.
\]

(A.9)

A perturbative solution to equation (A.4) in increasing powers of the matrix \( \vec{M}_E \) (i.e., in increasing powers of the electric field strength) may be identified from inspection to be

\[ \vec{x} = \left( \vec{1} + \vec{M}_E + \vec{M}_E^2 + \vec{M}_E^3 + \ldots \right) \vec{b} \]  

(A.10)

(to verify, substitute into both sides of equation (A.4) and observe equality to all orders in \( \vec{M}_E \)).

We are now in a position to argue that our decision to truncate summations over \( \bar{m} \) about
the central value $\bar{m} = 0$ in Section 2.2.5 was made in order to maintain consistency with results of first order perturbation in the field strength. Consider equation (A.10) truncated to first order in $\vec{M}_E$. For pumps constant in the time domain, only their zeroth modes vanish; i.e., $\mathbb{L}_m^{(N/P)} = \mathbb{L}_0^{(N/P)} \delta_{m,0}$. Thus each of expressions (A.7)–(A.9) contains a non-vanishing value in only its $m = 0$ element.

To first order, equation (A.10) acts on each of $\vec{b}_{\text{upper}}, \vec{b}_{\text{middle}},$ and $\vec{b}_{\text{lower}}$ with matrix multiplication by first powers of the submatrices $\vec{E}, \vec{E}^*, \vec{E}^\dagger$ and $\vec{E}'$. Such multiplication involves summation over the $\bar{m}$ column index of expressions (A.6). As per the observation of the prior paragraph, the $\vec{b}$ vectors possess non-vanishing values in only their $m = 0$ elements; consequently, matrix multiplication upon them remains correct when the column summation of the matrix multiplication operation is truncated to only the $\bar{m} = 0$ column. Comparing expressions (A.6) to equations (A.1) and (A.2), we see that this statement concerning matrix multiplication is equivalent to the statement that summations over $\bar{m}$ in equations (A.1) and (A.2) be truncated to $\bar{m} = 0$.

This constitutes the formal argument that the LMI approximation’s summation truncation range should be centered about $\bar{m} = 0$ in order to maintain consistency with first order perturbation results; this observation removes the centering ambiguity in the introduction of the LMI approximation in Section 2.2.5. Suppose, for example, that we had alternatively expressed the first term on the right side of equation (2.13) by a less judicious choice of summation variable $m' = \bar{m} - m$, such that

\begin{align}
(\text{imd}\omega) N_{p,m} = & \frac{i}{\hbar} \sum_m \left( \vec{F}_{p,m}\vec{B}_{p+m-m}^+ - \cdots \right) \\
= & \frac{i}{\hbar} \sum_{m'} \left( \vec{F}_{p,m'+m}\vec{B}_{p+m'}^+ - \cdots \right). 
\end{align}

(A.11) (A.12)

Had we naively proposed that the LMI approximation be achieved by truncating summation about $m' = 0$ (i.e., about field modes centered upon the natural Doppler shifted resonance of the velocity channel—a very reasonable proposition), we would have violated consistency with first order perturbation results.
Appendix B

Integral Fourier Mode Relations in their Real and Imaginary Parts

We present below the full expression of the IF mode relations in their real and imaginary parts, where all algebra eliminating negative modes of the inversion has been completed. In the summations over $\bar{m}$, the symbol $\mathcal{T}$ denotes the truncated range $[-N_{\text{int}}, +N_{\text{int}}]$ and the symbol $\mathcal{T}^+$ denotes the positive truncated range $[+1, +N_{\text{int}}]$. Indices $m$ and $p$ span those ranges described in Section 2.5.2.

\begin{equation}
md\omega N^R_{p,0} = \frac{1}{\hbar} \sum_{\bar{m} \in \mathcal{T}} \left[ \frac{\bar{N}^R_{p,\bar{m}}}{\bar{m}} \left( E^R_{\bar{m} + m + p} - E^R_{\bar{m} - m + p} \right) + \frac{\bar{N}^I_{p,\bar{m}}}{\bar{m}} \left( E^I_{\bar{m} + m + p} - E^I_{\bar{m} - m + p} \right) \right] - \frac{N^R_{p,0}}{T_1} + L_m^{(N)R} \tag{B.1}
\end{equation}

\begin{equation}
md\omega N^I_{p,0} = \frac{1}{\hbar} \sum_{\bar{m} \in \mathcal{T}} \left[ \frac{\bar{N}^R_{p,\bar{m}}}{\bar{m}} \left( E^R_{\bar{m} + m + p} - E^R_{\bar{m} - m + p} \right) + \frac{\bar{N}^I_{p,\bar{m}}}{\bar{m}} \left( E^I_{\bar{m} + m + p} - E^I_{\bar{m} - m + p} \right) \right] - \frac{N^R_{p,0}}{T_1} + L_m^{(N)I} \tag{B.2}
\end{equation}

\begin{equation}
md\omega N^I_{p,m} = \frac{1}{\hbar} \sum_{\bar{m} \in \mathcal{T}} \left[ \frac{\bar{N}^R_{p,\bar{m}}}{\bar{m}} \left( E^I_{\bar{m} + m + p} + E^I_{\bar{m} - m + p} - 2E^I_{\bar{m} + m + p} \right) + \frac{\bar{N}^I_{p,\bar{m}}}{\bar{m}} \left( E^R_{\bar{m} + m + p} + E^R_{\bar{m} - m + p} - 2E^R_{\bar{m} + m + p} \right) \right] + \frac{1}{T_1} \left( N^R_{p,m} - N^R_{p,0} \right) + L_0^{(N)R} - L_m^{(N)R} \tag{B.3}
\end{equation}
\[ d\omega \left( \tilde{\Phi}_p(0) - \Phi_p(0) \right) = \frac{2d^2d\omega}{\hbar} \left\{ N_{R,0}^{\Xi} + \sum_{i \in T^+} \left[ N_{R,i}^{\Xi} \left( \Xi_{p-i}^{\Xi} + \Xi_{p+i}^{\Xi} \right) + N_{R,i}^{\Xi} \left( \Xi_{p-i}^{\Xi} - \Xi_{p+i}^{\Xi} \right) \right] \right\} \]

\[- \frac{\pi}{T_2} \tilde{\Phi}_p(0) + \sum_{i \in T^+} \frac{1}{mT^2} \tilde{\Phi}_{p,i} + \pi^{(P)\Xi}_{0} \sum_{m \neq 0} \frac{1}{m} \left\{ \begin{array}{l} \Xi_{p-i}^{\Xi} - \Xi_{p+i}^{\Xi} \end{array} \right\} \]

(B.4)

\[ d\omega \left( \tilde{\Phi}_p(0) - \Phi_p(0) \right) = \frac{2d^2d\omega}{\hbar} \left\{ N_{R,0}^{\Xi} + \sum_{i \in T^+} \left[ N_{R,i}^{\Xi} \left( \Xi_{p-i}^{\Xi} + \Xi_{p+i}^{\Xi} \right) + N_{R,i}^{\Xi} \left( \Xi_{p-i}^{\Xi} - \Xi_{p+i}^{\Xi} \right) \right] \right\} \]

\[- \frac{\pi}{T_2} \tilde{\Phi}_p(0) + \sum_{i \in T^+} \frac{1}{mT^2} \tilde{\Phi}_{p,i} + \pi^{(P)\Xi}_{0} \sum_{m \neq 0} \frac{1}{m} \left\{ \begin{array}{l} \Xi_{p-i}^{\Xi} - \Xi_{p+i}^{\Xi} \end{array} \right\} \]

(B.5)

\[ m \omega \tilde{\Phi}_{p,m} = \frac{2d^2}{\hbar} \left\{ N_{R,0}^{\Xi} \left( \Xi_{m+p}^{\Xi} - \Xi_{p}^{\Xi} \right) + \sum_{i \in T^+} \left[ N_{R,i}^{\Xi} \left( \Xi_{m+i+p}^{\Xi} + \Xi_{m-i+p}^{\Xi} - \Xi_{m+i}^{\Xi} - \Xi_{m-i}^{\Xi} \right) \right] \right\} + \frac{1}{T_2} \left( \tilde{\Phi}_{p,0}^{\Xi} - \tilde{\Phi}_{p,m}^{\Xi} \right) + \pi^{(P)\Xi}_{m} \pi^{(P)\Xi}_{0} \]

(B.6)

\[ m \omega \tilde{\Phi}_{p,m} = \frac{2d^2}{\hbar} \left\{ N_{R,0}^{\Xi} \left( \Xi_{p}^{\Xi} - \Xi_{m+p}^{\Xi} \right) + \sum_{i \in T^+} \left[ N_{R,i}^{\Xi} \left( \Xi_{m+i+p}^{\Xi} + \Xi_{m-i+p}^{\Xi} - \Xi_{m+i}^{\Xi} - \Xi_{m-i}^{\Xi} \right) \right] \right\} + \frac{1}{T_2} \left( \tilde{\Phi}_{p,0}^{\Xi} - \tilde{\Phi}_{p,m}^{\Xi} \right) + \pi^{(P)\Xi}_{m} \pi^{(P)\Xi}_{m} \]

(B.7)
Chapter 2 Bibliography


Chapter 3

Transient Structure in the Non-linear Superradiance Regime of Widely Doppler Broadened Media

A version of this chapter has been submitted to the Monthly Notices of the Royal Astronomical Society as


3.1 Introduction

In the quantum phenomenon of superradiance (SR) a population of two-level molecules evolves, through its interaction with the quantized radiation field, into a composite excitation state possessing a high degree of entanglement. This highly entangled state couples strongly to the radiation field and produces cooperative emission of much greater intensity than would otherwise be generated by the independent spontaneous emission of the constituent molecules. SR was first described theoretically by Dicke [4] and was verified experimentally in an HF gas by Skribanowitz et al. in 1973 [17]. It has more recently been applied to astrophysical processes [13, 7, 8, 12], including the description of flux transient features observed in maser-harbouring regions [14, 15, 16, 11] where velocity coherence conditions within the environment foster
evolution into the entangled state prerequisite to SR emission.

Indeed, velocity coherence is of paramount importance to the development of SR. A collection of stationary two-level molecules is able to evolve into a state of high entanglement because of the molecules’ interactions with a common radiation field; molecules in motion, however, interact only with those field excitation states near their Doppler shifted spontaneous emission modes. Molecules of excessive velocity differences thus do not see a common radiation field and cannot be a priori assumed to evolve into a highly entangled state. In the large velocity separation limit molecules radiate as completely independent, statistically uncorrelated emitters.¹

Velocity coherence plays a central role in astrophysical applications of SR, but it has thus far been introduced in the literature at only a high level based upon back-of-the-envelope physical arguments. Previous studies have reduced the populations of molecules proposed to interact coherently to those sub-populations of the full velocity distribution which share a neighbourhood of bandwidth established by the SR transient pulse duration. Despite the fact that such a velocity width represents an exceedingly small fraction of the total population, the large astronomical scales (which are many orders of magnitude larger than those realised in typical laboratory experiments) have been considered sufficient to produce such a large total number of molecules that the sub-population of a narrow velocity slice can feasibly exceed the critical inverted column density threshold required to initiate SR.

In order to distinguish the present work from existing results in the literature, let us quantify the extent of Doppler broadening in astrophysical SR processes. The coherent SR emission transients described in [16], for example, were generated over a timescale \( T \sim 5 \times 10^8 \) s by molecules possessing a natural emission frequency \( f_0 = 6.7 \text{ GHz} \). The spectrum of the SR process therefore covered a bandwidth \( \Delta f \) on the order of \( 1/T \), such that the electric field generated would interact with molecules over a velocity range \( \Delta v \sim c \Delta f / f_0 \sim 10^{-8} \) m/s. Any realistic velocity distribution is certainly many orders of magnitude wider than such a velocity coupling width, and numerical simulations of such systems must therefore invoke extremely fine velocity channel discretisation in order to capture the important physics of the problem.

¹Such independence can be easily demonstrated analytically in the Schrödinger picture of two-level molecules [2], but we choose to demonstrate it in Section 3.4.1 within the framework of SR suited to the physical configurations of interest to this paper.
Were the channel discretisation width in a simulation of the aforementioned sample to exceed $10^{-6}$ m/s, for example, all channels would be a priori decoupled and the response would be the trivial uncorrelated combination of independent SR processes occurring within each velocity channel.

If a sample’s Doppler broadening is multiple orders of magnitude wider than the velocity width defined by the SR timescale, we refer to it as widely Doppler broadened (WDB); conversely, if a sample’s Doppler broadening is on the order of the velocity width defined by the SR timescale, we refer to it as narrowly Doppler broadened. We refer to a collection of molecules entirely on resonance (without velocity extent) as a resonant sample. Numerical simulations of SR under Doppler broadening exist in the literature, but have thus far been restricted to narrowly broadened processes. Such a restriction is of course sufficient for laboratory realisations of SR; MacGillivray and Feld [9], for example, modelled SR processes in an HF gas of velocity width $10^4$ cm/s, occurring over a duration of about $10^{-7}$ s. Such a duration corresponds to a velocity spread of about $3 \times 10^3$ cm/s for HF emission. The narrowly broadened system demonstrated SR with minor quantitative deviations from the peak intensity and timescales of the resonant case.

The objective of the present work is to model WDB distributions of two-level molecules with such high column densities that even narrow slices of the velocity distribution can possess population inversion levels sufficient to initiate non-linear SR processes. Such degrees of saturation differ dramatically from those of typical laboratory experiments [17] or of existing analytical solutions [2, 6, 9] (as discussed in more detail in Section 3.2.2) and are found, through numerical simulation, to yield rich features beyond minor corrections to the resonant theory. Whereas one might naturally expect the emergence of multiple locally coherent groups of velocity channels in some globally decoherent combination, we show that certain inversion processes lead to the development of wide velocity bandwidth polarisation correlation and the consequent quenching of temporal features. We demonstrate that a WDB sample responding to a transverse inversion can, however, sustain SR temporal structure if its length exceeds a critical threshold; while one responding to a swept inversion may sustain SR temporal structure only if its distribution obeys certain statistical properties.
3.1.1 Structure of the paper

The paper is structured as follows. In Section 3.2 we summarise the theoretical models used to simulate WDB SR processes and place the regime of the present work in the context of existing theoretical and experimental results for Doppler broadened SR at laboratory scales. We discuss numerical complexity scaling with distribution size and introduce efficient solution methods. One of these methods is unique to the present work; however, we summarise only its main features in Section 3.2 while leaving its mathematical derivation and details concerning its numerical implementation to Appendix C.

We commence our numerical study of WDB SR processes by first simulating transversely inverted samples in Section 3.3, whose responses are found to most closely resemble that of a resonant sample (though with some important modifications). We argue that a transverse pumping mechanism may inhibit the formation of wide bandwidth polarisation phase correlation via the so-called Arrechi-Courtens condition, whereby we recognise an autoregulation mechanism on the number of molecules cooperatively participating in the SR emission process during the transition to the WDB limit. We confirm finite temporal duration in the transversely pumped WDB SR regime via simulation, and we demonstrate it to be proportional to the sample’s length.

Conversely, a WDB SR process initiated by a swept inversion mechanism is found to deviate substantially from the resonant case; in light of its nuanced response, we therefore analyse a swept SR process in two steps. First, we simulate in Section 3.4 the swept inversion of simple (though physically unrealistic) discrete distributions; second, we simulate in Section 3.5 the more complicated (and physically realistic) case of a continuum distribution. This order of presentation allows us to build important intuition from simple models before presenting the unexpected response of a real system. We demonstrate in Section 3.5.1 the formation of wide bandwidth polarisation phase correlation in the continuum limit which we quantify as a function of the initial population inversion of the sample. We introduce in Section 3.5.2 an intuitive picture for the formation of correlation which motivates our study in Section 3.5.3 of the relationship between the stochastic characteristics of a velocity distribution and the emergence of SR temporal structure in the WDB limit.
Finally, we conclude in Section 3.6 by placing these results in the context of realistic astrophysical processes. We emphasize that the order of magnitude of an SR process timescale dictates its ability to demonstrate finite temporal features in the non-linear SR regime. We recommend candidate astrophysical phenomena for future research including, for example, events within turbulent environments or fast radio bursts.

3.2 Theory and Numerical Methods

3.2.1 The Maxwell-Bloch equations

Superradiance may be described by a semi-classical set of partial differential equations called the (Doppler broadened) Maxwell-Bloch (MB) equations. The MB equations may be derived from the Schrödinger picture density operator analysis of a collection of two-level molecules or from the Heisenberg picture of polarisation and inversion operators evolving in the presence of a semi-classical electromagnetic field. The MB equations describe the evolution of the expectation values of the coarse-grained molecular population inversion operator $\hat{N}$, the coarse-grained raising (lowering) operators $\hat{P}^+ (\hat{P}^-)$, and the photon annihilation (creation) operators $\hat{E}^+ (\hat{E}^-)$. In the classical limit such operators correspond to, respectively, the population inversion density, the forward (reverse) rotating parts\(^2\) of the polarisation, and the forward (reverse) rotating parts of the electric field. For a detailed derivation, see [6].

For simplicity and without loss of generality, we refer in all that follows to a medium interacting with the electromagnetic field through an electric dipole transition. We assume that the molecular dipole moments, the molecular polarisations, and the electric field polarisation are all aligned in a shared direction perpendicular to the $z$ axis of a one-dimensional sample.

\(^2\)If $f(z,t) = \int_{-\infty}^{\infty} f(z,\omega) e^{i\omega t} d\omega$, then $f^\pm (z,t)$ are defined as $f^- (z,t) = \int_{-\infty}^{0} f(z,\omega) e^{i\omega t} d\omega$ and $f^+ (z,t) = \int_{0}^{\infty} f(z,\omega) e^{i\omega t} d\omega$. 
The MB equations across a velocity distribution then read [6]

\[
\frac{\partial}{\partial t} + v \frac{\partial}{\partial z} N_v = \frac{i}{\hbar} (E^+ + E^-) (P^+_v - P^-_v) \tag{3.1}
\]

\[
\frac{\partial}{\partial t} + v \frac{\partial}{\partial z} P^+_v = i \omega_0 P^+_v + 2i \frac{d^2}{2 \hbar} E^- N_v \tag{3.2}
\]

\[
\left[ \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial z^2} \right] E^+ = - \frac{1}{\epsilon_0} \int dv' F_{v'} \frac{\partial^2 P^+_{v'}}{\partial t^2}, \tag{3.3}
\]

where we have partitioned molecules according to their velocity \(v\) within a distribution \(F_v\) defined such that the fraction of molecules of velocity between \(v\) and \(v + dv\) is \(F_v dv\) (with \(\int F_v dv = 1\)), and where \(d\) is the molecular dipole moment and \(\omega_0\) the angular frequency of spontaneous emission from a single molecule. Operator “hats” have been removed from all quantities above, since the MB equations describe the evolution of expectation values of all operators. Note that the formal definition of \(\hat{N}\) (omitted above for brevity) implies that the expectation value \(N_v\) corresponds, physically, to half the population inversion density.

We make a change of variables to the retarded time \(\tau = t - z/c\) and factor the polarisations and electric field functions with envelopes (Doppler shifted for the polarisations) as

\[
P^\pm_v (z, \tau) = \tilde{P}^\pm_v (z, \tau) e^{\pm i \omega_0 (1 + v/c) \tau} \tag{3.4}
\]

\[
E^\pm (z, \tau) = \tilde{E}^\pm (z, \tau) e^{\mp i \omega_0 \tau} \tag{3.5}
\]

(the primes to be removed when we normalise momentarily). The conventional theory is to eliminate the fast-rotating terms \(E^+ P^-_v\) and \(E^- P^+_v\) (the so-called “rotating wave approximation”) in equation (3.1) and recognise that \(\partial/\partial z \ll \omega_0 / c\) and \(\partial/\partial \tau \ll \omega_0\) when acting on the envelope functions, to arrive at the so-called slowly-varying envelope approximation (SVEA) of the MB equations [6],

\[
\frac{\partial N_v}{\partial \tau} = i \frac{\hbar}{2} \left( \tilde{P}^+ \tilde{E}^+ e^{i \omega_0 \tau} - \tilde{P}^- \tilde{E}^- e^{-i \omega_0 \tau} \right) - \frac{N_v - N_0}{T_1} + \Lambda^{(N)} \tag{3.6}
\]

\[
\frac{\partial \tilde{P}^+}{\partial \tau} = i \frac{2 \hbar}{\omega_0} \tilde{E}^- N_v e^{-i \omega_0 \tau} - \frac{\tilde{P}^+}{T_2} + \Lambda^{(P)} \tag{3.7}
\]

\[
\frac{\partial \tilde{E}^+}{\partial z} = i \frac{\omega_0}{2 \epsilon_0 c} \int dv' F_{v'} \tilde{P}^+_{v'} e^{-i \omega_0 \tau} \tag{3.8}
\]
for a molecular number density $2N_0$, where we have introduced population inversion and polarisation pumping sources $Λ^{(N)}(τ)$ and $Λ^{(P)}(τ)$, respectively, as well as relaxation and dephasing time scales $T_1$ and $T_2$, respectively. Additionally, we made the approximation $(1 + v'/c) \approx 1$ in our derivation of (3.8).

We may simplify equations (3.6)–(3.8) by recognising an intrinsic SR timescale

$$T_R = \frac{2ε_0hc}{d^22N_0ω_0L} = τ_{sp} \frac{8π}{3λ^22N_0L}$$

for an initial population inversion density $2N_0$, where $τ_{sp}$ is the spontaneous emission timescale of a free molecule and $L$ the length of the sample. It can be shown [2, 6] that, at least in the resonant case, both the delay $τ_D$ and duration $τ_P$ of peak SR emission intensity are proportional to $T_R$. Furthermore, if we work in the dimensionless quantities

$$Z_ω = \frac{1}{2N_0N_ω}, \bar{P}_ω^{±} = \frac{1}{d2N_0} \bar{P}_ω^{±}, \text{and } E_ω^{±} = \frac{d}{\hbar}T_RE_ω^{±};$$

whose velocity channels are now indexed by angular frequencies $ω = ω_0v/c$, then the MB equations read

$$\frac{∂Z_ω}{∂τ} = i \frac{T_R}{Z_ω} (\bar{P}_ω^{+}E_ω^{+}e^{iωτ} - \bar{P}_ω^{-}E_ω^{-}e^{-iωτ}) - \frac{Z_ω-1}{T_1} + Λ^{(Z)}$$

$$\frac{∂\bar{P}_ω^{+}}{∂τ} = 2i \frac{T_R}{T_2} E_ω^{-}Z_ω e^{-iωτ} - \bar{P}_ω^{+} + Λ^{(P)}$$

$$\frac{∂E_ω^{+}}{∂τ} = \frac{i}{2L} \int dω' F_ω' \bar{P}_ω' e^{-iω'τ}.$$  

For a long cylindrical sample of cross-sectional area $A$ and length $L$ having fresnel number $A/(Lλ)$ on the order of unity, the resonant case of equations (3.11)–(3.13) can be solved analytically [2]. The resulting theoretical transient intensity response to an initially entirely inverted total number of molecules $n_{tot}$ is characterised by peak intensity bursts of decreasing amplitude. In what follows we will scale our intensity plots to the theoretical peak intensity $I_p$ of the first burst in said analytical response given by [9, 5]

$$I_p = \frac{4n_{tot}hω_0/(AT_R)}{|ln [θ_0/(2π)]|^2}.$$
where $\theta_0 = 2/\sqrt{n_{\text{tot}}}$ as discussed further in Section 3.2.6.

### 3.2.2 The present regime in the context of existing results

This paper investigates Doppler broadening of the MB equations, but in fact any inhomogeneous broadening mechanism\(^3\) is described by equations (3.11)–(3.13), for which analytical solutions as well as numerical results exist in the literature for a variety of regimes. It is important to place the present work in the context of these known results.

One may distinguish between the linear and non-linear regimes of SR. Below a certain population inversion density threshold a sample evolves with only negligible loss of population inversion, such that the $Z_{\omega}$ may be considered constants with respect to time. For such a process the MB equations are linearised and the sample is said to be in the linear regime. Conversely, if the initial population inversion density is sufficient to cause a loss of population inversion (potentially after some delay) the system is said to be in the non-linear SR regime.

We are interested in conditions within which a WDB sample may generate high intensity emission over a finite timescale via non-linear SR processes. As investigated in [16] and [5], the quasi-steady state regime of the MB equations describes maser processes, while the fast flux rises and transient structures modelled in [15, 16] result from the non-linear SR regime of the MB equations. Although there exist analytical solutions to equations (3.11)–(3.13) for an inhomogeneously broadened system in the linear regime [2], the non-linear case must be treated numerically.

Let us make our introductory comments of Section 3.1 regarding the column densities per velocity interval within our WDB astronomical samples more precise in the context of existing results in the literature. We imagine commencing the construction of a WDB sample with a rather narrow velocity extent $\delta v$ and an initial population inversion defining a characteristic SR timescale $T_{R,i}$ (here and in the remainder of this section a subscript $i$ denotes “initial”) much less than the characteristic inhomogeneous broadening time $T_{2,i}^* = 2\pi c / (\omega_0 \delta v)$. This initial configuration is able to generate a transient SR intensity pulse of some delay $\tau_{d,i}$ and duration $\tau_{p,i}$, but as of yet represents only a negligible fraction of the full WDB velocity extent.

\(^3\)That is, any mechanism by which a distribution of spontaneous emission frequencies exists within the sample.
A conventional study of inhomogeneous broadening [2] and its effect upon laboratory systems would proceed, from this starting configuration, to redistribute the entire population of molecules over a wider $\Delta v > \delta v$ corresponding to a smaller $T_{2}'$. As $T_{2}'$ is reduced below $T_R$, both a reduction in peak intensity and an elongation of timescales to $\tau_d' > \tau_{d,i}$ and $\tau_p' > \tau_{p,i}$ are then observed. See, for example, [2, 6].

In our present investigation, however, we build upon the initial distribution of extent $\delta v$ by adding molecules further and further from resonance. This addition of molecules decreases the formal quantity $T_R$ with the increase in $N_0$ (as per equation (3.9)); however, we are interested in how the observed timescales $\tau_d$ and $\tau_p$ respond to the simultaneous reduction of $T_{2}'$ (caused by the widening of the distribution) and of $T_R$ (caused by the increase in total number of initially inverted molecules). While continuously adding molecules to the outer ends of the velocity distribution, the development of correlation between velocity channels deep inside the distribution will tell us the degree to which a sample’s SR temporal features will be reduced in the WDB limit, and will help us make realistic quantitative conclusions regarding SR emission intensities from full distributions.

### 3.2.3 Numerical complexity

Equations (3.11)–(3.13) in the time domain are $O(n^2)$ complex in the number of velocity channels $n$ simulated. As the velocity channel count increases, the temporal step size must concurrently decrease in order to avoid aliasing of the oscillating exponentials $e^{i k_0 \omega c \tau}$. For a detailed discussion see [18]. We herein refer to the simulation of the MB equations via time stepping\footnote{Via a fourth order Runge-Kutta scheme.} of the unmodified equations (3.11)–(3.13) as the conventional time domain (CTD) method.

There exist, however, (at least) two algorithms for solving equations (3.11)–(3.13) which are $O(n)$ complex under certain approximations. The first, developed in [18], is known as the integral Fourier (IF) algorithm and is summarised in Section 3.2.4. The second is unique to the present work and offers improved performance over the IF algorithm for the SR regimes investigated by this paper; it is referred to as the supplementary fields (SF) algorithm and is introduced in Section 3.2.5 but derived in Appendix C. The two algorithms share a fundamental physical approximation which enables their advantageous complexity scaling, but each has its
advantages in differing applications. We will only briefly describe the IF algorithm for the purpose of identifying its advantageous regimes; however, only the CTD and SF algorithms will be used throughout this paper.

3.2.4 The integral Fourier algorithm

The IF algorithm established in [18] is based upon a Fourier series representation of the (time) integral equations equivalent to equations (3.11)–(3.13), and is able to accurately model transient processes described by the MB equations (a novel feature over other Fourier domain methods [10] which are suited to quasi-steady state modelling). We briefly summarise the IF algorithm’s main features now, leaving its full expression to Appendix D. Its detailed derivation may be found in [18]. The purpose of this brief introduction is only to place the IF algorithm in the context of the SF algorithm to be developed in Section 3.2.5, in order to guide future research in selecting the most efficient algorithm for a particular regime of WDB SR under investigation.

In an IF simulation of temporal duration $T$, the population inversion, the polarisation envelopes, the pumping sources, and the field envelopes are expanded in temporal Fourier series of mode separation $d\omega = 2\pi/T$. The velocity distribution is partitioned with a granularity $dv$ of the equivalent Doppler shift $d\omega$ between adjacent channels (namely, $dv = c d\omega/\omega_0$), such that each velocity slice of the material quantities ($N$ and $\vec{P}^\pm$) possesses its own Fourier series expansion. The population inversion of the sample is described by a spatially dependent two-dimensional array $N_{p,m}(z)$, where $p$ indexes the velocity channel and $m$ its Fourier expansion coefficient. Similarly, the polarisation is described by a spatially dependent two-dimensional array $\vec{P}^\pm_{p,m}(z)$. The electric field, on the other hand, is not a velocity-partitioned material quantity and is defined by a one-dimensional array of Fourier expansion coefficients $E^\pm_m(z)$. The precise Fourier expansions of $N$, $\vec{P}^\pm$, and $E^\pm$ in terms of the coefficient arrays $N_{p,m}$, $\vec{P}^\pm_{p,m}$, and $E^\pm_m$ may be found in Appendix D.

The IF simulation of the MB equations propagates solutions for the Fourier coefficients down the length of the sample in two looping steps: (1) the $E^\pm_m$ array at a given $z$ establishes, for each velocity channel, a system of linear equations to be solved in $N_{p,m}$ and $\vec{P}^\pm_{p,m}$; and (2)
said solution is used to propagate forward $\mathcal{E}_m^\pm$ to $z + dz$. In practice, Step 2 is executed with a fourth-order Runge-Kutta scheme.

The computational scaling advantage of the IF method is achieved through truncation of interactions between material quantities over a finite neighbourhood of electric field modes $\mathcal{E}_m^\pm$ centred upon each velocity channel’s natural frequency; such neighbourhoods are set to a fixed bandwidth, regardless of the total number of velocity channels added to the simulation. This interaction bandwidth truncation is known as the local mode interaction (LMI$_{\text{IF}}$) approximation, and a very similar (though slightly distinct) approximation will be made in the SF algorithm (introduced in Section 3.2.5) and referred to as the LMI$_{\text{SF}}$ approximation. For physical motivation and numerical validation of the LMI$_{\text{IF}}$ approximation, as well as detailed discussion of the IF algorithm’s implementation, see [18].

In Sections 3.4.2 and 3.5.1 we will observe extremely wide polarisation correlation across a Doppler broadened SR system, over a bandwidth many times greater than that of the transient SR pulse which a resonant sample in the same non-linear SR regime would produce. The IF algorithm under its LMI$_{\text{IF}}$ approximation is in fact able to recover wide correlation, even if that correlation bandwidth exceeds the LMI$_{\text{IF}}$ approximation truncation width used. The physical correlation of far distant frequency modes is a higher-order effect of the MB equations, and the IF algorithm is immune to the removal of formally manifest mathematical coupling between said modes. Conversely, the algorithm introduced next in Section 3.2.5 must be executed with full fidelity to model such systems, which renders the IF algorithm potentially advantageous in such circumstances. We therefore compare in more detail the performance of all algorithms as function of the distribution size and emergent correlation bandwidth in Appendix E.

### 3.2.5 The method of supplementary fields

We derive in Appendix C a novel algorithm built upon the physical principles of the IF algorithm (namely, the concept of local spectral interaction between the field and the medium) but operating entirely within the time domain, which we call the method of supplementary fields (SF). Its name is rooted in our introduction of an array of electric field envelopes

\[ \mathcal{E}_{\omega}^\pm \equiv \mathcal{E}^\pm e^{\pm i\omega t}, \]

(3.15)
such that equations (3.11)–(3.13) may be expressed as

\[
\begin{align*}
\frac{\partial Z_\omega}{\partial \tau} &= \frac{i}{T_R} (\tilde{P}_{\omega}^+ \tilde{E}_{\omega}^- - \tilde{P}_{\omega}^- \tilde{E}_{\omega}^+) - \frac{Z_\omega - 1}{T_1} + \Lambda^{(N)} \\
\frac{\partial \bar{P}_{\omega}^+}{\partial \tau} &= \frac{2i}{T_R} \tilde{E}_{\omega} \bar{Z}_{\omega} - \frac{\bar{P}_{\omega}^+}{T_2} + \Lambda^{(P)} \\
\frac{\partial \bar{E}_{\omega}^+}{\partial \tau} &= \frac{i}{2L} \int d\omega' F_{\omega - \omega'} \tilde{P}_{\omega - \omega'} e^{i\omega' \tau} s_\delta (\omega').
\end{align*}
\] (3.16) (3.17) (3.18)

The function \( s_\delta (\omega') \) is referred to as the local mode interaction kernel. If \( s_\delta = 1 \) everywhere, then equations (3.16)–(3.18) are mathematically equivalent to equations (3.11)–(3.13); however, we permit \( s_\delta \) to be a function non-vanishing on only a finite domain \( |\omega'| \preceq \delta / 2 \). We refer to this act, which effectively limits the convolution bandwidth in equation (3.18), as the local mode interaction approximation in the SF representation (abbreviated as the LMI \( \text{SF} \) approximation). The LMI \( \text{SF} \) approximation is similar in character to the LMI \( \text{IF} \) approximation of the IF algorithm [18] as described previously in Section 3.2.4, but with some important distinctions detailed in Appendix E.

Equations (3.16)–(3.18) together form an \( O(n) \) complex algorithm in the number \( n \) of velocity channels. The problematic aliasing of fast oscillating exponentials is removed by effectively truncating integration ranges over a finite velocity bandwidth via the finite width of \( s_\delta \), such that additional velocity channels may be indefinitely added to the system without concurrently reducing the temporal step size in a Runge-Kutta temporal propagation. Although equation (3.18) has a convolution structure (which suggests greater than \( O(n) \) complexity), the convolution is in fact \( O(n) \) since the kernel \( e^{i\omega' \tau} s_\delta (\omega') \) does not increase in width with increasing channel count. In fact, we demonstrate in Appendix C that a rectangular kernel has the special status of circumventing the convolution calculation altogether; however, we detail also in Appendix C a fast Fourier transform (FFT) method which performs optimally in the regimes of interest to this paper and is therefore used wherever the SF algorithm is applied in what follows.
3.2.6 The initial quantum mechanical stage and considerations for widely Doppler broadened superradiance

The MB equations describe the self-consistent evolution of an invertible, polarisable medium exposed to and interacting with an electromagnetic field; however, they do not describe quantum fluctuations in any of these quantities. Such fluctuations are critical to the initial stages of SR. In the fully quantum mechanical analysis of the combined molecular and radiative system, an initially inverted, non-polarised, zero photon state is not an eigenstate of the Hamiltonian. Such an initial state therefore evolves very quickly, as described by perturbation theory, into a state of non-zero photons and non-zero polarisation.

One can show [6] that the important transient features of SR—including the time delay to peak emission intensity, the scaling of peak intensity with the square of the initial population inversion, and the duration of peak intensity—are all modelled by the MB equations, so long as they are prescribed initial conditions which capture the initial stages of quantum fluctuations. The initial condition prescription is derived from a statistical analysis of the expectation values of the initial evolution of the population inversion, the polarisation, and the electric field. The derivation finds that, at a certain time $\tau_{\text{classical}}$, commutators associated with the classical limit become negligible and the system can thereafter be evolved from a statistical distribution of states at $\tau_{\text{classical}}$ along trajectories for $\tau > \tau_{\text{classical}}$ defined by the MB equations.

The statistical distribution of states at $\tau_{\text{classical}}$ is prescribed as follows. Suppose the system is first prepared at $\tau = 0$ with an initial inversion $N_0$ and with no polarisation. The state of the inversion and the polarisation may be plotted in three dimensional space, such that the $x$ and $y$ axes correspond to the real and imaginary parts of the polarisation and the $z$ axis to the dipole moment $d$ times the inversion $N$. The conventional “Bloch sphere” is defined as a sphere of radius $N_0 d$ centred at the origin, and the initial state of the system is located at the North pole of the Bloch sphere. A detailed quantum mechanical analysis [6] demonstrates that the state of the system quickly tips, at a small time $\tau = \tau_{\text{classical}}$, into a statistical distribution of states on the surface of the Bloch sphere. Specifically, the distribution is uniform about the azimuthal axis (the polarisation phase) and Gaussian about a mean polar angle (the angle between $Nd$ and $|P^+|$) of $\theta_0 = 2/\sqrt{n_{\text{tot}}}$, where $n_{\text{tot}}$ is the total number of interacting atoms in the sample. A
Figure 3.1: The Bloch sphere with example statistical distribution (scatter points) of initial tipping angles and polarisation phases. Expectation value shown in solid. Note that this example is greatly exaggerated in its standard deviation and in its expectation value of $\theta_0$; the initial tipping angles are highly localised near $\theta = 0$ (the top pole) for all simulations of this paper.

The initial Bloch tipping angle affects the transient system behaviour, but its value is not well-defined in the WDB limit. The mean value of $\theta_0 = 2/\sqrt{n_{\text{tot}}}$ is usually (in resonant or narrowly Doppler broadened samples) computed from the total number of atoms in the sample; however, the quantum mechanical derivation of this result [6] assumes that all $n_{\text{tot}}$ interact with a common quantized radiation field. Conversely, in a widely Doppler broadened sample we expect some degree of independence between distant velocity channels. It is beyond the scope of this paper to generalise the quantum mechanical calculation of [6] to the complicated problem of an ensemble of molecules possessing a continuum velocity distribution; instead, we make the “best guess” that $\theta_0 = 2/\sqrt{n_{\text{atoms, } \Delta \omega}}$, where $n_{\text{atoms, } \Delta \omega}$ is the number of atoms inside the bandwidth $\Delta \omega$ of the transient SR pulse generated by the simulation. This definition may appear circular; however, it is in fact well-defined, as the timescale of the SR transients generated by WDB samples are found to be affected by $\theta_0$ only if it is modified by multiple orders of magnitude.
3.2.7 Velocity channel discretisation

We are investigating the effect of a continuum velocity distribution upon transient SR features. In addition to the initial Bloch tipping angle prescription, we must also choose a discretisation to our velocity distribution which sufficiently captures the physical coupling of channels in the continuum while remaining numerically manageable. If the velocity channel separation is too coarse, the physics of the continuum will be lost and the channels will behave as physically inaccurate, individually resonant Dirac delta-like distributions; on the other extreme, excessively fine discretisation comes at the cost of numerical complexity.

In all simulations we establish our distribution discretisation from the simulation duration $T$; i.e., we separate velocity channels by $dv = 2\pi c / (\omega_0 T)$. Achieving a physically accurate discretisation under this prescription requires that transient SR processes evolve over a period less than $T$. We typically meet this constraint by commencing with a resonant sample demonstrating SR transient timescales less than $T$; we then regulate initial population inversion levels as the Doppler broadened system is constructed, ensuring that transient features never exceed a moderate fraction of $T$. In fact, since in the resonant case $T_R$ is inversely proportional to the total initial population inversion density $N_0$, we expect transient SR timescales to only possibly reduce during the addition of molecules to the ends of our distribution (though the exact degree of that reduction is as of yet unknown for the broadened system). We may therefore assume that our starting duration $T$ for a resonant sample yields physically accurate velocity discretisation granularity under our Doppler broadened distribution construction method described near the end of Section 3.2.2.

3.3 Transverse Inversion Processes

The WDB SR processes analysed in this paper are initiated by the sudden inversion of a sample via some pumping process. Mathematically speaking, the most simple$^5$ of such processes is a swept inversion, which corresponds to the simultaneous onset of population inversion everywhere at retarded time $\tau = \tau_0$. A swept inversion is realised by a pumping action propagating at the speed of light $c$ down the length of the sample such as, for example, the sudden introduc-

$^5$That is, in our retarded time representation of the MB equations (3.11)—(3.13).
tion of transition-inducing radiation at \( z = 0 \) in the direction of the sample’s longitudinal axis. Alternatively, radiation incident in a direction perpendicular to the longitudinal axis of a sample may initiate a population inversion everywhere at simultaneous non-retarded time \( t = t_0 \); such a process is conventionally referred to as a transverse inversion.

If the length of the sample is small in comparison to \( c \) times any relevant timescales, a transverse inversion will yield the same response as a swept inversion. Conversely, if the length of the sample is greater than the distance which light can travel during the characteristic SR timescale, then the type of pumping process will dramatically alter the intensity transient at the end of the sample. In such a configuration, transversely inverted molecules separated by a distance greater than \( c \tau_d \) (for \( \tau_d \) the delay to peak SR emission) will not become correlated before the onset of peak SR emission. The number of molecules cooperatively contributing to the SR intensity pulse generated by a transverse inversion will thus be limited to a fraction of the total population within the so-called cooperation length \( L_c \propto c \tau_d \), and non-linear SR emission may only develop after transverse inversion if the number of molecules within \( L_c \) exceeds the critical threshold for that SR regime. This requirement, known as the Arecchi-Courtens condition [1], causes the response to a WDB transverse inversion to more closely resemble the resonant case than does the response to a WDB swept inversion.

The Arecchi-Courtens condition inhibits the formation of global polarisation phase correlation during a transverse WDB SR process via an autoregulation mechanism on the number of molecules acting cooperatively; we justify this claim via the following thought experiment. Suppose that a sample of length \( L \) possesses a smooth distribution of some initial total velocity bandwidth \( \Delta v_i \), such that the resulting SR endfire intensity transient possesses a peak duration \( \tau_{p,i} \approx L/c \). This duration is inversely proportional to the total number of molecules participating in the SR process [6] which is, at present, the population of the entire velocity distribution. Suppose that this distribution is now widened by adding molecules adjacent to its lower and upper bounds. In a short sample of length \( L \ll L_c \)—or in response to a swept inversion—the added molecules can correlate with the existing distribution and reduce the peak intensity duration to some \( \tau'_p < \tau_{p,i} \). In a transverse SR process of length \( L \approx L_c \), however, if the added

---

6In Sections 3.4.2 and 3.5 we will in fact demonstrate that the duration of a swept SR process reduces arbitrarily as molecules are added to the distribution wings.
velocity population were to reduce the SR timescale, then the cooperation length $L_c$ would simultaneously reduce to $L_c < L$. This shorter cooperation length would, in turn, contain a smaller total number of molecules and thereby increase the SR timescale and decrease the velocity bandwidth of molecules cooperatively participating in the SR process. This feedback loop forms an autoregulation mechanism which inhibits arbitrarily wide bandwidth correlation and acts to sustain finite temporal structure. If a sample’s length is on the order of or exceeds $L_c$, then, we expect its transversely initiated SR intensity response to retain finite temporal features in the WDB limit.

We show in Figure 3.2 the transient endfire intensity responses to the transverse inversion of three samples of differing lengths and identical column densities, such that only the final 10% of their lengths enter the non-linear SR regime. For a simulation duration $T = 1 \times 10^8$ s, the samples of lengths $cT/32$, $cT/8$, and $cT/2$ are suddenly inverted everywhere at simultaneous non-retarded time $t = 0$. Each sample possesses a Gaussian velocity distribution of 4,095 channels spanning $\pm \sqrt{2}\sigma$ for a Gaussian characteristic width $\sigma = 2,047d\omega/\sqrt{2}$ (where $d\omega = 2\pi/T$ as usual). The SR emission reaches peak intensity approximately after the delay required for radiation to propagate down the length of each sample.

Figure 3.2 presents both the raw intensity transients generated by simulation (jagged plots) as well as envelopes (smooth plots) which more accurately represent the intensity measured by a real observation. Although the number of channels simulated is large relative to the fundamental frequency differential $1/T$, the total velocity extent remains small on the order of the bandwidth of the observing instrument. A real telescope would gather incident radiation generated by the decoherent combination of many versions of the above transient across the observing bandwidth. Furthermore, we expect a real astrophysical sample to be composed of a large number of unresolved independent cylindrical samples generating SR radiation. More precisely, the envelopes of Figure 3.2 were constructed by (1) taking the full FFT of the signal; (2) masking out a 20-channel band at the lower end of the spectrum, inverting back to the time domain, and computing its intensity; and (3) repeating Step 2 for a sliding band across the entire FFT, and averaging all intensity transients so generated.

Each transient of Figure 3.2 contains a finite duration intensity pulse, demonstrating that transverse inversion indeed sustains temporal structure in the WDB limit; additionally, the
Figure 3.2: Endfire intensity transients resulting from the sudden transverse inversion at $t = 0$ of three samples of lengths $cT/32$ (solid, leftmost), $cT/8$ (dashed), and $cT/2$ (dotted, rightmost) for a simulation of duration $T = 10^8$ s. All transients normalised to $I_p$ with peak values provided in text overlays. See accompanying text for discussion of low-pass envelope overlays. Velocity extent of 4,095 channels total, simulated with the CTD algorithm.
duration of a pulse is observed to be proportional to the sample’s length. This proportionality can be explained by the autoregulation mechanism as follows. Suppose we construct a WDB distribution by appending velocity channels to the boundaries of an initially narrow velocity extent. As the distribution is widened, the SR emission duration and the cooperation length \( L_c \) decrease as more molecules enter the SR process, but \( L_c \) at first remains above the sample length \( L \). Eventually, however, \( L_c \) reduces below the order of the length of the sample. At this point the temporal duration attains a critical value \( \tau_{p, \text{crit}} \) proportional to the length of the sample. If the velocity distribution is widened further, then any potential reduction in temporal duration and corresponding cooperation length would also decrease the number of molecules participating in the SR process, thereby contradictorily extending the temporal duration. The reduction in SR intensity duration is thus arrested at \( \tau_{p, \text{crit}} \propto L \).

We can verify that the finite temporal duration \( \tau_p \) of, for example, the rightmost transient intensity pulse in Figure 3.2 is indeed a critical limiting value by repeating the simulation at a revised total velocity width. The 4,095 velocity channel extent is in fact an order of magnitude greater than the critical velocity extent for that sample, so we choose for numerical convenience to reduce the width of the distribution by a factor of two, to span 2,047 channels total. The resulting transient intensity pulse, shown in Figure 3.3, indeed possesses the same temporal duration as its earlier counterpart in Figure 3.2.

In the opening to this section we discussed the role of the Arecchi-Courtens condition in a transverse inversion SR process, where we preemptively stated an upcoming result of Sections 3.4 and 3.5; namely, that a swept inversion leads to extremely wide bandwidth polarisation phase correlation and the quenching of temporal structure in the WDB limit. For a pump propagating with speed \( c \) at an angle \( \theta \) to the longitudinal axis of the sample, the SR emission duration \( \tau_p \) of a WDB sample is only at present known in two cases; i.e.,

\[
\tau_p (\theta) \approx \begin{cases} 
\tau_{p, \text{coh}} & \theta = 0 \\
\max \left( \alpha \frac{L_c}{c}, \tau_{p, \text{coh}} \right) & \theta = \pi/2 
\end{cases}
\]  

where \( \tau_{p, \text{coh}} \) is the duration of SR emission that a resonant ensemble of the same total initial population inversion would generate (\( \tau_{p, \text{coh}} \to 0 \) in the infinitely WDB limit) and \( \alpha \) is a cal-
Figure 3.3: Endfire intensity transient resulting from the sudden transverse inversion at \( t = 0 \) of a sample of length \( cT/2 \) for a simulation of duration \( T = 10^8 \) s. Normalised to \( I_p \) with peak value provided in text overlay. Velocity extent of 2,047 channels total, simulated with the CTD algorithm; cf. the rightmost intensity transient of Figure 3.2.
We estimate from Figure 3.2 that $\alpha \approx 0.2$.

We may make an educated guess as to the generalisation of expression (3.19) to intermediate angles $0 < \theta < \pi/2$ if we first recognise the quantity $L/c$ appearing in the transverse case ($\theta = \pi/2$) as the amount of time required for the initial onset of inversion to propagate back up the length of the sample in the retarded frame. For an arbitrary $\theta$, the duration of said propagation becomes $(L/c) \sin \theta$; it is therefore natural to propose that

$$\tau_p(\theta) = \max \left( \alpha \frac{L}{c} \sin \theta, \tau_{p, \text{coh}} \right).$$

(3.20)

### 3.4 Swept Inversion Processes Part I: Discrete Distributions

We now shift our attention to swept inversion processes; unless otherwise mentioned, every WDB SR process in this section as well as in Section 3.5 will be initiated by a swept inversion occurring at simultaneous retarded time $\tau$. Before investigating the development of WDB SR transients in response to the swept inversion of a continuum velocity distribution, it is important to first quantify the degree of interaction between discrete velocity populations at various Doppler offsets. In this section we simulate distributions comprising multiple Dirac delta-like contributions at varying velocity offsets and degrees of SR saturation. The so-called spectral interaction distance evaluated in this section will guide our physical intuition of the continuum in later sections and will serve three quantitative purposes: first, it will determine the required fidelity of our later LMI\(^{\text{SF}}\) approximations; second, it will allow us to quantify the degree of SR saturation in the continuum; and third, it will determine characteristic scales of stochastic distribution variations which become important in Section 3.5.3.

#### 3.4.1 Two discrete samples and spectral interaction distance

We now execute a series of simulations of varying distributions, each comprising two Dirac delta sub-populations at successively reducing Doppler offsets as depicted in Figure 3.4. Our simulations merge the two sub-populations together while evaluating, for each separation, the degree of interaction between them. Our goal in this exercise is to quantify the relationship
between velocity separation and SR coherence modelled by the MB equations.

In order to establish the dependency of SR coherence upon velocity separation, we must first establish a meaningful metric for the degree of SR cooperation between two populations. Recalling that SR is characterised by an \( n^2 \) peak intensity dependency upon population size \( n \) (see equation 3.14 and note there that \( T_R \) is inversely proportional to \( n \)), it may be tempting to simply evaluate peak intensity as a function of velocity separation; however, such measurements are sensitive to the beating effect of interfering SR radiation from two carrier frequencies. The time of occurrence of peak intensity in the SR transient does not necessarily coincide, for different velocity separations, with a repeatably constructive or destructive phase of the two populations’ interference pattern. Peak SR intensity therefore does not vary smoothly with the velocity separation between two samples.

A better metric for the degree of interaction between two samples is, instead, the total energy released by the system during the transient process. The justification of such a metric is not a priori apparent from simple physical intuition regarding a small sample SR system, since the total energy released in such a sample scales linearly with its population;\(^7\) however, in the large sample case the degree of SR saturation determines the length fraction entering

\(^7\)Although the peak intensity of a small sample scales quadratically with its population, the intensity timescale concurrently reduces inversely [6].
the non-linear regime (see, for example, [12]) and thereby converting its initial population inversion into radiated energy. The total energy released by the sample is dominated by that length fraction which loses its inversion and is, therefore, greater at higher SR saturations. If two large sample populations are interacting during their emission of radiation, then, we should observe greater total radiated energy over the transient duration than in their far-separated limit.

We simulate the CTD representation of the MB equations (3.11)–(3.13) for a one-dimensional sample of methanol molecules ($\omega_0 = 2\pi \times 6.7 \text{ GHz}, d = 0.7 \text{ D}$) over a duration $T = 10^8 \text{ s}$, having length $L = 2 \times 10^{13} \text{ m}$, radius $w = 5.4 \times 10^5 \text{ m}$, population inversion relaxation time constant $T_1 = 1.64 \times 10^7 \text{ s}$, and polarisation dephasing time constant $T_2 = 1.55 \times 10^6 \text{ s}$. This system is similar to that of [12], but differs in its dimensions and—most importantly—in that we now simulate two velocity distributions each with initial population inversions of $2N_0 = 3.0 \times 10^{-6} \text{ m}^{-3}$ at $\tau = 0$. At present, the value of $2N_0$ is considered to represent the total number of molecules inside the bandwidth of the transient process generated which is found, momentarily (see Figure 3.5), to be on the order of $10^{-7} \text{ Hz}$. Extrapolating this to the population of a sample spanning, say, 1 km/s, this value of $2N_0$ would represent a total Doppler-broadened population of $2N_0(1 \text{ km/s})(6.7 \text{ GHz})/\left[c(10^{-7} \text{ Hz})\right] = 6.7 \times 10^5 \text{ m}^{-3}$. We apply a constant restoring population inversion pump equal to the relaxation rate; i.e., $\Lambda^{(N)}(\tau) = 2N_0/(2T_1)$.

Note that these dimensions correspond to a Fresnel number $\pi w^2/L\lambda$ of unity.

The velocity distribution is, for each simulation, a pair of Dirac delta functions separated by some $v_{\text{sep}}$; i.e.,

$$F_v = \frac{1}{2} \delta(v + v_{\text{sep}}/2) + \frac{1}{2} \delta(v - v_{\text{sep}}/2). \quad (3.21)$$

Note that we choose to normalise $F_v$ which implies that, actually, $2N_{-v_{\text{sep}}/2}(\tau = 0, z) = 2N_{v_{\text{sep}}/2}(\tau = 0, z) = 6.0 \times 10^{-6} \text{ cm}^{-3}$ in order for each of the two sub-samples to correspond to an initial population inversion density

$$\int_{\pm v_{\text{sep}}-\epsilon}^{\pm v_{\text{sep}}+\epsilon} \text{d}v2N_v(\tau = 0, z) = 3.0 \times 10^{-6} \text{ cm}^{-3}. \quad (3.22)$$

Defining the fundamental velocity step $dv = (c/\omega_0)2\pi/T$, we commence our series of simula-

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8Recall that $N_v$ of equation (3.10) is half the population inversion, hence the factor of 1/2 in the restoring pump.
Figure 3.5: Normalised endfire intensity from two far-separated samples, \( v_{\text{sep}} = 64dv \). Fast-varying curve: endfire intensity; slowly-varying curve: low-pass filtered intensity. Normalised to \( I_p \).

The endfire intensity is, as expected, an interference of two Doppler offset SR processes, each precessing positively or negatively relative to the reference envelope frequency. The transient profile of the underlying SR processes occurring in either independent sample may be visualised by applying a low-pass filter: the orange slowly-varying curve superimposed on Figure 3.5 is the result of convolution with a Gaussian of width \( T/40 \) and clearly demonstrates the salient SR transient features of a delay to peak intensity and a ringing thereafter. These results are not surprising, but they do demonstrate two physically accurate properties of the MB equations; first, that the SVEA of the MB equations is not preferential to the central carrier frequency—the MB equations under the SVEA model SR equally well anywhere within the velocity distribution, and not only at the central factorisation frequency \( \omega_0 \). The choice of \( \omega_0 \) introduces a visualisation asymmetry between velocities, but not a physical one. Second,
the independence of sufficiently Doppler-offset SR processes is properly modelled by the MB equations.

We now proceed to merge the two Dirac delta velocity sub-components by simulating at separations of \( \nu_{\text{sep}}/d\nu = \pm \{64, 32, 16, 15, 14, 13, 12, \ldots, 1, 0\} \). The terminal case \( \nu_{\text{sep}} = 0 \) is shown in Figure 3.6. For each separation we integrate the intensity for the first quarter of the simulation duration, which roughly covers the transient SR pulse visible in the first \( 2.5 \times 10^7 \) s of Figure 3.6. We subtract from this the total energy emitted in the independent limit \( \nu_{\text{sep}} \to \infty \) (practically established from \( \nu_{\text{sep}} = 256d\nu \)) and normalise to the zero separation case. The resulting dependency upon velocity separation is plotted in Figure 3.7.

Some spurious features are visible in Figure 3.7. Around \( \nu_{\text{sep}} = 15d\nu \) the total energy dependency varies erratically. Near such separations, the two channels remain locked in phase during the SR transient phase for \( \tau \lesssim 2.5 \times 10^7 \) s but separate immediately thereafter, at which point they engage in constructive or destructive interference for \( \tau > 2.5 \times 10^7 \) s. The total integrated intensity therefore varies quickly with velocity separation as the constructive or de-
Figure 3.7: Normalised total SR energy emitted as a function of two-sample velocity separation (separation in counts of the fundamental velocity differential $d\nu$). Orange curve: least-squares Lorentzian fit.

Structive phases traverse the exit of the integration window. Such behaviour is similar to the issues alluded to earlier regarding peak intensity, where constructive or destructive phasing relative to the time of peak onset corrupts the smooth dependency of peak intensity upon velocity separation. In the present case, however, this spurious effect is minor and a Lorentzian-like profile is clearly apparent in the data.

We perform a least-squares Lorentzian fit on the central ±16 velocity separations (in units of $d\nu$), under the assumption that the central lobe behaviour is most representative of the interaction between the two-samples. The resulting fit shown in Figure 3.7 has $\Gamma = 23$ for a functional form

$$G(v_{\text{sep}}) = \frac{\Gamma^2}{\left(v_{\text{sep}}/d\nu\right)^2 + \Gamma^2}. \quad (3.23)$$

For the sample of the present simulation we therefore identify the so-called spectral interaction distance as $v_{\text{int}} = \Gamma d\nu = 23d\nu$.

The factor of 23 should not be surprising. In Figure 3.5 the fastest transient SR feature (the first peak of the low-pass filtered SR transient pulse) generated by either separate sub-sample
occurs over a duration $T_d \approx T/23$. We expect the electric field generated by a sample to interact with channels Doppler shifted by $23d\omega$, an angular frequency extent which corresponds to 23 velocity channels under our discretisation scheme. The present result is the quantitative verification of this physical intuition that two samples should interact only if they are Doppler offset by a velocity on the order of their transient spectral extent. Though unsurprising, this spectral interaction distance result was not mathematically guaranteed in the non-linear regime of the MB equations.

### 3.4.2 Comb distributions

We now conduct three experiments on so-called comb distributions, which elucidate the connection between discrete distributions and continuum distributions. We define a comb distribution as a chain of discrete channels of adjacent separation $\Delta v_{\text{sep}}$, and we identify an experiment as a series of simulations of progressively smaller $\Delta v_{\text{sep}}$ for a fixed initial population inversion. In all simulations the channel polarisation phases and initial Bloch tipping angles are initiated in accordance with the discussion of Section 3.2.6.

The three experiments differ only in their total initial population inversions $N_0$, and each experiment proceeds as follows. For a given $N_0$ we simulate 161 channels separated by successively decreasing $\Delta v_{\text{sep}}$. We commence with a simulation of $\Delta v_{\text{sep}} = 24dv$, and proceed to compress the comb distribution through simulations of finer $\Delta v_{\text{sep}}$ until the terminal case of $\Delta v_{\text{sep}} = 1dv$. The first two iterations of the process are depicted in Figure 3.8.

We classify each experiment relative to the so-called “critical threshold,” identified as follows. In the case of a single resonant velocity channel, the critical initial population inversion threshold $N_{0 \text{ crit (coh)}}$ is defined as that value of $N_0$ above which the inversion at the end of the sample is lost (i.e., $N(z = L) \rightarrow 0$ and the system therefore attains the non-linear regime) at some time during the transient response of the system. Similarly, there should exist some critical per-channel initial population inversion threshold $N_{0 \text{ crit (non-coh)}}$ above which a continuum velocity distribution will lose its inversion at $z = L$. If the transient response of a resonant sample at the critical initial population inversion level has bandwidth $\Delta f$, then we roughly expect
Figure 3.8: Successive comb distributions for a given comb experiment, which proceeds multiple iterations further than depicted and terminates at $\Delta v_{sep} = 1dv$.

that

$$N_{0 \text{ crit (non-coh)}} \approx N_{0 \text{ crit (coh)}} \frac{dv}{c\Delta f/f_0}$$  \hspace{1cm} (3.24)

for a natural emission frequency $f_0$; however, we practically identify $N_{0 \text{ crit (non-coh)}}$ by experiment.

**First experiment: initial population inversion well above the critical threshold**

Our first experiment begins with 161 channels, each possessing an initial population inversion of $N_{0, \text{high}} = 8.0 \times 10^{-7}$ m$^{-3}$ and separated by $\Delta v_{sep} = 24dv$. Such a population inversion is well above the critical threshold in each channel (more precisely, 50% of the sample length loses its population inversion during the SR process) and $\Delta v_{sep}$ is sufficiently wide such that each of the 161 channels evolves independently. The transient response of the top panel of Figure 3.9 is, therefore, the decoherent combination of 161 SR transients, each modulated by an individual channel’s natural frequency. Note that the intensity peaks of periodicity $T/24$ in the top panel of Figure 3.9 (and similarly in other panels) are spurious features of the artificial comb distribution which would not be generated by a real continuum distribution; however, their emergence indicates a small degree of correlation between neighbouring channels. For
further discussion and quantification of this remark, see Appendix F.

We proceed to simulate combs of decreasing velocity separations, each for the same initial population inversion $N_{0, \text{high}}$. The intensity responses at the end of the sample are shown in Figures 3.9 for $\Delta v_{\text{sep}}/dv = 24$ (top) and $\Delta v_{\text{sep}}/dv = 18$ (bottom); and in Figure 3.10 for $\Delta v_{\text{sep}}/dv = 12$ (top) and $\Delta v_{\text{sep}}/dv = 1$. In each panel we overlay a low-pass filtered intensity generated by convolution against a Gaussian of $\sigma$-width $T/20$ (both panels of Figure 3.9 and top panel of 3.10) or $T/200$ (bottom panel of Figure 3.10).

The initial simulation (Figure 3.9, top) generates a transient SR pulse of full-width-half-maximum (FWHM) duration in its first lobe equal to $\tau_p \approx T/10$. We therefore expect channel coupling when $\Delta v_{\text{sep}}$ reduces below the order of $\sim 10dv$. In the second panel of Figure 3.9, $v_{\text{sep}}$ has been reduced to $18dv$ and demonstrates only minor deviation from the independent channel limit; however, at $\Delta v_{\text{sep}} \approx 12dv$ (Figure 3.10, top) we observe the onset of an initial intensity pulse near $\tau = 0$.

As the comb is compressed further, such that $\Delta v_{\text{sep}} < 12dv$, groups of adjacent channels couple together to form more populated SR ensembles. This cooperative effect is a non-linear one, as the larger populations of these groups will generate SR processes of shorter $\tau_p$ [6], and thus of wider spectral interaction width; this wider spectral interaction width in turn couples more channels together, reducing $\tau_p$ even further. In the terminal case $\Delta v_{\text{sep}} = 1dv$ shown in the bottom panel of Figure 3.10, the population inversion is lost extremely quickly and the temporal duration of the peak intensity collapses to a very short pulse near $\tau = 0$.

We numerically demonstrated the formation of wide velocity bandwidth correlation in a highly saturated sample ($N_0 \gg N_{0, \text{crit (non-coh)}}$), which we intuitively understood via a theoretical estimate of the total number of molecules $n_{\text{tot}}$ participating in the SR process. This estimate assumed $\tau_p$ to be inversely proportional to $n_{\text{tot}}$; however, such a statement is theoretically derived [6] for molecules of identical velocities. We thus consider this picture an intuitive one only, and one which does not a priori predict WDB SR intensity behaviour at all degrees of saturation. We proceed next to determine numerically if the wide velocity coherence demonstrate in Figure 3.10 (bottom) is realised for lower initial population inversions $N_0$, or if $N_0$ may be tuned to sustain finite SR temporal structure.
Figure 3.9: Intensity transients at the endfire of a comb distribution of 161 channels and initial population inversion $N_{0,\text{high}} = 8.0 \times 10^{-7} \text{ m}^{-3}$ for velocity separations $\Delta v_{\text{sep}}/dv = 24$ (top) and $\Delta v_{\text{sep}}/dv = 18$ (bottom).
Figure 3.10: Intensity transients at the endfire of a comb distribution of 161 channels and initial population inversion $N_{0,\text{high}} = 8.0 \times 10^{-7} \text{ m}^{-3}$ for velocity separations $\Delta v_{\text{sep}}/dv = 12$ (top) and $\Delta v_{\text{sep}}/dv = 1$ (bottom).
Second experiment: initial population inversion slightly above the critical threshold

We are interested in determining if a swept WDB sample is able to sustain an SR process of finite temporal structure. The initial population inversion used in the previous experiment of Section 3.4.2 was found to introduce coupling between channels when $\Delta v_{\text{sep}}$ crossed below a critical threshold of $\sim 12d\nu$, such that further reduction of $\Delta v_{\text{sep}}$ down to the continuum case $\Delta v_{\text{sep}} = 1d\nu$ resulted in a wide coupling of velocity channels and the quenching of temporal structure. We now use this $\Delta v_{\text{sep}} = 12d\nu$ coupling threshold of the previous experiment to inform our attempt to configure a sample which may sustain finite temporal SR structure down to the continuum limit, by setting our initial population inversion to $N_{0,\text{mod}} = N_{0,\text{high}}/12 = 6.7 \times 10^{-8}$ m$^{-3}$.

The top panel of Figure 3.11 depicts the intensity at the endfire of the sample for $\Delta v_{\text{sep}} = 12d\nu$, and demonstrates temporal structure of very long duration (in fact, $\tau_d > T$). The temporal structure is sustained through successively finer $\Delta v_{\text{sep}}$; in fact, the retention of finite temporal structure is apparent even at $\Delta v_{\text{sep}} = 2d\nu$, depicted in the second panel of Figure 3.11. Our earlier discussion of Section 3.4.2 regarding the spurious periodic modulation of the slow transient is relevant here again, where now the channel separation of $2d\nu$ explains (see Appendix F) the sharp rise at $T/2$ in the time domain. Ignoring this non-physical artifact of the comb distribution, the case $\Delta v_{\text{sep}} = 2d\nu$ is seen to yield finite temporal duration.

If we cross below the coupling threshold which we configured to occur (via our choice of $N_{0,\text{mod}} = N_{0,\text{high}}/12$) at $\Delta v_{\text{sep}} = 1d\nu$, however, the response changes dramatically. In the bottom panel of Figure 3.11, the reduction to $\Delta v_{\text{sep}} = 1d\nu$ yields extremely wide bandwidth velocity coupling with a collapse of the SR peak intensity duration to a very short duration. Additionally, we observe a loss of population inversion in only the final 10% of this sample’s length. Our act of configuring the sample to yield channel coupling precisely when $\Delta v_{\text{sep}} = 1d\nu$ has, in fact, also selected an initial population inversion which is just above the critical threshold required to initiate a non-linear SR process. This is not a coincidence, but an important result; we demonstrate next in Section 3.4.2 that the development of wide velocity spectral coherence only occurs (for a smooth distribution with a swept inversion) when the sample enters the non-linear SR regime.
Figure 3.11: Intensity transients at the endfire of a comb distribution of 161 channels and initial population inversion $N_{0,\text{med}} = 6.7 \times 10^{-8} \text{ m}^{-3}$ for velocity separations $\Delta v_{\text{sep}}/d\nu = 12$, 2, and 1 (top to bottom).
**Third experiment: initial population inversion below the critical threshold**

We now set our initial population inversion in each velocity channel to \( N_{0, \text{low}} = \frac{N_{0, \text{med}}}{2} = 3.3 \times 10^{-8} \text{ m}^{-3} \), which is below the critical threshold required to initiate a non-linear SR process. The resulting endfire intensities generated by comb distributions of \( \Delta v_{\text{sep}}/d\nu = 12, 2, \text{ and } 1 \) are shown from top to bottom in Figure 3.12, respectively. In the terminal continuum case, where \( \Delta v_{\text{sep}} = 1d\nu \) (Figure 3.12, bottom), the transient response possesses a finite temporal duration in its exponential decay with timescale equal to \( T/2 \). Additionally, the population inversion never departs from the linear regime \((N_\omega \approx N_{0, \text{low}} \text{ for all } \omega \text{ and } z \text{ throughout the simulation})\). This result suggests that increasing the initial population inversion above that critical threshold which initiates a non-linear SR process induces not only a change in radiative process, but also a “phase transition” in the sample’s ability to develop wide velocity bandwidth phase correlation. We make this concept more precise and study it further in Section 3.5.1.

### 3.5 Swept Inversion Processes Part II: Continuum Distributions

The terminal cases of Sections 3.4.2, 3.4.2, and 3.4.2 with \( \Delta v_{\text{sep}} = 1d\nu \) described continuum distributions, but represented exceedingly small fractions of realistic velocity distributions as discussed in our introductory remarks of Section 3.1. We now turn to simulate much wider velocity distributions which capture the behaviour of a system in its transition to the WDB limit. Our objective is to reach conclusions regarding a sample’s peak intensity scaling, its transient profile, and its ability to sustain temporal SR structure in the WDB limit under various configurations.

#### 3.5.1 Polarisation phase correlations

The quenching of SR temporal duration is related to the bandwidth of polarisation phase correlation which develops across velocity channels during the transient SR process. In Figure 3.13 we show the polarisation phases across all velocity channels at three successive moments \((\tau = 0, \tau = T/600, \text{ and } \tau = T/300)\) during the initial stages of the simulation executed in Sec-
3.5. Swept Inversion Processes Part II: Continuum Distributions

Figure 3.12: Intensity transients at the endfire of a comb distribution of 161 channels and initial population inversion $N_{0,\text{low}} = 3.3 \times 10^{-8}$ m$^{-3}$ for velocity separations $\Delta v_{\text{sep}}/dv = 12$, 2, and 1 (top to bottom).
Figure 3.13: Endfire polarisation phases at three successive times for a sample slightly above the SR critical threshold (Section 3.4.2). Scatter: $\tau = 0$; dashed: $\tau = T/600$; solid: $\tau = T/300$. 

We quantify the development of phase correlation in a simulation as a function of the initial population inversion by first defining a so-called critical coherence time $T_{cc}$, as follows. Suppose that a distribution is simulated which spans some total velocity extent $\Delta v_{\text{tot}}$ that is a multiple $n_{\text{ch}}$ of the fundamental velocity interval $dv = (c/\omega_0)(2\pi/T)$ (for example, the experiments of Sections 3.4.2, 3.4.2, and 3.4.2 each simulated distributions with $n_{\text{ch}} = 160$). If all channel polarisations are initiated in phase with each other, and if all channels evolve independently, then channel polarisations will interfere constructively until approximately $\tau = T/(2n_{\text{ch}})$. We therefore define a critical coherence time for a simulation of duration $T$ and distribution extent $\Delta v_{\text{tot}} = n_{\text{ch}}dv$ as $T_{cc} = T/(2n_{\text{ch}})$. 

Although neither of the aforementioned conditions of in-phase polarisation initiation or channel independence are true of our simulations, $T_{cc}$ defines a meaningful timescale over which to assess the development of polarisation phase correlation. If wide polarisation phase correlation develops within a sample at or before $T_{cc}$, then we expect the system dynamics to be dominated by a strong, coherent electric field. Figure 3.14 shows the polarisation phases at $T_{cc}$ for each of the three experiments of Sections 3.4.2, 3.4.2, and 3.4.2. It is clearly apparent
that the polarisation phases are widely correlated at $T_{cc}$ only for those experiments above the critical threshold (Sections 3.4.2 and 3.4.2).

The introduction of $T_{cc}$ helps us visualise a transition in the polarisation phase correlation as the initial population inversion density is increased. Referring to Figure 3.14, we identify the development of wide polarisation phase correlation with a low chi-squared error in a linear fit to the polarisation phases across velocity channels. In Figure 3.15 we therefore plot, as a function of initial population inversion density, the mean chi-squared error in a linear fit to the polarisation phases at $T_{cc}$ across all velocity channels. Figure 3.15 is in fact generated from simulations of distributions possessing 1281 velocity channels; remarkably, polarisation phase correlation develops across the entire velocity extent as long as the initial population inversion density exceeds the non-linear SR critical threshold of approximately $6 \times 10^{-8}$ m$^{-3}$. 

![Figure 3.14: Endfire polarisation phases at the critical coherence time $T_{cc}$ for initial population inversion density below the SR critical threshold (scatter, see Section 3.4.2), slightly above the SR critical threshold (dashed, see Section 3.4.2), and well above the SR critical threshold (solid, see Section 3.4.2).](image)
Figure 3.15: Chi-squared error in a linear fit to the polarisation angles across velocity channels as a function of initial population inversion density.
3.5.2 Candidate stochastic distributions for sustaining temporal structure

It is important to recognise that the transition preceding the onset of the non-linear SR regime when moving from left to right in figure 3.15 describes a global correlation emerging from a system with only local physical interactions. In Section 3.5.1 we demonstrated the emergence of global polarisation phase correlation when $N_0$ exceeded $N_{0, \text{crit}} \approx 6 \times 10^{-8} \text{ m}^{-3}$ in each velocity channel, and in Section 3.4.1 we demonstrated that the Doppler-broadened MB equations do not explicitly couple distant velocity channels. The global phase correlation in Figure 3.15 must therefore be a consequence of correlations propagating transitively through the connectivity of the full velocity distribution.

The picture is somewhat analogous to a phase transition in a crystal lattice, although such an analogy is not strictly true due to the finite length over which the transition takes place. Still, the analogy of defects in a crystal lattice suggests potential WDB configurations which may sustain SR temporal structure: because the global phase correlation is a transitive phenomenon, it may be eliminated by “breaking” the connectivity of the velocity distribution. We therefore seek to establish discontinuities in our velocity distribution analogous to defects in a crystal lattice; such defects facilitate the development of numerous correlated sub-regions in a lattice which are globally decoherent; similarly, we expect “defects” in a WDB velocity distribution to limit the correlation between velocity channels to finite neighbourhoods.

We introduce discontinuities into our WDB distribution by considering a stochastic velocity distribution; however, $F_\omega$ must possess certain statistical properties if it is to inhibit the connectivity of the distribution and thereby sustain finite SR temporal structure. If typical variations in $F_\omega$ occur over too short of velocity bandwidths, then coupling between velocity channels will simply extend over the variations and the distribution will continue to develop global correlations; on the other hand, if typical variations in $F_\omega$ occur over very wide velocity bandwidths, then sub-correlated regions which develop within a typical variation width will be of such wide bandwidths that their temporal structures will each individually collapse. Finally, standard deviations in $F_\omega$ must be of sufficient amplitude to facilitate SR in some regions while eliminating it in others; that is, they must be on the order of the phase transition length such as,
for example, the region from $2.5 \times 10^{-8}$ m$^{-3}$ to $5.0 \times 10^{-8}$ m$^{-3}$ on the horizontal axis of Figure 3.15.

These constraints are admittedly fine-tuned, and it is not our present objective to investigate the feasibility of such configurations occurring in nature; however, three comments may be made in this regard. First, turbulence in the ISM may provide a physical mechanism for developing a stochastic $F_{\omega}$, although the bandwidth and amplitude of velocity distribution variations induced by the turbulent property of intermittency requires further research. Second, although the extremely long time scales of the examples provided in this paper imply a demand for very small bandwidth velocity variations, they also provide very wide velocity extents over which such variations could somewhere occur. That is, although the vast majority of a WDB velocity distribution might not possess the specific statistical characteristics outlined above, we pointed out in our opening remarks of Section 3.1) that simulations of this paper describe only a negligible fraction of a full distribution; thus, even if the majority of a real distribution is incapable of supporting SR transient processes, it is not (as of yet) unreasonable to suppose that the fine-tuned statistical requirements might be realised at least somewhere within the distribution. Third, in the event that further research into the particular long time scale examples of this paper should find these fine-tuned constraints practically unfeasible, the following results are applicable to other WDB systems which may demonstrate SR processes over shorter time scales and thereby demand less finely tuned noise statistics.

### 3.5.3 Stochastic velocity distribution simulations

Motivated by the discussion of Section 3.5.2, we investigate now the response of a noisy WDB velocity distribution. We suspect that a distribution possessing appropriate statistical characteristics will inhibit the development of global phase correlations and thereby justify the application of the SF algorithm, but we must verify this result with a CTD simulation first. We therefore begin with a CTD simulation of a noisy distribution of moderate velocity extent (1,023 channels) constructed to satisfy the statistical requirements of Section 3.5.2 as follows. First, the distribution $F_{\omega}$ is a normal distribution of mean $N_{0 \text{crit}}$ and standard deviation $(2/3)N_{0 \text{crit}}$; second, the bandwidth of the noise is reduced via convolution with a Gaussian of
Figure 3.16: Endfire intensity transient generated from a CTD simulation of a stochastic distribution 1,023 channels wide. Normalised to $I_p$. See Section 3.3 for discussion of low-pass envelope overlay (dashed).

width $\sigma$ equal to 20 velocity channels. The resulting intensity shown in Figure 3.16 indeed sustains finite temporal structure as expected.

The distribution generating Figure 3.16 does not represent a statistically significant total velocity extent. The reduction of noise bandwidth in $F_\omega$ by $\sigma = 20$ effectively modifies the distribution to represent on the order of $n_{\text{total channels}}/\sigma \approx 50$ sub-groups of varying population sizes. Although the sustained temporal structure demonstrated in Figure 3.16 implies the independence of these sub-groups, accurate statistical statements regarding intensity transients in the WDB limit cannot be made without simulating a much wider distribution. We therefore now widen our total velocity extent by a factor of 32 to the distribution $F_\omega$ of 32,767 total velocity channels shown in Figure 3.17. The zoomed inset panel shows the limited bandwidth structure to the statistics in $F_\omega$.

The CTD simulation of Figure 3.16 was generated in approximately 16 minutes on a 2018
Figure 3.17: Wide distribution of 32,767 total velocity channels. Inset: zoom into the portion of the stochastic distribution highlighted, showing the finite bandwidth structure to its statistics.
CPU, while the same distribution would take only 11 minutes to simulate with the SF algorithm. This performance gap widens dramatically as we expand to 32,767 channels: the CTD algorithm would take more than 11 days to simulate such a distribution, while the SF algorithm simulates it in under 6 hours. The resulting intensity transient is shown in Figure 3.18.

In order to investigate the behaviour of temporal structure and peak intensity in the WDB limit, we define a simple functional form with three free parameters to fit against the intensity transients generated. We expect the intensity to decay at large times with time constant $T_2/2$, and therefore choose the envelope factorisation

$$I_{\text{fit}} = I_{\text{amp}} S_w (\tau - \tau_0) e^{-2\tau/T_2},$$

(3.25)

where $I_{\text{amp}}$, $w$, and $\tau_0$ are fitting parameters. The envelope $S_w$ should smoothly merge into a constant value after some time and should tend to zero as $\tau \to -\infty$. A simple choice of $S_w$ is

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Figure 3.18: Endfire intensity transient generated from the distribution shown in Figure 3.17. The FFT-based SF algorithm was used with a rectangular kernel of width $160 \delta v$. Normalised to $I_p$, smooth curve corresponds to functional fit (see text).
the “S-curve” generated by a hyperbolic tangent of varying offset $\tau_0$ and characteristic width $w$; specifically,

$$S_w(\tau - \tau_0) = 1 + \tanh\left(\frac{\tau - \tau_0}{w}\right). \quad (3.26)$$

A fit of this form is superimposed on the transient of Figure 3.18 and provides a peak intensity estimate of $1.7 \times 10^{-6} I_p$ at a time of $1.5 \times 10^7$ s for the simulation of 32,767 total velocity channels.

The finite delay to peak intensity is an important statistical result. The number of velocity channels simulated may be very large as a number of fundamental interacting velocity slices, but remains small on the scale of observation bandwidth. For the present system, 32,767 velocity channels corresponds to a measurement bandwidth of only $32,767/T \approx 3 \times 10^{-4}$ Hz (recall that the velocity slices, indexed by their Doppler-shifted natural angular frequencies, are discretised by the fundamental angular frequency differential $d\omega = 2\pi/T$). An observer averaging the power across some finite bandwidth about some central observation frequency over some observation time would measure a pulse emerging after a delay on the order of $1.5 \times 10^7$ s. This delay is the statistical outcome of many decoherently interfering SR processes developing in small, independent sub-groups of velocity channels across the bandwidth observed.

The large number statistics of the ensemble of fine bandwidth SR processes of varying amplitudes and delays ultimately defines the total transient of Figure 3.18; however, due to the finite bandwidth of the measuring apparatus these sub-processes would be unidentifiable to an observer (at least for the present system). It is nonetheless informative to inspect the transients of narrow bandwidth windows at various locations across the velocity distribution for two reasons. First, observation of SR sub-processes within the distribution confirms the physical intuition we have built regarding independent sub-samples decoupled by the discontinuity of the distribution. Second, these results would be applicable to other WDB SR systems of much shorter time scales, wherein the observation bandwidth required to observe the SR sub-processes may widen to practically achievable levels. For example, the fast radio bursts modelled in [7, 8] are of durations on the order of milliseconds.

We choose four central channel indices $-9,000$, $-3,000$, $3,000$, and $9,000$ (well separated across the distribution) and truncate the FFT of the electric field to a neighbourhood of angular
3.5. Swept Inversion Processes Part II: Continuum Distributions

Figure 3.19: Specific endfire intensities from finite bandwidths centred upon four arbitrary frequencies across the total spectrum. Extracted from the simulation of Figure 3.18. Linetypes vary for visual differentiation and not for identification.

frequencies $\pm 10d\omega$ about the natural frequency of each central channel. We plot the resulting four intensity transients in Figure 3.19. The observed transients are clearly SR processes, each demonstrating the salient SR features of a delay to peak intensity, a peak intensity duration proportional to the particular delay realised, and an ensuing ringing thereafter.

Although the pulse of Figure 3.18 does not demonstrate ringing, we can investigate other scaling features against those of a resonant (single velocity) SR system. First, we expect the system to be decoherent in the total velocity width simulated; we therefore reduce the distribution width by a factor of two while leaving the number of molecules per velocity differential of the distribution constant. The resulting intensity transient is shown in Figure 3.20 and possesses a peak intensity that is indeed half that of Figure 3.18. This is a trivial result in light of our construction of the distribution, which intentionally eliminated wide correlations between velocity channels; still, it provides a good consistency check.

We next investigate the dependency of the intensity transient upon increasing initial popu-
Figure 3.20: Endfire intensity transient generated from a distribution of 16,383 channels. The FFT-based SF algorithm was used with a rectangular kernel of width 160\textit{dv}. Normalised to \textit{I}_p, smooth curve corresponds to functional fit (see text).
lation inversion with fixed total velocity extent. We simulate four successive systems of 8,191
channels, starting from a distribution possessing the same density of population inversion per
velocity differential and statistical characteristics as simulated previously (a cropped portion of
Figure 3.17). We define \( I_{p,0} \) as that \( I_p \) corresponding to the first simulation, shown in the top
panel of Figure 3.21. We proceed to increase the initial population inversion by successive fac-
tors of \( \sqrt{2} \), plotting the resulting intensity transients in units of \( I_{p,0} \). The first three experiments,
depicted in Figure 3.21 and in the top panel of Figure 3.22, in fact demonstrate greater than
quadratic scaling in their peak intensities. This is due to the fact that the starting distribution \( \mathcal{F}_\omega \)
only slightly exceeds the critical threshold in rare locations; increasing the initial population
inversion thus increases not only the saturation, but also the number of sub-groups within the
SR regime. The fourth transient seen in the bottom panel of Figure 3.22, where the majority
of sub-groups within the distribution are now in the SR regime, indeed demonstrates quadratic
scaling in its peak intensity equal to twice that of the prior transient above it.

The progression of intensity transients in Figures 3.21 and 3.22 demonstrates two important
temporal structure features. First, the delay to peak intensity is inversely proportional to the
initial population inversion, just as it is in a resonant SR sample. Second, there appears to be an
abrupt transition in the profile shape of the fourth plot: whereas the top three transients are rel-
tively slowly and smoothly increasing pulses (ignoring stochastic noise), the fourth transient
shows a delay followed by an abrupt onset of intensity which thereafter decays exponentially
with time constant \( T_2/2 \). These features are most clearly visible in the functional fits to the
noisy transients, which possess sufficient degrees of fitting freedom to capture the change in
profile shape and are indeed seen to abruptly change between the third and fourth plots.

The abrupt transition in Figure 3.22 (bottom) may be understood in the context of the
statistical properties of \( \mathcal{F}_\omega \). The stochastic velocity distribution of Figure 3.17 was constructed
via convolution with a Gaussian of width \( \sigma \) on the order of the bandwidth of an intensity
transient generated by a sample just above the non-linear SR threshold (see, for example, Figure
3.5). In the phase transition discussion closing Section 3.5.2, we proposed that “defects” in
the continuity of \( \mathcal{F}_\omega \) should occur over the spectral interaction distance (see Section 3.4.1) if
temporal structure is to be retained in the WDB limit. As \( N_0 \) is raised from the top to the
bottom panel of Figure 3.22, however, the higher saturation extends the bandwidth of typical
Figure 3.21: Endfire intensity transients generated from simulations of 8,191 channels. Initial population inversion per velocity differential increased by a factor of $\sqrt{2}$ in the bottom panel. The FFT-based SF algorithm was used with a rectangular kernel $\delta$ of width $\delta = 160dv$ in both panels of this figure. Normalised to $I_{p,0} = I_p$ of the top panel, smooth curve corresponds to functional fit (see text).
Figure 3.22: Endfire intensity transients generated from simulations of 8,191 channels. Initial population inversion per velocity differential increases by a factor of $\sqrt{2}$ from the bottom panel of Figure 3.21 to the top panel of this figure, and from the top panel of this figure to the bottom panel of this figure. The FFT-based SF algorithm was used with a rectangular kernel $s_\delta$ of width $\delta = 160dv$ in the top panel, and increased to $240dv$ in the bottom panel (see discussion). Normalised to $I_{p,0} = I_p$ of the top panel of Figure 3.21, smooth curve corresponds to functional fit (see text).
SR transients developing within sub-groups of the distribution to become larger than the defect scale $\sigma$. In other words, the distribution is nearly smooth on the scale relevant to the higher saturation SR processes of the fourth simulation. Note that the SF algorithm required a wider interaction kernel and finer temporal step size in order to converge to a solution in the bottom transient of Figure 3.22 (see caption), which is consistent with this concept of wider spectral interaction distance occurring within the system.

3.6 Summary and Future Work

We demonstrated that a sample inverted by a transverse pump will sustain a finite intensity duration in the WDB limit whenever its length is comparable to the corresponding cooperation length of the non-linear SR emission process. At such a length scale and under such a pumping process, the emission duration was found to be proportional to the length of the sample.

We demonstrated in Sections 3.4 and 3.5 that a velocity distribution inverted above the critical threshold $N_{0,\text{crit}}$ yielding non-linear dynamics in the MB equations cannot generically sustain temporal structure in the WDB limit. A smooth distribution inverted above $N_{0,\text{crit}}$ by a swept pumping mechanism will develop a global polarisation phase correlation, which has the effect of quenching temporal structure. During a realistic finite duration swept pumping event the endfire intensity generated by non-linear SR emission within a WDB sample will follow the pump’s transient profile.

Having recognised in Section 3.4.1 that the MB equations indeed capture the independence of SR samples separated by at least the bandwidth of the transient process, we attributed the formation of global polarisation correlation in a WDB distribution to transitive coupling across its velocity channels. This interpretation suggested certain statistical characteristics of a velocity distribution which might inhibit the transitive formation of correlation; namely, that variations in the distribution exist over the bandwidth of the transient SR pulse and that the amplitude of said variations be on the order of the saturation phase transition distance. We demonstrated in Section 3.5.3 that such distributions do indeed sustain a finite time delay and duration in their peak intensity response in the WDB limit. Such constructions were admittedly fine-tuned for the long timescale processes described in this paper, but these results also hold for shorter
timescale processes which could reasonably realise the necessary statistical characteristics. A turbulent medium demonstrating intermittency in its velocity distribution, or a population not in thermal equilibrium, for example, could both demonstrate finite temporal structure in the WDB limit under a swept inversion above the non-linear SR threshold.

The order of magnitude of an astrophysical SR process timescale strongly affects the application of this work. Even under modest thermal broadening, very long timescale SR processes (such as those modelled in Sections 3.3, 3.4, and 3.5) must be classified as WDB. Consequently, the temporal duration of such processes is determined by the swept or transverse character of the pumping mechanism. In such environments the temporal duration of SR emission from a fixed population inversion can vary dramatically, depending upon the orientation of the triggering mechanism; i.e., both short duration radiation flares and long duration emission events can be generated in the same region via different SR triggering mechanisms. Astrophysical processes possessing shorter timescales (not modelled in this paper but existing in the literature as described in Section 3.1), on the other hand, could permit the observation of temporal SR structure in response to even a swept inversion mechanism, as the bandwidth of SR features may extend over a bandwidth resolvable by the measuring apparatus. Finally, in the most trivial case, an SR process may be of such short temporal duration that its bandwidth becomes broader than its thermal broadening extent; such a sample is accurately modelled by the resonant case of the MB equations and thereby renders the methods of this work unnecessary.

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9 We assume for this discussion a smooth velocity distribution.
Appendix C

Derivation of the Method of Supplementary Fields

We provide in this appendix the derivation and a brief study of the SF method. To motivate this new approach, notice that the \( O\left(n^2\right) \) complexity of equations (3.11)—(3.13) is rooted in the channel asymmetry introduced by the choice of envelope factorisation \( \omega_0; \) that is, channels of natural frequencies far removed from the central envelope introduce fast oscillating exponentials which demand fine time stepping to avoid their aliasing.

Motivated by a desire to symmetrise equations (3.11)—(3.13) across velocity channels, let us introduce supplementary electric field envelopes: one electric field factorisation for each natural frequency available across velocity channels. Explicitly, we define the array of electric field envelopes

\[
\vec{E}_\omega^\pm \equiv E_e^\pm e^{i\omega \tau} \quad (C.1)
\]

such that equations (3.11)—(3.13) now read

\[
\frac{\partial Z_\omega}{\partial \tau} = \frac{i}{T_R} \left( \vec{P}_\omega^+ \vec{E}_\omega^+ - \vec{P}_\omega^- \vec{E}_\omega^- \right) - \frac{Z_\omega - 1}{T_1} + \Lambda^{(N)} \quad (C.2)
\]

\[
\frac{\partial \vec{P}_\omega^+}{\partial \tau} = \frac{2i}{T_R} \vec{E}_\omega Z_\omega - \frac{\vec{P}_\omega^+}{T_2} + \Lambda^{(P)} \quad (C.3)
\]

\[
\frac{\partial \vec{E}_\omega^+}{\partial \zeta} = \frac{i}{2L} e^{i\omega \tau} \int d\omega' F_{\omega'} \vec{P}_{\omega'} e^{-i\omega' \tau}. \quad (C.4)
\]

We have thus far only added to the complexity of the problem by simulating an array of supplementary fields described by the many instances of equation (C.4) which are, at present, not
C.1 Local mode interaction kernels in the SF representation

Independent: they are algebraically related by a simple exponential multiplication factor. We will shortly break this algebraic relation, however, and evolve each $\bar{E}_\omega^+$ independently; doing so will ultimately lead to a final algorithm of reduced complexity in many sample configurations.

Equation (C.4) is not yet manifestly symmetric across velocity channels, as higher Doppler shifted velocity channels are driven by faster oscillating exponentials appearing in the term multiplying the integral. We now make a change of integration variable $\omega' \rightarrow \omega' + \omega$ in equation (C.4) in order to cast the entire problem into a channel-symmetric form, viz.,

$$\frac{\partial \bar{E}_\omega^+}{\partial z} = \frac{i}{2L} \int d\omega' F_{\omega - \omega'} \bar{P}_{\omega - \omega'} e^{i\omega' \tau}. \quad (C.5)$$

Equations (C.2), (C.3), and (C.5) are manifestly symmetric across velocity channels, as equation (C.5) couples all field modes to the medium in identical form. For any given instance $\omega$ of equation (C.5), a common kernel $e^{i\omega' \tau}$ is integrated against medium properties of that velocity neighbourhood possessing central natural frequency matching the envelope factorisation frequency of $\bar{E}_\omega^+$. We herein refer to this system as the supplementary fields (SF) representation, and any of the solution methods listed to follow as SF algorithms.

C.1 Local mode interaction kernels in the SF representation

The channel-symmetric form of equations (C.2), (C.3), and (C.5) allow us to apply a natural, physically motivated approximation which achieves $O(n)$ numerical complexity whenever the velocity coherence developing within an SR process is of a moderate bandwidth.

As a WDB distribution is constructed via addition of velocity channels to the wings of an initially narrow distribution, the extent to which polarisation phase correlation (and an associated reduction in transient timescales) will develop is not a priori known. Indeed, it is the task of the following sections to determine just how temporal SR features collapse as various distributions are constructed.

In many cases, however, phase correlation across velocity channels will be limited to some bandwidth $\delta$; that is, the continued addition of velocity channels to the wings of the velocity distribution will introduce independent SR processes within finite groups of coherent velocity
channels, each group being randomly phased relative to another. In the WDB limit, the total electric field at the endfire of such a sample will become the random interference of multiple coherent signals. Such an electric field transient will appear, on coarser timescales, to be a slow envelope with a timescale of variation on the order of the SR transient timescales characteristic of the separate coherent velocity groups.\footnote{Looking back, for example, one may refer to Figure 3.18.}

In such systems the polarisation of a velocity channel of angular frequency $\omega$ is correlated only to fields of angular frequency $\omega'$ within $|\omega' - \omega| \lesssim \delta$; i.e., we do not expect $\tilde{\mathcal{E}}^{+}_\omega$ of equation (C.5) to be correlated to the integrand $\tilde{P}^{-}_{\omega' - \omega}$ for $|\omega'| > \delta$. Let us introduce a kernel $s_\delta(\omega')$ of width $\delta$ to the integrand such that equation (C.5) becomes

$$\frac{\partial \tilde{\mathcal{E}}^{+}_\omega}{\partial z} = \frac{i}{2L} \int d\omega' F_{\omega' - \omega} \tilde{P}^{-}_{\omega' - \omega} e^{i\omega'\tau} s_\delta(\omega').$$

(C.6)

Note that the kernel introduction to equation (C.6) has the effect of breaking the algebraic relationship between differing $\tilde{\mathcal{E}}^{+}_\omega$; i.e., it is no longer true that $\tilde{\mathcal{E}}^{+}_\omega = \mathcal{E}^+ e^{i\omega\tau}$. Instead, $\tilde{\mathcal{E}}^{+}_\omega$ is $\mathcal{E}^+$ shifted in the Fourier domain by an amount $\omega$, and then truncated to some finite bandwidth neighbourhood of width $\delta$.

Equations (C.2), (C.3), and (C.6) together form an $O(n)$ complex algorithm. The problematic aliasing of fast oscillating exponentials has been removed by effectively truncating integration ranges over a finite velocity bandwidth due to the finite width of $s_\delta$, such that additional velocity channels can be indefinitely added to the system without concurrently reducing the temporal step size in a Runge-Kutta temporal propagation. Although equation (C.6) has a convolution structure (which suggests greater than $O(n)$ complexity in the number of velocity channels simulated), the convolution is in fact $O(n)$ since the kernel $e^{i\omega'\tau} s_\delta(\omega')$ does not increase in width with increasing channel count. In fact, we demonstrate in Section C.4 that a rectangular kernel has the special status of circumventing the convolution calculation altogether; however, the FFT method detailed next in Section C.2 performs optimally in our regimes of interest and is therefore used wherever the SF algorithm is applied in this paper.

The functional form $s_\delta(\omega')$ is at this point arbitrary; in fact, it can be any kernel with a central value of 1 extending over a width of at least $\delta$. Its purpose is not to describe the
lineshape of the SR emission process, which is determined by simulation; instead, it is a tool to reduce the numerical complexity of the problem. Any choice of kernel shape may be arbitrarily widened\(^2\) to achieve accurate results, and its width should be tuned against a full fidelity CTD simulation at a moderate velocity distribution width prior to applying it to the simulation of a WDB distribution.

C.2 Fourier domain acceleration

The FFT provides a further speedup over computing the raw convolution of equation (C.6), but only gains supremacy when the polarisation correlation width $\delta$ exceeds some threshold. If $\delta$ covers a small number of velocity channels, a raw convolution method will be faster than the FFT method; however, if $\delta$ covers a large number of velocity channels, the FFT method may overtake the raw convolution. Additionally, the introduction of the kernel $s_\delta$ further accelerates the computation of the FFT, as described momentarily.

Let $\mathcal{F}_{\omega \rightarrow \xi}$ denote a Fourier transform (or FFT) operation which transforms the space indexed by $\omega$ to a space indexed by $\xi$, and $\mathcal{F}_{\omega \rightarrow \xi}^{-1}$ its inverse operation. We denote the transform of a quantity with a circumflex; for example,

$$\hat{F^c\mathcal{P}}_{\xi} = \mathcal{F}_{\omega \rightarrow \xi} \left\{ F^c\mathcal{P}_{\omega} \right\}$$

(C.7)

(which, importantly, is not the same as $\hat{F\mathcal{P}}$). It is essential to note that the transform is performed with respect to the indices over which the convolution of equation (C.6) occurs, and not with respect to either $\tau$ or $z$. Furthermore, the original space’s index $\omega$ is an angular frequency index, making the transformed space’s index $\xi$ a time-like index; this is opposite to traditional usages of a Fourier transform to move from the time domain to the frequency domain, and so we will not use terms such as “bandwidth” or “duration” in references to the ranges of either index (in order to avoid confusion).

\(^2\) At the cost of numerical complexity, see discussion below.
We now write the convolution of equation (C.6) as

\[
\left( F\hat{\mathcal{P}} \right) \star \left( e^{i\omega \tau} \delta \right) = F^{-1} \left\{ \left( e^{i\omega \tau} \delta \right) \star \left( s_\delta (\xi) \right) \right\} 
\]

\[
= F^{-1} \left\{ \left[ F\hat{\mathcal{P}} \left( \delta (\xi - \tau) \star s_\delta (\xi) \right) \right] \right\}.
\]

Expression (C.9) demonstrates further acceleration over a conventional FFT-based approach (which itself is already \( O(n) \) complex) to the convolution of equation (C.6). The finite width of \( \delta \) in \( \xi \)-space (proportional to \( 1/\delta \)) means that its multiplication against \( F\hat{\mathcal{P}} \) need only be performed over a finite width centred upon \( \tau \), which offers the advantage of calculating a band-limited FFT of \( F\hat{\mathcal{P}} \) (a sub-region of the full FFT). A band-limited FFT may be computed by multiplying \( F\hat{\mathcal{P}} \) by \( e^{i\omega \tau} \), downsampling, and performing an FFT on the resulting smaller array; the steps are reversed for a band-limited inverse FFT (IFFT). Numerical libraries also exist for computing limited bandwidth FFTs and IFFTs. In Python3, for example, the routine \texttt{scipy.signal.zoom_fft} performs a band-limited FFT based upon the chirp z-transform; note, however, that a full bandwidth \texttt{numpy.fft.fft} call remains faster below some critical size of \( F\hat{\mathcal{P}} \).

A full simulation may now proceed according to the following steps:

1. At the start of the simulation, compute the FFT \( \hat{s}_\delta \) for use in each looping time step.

2. Propagate all population inversion densities \( Z_\omega \) and polarisations \( \hat{\mathcal{P}}_\omega \) forward in time from the electric fields \( \hat{E}_\omega \) via equations (C.2) and (C.3).

3. Compute the band-limited FFT \( \hat{F\mathcal{P}} \) centred upon \( \xi \)-space index \( \tau \) (i.e., a different region of \( \xi \)-space is required at each time step, and slides forward throughout the simulation),

4. Multiply the result by the pre-computed \( \hat{s}_\delta \),

5. Perform the band-limited IFFT, and

6. Use this result to propagate the electric fields \( \hat{E}_\omega \) down the length of the sample via equation (C.6).

7. Repeat at step 2.
Steps 1—7 make more precise our earlier remarks concerning the supremacy of an FFT-based convolution over a raw convolution. As $\delta$ is increased, the smaller width (proportional to $\delta^{-1}$) of $\delta$ in $\xi$-space narrows the band-limited FFT and IFFT employed in steps 3 and 5, and increases the complexity of a raw convolution of equation (C.6).

As a final remark on the FFT-based SF algorithm we point out that a kernel of the form $s_\delta(\omega) = \text{sinc}(\omega)$ is particularly efficient, as its FFT is a rectangular function and therefore eliminates step 4 above. For such a sinc kernel, the convolution of equation (C.6) is realised by the operations of applying a band-limited FFT (centred on $\tau$) followed by a band-limited IFFT. For this reason, the sinc kernel is used wherever the SF algorithm is applied in this paper.

## C.3 High temporal fidelity reconstruction

The LMI$^{\text{SF}}$ approximation established by the finite kernel introduced in equation (C.6) essentially means that each transient $\tilde{E}^+_{\omega}(z, \tau)$ is a bandwidth-limited window into the full transient $E^+(z, \tau)$, centred upon angular frequency $\omega$. In many cases this is advantageous: by generating transients of every $\tilde{E}^+_{\omega}(z, \tau)$ during the SF algorithm execution, we have immediate access to a bandwidth-limited transient response of the electric field in any desired region of the spectrum. In the original CTD algorithm, such responses would have to be filtered out by band masking and frequency shifting the Fourier transform of the electric field transient.

On the other hand, the SF algorithm does not (immediately) provide a high fidelity, wide-bandwidth transient of the full electric field $E^+(z, \tau)$. Suppose, for instance, that we were to plot as a function of $\tau$ the central supplementary field $\tilde{E}^+_{\omega=0}(z', \tau)$ at some position $z'$. The fine detail of the transient so generated would be limited to time steps defined by the kernel width $\delta$. If $\delta$ is $m$ multiples of the fundamental angular frequency differential $d\omega$, then the present transient will only resolve features down to temporal width $T/m$.

Higher temporal resolution is, however, available from the results of the SF algorithm by assembling together all $\tilde{E}^+_{\omega}$ responses. We begin the reconstruction in the Fourier domain by recognising that

$$
\mathcal{F}\{E^+(\omega)\} = \int d\tau e^{-i\omega \tau} E^+(\tau) = \int d\tau \tilde{E}^+_\omega(\tau),
$$

(C.10)
where $\mathcal{F}$ denotes the Fourier transform with respect to $\tau$ and we have suppressed all $z$ dependence. The $\omega$ mode of the Fourier transform of $\mathcal{E}^+$ is, therefore, the integral over all time of the $\tilde{\mathcal{E}}^+_{\omega}$ transient.

At present we have in our possession the Fourier transform of $\mathcal{E}^+$ between $\omega_{\text{min}} = \omega_0 v_{\text{min}} / c$ and $\omega_{\text{max}} = \omega_0 v_{\text{max}} / c$ for a distribution $F_\omega$ defined on $\omega_{\text{min}} \leq \omega \leq \omega_{\text{max}}$; however, we can also extend beyond the trailing and leading ends of the current spectrum by an amount $\delta/2$ by considering the Fourier transforms of $\tilde{\mathcal{E}}^+_{\omega_{\text{max}}}$ and $\tilde{\mathcal{E}}^+_{\omega_{\text{min}}}$, respectively. For example, for a frequency a distance $\eta \leq \delta/2$ beyond the leading end of the current spectrum, we compute

$$
\mathcal{F}\{E^+\}(\omega_{\text{max}} + \eta) = \int d\tau \tilde{E}^+_{\omega_{\text{min}}} (\tau) e^{-i \eta \tau} \tag{C.11}
$$

which is the $\eta$ mode of the Fourier transform of the minimum velocity supplementary field.\footnote{This assumes that $F_\omega$ is equally distributed about $\omega = 0$; i.e., that $\omega_{\text{min}} = -\omega_{\text{max}}$.} A similar result holds at the trailing end of the spectrum.

The previous discussion is admittedly high-level, and should be made precise by (i) discretising all quantities over $\omega$ into arrays of velocity separation $d\omega$ and (ii) re-expressing all (continuous) Fourier transforms as discrete Fourier transforms (DFTs) of mode separation $d\omega = 2\pi/T$. We omit these trivial details, but point out that an FFT is used to compute the DFTs of $\tilde{\mathcal{E}}^+_{\omega_{\text{min}}}$ and $\tilde{\mathcal{E}}^+_{\omega_{\text{max}}}$ for extending the spectrum, and that an IFFT is used to finally recover the full high fidelity electric field transient. The entire reconstruction procedure is illustrated in Figure C.1.

### C.4 Special status of the rectangular kernel

A rectangular kernel

$$
s_\delta (\omega) = \begin{cases} 
1 & |\omega| < \frac{\delta}{2} \\
0 & \text{otherwise}
\end{cases} \tag{C.13}
$$

leads to an especially efficient algorithm for computing the convolution of equation C.6. Inserting expression (C.13) for $s_\delta$ into equation C.6 and changing the variable of integration, we
Figure C.1: Schematic of the high temporal fidelity reconstruction procedure from the output of the SF algorithm.
return to the asymmetric formulation (but now with a finite range of integration)

\[
\frac{\partial \tilde{E}_\omega^+}{\partial z} = \frac{i}{2L} e^{i\omega \tau} \int_{\omega-\delta/2}^{\omega+\delta/2} d\omega' F_{\omega'} \tilde{P}_{\omega'} e^{-i\omega' \tau}. \tag{C.14}
\]

Equations (C.14) contain what appear to be fast oscillating exponentials; however, said form being mathematically identical to equation (C.6), it is nonetheless physically correct for large temporal step sizes (even if those step sizes alias the terms explicit in this formulation).

Equation (C.14) is much more computationally efficient than equation (C.6), if only a numerical solution implements the following order of operations:

1. Compute an array \( I_{\omega'} \) representing the integrand of equation (C.14); i.e.,

\[
I_{\omega'} = F_{\omega'} \tilde{P}_{\omega'} e^{-i\omega' \tau}. \tag{C.15}
\]

2. Defining the following array pending computation,

\[
J_{\omega} = \int_{\omega-\delta/2}^{\omega+\delta/2} d\omega' I_{\omega'}, \tag{C.16}
\]

compute its first element \( J_{\omega, \text{min}} \) by summing over the appropriate number of starting elements of \( I_{\omega'} \).

3. Proceed by induction to compute \( J_{\omega+d\omega} \) from \( J_{\omega} \) by simply adding to \( J_{\omega} \) that element of \( I_{\omega'} \) which is introduced by advancing the integration window forward one step, and subtracting from it that element of \( I_{\omega'} \) which is lost by advancing the integration window forward one step.

4. Compute \( \frac{\partial \tilde{E}_\omega^+}{\partial z} \) by element-wise multiplication between the array preceding the integral of equation (C.14) and the array \( J_{\omega} \).
Appendix D

The Integral Fourier Representation of the Maxwell-Bloch Equations

The integral Fourier (IF) representation of a system of partial differential equations provides a system of mode relations which, when solved, yield a convergent Fourier series representation of a system’s transient behaviour. Such a representation is importantly distinct from the typical Fourier mode relations of, for example, [10], which describe only the steady-state regime of a dynamical system. The derivation of the IF representation can be found in [18].

The inversion and polarisation of each velocity channel, as well as the electric field, are expanded in the Fourier series

\[ N_p = \sum_m \bar{N}_{p,m}(z) e^{im\omega \tau} \]  

(D.1)

\[ \bar{P}_p = \sum_m \bar{P}_{p,m}(z) e^{\pm im\omega \tau} \]  

(D.2)

\[ E^\pm = \sum_m \bar{E}^\pm_m(z) e^{\mp im\omega \tau} \]  

(D.3)

\[ \Lambda^{(N/P)} = \sum_m \bar{\Lambda}_{m}^{(N/P)} e^{im\omega \tau}. \]  

(D.4)

Under such a Fourier series expansion the IF representation of the MB equations yields a set of mode relations which is fundamentally distinct from the typical steady state Fourier
mode relations. Specifically, the IF relations read

\[ N_{p,0} = N_p(0) + \frac{i}{\hbar} \sum_m \left( \Xi^+_{\bar{m}+p} \bar{\Phi}^+_{\bar{p},\bar{m}} - \Xi^-_{\bar{m}+p} \bar{\Phi}^-_{\bar{p},\bar{m}} \right) + \sum_m T_m \left( \frac{L_m^{(N)}}{T_1} - \frac{N_{p,m}}{T_1} \right) \]  

(D.5)

\[ N_{p,m\neq 0} = T_m \left\{ \frac{i}{\hbar} \sum_m \left[ \bar{\Phi}^+_{\bar{p},\bar{m}} \left( \Xi^+_{\bar{m}+p} - \Xi^+_{\bar{m}-m+p} \right) - \bar{\Phi}^-_{\bar{p},\bar{m}} \left( \Xi^-_{\bar{m}+p} - \Xi^-_{\bar{m}+m+p} \right) \right] + \frac{1}{T_1} \left( N_{p,m} - N_{p,0} \right) + \left( L_0^{(N)} - L_m^{(N)} \right) \right\} \]  

(D.6)

\[ \bar{\Phi}^+_{p,0} = \bar{\Phi}^+_{p}(0) + \sum_m T_m \left[ \frac{2id^2}{\hbar} \sum_m \left( N_{p,m} \Xi^-_{m-\bar{m}+p} \right) - \frac{\bar{\Phi}^+_{p,m}}{T_2} + \frac{L_m^{(P)}}{T_2} \right] \]  

(D.7)

\[ \bar{\Phi}^+_{p,0} = \bar{\Phi}^+_{p}(0) + \frac{2id^2}{\hbar} \sum_m \Xi^-_{p-\bar{m}} N_{p,\bar{m}} + \sum_m T_m \left( \frac{L_m^{(P)}}{T_2} - \frac{\bar{\Phi}^+_{p,m}}{T_2} \right), \]  

(D.8)

where the arrays \( T_m \) and \( \Xi^\pm_a \) are defined as

\[ T_m = \begin{cases} \frac{\pi}{d\omega}, & m = 0 \\ \frac{i}{md\omega}, & m \neq 0 \end{cases} \] 

and

\[ \Xi^\pm_a = \sum_m T_m \Xi^\pm_{a,m}. \] 

(D.9)
Appendix E

Algorithm Comparisons

E.1 Approximation distinctions and domains of validity

The LMI_{IF} and LMI_{SF} approximations, though motivated by similar physical arguments, are not formally identical mathematical operations. The LMI_{IF} approximation asserts, in the Fourier domain, that a limited bandwidth of the electric field be considered in the evolution of each of the material quantities; conversely, the LMI_{SF} approximation asserts that a limited velocity width of all available polarisation channels be used to generate local regions of the electric field spectrum. Both LMI approximations are valid in that they generate signals matching those of the full fidelity time domain MB equations. Successful simulation of transient SR processes by the IF algorithm under the LMI_{IF} approximation is demonstrated in [18] and the successful simulation of transient SR processes by the SF algorithm under the LMI_{SF} are demonstrated in Section 3.5.

The two algorithms are found, however, to converge differently depending upon the degree of spectral correlation developing across the velocity distribution. Under a set of simulation parameters yielding polarisation phase correlation across an extremely wide velocity bandwidth, the IF algorithm converges with a moderate fidelity LMI_{IF} approximation [18]. In such cases the SF algorithm, on the other hand, requires a higher fidelity LMI_{SF} approximation and thus loses its computational complexity advantage in such systems. Extremely wide distributions demonstrating wide bandwidth correlations should therefore be simulated with the IF algorithm under the LMI_{IF} approximation; however, such regimes are not so deeply explored in
Figure E.1: Log-log plot of simulation execution times for the three algorithms as a function of velocity distribution size. In these particular samples wide spectral correlation does not develop. Note that only the first \(\sim 60\) seconds of each simulation were executed, from which total execution time was extrapolated. SF > CTD denotes, for example, that the SF algorithm is superior to (shorter execution time than) the CTD algorithm. Note the logarithmic axes.

This paper as to warrant implementation of the IF algorithm in any of the following sections.

### E.2 Execution speeds

Figure E.1 is a log-log plot of total execution times for the full fidelity CTD algorithm, the SF algorithm, and the IF algorithm as a function of the total number of velocity channels simulated. Of critical importance in the context of the closing remarks of Section E.1, the samples simulated were not configured so as to develop wide bandwidth polarisation correlations; the SF algorithm therefore overtakes the full fidelity CTD algorithm when the distribution size exceeds about 640 total velocity channels.

Conversely, were the system to demand fine time stepping of the SF algorithm due to the
E.2. Execution speeds

Figure E.2: Schematic ranking algorithm efficiency (inequality key: faster > slower) across various sample configurations. Note that the schematic is only coarsely partitioned and does not precisely identify all inequality ordering transition boundaries.

development of wide bandwidth polarisation phase correlations, the SF complexity dependency would be quadratic and would plot along a line parallel to the CTD complexity dependency; the IF algorithm would then achieve the fastest execution times above about 1500 total velocity channels simulated. All of these results are summarised schematically in Figure E.2, where speed ranking of the three algorithms is shown for various distribution sizes and degrees of polarisation phase correlation development.

The simulations of Section 3.4 fall in region A0 of Figure E.2 and are simulated exclusively with the full fidelity CTD algorithm. These simulations lay the intuitive foundations for broader velocity distributions simulated in Section 3.5, which fall primarily in regions A1, A2 and B2 of Figure E.2. A few simulations will fall in regions B1, C1, and C2; however, Section 3.5 made exclusive use of the SF algorithm, as the regions B1, C1, and C2 will not be explored to such a depth as to render the SF algorithm dramatically inferior to either the CTD or the IF algorithms.
Appendix F

Spurious Periodicity in the Comb Distribution Intensity Transients

In Section 3.4.2 we observed spurious peaks of periodicity $T/24$ in the intensity transient generated from a comb distribution of channel separation $24dv$. We remarked that this structure implied the existence of at least some degree of correlation between neighbouring channels. To make this statement more precise, consider a reference signal $h(t)$ of duration $T$ that is generated by some physical process. Suppose that this signal is modulated by an array of carrier frequencies $f_n = n(M/T)$ for $n \in [-N, N]$, where $M$ is the number of fundamental frequencies between neighbouring comb channels; i.e., in the aforementioned example, $M = 24$. Mathematically, we have a total signal

$$h_{\text{tot}}(t) = \sum_{n=-N}^{N} h(t)e^{i[2\pi(nM/T)t+\phi(n)]}$$

or

$$h_{\text{tot}}(t) = h(t) \sum_{n=-N}^{N} e^{i[2\pi(nM/T)t+\phi(n)]}$$

for $\phi(n)$ a (potentially varying) phase angle of the $n^{th}$ signal.

Suppose first that $\phi(n)$ does not vary across channels; say, $\phi(n) = 0$. Let $S$ denote the sum of equation (F.2) which obeys the recurrence relation

$$S = e^{2\pi(M/T)t}S + e^{-Ni2\pi(M/T)t} - e^{(N+1)2\pi(M/T)t}$$

(F.3)
such that

\[ S = \begin{cases} \frac{\sin[2\pi(N + 1/2)(M/T)t]}{\sin[\pi(M/T)t]} & t \neq \text{integer multiple of } T/m \\ 2N + 1 & \text{otherwise.} \end{cases} \]  

(F.4)

Notice that \( S \) is periodic in \( T/M \). Inspecting (without loss of generality) the first cycle of \( S \) and assuming large \( N \), for \( t \) on the order of \( T/[(N + 1)M] \) we may Taylor expand the denominator in expression (F.4) to obtain

\[ S \approx (2N + 1) \text{sinc} [(N + 1/2)Mt] \text{ for } t < T/[(N + 1)M] \]  

(F.5)

which makes precise our statement that the spurious intensity spikes of periodicity \( T/24 \) are expected of a comb distribution possessing correlation between its channels.

At the other limit, suppose that the channel phases are now completely uncorrelated; i.e., that \( \phi(n) = \text{Rand}[0, 2\pi] \). In this case \( S \) may be viewed as the DFT of periodicity \( T/M \) of an array of unit norm and randomly phased elements \( \exp[i\phi(n)] \). By a familiar result of Fourier theory [3], the squared norm of such a DFT is a random signal; thus, if the \( \phi(n) \) are completely uncorrelated, then \( h_{\text{tot}}(t) \) will be the signal \( h(t) \) modulated by random noise.

In-between the two limits of completely correlated and completely un-correlated channel phases, we may investigate the signal generated by partially correlated channels numerically. For this purpose, let us construct \( \phi_l(n) \) of varying finite channel phase correlation length \( l \) via convolution of a completely random phase angle array \( \phi^R(n) \) against a Gaussian \( g_l(n) \) of characteristic width \( l \):

\[ \phi_l(n) = \{g_l \star \phi^R\}(n). \]  

(F.6)

We now consider, for example, the reference signal \( h(t) \) pictured in the top-left panel of Figure F.1. We multiply \( h(t) \) by the sum \( S \) resulting from three different degrees of channel phase correlation; namely, \( l = 0.1, 1.0, \) and \( 2.0 \) for 161 channels separated by \( 24dv \) (\( M = 24 \) in the present notation). The results, shown in the top-right, bottom-left, and bottom-right panels of Figure F.1, can be compared against Figures 3.9 and 3.10. Roughly speaking, the bottom-left panel with \( l = 0.7 \) appears most similar in character to the spurious features of Figures 3.9 and 3.10, hence our qualitative assertion in section 3.4.2 that those features were indicative of
Figure F.1: Demonstration of the effect of modulating a reference signal at offset frequencies possessing varying degrees of phase correlation. Top-left: reference signal. Top-right, bottom-left, bottom-right: phase correlation length $l = 0.1, 0.7, 1.4$ (respectively).

a small degree of correlation between neighbouring channels.
Chapter 3 Bibliography


Chapter 4

Superradiance from a Relativistic Source

A version of this chapter has been submitted to the New Journal of Physics as C. M. Wyenberg, F. Rajabi, and M. Houde. “Superradiance from a Relativistic Source.” Submitted to: NJoP, submission reference NJP-115306; available on the ArXiv, identifier 2207.05570 (2022).

4.1 Introduction

In the quantum optics phenomenon of superradiance (SR) a collection of excited particles evolves into highly entangled states which generate greatly enhanced radiative emission over that predicted by the spontaneous decay of its constituent particles. In his seminal theoretical treatment of SR, Robert Dicke [3] first described the enhanced emission process as a cascade through symmetric states isomorphic to the maximal total angular momentum states from the elementary theory of the addition of spin-1/2 particles. If $|S(N,m)\rangle$ denotes the symmetric superposition of the possible quantum states of $N$ particles having $m$ excited and $N-m$ in the ground state (for example, $|S(2,1)\rangle = (|e,g\rangle + |g,e\rangle) / \sqrt{2}$), then the Dicke model describes the SR emission process by the cascade

$$|S(N,N)\rangle \rightarrow |S(N,N-1)\rangle \rightarrow \cdots \rightarrow |S(N,0)\rangle. \quad (4.1)$$

The probability of transition $P_{kk-1} = |S(N,k)\rangle \rightarrow |S(N,k-1)\rangle$ in the Dicke model is computed from a perturbation analysis of the Hamiltonian coupling the particle to the quan-
tized radiation field via the relevant transition operator (electric dipole, magnetic dipole, or other), from which the central state \( k = N/2 \) \(^1\) is found to couple most strongly to the radiation field. The full cascade through all states is characterized by the two salient SR features of a delay and enhancement in peak emission intensity. These theoretical predictions have been well verified by laboratory demonstrations of SR [16] since their original prediction by Dicke. Additionally, more sophisticated theoretical models of SR have been developed which generalize the Dicke model to extended volumes; which introduce realistic dephasing, relaxation, and pumping terms; and which lend themselves to more efficient numerical simulation. SR indeed possesses a rich experimental and theoretical history at the scale of laboratory quantum optics.

SR has also been recently applied to astrophysical environments, where it has been used to model maser flares [11, 12, 14, 10] and fast radio bursts [13, 7]; however, such environments lead to regimes of SR not realized in the laboratory and thus as of yet untreated theoretically. Some initial progress has been very recently made towards the theory of wide inhomogeneous broadening of SR, and two \( O(n) \) complex algorithms for simulating transients of \( n \) velocity channels have been derived [20, 19] and applied [19] to astrophysical environments; however, there presently exists in the literature no relativistically covariant formulation of SR. Such a model could prove essential to analyzing, for example, the proposed highly relativistic SR events of [9]. The objective of this work is to develop a simple model of SR emission from a small sample of particles travelling with differing relativistic velocities, and with it to determine the degree of velocity coherence required in order to measure enhanced SR emission in the observer’s frame.

There exist various representations of SR which suggest differing approaches to this problem. In the so-called Maxwell-Bloch model of a large population in an extended medium, for example, the complicated quantum mechanical time evolution is shown [5, 1, 8] to very quickly transition into a stage described by an ensemble of population inversion, polarization, and electric field trajectories determined by a relatively simple set of partial differential equations. In this later stage a correspondence to the classical electric and magnetic fields may be made, which could potentially be transformed to the observer’s frame by the usual relativistic transformation of the electromagnetic field tensor [9].

\(^1 k = (N \pm 1)/2 \) for \( N \) odd
4.1. Introduction

As a simple and fundamental first step towards a relativistic description of SR, however, this work does not seek to model a large population. Moreover, it is not a simple exercise to generalize the aforementioned result [5, 1, 8] regarding the emergence of an ensemble of trajectories to observers of a relativistic source. We seek rather to develop a relativistic model of the proper quantum mechanical evolution of a small number of particles capable of SR emission.

Constructing a simple relativistic model of the quantum evolution of even a small number of particles presents numerous challenges. First, observer frame measurements cannot be described by the classical relativistic transformation of the electromagnetic field tensor. Instead, the time evolution of the observer frame quantum state must be computed from a Hamiltonian built upon a covariant formulation of the matter-field interaction. For this purpose we first introduce in Section 4.3 an existing covariant model [2] of a two-level particle established in the literature.

The second challenge faced by a relativistic model is the interpretation of the cascade through symmetric states. As detailed momentarily in Section 4.2, the small sample Dicke model [3] identifies the intensity transient with the average emission rate over a distribution of stochastic cascades, where each realization involves transitions between the symmetric states at definite times.\(^2\) In the case of photons emitted from a relativistic source, however, Dicke’s interpretation becomes nuanced when considerations of time dilation or other relativistic effects are introduced into the stochastic cascade analysis. We therefore use the full diagrammatic method of quantum optics to evolve the matter-field system from the covariantly-derived Hamiltonian, and depart from the concept of stochastic discrete-time transition events in favour of a purely mathematical description of the system’s time evolution operator.

The third challenge involves the choice of basis for the Hilbert space describing each particle’s center of mass coordinate. SR is highly dependent upon the relative position of the particles, which therefore suggests the use of a position basis; however, we are interested in velocity coherence effects which are more naturally realized in a momentum basis. We define a compromised representation of the problem in Section 4.4.1, and proceed to apply the

\(^2\)This method is, strictly speaking, not theoretically rigorous; nonetheless, the more sophisticated diagrammatic methods of quantum optics confirm its results.
diagrammatic method to a relativistic two-particle SR system in Section 4.4.3.

After building these foundations we turn in Section 4.4.4 to apply our revised diagrammatic method to the Hamiltonian introduced earlier in Section 4.3. We finally solve the simple case of two particles travelling with offset relativistic velocities in Section 4.5, where we establish a metric for SR enhancement as a function of velocity separation in the observer frame. We evaluate this metric for samples travelling at various highly relativistic speeds and we analyze the resulting velocity coherence requirements in the observer frame in the context of the standard relativistic transformation of velocity differentials.

4.2 Existing Models of Superradiance

Dicke [3] first analyzed the so-called small sample limit wherein all particles are contained within a volume of dimensions much less than the emission wavelength. He considered the expectation values $\rho_m$ of populations of all states $|S(N,m)\rangle$ as evolving according to the transition rates $\{\mathcal{P}_{m,m-1}\}$. The resulting responses of the populations $\rho_m$ were used to determine the total intensity transient which peaked after some delay at a maximum emission rate greatly exceeding that predicted by spontaneous emission. Dicke continued in his foundational work to describe SR within an extended volume, to make realistic revisions to his model, and to analyze the effect of momentum back-reaction during photon emission.

Later models of SR approached the problem from either the master equation of quantum optics or the Heisenberg picture evolution of coarse grained inversion and polarization operators coupled to the quantized radiation field. These approaches lead to the Maxwell-Bloch equations briefly mentioned in Section 4.1. The Maxwell-Bloch equations are extremely useful for large samples which are intractable with purely quantum mechanical methods; however, they do not describe the initial quantum mechanical stage of SR. Moreover, our objective is to establish the most simple model possible for asking fundamental questions of a relativistic SR sample. The Maxwell-Bloch model is overly complicated for our present purposes.

We choose instead to generalize the small sample Dicke model [3] to a relativistic sample; however, as briefly discussed in Section 4.1, we deviate from his original methods. The Dicke model imagines the system evolving via discrete transitions through definite quantum states,
essentially constructing a stochastic time sequence of events. Such an analysis intrinsically involves a non-relativistic observer and it is not at all clear how to translate this stochastic time sequence into a relativistic framework. Rather than attempt to analyze the role of relativistic effects in the cascade, we choose to apply the rote diagrammatic method of quantum optics to compute the fully quantum mechanical time evolution operator from our relativistically-derived Hamiltonian, and to introduce observer frame intensities only at the end of the analysis as measurements upon the formally evolved quantum state of the system.

4.3 The Boussiakou, Bennett, and Babiker Relativistic Model of a Two-Level Particle

There exists in the literature a quantum mechanical model of a two-level particle [2] derived from a relativistically covariant Lagrangian describing the matter-field interaction. Canonical quantization of the resulting observer frame Hamiltonian yields a Klein-Gordon-like description of the particle’s center of mass, as well as relativistic $\gamma$ factor corrections to certain matter-field interaction terms. We refer to this Hamiltonian as the Boussiakou, Bennett, and Babiker (BBB) model of a relativistic two-level particle. We state the Hamiltonian below in this Section but leave its detailed synopsis to Appendix G.

In a confirmation of the BBB model’s relativistic foundations, it is shown in Section 4.4 to properly describe the relativistic time dilation of the spontaneous decay of an initially excited particle, now in our diagrammatic framework (a recovery of the result already obtained in [2]). The BBB model is well suited to the task of constructing a relativistic SR model. We describe in Appendix G its derivation at a high level as well as features and slight revisions pertinent to our work; for its detailed derivation, see [2]. The BBB Hamiltonian in the observer frame
reads,

\[ \hat{H} = \hat{H}^0_a + \hat{H}^0_f + \hat{V} \]  
\[ \hat{H}^0_a = \hat{H}_{KG, \text{com}} + \frac{1}{\gamma} \hbar \omega^0 \hat{\pi}^\dagger \hat{\pi} \]  
\[ \hat{H}^0_f = \sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k \]  
\[ \hat{V} = \sum_k \left[ -\left( \frac{1}{\gamma} \vec{d}_k^\parallel + \vec{d}_k^\perp \right) \cdot \vec{\epsilon}_k + \left( \vec{d}_k^\times \vec{\epsilon}_k^\perp \right) \cdot \frac{\vec{V}}{c} \right] \left( \xi_k \hat{L}_k^\dagger \hat{\pi}^\dagger \hat{a}_k + \xi_k^* \hat{L}_k \hat{\pi} \hat{a}_k^\dagger \right) \]

where \( \omega^0 \) is the rest frame spontaneous emission (angular) frequency, \( \gamma \) is the Lorentz factor for the center of mass velocity, and numerous additional terms require definition. The center of mass motion is described by the Klein-Gordon term

\[ \hat{H}_{KG, \text{com}} = \int d^3P E_P |\langle P| \]  

for \( P = \hbar \vec{K} \) the center of mass three-momentum of a Klein-Gordon particle of energy

\[ E_P = \sqrt{M^2 c^4 + \hbar^2 c^2 K^2} = \gamma_P M c^2. \]

The operators \( \hat{\pi}^\dagger \) and \( \hat{\pi} \) raise and lower the internal energy state of the particle, respectively.

Within the free field Hamiltonian \( \hat{H}^0_f \) the operators \( \hat{a}_k^\dagger \) and \( \hat{a}_k \) create and annihilate (respectively) photons of mode \( k \) and linear polarization \( \vec{\epsilon}_k \), where \( \vec{\epsilon}_k^\perp = (\vec{k}/|k|) \times \vec{\epsilon}_k \). The single-photon electric field coefficient is

\[ \xi_k = i \sqrt{\frac{\hbar \omega_k}{2 \epsilon_0 V_Q}} \]

for \( V_Q \) a fiducial field quantization volume and \( \epsilon_0 \) the permittivity of free space. We imply in all summations over field modes \( k \) a summation also over two orthogonal polarizations.

Within the interaction term \( \hat{V} \) the operators \( \hat{L}_k^\dagger \) and \( \hat{L}_k \) raise and lower the Klein-Gordon
center of mass momentum by $\hbar k$, respectively; i.e.,

$$\hat{L}_k^+ = \int d^3 P |P + \hbar k\rangle \langle P|$$

and

$$\hat{L}_k = \int d^3 P |P - \hbar k\rangle \langle P|.$$

The notation $\hat{L}_k$ ($\hat{L}_k^+$) is unique to this work. The corresponding terms in [2] read $\exp(\pm i k \cdot \hat{Q})$ for a center of mass position operator $\hat{Q}$, i.e.,

$$e^{\pm i k \hat{Q}} |P\rangle = |P \mp \hbar \rangle.$$

For simplicity we have suppressed magnetization terms and restrict our work to the case of a particle possessing a rest frame electric dipole moment $\vec{d}'$. The dipole moment vector is composed of $\vec{d}'_\parallel$ and $\vec{d}'_\perp$ parallel and perpendicular to the particle’s velocity, respectively.

### 4.4 Relativistic Two-Particle SR: Theory

We develop in this section the diagrammatic representation of a relativistic two-particle SR sample. We assume familiarity with the diagrammatic representation of the SR time evolution operator; however, we provide a summary of the method in Appendix H.

The velocity dependent small sample SR problem is complicated by the localization of particles to within the wavelength of emission. Our development of the diagrammatic method therefore deviates from conventional methods, as we introduce in Section 4.4.1 a compromise between the position basis and the momentum basis which departs from a standard convolution structure [1].

#### 4.4.1 Center of mass representation and localization considerations

Seeking in this work a simple model of relativistic SR, we restrict our analysis to the small sample limit defined such that all particles are contained within a volume of dimensions much smaller than the wavelength of emission $\lambda$. The Heisenberg uncertainty relation then implies that $\Delta P_i \gg \hbar / (2\lambda)$ in the initial momentum of any particle. This uncertainty could be realized,
for example, by an initial Gaussian superposition in the center of mass Hilbert subspace. A particle initially excited with no photon in the radiation field could be described by the (unnormalized) state

$$|\Psi(t = t_0)\rangle = \int d^3 P' e^{-|P' - P_i|^2/(2\lambda^2)} |P'\rangle \otimes |e, 0\rangle.$$  \hspace{1cm} (4.12)

Alternatively, the center of mass could be considered well-localized at some position $R_i$ such that

$$|\Psi(t = t_0)\rangle = |R_i\rangle \otimes |e, 0\rangle.$$  \hspace{1cm} (4.13)

The literature treatment of diagrammatic SR [1] conventionally adopts the latter approach, working in the position basis and choosing to neglect the changes in momentum caused by the actions of $\hat{L}_k$ or $\hat{L}_k^\dagger$ in the interaction term $\hat{V}$. Alternatively, in a perfectly formal execution of such calculations, the initial state of the system would be defined by a construction similar to equation (4.12), and modifications by $\hat{L}_k$ and $\hat{L}_k^\dagger$ to all $P'$ in the superposition would be tracked during the system’s time evolution. This exercise would greatly complicate calculations.

In our study of velocity dependent relativistic SR the initial particle momenta are critical to the velocity coherence analysis. We choose the following compromise between the strictly formal use of a superposition (such as equation (4.12)) and the literature approach of working in the position basis without tracking the particle momenta. We work in the momentum basis but we implicitly assume always that a center of mass momentum state $|P\rangle$ is in fact a coarse distribution about a central value $P$. The intuitive picture is that momentum states overlap with coarseness much greater than $\hbar/(2\lambda)$.

In practice the coarse momentum assumption is important to transition matrix elements. Suppose two particles starting with momenta $P_1$ and $P_2$ evolve according to the BBB Hamiltonian. We omit their momenta in our description of the state of the system, but imply always that their values are consistent with conservation of momentum through transition matrix elements. For example, consider the matrix element $\tilde{V}_{g,k;1,0}(t)$ for a two-particle system, where the particle energy state $|1\rangle$ corresponds to only the first particle excited, $|g\rangle$ to both particles in the ground state, and $|0\rangle/|k\rangle$ to the zero photon / single photon of mode $k$ states. This matrix element is implied to denote a transition between any two coarse momentum states consistent
with momentum conservation in their central values; i.e.,

\[ \tilde{V}_{g,k;1,0}(t) = \langle g, \mathbf{k}, \mathbf{P}_1 - \hbar \mathbf{k}, \mathbf{P}_2 | \tilde{\mathbf{V}}(t) | 1, 0, \mathbf{P}_1, \mathbf{P}_2 \rangle, \] (4.14)

which is related to the non-interaction picture matrix element by

\[ \tilde{V}_{g,k;1,0}(t) = e^{-i\left(\omega'_0 t - \alpha_{k1}\omega_k\right)} V_{g,k;1,0}(t), \] (4.15)

where \( \alpha_{k1} = 1 - \cos \theta_{k1} V_1/c \) for \( \theta_{k1} \) the angle between \( \mathbf{P}_1 \) and \( \mathbf{k} \) (this result is obtained in the next section, viz. equation (4.24)). Whether momentum conservation is expressed by modifying the momentum in the bra or the ket of equation (4.15) does not affect the results of later calculations.

The situation is further nuanced by considerations of the cascade history. It is argued in the literature [1] that the density operator formalism used in Section 4.4.4 below does not need to reference multi-photon states; however, when particle momenta are carefully tracked (which is not the case in [1]), prior photons emitted during the SR cascade affect later transition matrix elements. For example, suppose that the sample begins doubly excited and that particle 2 first emits a photon of mode \( \mathbf{k}' \), such that its momentum is equal to \( \mathbf{P}_2 - \hbar \mathbf{k}' \) prior to the transition process of equation (4.14). The resulting matrix element of equation (4.15) would then be modified. In [1] the particle momenta are suppressed and such complications are avoided; this suggests that their methods are deficient for our present (relativistic) velocity dependent purposes, which necessarily reference the particle momenta.

We take the following approach. Instead of complicating the diagrammatic methods of the literature by tracking the particle momenta and generalising existing techniques to multi-photon states, we choose to suppress the particle momenta everywhere, save for transition matrix elements such as shown in equation (4.15). This is admittedly ad-hoc, but we are reassured by the success of the approach demonstrated in later sections. The model will be found to properly describe the continuum between the totally coherent limit where the particle velocity difference \( \Delta v \) vanishes and the independent spontaneous emission limit \( \Delta v \to \infty \); additionally, it will be found to accurately recover time dilation in the totally coherent case.
4.4.2 Convolution structure and the self-energy term under the BBB Hamiltonian

We now establish matrix elements for the time evolution operator, beginning from its standard series expansion identities of Appendix H. As demonstrated in the literature [1], Eqs. (H.6) and (H.7) possess convolution structures. For example, the \( U_{e,0;e,0} \) matrix element in the non-interaction picture may be expressed via equation (H.6) as

\[
U_{e,0;e,0}(t, t_0) = F_{e,0}(t, t_0) - \frac{1}{\hbar^2} \sum_k |f_k|^2 \left[ F_{e,0} \star \left( F_{g,k} \star U_{e,0;e,0} \right) \right](t, t_0)
\] (4.16)

where \( f_k \) depends upon the initial velocity and dipole moment orientation of the particle according to

\[
f_k = \xi_k \left[ -\left( \frac{1}{\gamma} \hat{d}_l + \hat{d}_l' \right) \cdot \epsilon_k + \hat{d} \times \epsilon_k^l \right] \cdot \frac{V_i}{c}.
\] (4.17)

\( F_{e,0} \) is the free theory propagator for a particle in the excited state and \( F_{g,k} \) the free theory propagator for a particle in the ground state with a single photon of mode \( k \) in the radiation field.

A Laplace transform replaces the convolution structure of equation (4.16) with an algebraic relation; denoting the propagator Laplace transforms as \( \mathcal{F}_{e,0}(s) \), \( \mathcal{F}_{g,k}(s) \), and \( \mathcal{U}_{e,0;e,0}(s) \), we have the relation

\[
\mathcal{U}_{e,0;e,0} = \mathcal{F}_{e,0} \left[ 1 - \frac{1}{\hbar^2} \sum_k |f_k|^2 \mathcal{F}_{g,k} \mathcal{U}_{e,0;e,0} \right].
\] (4.18)

In the interaction picture equation (4.18) reads

\[
\tilde{U}_{e,0;e,0} = \mathcal{F}_{e,0} \left[ 1 - \frac{1}{\hbar^2} \sum_k |f_k|^2 \mathcal{F}_{g,k} \mathcal{U}_{e,0;e,0} \right].
\] (4.19)

The summation

\[
\Xi(s) = \frac{1}{\hbar^2} \sum_k |f_k|^2 \frac{1}{s + i(\omega_{g,k} - \omega_{e,0})}
\] (4.20)

is conventionally referred to as the (Laplace domain) self-energy term, and it now deviates from the literature due to the velocity dependence as well as our use of the BBB Hamiltonian.

We require an expression for \( \Xi \) for two different orientations of the dipole moment \( \hat{d} \) rela-
tive to the initial velocity of the particle. Recall that the mode identifier $k$ is implied to extend over two polarization directions, which we now explicitly identify as 

$$\epsilon_{k\theta} = \epsilon_{k\phi} \times \frac{k}{|k|}$$

and 

$$\epsilon_{k\phi} = \frac{\vec{d}' \times k}{|\vec{d}' \times k|},$$

(4.21)

with associated perpendiculars 

$$\epsilon_{k\theta}^\perp = \epsilon_{k\phi}$$

and 

$$\epsilon_{k\phi}^\perp = -\epsilon_{k\theta}$$

(4.22)

by our earlier definition $\epsilon_{k\phi}^\perp = (k/|k|) \times \epsilon_k$. When $k$ is parallel to $V_i$ (and therefore also to $\vec{d}'$) equation (4.21) becomes undefined, but by the symmetry of such a situation two orthogonal polarization directions may be arbitrarily assigned (in the plane perpendicular to $k$) without affecting what follows.

From the BBB Hamiltonian we have that $\omega_{e,0} = \gamma Mc^2/\hbar + \omega'_0/\gamma$ and that

$$\omega_{g,k} = \frac{1}{\hbar} \sqrt{M^2c^4 + \hbar^2c^2 (K_i - k)^2 + \omega_k}$$

$$\approx \frac{\gamma Mc^2}{\hbar} - V_i \cdot k + \omega_k$$

(4.23)

for an initial center of mass velocity $V_i$ (with which the $\gamma$ factor is identified). We thus have

$$\omega_{e,0} - \omega_{g,k} = \omega'_0/\gamma - \omega_k \left[ 1 - \frac{V_i}{c} \cos(\theta_k) \right]$$

(4.24)

for an angle $\theta_k$ between $k$ and $V_i$. Let us define

$$\alpha_k = 1 - \beta_i \cos(\theta_k)$$

(4.25)

for $\beta_i = V_i/c$.

We consider first the case where $V_i$ is parallel to $\vec{d}'$, in which case only the $\epsilon_{k\theta}$ polarizations yield non-vanishing interaction matrix elements

$$\langle e, 0|V|g, k\theta \rangle = \frac{d'}{\gamma} \sin(\theta_k) \xi_k,$$

(4.26)
such that the self-energy term of equation (4.20) becomes

$$\Xi(s) = \frac{d'^2}{\hbar^2\epsilon_0\gamma^2 V_Q} \sum_k \omega_k \sin^2(\theta_k) \frac{1}{s + i\left(\omega_0'/\gamma - \alpha_k \omega_k\right)}.$$  \hspace{1cm} (4.27)

Repeating the arguments of [1] (now for our relativistic Hamiltonian), in the absence of interaction equation (4.19) possesses a pole at \(s = 0\). We therefore expect, in the perturbed system, that the neighbourhood of \(s = 0\) gives the main contribution to the sum over \(k\) such that we may approximate the self-energy factor of expression (4.27) by evaluating it at \(s = \epsilon\) and taking the limit \(\epsilon \to 0\) [1]. Using the Sokhotski–Plemelj identity [6]

$$\lim_{\epsilon \to 0^+} \frac{1}{i\epsilon + \epsilon} = \pi\delta(x) - i\text{Pr}\frac{1}{x},$$  \hspace{1cm} (4.28)

(where \(\text{Pr}\) denotes the Cauchy principal value) and expressing the sum over \(k\) as an integral via

$$\sum_k \to \frac{V_Q}{(2\pi)^3} \int d^3k = \frac{V_Q}{(2\pi)^3} \int d\Omega \int_0^\infty dk k^2$$  \hspace{1cm} (4.29)

equation (4.27) becomes

$$d'^2 \frac{\omega_0^3}{\hbar^2\epsilon_0\gamma^2 V_Q} \int d\Omega \sin^2(\theta_k) \int_0^\infty dk k^2 \omega_k \delta\left(\frac{\omega_0'}{\gamma} - \alpha_k \omega_k\right)$$

$$= \frac{d'^2 \omega_0^3}{8\pi\hbar^2\epsilon_0\gamma^5 c^3} \int_0^\pi d\theta_k \frac{\sin^3(\theta_k)}{\alpha_k^4}$$

$$= \frac{d'^2 \omega_0^3}{8\pi\hbar^2\epsilon_0\gamma^5 c^3} \int_0^1 dx \left(1 - x^2\right) \left[1 - x\beta\right]^4 = \frac{\Gamma'_0}{2\gamma}$$  \hspace{1cm} (4.30)

for \(\Gamma'_0\) the rest frame spontaneous decay rate. In the time domain equation (4.30) reads

$$\Xi(t - t_0) = \frac{\Gamma'_0}{2\gamma} \delta(t - t_0).$$  \hspace{1cm} (4.31)

We have discarded the imaginary principal value part leading to the Lamb-like energy level shift, which we assume has been included in the definition of \(\omega_0'\).

The fact that the self-energy \(\Xi\) obtained by our diagrammatic approach corresponds to half the (time dilated) spontaneous emission rate is understandable if we now use \(\Xi\) to solve for
4.4. Relativistic Two-Particle SR: Theory

\( \tilde{U}_{e,0,e,0} \) in equation (4.19) and obtain

\[
\tilde{U}_{e,0,e,0}(s) = \frac{1}{s + \Xi} = \frac{1}{s + \Gamma'_0/(2\gamma)} \tag{4.32}
\]

which has inverse Laplace transform

\[
\tilde{U}_{e,0,e,0}(t,t_0) = e^{-\Delta t \Gamma'_0/(2\gamma)}. \tag{4.33}
\]

Since \( |\tilde{U}_{e,0,e,0}(t,t_0)|^2 \) corresponds to the probability of finding the particle excited after a time \( \Delta t = t - t_0 \), equation (4.33) tells us that the particle decays with time constant \( \Gamma'_0/\gamma \), which is the same decay behaviour calculated in [2]. Our time evolution operator approach, necessary to our SR application, indeed recovers the results obtained by Boussiakou, Bennett, and Babiker [2]. Note that this result is their demonstration that the BBB Hamiltonian successfully describes time dilation of spontaneous emission, which was the driving motive for our use of the BBB Hamiltonian in the present work.

It remains to yet evaluate \( \Xi \) when \( \mathbf{V}_i \) is perpendicular to \( \vec{d}' \). The calculation is similar to the above work and ultimately leads again to \( \Xi = \Gamma'_0/(2\gamma) \). This is not a new result, but rather a reproduction of the conclusion of [2], unique only in our use of the time evolution operator and Laplace domain methods.

4.4.3 Relativistic two-particle SR propagators

We now have at our disposal all of the tools necessary to compute the propagators of a relativistic two-particle SR sample. The methods of this section depart from the standard diagrammatic methodology of the literature, as we shortly demonstrate that the velocity dependent problem in our coarse momentum framework (necessary to the small sample SR problem) no longer possesses a convolution structure in its propagator expansions.

We denote the doubly excited particle state \( |e\rangle \), the state of only the first particle excited \( |1\rangle \), the state of only the second particle excited \( |2\rangle \), and the doubly ground state \( |g\rangle \). In this section we in fact compute only the zero photon propagators, which will prove sufficient for our density operator work in Section 4.4.4; however, we do provide the relativistic propagators
for single particle transitions into single photon states in Appendix J. The results of Appendix J can be used to compute multi-particle photon emission propagators where future research demands. Because no photons are present in this section, we will omit the radiation field state in our notation by writing, for example, $\tilde{U}_{e,e}$ instead of $\tilde{U}_{e,0;e,0}$.

**Doubly excited propagator**

Let us first find the propagator for the two particles to both remain in the excited state. We obtain from the recurrence relation of equation (H.6) the relation

$$
\tilde{U}_{e,e}(t, t_0) = 1 + \left[-\frac{i}{\hbar}\right]^2 \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \sum_k \tilde{V}_{e,1,k}(t_2) \tilde{V}_{1,k,e}(t_1) \tilde{U}_{e,e}(t_1, t_0)
+ \left[-\frac{i}{\hbar}\right]^2 \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \sum_k \tilde{V}_{e,2,k}(t_2) \tilde{V}_{2,k,e}(t_1) \tilde{U}_{e,e}(t_1, t_0).
$$

(4.34)

We would be repeating ourselves to calculate the self-energy parts within these convolutions: upon expanding the interaction matrix elements of equation (4.34) the central summations become the time domain equivalents to the calculation which followed equation (4.27) and yielded $\Xi(t_2 - t_1) = \left[\Gamma_0/(2\gamma)\right] \delta(t_2 - t_1)$. Equation (4.34) thus becomes

$$
\tilde{U}_{e,e}(t, t_0) = 1 - \frac{\Gamma_0}{2\hbar^2} \left(1 + \frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) \int_{t_0}^t dt_1 \tilde{U}_{e,e}(t_1, t_0),
$$

(4.35)

which has solution

$$
\tilde{U}_{e,e}(t, t_0) = e^{-\Gamma_0(t-t_0)/(\gamma_1+1/\gamma_2)\Delta v/c}/2
$$

(4.36)

for $\gamma_1$ and $\gamma_2$ the relativistic factors of particle 1 and 2, respectively. We can approximate

$$
\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) \approx \frac{2}{\gamma}
$$

(4.37)

for $\gamma$ the Lorentz factor of the average of the velocities, where the above approximation is correct to first order in $\Delta v/c$ for a velocity difference $\Delta v$ between the two particles. We finally have that

$$
\tilde{U}_{e,e}(t, t_0) = e^{-\Gamma_0(t-t_0)/\gamma}.
$$

(4.38)
Note that squaring the above result confirms that the doubly excited state again decays as expected by time dilation.

**Singly excited propagators**

We now calculate the propagator for the system to commence in the singly excited state of either particle and remain in that state at a later time. This situation differs from existing propagator calculations and results in the literature, as we shortly demonstrate a loss of convolution structure rooted in our use of the coarse momentum framework necessary to velocity dependent SR. We compute the propagator for the state \(|1\rangle\) where only the first particle is excited; the propagator for \(|2\rangle\) may be easily obtained by symmetry.

The recurrence relation of equation (H.6) reads, for this matrix element,

\[
\tilde{U}_{1;1}(t, t_0) = 1 + \left[ -\frac{i}{\hbar} \right]^2 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_1 \sum_k \tilde{V}_{1,0;g,k}(t_2) \tilde{V}_{g,k;1,0}(t_1) \tilde{U}_{1;1}(t_1, t_0) + \left[ -\frac{i}{\hbar} \right]^2 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_1 \sum_k \tilde{V}_{1,0;g,k}(t_2) \tilde{V}_{g,k;2,0}(t_1) \tilde{U}_{2;1}(t_1, t_0).
\] (4.39)

The first integration term on the right side of equation (4.39) describes the process of propagating from singly excited to singly excited, transitioning to the ground state, and transitioning back to the singly excited state. This term is familiar to us, becoming in the non-interaction picture a simple convolution structure containing the usual self-energy expression \(\Xi(t_2 - t_1)\). It simplifies to

\[
-\frac{\Gamma_0'}{2\gamma} \int_{t_0}^{t_1} dt_1 \tilde{U}_{1;1}(t_1, t_0).
\] (4.40)

The second line of equation (4.39) departs from the convolution structure. This line describes a process by which (i) the system transitions from the first particle singly excited state to the second particle singly excited state; (ii) the second particle emits a photon; and (iii) the first particle absorbs the photon. This physical picture of transferring a photon between the two particles suggests that this term will play a central role in SR velocity coherence.

It will greatly simplify what follows if we now choose a relative orientation between the initial velocities and dipole moments of the particles. Let us suppose that the two particles are travelling in the same direction with mean relativistic velocity \(v (|v| = v = \beta c)\) and velocity
difference $\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$ ($|\Delta \mathbf{v}| = \Delta \nu$) as measured in the observer frame, and that their dipoles are oriented parallel to each other but perpendicular to their shared velocity direction. The two interaction terms within the central summation in the second line of equation (4.39) then become

$$e^{i\omega_0 t_1/\gamma} \xi_k^* d' (\sin \theta_k - \beta \cos \phi_k) e^{-i\omega_1 t_2} e^{i\omega_2 t_1} \xi_k d' (\sin \theta_k - \beta \cos \phi_k) e^{-i\omega_0 t_1/\gamma} \Delta \nu$$

where $\alpha_{k1/2}$ are the $\alpha_k$ factors for $v_{1/2}$, respectively.

The central two exponentials of equation (4.41) form the convolution violating term

$$e^{-i(\alpha_{k1} \omega t_2 - \alpha_{k2} \omega t_1)}$$

which cannot be expressed as a function of $t_2 - t_1$. The occurrence of this term may be traced back to our use of the coarse momentum framework, wherein we chose not to evolve the distribution of momentum eigenstates describing a localized particle. Had we evolved such a distribution, we would have indeed retained a convolution structure between every adjacent occurrence of $V$ in the series representations of propagators; however, we would also have been required to track the center of mass momenta in our kets, which would have dramatically increased the number of matrix elements $\tilde{V}_{nm}$. Despite removing the convolution structure, then, our approach remains more efficient than describing the state of the particle’s center of mass via a superposition such as shown, for example, in equation (4.12).

We next evaluate the summation over $k$ in the second line of equation (4.39). The details of this lengthy calculation are left to Appendix I, where we find the entire second line of equation (4.39) to be equal to

$$-\frac{\Gamma'}{2\gamma} \int_{t_0}^{t'} dt' C_{\beta,\Delta \nu}^{SR, rel} (t') \tilde{U}_{2,1} (t', t_0),$$

where

$$C_{\beta,\Delta \nu}^{SR, rel} (t') = \frac{3}{8\gamma^2} \int_{-1}^{1} dx \frac{(1 + \beta^2)(1 + x^2)}{[1 - \beta x]^4} e^{i\omega_0 (\Delta \nu/c) x t'/[\gamma(1 - \beta x)]}. \quad (4.44)$$

We refer to $C_{\beta,\Delta \nu}^{SR, rel}$ as the relativistic SR velocity coherence kernel, which is an important tool in this work. A plot of $|C_{\beta,\Delta \nu}^{SR, rel}|^2$ is shown in Figure 4.1 for three values of $\beta$; we highlight
4.4. Relativistic Two-Particle SR: Theory

Figure 4.1: The squared norm of the relativistic SR velocity coherence kernel for $\beta = 0, 0.8, 0.95$.

features of $C_{\beta, \Delta v}^{SR, rel}$ at the end of this section after establishing its relationship to the two-particle propagators.

Equation (4.39) now reads

$$\tilde{U}_{1;1} (t, t_0) = 1 - \frac{\Gamma'}{2\gamma} \int_{t_0}^{t} dt_1 \tilde{U}_{1;1} (t_1, t_0) - \frac{\Gamma'}{2\gamma} \int_{t_0}^{t} dt' C_{\beta, \Delta v}^{SR, rel} (t') \tilde{U}_{2;1} (t', t_0), \quad (4.45)$$

which may be differentiated with respect to $t$ to obtain the first order differential equation

$$\frac{d}{dt} \tilde{U}_{1;1} = -\frac{\Gamma'}{2\gamma} \tilde{U}_{1;1} - \frac{\Gamma'}{2\gamma} C_{\beta, \Delta v}^{SR, rel} \tilde{U}_{2;1}, \quad (4.46)$$

Repeating all of the above procedure from equation (4.39) onward for each of the possible
single excitation propagators, we ultimately arrive at the set of coupled differential equations

\[
\frac{d}{dt} \tilde{U}_{1;1} = -\frac{\Gamma_0}{2\gamma} \tilde{U}_{1;1} - \frac{\Gamma_0}{2\gamma} C_{\beta,\Delta v}^{SR, rel} \tilde{U}_{2;1} \\
\frac{d}{dt} \tilde{U}_{2;2} = -\frac{\Gamma_0}{2\gamma} \tilde{U}_{2;2} - \frac{\Gamma_0}{2\gamma} C_{\beta,\Delta v}^{SR, rel} \tilde{U}_{1;2} \\
\frac{d}{dt} \tilde{U}_{1;2} = -\frac{\Gamma_0}{2\gamma} \tilde{U}_{1;2} - \frac{\Gamma_0}{2\gamma} C_{\beta,\Delta v}^{SR, rel} \tilde{U}_{2;2} \\
\frac{d}{dt} \tilde{U}_{2;1} = -\frac{\Gamma_0}{2\gamma} \tilde{U}_{2;1} - \frac{\Gamma_0}{2\gamma} C_{\beta,\Delta v}^{SR, rel} \tilde{U}_{1;1}.
\]

(4.47) (4.48) (4.49) (4.50)

After first numerically calculating the relativistic SR velocity coherence kernel, Eqs. (4.47)–(4.50) may be solved by any forward stepping numerical method from the initial conditions \( \tilde{U}_{j,i}(t_0, t_0) = \delta_{ij} \). We use a fourth-order Runge Kutta stepping scheme for this purpose in later sections.

We point out the following reassuring properties of the relativistic SR velocity coherence kernel (see Figure 4.1):

- In the limit as \( \Delta v \to \infty \) the kernel \( C_{\beta,\Delta v}^{SR, rel} \to 0; \) in this case Eqs. (4.47)–(4.50) are decoupled across the particles’ identities such that they evolve independently.

- In the limit as \( \Delta v \to 0 \) the kernel \( C_{\beta,\Delta v}^{SR, rel} \to 1; \) this converts Eqs. (4.47)–(4.50) into the familiar coherent (zero velocity separation) propagator relations [1].

- For non-relativistic \( \beta, C_{\beta,\Delta v}^{SR, rel} \) has characteristic width \( \omega_0'(\Delta v/c)t' \approx 1 \) (to order of magnitude); i.e., it yields SR enhancement between two particles over an SR timescale \( T_d \) only if their Doppler separation is less than the characteristic frequency \( 1/T_d \) established by said timescale.

- The width of \( C_{\beta,\Delta v}^{SR, rel} \) narrows with increasing \( \beta; \) i.e., velocity coherence requirements become stricter at increasingly relativistic velocities.

### 4.4.4 Diagrammatic density operators for relativistic two-particle SR

Having in our possession propagators for the relativistic two-particle SR sample, we are now able to describe the system’s density operator evolution. In this section we parallel the steps
4.4. Relativistic Two-Particle SR: Theory

followed by Benedict, Ermolaev, Malyshev, Sokolov, and Trifonov [1], but modify the standard method as our velocity dependent relativistic SR model requires.

A sample commencing in the initial state $|\Psi (t_0)\rangle$ has a density operator $\hat{\rho} (t) = |\Psi (t)\rangle\langle \Psi (t) |$ which evolves according to

$$\rho (t) = \hat{U} (t, t_0) \rho (t_0) \hat{U}^\dagger (t, t_0). \quad (4.51)$$

We will be interested shortly in the diagonal elements $\rho_{m,m} (t)$. Consider for a moment the simpler case of the single photon diagonal matrix elements from a single-particle system evolving according to the BBB Hamiltonian’s time evolution operator. Using the non-interaction picture version of equation (H.5) to obtain the propagator

$$U_{g,k;e,0} = - \frac{i}{\hbar} \int_k \left[ F_{g,k} \star U_{e,0,e,0} \right] (t - t_0), \quad (4.52)$$

we have that

$$\rho_{g,k;g,k} (t) = \frac{|f_k|^2}{\hbar^2} \left[ F_{g,k} \star U_{e,0,e,0} \right] (t - t_0) \rho_{e,0,e,0} (t_0) \left[ F^*_{g,k} \star U^*_{e,0,e,0} \right] (t - t_0). \quad (4.53)$$

Equation (4.53) is not a convolution between the first and second lines; however, we are interested not in the diagonal single photon elements themselves, but in the trace over them. By a standard result of the literature [1] the trace operation restores the convolution structure. We now recover this result in our present BBB Hamiltonian and see how it is modified.
Figure 4.2: Diagrammatic representation of the standard vertical photon result of the literature for a single particle system which, in our present study of relativistic SR, now introduces a factor of $\Gamma_0'/\gamma$.

The trace over equation (4.53) becomes

$$
\sum_k \left| f_k \right|^2 \frac{1}{\hbar^2} \int_{t_0}^t \int_{t_0}^{t'} \mathrm{d}t_1 \mathrm{d}t_1' e^{i\omega k (t_1-t_1')} U_{e,0,e,0} (t_1, t_0) U^*_{e,0,e,0} (t_1', t_0) \rho_{e,0,e,0} (t_0)
$$

$$
= \int_{t_0}^t \int_{t_0}^{t'} \mathrm{d}t_1 \mathrm{d}t_1' e^{-i\omega_0' (t_1-t_1')/\gamma} \Xi (t_1 - t_1') U_{e,0,e,0} (t_1, t_0) U^*_{e,0,e,0} (t_1', t_0) \rho_{e,0,e,0} (t_0)
$$

$$
= 2 \int_{t_0}^t \int_{t_0}^{t'_1} \mathrm{d}t_1 \mathrm{d}t_1' \Xi (t_1 - t_1') \left| U_{e,0,e,0} (t_1', t_0) \right|^2 \rho_{e,0,e,0} (t_0)
$$

$$
= 2 \int_{t_0}^t \mathrm{d}t_1 \left| \Xi \star \left| U_{e,0,e,0} \right|^2 \right| (t_1, t_0) \rho_{e,0,e,0} (t_0)
$$

$$
= \frac{\Gamma_0'}{\gamma} \int_{t_0}^t \mathrm{d}t_1 \left| U_{e,0,e,0} \right|^2 (t_1, t_0) \rho_{e,0,e,0} (t_0)
$$

$$
= \frac{\Gamma_0'}{\gamma} \left| U_{e,0,e,0} \right|^2 \star \left| U_{g,0,g,0} \right|^2 (t, t_0) \rho_{e,0,e,0} (t_0),
$$

where we moved from the first line to the second by recognizing the presence of the time domain version of equation (4.27) and using the result which followed it in equation (4.31); from the second line to the third by limiting to half the integration domain and exploiting the $t_1 \leftrightarrow t_1'$ symmetry, as well as recognizing that the integrand containing $\Xi (t_1 - t_1') = \delta (t_1 - t_1') \Gamma_0'/(2\gamma)$ will vanish except where $t_1 = t_1'$; and from the fourth line to the fifth by exploiting the delta distribution in $\Xi$. Note that $U_{g,0,g,0} (t, t_0) = e^{-iE_P(t-t_0)/\hbar}$ for $E_P$, the kinetic energy of the Klein-Gordon particle possessing momentum $P_P$.

Equation (4.54) tells us that the so-called vertical photon result is retained in the BBB Hamiltonian, but that it now introduces a factor of $\Gamma_0'/\gamma$. This result is critical to our density
operator work to follow and is depicted pictorially in Figure 4.2. On the top of the diagram are propagators appearing without conjugation in equation (4.53), and on the bottom are propagators appearing with conjugation in equation (4.53). The photon is depicted by a squiggly line. The left diagram depicts the original equation (4.53) prior to the trace operation and contains a convolution between top propagators multiplied by a convolution between bottom propagators. The right diagram depicts the trace result of equation (4.54), which transforms the structure into the multiplication of the leftmost top and bottom propagators, convolved with the multiplication of the rightmost top and bottom propagators, and introducing an overall factor of $\Gamma'_0/\gamma$. Note also that the rightmost propagators change from propagators with a photon in the field to no-photon propagators.

Returning now to two-particle relativistic SR, the vertical photon result with accompanying $\Gamma'_0/\gamma$ factor is ubiquitous. Consider, for example, the quantity

$$\rho_{1;1}(t) = \text{Tr}_{\text{rad}} \{ \langle 1, k | \hat{\rho}(t) | 1, k \rangle \} = \sum_k \rho_{1,k;1,k}(t)$$  \hspace{1cm} (4.55)

which represents the probability of finding the system at time $t$ in the state with only the first particle excited. Similar calculations as preceding the single particle emission result of equation (4.54) find that the four diagrams of Figure 4.3 together describe the quantity $\rho_{1;1}(t)$. These four diagrams exhaust the possible permutations of propagators obeying the following constraints: first, the leftmost side of a diagram must commence in both its top and bottom propagators in the excited state, corresponding to the starting state of the system at $t = t_0$; second, transition across any vertical photon line introduces a factor of $\Gamma'_0/\gamma$; third, in any region demarcated by vertical photon partitioning, propagators must couple states of like total number of excited particles, decreasing by one when crossing a vertical photon partition; and fourth, the singly excited propagators must terminate in the state $|1\rangle$ where the first particle in particular is excited, at least for this particular calculation of $\rho_{1;1}(t)$. Note when interpreting Figure 4.3 that a propagator matrix element $U_{b,a}$ describes transition from $|a\rangle$ to $|b\rangle$; i.e., from right to left in its indices, even though time flows from left to right in all diagrams.

The diagrammatic rules outlined above generalize to more complicated diagrams describing the quantity $\rho_{g,g}(t)$. One such (randomly chosen) diagram is shown in Figure 4.4. Propa-
Figure 4.3: Diagrams corresponding to the quantity $\rho_{1;1}(t)$ for a two-particle relativistic SR sample starting in the doubly excited state $|e\rangle$. Note that time flows from left to right in each diagram, but that a propagator conventionally denotes transition from the state of its right index to the state of its left index.

Propagators within the second (central) region of Figure 4.4 are no longer constrained to terminate in any particular state, which increases the number of total possible diagrams contributing to $\rho_{g;g}(t)$ to 16.

Actually, there is a great deal of symmetry to the set of differential Eqs. (4.47)–(4.50) and their initial conditions. From inspection, we see that $U_{1;2} = U_{2;1}$ and that $U_{1;1} = U_{2;2}$. Let us refer to the result of multiplying a top propagator with a bottom propagator within a region between vertical photons as a block propagator. In light of these symmetries and the constraints placed upon diagrams outlined above, the only possible block propagators for our system are

\begin{align}
B_e &= \left|U_{e;e}\right|^2 \\
B_a &= \left|U_{1;1}\right|^2, B_b = U_{1;1}^* U_{1;2}, B_c = \left|U_{1;2}\right|^2 \\
B_g &= \left|U_{g;g}\right|^2 = 1,
\end{align}

as well as $B_b^*$. 

\[ B_e = \left|U_{e;e}\right|^2 \] 
\[ B_a = \left|U_{1;1}\right|^2, B_b = U_{1;1}^* U_{1;2}, B_c = \left|U_{1;2}\right|^2 \] 
\[ B_g = \left|U_{g;g}\right|^2 = 1, \] 

\[ B_b^* \]
4.4. Relativistic Two-Particle SR: Theory

We are finally prepared to assess velocity coherence in a relativistic two-particle SR sample possessing some mean $\beta$ and observer frame $\Delta v$, by the following steps:

1. Solve the set of differential Eqs. (4.47)–(4.50) for the desired $\beta$ and $\Delta v$,

2. Compute the diagram for each of $\rho_{1;1} (t)$, $\rho_{2;2} (t)$ and $\rho_{g;g} (t)$ as follows:

   (a) For each region (dotted rectangles in Figs. 4.3 and 4.4), compute the block propagator by multiplying the top propagator by the bottom propagator,

   (b) Convolve all adjacent block propagators, and

   (c) Multiply the result by a factor of $\Gamma'_0/\gamma$ for each vertical photon in the diagram

3. Sum all diagram results for a given $\rho_{a,a}$.

4. In order to interpret a total emission power from the particle excitation time dependence, compute the time derivative of the total particle energy in the observer frame; i.e., $(2\rho_{ee} + \dot{\rho}_{1;1} + \dot{\rho}_{2;2}) \hbar \omega'_0/\gamma$. 

![Diagram](image-url)
In terms of our block propagators and symmetries, we have that

\[ \rho_{ee} (t) = B_e (t) \]  
(4.59)

\[ \rho_{1:1} (t) = \frac{\Gamma'}{\gamma} [B_e \star [B_a + 2\text{Re} (B_b) + B_c]] (t) \]  
(4.60)

\[ \rho_{2:2} (t) = \rho_{1:1} (t) \]  
(4.61)

\[ \rho_{gg} (t) = 4 \left( \frac{\Gamma'}{\gamma} \right)^2 \left\{ B_e \star [B_a + 2\text{Re} (B_b) + B_c] \star B_g \right\} (t). \]  
(4.62)

The above results are differentiated in step 4 above, which allows us to skip one convolution from any of the above calculations; for example,

\[ \dot{\rho}_{gg} (t) = 4 \left( \frac{\Gamma'}{\gamma} \right)^2 B_e (t) \left\{ [B_a + 2\text{Re} (B_b) + B_c] \star B_d \right\} (t) \]

\[ = 4 \left( \frac{\Gamma'}{\gamma} \right)^2 e^{2\Gamma_0 (t - t_0)/\gamma} \left\{ [B_a + 2\text{Re} (B_b) + B_c] \star B_g \right\} (t). \]  
(4.63)

### 4.5 Relativistic Two-Particle SR: Results

The first result of our theory is a confirmation of the time dilation of two-particle SR in the \( \Delta v \rightarrow 0 \) limit. In this limit the relativistic coherence kernel goes to 1, so that all factors of \( \gamma \) in Eqs. (4.47)–(4.50) may be removed by transforming to the variable \( \tau = t/\gamma \). Whatever the result of performing all steps required to compute a total emission power transient, then, that result will be a function of \( t/\gamma \) and thus time-contracted in the observer frame. This is not so much a useful result as it is a reassurance that our model behaves as expected.

The quantity of interest to this work is the behaviour of SR coherent enhancement as a function of the velocity difference of highly relativistic particles. Let us work in the variable \( q = \Gamma_0' \gamma / (2\gamma) \) such that the only non symmetry-redundant quantities of Eqs. (4.47)–(4.50) are related by

\[ \frac{d}{dq} \tilde{U}_{1:1} = -\tilde{U}_{1:1} - C_{\beta, \Delta v}^{\text{SR, rel}} \tilde{U}_{2:1} \]  
(4.64)

\[ \frac{d}{dq} \tilde{U}_{1:2} = -\tilde{U}_{1:2} - C_{\beta, \Delta v}^{\text{SR, rel}} \tilde{U}_{1:1}. \]  
(4.65)
4.5. Relativistic Two-Particle SR: Results

We have also at the outset that $\tilde{\mathcal{U}}_{ee}(q) = e^{-2q}$.

A Python script was written to compute $C^{\text{SR, rel}}_{\beta, \Delta \nu}$, to solve Eqs. (4.64) and (4.65), to compute the block propagators, to convolve the appropriate diagrams for the density operator traces, and finally to obtain the total power transient for various particle velocity differences and relativistic $\beta$ values.

In the simplest case, we first verify that our model demonstrates a transition of SR enhancement between the far separated $\Delta \nu = \infty$ limit and the coherent $\Delta \nu = 0$ limit. The photon emission rate at $\beta = 0$ is shown in Figure 4.5 for the coherent case $\Delta \nu = 0$ and for the well separated case $\Delta \nu = 100c\Gamma_0'/\omega_0'$. The former demonstrates SR enhancement, while the latter recovers the spontaneous emission rate from two independent particles.

Figure 4.5 does not differ substantially between the coherent SR ($\Delta \nu = 0$) and the independent spontaneous emission ($\Delta \nu \rightarrow \infty$) cases. In order to obtain a more dramatic SR pulse demonstrating enhanced and delayed peak intensity one would need to simulate a larger

![Figure 4.5: Photon emission rate as a function of time for two velocity separations $\Delta \nu = 0$ and $\Delta \nu = 100c\Gamma_0'/\omega_0'$ for $\beta = 0$.](image)
number of particles. Such an exercise becomes very complicated very quickly, as the number of diagrams requiring evaluation grows extremely fast with the number of particles. Combinatorial considerations or computer methods could speed the process up and make a slightly larger population manageable, but we leave such exercises to future work. Instead, we focus now on extracting meaningful information from the two-particle case.

For the purpose of quantifying the departure of an SR system from the independent limit as a function of observer frame velocity difference, let

\[ I_{\Delta v} (t) = 2\dot{\rho}_{e,e} + \dot{\rho}_{1:1} + \dot{\rho}_{2:2} \]  

(4.66)

for a sample having particle velocity difference \( \Delta v \), and define the relativistic velocity coherence metric

\[ G(\Delta v) = \lim_{T \to \infty} \frac{1}{A} \int_0^T \left[ I_{\Delta v} (t) - 2\frac{\Gamma'}{\gamma} e^{-\frac{\Gamma'_0}{\gamma}} \right]^2 \]  

(4.67)

which measures total departure of a given power transient from that transient which would be generated by independently emitting particles. The \( 1/A \) out front is a normalisation factor such that \( G(0) = 1 \). Figure 4.6 shows \( G(\Delta v) \) for three values of \( \beta \). The full-width half maxima (FWHM) of the various \( G(\Delta v) \) provide us with a measure of the velocity coherence required for two relativistic particles to demonstrate enhanced SR emission. For \( \beta = 0 \) the FWHM is approximately \( 9.1c\Gamma'_0/\omega'_0 \), while for \( \beta = 0.95 \) the FWHM is approximately \( 0.52c\Gamma'_0/\omega'_0 \).

There are numerous ways to interpret the various FWHM. In the simplest approach, let us compare their values in the context of the relativistic transformation of velocity differentials. The usual relativistic velocity composition rules state that small velocity differences transform as \( \Delta v' \approx \gamma^2 \Delta v \) between the observer and rest frames. The \( \beta = 0.95 \) plot of Figure 4.6 corresponds to \( \gamma^2 \approx 10.3 \); however, that plot’s FWHM differed from the \( \beta = 0 \) case by a factor of 17.5. This suggests that the velocity coherence required between two particles to demonstrate enhanced SR emission in the observer frame is more restrictive than accounted for by simple considerations of the relativistic transformation of velocity spreads.

The situation is further nuanced if we ask which velocity scale should be assigned meaning when quantifying velocity coherence requirements. We have thus far compared velocity differences in terms of the relativistically invariant unit \( c\Gamma'_0/\omega'_0 \); however, the researcher may be
Figure 4.6: The relativistic velocity coherence metric $G(\Delta v)$ for $\beta = 0, 0.8, 0.95$. 
more interested in absolute velocity differences in a selected reference frame. Whether velocity coherence requirements become more or less restrictive from a relativistic source is thus not well-defined in a general sense, and will depend upon what other interactions are of interest to the researcher. Moreover, many other relativistic effects may enter into the picture. We discuss some of these effects in Section 4.6.

4.6 Summary and Future Work

We constructed a model of velocity dependent relativistic two-particle SR based upon an existing model [2] of relativistic spontaneous emission from a two-level particle. We applied the diagrammatic method of quantum optics to our relativistic SR Hamiltonian and found that certain standard results and techniques of the literature required revision. The introduction of velocity dependence caused a loss of convolution structure in zero-photon propagator calculations. We introduced a set of coupled differential equations, as well as the so-called relativistic velocity coherence kernel, which together enabled the calculation of SR propagators. The standard vertical photon result of the literature was re-proven and was shown to introduce an overall Lorentz factor into the density operator diagrams of SR.

We solved our relativistic SR model in Section 4.5 for two particles and characterized velocity coherence requirements for SR emission enhancement from relativistic sources at various $\beta$ factors. Initial results suggested that velocity coherence requirements for enhanced SR emission in the observer frame become more restrictive as a source moves with higher relativistic velocity. Some caveats were made in Section 4.5 regarding the interpretation of velocity coherence requirements, which in reality becomes a nuanced exercise.

The model constructed in this paper forms a foundation for relativistic SR work. Future research could proceed in numerous directions. First, it remains to yet solve the present model for (i) dipoles aligned parallel to the particles’ velocities, (ii) dipoles not oriented parallel to each other, and (iii) non co-linear particle velocities. Second, the diagrammatic method may also be used to calculate the spectrum and angular photon distribution from an SR source. This procedure could be applied to the present model and radiation intensity could be characterized across different directions from a relativistic SR source. Some initial calculation steps for this
purpose are provided in Appendix J. Third, this model could be expanded to a larger number of particles. Although the diagrammatic method becomes quickly cumbersome with increasing population, computer methods exploiting combinatoric considerations could certainly simulate a moderately sized sample. Despite remaining physically negligible, a sample of four or more particles begins to demonstrate the salient SR features of delayed and enhanced peak intensity. A relativistic model of these features would be especially helpful to astrophysicists. Fourth, this model could be generalized to an extended sample by paralleling existing derivations of the Maxwell-Bloch equations, now with our relativistic SR Hamiltonian. Considerations of relativistic length contraction or time dilation of relaxation processes in such an extended model could lead to richer relativistic SR behavior.
Appendix G

Synopsis of the Original BBB Hamiltonian Derivation [2] and Manipulation into its Present Form

In this Appendix we provide a synopsis of the physical motivation, derivation, and manipulation of the BBB Hamiltonian first presented by Boussiakou, Bennett, and Babiker [2] and slightly revised in form for the purpose of this work.

G.1 The BBB Hamiltonian

Prior to canonical quantization, construction of the BBB model starts from the usual Lagrangian of a positive and a negative charge, with generalized coordinates corresponding to the center of mass coordinate and to the charge separation in a frame co-moving with the particle’s center of mass. Boussiakou, Bennett, and Babiker [2] select the Power-Zienau-Woolley (PZW) gauge for the electromagnetic field, which permits them to express the Lagrangian in the particle rest frame in a manifestly covariant form by contracting the field tensor with itself (the usual free field contribution to the Lagrangian) and contracting the field tensor with the polarization-magnetization tensor (a covariant description of the matter-field interaction). The polarization and magnetization fields comprising the latter tensor in the PZW gauge are constructed from the charge separation coordinate and its velocity via a prescription provided by
The covariant Lagrangian formulation enables Boussiakou et al. to determine, via careful Lorentz transformations, an expression for the observer frame Hamiltonian in terms of the observer frame center of mass coordinates, charge separation coordinates, and radiation field; these observer frame quantities are coupled to the particle’s rest frame polarization and magnetization properties. This form of the Hamiltonian ultimately enables the BBB model to relate observer frame behaviour to transition operators intrinsic to the particle. The BBB Hamiltonian in the observer frame reads,

$$\hat{H} = \hat{H}_0^a + \hat{H}_0^f + \hat{V}$$  \hspace{1cm} (G.1)

$$\hat{H}_0^a = \hat{H}_{KG, \text{com}} + \frac{1}{2\gamma\mu} \left( \hat{\rho}_\perp^2 + \frac{\hat{\rho}_\parallel^2}{\gamma^2} \right) - \frac{e^2}{4\pi\varepsilon_0 \gamma \hat{q}'}$$  \hspace{1cm} (G.2)

$$\hat{V} = \int \mathrm{d}^3 r \left\{ \frac{1}{\varepsilon_0} \left[ \hat{\mathcal{P}}_\parallel' (\mathbf{r}) + \gamma \hat{\mathcal{P}}_\perp' (\mathbf{r}) \right] \cdot \hat{\mathbf{E}} (\mathbf{r}) + \frac{\hat{\mathbf{P}}}{2M} \cdot \left[ \hat{\mathcal{P}}_\parallel' (\mathbf{r}) \times \hat{\mathbf{B}} (\mathbf{r}) \right] + \left[ \hat{\mathcal{P}}_\perp' (\mathbf{r}) \times \hat{\mathbf{B}} (\mathbf{r}) \right] \cdot \frac{\hat{\mathbf{P}}}{2M} \right\}$$  \hspace{1cm} (G.3)

where a few outstanding terms have not yet been defined within the body of the paper. The position operators $\hat{q}_\parallel$ and $\hat{q}_\perp$ describe the charge separation coordinates (unprimed in the observer frame, primed in the rest frame) parallel and perpendicular to the center of mass velocity, the operators $\hat{p}_\parallel$ and $\hat{p}_\perp$ are their conjugate momenta operators, and the center of mass has an observer frame momentum operator $\hat{P}$. The Lorentz factor $\gamma$ corresponds to the initial center of mass momentum of the particle. The operator $\hat{q}'$ is the charge separation distance operator in the rest frame coordinates, $\mu$ is the rest frame reduced mass of the particle, and $\hat{\mathcal{P}}'$ is the particle’s rest frame polarization operator. For simplicity we have suppressed magnetization terms, consistent with the paper’s restriction to the case of an electric dipole which couples to the observer frame electric ($\hat{\mathbf{E}}$) and magnetic ($\hat{\mathbf{B}}$) fields. The second line of Eq. (G.3) is the so-called Röntgen interaction term.

\footnote{namely, Eqs. (6) and (7) of [2] for reference}
G.2 Eigenfunctions of the unperturbed BBB Hamiltonian

We require the unperturbed eigenfunctions of the atomic Hamiltonian term defined by Eq. (G.2). We present here the main results of the solution provided in [2]; the eigenfunction work below is adapted directly from their work.

Projecting onto the observer frame center of mass and charge separation coordinate basis and factoring the wavefunction as \( \Psi(Q, q) = e^{i K \cdot Q} \psi(q) \), one obtains

\[
\left[ -\frac{1}{2\mu} \left( \nabla_{q,\perp}^2 + \frac{1}{\gamma^2} \frac{\partial^2}{\partial q_{\parallel}^2} \right) - \frac{e^2}{4\pi \epsilon_0 q'} \right] \psi(q) = \gamma \epsilon \psi(q), \tag{G.4}
\]

for \( \epsilon \) the (observer frame) energy of the state. Eq. (G.4) may be expressed in rest frame coordinates as

\[
\left[ -\frac{1}{2\mu} \nabla_{q'}^2 - \frac{e^2}{4\pi \epsilon_0 q'} \right] \psi(q') = \gamma \epsilon \psi(q'), \tag{G.5}
\]

which is the usual hydrogenic Schrödinger equation, but where the energy eigenvalues of the conventional solutions correspond now to \( \gamma \) times the energy eigenvalues in the observer frame. Additionally, reversion back to the observer frame will introduce a length contraction to the conventional hydrogenic wavefunction shapes. Both of these results [2] are consistent with familiar results from the special theory of relativity [4].

It is important to keep in mind that the primed coordinates are simply a mathematical convenience for working with a wavefunction which is physically relevant to the observer frame. Although the unprimed and primed coordinates are related by the familiar Lorentz transformation between the observer and rest frame position coordinates, the primed coordinates serve only as a tool for solving Eq. (G.4). The physics thus far developed has been based upon canonical quantization in the observer frame and meaningful conclusions from the present wavefunction can only be made with respect to that frame.

We now make an important point regarding normalization which affects our diagrammatic calculations in this work. The energy eigenfunction relation of Eq. (G.4) becomes hydrogenic in the primed coordinates of (G.5); we choose therefore to use the conventional hydrogenic wavefunctions in the primed coordinates, or their transformed version in the unprimed coordinates. Such functions are no longer normalized in the unprimed (observer) frame due to the
factor of $\gamma$ relating the coordinates parallel to the particle’s motion. We might alternatively re-normalize the eigenfunctions in the observer frame; however, we will later be interested in expanding operators in basis states indexed by the primed (rest frame) coordinates. We therefore retain the conventional normalization of the hydrogenic wavefunctions relative to the primed coordinates, but remembering always that the completeness must then be asserted relative to the primed basis states; i.e.,

$$\hat{I} = \int d^3 q' |q'(q')\rangle \langle q'|.$$  \hspace{1cm} (G.6)

### G.3 The BBB Hamiltonian in its form useful to this work

In this section we summarize the work of [2] to express the BBB Hamiltonian in terms of operators and properties intrinsic to the particle, including internal raising and lowering operators $\hat{\pi}^\dagger$ and $\hat{\pi}$, the internal dipole moment $\vec{d}'$, and the rest frame transition energy $\hbar \omega_0'$. We also introduce a simplified form of the interaction Hamiltonian useful to this work.

The operator $\hat{\mathcal{P}}'$ appearing in the interaction operator $\hat{\mathcal{V}}$ of Eq. (G.3) refers to the polarization vector operator in the rest frame, and it couples to field operators in the observer frame. Boussiakou, Bennett, and Babiker [2] assert the dipole approximation to express the rest frame polarization operator as $\hat{\mathcal{P}}'(r') = \hat{\mathcal{d}}' \delta(\mathbf{r}' - \mathbf{Q}')$, where $\hat{\mathcal{d}}' = e \hat{q}'$. With this approximation, we perform first the integration over $d^3r$ of Eq. (G.3) in the observer frame; however, we must recognize that the arguments of the delta distribution are in the rest frame coordinates, and thus its support is length-contracted in the observer frame; i.e., the integration picks up a factor of $1/\gamma$ and we are left with

$$\hat{\mathcal{V}} = -\left(\frac{1}{\gamma} \hat{\mathcal{d}}'_\parallel + \hat{\mathcal{d}}'_\perp\right) \cdot \mathbf{E}(Q) + \frac{\mathbf{P}}{2M\gamma} \cdot \left[ \hat{\mathcal{d}}' \times \mathbf{B}(Q) \right] + \left[ \hat{\mathcal{d}}' \times \mathbf{B}(Q) \right] \cdot \frac{\mathbf{P}}{2M\gamma}. \hspace{1cm} (G.7)$$

Let us prepare ourselves for the theory of dipole emission between two energy eigenstates.
\[ |E_{1,2}\rangle \text{ of the particle. We expand the dipole operator as} \]
\[ \hat{d}' = e\hat{q}' \]
\[ = e \sum_{m,n} |E_m\rangle\langle E_n| \hat{q}' \langle E_n| \langle E_m|. \]  
(G.8)

The matrix elements are computed as
\[ \langle E_m|\hat{q}'|E_n\rangle = \langle E_m|\hat{q}' \int d^3\hat{q}' \langle \hat{q}'|E_n\rangle \]
\[ = \int d^3\hat{q}' \psi^*_m (\hat{q}') \hat{q}' \psi_n (\hat{q}'), \]  
(G.9)

where we intentionally introduced the completeness relation over the primed coordinates as per our earlier discussion. This final expression is mathematically the same as the conventional dipole matrix element calculation for a stationary particle, having nonzero elements only when \( m \neq n \). We need not perform the actual integration over the hydrogenic wavefunctions, since we are concerned only with relating our theory to that of a stationary particle; we thus have
\[ \hat{d}' = \vec{d}' \left( \hat{\pi}^\dagger + \hat{\pi} \right) \]  
(G.10)

for \( \hat{\pi}^\dagger \) and \( \hat{\pi} \) the internal raising and lowering operators of the particle, and where \( \vec{d}' \) is the rest frame dipole moment.

Summarizing the above results and expanding the field operators in terms of creation and annihilation operators, the BBB Hamiltonian is expressed using rest frame quantities and operators as
\[ \hat{H}_a^0 = \hat{H}_{KG,\text{com}} + \frac{1}{\gamma} \hbar \omega' \hat{\pi}^\dagger \hat{\pi} \]  
(G.11)

\[ \hat{V} = \sum_k \left[ -\left( \frac{1}{\gamma} \hat{d}'^\dagger + \hat{d}' \right) \cdot \xi_k \left( \hat{\xi}^\dagger_k \hat{\pi}^\dagger \hat{a}_k + \hat{\xi}_k \hat{\pi} \hat{a}_k^\dagger \right) \right] \]
\[ + \frac{1}{2M\gamma c} \left( \vec{d}' \times \vec{e} \right) \cdot \left\{ \hat{P}_r, \xi_k \hat{L}_k^\dagger \hat{\pi}^\dagger \hat{a}_k + \xi_k^\ast \hat{L}_k \hat{\pi} \hat{a}_k^\dagger \right\}. \]  
(G.12)

In this work we consider the evolution of an initially excited particle starting with a center
of mass momentum $P_i$. This initial state is coupled by Eq. (G.12) only to single photon states differing in center of mass momentum from $P_i$ by the momentum of the photon present; we may therefore simplify the anti-commutator of Eq. (G.12) by investigating its action upon this restricted subspace of (zero- or single-photon) states. For example, consider the term

$$\{ P, \xi_k \hat{L}_k^\dagger \pi^\dagger \hat{a}_k \} |P_i - \hbar k, g, k\rangle = \xi_k (2P_i - \hbar k) |P_i, e, 0\rangle$$

where $e$ or $g$ denote the excited or ground states of the particle and 0 or $k$ denote the zero photon or single photon of mode $k$ states. A similar result holds for the second term in the anti-commutator applied to a state $|P_i, e, 0\rangle$. For a relativistic system the photon momentum is much less than the initial particle momentum, i.e., $\hbar k \ll P_i$. Recognizing that $P_i = \gamma MV_i$ we may thus write for the interaction term,

$$\hat{V} = \sum_k \left[ \left( -\frac{1}{\gamma} \vec{d}_i + \vec{d}_\perp \right) \cdot \vec{\epsilon}_k \right] \hat{L}_k \hat{a}_k + \left( \frac{\vec{d}_i \times \vec{\epsilon}_k}{c} \right) \cdot \frac{V_i}{c} \left( \xi_k \hat{L}_k \pi^\dagger \hat{a}_k + \xi_k^* \hat{L}_k \hat{a}_k \pi^\dagger \right).$$

This is the form of the BBB interaction term used within the paper.
Appendix H

Diagrammatic Representation of the Time-Evolution Operator

This appendix follows closely the standard procedures of the literature; see, for example, the summary work of [1] based upon the original papers [15, 17, 18]. We work with the state of the system \( |\tilde{\Psi}(t)\rangle \) in the interaction picture defined as

\[
|\tilde{\Psi}(t)\rangle \equiv e^{i\hat{H}_0(t-t_0)/\hbar} |\Psi(t)\rangle.
\] (H.1)

The state of the system at time \( t \) is related to its original state at time \( t_0 \) via the interaction picture time evolution operator \( \tilde{U}(t, t_0) \); i.e.,

\[
|\tilde{\Psi}(t)\rangle = \tilde{U}(t, t_0) |\tilde{\Psi}(t_0)\rangle.
\] (H.2)

The Schrödinger equation for \( \tilde{U}(t, t_0) \) in the interaction picture reads

\[
i\hbar \frac{d\tilde{U}}{dt} = \tilde{V} \tilde{U}
\] (H.3)

where

\[
\tilde{V}(t) = e^{i\hat{H}_0(t-t_0)} \tilde{V} e^{-i\hat{H}_0(t-t_0)}.
\] (H.4)
We require in this work the exact first-order recurrence relation
\[
\widetilde{U}(t, t_0) = 1 + \left[ -\frac{i}{\hbar} \right] \int_{t_0}^{t} dt_1 \widetilde{V}(t_1) \widetilde{U}(t_1, t_0),
\]
(H.5)
as well as the exact second-order recurrence relation
\[
\widetilde{U}(t, t_0) = 1 + \left[ -\frac{i}{\hbar} \right] \int_{t_0}^{t} dt_1 \widetilde{V}(t_1) + \left[ -\frac{i}{\hbar} \right]^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_2} dt_2 \, \text{d} t_1 \widetilde{V}(t_2) \widetilde{V}(t_1) \widetilde{U}(t_1, t_0).
\]
(H.6)

We also make use of the infinite series expansion
\[
\widetilde{U}(t, t_0) = 1 + \left[ -\frac{i}{\hbar} \right] \int_{t_0}^{t} dt_1 \widetilde{V}(t_1) + \left[ -\frac{i}{\hbar} \right]^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_2} dt_2 \, \text{d} t_1 \widetilde{V}(t_2) \widetilde{V}(t_1) + \ldots
\]
(H.7)

A matrix element \( \tilde{U}_{nm} = \langle E_n | \tilde{U} | E_m \rangle \) coupling two free theory energy eigenstates \( | E_n \rangle \) and \( | E_m \rangle \) may be interpreted by momentarily leaving the interaction picture. The composition of adjacent occurrences of \( \tilde{V} \) within each integrand translates to the \((k, l)^{th}\) non-interaction picture matrix element
\[
\tilde{V}(t_{n+1}) \tilde{V}(t_n) \rightarrow e^{i\hbar \omega_{pq} t_{n+1}} V_{kp} e^{-i\hbar \omega_{pq} t_n} V_{pl} e^{-i\hbar \omega_{pq} t_n} = e^{i\hbar \omega_{pq} t_{n+1}} V_{kp} F_p(t_{n+1}, t_n) V_{pl} e^{-i\hbar \omega_{pq} t_n}
\]
(H.8)

where summation over \( p \) is implied and where \( \hbar \omega_{pq} \) is the energy of a free theory energy eigenstate \( | E_q \rangle \). The term
\[
F_p(t_{n+1}, t_n) = e^{-i\hbar \omega_{pq} (t_{n+1} - t_n)}
\]
(H.9)
is the free theory propagator of the system in state \( | E_p \rangle \) from time \( t = t_n \) to time \( t = t_{n+1} \).

The sequence \( V_{kp} F_p(t_{n+1}, t_n) V_{pl} \) within expression (H.8) is interpreted from right to left as the physical process wherein the system (i) transitions from state \( | E_l \rangle \) to state \( | E_p \rangle \) at time \( t_n \), (ii) propagates freely in state \( | E_p \rangle \) from time \( t_n \) to time \( t_{n+1} \), and (iii) transitions from state \( | E_p \rangle \) to state \( | E_k \rangle \) at time \( t_{n+1} \). Each term in Eq. (H.7) may be interpreted in this manner, where the order of a term in the expansion corresponds to the number of transition events. For each event order sequence, we are directed to integrate over every possible timing of the transition events.
Figure H.1: The first two non-vanishing terms in the diagrammatic representation of the time evolution operator matrix element $\tilde{U}_{e,0,e,0}$. Transition events in the second order term occur at the vertical dashed lines.

which satisfies the ordering $t > t_n > t_{n-1} > \cdots > t_0$. The summation over transition matrix elements during matrix multiplication of adjacent $V_{kl}$ terms directs us to sum over all possible intermediate transition states.

The particular form of the interaction of interest in this work, given by Eq. (G.14), yields transitions from a state $|e, 0\rangle$ (excited particle with no photon in the radiation field) only to states of the form $|g, k\rangle$ (ground state particle with one photon of mode $k$ in the radiation field); and transitions from a state of the form $|g, k\rangle$ only to the state $|e, 0\rangle$. This interaction therefore greatly restricts the collection of possible transition event sequences; for example, a transition event of the form $|e, k\rangle \rightarrow |e, k'\rangle$ for $k \neq k'$ is strictly forbidden 1.

Terms in the series expansion of a time evolution operator matrix element are conventionally brought into correspondence with diagrams depicting sequences of transition events. In light of our previous comments, all possible transitions of a single particle system evolving with the interaction defined by Eq. (G.14) may be found in Fig. H.1, which shows the first two non-vanishing terms in the diagrammatic representation of the matrix element corresponding to an initially excited particle being found in the excited state after time $t - t_0$.

---

1We refer in such statements to single order processes; in this case, for example, a higher order process involving $|e\rangle \rightarrow |g\rangle \rightarrow |e\rangle$ particle energy level transitions could still yield such a transition
Appendix I

Calculation of the Relativistic Velocity Coherence Kernel

We evaluate in this section the convolution violating term appearing in Eq. (4.39):

\[
\left[-\frac{i}{\hbar}\right]^2 \int_{t_0}^{t_2} dt_1 \int_{t_0}^{t_2} dt_2 \sum_k \tilde{V}_{1,0;\mathbf{k}}(t_2) \tilde{V}_{g,\mathbf{k};2,0}(t_1) \tilde{U}_{2;1}(t_1, t_0) = \left[-\frac{i}{\hbar}\right]^2 \int_{t_0}^{t_2} dt_1 \int_{t_0}^{t_2} dt_2 \sum_k e^{i(\omega_0'/\gamma-\alpha_k \omega_k)t_2} V_{1,0;\mathbf{k}}(t_2) e^{-i(\omega_0'/\gamma-\alpha_k \omega_k)t_1} V_{g,\mathbf{k};2}(t_1) \tilde{U}_{2;1}(t_1, t_0). \tag{I.1}
\]

by our prescription of Eq. (4.15). Let us assume that both dipoles are aligned with the \(x\) axis, that both velocities are in the \(z\) direction, and that the mode \(\mathbf{k}\) has length \(k\) and direction \(\Omega\) (with polar and azimuthal parts). Suppressing for a moment the \(t_1, t_2\) integrals and \(\tilde{U}_{2;1}\), the sum over \(\mathbf{k}\) (including the leading \([-i/\hbar]^2\) multiplier) becomes

\[
-\frac{\Gamma_0' \gamma c^4}{\omega_0^3} \frac{3}{16\pi^2} \int d\Omega \left[ \cos^2 \phi (\beta - \cos \theta)^2 + \sin^2 \phi (1 - \beta \cos \theta)^2 \right] \times \int_0^{\infty} dk k^3 e^{i(\omega_0'/\gamma-\bar{\omega}_k)(t_2-t_1)} e^{i k \Delta v \cos \theta (t_1+t_2)/2} = -\frac{\Gamma_0' \gamma c^4}{\omega_0^3} \frac{3}{16\pi} \int_{-1}^{1} dx \left[ (1 + \beta^2)(1 + x^2) - 4\beta x \right] \int_0^{\infty} dk k^3 e^{i(\omega_0'/\gamma-\bar{\omega}_k)(t_2-t_1)} e^{i k \Delta v x(t_1+t_2)/2} \tag{I.2}
\]

where \(\bar{\omega}_k = \omega_k \alpha_k\) with \(\alpha_k = 1 - \beta \cos (\theta)\) for \(\theta\) the angle between the mode propagation direction and mean velocity \(\mathbf{v} = (\mathbf{v}_1 + \mathbf{v}_2)/2\) (or \(|\mathbf{v}| = \beta c\)). The velocity difference is \(\Delta \mathbf{v} =\)
\[ v_2 - v_1. \]

We now assume that the integration over \( k \) contributes predominantly near where \( \omega'_0/\gamma - \bar{\omega}_k = 0 \). We can therefore pull the slowly-varying part \( k^3 e^{i k \Delta x (t_1 + t_2)/2} \) out of the integral over \( k \) and evaluate it at \( k'_0/(\gamma a_k) \), make a change of integration variable, and formally extend our integration limits to obtain,

\[
\begin{align*}
&= -\Gamma'_0 \frac{3}{16\pi} \int_{-1}^{1} dx \frac{(1 + \beta^2)(1 + x^2) - 4\beta x}{\gamma^4 \alpha_k^4} e^{ik'_0 \Delta x (t_1 + t_2)/(2\gamma a_k)} \int_{-\infty}^{\infty} d\bar{\omega} e^{-i\bar{\omega}(t_2 - t_1)} \\
&= -\frac{\Gamma'_0}{\gamma^3} \frac{3}{8} \int_{-1}^{1} dx \frac{(1 + \beta^2)(1 + x^2) - 4\beta x}{(1 - \beta x)^4} e^{ik'_0 \Delta x (t_1 + t_2)/[2\gamma(1 - \beta x)]} \delta(t_2 - t_1)
\end{align*}
\]

Plugging this result back into our time integrals above, and recalling that \( t_1 \) only integrates up to \( t_2 \) (and thus only picks up half of the support of the Dirac delta distribution), we can immediately execute the integral over \( t_1 \) to obtain

\[
\begin{align*}
&= -\frac{\Gamma'_0}{\gamma^3} \frac{3}{16} \int_{t_0}^{t'} d't' \int_{-1}^{1} dx \frac{(1 + \beta^2)(1 + x^2) - 4\beta x}{(1 - \beta x)^4} e^{ik'_0 \Delta x x'/[\gamma(1 - \beta x)]} \tilde{U}_{2;1} (t', t_0),
\end{align*}
\]

which is our desired expression.
Appendix J

Photon Emission Propagator for the BBB Hamiltonian

We provide in this appendix the single particle propagators corresponding to evolution into non-zero photon states in the BBB Hamiltonian. These propagators are not necessary for the relativistic coherence study of the present work, but would be essential to potential future research into the emission spectrum from a relativistic SR system. We require for this purpose the elements

\[ V_{g,k:e,0} = \langle g, k|V|e, 0 \rangle \]

\[ = \left[ -\frac{1}{\gamma} \vec{d}_x + \vec{d}_z \right] \cdot \epsilon_k + \left( \vec{d}_x \times \epsilon'_k \right) \cdot \frac{V_j}{c} \xi_k. \]  

(J.1)

Let us consider first the case where \( V_i \) is parallel to \( \vec{d}' \). Using the \( \epsilon_{k\theta} \) and \( \epsilon_{k\phi} \) polarizations of Section 4.4.2, only the former yields a non-vanishing interaction matrix element

\[ V^\parallel_{g,k\theta:e,0} = \xi^*_k d' \sin \theta_k / \gamma \]  

(J.2)

(\text{where } V^\parallel \text{ denotes that this matrix element is computed for } V_i \parallel \vec{d}') \text{ or, in the interaction picture,}

\[ V^\parallel_{g,k\theta:e,0} = e^{i\omega_k \theta t} (\xi^*_k d' \sin \theta_k / \gamma) e^{-i\omega_k \theta t} \]

\[ = e^{-i(\omega_0^p / \gamma - \alpha_k \omega_k)} \xi^*_k d' \sin \theta_k / \gamma \]  

(J.3)
which has Laplace transform

\[
\tilde{V}_{Q,k\theta,e,0}^\parallel = \frac{\xi_k^* d' \sin \theta_k / \gamma}{s + i \left( \omega_0' / \gamma - \alpha_k \omega_k \right)}.
\]  

(J.4)

The emission matrix element \( \tilde{U}_{g,k\theta,e,0} \) of the time evolution operator may be found from the first order exact relation,

\[
\tilde{U}_{g,k\theta,e,0}^\parallel (t, t_0) = -\frac{i}{\hbar} \int_{t_0}^{t} dt_1 \sum_m \tilde{V}_{g,k\theta,m}^\parallel (t_1) \tilde{U}_{m,e,0} (t_1, t_0).
\]  

(J.5)

The sum over \( m \) picks up only the \( |e, 0 \rangle \) state, yielding

\[
\tilde{U}_{g,k\theta,e,0}^\parallel (t, t_0) = -\frac{i}{\hbar} \int_{t_0}^{t} dt_1 \tilde{V}_{g,k\theta,e,0}^\parallel (t_1) \tilde{U}_{e,0,e,0} (t_1, t_0)
\]  

(J.6)

which becomes in the Laplace domain

\[
\tilde{U}_{g,k\theta,e,0}^\parallel = -\frac{i}{\hbar} \begin{bmatrix}
\xi_k^* d' \sin \theta_k / \gamma \\
\xi_k d' \sin \theta_k / \gamma
\end{bmatrix} \begin{bmatrix}
\frac{1}{s + \Gamma_0'/ (2\gamma)} \\
\frac{1}{\Gamma_0'/ (2\gamma) + i (\alpha_k \omega_k - \omega_0') / \gamma}
\end{bmatrix}
\]  

(J.7)

such that in the time domain,

\[
\tilde{U}_{g,k\theta,e,0}^\parallel (t, t_0) = -\frac{i}{\hbar} \begin{bmatrix}
\xi_k^* d' \sin \theta_k / \gamma \\
\xi_k d' \sin \theta_k / \gamma
\end{bmatrix} \begin{bmatrix}
\frac{1}{s + \Gamma_0'/ (2\gamma)} \\
\frac{1}{\Gamma_0'/ (2\gamma) + i (\alpha_k \omega_k - \omega_0') / \gamma}
\end{bmatrix} e^{i(\alpha_k \omega_k - \omega_0') \Delta t} - e^{-\Gamma_0' \Delta t / (2\gamma)}.
\]  

(J.8)

In the limit as \( t \to \infty \) the probability of a photon having been emitted of mode \( k\theta \) is

\[
\mathcal{P}_{k\theta}^\parallel (t = \infty) = \frac{|\xi_k|^2}{\hbar^2} \frac{d^2 \sin^2 \theta_k / \gamma^2}{\left( \omega_0' / \gamma - \alpha_k \omega_k \right)^2 + \Gamma_0'^2 / (2\gamma)^2}
\]  

(J.9)

which is a Lorentzian centered at the expected relativistic Doppler shifted frequency \( \omega = \omega_0' / \left( \gamma(1 - \cos \theta V / c) \right) \). Although this result constitutes another validation of our methods and of the BBB Hamiltonian proposed in [2], it is the propagator of Eq. (J.8) that is of use to
future researchers. Note that the arbitrary fiducial quantisation volume present in $|\xi_k|^2$ may be removed by changing over to a mode probability density in $\{\Omega, \omega\}$-space and working with the physically significant intensity, proportional to the probability density per solid angle.

The above was computed for the case where $V_i$ was parallel to $\vec{d}'$; if $V_i$ is now perpendicular to $\vec{d}'$, we obtain the propagators

$$
\tilde{U}^+_{g,k\theta,e,0}(t, t_0) = -\frac{i}{\hbar} \frac{\xi_k^* d'(\sin \theta_k - \cos \phi_k V_i/c)}{\Gamma_0'/(2\gamma) + i(\alpha_k \omega_k - \omega_0'/\gamma)} \left[ e^{i(\alpha_k \omega_k - \omega_0'/\gamma)\Delta t} - e^{-\Gamma_0'\Delta t/(2\gamma)} \right]
$$

(J.10)

$$
\tilde{U}^+_{g,k\phi,e,0}(t, t_0) = -\frac{i}{\hbar} \frac{\xi_k^* d' \cos \theta_k \sin \phi_k}{\Gamma_0'/(2\gamma) + i(\alpha_k \omega_k - \omega_0'/\gamma)} \left[ e^{i(\alpha_k \omega_k - \omega_0'/\gamma)\Delta t} - e^{-\Gamma_0'\Delta t/(2\gamma)} \right]
$$

(J.11)

where $\phi_k$ is the angle between $V_i$ and the projection of $k$ onto the plane perpendicular to $\vec{d}'$. 
Chapter 4 Bibliography


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Chapter 5

Conclusion and Future Plans

5.1 Summary

In this thesis the role of Dicke’s superradiance (SR) in astrophysics was refined by modelling realistic, wide-bandwidth velocity distributions as well as relativistic source velocities in populations of SR particles. In Chapter 2 we commenced with a numerical study of the velocity dependent Maxwell-Bloch (MB) partial differential equations, and found their complexity to scale as $O(n^2)$ in the total number of simulated velocity channels $n$. Prior to the work of this thesis it was believed that realistic conclusions regarding wide bandwidth SR could be extrapolated from simulations of distributions possessing relatively moderate bandwidths; however, the emergence of extremely wide bandwidth polarisation phase correlation established the need for a new simulation algorithm of superior complexity scaling over conventional methods. Chapter 2 therefore sought inspiration from the efficient maser modelling methods of Mene-gozi and Lamb [7]; however, upon demonstrating their approach inadequate to the transient SR regime, we derived the novel so-called Integral Fourier (IF) algorithm which achieves $O(n)$ complexity scaling. We demonstrated its convergence to the benchmark solution determined by a conventional time-domain (CTD) algorithm, and prescribed approximation fidelity demands as a function of the bandwidth of the emergent SR transient.

Chapter 3 applied novel numerical methods to the study of extremely wide bandwidth velocity distributions in the transient SR regime. A second novel algorithm—namely, the Supplementary Fields (SF) algorithm—was derived which achieves $O(n)$ complexity scaling, but
which operates entirely within the time domain and thereby achieves superior baseline performance over the IF algorithm. Notably, however, the SF algorithm is computationally advantageous over either the CTD or the IF algorithms only if the simulated transient SR process demonstrates limited emergent polarisation phase correlation.

It was found that swept inversion processes lead to extremely wide bandwidth polarisation phase correlation and, therefore, that the CTD algorithm is best suited to their simulation. The CTD algorithm was applied to numerous swept inversion experiments, and it was demonstrated that in such configurations the salient transient SR timescales of peak intensity delay and duration are quenched in the widely Doppler broadened limit. It was hypothesised and confirmed that a velocity distribution possessing certain stochastic characteristics can, alternatively, support transient SR features (though with modification) from the swept inversion of an arbitrarily wide bandwidth population.

Conversely, a transverse inversion was demonstrated to be able to sustain temporal SR features in the widely Doppler broadened limit, so long as it is of sufficient length to enter the so-called Arecchi-Courtens limited regime. In said regime, the increasing correlation of velocity channels with increasing distribution width is arrested by the light-speed propagation of correlation information down the length of the sample. The temporal duration of a transversely initiated SR pulse in an arbitrarily widely Doppler broadened sample was shown to be proportional to the sample’s length.

In Chapter 4 we developed a relativistic model of SR built upon an existing relativistic model [2] of spontaneous emission from a single particle. We sought a strictly formal approach to solving the resulting Hamiltonian which invoked the diagrammatic methods of the literature; however, introducing velocity dependence to the problem incurred a deviation from conventional convolution structures and introduced the so-called relativistic coherence kernel and an associated set of differential equations requiring solution alongside typical diagrammatic techniques. We applied the method to two relativistic particles and found that the observation of SR from such a sample demands finer velocity coherence than suggested by simplistic considerations of the relativistic transformation of velocity differentials.
5.2 Future Work

Both the non-relativistic and relativistic components of this thesis present opportunities for future research. For transversely inverted, widely Doppler broadened non-relativistic distributions, even the extremely wide bandwidths simulated in Chapter 3 represent only a negligible statistical fraction of a realistic astrophysical environment. It is not yet clear if temporal ringing structure would emerge in the extremely large number statistics generated by a full velocity bandwidth as well as a large collection of SR samples comprising a solid angle of astronomical observation. Earlier work [8] strongly suggests the presence of SR ringing in certain astrophysical processes of very long temporal durations. We suspect that there exists some transformation relating the transverse simulation results to expected real telescope observations and which recovers the ringing phenomenon verified to exist within fine bandwidths of the simulation outputs; however, no such transformation has yet been rigorously established.

The early time evolution of SR in a widely Doppler broadened sample is not, strictly speaking, formally justified by the results in [5, 6, 1], where the derivation of the initial tipping angle prescription assumed velocity coherence. In Chapter 3 we made the judicious choice that the tipping angle of a velocity channel be equal to the initial Bloch angle defined for that total population of particles existing within a velocity width equivalent to the bandwidth of the transient SR process. Alternatively, a proper theoretical analysis of the initial stages of widely Doppler broadened SR should proceed via canonical quantization of a continuum velocity distribution, in order to accurately generalise the initial tipping angle result of Gross and Haroche [5] to the non-coherent case.

The results of Chapter 3 regarding the ability of a statistical velocity distribution to support SR temporal structure (including a finite delay to peak intensity and a finite emission duration) should be analysed in the context of realistic turbulent astrophysical (or other) environments. Future work could investigate the relationship between the turbulent properties of the ISM or other media [9, 4] and the statistical velocity distribution properties prescribed in Chapter 3 in order to assess the feasibility of generating transient astrophysical SR pulses via a swept inversion mechanism.

The relativistic work of Chapter 4 is simplistic and forms a foundational model upon which
relativistic research may proceed in more depth. The most natural next research step would be to generalise the model to a moderate number of particles; such a task would be primarily an exercise in combinatorics, as the resulting block propagators would present a large number of particle interchange symmetries. Additionally, Chapter 4 contains all of the tools necessary to compute SR emission intensity from a two- or moderate-particle number relativistic SR sample; we have provided in Appendix J the matrix elements of the photon emission propagator necessary to the evaluation of relevant emission diagrams. A natural next application of the relativistic SR model would therefore be to compute the observer frame angular distribution of radiation intensity generated from a relativistic SR source.

Perhaps more relevant to practical research, however, would be an analysis of familiar foundational SR results in the context of this novel relativistic model. For example, repetition of the derivation steps performed in [5, 6, 1] for the SR master equation or for the MB equations could lead to their relativistic generalisations. The interplay between a more realistic relativistic model of an extended sample (a relativistic generalisation of the MB equations, for example) and the results of Chapter 3 could present a very nuanced research project. Relativistic length contraction [3] would certainly affect our results regarding transverse versus swept inversion mechanisms, the role of the Arecchi-Courtens condition in regulating the extent of polarisation phase correlation bandwidth, and the quantifying of SR emission intensity.
Chapter 5 Bibliography


Curriculum Vitae (Abbr)
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EDUCATION

**Doctor of Philosophy, Physics, The University of Western Ontario (London, Ontario)** Sep 2018–Aug 2022
*Department of Physics and Astronomy: cumulative course grades 97%, comprehensive examinations 95%*  
**Thesis:** "Wideband and Relativistic Superradiance in Astrophysics," under the supervision of Dr. Martin Houde

**Master of Science, Theoretical Physics, The University of Western Ontario** Sep 2007–May 2009
*Department of Applied Mathematics: cumulative course grades 92%*  
**Thesis:** "Edge States of a Graphene Ribbon in a Magnetic Field," under the supervision of Dr. Vladimir Miransky

**Bachelor of Science, Electrical Engineering, Physics, Mathematics, Dordt University (Iowa, USA)** Sep 2002–May 2007  
*Departments of Engineering; Physics & Astronomy; Mathematics, Statistics, & Computer Science (Five years, triple major)*  
**Engineering major cumulative GPA 3.96/4.00**  
**Senior design project:** "Design of a WWVB-Synchronized Electronic Controller for a Dordt Clock Tower," under the supervision of Dr. Douglas DeBoer  
**Physics major cumulative GPA 4.00/4.00**  
**Capstone project:** Independent Study of General Relativity, under the supervision of Dr. Arnold Sikkema  
**Mathematics major cumulative GPA 4.00/4.00**  
**Capstone project:** Independent Study of Topology, under the supervision of Dr. Calvin Jongsma

GRADUATE EDUCATION DETAILS

**Visiting Graduate-Level Courses**  
Modern Quantum Optics and Nanophotonics, *The Institute for Quantum Computing (Waterloo)*, grade 100%  
Fall 2019  
Introduction to Quantum Field Theory, *The Perimeter Institute (Waterloo)*, grade 93%  
Fall 2008  
Advanced General Relativity, *The Perimeter Institute (Waterloo)*, grade 90%  
Winter 2008

**University of Western Ontario Graduate-Level Courses**, average grade of all courses listed 92%  
Advanced Numerical Analysis  
Functional Analysis  
Non-linear Partial Differential Equations  
Advanced Quantum Mechanics  
Fundamentals of Physics  
Quantum Mechanics  
Classical Electrodynamics  
Fundamentals of Radiative Processes  
Statistical Physics  
Classical Field Theory  
Introduction to Radiative Transfer  
Computational Electromagnetics  
MRI Physics

**Additional Studies:**  
*Doctor of Philosophy, Physics (Medical), The University of Western Ontario* Sep 2009–Feb 2011  
*Partial work, departed program in excellent standing for career opportunity with JMP Engineering*  
Cumulative grade percentage 91%  
**Thesis:** "Reliable Operation of Active Implants in the MR Environment," under the supervision of Dr. Blaine Chronik