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# Automorphism-preserving color substitutions on Profinite Graphs

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# Abstract

Profinite groups are topological groups which are known to be Galois groups. Their free product was extensively studied by Luis Ribes and Pavel Zaleskii using the notion of a profinite graph and having profinite groups act freely on such graphs. This thesis explores a different approach to study profinite groups using profinite graphs and that is with the notion of automorphisms and colors. It contains a generalization to profinite graphs of the theorem of Frucht (1939) that shows that every finite group is a group of automorphisms of a finite connected graph, and establishes a profinite analog of the theorem of Sabidussi (1959) that states that every abstract group is a group of automorphisms of a connected graph. The profinite version of these theorems is: Every finitely generated profinite group is a group of continuous automorphisms of a profinite graph with a closed set of edges and every profinite group is a group of continuous automorphisms of a connected profinite graph. The thesis contains an application of these theorems, which is a solution to the conjecture of Sidney Morris and Karl Hoffmann stating that every profinite group is a group of autohomeomorphisms of a connected compact Hausdorff space.

**Keywords:** profinite graphs, profinite groups, profinite topology, graph automorphisms

# Summary for Lay Audience

Mathematicians often try to find links between seemingly unrelated topics. This can help in solving difficult problems. For example a problem in one theory, like geometry, could be very difficult to solve by itself, but if one looks at it from the point of view of algebra, it suddenly becomes much easier.

The goal of this thesis is to study relations between an old theory invented to solve equations: Galois theory and a much more recent theory that is used to represent connections: graph theory. At first glance the two topics are seemingly unrelated: on one hand we get equations like $x^5 - x^3 + 12x + 1 = x^2 - 2$ and on the other we get a list of points (vertices) and connections between them. The key that binds them together is the notion of automorphism group, which one can think of as a list of swaps that preserve certain properties. In the case of equations, we swap their solutions, in the case of graphs, we swap the vertices in such a way that, after the swaps are done, the connections remain the same. In this thesis I explore the different ways in which such swaps on solutions of equations (Galois theory) can be represented as swaps on vertices (Graph theory).

# Acknowledgments

I would like to thank my supervisor, Professor Ján Mináč for his wonderful and enthusiastic support. Thanks to his deep knowledge in many areas of algebra, he guided me systematically to the core topic of my PhD: the relations between graph theory and Galois theory.

I would also like to thank my second supervisor Professor Lyle Muller for helping me to learn new areas in applied graph theory. His lectures on computational neuroscience were fascinating and the multidisciplinary project on Covid 19 he started in the summer 2020 made it possible for me to meet people from different disciplines including physics, applied mathematics and computing and be to able to learn from them and teach them as well.

I would like to thank as well Professor Luis Ribes. His book was a great inspiration for my work and his comments and suggestions on my thesis were very valuable to me.

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# Contents

# Introduction

Galois theory is the study of separable field extensions using groups. Due to the fundamental theorem of Galois theory, there is a correspondence between subextensions of a given finite normal extension and subgroups of its group of automorphisms better known as the Galois group. The correspondence works in the following way: to a subextension, we associate the subgroup of automorphisms that fix that subextension and to a subgroup of automorphisms we associate its fixed field. When dealing with the infinite case a problem arises that there is no longer just one subgroup that corresponds to the fixed field. There exists however a solution that consists of putting a topology on the Galois group known as the profinite topology and then consider only closed subgroups. Closed subgroups of the Galois group give us a natural choice for a group associated to a subextension and reestablishes the bijectivity of the Galois correspondence. The most systematic way to study the infinite Galois groups is to study the absolute Galois group. That is the Galois group of the normal closure: the largest normal extension of a given field. The difficulty with the absolute Galois group is however that it is complicated and outside of a few well known examples like finite fields and the field of real numbers it is not well known.

A simpler case to study would be the pro-p groups, where we only consider normal extensions of degree that is a power of some prime number p, or towers of these extensions. The introduction to this theory is well covered by Helmut Koch in [29].

Examples of work in this field include the article by Ján Mináč and Michel Spira [35] on relations between Galois groups of extensions that are powers of 2 and quadratic forms on a given field of characteristic distinct from 2 and the article by Ján Mináč, Andrew Schultz and John Swallow [34] that studies the structure of the $\mathbb{F}_p[G]$ modules, with $G$ being the Galois group of a cyclic extension of degree $p^n$. During my thesis, I have worked on the topic of pro-$p$ groups as well together with my fellow PhD students Ali Alkhairy and Oussama Hamza. We have worked on the theorem discovered by Serre that for a given finitely generated pro $p$ group open subgroups are exactly those of finite index as in: exercise 6 page 32 of [45]. We used it to establish Galois theory of infinite extensions whose Galois group is finitely generated without having to resort to topology since for subgroups being open is simply characterized by being of finite index. For further details, see [3]. It is worth noting that this result was generalized for finitely generated profinite groups that are not necessarily pro p by Nikolay Nikolov and Dan Segal in [40].

An important tool for studying Galois groups is the cohomology. To any group

$G$ we can associate a certain projective exact sequence, which we then tensor with the group ring $RG$, with $R$ the desired ring, often $\mathbb{Z}$ or $\mathbb{F}_p$. After the tensoring, the exact sequence is no longer exact and so we can study its cohomology. The first two cohomologies give us important information about the group: the first cohomology classifies derivations on a group and the second classifies group extensions with Abelian kernel. More details on this topic are for example covered by Kenneth Brown in [8] Chapter 4. The higher order cohomologies are often useful to provide information on the first two: using tools like dimension shifting. A question that has been studied in Galois theory is the question of formality: that means how much is the given chain complex associated to the Galois group determined by its cohomology and when can it be recovered. A sufficient condition was found using a certain product coming from topology called the Massey product that is a generalization of a cup product. A Massey product of order $n$ associates to $n$ elements in the cohomological algebra a subset of the cohomological algebra of elements that are solutions to certain equations. It is worth pointing out that it is not always defined and it is not an internal operation, since rather than associating an element in the cohomology it associates a set. For more details on how the Massey products can be used in topology: one can check [30]. An important question when it comes to Massey products is when they vanish. The definition of vanish in this context is that they contain zero. If a chain complex is formal that is it's structure can completely be recovered from its cohomology, then the Massey products vanish whenever they are defined. In the case of Galois groups, an extensive research has been done on the triple Massey products. Examples of such work field include the work of Ido Efrat and Eliyahu Matzri [12], showing that the triple Massey product restricted to $H^1(G_F)^3$ in the case of Absolute Galois group of a field containing a $p$th root of unity vanish whenever it is defined. In 2015 Ján Mináč and Nguyen Tan showed in [37] that triple Massey products always vanish for the Absolute Galois group of any field. An important part in this proof was the existence of unipotent extensions: that is Galois extension whose Galois group is $\mathbb{U}_4(\mathbb{F}_p)$: the group of upper triangular matrices over the field $\mathbb{F}_p$ for $p$ a prime. An explicit construction of such extensions is given in [38]. As showed in [36] Massey products can as well be be used in counting the $\mathbb{U}_4(\mathbb{F}_p)$- extensions. The case of Massey products for $n \geq 4$ is still an open problem. In 2019 a significant advancement has been made by Yonatan Harpaz and Olivier Wittenberg in [22], proved that for any number field $k$, any prime number $p$ and any natural number $n \geq 3$, the Massey product of classes in $H^1(k, \mathbb{Z}/p\mathbb{Z})$ vanishes when defined.

The Massey products and cohomology in general open the possibility of using combinatorial approaches to study Galois group. A very general open question is how much can one recover from the Galois group using combinatorics. My thesis focuses on using a very specific combinatorial tool to study Galois groups and that is graph theory. There is a well known relation between graphs and groups that has been established by Robert Frucht in 1939 [17]. Any finite group is isomorphic to the group of automorphisms of a graph. In the case of Galois theory, this theorem was used by Ervin Fried and János Kollár [15] [With corrections done by Michael

Fried in [16]] to prove a weaker version of the inverse Galois problem, where instead of requiring that any finite group is a Galois group of a normal separable extension, we require that it is just a Galois group of a separable extension, which proves that using Graph theory to solve Galois theory question can be a viable approach.

In this thesis, we will look at two ways of studying Galois theory using graph theory. In the first chapter, we will establish a few generalities on finite graphs and show a proof of Frucht's theorem invented by László Lovász in [31] and offer a generalization of the theorem that I came up with. This generalization consists of representing a group action on a finite set by a graph that I called the Group action Cayley graph. We will then reinterpret the classical notion of group actions in terms of graph properties. The importance of group action on a finite set is that if the group is a Galois group of a field, then this group action corresponds to a certain generalization of a notion of a separable extension, which is called Etale Algebra.

In the second chapter we will go over another possible approach to representing Galois theory using graph theory and that is the notion of covering graphs. We will also establish a few useful tools in graph homology to detect cycles and connectedness.

The third chapter is dedicated to the profinite structures that will be used in this thesis. I used the following profinite structures in my thesis: Profinite groups, rings, modules, graphs and covering graphs. Since certain notions like projective limits or compactness were common for all these structures, I decided to group them under a certain general category-theoretical notion which I called a profinite structure and proved their common properties at once, rather than establishing them separately. The last part of the third chapter is about Etale algebras and showing why they correspond to the action of the absolute Galois group on finite sets.

The fourth Chapter ties all the notions seen previously together and generalizes them for profinite graphs. We will first give a definition of a profinite graph, then generalize the notion of connectedness and see how they are different between the abstract and the profinite case. We will also see a weaker notion of connectedness that I came up with and call superpath-connectedness and show why this notion is weaker. We will then generalize the graph homology to profinite graphs and define with it the notion of a profinite tree. We will then go over the notion of profinite covering graphs and see why they are a profinite structure in the sense of the definition given in chapter 3 and then prove the fundamental theorem of Galois theory of profinite covering graphs. Finally we will generalize the notion of the group action Cayley graph seen in Chapter 1. We will establish a generalization of a notion of color to the profinite graph, then use it to construct the profinite group action Cayley graphs and finally give give two procedures for dropping the colors. The first one is in the case of finite sets of colors. That procedure is in fact a generalization for profinite graphs of the procedure given by Lovász in [31]. The second method drops colors for any graph with closed set of edges, but with the difference that the resulting graph after getting rid of the colors will no longer have closed edges. At the end of the fourth chapter we will work on an application of this color substitution to the study of the topology of profinite groups. We will prove a generalization of a result established by Karl Hoffmann and Sidney Morris, proving in [25] that every profinite group with one topological generator is a group of autohomeomorphisms of a compact Hausdorff connected space. We will generalize the method exposed in the article and prove that every profinite group is a group of autohomeomorphisms of a compact Hausdorff connected space. Given that this proof uses an old theorem proved by de Groot in [20] and that the proof lacked certain details that is completed in this thesis using convex geometry.

# Contributions

Since this thesis contains a substantial amount of background material, I decided to dedicate a small section summing up the original contributions of this thesis.

In chapter 1 I examine the theorem of Frucht as proven by Lászlo Lovász in [31] and I get an alternative construction to his substitution of colors by graphs, which works for more general graphs than just Cayley graphs and I also examine the difference between the groups of automorphisms we obtain between the two substitutions. I then give in 1.5.1 a theorem that generalizes the construction for Cayley graphs and captures action of a finite group on a finite set. On pages 16 and 17, I examine the reinterpretation of an group action properties in terms of their associated group action Cayley graphs.

Throughout the chapter 3, I introduce an original notion that unifies structures such as profinite groups, profinite rings and profinite modules into a single category called profinite structure as defined in 3.1.4

**Finite and infinite color substitution** In chapter 4 I introduce the notion of color on profinite graphs in 4.6.1 and I use it to construct profinite group action Cayley graphs, which capture the action of a profinite group on a profinite set. I then give two distinct theorems for substituting colors in graphs by sugraphs that preserve the groups of automorphisms. The first one in 4.6.6 works for a profinite graph with closed and colored set of edges and a finite set of colors. The second approach in 4.6.8 works for a profinite set of colored edges with this time the set of colors being profinite possibly infinite. The drawback of the second approach is that after doing the substitution, the set of edges is no longer closed.

**Main result:** In the section 4.7 I then give an application of the color substitution method, where I prove a generalization of a result proven in [25]. One can observe that if we take a compact Hausdorff space $X$, the group of autohomeomorphism of $X$ can be equipped with open compact topology to make it into a topological group. While $X$ is compact the group of automorphisms isn't necessarily compact. In 2012 in [24] Sidney Morris and Karl Hofmann showed that if the group is compact then it is profinite. Naturally one asks the question whether conversely a profinite group is isomorphic to a group of autohomeomorphisms of a compact Hausdorff space $X$. Morris and Hofmann conjectured it is the case and in [25], they showed it for groups with single generator using topological Cayley graph.

I generalise their approach for all profinite groups using the profinite color substution theorems. In order to do such a proof, I use an old theorem shown by de Groot in [20] constructing the de Groot continua. While in the publication, the precise way of constructing the continuum is shown, it is missing a substantial amount of details, which I complete in my thesis using convex geometry. It is important to mention that originally I proved this result for only finitely generated profinite groups. Since finitely generated profinite groups are metric as shown in 4.8, there is actually a more general result by Gartside and Aneirin proven in [18], showing that every metric profinite group is a group of autohomeomorphisms of a compact connected Hausdorff space. The difference with my approach is that it generalizes the one done by Hoffmann and Morris and uses the theory of profinite graphs and the color substitution tools and ultimately led to the valid case for all profinite groups without any restrictions.

# Chapter 1

# Graph theory

## 1.1 Introduction

In this first chapter, we will give a simple definition of a graph that we will generalize in the chapters that follow.

**Definition 1.1.1.** *A graph $G$ is defined as a pair $(V, E)$, where $V$ is a nonempty set called the set of vertices and $E \subset V^2$ is called the set of edges.*

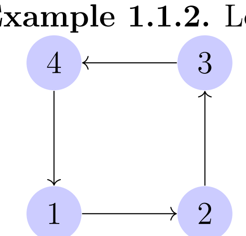
*If $G = (V, E), G' = (V', E')$ are two graphs, then a morphism between $G$ and $G'$ is a map $\phi$ from $V$ to $V'$, such that if $(x, y) \in E$ is an edge, then $(\phi(x), \phi(y)) \in E'$ is an edge as well.*

*An isomorphism between $G$ and $G'$ is a morphism $\phi$, such that $\phi$ is bijective and $\phi^{-1}$ is also a morphism.*

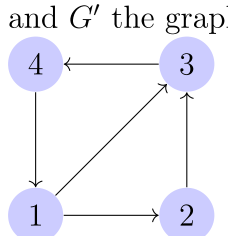
*A graph is called undirected loopless if and only if the set of edges $E$ has no diagonal elements $(\forall a \in V, (a, a) \notin E)$ and $E$ is symmetric i.e $\forall a, b \in V, (a, b) \in E \Rightarrow (b, a) \in E$.*

*Remark.* For a bijection of vertices $\phi$ it is necessary to check that both $\phi$ and $\phi^{-1}$ are morphisms to prove that $\phi$ is an isomorphism, as shown in the example below:

**Example 1.1.2.** Let $G$ be the graph




and $G'$ the graph



Identity on the set $\{1,2,3,4\}$ is a morphism from $G$ to $G'$, but not from $G'$ to $G$ since $G'$ has an extra edge $(1,3)$. However we have the following result:

**Proposition 1.1.3.** *Let $G=(V,E), G'=(V',E')$ be two finite graphs with the same number of edges. Let $\phi$ be a bijection from $V$ to $V'$ that is a morphism from $G$ to $G'$. Then $\phi$ is an isomorphism.*

*Proof.* All we have to prove is that if $(v,v')\in E'$, then $(\phi^{-1}(v),\phi^{-1}(v'))\in E$. Consider the map $\Phi=\begin{cases} E\longrightarrow E' \\ (u,u')\mapsto(\phi(u),\phi(u')) \end{cases}$. It is injective, since $\phi$ is injective. By assumption $E$ and $E'$ have the same number of elements, hence $\Phi$ is bijective and its inverse is: $\begin{cases} E'\longrightarrow E \\ (v,v')\mapsto(\phi^{-1}(v),\phi^{-1}(v')) \end{cases}$ □

**Definition 1.1.4** (automorphism of graphs)**.** *For a graph $G$ we define an automorphism as an isomorphism from $G$ to itself.*

Observe that the set of automorphisms is a subgroup of the group of permutations on the set of vertices of $G$.

Since we know that every group can be realized as a subgroup of the group of permutations, an interesting question concerning graphs is whether a certain given group can be represented as automorphisms of a graph. It is a well known case for finite groups, which we shall show here using an exercise in Lászlo Lovász textbook [31] and afterwards I will expand on this construction. It is worth mentioning that infinite groups can be represented as an automorphism group of a graph as well as proved independently by Johannes Groot in [21] and Gert Sabidussi in [44].

*Remark.* Given a morphism of finite graphs, one can prove in polynomial time that it is an isomorphism (simply check its injectivity), however finding explicit isomorphisms in polynomial time is currently an open problem. Provided there is no polynomial time algorithm, there are possible applications to cryptography in form of zero knowledge proofs protocol explained for example in [11]. Major progress on this problem has been made using group theory by László Babai in [4] and [5], proving this problem can be solved in quasi polynomial time.

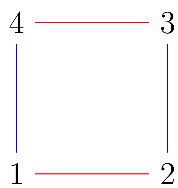
## 1.2 Cayley graphs and groups

**Definition 1.2.1** (edge colored graphs)**.** *An edge colored graph $G$ is the quadruplet: $(V,E,C,c)$ with $(V,E)$ a graph, $C$ a non empty set and $c$ a map from $E$ to $C$. The function $c$ is called a coloring and $C$ is called the set of colors.*

*If $(V,E,c,C)$ and $(V',E',c',C')$ are two colored graphs a morphism is a map $f$ from $V$ to $V'$, such that $f$ is a morphism from the graph $(V,E)$ to the graph $(V',E')$ and $\forall(u,v)\in E,\forall(w,z)\in E, c((u,v))=c((w,z))\Rightarrow c'((f(u),f(v)))=c'((f(w),f(z)))$, i.e $f$ sends edges of the same color, to the edges of the same color.*

There is a very particular kind of colored graphs that makes it possible to represent a group as a group of automorphisms. It is called a Cayley graph.

When we talk about the automorphism group here, we have to be careful. We are talking about automorphisms that fix colors. They are not automorphisms in a category theoretical sense. Remember: in the category of edge colored graphs, the automorphisms are in fact potentially permuting colors as well. Take for example the following graph:

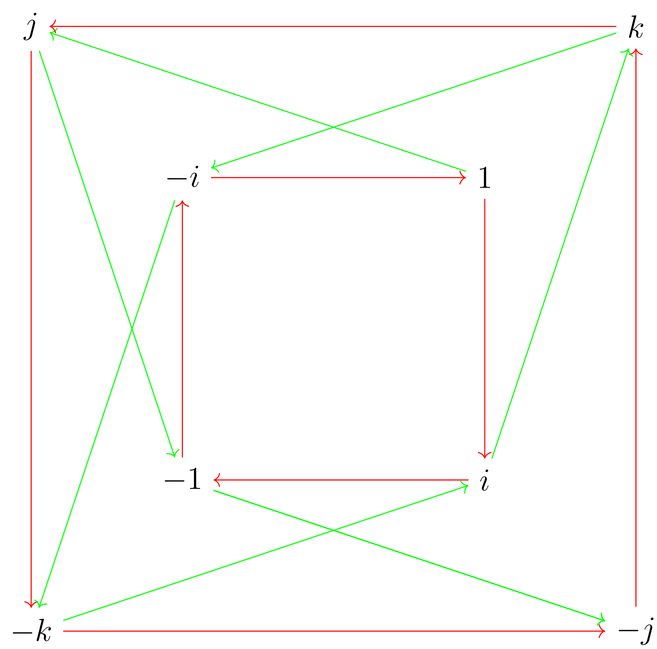



The permutation (1 2)(4 3) preserves both edges and their colors. It is what we call a color-preserving automorphism. On the other hand the cycle (1 2 3 4) preserves the edges, but permutes the colors: it is what we call a color-permuting automorphism. It is clear that the color-fixing automorphisms form a subgroup of the color-permuting automorphisms and the example above shows that it can be a proper subgroup.

Henceforth for an edge colored graph $G$, we will denote $Aut(G)$ the set of color-fixing automorphisms and simply call it 'automorphisms' unless stated otherwise. Just keep in mind that only the color-permuting automorphisms are automorphisms in the category theoretical sense.

**Definition 1.2.2** (Cayley graph)**.** *Let $G$ be a group generated by a set $S$ not containing $1_G$. We call the Cayley graph of $G$, the edge colored graph $(V, E, C, c)$, with $V = G$, $E = \{(g, gs) | s \in S, g \in G\}$, $C = S$ and $c(g, gs) = s$.*

**Example 1.2.3.** Take $Q_8 =< 1, i, j, k >$ the group of quaternions with generators $i, j$. Its Cayley graph is:




With red being the multiplication by $i$ and green by $j$.

The reason why Cayley graphs represent groups well comes from the following theorem.

**Theorem 1.2.4.** *Let $G$ be a group generated by a set $S$ with $1_G \notin S$ and $\mathfrak{G} = (V, E, C, c)$ an associated Cayley graph. Then there exists an isomorphism $\Phi$ from $G$ to $Aut(\mathfrak{G})$, such that the composition of $\Phi$ with the natural action of $Aut(\mathfrak{G})$ on $\mathfrak{G}$ is the multiplication on the left by elements of $G$.*

*Proof.* Take $\Psi$ the map

$$\begin{cases} Aut(\mathfrak{G}) \longrightarrow G \\ \sigma \mapsto \sigma(1_G) \end{cases}$$

Let us show that $\Psi$ is an isomorphism of groups.

Take $\sigma \in Aut(\mathfrak{G})$ and let $g = \sigma(1_G)$. For a $g' \in G$, we denote $l(g')$ the minimal $n$ such that there exists a sequence $s_1, \cdots, s_n \in S$, for which $g' = s_1 \cdots s_n$. Let us prove by induction on $l(g')$ that $\sigma(g') = gg'$ for all $g' \in G$.

If $l(g') = 0$, then $g' = 1_G$ and the statement is vacuously true. Now suppose the statement true for all $g'$, such that $l(g') < n$. Let us prove that it is also true for all $g'$, such that $l(g') = n$. Write $g' = g''s$ with $l(g'') < n$. Then we know that $\sigma(g'') = gg''$. Now $(g'', g''s)$ is an edge of color $s$ and $\sigma$ is an automorphism of $\mathfrak{G}$, therefore $(\sigma(g''), \sigma(g''s)) = (gg'', \sigma(g''s))$ is also an edge of color $s$. There is only one edge of that color coming out of $gg''$, therefore $\sigma(g''s) = gg''s = gg'$, proving the statement true for $g'$. We conclude that for all $g' \in G$, $\sigma(g') = gg'$. Now let us prove that $\Psi$ is a group morphism. Take $\sigma, \sigma' \in Aut(\mathfrak{G})$ and write $g = \sigma(1_G)$ and $g' = \sigma'(1_G)$. Then by what we have shown $\sigma(\sigma'(1_G)) = gg'$, proving that $\Psi$ is indeed a morphism.

Now let us prove that $\Psi$ is injective. If $\Psi(\sigma) = 1_G$, then

$$\forall g \in G, \sigma(g) = 1_G \cdot g = g$$

therefore $\sigma = id_G$ :

Now let us prove that $\Psi$ is surjective. Let $g \in G$. Define

$$\sigma = \begin{cases} G \longrightarrow G \\ g' \mapsto gg' \end{cases}$$

Let us show that $\sigma$ is an automorphism of $Aut(\mathfrak{G})$. Take $(g', g's)$ an edge. Then we have $(\sigma(g'), \sigma(g's)) = (gg', gg's)$ an edge of the same color: $s$. We conclude that $\Psi$ is an isomorphism.

Finally we put $\Phi = \Psi^{-1}$ it is clear that $\Phi$ composed with the natural action of automorphisms gives the multiplication on the left by $G$. □

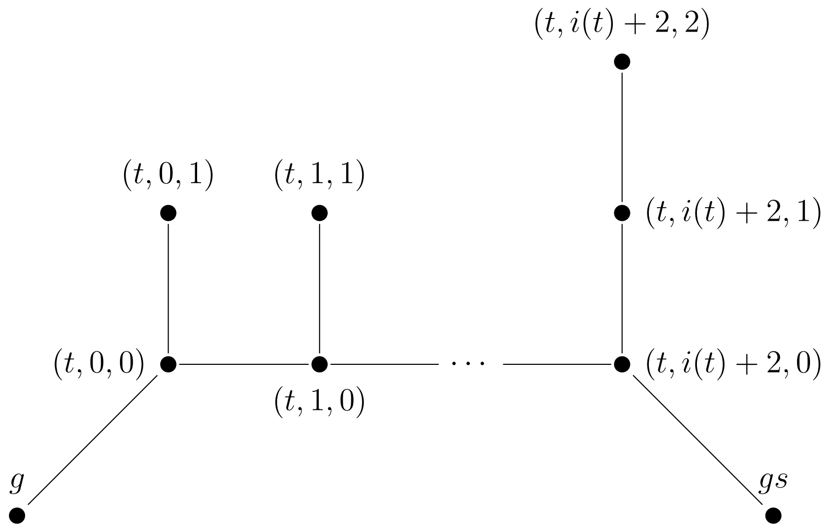
We have shown here the well known fact that a group can be represented as the automorphism group of a Cayley graph. However we have to remember that the automorphism group we are considering is in fact the group of color-preserving automorphisms and a natural question is whether for Cayley graphs this group is the same as the color-permuting automorphisms. The answer to this question is negative and it is explored in depth in the article [2].

Using a construction from an exercise in Lovász's book [31] page 495 exercise 5, we can prove then that any finite group is a group of automorphisms of a certain undirected graph. This theorem was for the first time established by Frucht in [17] and is as follows:

**Theorem 1.2.5** (Frucht). *Let $H$ be a finite group. Then there exists a graph $G$, such that $H$ is isomorphic to the group of automorphisms of $G$.*

Proof sketch:

Take $S$ a set of generators of a group $G$. Take $i$ an injective map from $S$ to the set of integers $\{1, \cdots, n\}$, with $n = \#S$, $(G, E, c, \{1. \cdots, n\})$ a colored graph with $E' = \{(g, gs) | g \in G, s \in S\}$ and $c((g, gs)) = i(s)$. The graph $(G, E', c, \{1, \cdots, n\})$ is then a Cayley graph and its group of automorphisms is therefore isomorphic to $G$. We will now transform it into an undirected loopless graph without colors in a way that preserves the automorphism group. We start from the original graph and we transform an edge of the type $t = (g, gs)$ into the following:



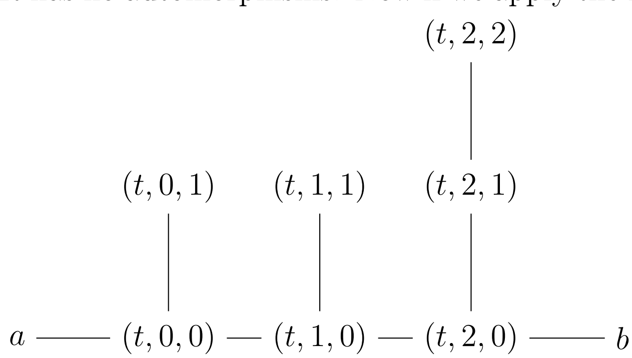

We will go into more details as to why this transformation preserves the automorphism groups later, but the general idea is that a path of a length $i(t)$ can only be sent by an automorphism on a path of the same length. The edge that was originally of color $s$ will become a path of length $i(s)$, which is how the automorphisms remain the same. The purpose of the 'flag' $(t, i(t), 2)$ is to preserve the orientation of the edges: it breaks the overall symmetry of the shape.

The new graph constructed from the Cayley graph has the desired property. However one small disadvantage of this transformation is that it does not preserve the automorphism group in the case of all colored graphs. The reason is because in some instances a vertex could be swapped with its 'flag'.

Take for example this simple graph with two vertices and one edge $t$:

$a$ — $b$

It has no automorphisms. Now if we apply the Lovász transformation, we obtain:




This artificially adds the transposition between $a$ and the flag $(t, 0, 1)$ as an automorphism.

In the part that follows, we will give a more general construction that preserves the automorphism group for all flags and that is also functorial provided that we fix a coloring system in $\mathbb{N}$. Since we mentioned functors, we will start by defining the category we are going to be working with.

## 1.3 An automorphism preserving transformation.

In this section I will give a variation on the transformation given by Lovász I invented. This new transformation preserves the automorphism group for all colored graphs provided that the number of colors is finite. I will then show by how much the automorphism group for the Lovász transform and my own differ.

**Definition 1.3.1** (pointed colored graphs)**.** *We define a category of pointed finite colored graphs as $\mathscr{C}\text{-}o\text{-}\ell$, a category with the objects being colored graphs:* $(V, E, c, \mathbb{N})$ *with* $V$ *a finite set. A morphism* $f$ *from* $(V, E, c, \mathbb{N})$ *to* $(V', E', c', \mathbb{N})$ *is a morphism of colored graphs, such that* $\forall t \in E, c(t) = c'(f(t))$.

Since the set of colors is always $\mathbb{N}$, we do not need to specify it and we will just denote a graph in our category as $(V, E, c)$.

For this type of graph, all the automorphisms have to be color-preserving rather than color-permuting.

**Definition 1.3.2.**

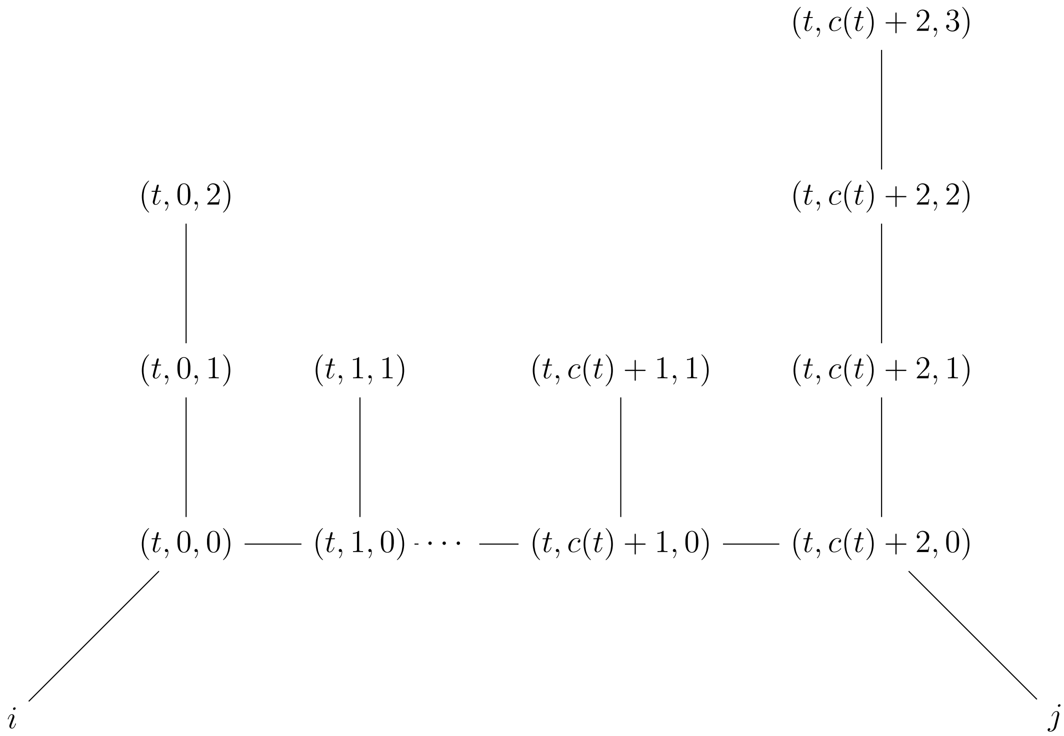
- *For a pointed finite colored graph* $G = (V, E, c)$, *we define* $\mathcal{T}(G)$ *the undirected graph* $T(G) = (V', E')$, *with* $V'$ *being the set:*

$$V \cup \{(t, 0, k) | t \in E, k \in \{0, 1, 2\}\} \cup$$
$$\{(t, k, k') | t \in E, k' \in \{1, \cdots, c(t) + 1\}, k \in \{0, 1\}\} \cup$$
$$\{(t, k, c(t) + 2) | t \in E, k \in \{0, 1, 2, 3\}\}$$

  *and* $E'$ *being symmetrization of the set:*

$$\{(i, (0, 0, t)) | t \in E, \exists j \in V, t = (i, j)\} \cup$$
$$\{((t, k, k'), (t, k, k' + 1)) | t \in E \text{ and } (t, k, k' + 1) \in V'\} \cup$$
$$\{((0, k, t), (0, k + 1, t)) | (0, k + 1, t) \in V'\} \cup$$
$$\{((t, c(t) + 2, 0), j) | t \in E, \exists i \in V, (i, j) = t\}$$

*The corresponding image one can think of for this construction is as follows: we start with the colored graph and we replace an edge* $(i, j)$ *of color* $c(t)$*, with a path of length* $c(t) + 2$ *and adding the corresponding flags as in:*




- *We shall now call the Lovász transform* $\mathscr{L}$ *the transformation that to a finite colored graph* $G = (V, E, c)$ *associates* $\mathscr{L}(G) = (V', E')$*, with* $V'$ *being the set:*

$$V \cup \{(t, k, k')|t \in E, k' \in \{0, 1\} \text{ and}$$
$$k \in \{0, \cdots, c(t) + 1\}\} \cup \{(t, c(t) + 2, k)|k \in \{0, 1, 2\}\}$$

*and* $E'$ *is the symmetrization of:*

$$\{((t, k, k'), (t, k, k' + 1))|(t, k, k' + 1) \in V'\} \cup$$
$$\{(i, (t, 0, 0))|t \in E, \exists j \in V, (i, j) = t\} \cup$$
$$\{((t, c(t) + 2, 0), j)|\exists i \in V, (i, j) = t\}$$

The illustration for how this transformation works is in 1.2.5. Now we will show that the transformation $\mathscr{T}$ preserves the automorphism group and we will compare how the transformation $\mathscr{L}$ differs from it.

**Proposition 1.3.3.** *The transformation* $\mathscr{T}$ *described above is a functor from the category of pointed finite colored graphs into the category of loopless undirected graphs, such that for any pointed finite colored graph* $G$*,* $Aut(\mathscr{T}(G))$ *is isomorphic to* $Aut(G)$*.*

*Proof.* Let us start with the functor part. First we need to define $\mathcal{T}$ on morphisms. Take $f$ a morphism from $G_0 = (V_0, E_0, c_0)$ to $G_1 = (V_1, E_1, c_1)$, we take $(E'_0, V'_0) = \mathcal{T}(G_0)$ and $(E'_1, V'_1) = \mathcal{T}(G_1)$. Now define

$$\mathcal{T}(f) = \begin{cases} V'_0 \longrightarrow V'_1 \\ x \mapsto \begin{cases} f(x) \text{ if } x \in V_0 \\ (f(t), k, k') \text{ if } x = (t, k, k') \end{cases} \end{cases}$$

With the convention that if $t = (a, b)$ is an edge in $G_0$, then $f(t) = (f(a), f(b))$. We need to show that $\mathcal{T}(f)$ is well defined and is a morphism.

If $(t, k, k')$ is a vertex in $V'_0$ that means that $k' \leq c_0(t)+2$. Since $c_1(f(t)) = c_0(t)$, by the virtue of $f$ being a morphism, we get that $k' \leq c_1(f(t)) + 2$, which makes $(f(t), k, k')$ of the correct format to be a vertex in $V'_1$, proving that $T(f)$ is well defined. The fact that $T(f)$ is a morphism is fairly clear. It is also clear that $\mathcal{T}$ is compatible with composition. Now that we know it is a functor, we will prove that it preserves the automorphism group.

Let $G = (V, E, c)$ be a pointed colored graph. Let $\mathcal{T}(G) = (V', E')$ Let us prove that $\Phi$ the functor $\mathcal{T}$ restricted to $Aut(G)$ is an isomorphism of groups from $Aut(G)$ to $Aut(\mathcal{T}(G))$.

Proof that $\Phi$ is injective:

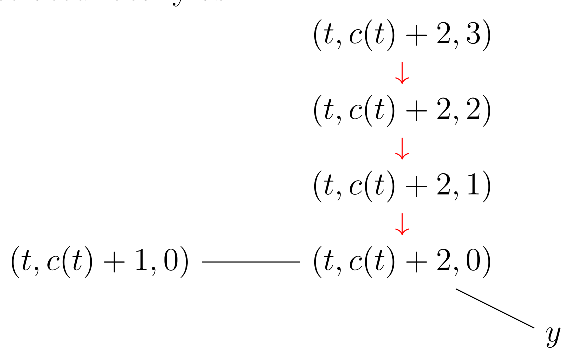
Suppose that for $g \in Aut(G)$, $\Phi(g) = id_{V'}$ Then since $\Phi(g)_{|V} = g$, we get that $g = id_V$.

Proof that $\Phi$ is surjective:

Let $g \in Aut(\mathcal{T}(G))$. We start by observing that the path

$$((t, c(t) + 2, 3), (t, c(t) + 2, 2), (t, c(t) + 2, 1), (t, c(t) + 2, 0))$$

illustrated locally as:




is of the form $(A, B, C, D)$ with $deg(A) = 1$,$deg(B) = 2$,$deg(C) = 2$, $deg(D) = 3$, so $(g(A), g(B), g(C), g(D))$ must also be a path of that form. The vertex $g(A)$ cannot be an element of $V$, because all neighbors of elements of $V$ are of degree 3, but $g(B)$ is a neighbor of $g(A)$ of degree 2. The vertex $g(A)$ is therefore of the form $(t, k, k')$. Since it is of degree 1, it can only be one of the following:

$$(t', 0, 2), (t', k, 1) \text{ with } k < c(t') + 2 \text{ or } (t', c(t') + 2, 3)$$

We then differentiate the three cases:

- In the first case, $g(C)$ would have to be $(t', 0, 0)$, which is impossible since $g(C)$ is of degree 2, while $(t, 0, 0)$ is of degree 3.
- In the second case $g(B)$ would have to be $(t', k, 0)$, which is again impossible for the degree reasons.
- We know therefore that $g(A)$ is of the form $(t', c(t') + 2, 3)$ and so $g(D) = (t', c(t') + 2, 0)$.

The next step to prove is that $t$ and $t'$ are of the same color, i.e that $c(t) = c(t')$. By induction on $k$, we shall prove that:

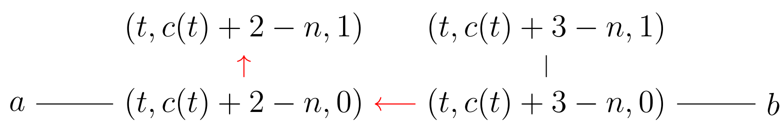
$$\forall k \in \{0, 1, \cdots, c(t)+1\}, c(t')+2-k > 0 \text{ and } g((t, c(t)+2-k, 0)) = (t', c(t')+2-k, 0)$$

Since we have already proved the initialization, we can now show the inductive step. Suppose that the statement is true for all $k$ strictly smaller than some $n \leq c(t) + 1$ (and $n > 0$). Let us show that it is also true for $n$.

Let $(A, B, C)$ be the path

$$((t, c(t) + 3 - n, 0), (t, c(t) + 2 - n, 0), (t, c(t) + 2 - n, 1))$$

illustrated in red as:

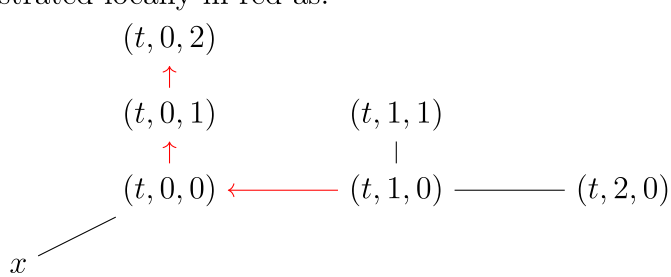



The degrees of the vertices in the path are respectively 3,3,1. The path $(g(A), g(B), g(C))$ has therefore the same degree values. The vertex $g(B)$ is of degree 3 and is connected to $g(C)$ that is of degree 1, hence it can only be of the form $(t'', k', 0)$. Since however its neighbor $g(A)$ is of the form $(t', c(t') + 3 - n, 0)$, therefore $t = t'$ and $g(B)$ is either $(t', c(t') + 4 - n, 0)$ (Provided such a vertex exists) or $(t', c(t') + 2 - n, 0)$, but it can only be the latter by the injectivity of $g$. Hence we indeed have that $c(t') + 2 - n > 0$ and $g((t, c(t) + 2 - n, 0)) = (t', c(t') + 2 - n, 0)$ This in particular means that $g((t, 1, 0)) = ((t', k, 0))$ with $k = c(t') - c(t) + 1$. Now let us prove that $g((t, 0, 0)) = (t', 0, 0)$.

Let $(A, B, C, D)$ be the path

$$((t, 1, 0), (t, 0, 0), (t, 0, 1), (t, 0, 2))$$

Illustrated locally in red as:



The degrees of the vertices in that path are respectively 3,3,2,1. $g(A) = (t', k, 0)$, so $g(B)$ is either $(t', k-1, 0)$ or $(t', k+1, 0)$, but it can't be the latter by the injectivity of $g$, so $g(B) = (t', k-2, 0)$. The vertex $g(C)$ is of degree 2, so it can either be in $V$, of the form $(t'', c(t'')+2, 1)$ or of the form $(t'', 0, 1)$.

- It can't be the first case, because elements of $V$ are only connected to vertices of degree 3, so don't have a neighbor of degree 1 like $g(C)$ does.
- It can't be the second case, because the neighbors of $(t'', c(t'')+2, 1)$ are of degree 2 and of degree 3, so it doesn't have a neighbor of degree 1 either.
- we therefore conclude that $g(C) = (t'', 0, 1)$.

Since $g(B)$ is a neighbor of $g(C)$ of degree 2, we conclude that $k-1 = 0$ and $t'' = t'$. This means that $c(t) = c(t')$.

So far we therefore know that

$$\forall (t, k, k') \in V', \exists ! t' \in E, g((t, k, k')) = (t', k, k')$$

and for such a $t'$, $c(t) = c(t')$.

Now we shall prove that if $t = (u, v) \in E$, then $t' = (g(u), g(v))$. We have that $g(u)$ is a neighbor of $(t', 0, 0)$, so it can only be $(t', 0, 1)$, $(t', 1, 0)$ or $u'$, such that there is a $w \in V$ for which $(u', w) \in E$. $(t', 0, 1) = g((t, 0, 1))$ and $(t', 1, 0) = g((t, 1, 0))$, so by injectivity of $g$, it can only be $u'$, with $t' = (u', w)$. With a similar reasoning done on $g((t', c(t')+2, 0))$, we get that $g(v) = w$, which proves that $t' = (g(u), g(v))$.

We get in the end that $g(V) = V$ and that if $(u, v) \in E$, $(g(u), g(v)) \in E$ and $c((u, v)) = c(g((u, v)))$, proving that $g_{|V}$ is an automorphism of the finite pointed colored graph $G$. Furthermore we have that $g = \mathscr{T}(G)(g_{|V})$, proving finally the surjectivity of $\mathscr{T}$. □

Thanks to the functoriality of the transformation $\mathscr{T}$, if we construct some colored graphs whose properties of automorphisms we wish to study, we can always make them into undirected non colored graphs with the same automorphisms. While $\mathscr{L}$ is also a functor, as we have already shown it doesn't preserve the group structure. There is however a precise answer as to how much larger the automorphism group of $\mathscr{L}(G)$ compared to that of $G$ is and we can also give its precise structure.

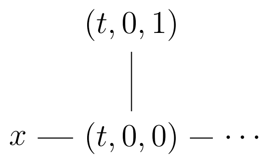
**Proposition 1.3.4.** *Let $G = (V, E, c)$ be a pointed colored graph. Let $X \subseteq V$ be the set of vertices with in/out degree (0,1). Then $Aut(\mathscr{L}(G))$ is isomorphic to the semidirect product $Aut(G) \rtimes {\mathbb{Z}/2\mathbb{Z}}^X$, with ${\mathbb{Z}/2\mathbb{Z}}^X$ being the free $\mathbb{Z}/2\mathbb{Z}$ module on the set $X$ and $Aut(G)$ acting on it in the following way:*

*If $f$ is a function from $X$ to $\mathbb{Z}/2\mathbb{Z}$, then $g \cdot f = f \circ g^{-1}$.*

*Proof.* We start by proving that $Aut(G)$ indeed acts on ${\mathbb{Z}/2\mathbb{Z}}^X$. That is a direct consequence of the fact that $Aut(G)$ has to preserve in/out degrees, hence $\forall g \in Aut(G), g(X) = X$ and therefore $f \circ g^{-1}$ is a well defined function and thus $Aut(G)$

acts on the left on ${\mathbb{Z}/_{2\mathbb{Z}}}^X$. Consider now the functor $\mathscr{L}$ as a morphism from $Aut(G)$ to $Aut(\mathscr{L}(G))$. Furthermore let $H$ be the image of that morphism. Let us show that $H$ is isomorphic to $Aut(G)$. It simply comes from the fact that the restriction of elements of $H$ on $V$ is an inverse on the right of $\mathscr{L}$. The map $\mathscr{L}$ is therefore injective and a bijection to is image $H$.

The next step is to prove that ${\mathbb{Z}/_{2\mathbb{Z}}}^X$ injects itself into $Aut(\mathscr{L}(G))$. Take $(V', E') = \mathscr{L}(G)$. Consider $H'$ the subgroup of $\mathcal{S}(V')$ (the group of permutations on $V'$), generated by the transpositions $(x\ (t, 0, 1))$ with $x \in X$ and $t \in E$. Let us prove that it is a subgroup of $Aut(\mathscr{L}(G))$. It is enough to prove that these transpositions are automorphisms of $\mathscr{L}(G)$. That is however trivial. Indeed since $x$ is of in-out degree (0,1), the graph will locally around $x$ look like this:




with $t$ being the unique edge in $G$ coming out of $x$. We can see on that image that swapping $x$ with $(t, 0, 1)$, preserves the structure of the graph. Since all these transpositions are of disjoint support they commute with each other and any element in $H'$ can be written as a unique product of them. From that we obtain an isomorphism from $H'$ to ${\mathbb{Z}/_{2\mathbb{Z}}}^X$, hence an injection $i$ from ${\mathbb{Z}/_{2\mathbb{Z}}}^X$ to $Aut(\mathscr{L}(G))$. We notice that

$$\forall g \in Aut(G), \forall f \in {\mathbb{Z}/_{2\mathbb{Z}}}^X, i(g \cdot f) = \mathscr{L}(G)(g) \circ f \circ \mathscr{L}(G)(g)^{-1}$$

thus $H$ acts on $H'$ by conjugation.

Then to prove that $H \rtimes H'$ is isomorphic to $Aut(\mathscr{L}(G))$, we need to prove that $HH' = Aut(\mathscr{L}(G))$ and that $H \cap H' = id$.

<u>Proof that $H \cap H' = \{id\}$:</u>

Let $h \in H \cap H'$. By contradiction, assume that $h$ is not the identity. Then since $h \in H'$, there exists $x \in X$ and $t \in E$, such that $h(x) = (t, 0, 1)$. However since $h \in H$, we have that $h(X) \subseteq X$, even though $h(x) \notin X$, which is a contradiction.

<u>Proof that $HH' = Aut(\mathscr{L}(G))$:</u>

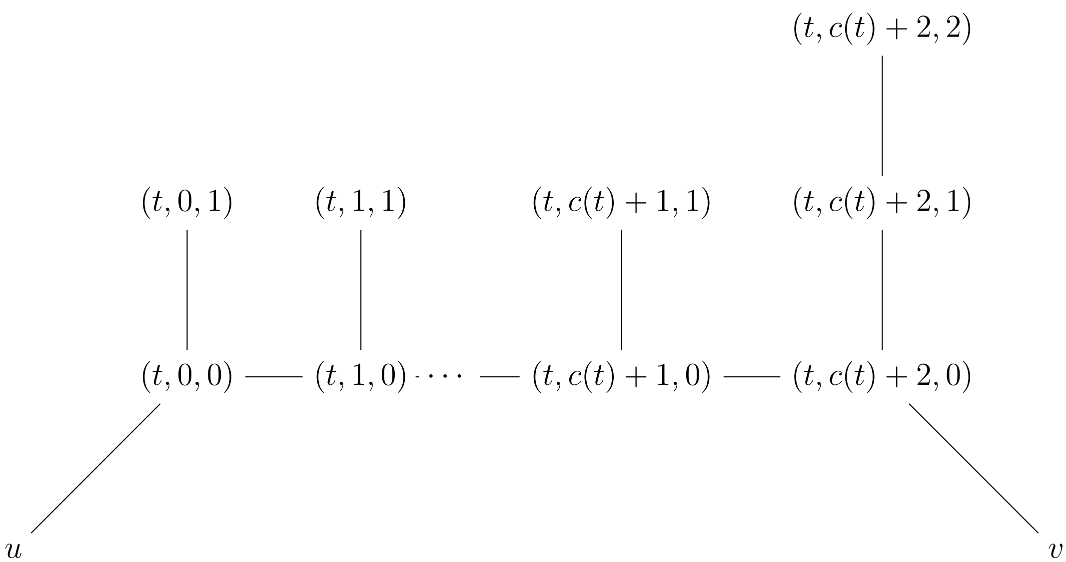
We write

$$mX = \{((x, y), 0, 1) | (x, y) \in E \text{ and } x \in X\}$$

And

$$Y = V' \setminus (X \cup mX)$$

Let $g \in Aut(\mathscr{L}(G))$. Let us prove that $g(Y) = Y$. We do it by induction similar to the method in 1.3.3. First we start with vertices $(t, c(t) + 2, 2)$. As a reminder they look locally like:




Now the vertices of the type $(c(t)+2, 2, t)$ are the only vertices of degree 1 that are connected to a vertex of degree 2. Therefore there exists $t' \in E$, such that $g((t, c(t)+2, 2))$ The path $((t, c(t)+2, 2), (t, c(t)+2, 1), (t, c(t)+2, 0))$ gets sent to $((t, c(t')+2, 2), (t, c(t')+2, 1), (t, c(t'), 0)$. Similarly as in 1.3.3,we prove using induction that $c(t) = c(t')$ and that $(t, k, 0)$ gets sent to $(t', k, 0)$ and $(t, k, 1)$ for $k > 0$ gets sent to $(t', k, 1)$. Now if $u \in V \setminus X$, it cannot be sent onto $X \cup mX$, because it's of degree at least 2 and all elements in $X \cup mX$ are of degree 1. We have now covered all elements that are in $Y$ and none of them are sent onto $X \cup mX$, hence $g(Y) = Y$ and $g(X \cup mX) = X \cup mX$.

Now for $u \in X$, we write $t_u$ the transposition that swaps $u$ with $(t, 0, 1)$, with $t$ being the only edge coming out of $u$. Let us denote for $u \in X$

$$\alpha_g(u) = \begin{cases} 1 \text{ if } g(u) \in mX \\ 0 \text{ else} \end{cases}$$

and write $g' = g \prod_{u \in X} t_u^{\alpha_g(u)}$. Let us show that $g' \in H$. All we really need to show is that $g'(V) = V$ and that is true, since we already showed it for $x \in V \setminus X$ and if we take $x \in X$, we only have to deal with the two following cases:

- Case 1:

  If $g(x) \in X$, then $x$ is not in the support of $\prod_{u \in X} t_u^{\alpha_g(u)}$ and hence $g'(x) = g(x)$, and so $g'(x) \in X$.

- Case 2:

  If $g(x) \in mX$, then $g'(x) = g(t_x(x))$. Since $x$ is sent onto some $(t', 0, 1)$, then there is no other option for $t_x(x)$ than to be sent onto the $u' \in X$ such that $t = (u', v')$ for some $v' \in V$.

From this, we can obtain that $g' = \mathscr{L}(G)(g'_{|V})$ and therefore $g' \in H$. We can finally conclude that $g \in HH'$, proving that $Aut(\mathscr{L}(G))$ is isomorphic to $H \rtimes H'$.

□

As a consequence of this proposition, we can see that the Lovász transformation, preserves the automorphism groups if and only if the graph has no vertices of in-out degree $(0,1)$. In particular that is true for the Cayley graphs, since for a Cayley graph $(G,S)$, there is always at least one edge exiting each vertex.

## 1.4 Using graphs to solve a Galois theory question

### 1.4.1 The context

In [15] Fried and Kollár show a weaker version of the inverse Galois problem. The inverse Galois problem is the following: Given a finite group $G$, is it possible to find $\mathbb{K}$ a Galois extension of $\mathbb{Q}$, for which $Gal(\mathbb{K}/\mathbb{Q})$ is isomorphic to $G$? This is a very difficult question, which is still unanswered, however [15] shows a weaker version of this problem using graph theory. The weaker theorem is as follows:

**Theorem 1.4.1.** *For every finite group $G$, there exists $\mathbb{K}$ an extension of $\mathbb{Q}$, such that $Aut(\mathbb{K}/\mathbb{Q})$ is isomorphic to $G$.*

We will first explain the main arguments of the proof of this theorem and then we will show why the field $\mathbb{K}$ constructed in this proof can never be Galois.

### 1.4.2 Construction of the field $\mathbb{K}$

To construct the field $\mathbb{K}$, the authors of the article rely on the following lemma:

**Lemma 1.4.2.** *Let $L$ be an algebraic number field and $R$ be the ring of integers in $L$. Let $f_1, \cdots, f_m \in R[x]$, be unitary polynomials. Assume that for some prime number $p$, none of the $f_i$ are a $p$-th power in $R[x]$, then there exists a $t \in \mathbb{Z}$, such that none of the $f_i(t)$ is a $p$-th power in $L$.*

The proof of this lemma in the original article [15] contained a mistake that was later on spotted and corrected by Michael Fried in [16].

We will now sketch the proof of the theorem of Fried and Kollár: We start by taking $G$ a finite group and represent it as a group of automorphisms of some undirected loopless graph $X = (V,E)$. We then denote $V = \{1, \cdots, n\}$ and we may assume that $n \geq 5$, since the result is well known for groups of lesser size. Then we choose a polynomial $F(x)$ of degree $n$, whose roots are algebraic integers and such that the Galois group of $F$ is isomorphic to the permutation group $\mathcal{S}_n$. We denote $a_1, \cdots, a_n$ these roots. Take $L$ the splitting field of $F$. Then using the lemma, Fried and Kollár chose a prime $p > 3$ and and integer $t$, such that if $(b_{i,j})$ is a $p$-th root of $a_i + a_j + t$ $(1 \leq i < j \leq n)$ and $\omega$ is a non trivial $p$-th root of unity, then for any $A \subseteq \{b_{(i,j)|1 \leq i<j \leq n}\}$, $b_{(i_0,j_0)} \notin A \Rightarrow \omega b_{(i_0,j_0)} \notin L(A)$.

Now take

$$E' = (b_{(i,j)} | i < j \text{ and } (i,j) \in E)$$

and write $\mathbb{K} = L(E')$. We shall show that its group of automorphisms is isomorphic to those of graph $X$.

If we take $\Phi \in Aut(X)$, one can by induction construct an automorphism $\hat{\Phi}$ on $L$, such that $\hat{\Phi}(b_{(i,j)}) = b_{(\Phi(i),\Phi(j))}$.

If on the other hand we take an automorphism $\Psi$ of $\mathbb{K}$, then for $(i,j) \in E$, $F(\Psi(b_{(i,j)})^p) = 0$, therefore there exists $(i',j')$ and $\omega$ a $p$ th root of unity, such that $\omega b_{(i',j')} \in \mathbb{K}$. Which by the property of $b$ means that $(i',j') \in E$ concluding the theorem.

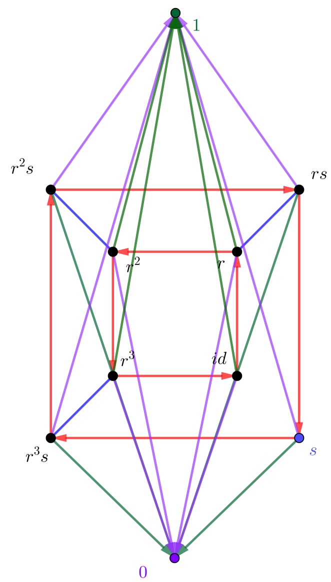
### 1.4.3 Why the field is not a Galois extension

It simply is not a Galois extension, because it contains none of the conjugates of $b_{(i,j)}$, which would be the multiples by $p$-th roots of unity. Moreover the Galois group of the Galois closure would have to be surjective on the group $\mathcal{S}_n$, so would have to be at least as large as $\mathcal{S}_n$, so it turns out the Galois group of the Galois closure is much larger than the group $G$.

While this construction does not make it possible to fully answer the inverse Galois problem, it still shows that connecting Galois theory and Graph theory can lead to interesting results. An essential property of field extensions is their action on groups, so in order to study links between Galois theory and Graph theory, I came up with a way to represent these actions. In fact the procedure will give a way to represent the action of any finite group on a set by a graph.

## 1.5 Group-action Cayley graphs

Consider $(G,X)$ a set $X$ together with an action of $G$. Furthermore suppose that $G$ is generated by a set $S$. We define the group action Cayley graph $Cay(G,S,X)$ as the colored graph whose vertices are $G \amalg X$, colors $S \amalg X$ and edges $\{(g,gs) | g \in G, s \in S\} \amalg \{(g,x) | x \in X, g \in G\}$. Finally we color the edges of the type $(g,gs)$ with $s \in S$, while the edges of the type $(g,x)$ with the color $g^{-1} \cdot x$. For example if we take the dihedral group $D_4$ acting on the set $\{0,1\}$ such that rotations fix $\{0,1\}$ and reflections swap them, the corresponding group action Cayley graph will look as follows:




These types of graphs give many edges, so they are in general difficult to draw in a clear manner. We now have the proposition that follows.

**Proposition 1.5.1.** *Let $G$ be a group generated by a subset $S$ (not containing identity) acting on a set $X$. Then $Cay(G, S, X)$ is a graph with the automorphism group isomorphic to $G$ and the action of $Aut(Cay(G, S, X))$ on $X$ by restriction is equivalent to the action of $G$.*

*Proof.* Take $V = G \amalg X$. Let $E', C', c'$ be such that $(G, E', C', c')$ is a Cayley graph associated to $G$, so that $Aut((G, E', C', c'))$ is isomorphic to $G$. Now expand $E'$ to $E$ by adding all the edges of type $\{(g, x) | g \in Gx \in X\}$. Now define $C = C' \amalg X$ and extend $c'$ into a map from $E$ to $C$ as follows:

$$c = \begin{cases} E \longrightarrow C \\ t \mapsto \begin{cases} c'(t)| \text{ if } t \in E' \\ g^{-1} \cdot x| \text{ if } t = (g, x) \end{cases} \end{cases}$$

$\mathscr{G} = (V, E, C, c)$ is a directed edge colored graph. Now let us show that it has the desired properties:

First let us show that $\forall \sigma \in Aut(\mathscr{G})$, $\sigma(G) \subseteq G$ and consequently $\sigma(X) \subseteq X$, since $G$ and $X$ are disjoint. By contradiction, assume that there exists $g \in G$, such that $x = \sigma(g) \in X$. We have $(g, x)$ that is an edge of color $g^{-1} \cdot x$. Therefore $(\sigma g, \sigma x)$ has to be of color $g^{-1} \cdot x$ as well. That can only happen if $\sigma x \in X$.We end up with an edge between two elements in $X$, which is impossible since there are no such edges. Therefore $Aut(\mathscr{G})$ can be restricted on $G$.

Now let us show that the restriction of $Aut(\mathcal{G})$ on $G$ is injective. Let $\sigma \in Aut(\mathcal{G})$ be such that $\sigma_{|G} = id_G$. Let us show that $\sigma = id_V$. Take $x \in X$. The edge $(id, x)$ is of color $x$, therefore $(g(id), g(x)) = (id, g(x))$ is of color $x$ as well. The only edge leaving $id$ of color $x$ is $(id, x)$, therefore $g(x) = x$. This being true for all $x$, we can conclude that $g = id_V$.

Now let us denote $H$ the image of $Aut(\mathcal{G})$ by the restriction. Now let us show that $H$ is the subgroup of the group of color preserving automorphisms of the Cayley graph $(G, E', C', c')$. Take $h \in H, (g, g') \in E'$. We have that $(h(g), h(g')) \in E$, but also both $h(g)$ and $h(g')$ are in $G$, thus $(h(g), h(g')) \in E$. Now let us find its color

$$c'((h(g), h(g'))) = c((h(g), h(g'))) = c((g, g')) = c'((g, g'))$$

We therefore conclude that $H$ is a subgroup of the automorphisms of the Cayley graph. Now let us show that it is equal to that group. Take $\sigma'$ an automorphism of the Cayley graph $(G, E', C', c')$. Let us show that it can be extended to an automorphism of $\mathcal{G}$ and therefore is in $H$. Write $g = \sigma'(id)$. Now define the function

$$\sigma = \begin{cases} V \longrightarrow V \\ v \mapsto \begin{cases} \sigma'(v) \text{ if } v \in G \\ g \cdot v \text{ if } v \in X \end{cases} \end{cases}$$

It is clear that $\sigma$ extends $\sigma'$. Let us show that it is an automorphism. We start by proving it is bijective. First it is clear that $\sigma G \subset G$ and $\sigma X \subset X$. Since $X$ and $G$ are disjoint, it is enough to show that $\sigma$ is bijective on both these sets. On $G$ it is bijective, since equal to the bijection $\sigma'$. On $X$ it is bijective as an action by $g$. Now we need to prove that $\sigma$ preserves edges and their colors. It is clear for edges in $E'$, since on $G$, $\sigma$ and $\sigma'$ are equal and $\sigma'$ preserves edges with their coloring on $G$. Now take $(g', x)$ an edge in $E$. Then $(\sigma(g'), \sigma(x)) = (\sigma'(g'), g \cdot x) = (gg', g \cdot x)$ is an edge in $E$ as well. Its color is: $(gg')^{-1} \cdot (g \cdot x) = (g')^{-1} \cdot x$, so the color is preserved as well. □

We shall call this graph the "Group action Cayley graph". This graph represents an action of a group on a finite set. We can therefore reinterpret the usual notions of groups actions in terms of graph theory.

**Proposition 1.5.2** (Translating group actions properties)**.** *Let $G$ be a group generated by a set $S$ not containing id and acting on a set $X$. Let $\mathcal{G}$ be the group action Cayley graph corresponding to the action. Then we have:*

*i. For every $x \in X$, the orbit of $x$ is the set of points in $X$ that can be accessed from any vertex in $G$ by a walk on $\mathcal{G}$ only using edges of colors in $S \cup \{x\}$. Alternatively it is the set of points in $X$ that can be accessed by a walk on $\mathcal{G}$ only using edges of colors in $S$ and exactly one color in the orbit of $X$. The color in question can be any arbitrary color in the orbit.*

*ii. For every $x \in X$, the stabilizer subgroup is the set of elements in $G$, such that $(g, x)$ is of color $x$. I.e to obtain the stabilizer, we look for all neighbors of $x$ connected by an edge of color $x$.*

iii. *The group action is transitive if and only if for every $x \in X$ $\mathscr{G}$ with only edges colored by $S$ and $\{x\}$ is (weakly) connected.*

iv. *The group action is faithful if and only for every $g \in G$, if for every $x \in X$ $(g, x)$ is of color $x$, then $g$ is identity or alternatively no two $g, g' \in G$ are connected to all $x \in X$ by the same color.*

v. *The group action is free if and only if id is the only $g \in G$, such that for there exists $x \in X$ $(g, x)$ is of color $x$, then $g$ is the identity. Or alternatively every $x$ has only one edge of a given color.*

*Proof.* i. Let $x \in X$. First let us fix a color $c \in X$ the orbit of $x$ . Let us take $y \in X$, such that there exists a path $(a_1, \cdots, a_m)$ with $a_1 = x$ and $a_m = y$ such that $(a_i, a_{i+1})$ is of color either in $S$ or equal to $c$. Let us prove that $y$ is in the orbit of $x$. Without loss of generality, we may assume that if $i \neq 1$ or $m$, then $a_i \in G$. indeed any step of the type $(g, x)$ has to be followed by a step $(x, g')$. We can replace that path by a path from $g$ to $g'$ only using colors in $S$ and vertices in $G$, since Cayley graphs are connected.

Now since we have only allowed colors in $S$ or colors equal to $c$, then $(a_2, x)$ has to be of color $c$, hence $(a_2)^{-1} \cdot x = c$. Also $(a_{m-1}, y)$ also has to be of color $c$. As such $a_{m-1}^{-1} \cdot y = c = a_2^{-1} \cdot x$, proving that $y$ is in the orbit of $x$.

Conversely assume $y$ in the orbit of $x$ and let us show that there exists a path from $x$ to $y$ with our constraints. First of all, we know that there exists $g \in G$, such that $gx = y$. Now since $c$ is in the orbit of $x$, we take $g' \in G$, such that $(g')^{-1} \cdot x = c$. We have then $(gg')^{-1} \cdot y = c$ and therefore the edge $(gg', y)$ is of color $c$. Finally we choose a path from $g'$ to $gg'$ $(g_1, \cdots, g_m)$ such that all the edges are of colors in $S$.

ii. Let $x$ be in $X$. Then $g$ is in the stabilizer of $x$ if and only if $g^{-1} \cdot x = x$, which is equivalent to the color of $(g, x)$ being $x$.

iii. First suppose the group action being transitive. Then if we fix $c = x$ a color, then every element in $X$ can be accessed by a walk from $x$ using only colors in $S$ and $c$. Proving that the graph is connected, since we can get from $x$ to any element in $G$.

Now conversely suppose that there exists a color $c$ such that the graph $\mathscr{G}$ is connected. Take $x = c$. Now take $y \in X$. Since $y$ can be accessed from $x$ using only edges of colors in $S$ and $c$, then $y$ is by *i.* in the orbit of $x$. Since $X$ has a single orbit, the action of $G$ is transitive.

iv. Suppose that the action of $G$ is faithful. Take $g, g' \in G$ two distinct elements of $G$. Then by definition of faithful there exists $x \in X$, such that $g^{-1} \cdot x \neq g'^{-1} \cdot x$ and therefore the edges $(g, x)$ and $(g', x)$ are of different colors. Conversely suppose that no two elements of $G$ are connected to every element in $X$ by the same color. Let $g, g' \in G$. Then there exists $x \in X$, such that $(g^{-1}, x)$ and $(g'^{-1}, x)$ have different colors, hence $g \cdot x \neq g' \cdot x$.

v. Suppose first that the action is free. Let $(g, x)$ and $(g', x)$ be of the same color: let us prove that $g = g'$. We have $g^{-1} \cdot x = g'^{-1} \cdot x$. Since the action is free then $g^{-1} = g'^{-1}$, hence $g = g'$, proving that $(g, x)$ and $(g', x)$ are the same edge. Now on the other hand suppose that no two edges coming from any $x$ have a same color. Let $g, g' \in G$, such that $g \cdot x = g' \cdot x$. Then the edges $(g^{-1}, x)$ and $(g'^{-1}, x)$ have a same color hence must be the same edge, proving that $g^{-1} = g'^{-1}$, so $g = g'$.

□

# Chapter 2

# Fundamental group and graph homology

## 2.1 Introduction

In this section, we will show a classical way of linking graph theory and group theory through topology: using paths and loops in graphs. While classically this subject is approached from a topological point of view (See for example Hatcher [23] Chapter 1), we will focus more on a purely algebraic approach. One of the reasons for this is that we will later on see how these notions get generalized for profinite graphs, where the profinite topology is very different from the one that graphs are typically equipped with. Our main reference for this chapter is [47].

In this chapter we will give a more general definition of a graph than the previous one.

**Definition 2.1.1** (graph). *We define a graph as a quadruplet* $(\Gamma, V, o, t)$ *with:*

- $\Gamma$ *a set and* $V \subseteq \Gamma$.
- $o, t$ *maps from* $\Gamma$ *to* $V$*, such that* $o_{|V} = t_{|V} = id_V$*. The letter* $o$ *stands for "origin" and* $t$ *stands for "terminus": and the two maps are called the incidence maps.*
- *The set* $V$ *is called the set of vertices and the set* $\Gamma \setminus V$ *is called the set of edges.*
- *A morphism of graphs* $(\Gamma, V, o, t)$ *and* $(\Gamma', V', o', t')$ *is a map* $f$ *from* $\Gamma$ *to* $\Gamma'$*, such that* $f(V) \subseteq V'$*,* $f(\Gamma \setminus V) \subseteq \Gamma' \setminus V'$ *and such that* $\forall x \in \Gamma$*,* $t'(f(x)) = f(t(x))$ *and* $o'(f(x)) = f(o(x))$*.*
- *For a graph* $\Gamma$*, we denote* $V(\Gamma)$ *the set of vertices and* $E(\Gamma)$ *the set of edges.*
- *By abuse of notation if there is no possible confusion, we will call a graph simply the set* $\Gamma$ *rather than the quadruplet and use the same* $o, t$ *maps notations for all graphs we encounter. In that case for example a morphism of*

*graphs* $\Gamma$, $\Gamma'$ *would be a map from* $\Gamma$ *to* $\Gamma'$ *that satisfies* $f(V(\Gamma)) \subseteq V(\Gamma')$, $f(E(\Gamma)) \subseteq E(\Gamma')$ *and* $\forall x \in \Gamma, o(f(x)) = f(o(x))$ *and* $f(t(x)) = t(f(x))$.

Now we will show that this definition extends the definition of a graph, we gave in the first chapter.

If $(V, E)$ is a couple of sets such that $E \subseteq V^2$, we define the map $o$ as the projection of $E$ onto its first component and $t$ the projection of $E$ onto the second component. We extend both maps on $V$, by putting $o_{|V} = t_{|V} = id_V$. Then $\Gamma = (V \amalg E, V, o, t)$ is a graph.

Furthermore if $(V, E)$ and $(V', E')$ are such that $E \subseteq V^2$ and $E' \subseteq V'^2$ and $f$ a map from $V$ to $V'$, then $f$ can be extended to a morphism of graphs $(V \amalg E, V, o, t)$ and $(V' \amalg E', V', o', t')$ if and only if $\forall (x, y) \in E$, $(f(x), f(y)) \in E'$, i.e if $f$ is a morphism of graphs in the sense of the definition given in the previous chapter.

The main advantage of extending a definition of a graph in this way is that we can now have multiple edges between two vertices. Sometimes graphs defined in this way are called multigraphs.

For the sake of convenience rather than necessity, we will work in this chapter with what is called an undirected graph. We will give the definition of an undirected graph that follows:

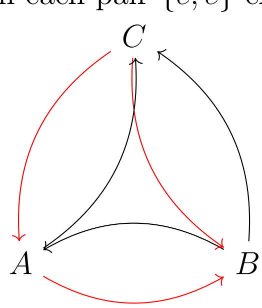
### 2.1.1 Definition of an undirected graph

We will take a variation of the definition given in [47]. An undirected graph is a graph $(\Gamma, V, o, t)$ together with a map from the set of edges $E(\Gamma)$ to itself that to an edge $e$ associates an edge $\overline{e}$ with the following properties:

- $\forall e \in E(\Gamma), \overline{e} \neq e$
- $\forall e \in E(\Gamma), \overline{\overline{e}} = e$
- $\forall e \in E(\Gamma), o(\overline{e}) = t(e)$

A morphism of undirected graphs $\Gamma, \Gamma'$ is then a morphism of graphs from $\Gamma$ to $\Gamma'$, such that $\forall e \in E(\Gamma)$, $f(\overline{e}) = \overline{f(e)}$. The reason we restrict ourselves to undirected graphs is because the notions of paths become easier: we do not have to worry about direction of edges, since if we have an edge $e$ between two vertices in one direction, $\overline{e}$ will have the opposite direction.

Finally we define an orientation of an undirected graph $\Gamma$ as a set $\mathbb{O} \subseteq E(\Gamma)$, such that $\forall e \in E(\Gamma)$, $e \in \mathbb{O}$ or $\overline{e} \in \mathbb{O}$ and $e \in \mathbb{O} \Rightarrow \overline{e} \notin \mathbb{O}$. It comes down to choosing from each pair $\{e, \overline{e}\}$ exactly one edge. Here is a simple example of an orientation



With the edges in $\mathbb{O}$ in red.

### 2.1.2 Definition of a path

Let $G$ be a graph: $x, y \in V$. We define a path $p$ from $x$ to $y$ as a following sequence: $(u_0, \cdots, u_{n+1})$ with:

- $u_0 = x$
- $u_{n+1} = y$
- $\forall k \in \{1, \cdots, n\}$, $o(u_{k+1}) = t(u_k)$ and $u_k$ is an edge.

We then call $n$ the length of a path, $x$ its origin and $y$ its terminus.

Note that for every $x$ with this definition, there exists a path of length 0 from $x$ to $x$.

A path is called a circuit if its initial and terminal vertex coincide. Furthermore if $p$ and $p'$ are paths such that $t(p) = o(p')$, they can be concatenated into a path from $o(p)$ to $t(p')$, which we denote $pp'$.

For a path $p = (u_0, \cdots, u_{n+1})$, we denote $\overline{p}$ the path $(u_{n+1}, \overline{u}_n, \cdots, \overline{u_1}, u_0)$.

We define a round trip as a path of the form $(x, e, \overline{e}, x)$, which we shall denote $e\overline{e}$.

Finally we define a connectedness equivalence relation on the set of vertices of a graph: $\sim$ by $x \sim y$ if and only if there exists a path from $x$ to $y$. It is reflexive due to the fact that there is the path of length zero from a point to itself, it is transitive, because we can concatenate two paths and it is symmetric, because if $p$ is a path from $x$ to $y$, then $\overline{p}$ is a path from $y$ to $x$.

We define the connected components as the equivalence classes for this relation and we say that a graph is connected if it has only one such a class.

## 2.2 Fundamental group and covering graphs

This section is based upon the work of J. Stallings in [47].

**Definition 2.2.1.** *Let $\Gamma$ be a connected graph and $x$ a vertex of $\Gamma$. The set $O$ of circuits starting at $x$ together with the concatenation forms a monoid (a set with an associative binary operation with an identity element). We can see it as a submonoid of the free monoid on $E(\Gamma)$, which we shall denote $F$. The injective morphism $i$ from $O$ into $F$ simply associates to a path $(x, e_1, \cdots, e_n, x)$ the word: $e_1 \cdots e_n$ and to the path of length 0 the identity. Now let $\mathbb{O}$ be an orientation on $E(\Gamma)$ and $A$ the free group on $\mathbb{O}$. We then define $T$ to be the unique morphism from $F$ to $A$, defined by the formula:*

$$T(e) = \begin{cases} e \text{ if } e \in \mathbb{O} \\ \overline{e}^{-1} \text{ else} \end{cases}$$

*We then define an equivalence relation on O:$\sim$ by $p \sim p'$ if and only if $T(i(p)) = T(i(p'))$. It is compatible with the monoid structure, since $T \circ i$ is a morphism. We then define $\pi_1(\Gamma, x)$ as the quotient monoid* $^{O}\!/\!_{\sim}$.

We will show the following facts:

- The equivalence relation $\sim$ is independent of the choice of orientation $\mathbb{O}$.
- $\pi_1(\Gamma, x)$ is a group.
- If $x'$ is another vertex in $\Gamma$, then $\pi_1(\Gamma, x)$ and $\pi_1(\Gamma, x')$ are isomorphic.

To prove that $\sim$ is independent of the choice of orientation, we will prove that it is the equivalence relation on $O$, generated by the subset

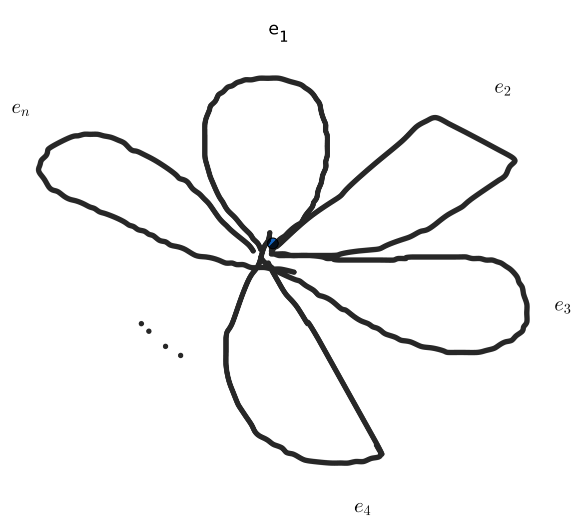
$$\{(pp', pe\overline{e}p')|p, p' \text{ are paths and } e \in E(\Gamma), \text{ such that: } \\ o(p) = x, t(p) = o(e), o(p') = o(e), t(p') = x\}$$

We call $R$ such an equivalence relation. We have that $T(e\overline{e}) = 1$ and therefore $T(i(pp')) = T(i(pe\overline{e}p'))$, so we conclude that $R \subseteq \sim$. To prove the other inclusion, we just observe that $T \circ i$ is injective on reduced paths (paths that don't contain any $e\overline{e}$), and the process of path reduction preserves both $\sim$ and $R$.

We know that $\pi_1(\Gamma, x)$ is a monoid, so to prove it is a group we just need to prove that every element in it has an inverse. Let $p = (x, e_1, \cdots, e_n, x)$ be a representative of an element in $\pi_1(\Gamma, x)$: one can then show that $\overline{p} = (x, \overline{e_n}, \cdots, \overline{e_1}, x)$ is a representative of an inverse in $\pi_1(\Gamma, x)$.

Finally for the isomorphism, we consider: $x'$ another vertex in $\Gamma$. Since $\Gamma$ is connected, there exists $p_0$ a path from $x$ to $x'$. The isomorphism between $\pi_1(\Gamma, x)$ and $\pi_1(\Gamma, x')$, then consists of taking a path $p$ in $\pi_1(\Gamma, x)$ and associate to it a path in $\pi_1(\Gamma, x')$ given by: $p_0 p \overline{p_0}$.

**Example 2.2.2** (bouquet)**.** Take an undirected graph $\Gamma$ with one one vertex $a$ and $e_1, \cdots, e_n$, and $\overline{e_1}, \cdots, \overline{e_n}$ as edges. A graphical representation of the graph is:



All paths start from $a$ and end at $a$, so the injection into the free monoid on $E(\Gamma)$ is in fact an isomorphism. The group $\pi_1(\Gamma, a)$ is then isomorphic to the free group on $e_1, \cdots, e_n$.

It turns out that $\pi_1$ is a group one can always calculate and is a free group as we will see. First however we need to define a notion of a maximal subtree.

**Definition 2.2.3.**
- *For $\Gamma$ an undirected graph , we call a subgraph of $\Gamma$ a set $\Gamma' \subseteq \Gamma$, such that*
$$\forall x \in \Gamma', o(x) \in \Gamma', t(x) \in \Gamma'$$
*and if $e \in \Gamma' \cap E(\Gamma)$, then $\overline{e} \in \Gamma'$.*
- *A graph $\Gamma$ is called a tree if it is connected and if every reduced circuit is of length 0.*
- *A subgraph $T$ of a connected graph $\Gamma$ is called a maximal subtree if $T$ is a tree and if $T \subseteq T' \subseteq \Gamma$ with $T'$ a subtree, then $T = T'$*

**Proposition 2.2.4.** *The following statements are true:*

i. *Every connected graph has a maximal subtree.*

ii. *If $T$ is a tree, then for every $x, y \in V(T)$, there exists a unique reduced path between $x$ and $y$.*

*Proof.* i. Let $\Omega = \{T \subseteq \Gamma | T \text{ is a tree}\}$ together with the relation of inclusion. Let us show that $\Omega$ is a non empty inductive set. The set $\Omega$ is non empty, because if we take $T$ reduced to one vertex and no edges, $T$ is in $\Omega$. Now take $\Delta$ a chain in $\Omega$. Let
$$T = \bigcup_{T' \in \Delta} T'$$
Let us show that $T$ is a tree. If $x \in T$, then there exists $T' \in \Delta$, such that $x \in T'$. Then $o(x)$ and $t(x)$ are both in $T' \subseteq T$ and if $x \in E(\Gamma)$, we get that $\overline{x} \in T' \subseteq T$. $T$ is connected: indeed if $x, y \in T$ are two edges, we take $x \in T' \in \Delta$ and $y \in T'' \in \Delta$. Without loss of generality, we may assume that $T' \subseteq T''$ and in that case both $x$ and $y$ are in $T''$ and by connectedness of $T''$, we get that there is a path in $T''$, so in $T$ between $x$ and $y$. Finally let $x$ be a vertex and $p = (x, e_1, \cdots, e_n, x)$ a reduced path in $T$ from $x$ to $x$: let us show that $n = 0$. By contradiction, if $n > 0$, consider $T_i \in \Delta$, such that $e_i \in T_i$. Then if we take $T'$ a maximum of the $T_i$, then $p$ is a reduced path in $T'$, which is a tree. We therefore get $n = 0$, which is a contradiction. $\Omega$ is therefore an inductive set and by Zorn's lemma it has a maximal element.

ii. Let $T$ be a tree and $x, y$ vertices in $T$. If $x = y$, then paths from $x$ to $y$ are circuits and by definition of a tree all circuits are of length 0. There is only one path from $x$ to $y$ of length 0. If $x \neq y$ and we take $p = (x, e_1, \cdots, e_n, y)$ and $p' = (x, e_1', \cdots, e_m', y)$ two paths, we observe that $p\overline{p'}$ is a path from $x$ to

$x$. Since both $p$ and $p'$ are reduced, we have to have that $e_n = \overline{\overline{e'_m}} = e'_m$ and we can then conclude by induction on $n$.

□

**Lemma 2.2.5.** *Let $p_1, \cdots, p_n$ be reduced paths in a tree $T$ that can be concatenated. Then $p_1 \cdots p_n$ is reduced if and only if for every $k < n$, $p_k \cdot p_{k+1}$ is reduced.*

**Theorem 2.2.6.** *Let $\Gamma$ be a connected graph, $T$ a maximal subtree, $\mathfrak{O}$ an orientation on $\Gamma$. Let $x$ be a vertex of $\Gamma$, then $\pi_1(\Gamma, x)$ is isomorphic to the free group on $\mathfrak{O} \setminus T$.*

*Proof.* For two vertices: $x, y$ we denote $p(x \to y)$ the unique path in $T$ from $x$ to $y$. Now we take $\Phi$ to be the unique morphism from the free group $F$ on $\mathfrak{O} \setminus T$ to $\pi_1(\Gamma.x)$ that to an edge $e \in \mathfrak{O} \setminus T$, associates the class of $p(x \to o(e)) \cdot e \cdot p(t(e) \to x)$. We shall prove that $\Phi$ is an isomorphism in the following steps:

- The path $p(x \to o(e)) \cdot e \cdot p(t(e) \to x)$ is reduced.
- Prove that if $e \in \mathfrak{O} \setminus T$, then $\Phi(e^{-1}) = p(x \to t(e)) \cdot \overline{e} \cdot p(o(e) \to x)$.
- Prove that for every $e, e'$ in $\mathfrak{O} \setminus T \cup (\mathfrak{O} \setminus T)^{-1}$, if $\Phi(ee') = 1$, then $e = e'^{-1}$.
- Prove that $\Phi$ is injective.
- Prove that $\Phi$ is surjective.

For the first part, we get that $p(x \to o(e))$ is reduced by definition and since $e$ is not in $T$, we cannot have $e$ to be equal to the inverse of the last edge in $p(x \to o(e))$. Since $e$ is not in $T$, we also can't have $e$ equal to the first edge in $p(t(e) \to x)$. Finally since $p(t(e) \to x)$ is reduced, we get that $p(x \to o(e)) \cdot e \cdot p(t(e) \to x)$ that is reduced.

Now take $e \in \mathfrak{O} \setminus T$. Then $\Phi(e^{-1}) = \overline{p(t(e) \to x)} \cdot \overline{e} \cdot \overline{p(x \to o(e))}$. The path $\overline{p(x \to o(e))}$ is a reduced path from $o(e)$ to $x$, so by uniqueness of reduced paths in a tree, we get that:
$\overline{p(x \to o(e))} = p(o(e) \to x)$. Similarly $\overline{p(t(e) \to x)} = p(x \to t(e))$, hence

$$\Phi(e^{-1}) = p(x \to t(e)) \cdot \overline{e} \cdot p(o(e) \to x)$$

By convention then from here on now, we just denote $e^{-1} = \overline{e}$, which is justified by the previous result and we can consider $e \in E(\Gamma) \setminus T$.

Now let $e, e'$ be in $\mathfrak{O} \setminus T \cup (\mathfrak{O} \setminus T)^{-1}$, such that $\Phi(ee') = 1$. We get that the path

$$p(x \to o(e)) \cdot e \cdot p(t(e) \to x) \cdot p(x \to o(e')) \cdot e' \cdot p(t(e') \to x)$$

can be reduced to the trivial path of length 0. By the previous lemma that means that one of the concatenations in that path can be reduced, since individual parts of the path are reduced. The only concatenation that can be reduced is
$p(t(e) \to x) \cdot p(x \to o(e'))$. After this reduction, we obtain the homotopy equivalent path:

$$p(x \to o(e)) \cdot e \cdot p(t(e) \to o(e')) \cdot e' \cdot p(t(e') \to x)$$

If $p(o(e) \to t(e))$ is not of length 0, we cannot reduce any further, so we end up with that part being of length 0 and we get that our initial path is homotopy equivalent to:

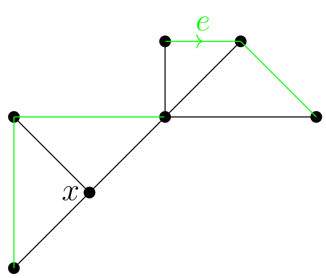
$$p(x \to o(x)) \cdot e \cdot e' \cdot p(t(e') \to x)$$

This can only be reduced further if $e = \overline{e'}$, hence the conclusion.

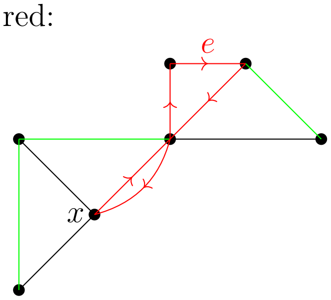
Now more generally, let us prove that $\Phi$ is injective. We then take $x \in \ker(\Phi)$. We write $x = e_1 \cdots e_n$ and we show that if $n > 0$, $x$ can be further reduced. Since the paths $\Phi(e_1), \cdots, \Phi(e_n)$ are all reduced, we get by the previous lemma that there exists $k < n$, such that $\Phi(e_k) \cdot \Phi(e_{k+1})$ can be reduced, which implies that $e_k = e_{k+1}^{-1}$. We therefore get that $x = e_1 \cdots e_{k-1} e_{k+2} \cdots e_n$. By induction, we can therefore conclude that $x = 1$. The map $\Phi$ is therefore injective.

Finally let us prove the surjectivity of $\Phi$. Let us write $p = (x, e_1, \cdots, e_n, x)$ a reduced path in $\pi_1(\Gamma, x)$. We shall prove by induction on the number of $i$ such that $e_i \in \Gamma \setminus T$,that $p$ is in the image of $\Phi$. If there are no edges $e_i$ in $\Gamma \setminus T$, then $p$ is a reduced path in $T$, so $p$ is of length zero and therefore equal to $\Phi(1)$. Now on the other hand assume that $k$ is the smallest number for which $e_k \in \Gamma \setminus T$. Then $p = p(x \to o(e_k)) \cdot e_k \cdot p'$, with $p' = (t(e_k), e_{k+1}, \cdots, e_n, x)$ a reduced path from $t(e_k)$ to $x$. $p$ is then homotopically equivalent to $\Phi(e_k) \cdot p(x \to t(e_k)) \cdot (t(e_k), e_{k+1}, \cdots, e_n, x)$. The path $p(x \to t(e_k)) \cdot (o(e_k), e_{k+1}, \cdots e_n, x)$ is then a reduced path from $x$ to $x$ with one less edge in $\Gamma \setminus T$, then $p$, so by induction it is in the image of $\Phi$, so there exists $u$ in $F$, such that this path is equal to $\Phi(u)$ and hence $p = \Phi(e_k)\Phi(u) = \Phi(e_k \cdot u)$, concluding the proof that $\Phi$ is an isomorphism. □

As an example , let us consider the following graph:




With the edges that are not part of the maximal subtree in green and a distinct edge $e$ with orientation and distinct point $x$. We will now illustrate the path $\Phi(e)$ in red:




If we come back to the bouquet: the maximal tree is just a vertex with no edges, so $\pi_1$ is indeed the free group on edges as we mentioned earlier.

Every group can be represented as a quotient of some free group and graphs give us a way of constructing free groups: the question now is, whether we can also obtain somehow the quotient through graph theory and the answer is yes. The tool for obtaining the quotients is the notion of a covering graph. In order to define a covering graph we first need to define a star.

**Definition 2.2.7** (star). *Let $\Gamma$ be a graph and $x$ a vertex in $\Gamma$. We define a star at $x$, the set*

$$St(\Gamma, x) = \{e \in E(\Gamma) | t(e) = x\}$$

*We call the degree of $x$ the cardinal of the star at $x$.*

Basically a star at $x$ is the set of edges connected to $x$. Trivially we get that if $f$ is a morphism from $\Gamma$ to $\Gamma'$ and $x$ a vertex in $\Gamma$, then $f(St(\Gamma, x)) \subseteq St(\Gamma, f(x))$, which justifies the definitions that follow:

**Definition 2.2.8.** *Let $\Gamma$ be a graph and $v$ a vertex in $\Gamma$ and $f$ a morphism from $\Gamma$ to some graph $\Gamma'$. We denote $f_v$ the map $f$ restricted to the star at $v$ and corestricted to the star at $f(v)$. We then have the following definitions:*

- *If $f_v$ is injective for all $v \in V(\Gamma)$, then we call $f$ an immersion.*
- *If $f_v$ is surjective for all $v \in V(\Gamma)$, then we call $f$ a locally surjective map.*
- *If $f_v$ is bijective for all $v \in V(\Gamma)$, then we call $f$ a covering.*

*If $f$ is a surjective covering map from $\Gamma$ to $\Gamma'$, then we call $\Gamma$ together with $f$ a covering graph of $\Gamma'$.*

The most fundamental properties of covering graphs are the lifting properties, which are completely analogous to the topological theory of covering graphs.

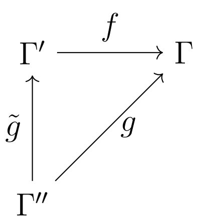
**Proposition 2.2.9** (Lifting properties). *Let $(\Gamma, f)$ be a covering graph of $\Gamma'$, then we have the following properties:*

*a) (Path lifting) Let $v$ be a vertex in $V(\Gamma)$ and $p$ a path in $\Gamma'$ with $f(v)$ as origin. Then there exists a unique path $\tilde{p}$ in $\Gamma$, with initial vertex $v$, such that $f(\tilde{p}) = p$.*

*b) (Homotopy lifting) If the path $p$ in $\Gamma'$ is homotopically equivalent to a path $p'$, then $\tilde{p}$ and $\tilde{p}'$ are homotopically equivalent.*

*c) (General lifting) If $g$ is a morphism from a connected graph $\Gamma''$ to $\Gamma$ and $u$ a vertex in $\Gamma'$, $v$ a vertex in $\Gamma''$, such that $f(u) = g(v)$, then $g$ can be factored through $f$ into a map $\tilde{g}$ from $\Gamma$ to $\Gamma$ if and only if $g(\pi_1(\Gamma'', v)) \subseteq f(\pi_1(\Gamma', u))$. If $\tilde{g}$ exists, it is unique.*

*Factored simply means that there exists a $\tilde{g}$, such that the following diagram commutes*




*Proof.* a) Existence:

We will do it by induction on the length of $p$. If $p$ is of length 0, the result is trivial, since by definition $f$ sends $v$ on $f(v)$, so the path of length 0 starting at $f(v)$ is equal to the image of the path of length 0 starting at $v$.

Now suppose that any path of length $n$ can be lifted. Let $p$ be a path of length $n+1$. Then $p$ can decomposed into a path of length $n$, $p'$ concatenated with some edge $e$. Now $p'$ can be lifted into a path $q$, so we write $p' = f(q)$. Let $x$ be the terminus of $q$. The map $f$ is then by definition of a covering map a bijection from $St(x, \Gamma)$ to $St(t(\overline{e}), \Gamma')$. In particular, there exists $e'$ an edge such that $o(e') = x$ and $f(e') = e$. Now define $\tilde{p}$ to be the path $q$ concatenated with the edge $e'$. Then $f(\tilde{p}) = p$ and we found the lift of $p$.

Uniqueness

We do the uniqueness proof again by induction of the length of the path. If $q, q'$ are two paths of length 0 lifting $p$ with origin $v$, then $q = q'$. Now assume that for all paths of length $n$ the lift by $f$ is unique. Let $p$ be a path of length $n+1$ at the origin $f(v)$. Then write $p = p' \cdot e$ with $p'$ being a path of length $n$. Let $q, q'$ be lifts of $p$. Write $q = q_0 \cdot e_0$ and $q' = q'_0 \cdot e'_0$. Then $f(q_0) = f(q'_0) = p'$ and therefore by induction, $q_0 = q'_0$. Take $x_0$ the terminus of the path $q_0 = q'_0$ and $x$ the terminus of the path $q'$. We have that $f$ is a bijection from $St(x_0, \Gamma)$ to $St(x, \Gamma')$, since $f$ is a covering morphism. Since $e_0, e'_0 \in St(x_0, \Gamma)$ and $f(e_0) = f(e'_0)$, then by injectivity of $f$, $e_0 = e'_0$ and we therefore conclude that $q = q'$.

b) To prove that lifts preserve the homotopy equivalence relation, it is enough to prove that it preserves the relation that generates it. Take $p$ a path with origin in $f(v)$ ending in $f(v)$ and $p'$ a path such that there exists an edge $e$ and paths $p_0$ and $p_1$, such that $p_0 \cdot e\overline{e} \cdot p_1 = p'$ and $p = p_0 \cdot p_1$: we need to show that then the lifts of the two paths are homotopically equivalent. Take $\tilde{p}$ the lift of $p$. We decompose it into $p'_0$ and $p'_1$. Write $x$ to be the origin of the path $p_1$, then $f(x) = o(e)$. Now let $e'$ be the unique edge in $St(x, \Gamma)$, such that $f(e') = e$. Then $f(\overline{e'}) = \overline{e}$ by virtue of $f$ being a morphism. Then we get that

$$f(p'_0 \cdot e'\overline{e'} \cdot p'_1) = f(p'_0) \cdot f(e')\overline{f(e')} \cdot f(p'_1) = p_0 \cdot e\overline{e} \cdot p_1 = p'$$

Therefore the lifts of $p$ and $p'$ are equivalent. Lifts therefore preserve the homotopy equivalence.

c) " $\Rightarrow$ "

Suppose that $g$ factors through $f$ into a map $\tilde{g}$ with $f\tilde{g} = g$. Let $p$ be a reduced path in $\pi_1(\Gamma'', v)$. Then $g(p)$ is a path in $\Gamma$ from $\tilde{g}(v)$ to itself. It therefore lifts into a reduced path starting at $u$, $\tilde{p}$ (with $f(\tilde{p}) = g(p)$). We therefore get that $f(\tilde{p}) = f(\tilde{g}(p))$. By uniqueness of the lift, we get that $\tilde{p} = \tilde{g}(p)$. As such, we get that the terminal point of $\tilde{p}$ is $u$ and $\tilde{p}$ is a reduced circuit, therefore an element of $\pi_1(\Gamma, u)$, hence we indeed have that $g(\pi_1(\Gamma'', v)) \subseteq f(\pi_1(\Gamma', u))$.

" $\Leftarrow$ "

Suppose that $g(\pi_1(\Gamma'', v)) \subseteq f(\pi_1(\Gamma', u))$.

For a vertex $x \in \Gamma''$, we define $\tilde{g}(x)$ as follows:

Take a path $p$ from $v$ to $x$, lift $g(p)$ by $f$ into a path $\tilde{p}$ starting at $u$ and define $\tilde{g}(x)$ as the terminus of the path $\tilde{p}$.

We need to show that $\tilde{g}$ is well defined, i.e does not depend on the chosen path. Take $p, p'$ two paths from $u$ to $x$ and $p_1, p_2$ the lifts of respectively $g(p)$ and $g(p')$ in $\Gamma'$ by $f$. We have that $\overline{g(p)} \cdot g(p')$ is a loop in $g(\pi_1(\Gamma'', v)) \subseteq f(\pi_1(\Gamma', u))$ therefore there exists a circuit $l \in \pi_1(\Gamma, u)$, such that $\overline{g(p)} \cdot g(p') = f(l)$. We then have that

$$f(l) = \overline{f(p_1)} \cdot f(p_2)$$

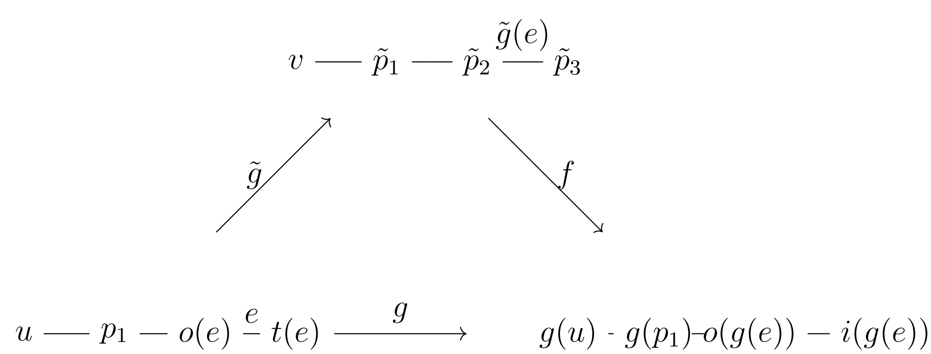
By uniqueness of path lifting, we then get:

$$l = \overline{p_1} \cdot p_2$$

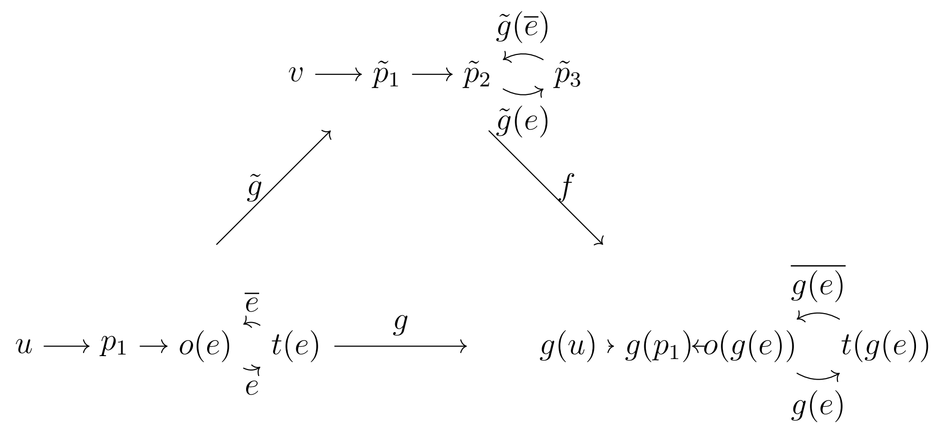
Since $l$ is a circuit, we get that $o(l) = t(l)$ and therefore $t(p_1) = t(p_2)$, proving that $\tilde{g}$ is well defined.

Now we need to define $\tilde{g}$ on edges. For an edge $e \in \Gamma''_E$, take $p$ a path from $v$ to $o(e)$ and concatenate it with $e$ to get a path $p'$. Lift $g(p')$ into a path $\tilde{p}'$ and define $\tilde{g}(e)$ as the last edge of the path $\tilde{p}'$. By the same reasoning as previously, we get that $\tilde{g}$ is well defined on the edges.

Let us now show that $\tilde{g}$ defined in this way is a morphism of graphs. It sends by definition edges on edges and vertices on vertices. Furthermore let $e$ be an edge in $E(\Gamma'')$. Take $p$ a path from $v$ to $o(e)$. Then take $\tilde{p}$ the lift of $g(p)$. Its terminal point is by definition $\tilde{g}(o(e))$. Now concatenate $p$ with the edge $e$ to get $p'$. Let $e'$ be the unique edge in $St(\tilde{g}(o(e)), \Gamma)$, such that $f(e') = g(e)$. Then $\tilde{p} \cdot e'$ is the lift of $p'$, therefore by definition $\tilde{g}(e)$ is the edge $e'$ and we have $o(e') = \tilde{g}(o(e))$ as expected.



Finally we need to show that $\tilde{g}$ preserves inverses of edges. Let $e$ be an edge in $E(\Gamma'')$. To calculate $\tilde{g}(e^{-1})$, we first take a path $p$ from $u$ to $t(e)$. Then define $p' = p \cdot e$, which is a path from $u$ to $o(e^{-1})$. Now take $\tilde{p}$ the lift of $g(p)$ and let $x$ be the terminus of $\tilde{p}$. Furthermore, take $e'$ the unique edge in $St(x, \Gamma)$, such that $f(e') = g(e)$. Then: $\tilde{p} \cdot e'\overline{e'}$ is a lift of $g(p \cdot e\overline{e})$. Its last vertex then is by definition $\tilde{g}(\overline{e})$, which is: $\overline{e'} = \tilde{g}(\overline{e})$, proving that $\tilde{g}$ is indeed a morphism of graphs.



□

## 2.3 Galois theory of covers

There is a very strong analogy between the classical Galois theory of field extensions and covers. As we will see, covers are the analogue of field extensions, while the fundamental group is the analogue of the Galois group.

**Definition 2.3.1** (universal cover)**.** *Let $\Gamma$ be a graph. We call a universal cover of $\Gamma$ a cover $(T, f)$, such that $T$ is a tree.*

**Theorem 2.3.2** (Existence of a universal cover)**.** *Every connected graph has a universal cover. A universal cover of a graph is unique up to isomorphism.*

*Proof.* Existence:

We will show here the classical proof that can be found for example in Hatcher [23] in Proposition 1.36.

We define the set of vertices of a graph $T$ as a set of reduced paths in the graph $\Gamma$ starting at a chosen vertex $v_0$. We take the set of edges of $T$ of the form $(\gamma, \gamma \cdot e)$, with $e$ an edge in $\Gamma$ starting at the vertex $v$ and $\gamma$ a path starting at $v_0$. The inverse of this edge is defined as: $(\gamma \cdot e), \gamma)$.

Let us show that a graph $T$ defined in this way is a tree.

First observe that there are no multiple edges between two vertices. To prove it is connected, we take $p$ an unreduced path in $\Gamma$. We may prove by induction on length of $p$ that there exists a path in $T$ from $(v_0)$ to $p$. For clarity we will denote paths in $T$ with upper case letters and the paths in $\Gamma$ with lowercase.

If $p$ is of length 0, $p = (v_0)$, so the path $((v_0))$ is a path from $(v_0)$ to $p$. Now assume that every path of length $n$ can be attained from $(v_0)$. Let us show it is the case for a path of length $n+1$. If $p$ is of length $n+1$, we decompose it into $p' \cdot e$ with $p'$ of length $n$ and $e$ an edge. Now apply induction to $p'$: by assumption there exists $P'$ a path from $(v_0)$ to $p'$. Then $P = P' \cdot (p', p)$ is a path from $(v_0)$ to $p$.

Now we need to show that there are no cycles in $T$. Take $P$ a reduced path starting at a $p$ and ending at the same $p$. If all the vertices visited by $P$ are paths in $\Gamma$ of the same length that implies that $P$ is of length 0, because the edges are only between paths whose lengths differ by 1. Now by contradiction assume that $P_1$ is of different length than $p$. The cases where the length is greater and lesser are very similar, so we will only consider $l(P_1) = l(p) + 1$. Since the path end back at $p$, there needs to be a step where length decreases by 1.

Let $k$ be the smallest integer where this happens. Now write

$$P_{k+1} = (v_0, e_1, \cdots, e_n)$$

We then get that

$$P_k = (v_0, (e_1, \cdots, e_{n+1}))$$

since $P_k$ and $P_{k+1}$ are connected by an edge, so by definition of edges in $T$, up until $n$-th step the paths must remain identical. Now $P_{k-1}$ is shorter than $P_k$ and since it is connected to $P_k$ by an edge, we have that $P_{k-1} = (v_0, (e_1, \cdots, e_n))$. We therefore end up with $P_{k-1} = P_{k+1}$ and since edges between two vertices in $T$ are in unique, we end up with a possible reduction, contradicting that the path $P$ is reduced. This concludes the proof that $T$ is a tree.

Now we need to define the projection $f$ to make $T$ into a universal cover of $\Gamma$. We send the vertex that is a reduced path $p$ to its terminus: $f(p)$. We send an edge of the form $(p, p \cdot e)$ to $e$ and an edge of the form $(p \cdot e, p)$ to $\overline{e}$. Now we need to show that $(T, f)$ is a covering graph.

We start by showing that $f$ is a morphism of graphs. It by definition sends edges on edges and vertices on vertices.

Now let us take an edge of the form $e' = (p, p \cdot e)$. We take $u = t(p)$. We get that $o(f(e')) = o(e) = u$. We have that $u$ is also the terminus of $p$, therefore $o(f(e')) = f(o(e'))$. Now if we take $e'$ an edge of the form $e' = (p \cdot e, p)$, we get that $o(f(e')) = o(\overline{f(e)}) = t(e)$. Now the terminus of $(p \cdot e)$ is $t(e)$, therefore we get again that: $o(f(e')) = f(o(e'))$. Finally we observe that

$$f(\ \overline{(p, p \cdot e)}\ ) = f((p \cdot e, p)) = \overline{e} = \overline{f(\ (p, p \cdot e\ )\ )}$$

making $f$ into a morphism.

Now we need to show that this morphism is in fact a cover. Let $p$ be a vertex in $T$. We get that

$$St(p, T) = \{(p \cdot e, p) | e \in E(\Gamma) \text{ and } o(e) = t(p)\}$$

Furthermore

$$St(f(p), T) = \{e \in E(\Gamma) | t(e) = f(p)\}$$

The map $f$ is a bijection between the two sets, making $f$ into a covering morphism.

<u>Uniqueness:</u>

Let $(T', f')$ be another universal covering graph. First of all $f'$ has to be surjective. Indeed take $v$ a vertex in $\Gamma$. Let us choose $x$ a vertex in $T'$. Then since $\Gamma$ is connected, there exists a path from $f'(x)$ to $v$: $p$. The path $p$ can then be lifted into a unique path starting at $x$, $\tilde{p}$ by $f'$. If then $u$ is the terminus of $\tilde{p}$, we get that $f(u) = v$. Since $f$ is surjective, we may pick an antecedent $u_0$ of $v_0$ by $f'$. $f'$ is a morphism from the connected graph $T'$ to $\Gamma$, such that $f'(u_0) = f((v_0))$, therefore by 2.2.9 c), there exists a unique $\Phi$ from $T'$ to $T$, such that $f' = f \circ \Phi$. Applying 2.2.9 c) again, we also get a unique morphism $\Psi$ from $T$ to $T'$, such that $f = f' \circ \Psi$. By an argument by universal property, $\Phi$ and $\Psi$ are inverses to each other, making $(T, f)$ and $(T', f')$ isomorphic. □

Now we shall see the theorem that shows the analogy between Galois theory and covers. In order to do that, we need to be able to define what the group of Galois group would be for graphs. In this case, we are interested in deck transformations.

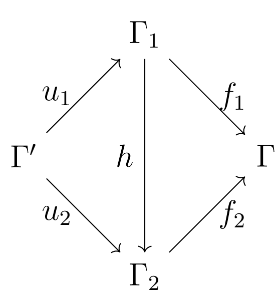
**Definition 2.3.3** (deck transformations)**.** *Let $\Gamma$ be a graph and $(\Gamma', f)$ a cover of $\Gamma$.*

*a) We call a deck transformation an automorphism of $\Gamma'$ that preserves the values of $f$. We will denote $Aut_f(\Gamma)$ the set of deck transformations.*

*b) We call a covering graph of $\Gamma$, $(\Gamma', f)$ normal, if there exists an $x$ a vertex in $\Gamma$, such that $Aut_f(\Gamma)$ acts transitively on the set $f^{-1}(x)$.*

*c) For a covering graph $(\Gamma', f)$, we call a triplet $(\Gamma'', u, f')$ a subcovering graph, if $u$ is a covering morphism from $\Gamma'$ to $\Gamma''$ $f$ a covering morphism from $\Gamma''$ to $\Gamma$ and $f = f' \circ u$.*

*d) For two subcovering graphs $(\Gamma_1, u_1, f_1)$ and $(\Gamma_2, u_2, f_2)$ of a covering graph $(\Gamma', f)$ of $\Gamma$, we call a morphism a map $h$ from $\Gamma_1$ to $\Gamma_2$ a morphism of graphs, such that the following diagram commutes:*




Note that classically the analogy of Galois theory is done for the universal covering space, see for example [14]. However with the right definition of "subcovering", we can extend this approach to any normal covering graph. In that case very loosely, one can think of separable field extension as a connected covering and of normal field extension a normal covering and a subextension then corresponds to a subcovering graph. Finally the group of deck transformations corresponds to the Galois group.

Another interesting remark is that if we omit the maps $u_1$ and $u_2$ from the definition of a morphism, then the Galois correspondence would be between conjugacy classes of subgroups

**Lemma 2.3.4.** *If $(\Gamma', f)$ is a normal covering graph of a connected graph $\Gamma$, then:*

*a)* $\forall x \in V(\Gamma)$, *$Aut_f(\Gamma)$ acts transitively on $f^{-1}(x)$.*

*b)* $\forall e \in E(\Gamma)$, *$Aut_f(\Gamma)$ acts transitively on $f^{-1}(e)$.*

*Proof.* a) By definition of normal cover, we take $x_0$ a vertex in $\Gamma$, such that $Aut_f(\Gamma)$ acts transitively on $f^{-1}(x_0)$. Let $x$ be a vertex in $\Gamma$. Let $u, u' \in f^{-1}(x)$. Take $p$ a path from $x$ to $x_0$. Then it lifts into a unique path $\tilde{p}$ starting at $u$ and a unique path $\tilde{p}'$ starting at $u'$. The terminal points $y, y'$ of respectively $\tilde{p}$ and $\tilde{p}'$ are in $f^{-1}(x_0)$, therefore there exists $g \in Aut_f(\Gamma)$, such that $y = g(y')$. Then $g(\tilde{p}^{-1})$ is a lift of $\overline{p}$ starting at $y'$ and so is $\tilde{\overline{p'}}$, therefore the two paths are equal. In particular their terminal point $g(u)$ and $u'$ are equal, so $g(u) = u'$.

b) Now we take $e$ an edge in $\Gamma$. Let $e'$ and $e''$ be two edges in $\Gamma'$, such that $f(e') = f(e'') = e$. Then we get that $f(o(e')) = f(o(e'')) = o(e)$. We then take $g$ a deck transformation, such that $g(o(e')) = o(e'')$. The map $f$ being a bijection from $St(o(e''), \Gamma')$ to $St(o(e), \Gamma)$ and $f(ge') = f(ge'')$, we get that $ge' = e''$, proving that the deck transformations act transitively on the edges.

□

**Theorem 2.3.5** (Fundamental theorem of Galois theory of graphs)**.** *Let $\Gamma$ be a connected graph and $(\Gamma', f)$ a normal covering of $\Gamma$. Choose a vertex $v$ in $\Gamma$ and $u$ its antecedent in $\Gamma'$ by $f$. Then $f(\pi_1(\Gamma', u))$ is a normal subgroup of $\pi_1(\Gamma, v)$, $Aut_f(\Gamma)$ is isomorphic to ${\pi_1(\Gamma, v)}/{\pi_1(\Gamma', u)}$ and there is a 1 to 1 inclusion reversing correspondence between subgroups of $Aut_f(\Gamma)$ and subcovers $(\Gamma'', f')$ (up to isomorphism) of $(\Gamma', f)$. Furthermore normal subextensions of $(\Gamma', f)$ correspond exactly to the normal subgroups of $Aut_f$.*

*Proof.* First let us start by proving that $f(\pi_1(\Gamma', u))$ is normal is $\pi_1(\Gamma, v)$. Let $p$ be a reduced path in $\Gamma'$ from $u$ to $u$ and $\alpha$ a reduced path from $v$ to $v$ in $\Gamma$. Let us show that there exists a reduced path $p'$ in $\Gamma'$, such that $[f(p')] = \alpha[f(p)]\alpha^{-1}$. The path $\alpha$ is starting at $v$, so by the lifting property, there exists a unique $\tilde{\alpha}$ path in $\Gamma'$, such that $f(\tilde{\alpha}) = \alpha$. Now let $u'$ be the terminus of $\tilde{\alpha}$. Let then $g$ be a deck transformation that sends $u$ to $u'$. The path $g(p)$ can then be concatenated on the left with $\alpha$ and on the right with $\alpha^{-1}$, since it starts at $u'$ and ends at $u'$. Furthermore $\alpha \cdot g(p) \cdot \alpha^{-1}$ starts at $u$ and ends at $u$, therefore has a representative in $\pi_1(\Gamma', u)$, which we shall call $\beta$. We have that $f(\beta) = f([\tilde{\alpha} \cdot g(p) \cdot \tilde{\alpha}^{-1}]) = [f(\alpha)] \cdot [f(g(p))] \cdot [f(\tilde{\alpha})]^{-1}$, with in this context for a circuit $P$, $[P]$ denoting its class in $\pi_1$. Now $f(g(p)) = f(p)$, since $g$ is a deck transformation and hence $f(\beta) = [\alpha][f(p)][\alpha]^{-1}$. This concludes the proof that $f(\pi_1(\Gamma', u))$ is normal in $\pi_1(\Gamma, v)$.

The next step is to prove that $Aut_f(\Gamma)$ is isomorphic to ${\pi_1(\Gamma, v)}/{f(\pi_1(\Gamma', u))}$

We define $\phi = \begin{cases} Aut_f(\Gamma) \longrightarrow {\pi_1(\Gamma, v)}/{f(\pi_1(\Gamma', u))} \\ g \mapsto [f([u, g \cdot u])] \end{cases}$ , with $[u, g \cdot u]$ denoting a path from $u$ to $g \cdot u$ and we shall prove that $\phi$ is an isomorphism of groups. We need to start by proving that $\phi$ is well defined, meaning it doesn't depend on the choice of the path from $u$ to $gu$. Suppose that $p_1$ and $p_2$ are two paths from $u$ to $g \cdot u$, then $f(p_1 \cdot \overline{p_2})$ is in $f(\pi_1(\Gamma', u))$ and therefore the classes of $f(p_1)$ and $f(p_2)$ are the same in the quotient group.

Now we need to prove that $\phi$ is a morphism of groups. Take $g, g' \in Aut_f(\Gamma)$. Let $p$ be a path from $u$ to $g(u)$ and $p'$ a path from $u$ to $g'(u)$. Then

$$f(p) \cdot f(p') = f(p) \cdot f(g(p')) = f(p \cdot g(p'))$$

and $p \cdot g(p')$ is a path from $u$ to $gg'(u)$. We therefore have that $\phi(gg') = \phi(g) \cdot \phi(g')$.

Now we need to prove that $\phi$ is injective. Let $g$ be a deck transformation and $p$ a path from $u$ to $g \cdot u$, such that the class $[f(p)]$ is equal to the class $[f(q)]$, with $[q] \in \pi_1(\Gamma', u)$. By uniqueness of lift, we get that $q$ and $p$ are the same path, therefore $p$ starts and ends at $u$, which means that $g(u) = u$. Since $g$ is a deck transformation that implies that $g$ is the identity.

Finally let us show that $\phi$ is surjective. Take $[p]$ a class in $\pi_1(\Gamma, v)$. Then there exists $\tilde{p}$ a path starting at $u$, lifting $p$. Let $u'$ be the terminus of $\tilde{p}$. By transitivity of the deck transformations, we get that there exists $g \in Aut_f(\Gamma)$, such that $g(u) = u'$. Then $\tilde{p}$ is a path from $u$ to $g(u)$, hence $p = \phi(g)$, proving the surjectivity of $\phi$.

Now is the time to prove the second part of the statement. We will prove that there is a one to one correspondence (up to isomorphism) between subcovers of $(\Gamma', f)$ and subgroups of $Aut_f(\Gamma)$. Let $H$ be a subgroup of $Aut_f(\Gamma)$. We define an equivalence relation on $\Gamma'$ as follows: two vertices $x, y$ are equivalent if there exists $h \in H$, such that $h(x) = y$ and two edges $e, e'$ are equivalent if there exists $h \in H$, such that $h(e) = e'$. We shall show that $f$ is compatible with this equivalence relation and therefore can factor into a morphism from the quotient graph ${}^{\Gamma'}\!/_{H}$ to $\Gamma$. We will then show that ${}^{\Gamma'}\!/_{H}$ together with the natural morphism is a covering graph. Let $x, y$ be equivalent edges or vertices. Then there exists $h \in H$, such that $h(x) = y$. Then $f(h(x)) = f(y)$, since $h$ is a deck transformation, showing that $f$ is indeed compatible. Now we define $\Gamma''$ as the quotient graph of $\Gamma'$ by our equivalence relation. Together with the induced map by $f$: $f' = \overline{f}^H$, $\Gamma''$ is a cover of $\Gamma$. To prove it, let us take $\overline{x}^H$ a vertex in $\Gamma''$. Now let us prove that $f'$ is a bijection from $St(\Gamma'', \overline{x}^H)$ in $St(\Gamma, f(x))$. Let us start with the surjectivity.

If $e \in E(\Gamma)$ is an edge, such that $o(e) = f(x)$, then by surjectivity of $f$, there exists $e' \in St(\Gamma', x)$, such that $f(e') = e$. Then we simply get $f'(\overline{e'}^H) = e$ and $\overline{e'}^H \in St(\Gamma'', \overline{x}^H)$. Now let us prove the injectivity of $f$. Suppose that $\overline{e}^H$ and $\overline{e'}^H$ are two edges in $St(\Gamma', x)$ mapping to the same edge. Now we have that $o(e) = y$ and $o(e') = y'$ and there are $h, h' \in H$, such that $h(y) = x$ and $h'(y') = x$. Then by injectivity of $f$ on the stars we get that $h(e) = h'(e')$ and therefore $e = h^{-1}h'(e')$, proving that the two classes $\overline{e}^H$ and $\overline{e'}^H$ are equal.

Finally we need to show that $\Gamma'$ together with the natural projection of $\Gamma''$: $\pi$ is a covering graph of $\Gamma''$. I.e we need to show that $\pi$ is a bijection on stars.

Let $\overline{x}$ be a vertex in $\Gamma''$. First let us show the injectivity of the projection. Let $y$ be such that $\overline{y} = \overline{x}$. Take $e, e'$ two edges in $St(\Gamma', y)$, such that $\overline{e} = \overline{e'}$. Then there exists by definition a $h \in H$, such that $h(e) = e'$. We also have that $h(x) = x$ and we know that if a deck transformation fixes one point, it is the identity. Therefore $e = e'$. The surjectivity of $\pi$ on the stars is trivial.

This proves that to a subgroup of $Aut_f(\Gamma)$, we can associate a subcovering graph, which we shall call $F(H)$. This map $F$ is inclusion reversing.

The next step is to prove that every subcovering graph $(\Gamma'', f')$ is isomorphic to some $F(H)$. Let $u$ be a morphism from $\Gamma'$ to $\Gamma''$, such that $(\Gamma', u)$ is a covering graph of $\Gamma''$ and $f = f' \circ u$. Let $H = Aut_u(\Gamma'')$. Let us show that $H$ is a subgroup of $Aut_f(\Gamma)$. Suppose that $h \in H$. Then for every $x \in \Gamma'$, $u(h(x)) = u(x)$ and therefore $f'(u(h(x))) = f'(u(x))$ and therefore $f(h(x)) = f(x)$. We therefore get that $h \in Aut_f(\Gamma)$.

Now consider the graph $F(H)$. First since $u$ is by definition constant on the equivalence classes of $H$, $u$ is compatible with the $H$-equivalence relation. It therefore factors into a morphism $\overline{u}^H$ from $F(H)$ to $\Gamma''$: i.e the diagram:

$$\begin{array}{ccc} \Gamma' & \xrightarrow{\;u\;} & \Gamma'' \\ {\scriptstyle \pi_H}\downarrow & \nearrow{\scriptstyle \overline{u}} & \\ F(H) & & \end{array}$$

commutes, with $\pi$ being the natural projection of $\Gamma'$ on $F(H)$. Let us show that $\overline{u}$ is an isomorphism of covering graphs of $\Gamma$. To prove it is a morphism, we need to show that $f' \circ \overline{u} = f_H$ with $f_H$ the natural covering morphism from $F(H)$ to $g$. That is however obvious, since if we pick $\overline{x}^H \in F(H)$, we get:

$$f'(\overline{u}(\overline{x}^H)) = f'(u(x)) = f(x) = f_H(\overline{x}^H)$$

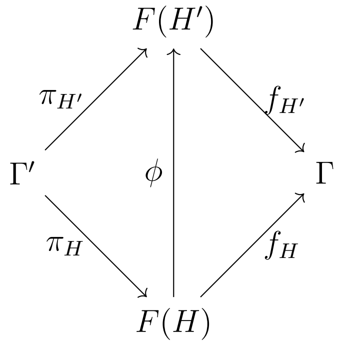
Now we need to prove that $\overline{u}^H$ is bijective. We start by proving that $\overline{u}$ is injective. Let $x, y$ be two vertices in $\Gamma'$, such that $u(x) = u(y)$. Then we get that $f(x) = f(y)$, therefore by the transitivity of the deck transformations, there exists $g \in Aut_f(\Gamma)$, such that $g(x) = y$. Now we need to prove that $g \in H$. By contradiction, suppose that there exists a vertex $z$ in $\Gamma'$, such that $u(z)$ is distinct from $u(g(z))$. Let $p$ be a path from $x$ to $z$. The paths $u(p)$ and $u(g(p))$ are two lifts of the path $f(p)$ starting at $u(x)$, therefore by uniqueness of the lift the two paths are equal. In particular their terminal points are equal. Hence we get that $u(z) = u(g(z))$ and therefore by definition $g \in H$. Since $x = g(y)$, the two vertices are equal in the quotient graph.

Take now $e, e' \in E(\Gamma')$ two edges, such that $u(e) = u(e')$. By transitivity of the deck transformations, there exist $g \in Aut_f(\Gamma)$, such that $g(e) = e'$. Then $g(o(e)) = o(e')$ and therefore by the previous part, we can conclude that $g \in H$. Then $e$ and $e'$ are equivalent in the quotient graph.

The surjectivity of $\overline{u}^H$ is a direct consequence of $u$ being surjective.

Now we need to show that $F$ is injective. Take $H$ and $H'$ subgroups of $Aut_f(\Gamma')$, such that $F(H)$ and $F(H')$ are isomorphic. Then there exists an isomorphism $\phi$ from $F(H)$ and $F(H')$. If then we denote the respective covering morphisms $f_H$ and

$f_{H'}$ and $\pi_H$ and $\pi_{H'}$ the natural projections onto the quotient graphs, we get the two following commutative diagram:




Now let us show that $H \subseteq H'$, which is enough to conclude that $H = H'$, since the two groups play a symmetric role. Let $h \in H$. In that case let us pick $x$ a vertex in $\Gamma'$. We get that:

$$\pi_{H'}(h(x)) = \phi(\pi_H(h(x))) = \phi(\pi_H(x)) = \pi_{H'}(x)$$

As such we get that there exists $h' \in H'$, such that $h(x) = h'(x)$. Since the group $Aut_f(\Gamma)$ acts freely on the vertices, we then conclude that $h = h'$ and so $h \in H'$.

We finally need to show that a subgroup $H$ of $Aut_f(\Gamma)$ is normal if and only if $F(\Gamma')$ is a normal covering graph of $\Gamma$.

Suppose first that $H$ is normal. Take $v$ a vertex in $\Gamma$ and $\overline{x}, \overline{y} \in F(\Gamma)$, such that $f(x) = f(y)$. Since $(\Gamma', f)$ is normal, there exists $g \in Aut_f(\Gamma)$, such that $y = g(x)$. Now let us show that $g$ induces a map $\overline{g}$ from $F(\Gamma)$ to $F(\Gamma)$. We need to show that if two vertices $w, z$ are equivalent that $g(z)$ and $g(w)$ are as well. We get that there exists $h \in H$, such that $z = h(w)$. Then: $g(z) = g(h(w)) = ghg^{-1}(g(w))$. Now since $H$ is normal, we get that $ghg^{-1} \in H$ and therefore $z$ and $w$ are equivalent.

The induced map $\overline{g}$ is trivially a deck transformation of $F(\Gamma)$, since $f'(\overline{g}(\overline{z})) = f(g(z)) = f(z)$. □

As we can see, we then get an analogue of Galois correspondence for fields. The larger the subgroup of symmetries: in this case the deck transformations, the smaller the subcover. The universal cover plays then the role of the algebraic closure. We are now ready to classify all the covering graphs of a graph.

**Theorem 2.3.6** (Classification of covering graphs)**.** *Let $\Gamma$ be a connected graph and a vertex $v$ in $\Gamma$. There is a bijection between connected covering graphs of $\Gamma$ up to isomorphism and subgroups of $\pi_1(v, \Gamma)$ up to conjugation.*

*Proof.* First take $(T, f)$ to be the universal cover of $\Gamma$. If we prove that any covering graph is isomorphic to a subcover of $T$, we can conclude by the fundamental theorem.

First of all the group of deck transformations of $T$ is isomorphic to $\pi_1(v, \Gamma)$, so we can just consider them to be equal by implicitly fixing an isomorphism. Now consider $F$ the Galois correspondence from the previous theorem. To a subgroup of $\pi_1(v, \Gamma)$ we associate the cover $(F(H), f_H)$. Let us now show that if $H$ and $H'$ are conjugate in $\pi_1(v, \Gamma)$, then $F(H)$ and $F(H')$ are isomorphic as covers. (But not as subcovers according to our definition!) Let $g \in \pi_1(v, \Gamma)$ such that $H' = gHg^{-1}$. In that case $\pi_{H'} \circ g$ is invariant by the action of $H$. Indeed if we pick $h \in H$ and $x$ a vertex in $T$:

$$\pi_{H'}(gh \cdot x) = \pi_{H'}(ghg^{-1}g \cdot x) = \pi_{H'}(g \cdot x)$$

Since $ghg^{-1} \in H'$. In that case by the universal property of the quotient, there exists a map $\phi$, such that the diagram

$$\begin{array}{ccc} T & \xrightarrow{\pi_{H'} \circ g} & F(H') \\ {\scriptstyle \pi_H}\downarrow & \nearrow {\scriptstyle \phi} & \\ F(H) & & \end{array}$$

commutes. We observe that $\phi$ is an isomorphism of coverings by constructing an inverse map that is obtained by factoring this time $\phi_{H'} \circ g^{-1}$. The map $F$ then factors into a map from conjugation classes of subgroups of $\pi_1(v, \Gamma)$ to isomorphism classes of coverings of $\Gamma$. We will still call that map $F$ and we shall prove that it is a bijection.

First let us prove that $F$ is injective. Suppose that $F(H)$ and $F(H')$ are isomorphic as covers of $\Gamma$. Let us show that $H$ and $H'$ are then conjugate. Let $\phi$ be the isomorphism of covers from $F(H)$ to $F(H')$.

We then get that the following diagram commutes:

$$\begin{array}{ccc} F(H) & \xrightarrow{f_H} & \Gamma \\ {\scriptstyle \phi}\downarrow & \nearrow {\scriptstyle f_{H'}} & \\ F(H') & & \end{array}$$

Now let $u$ be a vertex in $\Gamma$, such that $f_{H'}(u) = \phi(f_H(v))$. In that case $f(u) = f(v)$ and therefore there exists $g \in \pi_1(v, \Gamma)$, such that $u = g(v)$, since $\pi_1(v, \Gamma)$ acts transitively on fibers. Let us show that $H' = gHg^{-1}$. First observe that we have the following commutative diagram

$$\begin{array}{ccccc} T & & \xrightarrow{\quad f \quad} & & \Gamma \\ & {\scriptstyle \phi \circ f_H} \searrow \searrow {\scriptstyle f_{H'} \circ g} & & \nearrow {\scriptstyle f_{H'}} & \\ & & F(H') & & \end{array}$$

Furthermore $f_H(u) = f_H \circ g(u)$, so by the uniqueness of the factorization given in 2.2.9 part $c)$ , we can conclude that $\phi \circ f_H = f_{H'} \circ g$. Using the fact that $\phi$ is an isomorphism, we also get that $f_H \circ g^{-1} = \phi^{-1} \circ f_{H'}$ Now let $h \in H$.We get that

$$f_{H'}(ghg^{-1} \cdot v) = \phi(f_H(hg^{-1} \cdot v)) = \phi(f_H(g^{-1} \cdot v)) = f_{H'}(v)$$

In that case there exists $h' \in H'$, such that $ghg^{-1} \cdot v = h' \cdot v$ and since the action of $\pi_1(v, \Gamma)$ is free then $ghg^{-1} = h'$ and so $ghg^{-1} \in H'$, concluding that $H \subseteq gHg^{-1}$.

Using the inverse relation $f_H \circ g^{-1} = \phi^{-1} \circ f_{H'}$, we can similarly conclude that $H' \subseteq g^{-1}Hg$ and therefore $H' = gHg^{-1}$. That concludes the injectivity proof.

To prove that $F$ is surjective, we take $(\Gamma', f')$ a covering of $\Gamma$. Then let $(T', t)$ be the universal cover of $\Gamma$. In that case $(T', f' \circ t)$ is a universal cover of $\Gamma$ and $(\Gamma', t, f')$ is a subcover of $T$. By the fundamental theorem and the fact that universal covers are isomorphic, we then conclude that there exists $H$ a subgroup of $\pi_1(v, \Gamma)$, such that $F(H)$ is isomorphic to $(\Gamma', f')$ as expected. □

Another consequence of the fundamental theorem is that we can represent any group as a set of deck transformations of a covering graph as we will see:

**Proposition 2.3.7.** *Let $G$ be a group. Then there exists a graph $\Gamma$ and a normal covering graph $(\Gamma', f)$ of $\Gamma$, such that $Aut_f(\Gamma)$ is isomorphic to $G$.*

*Proof.* Take $\Gamma$ a graph with a single vertex $v$ And the set of edges equal to

$$E = \{[g] | g \in G\} \cup \{[g]^{-1} | g \in G\}$$

with $[g]$ just being a formal letter of an alphabet on $G$. Let $(T, u)$ be the universal cover of $\Gamma$. Its set of deck transformations is then the free group on letters in $E$. Now consider $\phi$ the morphism from $Aut_u(T)$ to $G$ that sends $[g]$ to $g$ and $[g]^{-1}$ to $g^{-1}$.

Let $H$ be its kernel. Let $F(H)$ be the subcover of $\Gamma$ associated to $H$. Then $F(H)$ is normal and the group of deck transformations on $F(H)$ is isomorphic to $Aut_f(\Gamma)/_H$, which itself is isomorphic to $G$ as expected. □

We can then imitate what happens in case of Galois theory for any kind of group. The problem is that infinite Galois groups are only interesting if we take into account their profinite topology, so we have to modify this approach to get profinite coverings, which we will cover in Chapter 4.

## 2.4 Graph homology

Homology is a tool coming from algebraic topology that as we will see is useful for finding the number of connected components as well as finding loops in graphs. In chapter 4, we will see how these tools are used in the profinite graphs case, but first we examine the abstract graphs. This part is loosely based on the work [10]. With a difference that we will use augmented chain complex, to stay consistent with the homology version in chapter 4 and definitions of graph that are consistent with this chapter: i.e we will be taking undirected graphs with the origin and terminus maps.

**Definition 2.4.1** (graph chain complex). *Let $R$ be a ring and $\Gamma$ an undirected graph. We choose an orientation $\mathfrak{O}$ on $\Gamma$.*

- *We denote $C_1(\Gamma, R)$ the free $R$-module on edges in $\mathfrak{O}$*
- *We denote $C_0(\Gamma, R)$ the free module on vertices of $\Gamma$*

- *We denote* $\partial$ *the differential from* $C_1$ *to* $C_0$ *that to an edge* $e$ *associates* $t(e) - o(e)$.
- *We denote* $\epsilon$ *the augmentation map on* $C_0$, *i.e the* $R$*-linear map from* $C_0$ *to* $R$ *that to any vertex* $v$ *associates* $1$.

*Then the* $R$*- complex associated to the graph* $\Gamma$ *is the complex:*

$$C_1(\Gamma, R) \xrightarrow{\partial} C_0(\Gamma, R) \xrightarrow{\quad\epsilon\quad} R$$

*We get that* $\epsilon \circ \partial = 0$ *and therefore we can speak of its homology. We write*

$$H_1(\Gamma, R) = \ker(\partial)$$

*and*

$$H_0(\Gamma, R) = \ker(\epsilon) / im(\partial)$$

*Both* $H_1$ *and* $H_0$ *are seen here is* $R$*-modules.*

It is worth mentioning that this complex is independent of the choice of orientation: if we choose any other orientation, we can see that the two complexes are isomorphic.

The three results we will show are that morphisms of graphs induce natural morphisms between homology rings that a graph is connected if and only if its zeroth homology with coefficients $\mathbb{Z}$ is zero and that a graph is a tree if an only if both $H_0(\Gamma, \mathbb{Z})$ and $H_1(\Gamma, \mathbb{Z})$ are zero.

**Proposition 2.4.2.** *Let* $\Gamma, \Gamma'$ *be two undirected graphs and* $f$ *a morphism from* $\Gamma$ *to* $\Gamma'$. *Let* $\Theta'$ *be an orientation on* $\Gamma'$ *and* $\Theta = f^{-1}(\Theta')$, *which is the unique orientation on* $\Gamma$, *such that* $f(\Theta) \subseteq \Theta'$. *Then there exist a unique morphism* $\Phi_1$ *from* $C_1(\Gamma, R)$ *to* $C_1(\Gamma', R)$, *such that*

$$\forall e \in \Theta, \Phi_1(e) = f(e)$$

*and a unique morphism* $\Phi_0$ *from* $C_0(\Gamma, R)$ *to* $C_0(\Gamma', R)$, *such that*

$$\forall v \in V(\Gamma), \Phi_0(v) = f(v)$$

*We also have that the diagram:*

$$\begin{array}{ccccc} C_1(\Gamma, R) & \xrightarrow{\partial} & C_0(\Gamma, R) & \xrightarrow{\epsilon} & R \\ \Big\downarrow \Phi_1 & & \Big\downarrow \Phi_0 & & \Big\downarrow id_R \\ C_1(\Gamma', R) & \xrightarrow{\partial'} & C_0(\Gamma', R) & \xrightarrow{\epsilon'} & R \end{array}$$

*commutes and therefore* $f$ *induces maps* $H_1(f)$ *and* $H_0(f)$ *on the homology.*

*Proof.* The existence of maps $\Phi_1$ and $\Phi_0$ simply comes from the fact that $C_1$ and $C_0$ are free modules and $f$ is a map from $\mathbb{O}$ to $\mathbb{O}'$ as well as a map from $V(G)$ to $V(G')$. To prove that the diagram commutes, we observe that for $e \in \mathbb{O}$, $t(f(e)) - o(f(e)) = f(t(e)) - f(o(e))$, since $f$ is a morphism. Since the diagram commutes on a basis of $C_1(G, R)$, it commutes everywhere by linearity of all maps involved. □

**Proposition 2.4.3.** *Let* $\Gamma$ *be an undirected graph. The graph* $\Gamma$ *then is connected if and only if* $H_0(\Gamma) = H_0(\Gamma, \mathbb{Z})$ *is equal to zero.*

*Proof.* Suppose first that $\Gamma$ is connected. Choose $\mathbb{O}$ an orientation on $\Gamma$. For $e \in E(\Gamma)$, we denote

$$\alpha(e) = \begin{cases} e \text{ if } e \in \mathbb{O} \\ -\bar{e} \text{ else} \end{cases}$$

One can show that $\ker(\epsilon)$ is generated by $v - u$, with $u, v \in V(\Gamma)$. To then prove that $\partial$ is surjective on $\ker(\epsilon)$, it is enough to prove that all the $v - u$ have an antecedent by $\partial$. Now let $(u, e_1, e_2, \cdots, e_n, v)$ be a path from $u$ to $v$. Notice that $\partial(\alpha(e)) = t(e) - o(e)$, hence
$\partial \sum_{k=1}^{n} \alpha(e_k) = v - u$, hence $\partial$ is indeed surjective onto $\ker(\epsilon)$.

Suppose now that $\partial$ is surjective onto $\ker(\epsilon)$. First of all, any $x \in \mathbb{Z}[\mathbb{O}]$ can be written as a sum $\sum_{k=1}^{n} \alpha(e_k)$. Now we will prove by induction on $n$ that if there exists $u, v \in V(\Gamma)$, such that $\sum_{k=1}^{n} \alpha(e_k) = v - u$, then there exists a permutation $\sigma$ on $\{1, \cdots, n\}$, such that $(o(e_{\sigma(1)}), e_{\sigma(1)}, \cdots, e_{\sigma(n)}, t(e_{\sigma(n)}))$ is a path from $u$ to $v$.

If $n = 1$ the result is trivial. Now suppose that it is true for $n \in \mathbb{N}$. Suppose that

$$\partial(\alpha(e_1) + \cdots + \alpha(e_{n+1})) = v - u$$

If we denote $m_i$ the projection of $\partial(\alpha(e_i))$ on $v$, we get that $m_i \in \{1, 0, -1\}$ and also that $\sum_{i=1}^{n} m_i = 1$. We then have to have that there exists $i_0$, such that $m_{i_0} = 1$, since otherwise the sum would have been negative or null. We have that $\partial(\alpha(e_{i_0})) = t(e_{i_0}) - o(e_{i_0})$. Then we must have that $t(e_{i_0}) = v$. If we then denote $x = o(e_{i_0})$ and $\tau$ the transposition $(i_0, n+1)$, we get that $\partial(\sum_{k=1}^{n} \alpha(e_{\tau(k)})) = x - u$. By induction, there exists then a permutation $\sigma'$ on $\{1, \cdots, n\}$, such that

$$(u, e_{\sigma'(\tau(1))}, \cdots, e_{\sigma'(\tau(n))}, x) \text{ is a path from } u \text{ to } x$$

We extend then $\sigma'$ into a permutation on $\{1, \cdots, n+1\}$, by fixing $n+1$ and we write $\sigma = \sigma' \circ \tau$. Then since $o(e_{\sigma(n+1)}) = o(e_{i_0}) = x$ and $t(e_{\sigma(n+1)}) = t(e_{i_0}) = v$, we get that

$$(u, e_{\sigma(1)}, \cdots, e_{\sigma(n+1)}, v) \text{ is a path from } u \text{ to } v$$

which is what we wanted to show. Now finally if we pick any $u, v \in V(\Gamma)$, then since we know that $H_0(\Gamma) = \{0\}$, we get that there exists $x \in \mathbb{Z}[E(\Gamma)]$, such that

$\partial(x) = v - u$ and by what we have shown earlier that implies that there is a path from $u$ to $v$. $\square$

**Proposition 2.4.4.** *Let $\Gamma$ be an undirected graph. Then $\Gamma$ is a tree if an only if $H_0(\Gamma) = \{0\}$ and $H_1(\Gamma) = \{0\}$.*

*Proof.* First suppose $\Gamma$ to be a tree. Since $\Gamma$ is connected, by the previous proposition, we get that $H_0(\Gamma) = \{0\}$. Now let us show that $H_1(\Gamma)$ is zero as well. Choose $\mathbb{O}$ an orientation on $\Gamma$.

By contradiction, assume that there exists $x \in \mathbb{Z}[\Theta]$, such that $\partial(x) = 0$, but $x$ is not zero. Let us then decompose $x$ into $\sum_{k=1}^{n} \alpha(e_k)$, with

$$\alpha(e) = \begin{cases} e \text{ if } e \in \mathbb{O} \\ -\overline{e} \text{ else} \end{cases}$$

Since $\alpha(e) + \alpha(\overline{e}) = 0$, we may suppose after simplifying that $\forall k, k'$, $\overline{e_k} \neq e_{k'}$. We furthermore have that $\partial(\alpha(e_1) + \cdots + \alpha(e_{n-1})) = o(e_n) - t(e_n)$. Since $\Gamma$ is a tree, it has no loops and therefore $o(e_n) \neq t(e_n)$. By what we have seen in the previous proposition, up to permutation, $(t(e_n), e_1, \cdots, e_{n-1}, o(e_n))$ is a path from $t(e_n)$ to $o(e_n)$. Then we get that $(t(e_n), e_1, \cdots, e_n, t(e_n))$ is a path from $t(e_n)$ to itself. Since $\forall k, k', \overline{e_k} \neq e_{k'}$, the path is reduced. It is then of length 0, since $\Gamma$ is a tree, which is a contradiction.

Now suppose on the other hand that $H_0(\Gamma)$ and $H_1(\Gamma)$ are both zero. Since $H_0(\Gamma)$ is zero, then $\Gamma$ is connected. Now let us show that $\Gamma$ is a tree. By contradiction, assume that there exists a non trivial reduced path from some $u$ to $u$. Let us denote that path $(u, e_1, \cdots, e_n, u)$ We may furthermore suppose that the path has no repetitions i.e that the map $i \mapsto o(e_i)$ is injective, simply by shortening the path if necessary. In that case: $\forall i$, $\overline{e_i} \neq e_1$. Indeed: suppose by contradiction that there is an $i$, such that $\overline{e_i} = e_1$. In that case $t(\overline{e_i}) = o(e_i) = o(e_2)$ and so $e_i = e_2$ by assumption that there are no repetitions in the path. In that case however $e_2 = \overline{e_1}$, which contradicts the assumption that our path is reduced. From this result we get that:

$\forall k$, $\alpha(e_k) \neq -\alpha(e_1)$ and therefore if we project orthogonally $\alpha(e_1) + \cdots + \alpha(e_n)$ onto $\alpha(e_1)$, we get a non zero number, hence $\alpha(e_1) + \cdots + \alpha(e_n)$ is non zero. However since $(u, e_1, \cdots, e_n, u)$ is a path from $u$ to $u$, we get that $\partial(\alpha(e_1) + \cdots + \alpha(e_n)) = u - u = 0$, contradicting the injectivity of $\partial$. $\square$

So far, we have seen two possible approaches to create analogies between Galois theory and graphs. The first one seen in the first chapter is to imitate an action of a Galois group on roots using the Cayley action graphs and the second is using the Galois theory of covers. These two representations however lack topology, which is necessary when we want to use infinite Galois theory rather than the finite one. The next step will then be to equip graphs with a profinite structure, but before doing that, we will see in more detail how infinite Galois theory works in the next chapter.

# Chapter 3

# Profinite structures and Etale algebras

## 3.1 Profinite structures

In this part we will define several profinite structures that we will be using. While these notions are typically defined separately, I decided to group them under the term of "Profinite structure" and prove their common properties together. I originally wanted to call them "Profinite categories", but this term is already used for a different concept in for example the publication: [27] that studies categories from the point of view of graphs.

**Definition 3.1.1** (directed set)**.** *We define a directed set as a nonempty ordered set $(I, \leq)$, such that*

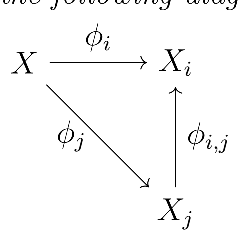
$$\forall x, y \in I, \exists z \in I, z \geq x \text{ and } z \geq y$$

A very simple example of a directed set is a set of finite subsets of a set $\Omega$ for the relation of inclusion. Observe that a union of two finite subsets is a finite subset and it is greater than both sets.

**Definition 3.1.2** (projective limits)**.** *For a category $\mathscr{C}$, we define a projective system as a collection of objects in $\mathscr{C}$ $(X_i)_{i\in I}$ indexed by a directed set $I$, together with morphisms $(\phi_{i,j})_{\substack{i,j\in I\\ i\leq j}}$ from $X_j$ to $X_i$, such that*

$$\forall i, j, k \in I, i \leq j \leq k \Rightarrow \phi_{i,k} = \phi_{i,j} \circ \phi_{j,k}$$

*If it exists, we define a projective limit of a projective system $((X_i)_{i\in I}, (\phi_{i,j})_{\substack{i,j\in I\\ i\leq j}})$ as an object $X$, together with morphisms $(\phi_i)_{i\in I}$ from $X$ to $X_i$ such that for $j \geq i$ the following diagram commutes:*



*Furthermore we require $X$ to be universal: that is if $(Y, (\psi_i)_{i\in I})$ is another object, such that:*

$$\begin{array}{ccc} Y & \xrightarrow{\psi_i} & X_i \\ & \searrow^{\psi_j} & \uparrow \phi_{i,j} \\ & & X_j \end{array}$$

*commutes for every $i, j \in I$, such that $i \leq j$, then there exists a unique morphism $f$ from $Y$ to $X$, such that $\forall i \in I, \psi_i = \phi_i \circ f$.*

**Example 3.1.3.** An example of such a projective limit would be taking a group $G$ and $I$ the set of normal subgroups of $G$ of finite index together with the order of reverse inclusion. Then take the projective system $({}^{G}\!/_{N})_{N\in I}$ with morphisms $\phi_{N,N'}$ being the natural projection of ${}^{G}\!/_{N'}$ on ${}^{G}\!/_{N}$ if $N' \subseteq N$. The limit of such a projective system exists in the category of groups and is called the profinite completion of $G$. Note that if $G$ is finite, then the profinite completion of $G$ is $G$.

One important result is that if a limit of a projective system exists, it is unique up to isomorphism, i.e if $Y$ together with maps $(\phi'_i)_{i\in I}$ is another limit, then there exists an isomorphism $f : X \longrightarrow Y$, such that $\forall i \in I, \phi'_i = \phi_i \circ f$.

So far these definitions are standard and are mentioned for example in [29] on the first page. For the sake of convenience I have decided to treat certain generalities about structures such as profinite rings, groups and modules and prove their common properties for a general structure I call a profinite structure.

**Definition 3.1.4.** *Let $\mathscr{C}$ be a category together with a functor $F$ from $\mathscr{C}$ into the category of sets. We make the following assumptions on $F$:*

- *The functor $F$ is faithful, i.e for all objects $X, Y$: $F$ as a map from $hom(X, Y)$ to $hom(F(X), F(Y))$ is injective.*
- *For any projective system in $\mathscr{C}$ $((X_i)_{i\in I}, (\phi_{i,j})_{j\geq i\in I})$, such that $F(X_i)$ is finite for all $i \in I$, there exists a limit $(X, (\pi_i)_{i\in I})$. Furthermore $(F(X), (\pi_i)_{i\in I})$ is isomorphic in the category of sets to the projective limit of the sets*

  $(F(X_i)_{i\in I}, (F(\phi_{i,j}))_{j\geq i})$.
- *For all objects $A, B, C$ in $\mathscr{C}$ and for all morphisms $u$ from $B$ to $A$, $v$ from $B$ to $C$ and $f$ a map from $F(A)$ to $F(C)$, if $F(u)$ is surjective and the diagram:*

  $$\begin{array}{ccc} F(A) & \xrightarrow{f} & F(C) \\ F(u) \uparrow & \nearrow_{F(v)} & \\ F(B) & & \end{array}$$

  *commutes, then there exists a morphism $w$ from $A$ to $C$, such that $f = F(w)$.*

*We call such category $\mathscr{C}$ together with $F$ a preprofinite category.*

*We then define $\mathscr{P}$ a profinite structure induced by $\mathscr{C}$ as a category whose objects are limits of projective systems of objects whose images by $F$ are finite. Objects in this category are called profinite objects. A morphism between $(X, (\phi_i)_{i\in I})$ and $(Y, (\psi_j)_{j\in J})$ is then defined as a morphism $f \in \hom(X, Y)$, such that for every $j \in J$, there exists an $i_0 \in I$ such that*

$$\forall x, y \in F(X), \forall i \geq i_0, F(\phi_i)(x) = F(\phi_i)(y) \Rightarrow F(\psi_j \circ f)(x) = F(\psi_j \circ f)(y)$$

This definition of a morphism doesn't seem to be very clear, but once we give a more topological interpretation of this category it will make more sense: as the morphisms between two profinite objects will essentially be morphisms that are continuous for a certain profinite topology. We will now show that $\mathscr{P}$ is indeed a category.

Let $(X, (\phi_i)_{i\in I})$ be an object in $\mathscr{P}$. Then observe that $id_X$ is a morphism from $X$ to $X$.

Now suppose that $(X, (\phi_i))_{i\in I}$, $(Y, (\psi_j))_{j\in J})$ and $(Z, (\omega_k))_{k\in K})$ are three objects, $f$ a morphism from $X$ to $Y$ and $g$ a morphism from $Y$ to $Z$. Let us show that $g \circ f$ is morphism from $X$ to $Z$. Let $k \in K$. Then since $g$ is a morphism, there exists $j_0 \in J$, such that

$$\forall j \geq j_0, \forall x, y \in F(Y), F(\psi_j)(x) = F(\psi_j)(y) \Rightarrow F(\omega_k \circ g)(x) = F(\omega_k \circ g)(y)$$

Now since $f$ is a morphism, there exists $i_0 \in I$, such that

$$\forall i \geq i_0, \forall x, y \in F(X), F(\phi_i)(x) = F(\phi_j)(y) \Rightarrow F(\psi_{j_0} \circ f)(x) = F(\psi_{j_0} \circ f)(y)$$

Now suppose that $i \geq i_0$ and $x, y \in F(X)$ are such that $F(\psi_i)(x) = F(\psi_i)(y)$. Then $F(\psi_{j_0})(f(x)) = F(\psi_{j_0})(f(y))$ and therefore

$$F(\omega_k \circ g \circ f)(x) = F(\omega_k \circ g \circ f)(x)$$

proving that $g \circ f$ is indeed a morphism.

The sole reason for this functor $F$ is that strictly speaking most profinite structures are not a subcategory of sets, but often do inject themselves in it. For example while groups are strictly speaking not sets: rather sets together with their internal operation, the morphisms of groups are maps between sets. In practice then by abuse of notation, we will identify $F(X)$ with $X$ and $F(\phi)$ with $\phi$, unless an ambiguity could arise.

We will show that profinite objects have interesting topological properties, but first we will prove a fundamental property of profinite objects that will serve as a foundation for the topology.

**Proposition 3.1.5.** *Let $(\mathscr{C}, F)$ be a preprofinite category and $(X, (\phi_i)_{i\in I})$ an object in the induced profinite structure $\mathscr{P}$. Then*

$$\forall x, y \in F(X), x = y \Leftrightarrow \forall i \in I, F(\phi_i)(x) = F(\phi_i)(y)$$

*Proof.* One implication is clear.
Now consider the set:

$$X' = \{(x_i)_{i\in I} | \forall i \in I, x_i \in F(X_i) \text{ and } \forall j \geq i \in I, F(\phi_{i,j})(x_j) = x_i\}$$

$X'$ together with $\pi_i$ the natural projections on $F(X_i)$ is then the limit of $((F(X_i)_{i\in I}, (F(\phi_{i,j}))_{j\geq i})$ in the category of sets. Since $\mathscr{C}$ is a preprofinite category, we know that $(F(X), (F(\phi_i))_{i\in I})$ is a limit of $(F(X_i)_{i\in I})$ in the category of sets. As such, there exists a bijection $u$ from $F(X)$ to $X'$ , such that $\forall i \in I, \pi_i \circ u = F(\phi_i)$. As such, if we take $x, y \in F(X)$, such that $\forall i \in I, F(\phi_i)(x) = F(\phi_i)(y)$, then we get that $\forall i \in I, u(x)_i = u(y)_i$ and therefore $u(x) = u(y)$. By injectivity of $u$, $x = y$. □

**Definition 3.1.6.** *Let $(\mathscr{C}, F)$ be a preprofinite category. Let $(X, (\phi_i)_{i\in I})$ be a profinite object. We call the profinite topology on $F(X)$ the coarsest topology that makes the maps $F(\phi_i)$ continuous, with $F(X_i)$ being equipped with its discrete topology.*

We then have a theorem that describes the topology that profinite objects can be equipped with. First we will need to prove a helpful lemma.

**Lemma 3.1.7.** *Let $(\mathscr{C}, F)$ be a preprofinite category, $((X_i)_{i\in I}, (\phi_{i,j})_{j\geq i})$ a projective system with $\forall i \in I, F(X_i)$ finite and $(X, (\phi_i)_{i\in I})$ its limit. Let then*

$$X' = \{(x_i)_{i\in I} | \forall i \in I, x_i \in F(X_i) \text{ and } \forall j \geq i \in I, F(\phi_{j,i})(x_j) = x_i\}$$

*together with its natural projections $\pi_i$ and equipped with the product topology (the coarsest topology making the projections continuous), then there exists a homeomorphism $f$ from $F(X)$ to $X'$, such that for all $i \in I$, $F(\phi_i) = \pi_i \circ f$.*

*Proof.* Since $(\mathscr{C}, F)$ is a preprofinite category, then if $X$ is a projective limit in $\mathscr{C}$. We then get that $F(X)$ is a projective limit in the category of sets. Since $F(X)$ and $X'$ are both limits, then by the universal property, there exists a natural isomorphism $f$ between them. We need to prove that $f$ and $f^{-1}$ are both continuous. For $i \in I$, we call $\pi_i$ the natural projection of $X'$ on $F(X_i)$. By Proposition 1 in Chapter 1 section 3 of Bourbaki General Topology [7], $f$ is continuous if and only if $\pi_i \circ f$ is continuous for all $i \in I$. We have that $\pi_i \circ f = F(\phi_i)$, which is continuous by the definition of the topology on $F(X)$, so $f$ is a continuous map. Again by the same proposition in Bourbaki, we get that $f^{-1}$ is continuous if and only if $F(\phi_i) \circ f^{-1}$ is continuous for all $i \in I$. Since $f^{-1}$ is a morphism of limits we get that $F(\phi_i) \circ f^{-1} = \pi_i$ and we get the continuity of $f^{-1}$. □

This lemma implies that if we want to study the topology of the set $F(X)$ induced by the projection maps $F(\phi_i)$, we can simply study the topology of the set $X'$. Finally before stating the theorem that describes the topology of $X'$, we will give here the definition of a uniform space.

**Definition 3.1.8** (uniform spaces)**.** *Let $X$ be a nonempty set. We call a uniform structure (or the set of entourages) on $X$ a set of relations $\mathscr{E}$ on $X$ with the following axioms:*

*i. For every $R \in \mathscr{E}$ and for every $R' \subseteq X \times X$, $R \subseteq R' \Rightarrow R' \in \mathscr{E}$.*

*ii. For every $R, R' \in \mathscr{E}$, $R \cap R' \in \mathscr{E}$.*

*iii. For every $R \in \mathscr{E}$, the diagonal $\Delta \subseteq X \times X$ is contained in $R$.*

*iv. For every $R \in \mathscr{E}$, $R^{-1} \in \mathscr{E}$.*

*v. For every $R \in \mathscr{E}$, there exists $R' \in \mathscr{E}$, such that $R' \subseteq R$.*

Axioms 1 and 2 imply that a uniform structure has to be in particular a filtration. The idea behind uniform structures is that they generalize metric spaces. In metric spaces we get a certain notion of closeness that is independent of where we are. (Just pick a small enough distance). Uniform structure with its relations provides also that independent notion of closeness. We observe that for a uniform structure, we obtain a topology defined as follows: a set of neighborhoods of $x$ is generated by $\Omega = \{\{y \in X | (x, y) \in R\} | R \in \mathscr{E}\}$.

**Theorem 3.1.9** (Properties of a profinite topology)**.** *Let $X = \{(x_i)_{i \in I}\}$ be a profinite set together with the natural projections $\pi_i$. Then the profinite topology $\tau$ on $X$ has the following properties:*

*i. $(X, \tau)$ is Hausdorff.*

*ii. $(X, \tau)$ is compact.*

*iii. $(X, \tau)$ is a uniform space.*

*Proof.* $X$ is Hausdorff, since the product Hausdorff spaces is Hausdorff.

By the theorem of Tychonov, we know that the product is a compact space, therefore $X$ is compact if and only if $X$ is closed in $\prod_{i \in I} X_i$. For $i, j \in I$, such that $i \leq j$, denote $\phi_{i,j}$ the transition map from $X_j$ to $X_i$. To prove that $X$ is closed, we define for $j \in I$

$$A_j = \{(x_k)_{k \in I} | \forall i \leq j, \phi_{i,j}(x_j) = x_i\}$$

We have that

$$A_j = \bigcup_{u \in X_j} \bigcap_{i \leq j} \pi_i^{-1}(\{\phi_{i,j}(u)\})$$

By continuity of the projections $\pi_i^{-1}(\{\phi_{i,j}(u)\})$ is closed for all $u \in X_j$ and all $i \leq j$. The set $A_j$ is then closed as a finite union of closed subsets. Finally $X = \bigcap_{j \in I} A_j$ is an intersection of closed subsets, therefore $X$ is closed, proving that $X$ is compact.

Finally, let us prove that $X$ is a uniform space. We define a collection of relations

$$(R_i)_{i \in I} = (\{(x, y) \in X \times X | \pi_i(x) = \pi_i(y)\})_{i \in I}$$

We then take $\mathscr{R}$ to be the set of all the $R_i$. We define the set of entourages as

$$\mathscr{E} = \{U \subseteq X \times X | \exists R \in \mathscr{R} | R \subseteq U\}$$

The relations in $\mathscr{R}$ are equivalence relations, therefore the set $\mathscr{E}$ of relations on $X$ generated by $R_i$ form a uniform structure on $X$.

We now need to prove that the topology coming from this uniform structure is the profinite topology on $X$.

Let $x \in X$ and $V$ a neighborhood of $x$ for the profinite topology. Then there exists a finite set $I_0 \subseteq I$ and a collection $(A_i)_{i \in I_0}$ with $\forall i \in I_0, A_i \subseteq X_i$, such that $V$ contains: $X \cap \prod_{i \notin I_0} X_i \times \prod_{i \in I_0} A_i$. Now let $i_0$ be an upper bound of $I_0$. In that case $V$ contains

$$\prod_{i \neq i_0} X_i \times \{\pi_{i_0}(x)\} \cap X$$

Indeed suppose that $y_{i_0} = x_{i_0}$. If we take $i \in I_0$, then

$$y_i = \phi_{i_0,i}(y_{i_0}) = \phi_{i_0,i}(x_{i_0}) = x_i \in A_i$$

Therefore we get that $y_i \in A_i$ for all $i \in I_0$ and hence $y \in V$. Now the set $X \cap \prod_{i \neq i_0} X_i \times \{x_i\}$ is exactly the set of $y$, such that $(x, y) \in R_i$, so $V$ is a neighborhood of $x$ for the uniform topology.

Now if we take $U$ a neighborhood of $x$ for the uniform topology, there exists $i \in I$, such that

$$\pi_i^{-1}(\{\pi_i(x)\}) = \{y \in X | (x, y) \in R_i\} \subseteq U$$

$U$ is therefore a neighborhood of $x$, by continuity of $\pi_i$. This concludes the proof that $X$ is indeed a uniform space.

□

For a profinite object $(X, (\phi_i)_{i \in I})$ in a profinite structure $\mathscr{P}$ induced by a pre-profinite category $(\mathscr{C}, F)$, the uniform relations on $F(X)$ will then be given by the maps $\phi_i$.

More often than not we do not have a metric structure on profinite spaces, but as the theorem shows, we will get a uniform structure given by the projections. If we replace then Cauchy sequences by Cauchy filters, then one can show that since profinite spaces are compact uniform, they are complete.

Structures like topological rings , groups are then uniform spaces and they can be completed into a profinite space. The example we saw in 3.1.3 is a case of such completion. We simply consider the group as a discrete topological group and we complete it into a profinite space where the original group is dense. We will see the details of it later. Before examining different profinite categories in detail, we will show one last generality on the profinite topology.

**Proposition 3.1.10.** *Let $(X, (\phi_i)_{i \in I})$ be a profinite object. Then $F(X)$ is totally disconnected. That is, if $Y \subseteq F(X)$ has only trivial clopen subsets, then $Y$ is reduced to a single element.*

*$F(X)$ is therefore a Stone Space: a totally disconnected, Hausdorff compact space.*

*Proof.* Let $Y \subseteq F(X)$ that has at least two distinct elements $x, y \in Y$. In that case, there exists $i \in I$, such that $F(\phi_i)(x) \neq F(\phi_i)(y)$. The set $U = F(\phi_i)^{-1}(\{F(\phi_i)(x)\})$ is a clopen in $X$ by continuity of $F(\phi_i)$. The set $U \cap Y$ is then a non trivial clopen in $Y$. It is non trivial since it contains $x$, but doesn't contain $y$. □

Depending on the categories Stone Spaces might not always be profinite objects in the same category, but it is true for the categories we will be working with.

We will give a sufficient condition for which Stone Spaces are profinite objects.

**Definition 3.1.11** (profinite-compatible objects)**.** *Let $(\mathscr{C}, F)$ be a preprofinite category and $X$ an object in $\mathscr{C}$. We then say that $X$ is a profinite-compatible object, if there exists a compact Hausdorff topology on $F(X)$, such that for every open equivalence relation $R$ on $F(X)$, there exists an open equivalence relation $R' \subseteq R$, an object $U$ in the category $\mathscr{C}$ and a morphism $f$ from $X$ to $U$, such that $F(X)$ together with $F(f)$ is isomorphic to the natural projection of $F(X)$ on ${F(X)}/{R'}$ as a map of sets. I.e: we have that $\forall x, x' \in F(X)$, if $xRx'$, then $F(f)(x) = F(f)(x')$ and if $g$ is a map from $F(X)$ to a set $Y$, such that $\forall x, x' \in F(X)$, $xRx' \Leftrightarrow g(x) = g(x')$, then there exists a unique map $u$ from $F(U)$ to $Y$, such that the following diagram commutes:*

$$\begin{array}{ccc} F(X) & \xrightarrow{g} & Y \\ {\scriptstyle F(f)}\big\downarrow & \nearrow{\scriptstyle u} & \\ F(U) & & \end{array}$$

*Remark.* Note that if $g$ is a morphism, i.e there exists $Y'$ an object in $\mathscr{C}$, such that $Y = F(Y')$ and $g'$ a morphism from $X$ to $Y'$, such that $g = F(g')$, then by the properties of a preprofinite category, we get that $u$ is a morphism i.e there exists $u'$ a morphism from $X$ to $Y'$, such that $F(u') = u$.

**Proposition 3.1.12.** *Let $(\mathscr{C}, F)$ be a preprofinite category, $\mathscr{P}$ the induced profinite structure and $X$ an object in $\mathscr{C}$. If $X$ is a profinite-compatible object, then the following statements are equivalent:*

*i. $X$ is a profinite object in $\mathscr{P}$, and the profinite topology on $F(X)$ is the same as the topology making $X$ into a profinite-compatible object.*

*ii. $F(X)$ together with its profinite-compatible topology is a Stone space: i.e compact, Hausdorff and totally disconnected.*

*iii. There exists $\Omega$ a directed set (for inverse inclusion) of open relations on $F(X)$, such that $\forall R \in \Omega$, $\exists X_R \in \mathcal{O}b_j(\mathscr{C}), \exists f_R \in \hom(X, X_R), (F(X), F(f_R))$ is isomorphic to the natural projection of $F(X)$ onto ${F(X)}/{R}$ and such that $\bigcap\limits_{R \in \Omega} R = \{\Delta\}$, where $\Delta$ is the diagonal of $F(X) \times F(X)$.*

*Proof.* *i.* $\Rightarrow$ *ii.* Assume $X$ to be a profinite limit of $((X_i)_{i\in I}),(\phi_{i,j})_{j\geq i})$. Equip $F(X)$ with its profinite topology. Then by 3.1.10, we get that $F(X)$ is a Stone-Space.

*ii.* $\Rightarrow$ *iii.*

Let $\Omega'$ be the set of open relations. We start by proving that $\bigcap\limits_{R\in\Omega'} R = \{\Delta\}$. We take $x \in F(X)$. We shall prove that for all $y \in F(X)$ such that $y \neq x$ there exists $R \in \Omega'$, such that $(x,y) \notin R$. To prove it we will use a similar method to the one used in the Lemma 1.1.11 (page 22) of [42].

First we consider $T$ the family of all clopen neighborhoods of $x$. We will show that $\bigcap\limits_{V\in T} V$ is connected. We write $A = \bigcap\limits_{V\in T} V$. Assume that $A = F_1 \cup F_2$ with $F_1 \cap F_2 = \emptyset$. By contradiction, assume that $F_1 \neq \emptyset$ and $F_2 \neq \emptyset$. Now let $a \in F_1$. For every $b \in F_2$, there exists $U_{(a,b)}$ an open neighborhood of $a$ and $V_{(a,b)}$ an open neighborhood of $b$, such that $U_{(a,b)} \cap V_{(a,b)} = \emptyset$, since $F(X)$ is Hausdorff. We have that $F_2 \subseteq \bigcup\limits_{b\in F_2} V_{(a,b)}$, therefore by compactness there exists a finite family $b(a)_1, \cdots, b_{n(a)} \in F_2$, such that $F_2 \subseteq \bigcup\limits_{k=1}^{n(a)} V_{(a,b(a)_i)}$. Now consider $U_a = \bigcap\limits_{k=1}^{n(a)} U_{(a,b(a)_k)}$ and $V(a) = \bigcup\limits_{k=1}^{n(a)} V_{(a,b(a)_l)}$. We get that $U(a)$ is an open neighborhood of $a$, $V(a)$ is an open containing $F_2$ and $U_a \cap \bigcup V(a) = \emptyset$. We have that $F_1 \subseteq \bigcup\limits_{a\in F_1} U_a$ , therefore by compactness, there exists a finite family : $a_1, \cdots, a_n \in F_1$, such that $F_1 \subseteq \bigcup\limits_{k=1}^{n} U(a)$. Now put $U = \bigcap\limits_{k=1}^{n} U_a$ and $V = \bigcap\limits_{k=1}^{n} V_a$. We get that $F_1 \subseteq U$, $F_2 \subseteq V$, $U \cap V = \emptyset$ and $U, V$ are open in $F(X)$. We then get that:

$$F(X) \setminus (U \cup V) \cap A - \emptyset$$

$F(X) \setminus (U \cup V)$ is closed, therefore by compactness, there exists a finite subfamily $T' \subseteq T$, such that:

$$F(X) \setminus (U \cup V) \cap \bigcap_{W\in T'} W = \emptyset$$

We have that $B = \bigcap\limits_{W\in T'} W$ that is a clopen neighborhood of $x$ as a finite intersection of clopen neighborhoods of $x$. Then we get that:

$$x \in (B \cap U) \cup (B \cap V) = B$$

Let us assume without loss of generality that $x \in B \cap U$. The set $B \cap U$ is open, as intersection of two opens. Its complement in $B$: $B \cap V$ is also open in $B$ and therefore in $F(X)$. The set $B \cap U$ is then a clopen neighborhood of $x$ and therefore $A \subseteq B \cap U$ and so $B \cap U = A$. We then get that $A \subseteq U$, so $F_2 \cap A = \emptyset$, proving that $F_2$ has to be empty which is a contradiction.

Now that we proved that $A$ is connected, we use the fact that $F(X)$ is totally disconnected to obtain that $A = \{x\}$ as announced. Now let $y \in F(X)$ distinct from $x$. Since $\bigcap\limits_{V\in T} V = \{x\}$, we get that $\{y\} \cap \bigcap\limits_{V\in T} V = \emptyset$. The singleton $\{y\}$ being closed in $F(X)$ we get by compactness that there exists $V_1, \cdots, V_n \in T$, such that

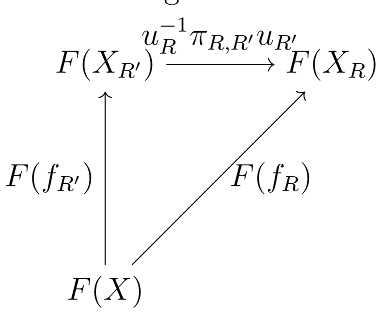
$y \notin V_1 \cap \cdots \cap V_n$. If then we put $V = V_1 \cap \cdots \cap V_n$, we get that $V$ is a clopen neighborhood of $x$ and $y \notin V$. Now consider $R$ an equivalence relation, whose equivalence classes are $V$ and $F(X) \setminus V$. It is then an open relation, since both $V$ and $F(X) \setminus V$ are open. We therefore get that $R \in \Omega'$ and $(x, y) \notin R$.

From that we conclude that $\bigcap_{R \in \Omega'} R = \Delta$.

Finally we will use that $X$ is a profinite-compatible object to reach our conclusion. For every $R \in \Omega'$, we take $\mathscr{P}(R) \subseteq R$ an equivalence relation, such that there exists an object $X_R$ in $\mathscr{C}$ and a morphism $f_R$ from $X$ to $X_R$, such that $(F(X), F(f_R))$ is isomorphic to the natural projection of $X$ onto ${X}/{\mathscr{P}(R)}$. We then define $\Omega$ as the set $\{\mathscr{P}(R) | R \in \Omega'\}$. Finally to conclude that $\bigcap_{R \in \Omega} R = \Delta$, we take $x \in F(X)$ and $y \neq x$. We then know that there exists $R \in \Omega'$, such that $(x, y) \notin R$. In that case we have that $(x, y) \notin \mathscr{P}(R)$ as well, proving indeed that $\bigcap_{R \in \Omega} R = \Delta$. Finally we need to prove that $\Omega$ is a directed set for the relation of inverse inclusion. That is however easy to see, since $\mathscr{P}(\mathscr{P}(R) \cap \mathscr{P}(R')) \subseteq \mathscr{P}(R) \cap \mathscr{P}(R')$ for any two relations $R, R' \in \Omega'$.

*iii.* $\Rightarrow$ *i.*
Consider the set $\Omega$ of the open equivalence relations compatible with $F$, such that $\bigcap_{R \in \Omega} R = \Delta$ and such that $\Omega$ is a directed set for the inverse inclusion.

For $R \in \Omega$, we take $X_R$ an object in $\mathscr{C}$ and $f_R$ a morphism from $X$ to $X_R$, such that $(F(X), F(f_R))$ is isomorphic to the natural projection $\pi_R$ onto ${F(X)}/{R}$. We denote $u_R$ the isomorphism between the two. Now consider for $R' \subseteq R$, $\pi_{R,R'}$, the map from ${F(X)}/{R'}$ to ${F(X)}/{R}$ that to a class $\overline{x}^{R'}$, associates the class $\overline{x}^{R}$. Now consider the diagram:




The map $F(f_{R'})$ is surjective and therefore there by the third axiom of preprofinite categories, there exists a $f_{R,R'}$ a morphism from $X_{R'}$ to $X_R$, such that $F(f_{R,R'}) = u_R^{-1}\pi_{R',R}u_{R'}$. Now let us show that if $R'' \subseteq R' \subseteq R$, then $f_{R,R''} = f_{R,R'} \circ f_{R',R''}$. We simply just have:

$$F(f_{R,R''}) = u_R^{-1}\pi_{R,R''}u_{R''} = u_R^{-1}\pi_{R,R'}u_{R'}^{-1}u_{R'}\pi_{R',R''}u_{R'} = \\ F(f_{R,R'}) \circ F(f_{R',R''}) = F(f_{R,R'} \circ f_{R',R''})$$

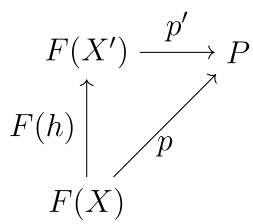
By injectivity of $F$, we then get that $f_{R,R''} = f_{R,R'} \circ f_{R',R''}$. Now let us show that $(X, (f_R)_{R \in \Omega})$ is the projective limit of the projective system $((X_R)_{R \in \Omega}, (f_{R,R'})_{R' \subseteq R \in \Omega})$.

First let us show, that $(F(X), F(f_R)_{R \in \Omega})$ is a projective limit of

$(F(X), F(f_{R,R'})_{R'\subseteq R})$ in the category of sets. First denote $P$ the projective limit of $(F(X).F(f_{R,R'})_{R'\subseteq R})$ in the category of sets and take $p$ the map from $F(X)$ to $P$, that to $x \in F(X)$, associates $(F(f_R)(x))_{R\in\Omega}$. Let us show, that $p'$ is a bijection. To prove that it is injective, we take $x, y \in F(X)$, such that $f_R(x) = f_R(y)$ for all $R \in \Omega$. That means that $(x, y) \in R$ for all $R \in \Omega$, hence $x = y$, since $\bigcap\limits_{R\in\Omega} R = \Delta$ by assumption on $\Omega$. Now let us prove that $p'$ is surjective. Let $(y_R)_{R\in\Omega}$ be a collection in $P$. For every $R \in \Omega$, $F(f_R)$ is isomorphic to the natural projection onto the quotient ${F(X)}/{R}$, hence the map $F(f_R)$ is surjective and therefore we can rewrite the collection $(y_R)_{R\in\Omega}$ as $(F(f)(x_R))_{R\in\Omega}$. Now by contradiction assume that $\forall x \in F(X)$, $\exists R \in \Omega, F(f_R)(x) \neq F(f_R)(x_R)$. That means that the set $\bigcap\limits_{R\in\Omega} x_R R$ is empty. Note that every $R \in \Omega$ is an open relation in a compact set, hence it is closed as well as we can write $F(X)$ as a finite union of open equivalence classes. We then get that $\bigcap\limits_{R\in\Omega} x_R R$ is an empty intersection of closed subsets, hence there exists $R_1, \cdots, R_n \in \Omega$, such that $x_{R_1}R_1 \cap \cdots \cap x_{R_n}R_n$ is empty. Now take $R_0 \in \Omega$ included in $R_1 \cap \cdots \cap R_n$(it exists since by assumption $\Omega$ is a directed set). We then get $x_{R_0} \in x_{R_1}R_1 \cap \cdots \cap x_{R_n}R_n$, which is a contradiction. As such there exists $x \in F(X)$, such that $\forall R \in \Omega, F(f_r)(x) = y_R$, proving that $p$ is a surjective map.

Using the fact that $(F(X), F(f_R)$ is a projective limit in the category of sets, we can now conclude that $(X, f_R)$ is a projective limit in our preprofinite category. Let $(X', g_R)$ be the limit of the projective system $((X_R)_{R\in\Omega}, (f_{R,R'})_{R'\subseteq R}$ (it exists, since the category $\mathscr{C}$ is preprofinite and we have shown earlier, that $F(X_R)$ has to be finite). Then we know that there exists a unique morphism $h$ from $X$ to $X'$, such that $\forall R \in \Omega, f_R = g_R \circ h$. We also know that $F(X')$ together with the $(F(g_R))_{R\in\Omega}$ is a projective limit in the category of sets, therefore there exists $p'$ an isomorphism between $F(X')$ and projective limit $P$ of the $F(X_R)$. We have shown that $F(X)$ together with the $F(f_R)$ is a projective limit of the system $(F(X_R)_{R\in\Omega}, F(f_{R,R'})_{R'\subseteq R\in\Omega})$ and we called $p$ the isomorphism from $F(X)$ to $P$. Now write $h' = p^{-1}p'$, which a map from $F(X')$ to $F(X)$.

Notice that we have:




Which can be checked by composing with the natural projections of $P$ onto $X_R$. In that case: $h' \circ F(h) = p^{-1} \circ p' \circ F(h) = p^{-1} \circ p = id_{F(X)}$. By similar arguments, one can also check that $F(h) \circ h' = Id_{F(X')}$. The map $F(h)$ is therefore bijective. Now we have that:

$$\begin{array}{ccc} F(X') & \xrightarrow{h'} & F(X) \\ {\scriptstyle F(h)}\uparrow & \nearrow {\scriptstyle F(id_X)} & \\ F(X) & & \end{array}$$

commutes. Therefore there exists $h''$ a map from $X'$ to $X$, such that $h' = F(h'')$. Using the injectivity of $F$, we conclude that $h''$ is the inverse of $h$ and therefore $X$ and $X'$ are isomorphic and $X$ is therefore a projective limit of finite objects in $\mathscr{C}$ hence an object in the the induced profinite structure $\mathscr{P}$.

Finally we need to show that the profinite topology on $F(X)$ is the one we started with. To prove that, consider $U$ an open neighborhood of $x \in F(X)$. Let us show that $U$ is a neighborhood of $x$ for the profinite topology. Since $\bigcap_{R \in \Omega} xR = \{x\}$, we have that $(F(X) \setminus U) \cap \bigcap_{R \in \Omega} xR = \emptyset$. By compactness of $F(X)$ there exists then a $R \in \Omega$, such that $xR \cap F(X) \setminus U = \emptyset$. In that case $xR \subseteq U$. The set $xR$ is equal to $\{y \in F(X) | F(f_R)(x) = F(f_R)(y)\}$, hence $xR$ is a neighborhood of $x$ in the profinite topology. The set $U$ then is a neighborhood of $x$ as well. On the other hand every neighborhood of $x$ for the profinite topology contains some $xR$, which itself is open in our starting topology, since $R$ is by assumption an open relation. This concludes the proof that $X$ is a profinite object in $\mathscr{P}$ and that its profinite topology is the one that makes $X$ profinite-compatible. □

**Proposition 3.1.13.** *Let $(\mathscr{C}, F)$ be a preprofinite category. Let $\mathscr{P}$ be the profinite structure associated to $\mathscr{C}$. Let $((X_i)_{i \in I}(\phi_{i,j})_{j \geq i})$ be a projective system in $\mathscr{P}$, such that for every $i \in I$, $X_i$ seen as a limit of finite sets has all of its natural projections surjective. Then $((X_i)_{i \in I}, \phi_{i,j})$ has a limit in $\mathscr{P}$.*

*Proof.* Every $X_i$ is by definition of profinite structure a limit of a projective system $((X_{i,j})_{j \in J_i}, (\psi^i_{j,k})_{j \leq k})$, with $J_i$ some directed set. We denote then $\psi^i_j$ the natural projection of $X_i$ onto $X_{i,j}$. By assumption, we have $F(\psi^i_j)$ surjective for all $i$ and $j$. Now let us define $\Omega = \coprod_{i \in I} I \times J_i$, together with the following relation: $(i,j) \leq (i',j')$ if and only if $i < i'$ and there exists a morphism $f$, such that:

$$\begin{array}{ccc} X_{i,j} & \xleftarrow{f} & X_{i',j'} \\ {\scriptstyle \psi^i_j}\uparrow & & \uparrow{\scriptstyle \psi^{i'}_{j'}} \\ X_i & \xleftarrow[\phi_{i,i'}]{} & X_{i'} \end{array}$$

commutes, or $i = i'$ and $j \leq j'$

We can observe that if such $f$ exists, it is unique. Indeed if $g$ is another such a morphism, let us take $a \in F(X_{i',j'})$. Then there exists $x \in F(X_{i'})$, such that $a = F(\psi^{i'}_{j'})(x)$. In that case:

$$F(f)(a) = F(f \circ \psi^{i'}_{j'})(x) = F(\psi^i_{i,j})(F(\phi_{i,i'})(x)) = F(g)(F(\psi^{i'}_{j'})(x)) = F(g)(a)$$

Since it is true for every $a \in F(X_{i,j})$, we get that $F(f) = F(g)$ and by injectivity of $F$, $f = g$.

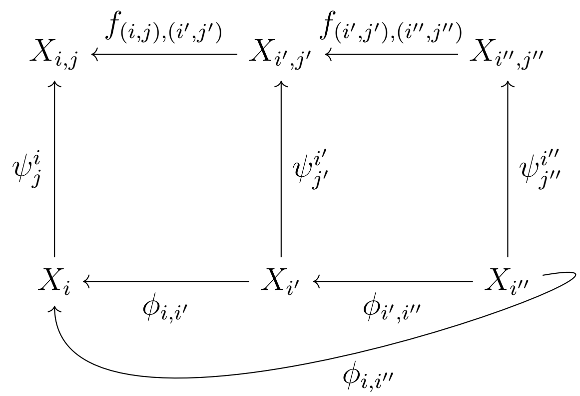
Since such an $f$ is unique, we will denote it $f_{(i,j),(i',j')}$. Now let us prove that $\Omega$ together with the relation $\leq$ is a directed set. We start by proving that $\leq$ is an order relation. Observe that it is reflexive.

Let us show that it is anti symmetric. Suppose that $(i,j) \leq (i',j')$ and $(i,j) \geq (i',j')$. We then have that $i \leq i'$ and $i' \leq i$ and so $i = i'$. Since $\leq$ is an order on $I$. We then are in the case where $j \leq j'$ and $j' \leq j$, so we have $j = j'$

For transitivity, take $(i,j) \leq (i',j')$ and $(i',j') \leq (i'',j'')$. We will now differentiate several cases:

- Case 1: $i < i' < i''$

  In that case we get the following commutative diagram:

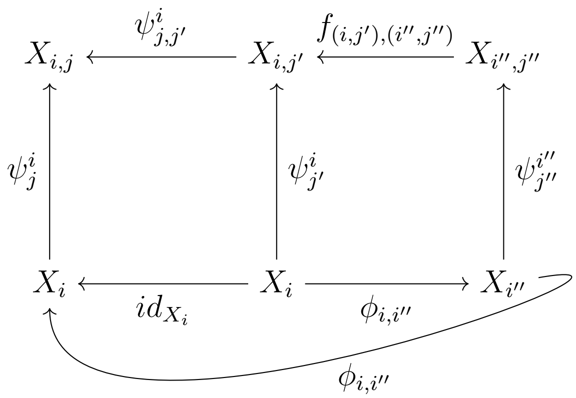



  Which means that if we complete this commutative graph with $f = f_{(i,j),(i',j')} \circ f_{(i',j'),(i'',j'')}$, then $f$ is the morphism required and so we have indeed $(i,j) \leq (i'',j'')$ We also proved at the same time the formula:

$$f_{(i,j),(i'',j'')} = f_{(i,j),(i',j')} \circ f_{(i',j'),(i'',j'')}$$

- Case 2: $i = i' < i''$

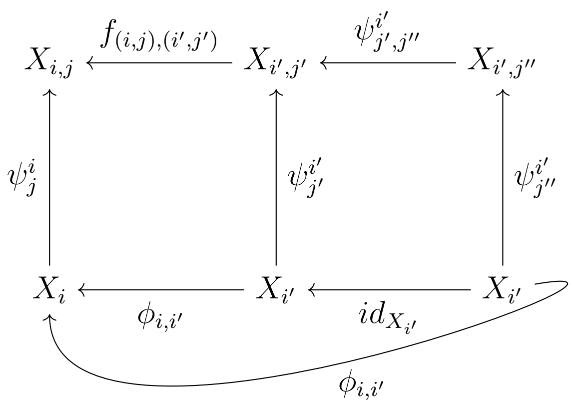
  We replace the previous diagram with:




Which proves both that $(i,j) \leq (i'',j'')$ and that $f_{(i,j),(i'',j'')} = \psi^i_{j,j'} \circ f_{(i',j'),(i'',j'')}$.

- Case 3: $i < i' = i''$

  Take this time the diagram:




It proves again that $(i,j) \leq (i'',j'')$ and also that $f_{(i,j),(i'',j'')} = f_{(i,j),(i',j')} \circ \psi^{i'}_{j',j''}$.

This concludes the proof that $\leq$ is an order relation. In light of what we have seen, we also denote for $j \leq j' \in J_i$, $f_{(i,j),(i,j')} = \psi^i_{j,j'}$ and we get $f_{(i,j),(i'',j'')} = f_{(i,j),(i',j')} \circ f_{(i',j'),(i'',j'')}$ for $(i,j) \leq (i',j') \leq (i'',j'')$.

Now we shall prove that $\Omega$ together with $\leq$ is a directed set.Let $(i,j),(i',j') \in \Omega$. Since $I$ is a directed set, we choose a $i''$ that is an upper bound of the set $\{i,i'\}$. The map $\phi_{i,i''}$ is a morphism of profinite spaces, therefore there exists $j_0 \in J(i'')$, such that:

$$\forall k \geq j_0, \forall x,y \in F(X_{i''}), F(\psi^{i''}_k)(x) = F(\psi^{i''}_k)(y) \Rightarrow F(\phi_{i,i''})(x) = F(\phi_{i,i''})(y) \quad (3.1)$$

For the same reason, there exists $j_1 \in J(i'')$, such that:

$$\forall k \geq j_1, \forall x,y \in F(X_{i''}), F(\psi^{i''}_k)(x) = F(\psi^{i''}_k)(y) \Rightarrow F(\phi_{i',i''})(x) = F(\phi_{i',i''})(y) \quad (3.2)$$

Since $J(i'')$ is directed,we take $j'' \in J(i)$, such that $j'' \geq j_0$ and $j'' \geq j_1$. In case that $i = i''$, we also require $j'' \geq j$. In case that $i' = i''$, we require that $j'' \geq j'$ as well. Since $(i, j)$ and $(i', j'')$ play symmetric roles, it is enough to show that $(i'', j'') \geq (i, j)$ to conclude that $\Omega$ is directed. If $i = i''$ the result is trivial since by construction $j'' \geq j$. Otherwise we define $g$ a function from $F(X_{i'',j''})$ to $F(X_{i,j})$ by the formula:

$$g(F(\psi^{i''}_{j''})(x)) = F(\psi^i_j)(F(\phi_{i,i''})(x))$$

Due to the continuity equation 3.1 and the surjectivity of $\Psi^{i''}_{j''}$, the function $f$ is well defined i.e independent of choice of $x$.

Furthermore $\psi^{i''}_{j''}$ is surjective and we have the following commutative diagram:

$$\begin{array}{ccc} F(X_{i'',j''}) & \overset{g}{\rightarrow} & F(X_{i,j}) \\ F(\psi^{i''}_{j''}) \uparrow & \nearrow F(\psi^i_j \circ \phi_{i,i''}) & \\ F(X_{i''}) & & \end{array}$$

therefore by the lifting property of the preprofinite category $\mathscr{C}$, we can conclude that there exists $f \in Hom(X_{i'',j''}, X_{i,j})$, such that $F(f) = g$. Using the injectivity

$$\begin{array}{ccc} X_{i,j} & \overset{f}{\longleftarrow} & X_{i'',j''} \\ \psi^i_j \uparrow & & \uparrow \psi^{i'}_{j'} \\ X_i & \underset{\phi_{i,i'}}{\longleftarrow} & X_{i'} \end{array}$$

of $F$, we get that the diagram:

commutes and therefore $(i, j) \leq (i'', j'')$ This leads us to define the following projective system $P = ((X_{i,j})_{(i,j)\in\Omega}, (f_{(i,j),(i',j')})_{(i,j)\leq(i',j')})$. Let us prove that the limit of $P$, which we shall call $X$, is a limit of our initial projective system $((X_i)_{i\in I}, (\phi_{i,j})_{i\leq j\in I})$.

Let us simply call $\pi_{(i,j)}$ the natural projection of $X$ onto $X_{i,j}$. For a fixed $i$, we have for all $j \leq j' \in J(i)$ the diagram:

$$\begin{array}{ccc} X & \overset{\pi_{i,j}}{\longrightarrow} & X_{i,j} \\ & \searrow^{\pi_{i,j'}} & \uparrow f_{(i,j),(i,j')} = \psi^i_{j,j'} \\ & & X_{i,j'} \end{array}$$

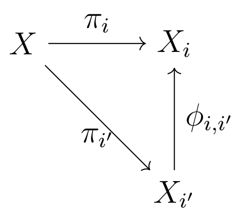
that commutes and therefore by the universal property of projective limit $X_i$, there exists a unique $\pi_i$ from $X$ to $X_i$, such that $\pi_{(i,j)} = \psi^i_j \circ \pi_i$.

Now let us prove that $\phi_{i,i'} \circ \pi_i = \pi_{i'}$ for all $i \leq i'$. By contradiction, assume that there exists $x \in F(X)$, such that $F(\phi_{i,i'})(F(\pi_{i'})(x)) \neq F(\pi_i)(x)$. In that case, there exists $j \in J(i)$, such that $F(\psi^i_j)(F(\phi_{i,i'}))(F(\pi_{i'})(x)) \neq F(\psi^i_j)(F(\pi_i)(x))$. By continuity of $F(\phi_{i,i'})$, there exists then a $j' \in J(i')$, such that if $F(\psi^{i'}_{j'}(y)) = F(\psi^{i'}_{j'}(y'))$, then $F(\psi^i_j(F(\phi_{i,i'})(y)) = F(\psi^i_j(F(\phi_{i,i'}))(y'))$. Using this implication, we can prove that $(i,j) \leq (i',j')$ We then have that

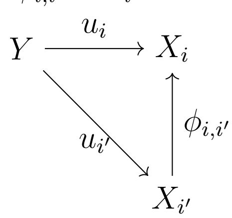
$$F(\psi_{j'}^{i'} \circ \phi_{i,i'})(F(\pi_i)(x)) = F(f_{(i,j),(i',j')} \circ \psi_j^i \circ \pi_i)(x) =$$
$$F(f_{(i,j),(i',j')} \circ \pi_{i,j})(x) = F(\pi_{i',j'})(x) = F(\psi_{j'}^{i'} \circ \pi_{i'})(x)$$

which is the contradiction.

We therefore have the following diagram that commutes:



We have now the first part of the definition of a projective limit. Now we need to prove that $X$ together with the maps $(\pi_i)_{i\in I}$ has the universal property. Let $Y$ be an object in $\mathscr{P}$, together with morphisms $u_i$ from $Y$ to $X_i$, such that for all $i > i'$, $u_{i'} \circ \phi_{i,i'} = u_i$ as illustrated in the commutative diagram below:



Let us prove that there exists a unique morphism $h$ from $Y$ to $X$, such that $\forall i \in I, u_i = \pi_i \circ h$. Consider for all $i \in I$ and $j \in J(i)$, maps $v_{i,j} = \psi_j^i \circ u_i$. Let $(i,j) \leq (i',j') \in \Omega$. Let us show that $v_{i,j} = f_{(i,j),(i',j')} \circ v_{i',j'}$. We have that

$$f_{(i,j),(i',j')} \circ v_{i',j'} = f_{(i,j),(i',j')} \circ \psi_j^i \circ u_i = \psi_{j'}^{i'} \circ \phi_{i,i'} \circ u_i = \psi_{j'}^{i'} \circ u_{i'} = v_{i',j'}$$

Then by the universal property of $X$, seen as a projective limit of the $(X_{i,j})$, we get that there exists a morphism $h$ from $Y$ to $X$, such that $\forall (i,j) \in \Omega, v_{i,j} = \pi_{i,j} \circ h$. The map $h$ is currently seen as morphism between objects in $\mathscr{C}$, we need to also show that it is continuous for the profinite topologies for it to be a morphism of profinite objects. This is equivalent to showing that for all $(i,j) \in \Omega$, $F(\pi_{(i,j)} \circ h)$ is continuous, but that is the case, since $F(\pi_{(i,j)} \circ h) = F(\psi_j^i) \circ F(u_i)$, which is a composition of two continuous maps.

Now that we know that $h$ is a morphism between profinite objects, we need to show that $\forall i \in I, \pi_i \circ h = u_i$. By contradiction, assume that there exists, $x \in F(Y)$, such that $F(\pi_i \circ h)(x) \neq F(u_i)(x)$. Then there exists $j \in J(i)$, such that $F(\Psi_j^i)(F(\pi_i \circ h)(x)) \neq F(\Psi_j^i)(F(u_i)(x))$. In that case, we get that $F(\pi_{i,j}) \circ F(h)(x) \neq F(v_{i,j})(x)$, which is a contradiction.

Finally we need to prove the uniqueness of $h$. Suppose that there is another morphism of profinite objects $h'$ from $Y$ to $X$, such that for every $i \in I$, $\pi_i \circ h' = u_i$. Then for every $(i,j) \in \Omega$, $\psi_j^i \circ \pi_i \circ h' = v_{i,j}$. By uniqueness of $h$ given by the fact that $X$ is a projective limit over $\Omega$, we get that $h = h'$. This concludes the proof that $X$ and object in $\mathscr{P}$ is indeed the projective limit of the $(X_i)_{i\in I}$. □

Now if we take $X$ a profinite space, one can define a topological group $Aut(X)$ that is going to be of interest in the next chapter.

### 3.1.1 Automorphisms of a profinite space

Let $X$ be a projective limit of $((X_i)_{i\in I}, (\phi_{j,i})_{j\geq i})$ and let $(\pi_i)_{i\in I}$ be the natural projections of $X$ to $X_i$. Now let $Aut(X)$ be the group of continuous bijective morphisms from $X$ to $X$. It is a topological group for the following topology: For $g \in Aut(X)$, we say that $V$ is a neighborhood of $g$, if and only if there exists $i \in I$, such that $\forall g' \in Aut(X)$, $\pi_i \circ g = \pi_i \circ g' \Rightarrow g' \in V$. One can prove that this topology is the compact open topology on $Aut(X)$ and is therefore in particular independent of the choice of limits projections and transition maps.

## 3.2 Profinite groups

As per definition of a profinite structure, given in 3.1.4, to show that there is a profinite structure on groups, we take $\mathcal{G}r$ the category of groups, together with the forgetful functor $F$.

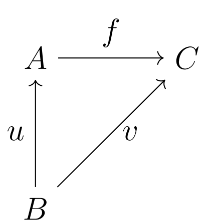
Let us show that $(\mathcal{G}r, F)$ is a preprofinite category.

- The forgetful functor is injective in morphisms.
- If $((G_i)_{i\in I}, (\phi_{i,j})_{j\geq i})$ is a projective system, then the set:

$$\{(g_i)_{i\in I} | \forall i \in I, g_i \in G \text{ and } \forall j \geq i \in I, \phi_{i,j}(g_j) = g_i\}$$

  together with a multiplication component by component is a limit in the category of groups. It is also a limit in the category of sets as well, hence the second property of preprofinite categories is true as well.

- Let $u, v$ be morphisms of groups and $f$ a map, such that $u$ is surjective and the diagram:




  commutes. Let us show that $f$ then is a morphism of groups. Let $x, y \in A$. Then by surjectivity of $u$, there exist $x', y'$, such that $x = u(x')$ and $y = u(y')$. Then

$$f(xy) = f(u(x'y')) = v(x'y') = v(x')v(y') = f(u(x'))f(u(y')) = f(x)f(y)$$

We have proved now that the category of groups is a preprofinite category and therefore we can associate to it a profinite structure of groups. As mentioned earlier we can "forget" the forgetful functor. We will just consider profinite groups as a structure on its own and we won't mention the forgetful functor,unless ambiguity between a profinite group seen as a set or as a group could arise.

**Proposition 3.2.1** (continuity of multiplication and inversion)**.** *Let $G$ be a profinite group together with its profinite topology. Then the multiplication and inversion are continuous operations. The group $G$ is therefore a topological group.*

*Proof.* Let us denote $(\pi_i)_{i\in I}$ the projections characterizing $G$. To show that the multiplication as a function $f$ from $G \times G$ to $G$ is continuous, we just need to prove that $\pi_i \circ f$ is continuous for all $i \in I$. Let $g, g' \in G$. Then $V = g\ker(\pi_i) \times g'\ker(\pi_i)$ is an open neighborhood of $(g, g')$, such that $\pi \circ f(V) = \{\pi_i(f((g, g')))\}$, proving the continuity of $f$.

Now consider $f$ as the function from $G$ to $G$ that maps $g$ to $g^{-1}$. Then

$$f^{-1}(g\ker(\pi_i)) = g^{-1}\ker(\pi_i)$$

proving the continuity of inversion as well. □

**Proposition 3.2.2.** *Let $G$ be a compact Hausdorff group. Then:*

*i. Every open subgroup is of finite index and closed.*

*ii. Every open subgroup contains an open normal subgroup.*

*iii. Every clopen neighborhood of 1 contains an open normal subgroup.*

*Proof.* i. This is a very straightforward result. If $H$ is an open subgroup, then $G$ is a union of $\overline{x} \in G/H$, which are all open since translates of $H$ and translation is a homeomorphism. By compactness of $G$, we can extract a finite subcover of $G$. However since all the classes are disjoint that subcover must be the initial cover itself, so our union $\bigcup_{\overline{x}\in G/H} \overline{x}$ had to be finite to begin with. The complement of $H$ is the union $\bigcup_{\overline{x}\in G/H\setminus\{\overline{1}\}} \overline{x}$, so it is open. As such, $H$ is closed.

ii. Let $H$ be an open subgroup of $G$. Consider $N = \bigcap_{g\in G} gHg^{-1}$. We start by proving that this intersection is finite. Let $g_1, \cdots, g_n$ be representatives of the cosets of $H$ on the left (by i. there are finitely many of them). Let us show that $N = \bigcap_{i=1}^{n} g_iHg_i^{-1}$. Let $x \in \bigcap_{i=1}^{n} g_iHg_i^{-1}$. To prove that $x \in N$, we take $g \in G$. Let us then show that $gxg^{-1} \in H$. We have that there exists $h' \in H$ and i such that $gg_i{}^{-1} \in H$. Furthermore by definition, $g_ixg_i^{-1} \in H$, therefore $gg_i^{-1}g_ixg_i^{-1}g_ig^{-1} \in H$. We conclude that $gxg^{-1} \in H$. The subgroup $N$ then is a finite intersection of open subsets, hence is itself open. It is also normal by construction.

iii. For this part, we shall use the proof in Ribes-Zaleskii Theorem 1.1.12 [42]. Let $U$ be a clopen neighborhood of 1. For a set $X$, we shall denote $X^n = \{x_1 \cdots x_n | x_1, \cdots, x_n \in X\}$.

Now write $F = (G \setminus U) \cap U^2$. By compactness of $U$ and continuity of the product, $U^2$ is compact and therefore $F$ is compact. For every $x \in U$, $x \in G \setminus F$ and so we can by continuity of multiplication choose an open neighborhood of $x$: $V_x$ and an open neighborhood of 1 $S_x$, such that $V_x S_x \subset G \setminus F$ and $V_x, S_x \subseteq U$. By compactness of $U$, we can extract a finite cover $V_{x_1}, \cdots V_{x_n}$ of $U$. Now write $S = \bigcap_{i=1}^{n} S_i$ and $W = S \cap S^{-1}$. Since $S_i$ are all open neighborhoods of 1, their intersection $S$ is an open neighborhood of 1. By continuity, $S^{-1}$ is also an open neighborhood of 1 and therefore $W$ is a symmetric open neighborhood of 1.

Let us observe that $UW \subseteq U^2$. Indeed: $UW \cap F \subseteq VS \cap F$. Now suppose that $u \in U$ and $s \in S$. We have that there exists $i$, such that $u \in V_{x_i}$. Since $s \in S_{x_i}$, we get that $us \in V_{x_i} S_{x_i} \subseteq G \setminus F$. As such, $UW \cap F = \emptyset$. Since we also have $UW \subseteq V^2$, we conclude from that that $UW \subset U$.

By induction we then get that $UW^n \subseteq U$ for all $n \in \mathbb{N}$. Since $W$ is symmetric, we have that the group generated by $W$ is the set $\bigcup_{n \in \mathbb{N}} W^n \subseteq V$. Given that $(W)$ contains an open subset:$W$, one can prove that $(W)$ is an open subgroup of $G$. Then by *ii.*, it contains a normal open subgroup $N$, which concludes the proof. □

**Theorem 3.2.3** (characterization of profinite groups). *Let $G$ be a compact Hausdorff topological group. The following statements are equivalent:*

*i. $G$ is profinite.*

*ii. $G$ is totally disconnected.*

*iii. There exists a set $\mathfrak{U}$ of open normal subgroups of $G$ that forms a neighborhood basis of 1.*

*Proof.* This theorem is a direct consequence of proposition 3.1.12, if we prove that all compact Hausdorff groups are profinite-compatible. Now if we assume $G$ to be a compact Hausdorff group and take $R$ an open relation on $G$, by *iii.* of the previous proposition, for each $x \in G$, there exists $N_x$ an open subgroup of $G$, such that $xN_x \subseteq xR$. We have that then the following open cover of $G$

$$G = \bigcup_{x \in G} xN_x$$

By compactness of $G$, we can then extract a finite subcover:

$$G = \bigcup_{k=1}^{n} x_k N_k$$

$N_1, \cdots, N_n$ are all open normal, hence $N = \bigcap_{k=1}^{n} N_k$ is open normal. Let $R_N$ be the equivalence relation on $G$ with $gR_Ng' \Leftrightarrow gg'^{-1} \in N$. Let us show that $R_N \subseteq R$. Suppose that $gR_Ng'$. Then there is a $k \in \{1, \cdots, n\}$, such that $g' \in x_kN_k$. In that case since $gR_Ng'$, we get that $g \in x_kN_k$. We get that $x_kN_k \subseteq x_kR$ and therefore $gRx_k$ and $g'Rx_k$, so $gRg'$ as expected.

Now if we take $f_N$ the natural product of $G$ onto ${}^{G}\!/_{N}$, we obtain that $f_N$ is a morphism in the category of groups and therefore we conclude that $G$ is profinite-compatible. The rest of the theorem follows from that as we explained. □

### 3.2.1 Galois theory of infinite extensions

Classical Galois theory studies the Galois group of finite Galois extensions. Let us denote $N/K$ a finite Galois extension of a field $K$ In that case, we get the fundamental theorem of Galois theory establishing an inclusion reversing bijection between subextensions of $N$ and subgroup of the group of automorphisms better known as the Galois group.

The bijection goes as follows:

$$\begin{cases} \{\text{Subextensions of } N\} \leftrightarrow \{\text{Subgroups of } Gal(N/K)\} \\ K' \mapsto Aut_{K'}(N, K') \\ \{x \in N | \forall g \in G, g(x) = x\} \leftarrow\!\shortmid G \end{cases}$$

When we however consider $N/K$ infinite, then the same map from the subgroups of $Gal(N/K)$ is no longer injective. However given that it still remains surjective, we could restrict it to only some subgroups of $Gal(N/K)$: the problem is we need some canonical way of choosing which subgroups we will restrict ourselves to. Fortunately the profinite topology on $Gal(N/K)$ provides us with one natural choice of subgroups upon which the map becomes injective: that natural choice is: we only consider closed subgroups for the profinite topology. In order to do this, we first need to prove that $Gal(N/K)$ is a profinite group.

**Theorem 3.2.4** (Galois groups are profinite)**.** *Let $K$ be a field and $N$ a Galois extension of $K$. Let*

$$\mathcal{N} = \{F \subseteq N | F \textit{ is a finite extension of } K\}$$

*denote the set of finite subextensions of $N$, together with the order relation. Let $G$ be the group $Gal(N/K) = Aut_K(N)$. Then*

$$\tau = \{U \subseteq G | \forall g \in U, \exists F \in \mathcal{N}, \forall g' \in G, \forall x \in F, g(x) = g'(x) \Rightarrow g' \in U\}$$

*as a set of opens makes $G$ into a topological group with a profinite topology.*

*Proof.* First step is to prove that $\tau$ defines a topology. Intuitively we can think of elements $g, g'$ as close if they are equal on many points, which is here the idea behind this topology.

We start by taking $(U_i)_{i\in I}$ a family of elements of $\tau$ and we shall prove that $\bigcup_{i\in I} U_i \in \tau$. That is relatively clear, since if $g$ is in some some $U_i$, we just take $F$ a finite subextension of $N$, such that

$$\forall g' \in G, \forall x \in F, g(x) = g'(x) \Rightarrow g' \in U_i \subseteq \bigcup_{i\in I} U_i$$

Now if we take $U, V \in \tau$, we need to prove that $U \cap V \in \tau$. Let $g \in U \cap V$. Let $F, F'$ be finite subextensions of $N$, such that if $g'$ is equal to $g$ on $F$, then $g'$ is in $U$ and if $g'$ is equal to $g$ in $F'$, then $g'$ is in V. If we then consider $F''$ the extension generated by $F'$ and $F$, it will still be a finite extension of $K$ of degree bounded by the product of the two degrees. Now suppose that $g'$ is equal to $g$ on $F''$. Then $g'$ is equal to $g$ on both $F$ and $F'$ and therefore $g'$ is both in $U$ and $V$ as expected. $G$ and $\emptyset$ are contained in $\tau$, which concludes the proof that $\tau$ defines a topology on $G$.

Now let us show that this topology makes $G$ into a topological group. Consider the map $u$ from $G \times G$ to $G$, with $u(g, g') = gg'$. Then let $U$ be a neighborhood of $gg'$. Then there exists $F$ a finite subextension of $N$, such that if $\forall x \in F, g''(x) \in F$, then $g''(x) \in U$. Now let $F'$ be the normal closure of $F$, $V = \{g'' \in G | \forall x \in F', g''(x) = g(x)\}$ and $V' = \{g'' \in G | \forall x \in F', g''(x) = g'(x)\}$, then we get that $V \times V'$ is a neighborhood of $(g, g')$. If we then take $(h, h') \in V \times V'$ and $x \in F$, then $h(x) = g(x)$. Since $g(x) \in F'$ ($F'$ is normal), then $h'(h(x)) = g'(g(x))$. This being true for all $x \in F$, we get that $hh' \in U$. Using the normal closure we prove similarly that the inverse function is continuous as well.

Now for a normal finite subextension $F$ of $N$, define $f_F(g) = g_{|F} \in Gal(F/K)$, which is a well defined morphism of groups since $g(F) = F$, due to $F$ being normal. For $F \subseteq F'$ two normal subextensions of $N$, define the transition map $f_{F,F'}$ as the morphism from $Gal(F', K)$ to $Gal(F, K)$ that to $g$ associates $g_{|F}$. For every $F \subseteq F'$, we get that the diagram:

$$\begin{array}{ccc} Gal(N/K) & \xrightarrow{f_{F'}} & Gal(F/K) \\ \Big\downarrow f_F & \swarrow f_{F',F} & \\ Gal(F'/K) & & \end{array}$$

Commutes and therefore if we denote $(G', (\pi_F)_{F\in\mathcal{N}})$ the limit of the projective system

$$((Gal(F/K))_{F\in\mathcal{N}}, (f_{F,F'})_{F\subseteq F'\in\mathcal{N}})$$

there exists a unique morphism from $G$ to $G'$, such that for every $F$ finite normal subextension of $K$, $\pi_F \circ f = f_F$. Let us show that $f$ is an isomorphism of topological groups, which will conclude the proof that $G$ is profinite. First $f$ is injective: indeed if for every normal finite subextension $F$ of $N$, $g_{|F} = id_F$, then if we take $x \in N$ and consider $F$ to be the normal closure of $K(x)$, we get that $g(x) = x$. To prove that $f$ is surjective, we take $(g_F)_{F\in\mathcal{N}} \in G'$. We then define

$$g = \begin{cases} N \longrightarrow N \\ x \mapsto g_F(x) \text{ if } x \in F \end{cases}$$

$g$ is well defined, due to the fact that $g_{|F}$ is compatible with restrictions. It fixes $K$, since all the $g_F$ fix $K$ and it is an automorphism since all the $g_F$ are. Now we need to prove that $f$ is continuous and open.

To prove that $f$ is continuous, we take $g \in G$ and $f(g)\ker(\pi_F)$ an open neighborhood of $f(g)$. Then simply take $U = \{g' \in G | \forall x \in F, g(x) = g'(x)\}$. The set $U$ is a neighborhood of $g$ and $f(U) \subseteq f(g)\ker(\pi_F)$.

Now let us finally show that $f$ is open. Consider $U$ a neighborhood of $g$. Then there exists $F$ a finite subextension of $N$, such that of $g'$ is equal to $g$ on $F$, then $g$ is in $U$. Let $F'$ be the normal closure of $F$: let us show that $f(U)$ contains $f(g)\ker(\pi_{F'})$ and thus is a neighborhood of $f(g)$. If we take $h \in f(g)\ker(\pi_{F'})$, then by surjectivity of $f$, there exists $g' \in G$, such that $f(g') = h$. Then we get that $g'_{|F'} = g_{|F}$ and therefore we get that $g' \in U$, thus $h \in f(U)$. Since as a topological group, $G$ is isomorphic to a profinite group, $G$ itself is profinite which concludes the proof of our theorem. □

Now that we have shown that Galois groups are profinite, we can now state the infinite version of the fundamental theorem of Galois theory.

**Theorem 3.2.5** (Fundamental theorem of infinite Galois theory)**.** *Let $N/K$ be a Galois extension. Let $\Omega$ be the set of subextensions of $N$ and $\Omega'$ the set of closed subgroups of $G = Gal(N/K)$. Then the function:*

$$\Phi = \begin{cases} \Omega \longrightarrow \Omega' \\ F \mapsto Aut_F(N) \end{cases}$$

*is an inclusion reversing bijection, with:*

$$\Psi = \begin{cases} \Omega' \longrightarrow \Omega \\ H \mapsto N_H = \{x \in N | \forall h \in H, h(x) = x\} \end{cases}$$

*as the inverse.*

*Furthermore $\Phi$ sends normal subextensions onto normal subgroups of $G$ and if $F/K$ is a normal subextension of $N$, then ${G}/{\Phi(F)}$ is isomorphic to $Gal(F/K)$.*

*Proof.* First we show that $\Phi$ and $\Psi$ are inverses. Let us start with $H$ a closed subgroup of $G$. First we observe that $H \subseteq \Phi(\Psi(H))$. Indeed: if $h \in H$ and $x \in N_H$, then by definition, $h(x) = x$ and therefore $h \in Aut_{N_H}(N)$. Now we need to show that $H$ is dense in $\Phi(\Psi(H))$. Let $g \in \Phi(\Psi(H))$ and let $U$ be a neighborhood of $g$. Then by the definition of the topology on $G$, there exists $F$ a finite normal subextension of $N$, such that every $g'$, if $g'_{|F} = g_{|F}$, then $g' \in U$.
Consider

$$H' = \{g' \in Gal(F/K) | \exists h \in H g' = h_{|F}\}$$

and

$$H'' = \{g' \in Gal(F/K) | \exists h \in \Phi(\Psi(H)), h_{|F} = g'\}$$

Now let us denote $F_{H'}$ the fixed field of $H'$ and $F_{H''}$ the fixed field of $H''$. We observe that $F_{H'} = F \cap N_H$: indeed: $x \in F_{H'} \Leftrightarrow (x \in F$ and $\forall h \in H, h_{|F}(x) = x)$. Furthermore, if $x \in F_{H'} = F \cap N_{F'}$, then $\forall g' \in \Phi(\Psi(H)), g'(x) = x$, since $\Phi(\Psi(H))$ fixes by definition $N_H$. We have that $F_{H'} \subseteq F_{H''}$ and using the fundamental theorem of finite Galois theory, we get that $H'' \subseteq H'$. As such, we get that $g_{|F} \in H'' \subseteq H'$ and therefore there exists $h \in H$, such that $h_{|F} = g_{|F}$ and therefore $h \in U$, proving that $H$ is dense in $\Phi(\Psi(H))$. Since $H$ is by assumption closed, we can conclude that $H = \Phi(\Psi(H))$.

Now let us prove that if $F$ is a subextension of $N$, $\Psi(\Phi(F)) = F$. First of all trivially, $F \subseteq \Psi(\Phi(F))$, since by definition, $\forall g \in \Phi(F) = Gal(N/F)$, $g(x) = x$ and therefore $x \in N_{\Phi(F)} = \Psi(\Phi(F))$. Now to prove the other inclusion, let $x \in N$ be fixed by all the elements of $\Phi(F)$ and $N'$ a finite normal extension of $F$ containing $x$ and contained in $N$. The restriction of $\Phi(F) = Gal(N/F)$ onto $Gal(N'/F)$ is surjective, since every automorphism of $N'$ fixing $F$ can be extended. Therefore $x$ is fixed by every element of $Gal(N'/F)$, which by the fundamental theorem of finite Galois theory implies that $x \in F$.

One small detail left to prove is that $\Phi(F) = Gal(N/F)$ is a closed subgroup of $G$. To prove that, we take $g$ that is not in $Gal(N/F)$. Since it does not fix $F$, there exists $x \in F$, such that $g(x) \neq x$. Now consider $U = \{g' \in G | \forall y \in K(x), g(y) = g'(y)\}$. It is an open neighborhood of $g$ and its intersection with $Gal(N/F)$ is empty, because none of the elements of $U$ can fix $x \in F$. Now we indeed have that $\Phi$ is a bijection from subextensions of $N$ into the set of closed subgroups of $G$.

The next step is to prove that $\Phi$ is inclusion reversing. That is however trivial since if $F \subseteq F'$, then any automorphism fixing $F'$ will fix $F$. In the same way, $\Psi$ is inclusion reversing as well.

Finally we need to show that the Galois correspondence maps normal subextensions onto normal closed subgroups and vice versa. Suppose that $F$ is a normal subextension of $N$. Let $g \in G$, $h \in \Phi(F) = Gal(N/F)$ and $x \in F$. Since $F$ is normal $g^{-1}(x) \in F$. In that case $h(g^{-1}(x)) = g^{-1}(x)$ and therefore $g(h(g^{-1})(x)) = g(g^{-1}(x)) = x$, proving that $\Phi(F)$ is normal. Now let $H$ be a normal subgroup of $G$ and $x \in \Psi(H) = N_H$. Let $x' \in N$ be another root of the minimal polynomial of $x$. In that case, there exists an automorphism $g \in G$, such that $x' = g(x)$. Now let us show that $x' \in N_H$. For that, we take $h \in H$. We have $h(x') = h(g(x)) = g(h(x))$ (since $H$ is normal) and therefore $h(x') = g(x) = x'$. Now to conclude the theorem, we need to prove that ${}^{G}\!/_{H}$ is isomorphic to $Gal(N_H/k)$. The restriction map of $G$ onto $N_H$ is surjective. It's kernel is $H$: indeed $g_{|N_H} = id_{N_H} \Leftrightarrow \forall x \in N_H, g(x) = h \Leftrightarrow h \in \Phi(N_H) = H$. □

## 3.3 Profinite rings and modules

Just like in the case of groups, projective limits exist in the category of rings, and therefore one can define a category of profinite rings. Just like for groups, one can prove that the operations of addition and multiplication are continuous, making profinite rings into topological rings.

**Example 3.3.1** (profinite completion of $\mathbb{Z}$)**.** Consider $\mathbb{N} \setminus \{0\}$ equipped with the relation of divisibility $|$. We say that $n \leq n'$ if $n|n'$. For $n|n'$, consider the transition map $\phi_{n,n'}$ that is the natural projection of $\mathbb{Z}/_{n'\mathbb{Z}}$ onto $\mathbb{Z}/_{n\mathbb{Z}}$ and then we call $\hat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$ the limit of the projective system in the category of rings.

### 3.3.1 Profinite modules

There are two kinds of modules we can make a profinite ring act on topologically. Either we just pick modules with discrete topology, or we pick profinite Abelian groups with a continuous action of a profinite ring, in which case we get the profinite modules. We will focus on the latter here and give the following definition:

**Definition 3.3.2.** *Let $R$ be a profinite ring. We call a profinite $R$-module a profinite Abelian group $M$ together with a continuous map $\rho$ from $R \times M$ (equipped with the product topology) into $M$, such that:*

- $\forall r, r' \in R, \forall m \in M, \rho(r + r', m) = \rho(r, m) + \rho(r', m)$
- $\forall r \in R, \forall m, m' \in M,\ \rho(r, m + m') = \rho(r, m) + \rho(r, m')$
- $\forall m \in M, \rho(1, m) = m$

A morphism of profinite modules would then be a continuous morphism of $R$-modules, which gives us a category. A natural question to ask is whether this category is a profinite structure. The answer is yes and more precisely we will see that every profinite $R$-module is a projective limit of finite $R$-modules equipped with discrete topology.

### 3.3.2 The category of $R$-modules is a profinite structure

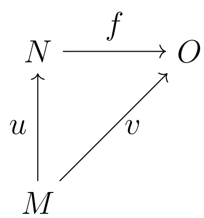
Let us denote $\mathscr{C}$ the category of profinite $R$-modules. We will show that this category is preprofinite and that every object in this category is a limit of finite objects, making it into a profinite structure.

We take $F$ the forgetful functor from $\mathscr{C}$ to the category of sets. Observe that the functor $F$ is faithful. Now we will show that the limits in the category $\mathscr{C}$ exist and that $F$ preserves them. Let $((M_i)_{i\in I}, (\phi_{i,j})_{j\geq i})$ be a projective system of profinite $R$-modules. Let $M$ be their limit in the category of Abelian groups. We will show that $M$ is also a profinite module. Define

$$\rho = \begin{cases} R \times M \longrightarrow M \\ (r, (x_i)_{i\in I}) \mapsto (r \cdot x_i)_{i\in I} \end{cases}$$

Let us show that $\rho$ is a continuous map. Since $M$ is equipped with the product topology of $\prod_{i\in I} M_i$, to prove continuity of $\rho$, it is enough to prove the continuity of $\phi_i \circ \rho$ for every $i \in I$ with $\phi_i$ being the natural projection on $M_i$. Now let $r \in R$ and $m \in M$ and $U$ an open neighborhood of $r \cdot m_i$ in $M_i$. Since the action of $R$ is continuous, there exists $V$ an open neighborhood of $r$ and $V'$ an open neighborhood of $m_i$, such that if $m' \in V'$ and $r' \in V$, then $rm \in U$. By continuity of $\phi_i$, $\phi_i^{-1}(V')$ is an open set in $M$ and therefore $V \times \phi_i^{-1}(V')$ is a neighborhood of $(r, v)$ and $\phi_i \circ \rho(V \times \rho^{-1}(V')) \subseteq U$, proving the continuity of $\rho$. Since limits in the category of profinite $R$ modules are limits in the category of Abelian groups, then they are also limits in the category of sets, hence the second axiom of preprofinite category is verified by $F$.

Now we need to show the last axiom of preprofinite categories. Let us pick $M, N, O$ three profinite modules, $u$ a surjective morphism from $M$ to $N$ , $v$ a morphism from $M$ to $O$ and $f$ a map from $N$ to $O$, such that the diagram:




commutes. Let us show that $f$ is a continuous linear map from $N$ to $O$. For that, we take $F$ a closed subset of $O$. The set $v^{-1}(F)$ is then a closed subset of $M$ by continuity of $v$. The set $u(v^{-1}(F))$ then is a closed subset of $N$, by continuity of $u$ and compactness of $M$. Now let us show that $u(v^{-1}(F)) = f^{-1}(F)$. Let $x \in f^{-1}(F)$. In that case by surjectivity of $u$, there exists $y \in M$, such that $x = u(y)$. We get that $v(y) = f(u(y)) = f(x) \in F$, therefore $y \in v^{-1}(F)$ and $x \in u(v^{-1}(F))$. Now in the other hand suppose that $x \in u(v^{-1}(F))$. In that case, there exists $y \in v^{-1}(F)$, such that $x = u(y)$. In that case: $f(x) = f(u(y)) = v(y) \in F$ and therefore $x \in f^{-1}(F)$. Since the inverse image by $f$ of every closed subset of $O$ is closed, then $f$ is continuous.

Now let us prove that $f$ is a morphism of modules. Let $r \in R$, $x, y \in N$. Then by surjectivity of $u$, there exists $x', y' \in M$, such that $u(x') = x$ and $u(y') = y$. We then get that

$$f(rx + y) = f(u(rx' + y')) = v(rx' + y') = rv(x') + v(y') = rf(x) + f(y)$$

$\mathscr{C}$ is therefore preprofinite category. To prove that it is actually profinite we need to prove that every profinite module $M$ is a profinite limit of finite modules with continuous action of $R$. For this part, we shall use the Lemma 5.1.1 b) in Ribes's and Zalesskii's book [42].

Consider $M$ a profinite module. Since it is equipped with a structure of a profinite Abelian group, it is of course compact Hausdorff and totally disconnected topological space. If we then manage to show that it is profinite-compatible, then it is a profinite limit by 3.1.12. Let $\sim$ be an open relation on $M$. Since $M$ is profinite-compatible as an Abelian group, there exists $U$ an open subgroup of $M$, such that

$$\forall x, y \in M, x - y \in U \Rightarrow x \sim y$$

Now we will show that $U$ contains an open submodule. By continuity of the action of $R$, for every $r \in R$, there exists $W_r$ a neighborhood of $R$ and $U_r$ an open subgroup of $U$, such that $W_r U_r \subseteq U$. By compactness of $R$, there exist then $r_1, \cdots, r_n$, such that $W_{r_1}, \cdots, W_{r_n}$ cover $R$. In that case we put:

$$V = \bigcap_{i=1}^{n} V_{r_i}$$

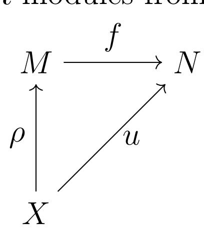
In that case, we get: $RV \subseteq U$. Now let $N$ be the module generated by $V$. Since $U$ is a subgroup, containing $RV$, we get that $N \subseteq U$. The set $N$ is a subgroup of $M$ containing an open subset $V$, therefore one can prove that $N$ is an open subgroup of $M$. The set $N$ is also a submodule of $M$, therefore let us take $f$ to be the natural projection of $M$ onto the $R$-module ${}^{M}\!/\!_{N}$. Let us show that ${}^{M}\!/\!_{N}$ is a finite module that the action of $R$ on ${}^{M}\!/\!_{N}$ with the discrete topology is continuous and that $f$ is a continuous map.

- ${}^{M}\!/\!_{N}$ is finite, since $N$ is an open subgroup of $M$ and therefore of finite index.
- Let us prove that the action of $R$ is continuous. Let $\overline{m} \in {}^{M}\!/\!_{N}$ and $r \in R$. Since $rm + N$ is open, there exists by continuity of the action of $R$ on $M$, $W$ a neighborhood of $r$ , such that $\forall r' \in W$, $r'm \in rm + N$. When we have that $W\{\overline{m}\} \subseteq \{rm\}$, proving the continuity of the action of $R$ on ${}^{M}\!/\!_{N}$.
- Observe that the map $f$ is continuous, since $f^{-1}(\{\overline{x}\}) = x + N$, which is open since $N$ is open.

From here on now, we can represent any profinite module as a projective limit of finite $R$- modules with a continuous action.

### 3.3.3 Free modules

If we take $X$ a profinite set and $R$ a profinite ring, then there exists up to isomorphism a unique $R$ module $M$ with a continuous map $\rho$ from $X$ to $M$, such that for any continuous map $u$ from $X$ to some $R$ module $N$, there exists a unique morphism of $R$ modules from $M$ to $N$, such that the diagram:




commutes. We denote such a module $R[[X]]$ and call it a free module on $X$.

*Proof.* The uniqueness just comes from the universal property of $R[[X]]$. To prove the existence, we will construct $R[[X]]$ as follows: Let $(\pi_i)_{i\in I}$ be the natural projections of $X$ onto a finite set $X_i$, with $(X,(\pi_i)_{i\in I})$ seen as a projective limit of the system $((X_i)_{i\in I},(\pi_{i,j})_{j\geq i})$. We assume furthermore that all the projections are surjective. (By restricting them to their image if necessary). We define $R[X_i]$ to be the free $R$-module on $X_i$ equipped with the product topology (for $x\in X_i$, we take the topology on $Rx$ to be the profinite topology on $R$). The set $R[X_i]$ is then an $R$ module, since multiplication by $R$ is continuous on every component of $R[X_i]$. For $j\geq i\in I$, we take the transition map from $R[X_j]$ to $R[X_i]$ to be the $R$-linear map $\phi_{i,j}$ induced by $\pi_{i,j}$. Let us show that $\phi_{i,j}$ is continuous. We write $X_j=\{u_1,\cdots,u_n\}$. Now let $x=\sum_{k=1}^{n} r_k u_k\in R[X_j]$ and let $U$ be an open neighborhood of $\phi_{i,j}(x)$. If we write

$$X_i=\{v_1,\cdots,v_m\}$$

and

$$\phi_{i,j}(x)=\sum_{k=1}^{m} r'_k v_k$$

then $U$ contains some $(r'_1+\ker(p))v_1+(r'_2+\ker(p))v_2+\cdots+(r'_m+\ker(p))v_m$ with $p$ a continuous ring morphism from $R$ to some finite ring $R'$ (by definition of the product topology). Now consider $V$ the neighborhood of $x$ defined by:

$$V=(r_1+\ker(p))u_1+\cdots+(r_n+\ker(p))u_n$$

let us show that $\phi_{i,j}(V)\subseteq U$.
We write

$$y=(r_1+q_1)u_1+\cdots+(r_n+q_n)u_n$$

Then

$$\phi_{i,j}(y)=\sum_{k=1}^{m}(r'_k+\sum_{k'=1}^{n} q_{k'}A_{k,k'})v_k$$

with $A_{k,k'}$ being the matrix of $\phi_{i,j}$ in the bases $u_1,\cdots,u_n$ and $v_1,\cdots,v_m$. Since for every $k'$, $q_{k'}$ is in the kernel of $p$, we get that $\sum_{k'=1}^{n} q_{k'}A_{k,k'}$ will be in the kernel of $p$ and therefore we conclude that $\phi_{i,j}(y)\in U$ as expected.

The morphisms $\phi_{i,j}$ are therefore continuous linear maps between profinite $R$-modules. $R[[X]]$: the limit of our projective system given by the transition maps $\phi_{i,j}$ is then a profinite module. Now we define the map $\rho$ that to $(x_i)_{i\in I}\in X$ associates $(x_i)_{i\in I}$ seen as elements of $R[X_i]$. We need to show that $\rho$ is continuous. To prove that, it is enough to prove that for every $i\in I$, $\phi_i\circ\rho$ is continuous, with $\phi_i$ being the natural projection of the projective system of $R[[X]]$ onto $R[X_i]$. Now let $x\in X$ and $U$ an open neighborhood of $\phi_i\circ\rho(x)$. If we consider

$$V=\{y\in X|\pi_i(y)=\pi_i(x)\}$$

a neighborhood of $X$, we get that $\phi_i(\rho(V)) = \{\phi_i(\rho(x))\} \subseteq U$, showing the continuity of $\phi_i \circ \rho$. Since the topology on $R[[X]]$ is given by the product topology of the $R[X_i]$, this is enough to prove that the map $\rho$ is continuous.

Now we need to prove that $(R[[X]], \rho)$ has the universal property. For that, we consider $N$ a profinite $R$-module and $u$ a continuous map from $X$ to $N$. We know by 3.3.2 that $N$ is a projective limit of finite $R$-modules. We take $J$ a directed set and $(N_j)_{j \in J}$ finite $R$ modules and $(\psi_{i,j})_{j \geq i \in J}$ transition morphisms, such that $N$ is the projective limit of the $N_j$. We then denote $\psi_j$ the natural projection of $N$ onto $N_j$. Consider the map $\psi_j \circ u$ from $X$ to $N_j$. It is then continuous, so there exists a $\delta(j) \in I$, such that

$$\forall i \geq \delta(j), \forall x, y \in X, \pi_i(x) = \pi_i(y) \Rightarrow \psi_j \circ u(x) = \psi_j \circ u(y)$$

Then we define a map $u_j$ from $X_{\delta(j)}$ to $N_j$, by the formula: $u_j(\pi_{\delta(j)}(x)) = \psi_j(u(x))$. Furthermore for $i \in I$, write $\rho_i$ the natural injection from $X_i$ to $R[X_i]$. For every $j \in J$, $R[X_{\delta(j)}]$ is a free module on $X_i$ in sense of abstract modules, therefore there exists a unique $R$-linear map (not necessarily continuous) $f_j$, such that the following diagram commutes:

$$\begin{array}{ccc} R[X_{\delta(j)}] & \xrightarrow{f_j} & N_j \\ \uparrow{\scriptstyle \rho_{\delta(j)}} & \nearrow{\scriptstyle u_j} & \\ X_{\delta(j)} & & \end{array}$$

Now for $x \in R[[X]]$, let $f(x)$ be the collection in $\prod_{j \in J} N_j$, defined by

$$f(x) = (\ f_{\delta(j)}(\phi_{\delta(j)}(x))\ )_{j \in J}$$

To show that this collection is in the limit $N$, we need to show that it is compatible with the transition maps. Let $j' \geq j$. Now let $i \in I$ be greater than both $\delta(j')$ and $\delta(j)$. We have that $f(x)_j = f_j(\phi_{\delta(j),i}(x_i))$ and that $f(x)_{j'} = f_{j'}(\phi_{\delta(j'),i}(x_i))$. Therefore to show that
$\psi_{j,j'}(f(x)_{j'}) = f(x)_j$, it is enough to show that $f_j \circ \phi_{\delta(j),i}$ and $\psi_{j,j'} \circ f_{j'} \circ \phi_{\delta(j'),i}$ are equal. Since $X_i$ forms a basis of $R[X_i]$, it is enough to show by linearity that they are equal on that set. Now if we take $x' \in X_i$, then there exists $x \in X$, such that $x_i = x'$.

In that case:

$$\psi_{j,j'} \circ f_{j'} \circ \phi_{\delta(j'),i}(x_i) =$$
$$\psi_{j,j'}(f_j(x_{\delta(j)})) = \psi_{j,j'}((u(x))_{j'}) = (u(x))_j = f_j(x_{\delta(j)}) = f_j(\phi_{\delta(j),i}(x_i))$$

This proves that $f$ is indeed a well defined map from $R[[X]]$ to $N$.

The next step is to prove linearity. If we take: $x, y \in R[[X]], r \in R$ and $j \in J$, we obtain:

$$f(x + ry)_j = f_j(\phi_{\delta_j}(x + ry_{\delta(j)})) = f_j(\phi_{\delta_j}(x)) + r f_j(\phi_{\delta_j}(y))$$

by linearity of $f_j$ and $\phi_{\delta_j}$. Since this is true for all $j \in J$, $f$ is indeed linear. Now we need to prove the continuity of $f$. To do that, we prove the continuity of the function $\psi_j \circ f$ for all $j \in J$. To prove that $\psi_j \circ f$ is continuous, it is enough to prove that $\ker(\psi_j \circ f)$ is a neighborhood of 0, since $\psi_j \circ f$ is linear. Since $R$ acts continuously on $N_j$, there exists a continuous ring morphism $p$ from $R$ to some finite ring $R'$, such that $\ker(p) \cdot N_j = \{0\}$. Now let us show that $\phi_{\delta(j)}^{-1}(\ker(p) \cdot R[[X_{\delta(j)}]])$ (which is an open neighborhood of 0 in $R[[X]]$ by continuity of $\phi_{\delta(j)}$) is contained in $\ker(\psi_j \circ f)$. Let $x \in R[[X]]$, such that there exists $r \in \ker(p)$ and $y \in R[[X_{\delta(j)}]]$, such that $\phi_{\delta(j)}(x) = ry$. We have that

$$\psi_j(f(x)) = f(x)_j = f_j(x_{\delta(j)}) = f_j(ry) = rf_j(y) = 0$$

$\ker(\psi_j \circ f)$ contains then an open neighborhood of 0, therefore is itself a neighborhood of 0, proving the continuity of $f$.

Now we need to show that $f$ factors through $\rho$ into $v$, but the is straightforward to check. Finally we need to show uniqueness of $f$. Suppose that $g$ is another continuous linear map from $R[[X]]$ to $N$, such that $g \circ \rho = v$. Let us show that $f = g$. We will prove it, by showing that for every $j \in J$, $\psi_j \circ f = \psi_j \circ g$. By continuity of $\psi_j \circ g$, there exists $i_1$, such that for all $i \geq i_1$ and for all $x, y \in R[[X]]$, if $\phi_i(x) = \phi_i(y)$, then $\psi_j(f(x)) = \psi_j(f(y))$. Now take $i$ that is greater than $i_1$ and $\delta(j)$. Now let us take an $x \in X$. Let us write $\phi_i(x) = \sum_{k=1}^{n} r_k e_k$, with $X_i = \{e_1, \cdots, e_n\}$. We have $f(x)_j = \sum_{k=1}^{n} r_k u(e_k)$. Now for $k$ in $\{1, \cdots, n\}$, let $y^k$ be an element in $X$, such that $(y^k)_j = e_k$. We have that $\phi_i(\sum_{k=1}^{n} r_k y^k) = \phi_i(x)$ and therefore by continuity, $\phi_j(g(x)) = \phi_j(g(\sum_{k=1}^{n} r_k y^k))$. We then have that $\phi_j(g(x)) = \sum_{k=1}^{n} r_k u(e_k) = \phi_j \circ f(x)$, which shows the uniqueness of $f$.

That concludes the proof that $R[[X]]$ is a free $R$-module on $X$. □

We will now prove the properties of profinite modules that we will make use of in the next chapter.

**Lemma 3.3.3.** *Let $M$ be a profinite module over a profinite ring $R$. Let $A$ be subset of $M$, such that the module generated by $A$ is dense in $M$ (i.e $A$ generates $M$ topologically). Let $f$ be a continuous $R$ linear form on $M$, such that $f(A) = \{1\}$. Then:* $\ker(f) \cap (A)$ *is dense in* $\ker(f)$ *,with*

$$(A) = \{\sum_{k=1}^{n} r_k a_k | n \in \mathbb{N} \text{ and } \forall k \in \{1, \cdots, n\}, r_k \in R \text{ and } a_k \in A\}$$

*Proof.* Let $(\phi_i)_{i \in I}$ be the natural projections of $M$ onto $M_i$ with $M$ seen as the limit of the finite $R$-modules $(M_i)_{i \in I}$. Let $(\pi_j)_{j \in J}$ be the natural projections of $R$ onto $R_j$ with $R$ seen as the projective limit of the finite rings $(R_j)_{j \in J}$. We take $u \in \ker(f)$. To prove that $u \in \overline{(A) \cap \ker(f)}$, we take $i \in I$ and show that there exists $v \in A \cap \ker(f)$, such that $\phi_i(u) = \phi_i(v)$. By continuity of the action of $R$ on $M$, there exists $j_0 \in J$, such that

$$\forall x \in M, \forall j \geq j_0, \forall r \in R, \pi_j(r) = 0 \Rightarrow \phi_i(r \cdot x) = 0 \tag{3.3}$$

The map $\phi_{j_0} \circ f$ is continuous and $\phi_{j_0} \circ f(u) = 0$, therefore there exists $i_0 \in I$, such that:

$$\forall i' \geq i_0, \forall x \in M, \phi_{i'}(x) = \phi_{i'}(u) \Rightarrow \pi_{j_0}(f(x)) = 0 \tag{3.4}$$

Now take $i'$ that is greater than both $i$ and $i_0$. Since $(A)$ is dense in $M$, there exists $a_0, \cdots, a_n \in A$ and $r_0, \cdots, r_n \in R$, such that $\phi_{i'}(u) = \phi_{i'}(r_0 a_0 + \cdots + r_n a_n)$. Now just take $a \in A$ and write $v' = r_0 a_0 + \cdots r_n a_n$ and $v = v' - (r_0 \cdots + r_n)a$. We get that $v \in \ker(f) \cap (A)$. Now let us show that $\phi_i(u) = \phi_i(v)$. We have that $\phi_{i'}(u) = \phi_{i'}(v)$ and therefore by (3.4), $0 = \phi_{j_0}(f(u)) = \phi_{j_0}(f(v))$. From that we obtain that $\phi_{j_0}(r_0 + \cdots + r_n) = 0$ and therefore by (3.3), $\phi_i((r_0 + \cdots + r_n)a) = 0$. We then get that $\phi_i(v') = \phi_i(v)$. However since $i' \geq i$ we also get $\phi_i(u) = \phi_i(v')$ and therefore $\phi_i(u) = \phi_i(v)$, which concludes the proof of the lemma. □

**Proposition 3.3.4.** *Let $u$ be a continuous injective map from a profinite set $X$ to a profinite set $Y$. Let $R$ be a profinite ring and let $f$ be the induced $R$-linear continuous map from $R[[X]]$ to $R[[Y]]$, then $f$ is injective.*

*Proof.* We consider $R[[Y]]$ to be the closed subset of $\prod_{i \in I} R[Y_i]$ with $Y$ being the projective limit of $Y_i$. By continuity of $u$, $A = u(X)$ is a closed subset of $Y$ and is therefore a profinite set. Furthermore $u$ is a homeomorphism between $X$ and $A$. Consider now the set

$$M = \{(x_i)_{i \in I} \in R[[Y]] | \forall i \in I, x_i \in R[\pi_i(A)]\}$$

With $\pi_i$ the natural projection of $Y$ on $Y_i$. It is a closed submodule of $R[[Y]]$. Let us show that it is the free module on $A$. For $i, j \in I$ with $j \geq i$, we denote $\phi_{i,j}$ the transition map from $Y_j$ to $Y_i$. Notice that $\phi_{i,j}$ then is a transition map from $\pi_j(A)$ to $\pi_i(A)$. One can then prove that $A$ is a projective limit of the $\pi_i(A)$, which give us that $M$ is indeed the free module over $A$, since it is the projective limit of the $R[\pi_i(A)]$.

Now consider the continuous map $u^{-1}$ from $A$ to $R[[X]]$ that to $u(x)$ associates $x$. It can then be extended to a unique continuous linear map from $M$ to $R[[X]]$, which we shall call $g$. Note that $f(R[[X]]) \subseteq M$ and therefore $g \circ f$ is a well defined linear map on $R[[X]]$ that sends every element of $X$ to itself and it is therefore the identity on $R[[X]]$. The map $f$ is invertible on the left and is therefore injective. □

### 3.3.4 Pointed free modules

In some cases we may want to collapse one point of the set $X$ on which the module is free to 0. In the case we get the following slight modification of the theorem of free modules:

**Theorem 3.3.5.** *Let $R$ be a ring and $(X, a)$ a pointed profinite space ( a profinite set $X$ together with an element $a \in X$). Then up to isomorphism, there exists a unique $R$-module called $R[[X, a]]$ with a continuous injective map $\rho$ from $X$ to $R[[X, a]]$, such that $\rho(a) = 0$ and for any continuous map $u$ from $X$ to an $R$ module $N$, such that $u(a) = 0$, we have that there exists a unique linear map $l$ from $R[[X, a]]$ to $N$, such that the diagram:*

$$\begin{array}{ccc} R[[X,a]] & \xrightarrow{\ l\ } & N \\ \rho \uparrow & \nearrow_{u} & \\ X & & \end{array}$$

*commutes.*

The proof of this theorem is very similar to the non pointed version, so will be omitted.

Another result that stays the same for pointed version is the following:

**Proposition 3.3.6.** *Let $(X, a)$ and $(Y, b)$ be two pointed profinite spaces and $u$ a continuous injective map from $X$ to $Y$, such that $u(a) = b$. Let $l$ be the induced $R$ linear map from $R[[X, a]]$ to $R[[Y, b]]$, then $l$ is injective.*

## 3.4 Etale algebras

Etale algebras are a generalization of finite separable extensions. In this section, we will see that they can be represented by an action of a group on a finite set and therefore by a group action Cayley graph as in 1.5.1. Etale algebras are algebras that can be written as a product of finite separable extensions. In order to study them, we will first prove the theorem that classifies finitely generated algebras over a field. This theory is very old one and **none of the material in this section is new.** The proofs found here are based on [6] Chapter V §6 and [28] Chapter V §18.

**Theorem 3.4.1.** *Let $A$ be a finite dimensional commutative algebra over a field $\mathbb{K}$. Then $A$ has only finitely many maximal ideals: $M_1, \cdots, M_k$ and there exist integers $a_1, \cdots, a_k$, such that $A$ is isomorphic to the algebra: $\prod_{k=1}^{n} A/M_k^{a_k}$.*

*Proof.* We start by proving that $A$ has only finitely many maximal ideals. Consider a sequence $M_1, \cdots, M_n$ of distinct maximal ideals. Then by using the Chinese remainder theorem, we know that the algebra $A/M_1 \cdots M_n$ is isomorphic to $A/M_1 \times \cdots \times A/M_n$. Now the dimension of $A/M_1 \cdots M_n$ is bounded by $\dim(A)$. The dimension of $A/M_1 \times \cdots \times A/M_n$ is at least $n$, therefore $\dim(A) \geq n$ and $A$ has at most $\dim(A)$ maximal ideals. Now let $M_1, \cdots, M_n$ be all the distinct maximal ideals of $A$. We shall prove that second part of the statement. Consider the sequence of $\mathbb{K}$-vector spaces $((M_1 \cdots M_n)^i)_{i \in \mathbb{N}}$. It is decreasing sequence of subspaces of $A$, therefore it is stationary.

Now we shall prove that if for an $m \in \mathbb{N}$, $(M_1 \cdots M_n)^{m+1} = (M_1 \cdots M_n)^m$, then $(M_1 \cdots M_n)^m = \{0\}$. We have $M_1 \cap \cdots \cap M_n = M_1 \cdots M_n$ by the Chinese remainder theorem and $(M_1 \cdots M_n)^{m+1} = (M_{1 \cdots M_n})^m$, so by Nakayama's lemma, $(M_1 \cdots M_n)^m = \{0\}$. Finally by the Chinese remainder theorem, $A/(M_1 \cdots M_n)^m$ is isomorphic to $A/M_1^m \times \cdots \times A/M_n^m$, proving the rest of the proposition. □

Notice that if the $m$ from the proof is equal to 1, then $A$ is isomorphic to a product of fields. We call these kinds of algebras "diagonalizable".

**Definition 3.4.2** (Etale algebras)**.** *We call an algebra $\mathbb{L}$ over a field $\mathbb{K}$ etale if $L$ is isomorphic to a finite product of separable extensions of $\mathbb{K}$.*

If $\mathbb{K}$ is a perfect field, then we can make the following simple observation.

**Proposition 3.4.3.** *Let $\mathbb{K}$ be a perfect field and $L$ a finite dimensional commutative algebra over $\mathbb{K}$, then $L$ is etale if and only if it has no non trivial nilpotent elements.*

*Proof.* If $L$ is a product of separable extensions of $\mathbb{K}$, we have $L = K_1 \times \cdots \times K_n$ and if we take $(a_1, \cdots, a_n) \in L$ and $m \in \mathbb{N}$, such that $(a_1^m, \cdots, a_n^m) = 0$, then $a_1^m = \cdots = a_n^m = 0$ and so $a_1 = \cdots = a_n = 0$.

Now on the other hand, suppose that $L$ has no nontrivial nilpotent elements. By the classifying theorem 3.4.1, we know that $L$ is a product $L/M_1^{a_1} \times \cdots \times M_n^{a_n}$ with $M_1, \cdots, M_n$ maximal ideals. Now let us prove that all the $a_i$'s are equal to 1. Suppose that $a_i > 1$ for some $i$. We may then simply assume that $M_i \neq M_i^{a_i}$ and take $u \in M_i^{a_i}$ that is not in $M_i^{a_i}$. Now consider the $t = (0, \cdots, 0, \overline{u}^{(M_i^{a_i})}, 0, \cdots, 0) \in L$. Then $u^{a_i} \in M_i$ and thus $t^{a_i} = 0$. Since $L$ has no non-trivial nilpotent elements, $t = 0$, but that means that $u \in M_i^{a_i}$, which is a contradiction. We therefore get that $L$ is a product of finite extensions of $\mathbb{K}$. Given the fact that $\mathbb{K}$ is perfect, we get that these extensions are separable, hence $L$ is an etale algebra. □

Etale algebras are a generalization of separable field extensions and there is a notion of Galois theory on them. We will in fact see that they are entirely characterized by their morphisms into the separable closure of the base-field. In the rest of this section, we will examine the properties that etale algebras have in common with field extensions. We will start by showing that all elements of etale algebras have separable minimal polynomials.

**Proposition 3.4.4.** *Let $L$ be an etale algebra over $F$. Let $\theta \in L$ and $\chi$ the minimal polynomial of $\theta$ over $L$, then $\chi$ is separable.*

*Proof.* We know that $L$ is isomorphic to a product $F_1 \times \cdots \times F_n$, with $F_i$ finite separable extensions of $F$. We denote $\pi_i$ the natural projection of $L$ onto $F_i$ and $\chi_i$ the minimal polynomial of $\pi_i(\theta)$. The polynomials $\chi_i$ are all irreducible with simple roots in $\overline{F}$(the algebraic closure of $F$), since $F_i$ are all separable. If then we denote $P$ the least common multiple of all the $\chi_i$, then $P$ is with simple roots in $\overline{F}$. Now let us prove that $\chi$ divides $P$. We have for every $i$, $\pi_i(P(\theta)) = P(\pi_i(\theta)) = 0$, since $\chi_i$ divides $P$. As such, we have that $P(\theta) = 0$, since $\pi_i$ are projections and therefore

$\chi$ divides $P$ as the minimal polynomial of $\theta$. The polynomial $\chi$ divides a separable polynomial and is therefore itself a separable polynomial. $\square$

We will show a criterion that determines whether a given finite-dimensional algebra is etale. Before that we will first show that just like in the case of separable field extensions, morphisms from etale algebras into the separable closure can be extended. We will need to make a use of two lemmas.

**Lemma 3.4.5.** *Let $L$ be an etale $K$-algebra and $f$ a morphism from $L$ to $K_{sep}$. Then $Im(f)$ is a field and thus* $\ker(f)$ *is a maximal ideal.*

*Proof.* It is a ring as an image of a morphism of rings. Now we need to prove that each element is invertible. Let $f(a) \in K_{sep}^{\times}$. Then $f(a)$ is algebraic over $K$ and therefore $K(f(a)) = K[f(a)]$. In particular $\frac{1}{f(a)}$ is a polynomial in $f(a)$ and therefore in the image of $f$. Since every non zero element is invertible in $Im(f)$, $Im(f)$ is a field and thus $\ker(f)$ is maximal. $\square$

**Lemma 3.4.6.** *Let $K$ be a field, let $L$ be an etale algebra, $L'$ a subalgebra such that $L = L'[\theta]$. Let $f$ be a morphism from $L'$ to $K_{sep}$, then $f$ can be extended to a morphism from $L$ to $K_{sep}$.*

*Proof.* Suppose that $L = \prod_{i=1}^{m} F_i$, with $F_i/K$ being a finite separable extension of $K$. Take $\pi_i$ the natural projection on $F_i$. For every $i$, $\pi_i$ restricted to $L'$ is a morphism into $F_{sep}$, hence its kernel is a maximal ideal. We therefore have that for every $i$, $\ker(\pi_i) \cap L'$ is a maximal ideal in $L'$. Let us now prove that there exists $i_0$, such that $\ker(\pi_{i_0}) \subseteq \ker(f)$.

By contradiction assume that

$$\forall i, \exists x_i \in \ker(\pi_i), f(x_i) \neq 0$$

In that case we have that $f(x_1 \cdots x_n) = f(x_1) \cdots f(x_n) \neq 0$. However $x_1 \cdots x_n = 0$: indeed for all $i$, we get that $\pi_i(x_1 \cdots x_n) = 0$. This contradicts that $f(x_1 \cdots x_n)$ is not zero.

Now consider $I$ the ideal in the polynomial ring $L'[X]$ defined by:

$$I = \{P \in L'[X] | P(\theta) = 0\}$$

By abuse of notation, we denote $f$ the extension of $f$ into a morphism from $L'[X]$ to $f(L')[X]$. Then it is a surjective morphism and therefore $f(I)$ is an ideal in $f(L')[X]$. Let us prove that $f(I) \neq f(L')[X]$. By contradiction assume that $f(I) = f(L')[X]$. In that case, there exists $a_0, a_1, \cdots, a_n \in L'$, such that

$$f(a_0) + f(a_1)X + \cdots + f(a_n)X^n = 1 \quad (3.5)$$

$$a_0 + a_1\theta + \cdots + a_n\theta^n = 0. \quad (3.6)$$

By (3.5), $f(a_0) = 1$ and for all $i > 0$, $f(a_i) = 0$. Since $\pi_{i_0}$ has the same kernel as $f$, we deduce from it that $\forall i > 0$, $\pi_{i_0}(a_i) = 0$. Applying $\pi_{i_0}$ to both sides of (3.6), we get that $\pi_{i_0}(a_0) = 0$ From this we get that $f(a_0) = 0$, which contradicts that $f(a_0) = 1$.

From this we deduce that $f(I)$ is not the whole $f(L')[X]$ and is therefore generated by a non constant polynomial $f(Q)$. Furthermore if $\chi$ denotes the minimal polynomial of $\theta$ over $F$, we get that $\chi \in f(I)$ and therefore $f(Q)$ divides $\chi$. The polynomial $\chi$ being separable (by 3.4.4), it implies that $f(Q)$ is separable as well. Since $f(Q)$ is non constant and separable, we can take $u \in F_{sep}$, a root of $f(Q)$. Now we extend $f$ with a formula $f(P(\theta)) = f(P)(u)$. Let us prove that $f$ is well defined. If we have that $P(\theta) = O(\theta)$, then $P - O(\theta) = 0$. In that case $P - O \in I$, therefore there exists $f(U) \in f(L')[X]$ such that $f(P - O) = f(U)f(Q)$. Then $f(P)(u) - f(O)(u) = 0$. Given the formula it is also clear that $f$ is a morphism. □

**Theorem 3.4.7** (extending morphisms of etale algebras)**.** *Let $L$ be an etale algebra and $L'$ a subalgebra of $L$ and $f : L' \to K_{sep}$ a morphism of $K$ algebras from $L'$ into the separable closure of $K$. Then there exists $g$ a morphism from $L$ to $K_{sep}$ that extends $f$.*

*Proof.* We shall do it by induction on the dimension of $L'$. If $\dim(L') = \dim(L)$, the statement is trivially true, since we may pick $g = f$. Now suppose that the statement is true for every $L'$ such that $\dim(L') > n$. Let us prove that it is true for $\dim(L') = n$ as well. Since $L' \subsetneq L$, we have that there exists a $\theta \in L$, such that $\theta \notin L'$.Therefore we have that $\dim(L'[\theta]) > \dim(L') = n$ and therefore $f$ can be extended to a map $h$ defined on $L'[\theta]$ using the previous lemma. By applying induction to $L'[\theta]$, we extend $h$ into a $g$ defined on $L$. Then $g_{|L'} = f$ and the theorem is proven. □

Finally we are going to prove that there is a correspondence between sets together with a continuous action of the absolute Galois group and etale algebras. As such, since we have seen that action of a group can be represented using the group action Cayley graph, we can represent these etale algebras using graphs.

For the proof, we are going to use the section 18 in Chapter 5 of [28].

**Theorem 3.4.8** (Dedekind's Lemma)**.** *Let $F/K$ be a field extension and $g_1, \cdots, g_n \in Aut(F/K)$ distinct automorphisms. Then $g_1, \cdots, g_n$ are $F$-linearly independent as maps from $F$ to $F$.*

**Lemma 3.4.9** (Galois descent)**.** *Let $F$ be a field, $F_{sep}$ its separable closure and $\Gamma = Gal(F_{sep}/F)$ its absolute Galois group. Suppose that $\Gamma$ acts continuously (for the discrete topology) on a $F_{sep}$-vector space $V$ by semi linear automorphisms: i.e*

$$\forall g \in \Gamma, \forall x, y \in V, g \cdot (x + y) = g \cdot x + g \cdot y$$

$$\forall g \in \Gamma, \forall \lambda \in F_{sep}, \forall x \in V, g \cdot \lambda x = g(\lambda) \cdot g(x)$$

*then*

$$V^\Gamma = \{v \in V | \forall g \in \Gamma, g \cdot v = v\}$$

*is a $F$- vector space and the map*

$$\begin{cases} F_{sep} \otimes_F V^\Gamma \longrightarrow V \\ x \otimes v \mapsto xv \end{cases}$$

*is an isomorphism of $F_{sep}$ vector spaces.*

*Proof.* One can think of the vector space $V$ in some sense as a field extension, while $V^\Gamma$ can be thought of as fixed field under $\Gamma$, which would correspond to the base field $F$. We are dealing here with vector spaces rather than fields, which slightly complicates matters.

The set $V^\Gamma$ is a $F$ vector space as a subspace of $V$ seen as a $F$-vector space. By continuity of the action of $\Gamma$, there exists a finite normal subextension of $F_{sep}$, $N$ , such that

$$\forall v \in V, \forall g, g' \in \Gamma, g_{|N} = g'_{|N} \Rightarrow g \cdot v = g' \cdot v$$

Since $N/F$ is finite, $Gal(N/F)$ is finite and therefore we write $Gal(N/F) = \{\gamma_1, \cdots, \gamma_n\}$, with $\gamma_1$ the identity map. We extend all these automorphisms into automorphisms of $F_{sep}$. Now take $\{m_1, \cdots, m_n\}$ a basis of $N$ seen as a $F$-vector space. ($[N : F] = |Gal(N/F)|$, since $N$ is separable). Now for $j \in \{1, \cdots n\}$ write

$$v_j = \sum_{i=1}^{n} \gamma_i(m_j)v$$

All these $v_j$ are in $V^\Gamma$: indeed, if we take $g \in \Gamma$, we have that there exists $i_0$, such that $g_{|N} = \gamma_{i_0|N}$. For that reason, we get that

$$g_{|N} \cdot v_j = \gamma_{i_0} \cdot v_j = \sum_{i=1}^{n} \gamma_{i_0} \cdot (\gamma_i(m_j)v) = \sum_{i=1}^{n} \gamma_{i_0}(\gamma_i(m_j))v$$

Now in the sum, we can do a bijective change of variable $\gamma'_i = \gamma_{i_0} \circ \gamma_i$ and therefore we can conclude that $g \cdot v_j = v_j$ for all $g \in \Gamma$ and all $j \in \{1, \cdots, n\}$. Now consider $M$ the $n \times n$ matrix with coefficient in $N$, given by: $M_{i,j} = \gamma_i(m_j)$. We will prove that $M$ is invertible, by proving its columns are linearly independent. Let $\lambda_1, \cdots, \lambda_n$ be such that

$$\sum_{i=1}^{n} \lambda_i M_{i,-} = 0 \tag{3.7}$$

Now if we consider $f$ the map from $N$ to $N$ that to $x$ associates $\sum\limits_{i=1}^{n} \lambda_i \gamma_i(x)$, it is $F$-linear. By 3.7, $f$ is zero on the $F$ basis of $N$, $m_1, \cdots, m_n$, therefore $f$ is a zero map. From that we conclude that $\sum\limits_{i=1}^{n} \lambda_i \gamma_i = 0$. By Dedekind's lemma, we then get that $\lambda_1 = \cdots = \lambda_n = 0$. Since the matrix $M$ is invertible, we shall denote $M'$ its inverse. Now we get that:

$$\sum_{j=1}^{n} M'_{j,1}v_j =$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n} M'_{j,1}\gamma_i(m_j)\gamma_i \cdot v =$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_i(m_j)M_{j,1}\gamma_i \cdot v =$$

$$\sum_{i=1}^{n} Id_{i,1}\gamma_i \cdot v =$$

$$\gamma_1 \cdot v = v$$

$v$ is therefore indeed in the image of the map from $F_{sep} \otimes_F V^{\gamma}$ to $V$.

Now to prove the injectivity of our map, we take $(e_i)_{i\in I}$ an $F$-basis of $V^{\Gamma}$. Now let $x \in F_{sep} \otimes_F V^{\Gamma}$ that gets mapped to $V$. Then there exists $i_1, \cdots, i_n \in I$ and $\lambda_1 \cdots \lambda_n \in F_{sep}$, such that

$$x = \lambda_1 \otimes e_{i_1} + \cdots + \lambda_n \otimes e_{i_n}$$

We then for the sake of simplicity denote the involved vectors in the basis as $e_1, \cdots, e_n$. The point $x$ then gets mapped to

$$y = \lambda_1 e_1 + \cdots + \lambda_n e_n$$

Assume that $y = 0$: let us show that all the $\lambda_i$ are zero. To prove that, we take $N$ a normal extension of $F$ containing all the $\lambda_i$ and we consider the non degenerate bilinear form coming from the trace. We will show that

$$\forall \mu \in N, \forall i \in \{1, \cdots, n\}, Tr(\mu\lambda_i) = 0$$

For $\gamma \in Gal(N/F)$, $\gamma \cdot \mu y = \gamma(\mu)\gamma \cdot y = 0$. Therefore we get that $\sum_{i=1}^{n} \gamma(\mu\lambda_i)e_i = 0$, since all the $e_i$ are fixed by $\gamma$. Summing these relations on $\gamma \in Gal(N/TF)$, we get that $\sum_{i=1}^{n} Tr(\mu\lambda_i)e_i = 0$. Since all the $e_i$ are linearly independent over $F$, we get that:

$$\forall i \in \{1, \cdots, n\}, Tr(\mu\lambda_i) = 0$$

This relation being true for all $\mu \in N$, we get that the $\lambda_i$ are indeed all zero. □

Note that the proof of injectivity worked regardless of assumption of the continuity of the action, therefore that map will be injective whether or not the action is continuous.

Now before stating the equivalence between etale algebras and finite $\Gamma$-sets, we will show that the number of morphisms from the algebra into the separable closure is its dimension.

**Proposition 3.4.10.** *Let $L$ be a finite-dimensional algebra over a field $F$. Let us denote $X(L)$ the set of morphisms of algebras from $L$ to $F_{sep}$, then: $|X(L)| \leq \dim(L)$ and we have equality if and only if $L$ is etale.*

*Proof.* First if $L$ is a finite dimensional algebra, then by 3.4.1, $L$ has only finitely many maximal ideals: $M_1, \cdots, M_n$ and $\sum_{k=1}^{n} \dim_F(L/M_k) \leq \dim_F(L)$. Now a morphism from $L$ to $F_{sep}$ can be considered as a morphism from $L$ to $\overline{F}$, where $\overline{F}$ is an algebraic closure of $F$ extending $F_{sep}$. If we then denote $X'(L)$ as the set of morphisms from $L$ to $\overline{F}$, we have that $X(L) \subseteq X'(L)$, therefore it is enough to prove the upper bound for $X'(L)$. For $i \in \{1, \cdots, n\}$, consider $\Phi_i$ an isomorphism between a subextension $K_i$ of $\overline{F}$ and $L/M_i$(which is an algebraic extension of $F$, because finite-dimensional over $F$). If $f$ is a morphism from $L$ to $\overline{F}$, then $\ker(f)$ is a maximal ideal of $L$, therefore $\ker(f)$ corresponds to one of the $M_i$. In that case $f$ factors to an isomorphism between $L/M_i$ and $K_i$, which by abuse of notation we shall still call $f$. In that case $f \circ \Phi_i^{-1}$ is an automorphism of $K_i$. We can then define a map $\Psi$ from $X'(L)$ to $\coprod_{i=1}^{n} Aut_F(K_i)$, by the formula:

$$\Psi(f) = f \circ \Phi_i^{-1}$$

The next step is to show that $\Psi$ is an injective map. To prove that, suppose that $\Psi(f) = \Psi(g)$. In that case $\ker(f) = \ker(g) = M_i$ for some $i$. Now if we take $x \in L$, we denote $\overline{x}$ its class in $L/M$. In that case, we have that $f(\overline{x}) = g(\overline{x})$ and so $f(x) = g(x)$. The morphism $f$ and $g$ being equal for all $x \in L$, we conclude that $f = g$. Since $\Psi$ is an injective map from $X'(L)$ to $\coprod_{i=1}^{n} Aut_F(F/M_i)$ we have the following inequalities:

$$|X(L)| \leq |X'(L)| \leq \sum_{i=1}^{n} |Aut_F(F/M_i)| \leq \sum_{i=1}^{n} [L/M_i : F] \leq \dim(L) \tag{3.8}$$

Now we need to prove the equivalence between $L$ being etale and $\dim(L) = |X(L)|$.

First assume that $L$ is etale. In that case $L$ is isomorphic to a product $\prod_{i=1}^{n} F_i$, where $F_i$ are separable finite extensions of $F$. Now let $\pi_i$ be the natural projection of $L$ on $F_i$. Then notice that for any $i$, the set $\{f \circ \pi_i | f \in Aut_F(F_i)\}$ has cardinal $[F_i : F]$ (since $F_i$ is separable) and it is contained in $X(L)$. Finally all these sets are disjoint, so the cardinal of $X(L)$ has to be at least $\dim(L)$, but by (3.8) it is at most $\dim(L)$, therefore it is equal to $\dim(L)$.

Now on the other hand assume that $|X(L)| = \dim(L)$. Then all the inequalities in (3.8) are in fact equalities. From that we deduce that $|Aut_F(F/M_i)| = [L/M_i]$ for all $i$, which proves that $L/M_i$ is separable for all $i$. Furthermore we have that $\dim(L/\bigcap_{i=1}^{n} M_i) = \sum_{i=1}^{n} [L/M_i : F] = \dim(L)$, which proves that $\bigcap_{i=1}^{n} M_i = \{0\}$. Now denote $\pi_i$ the natural projection of $L$ onto ${}^{L}\!/\!_{M_i}$. Let then consider the morphism of algebras:

$$\Phi = \begin{cases} L \longrightarrow \prod_{i=1}^{n} {}^{L}\!/\!_{M_i} \\ \theta \mapsto (\pi_i(\theta))_{i\in\{1,\cdots,n\}} \end{cases}$$

Since $\bigcap_{i=1}^{n} M_i = \{0\}$, we get that $\Phi$ is injective. Since the dimensions of the two algebras are equal, then $\Phi$ is an isomorphism. □

### 3.4.1 Equivalence between etale algebras and $\Gamma$ sets

To an etale algebra $L$, we associate the finite set $X(L)$, together with the action of $\Gamma = Aut(F_{sep}/F)$ given by $g \cdot f = g \circ f$. To a morphism of etale algebras $l$ $L_1 \to L_2$, we associate the $\Gamma$-equivariant map

$$X(l) = \begin{cases} X(L_2) \longrightarrow X(L_1) \\ f \mapsto f \circ l \end{cases}$$

$X$ is then a contravariant functor from the category of etale algebras to the category of finite sets together with a $\Gamma$-action. We will now show that this functor has an inverse. For a finite set $X$ together with a continuous action of $\Gamma$, we take the algebra $Map(X, F_{sep})$. We equip it with the following action: to a map $f$ from $X$ to $F_{sep}$, we associate the map

$$g \cdot f = \begin{cases} X \longrightarrow F_{sep} \\ x \mapsto g(f(g^{-1} \cdot x)) \end{cases}$$

We then take the algebra $Map(X, F_{sep})^{\Gamma}$. To a $\Gamma$-equivariant map $l$ from $X_1$ to $X_2$, we associate the morphism of algebras $Map(X_2, F_{sep})^{\Gamma}$ and $Map(X_1, F_1)^{\Gamma}$

$$M(l) = \begin{cases} Map(X_2, F_{sep})^{\Gamma} \longrightarrow Map(X_1, F_{sep})^{\Gamma} \\ f \mapsto f \circ l \end{cases}$$

Let us now show that $L = Map(X, F_{sep})^{\Gamma}$ is an etale algebra. By the Galois descent lemma, we get that $L \otimes_F F_{sep}$ is isomorphic to $Map(X, F_{sep})$, which itself is isomorphic to $F_{sep}^n$, with $n = |X|$. Then we know that $L$ injects itself to the product $F_{sep}^n$. Now let $\pi_1, \cdots, \pi_n$ be the natural projections of $L$ on $F_{sep}$. For all $i$, $\ker(\pi_i)$ is a maximal ideal in $L$ and we also have $\bigcap_{i=1}^{n} \ker(\pi_i) = \{0\}$. By the Chinese remainder theorem, we then get ${}^{L}\!/\!_{\bigcap_{i=1}^{n} \ker(\pi_i)} \cong L$ that is isomorphic to $\prod_{i=1}^{n} im(\pi_i)$, which is a product of subextensions of $F_{sep}$, therefore separable extension of $F$. We can therefore conclude that $L$ is etale.

We have therefore two contravariant functors $X$ and $M$: we shall show that they are an anti-equivalence of categories.

**Theorem 3.4.11.** *The functors $X$ and $M$ define an anti equivalence of categories between $F$ etale algebras and finite $\Gamma$-sets with a continuous action and we have the following table of correspondences:*

| *Etale algebras* | *Finite $\Gamma$-sets* |
|---|---|
| *Dimension* | *Cardinal* |
| *Tensor product (over $F$)* | *Direct product* |
| *Direct product* | *Disjoint union* |

Before we prove the theorem, note that in category theory terms the anti equivalence functors transforms products into coproducts and vice versa. Now let us begin the proof.

*Proof.* Before proving that the functors define an anti equivalence of categories, we first prove the second entry in the table: i.e that dimension gets transformed into cardinal and vice versa. If $L$ is an etale algebra, then by 3.4.10 $X(L)$ has the cardinal equal to the dimension of $L$. If $A$ is a finite $\Gamma$-set, then since $Map(A, F_{sep})$ is isomorphic to $F_{sep}^{|A|}$, then the dimension of $Map(A, F_{sep})$ as a $F_{sep}$-vector space is $|A|$. Since $Map(A, F_{sep})^\Gamma \otimes_F F_{sep}$ is isomorphic to $Map(A, F_{sep})$, then the dimension of $Map(A, F_{sep})^\Gamma$ as a $F$-vector space is $\dim(Map(A, F_{sep})) = |A|$. This concludes the first part of the proof. Now let us use it to prove that the two functors define an anti equivalence of categories.

Let $L$ be an etale algebra over $F$. Let us show that $L$ is naturally isomorphic to $M(X(L))$. To an element $x$, we associate the unique map in $M(X(L))$ $e_x$ that to $f \in Hom(L, F_{sep})$ associates $f(x)$. Let us show that $e_x$ is fixed by the action of $\Gamma$ for all $x$. Let $x \in L$,$g \in \Gamma$ and $f \in X(L)$. We have that

$$g \cdot e_x(f) = g((g^{-1} \cdot f)(x)) = g(g^{-1}f(x)) = f(x)$$

This relation being true for all $g \in \Gamma$ and $f \in X(L)$, we get that $e_x$ is indeed fixed by the action of $\Gamma$. Now consider

$$\Phi_L = \begin{cases} L \longrightarrow M(X(L)) \\ x \mapsto e_x \end{cases}$$

a morphism from $L$ to $M(X(L))$. Let us prove that is is an isomorphism. First to prove the injectivity, we consider $x \in L$, such that $\forall f \in X(L), f(x) = 0$. Let us show that $x = 0$. By contradiction, assume that it is non zero. Then let $\chi$ be the minimal polynomial of $x$ over $F$. By 3.4.4, $\chi$ is separable. Now take $\alpha \in F_{sep}$ a root of $\chi$. Consider the unique morphism $f$ from $F(x)$ to $F_{sep}$ that to $x$ associates $\alpha$. By 3.4.7 $f$ then can be extended to a morphism from $L$ to $F_{sep}$, which is a contradiction, since $f(x) = 0$ by assumption. Since $\Phi_L$ is an injective morphism and $L$ and $M(X(L))$ have a same dimension, then $\Phi$ is an isomorphism of algebras. Now let us show that the map $\Phi_L$ is natural. We have to be careful what that term means here. Since we are working with an anti-equivalence of categories, rather than an equivalence, we need to consider *op*: the functor from the category of etale algebras to its opposite. Then we shall show that if $l$ is a morphism from $L$ to $L'$, then the diagram:

$$\begin{array}{ccc} op(L) & \xleftarrow{op(l)} & op(L') \\ \uparrow{\scriptstyle op(\Phi_L)} & & \uparrow{\scriptstyle op(\Phi_{L'})} \\ M(X(L)) & \xleftarrow[M(X(l))]{} & M(X(L')) \end{array}$$

commutes. The composition of morphisms is then done in the opposite way compared to the category of Etale algebras.

To check the commutativity of the diagram, we then can simply check it for each $x \in L$. We have that $\Phi_{L'} \circ l$ is the unique map in $Map(X(L'), F_{sep})$ that to $\mu \in X(L')$ associates $\mu(l(x))$. Now on the other hand if we consider $M(X(l))(\Phi_L(x))$, it is by definition the map from $X(L')$ to $F_{sep}$ that to $\mu$ associates $\Phi_L(x)(\mu \circ l) = \mu(l(x))$. This concludes the proof that the diagram above commutes.

Now let $A$ be a finite set together with a continuous action of $\Gamma$. Let us show that $X(M(A))$ is isomorphic to $A$ as a $\Gamma$ set. Consider $\delta$ a map from $A$ to $X(M(A))$ that to an $a \in A$, associates the map $\delta_a = \begin{cases} M(A) \longrightarrow F_{sep} \\ f \mapsto f(a) \end{cases}$. The map $\delta_a$ is a morphism from $M(A)$ to $F_{sep}$. Let us show that it is injective. We take $x, y \in A$, such that $x \neq y$. In order to show that $\delta_x \neq \delta_y$, we differentiate two cases:

- Case 1:

  $x, y$ are in distinct orbits of $\Gamma$. In that case let

$$u = \begin{cases} A \longrightarrow F_{sep} \\ a \mapsto \begin{cases} 1 \text{ if } a \text{ is in the orbit of } x \\ 0 \text{ else} \end{cases} \end{cases}$$

  In that case, $u$ is in $Map(A, F_{sep})^\Gamma$, since constant on orbits of $G$ and with values fixed by $G$. Furthermore we have that $\delta_x(u) \neq \delta_y(u)$, hence $\delta_x \neq \delta_y$.

- Case 2: $x$ and $y$ are in the same orbit of $\Gamma$. Let then $g_0 \in \Gamma$ be such that $x = g_0^{-1} \cdot y$. Since $x$ and $y$ are distinct, the stabilizer of $x$, which we shall denote $H$ cannot be the whole $\Gamma$. Let $F^H$ be the fixed field of $H$ included in $F_{sep}$. Since the action of $\Gamma$ is continuous, $H$ is a closed subgroup of $\Gamma$. Therefore by the fundamental theorem of infinite Galois theory, $Gal(F_{sep}, F^H) = H$. Since $g_0 \notin H$, we get that there exists $\theta \in F^H$, such that $g_0(\theta) \neq \theta$. Now define

$$u = \begin{cases} A \longrightarrow F_{sep} \\ a \mapsto \begin{cases} g(\theta) \text{ if } a = g \cdot x \text{ and } g \in \Gamma \\ 0 \text{ else} \end{cases} \end{cases}$$

Let us show that $u$ is well defined and fixed by the action of $\Gamma$. To show that $u$ is well defined, we prove that if $g \cdot x = g' \cdot y$, then $g(\theta) = g'(\theta)$. If $g \cdot x = g' \cdot x$, then $g^{-1}g' \in H$. Since $x \in F^H$, then $g^{-1}g'(x) = x$ and therefore $g(x) = g'(x)$, proving that $u$ is well defined. Now we need to prove that $u$ is fixed by the action of $\Gamma$. Let $a \in A$ and $g \in \Gamma$. If $a$ is not in the orbit of $x$, we get that $g(u(g^{-1}a)) = g(0) = 0$. If on the other hand there exists $g' \in \Gamma$, such that $a = g' \cdot x$, then $(g \cdot u)(a) = g(u((g^{-1}g') \cdot x)) = g(g^{-1}g'(\theta)) = g'(\theta) = u(a)$. We then get that $u$ is indeed in $Map(A, F_{sep})^\Gamma$. Also $\delta_x(u) = u(x) = \theta \neq g_0(\theta) = u(y)$.

This concludes the proof of injectivity of $\delta$. Since $\delta$ is an injective map between the two sets $A$ and $Map(A, F_{sep})^\Gamma$ with the same cardinal, then $\delta$ is a bijection. Finally we need to prove that $\delta$ is equivariant. For that, let $g \in \Gamma$, $a \in A$ and $u \in Map(A, F_{sep})^\Gamma$. We get that

$$g \cdot \delta_a(u) = g(u(a)) = g(u(g^{-1}g \cdot a)) = g \cdot u(g \cdot a)$$

Since $u$ is fixed by the action of $\Gamma$, we conclude that $g \cdot \delta_a(u) = \delta_{g \cdot a}(u)$ and hence $\delta$ is an equivariant bijection between the two $\Gamma$-sets and it is therefore an isomorphism of $\Gamma$-sets. Just like in the previous case, one can prove that $op(\delta)$ is a natural transformation from $op(A)$ to $X(M(A))$.

This concludes the proof that $X$ and $M$ define an anti equivalence of categories. The next step in the proof is to show that products get sent on coproducts and vice versa. One can already conclude that from the fact that $X$ and $L$ are anti equivalences of categories, but it is worth seeing explicitly the isomorphisms constructed.

Let $L$ and $L'$ be two etale algebras. Let us show that $X(L \otimes L')$ is isomorphic to $X(L) \times X(L')$. To a morphism $f$ from $L \otimes L'$ to $F_{sep}$ we associate $(f_L, f_{L'})$, where $f_L(x) = f(x \otimes 1)$ and
$f_{L'}(y) = f(1 \otimes y)$. Such a map is a bijection from $X(L \otimes L')$ to $X(L) \times X(L')$, with the inverse the map that to $(f, g)$ associates the unique morphism of algebras $u$ defined by the formula
$u(x \otimes y) = f(x)g(y)$.

Now let us show that $X(L \times L')$ is isomorphic to $X(L) \amalg X(L')$. Consider the map from $X(L) \amalg X(L')$ that to $f$ associates the morphism

$$\phi(f) = \begin{cases} F \times F' \longrightarrow F_{sep} \\ (x, y) \mapsto \begin{cases} f(x) \text{ if } f \in X(L) \\ f(y) \text{ if } f \in X(L') \end{cases} \end{cases}$$

$\phi$ is injective and $X(L) \amalg X(L')$ and $X(L \times L')$ have the same amount of elements, therefore $\phi$ is a bijection.

Now suppose instead that $A, B$ are two finite $\Gamma$-sets. Consider the map from $M(A) \otimes M(B)$ to $M(A \times B)$, given by:

$$\Phi(f \otimes g) = \begin{cases} A \times B \longrightarrow F_{sep} \\ (a, b) \mapsto f(a)g(b) \end{cases}$$

Let us show that this map is surjective. One can show that $M(A \times B)$ is generated by maps that are null outside of one orbit in $A \times B$. If we can then find an antecedent of every such a map by $\Phi$, we'd prove that $\Phi$ is surjective. Let $O$ be an orbit by the action of $\Gamma$. Let us pick $(a_0, b_0)$ a representative in $O$. We write $H^A$ the stabilizer of $a$ by the action of $\Gamma$ on $A$,$H^B$ the stabilizer of $b$ by the action of $\Gamma$ on $B$ and finally $H$ the stabilizer of $(a, b)$. We have $H^A \cap H^B = H$. Let $f$ be a function in $M(A \times B)$ that is 0 outside of $O$. Let us show that $f(a_0, b_0) \in F_{sep}^H$. In order to do that, we take $\gamma \in F_{sep}^H$. Since $f \in M(A \times B)$, we get that $\gamma(f(a_0, b_0)) = f(\gamma \cdot (a_0, b_0)) = f(a_0, b_0)$. This being true for every $\gamma \in H$, we get that $f(a_0, b_0) \in F_{sep}^H$. Now let $u_1, \cdots, u_n$ be an $F$-basis of $F_{sep}^{H^A}$ and $v_1, \cdots, v_m$ an $F$-basis of $F_{sep}^{H^B}$. It is clear that the family $(u_i v_j)$ then generates the field $F_{sep}^{H(A)} F_{sep}^{H(B)} = F_{sep}^H$. Now let $\chi_i$ for $i$ between 1 and $n$ be the map :

$$\chi_i = \begin{cases} A \longrightarrow F_{sep} \\ a \mapsto \begin{cases} \gamma(u_i) \text{ if } a = \gamma a_0 \text{ with } \gamma \in \Gamma \\ 0 \text{ else} \end{cases} \end{cases}$$

and for $j$ between 1 and $m$,

$$\xi_j = \begin{cases} B \longrightarrow F_{sep} \\ b \mapsto \begin{cases} \gamma(v_j) \text{ if } b = \gamma b_0 \text{ with} \end{cases} \end{cases}$$

One can prove that $\chi_i$ are well defined maps in $M(A)$ and $\xi_j$ well defined maps in $M(B)$. Now we write

$$f(a_0, b_0) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{i,j} u_i v_j$$

Then we have that

$$f = \Phi(\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{i,j} \chi_i \otimes \xi_j)$$

Since the map $\Phi$ is surjective and the dimensions are the same (both equal to $|A| \times |B|$), then $\Phi$ is an isomorphism of algebras, which is what we wanted to show.

Finally we want to prove that $M(A \amalg B)$ is isomorphic to $M(A) \times M(B)$. For that we consider the morphism

$$\Phi = \begin{cases} M(A) \times M(B) \longrightarrow M(A \amalg B) \\ (f,g) \mapsto \begin{cases} A \amalg B \longrightarrow F_{sep} \\ x \mapsto \begin{cases} f(x) \text{ if } x \in A \\ g(x) \text{ if } x \in B \end{cases} \end{cases} \end{cases}$$

and the morphism

$$\Psi = \begin{cases} M(A \amalg B) \longrightarrow M(A) \times M(B) \\ f \mapsto (f_{|A}, f_{|B}) \end{cases}$$

Observe that these morphisms are inverse to each other, which concludes the proof. □

An interesting question is when does a Γ set correspond to a field extension rather than a product of field extensions. The following proposition answers that question:

**Proposition 3.4.12** (Field extensions are Γ-sets with transitive action)**.** *Let $F$ be a field with the absolute Galois group $\Gamma$. Let $A$ be a $\Gamma$ set. Then $A$ corresponds to a field extension of $F$ if and only if the action of $\Gamma$ is transitive.*

*Proof.* Suppose first that $\Gamma$ acts transitively on $A$. Let us show that $M(A)$ then is a field. Let $f \in M(A)$ a non zero map. Then there exists $a \in A$, such that $f(a)$ is non zero. Then for every $\gamma \in \Gamma$, $\gamma f(a) \neq 0$. Since $f \in M(A)$, this implies that for every $\gamma \in \Gamma$, $f(\gamma a) \neq 0$. If then we define $g = \begin{cases} A \longrightarrow F_{sep} \\ a' \mapsto \frac{1}{f(a')} \end{cases}$, $g$ is in $M(A)$ and $g$ is the inverse of $f$, proving that $M(A)$ is a field and therefore a separable extension of $F$.

Now on the other hand assume the action of $\Gamma$ is not transitive. Let us show that $M(A)$ is not a field. Let $O$ be an orbit in $A$. Then the function $f$ that is equal to 1 on $O$ and zero outside of $O$ is in $M(A)$ and is clearly non invertible. □

We have seen that etale algebras can be represented as Γ-sets. Now if Γ is finite then we know how to represent an action on a set by the Cayley action graph. If Γ is infinite, since we assumed the action continuous, we can still represent it using a finite quotient of Γ, rather than Γ itself. Another possible approach would be to somehow represent the profinite structure of the infinite group on a graph, which is going to be the goal of the next chapter.

# Chapter 4

# Profinite graphs

As we have seen, if a category $\mathscr{C}$ has limits of finite objects, we can define from it a profinite structure. Now for abstract graphs, the morphisms we typically consider are the ones that send edges on edges. While this would give us a profinite structure on graphs: it is a limited one and lots of interesting cases couldn't be explored. The more appropriate notion for our morphisms are the so called qmorphisms. The difference is that while morphisms send edges on edges, qmorphisms can contract edges to vertices. A trivial example of a qmorphism would be the map that contracts all the connected components of a graph onto a single vertex. Using these qmorphisms, we will construct the category of profinite graphs and generalize notions we have seen for finite graphs on this structure.

## 4.1 Basic notions

**Definition 4.1.1.**

- *We call an abstract graph a quadruplet: $(X, V, o, t)$, such that $X$ is a set and $V$ is a subset of $X$ called the set of vertices and $o, t$ are maps from $X$ to $V$, whose restriction on $V$ is identity. They are called origin and terminus and are also known as incidence maps. Finally for a graph $G = (X, V, o, t)$, we denote $V(G) = V$ and call it the set of vertices and $E(G) = X \setminus V$ and call it the set of edges.*

- *We call a qmorphism of graphs $(X, V, o, t)$ and $(X', V', o', t')$ a map from $X$ to $X'$, such that*

$$\forall x \in X, f(o(x)) = o'(f(x)) \text{ and } f(t(x)) = t'(f(x))$$

- *We call a morphism of graphs $(X, V, o, t)$, $(X', V', o', t')$ a qmorphism, such that*

  *$f(X \setminus V) \subseteq X' \setminus V'$, i.e it sends edges on edges.*

Note that a qmorphism always sends vertices to vertices.

Now to prove that graphs have a profinite structure we need to check the three axioms of preprofinite categories. We will start by proving that the projective limits in the category of graphs together with their qmorphisms exist. By abuse of notation, we shall often identify a graph $\Gamma$ with the set representing it and we just use the maps $o$ and $t$ universally for all graphs.

**Proposition 4.1.2** (Existence of projective limits)**.** *The category $\mathscr{C}$ of abstract graphs is a preprofinite category.*

*Proof.* First we prove that the projective limits exist. Let $((G_i)_{i\in I}, (f_{i,j})_{j\geq i})$ be a directed system. Write $G_i = (X_i, V_i, o_i, t_i)$ Let

$$X = \{(x_i)_{i\in I} \in \prod_{i\in I} X_i | \forall i, j \in I, j \geq i \Rightarrow f_{i,j}(x_j) = x_i\}$$

Define

$$V = \{(x_i)_{i\in I} \in \prod_{i\in I} V_i | \forall i, j \in I, j \geq i \Rightarrow f_{i,j}(x_j) = x_i\}$$

Now consider the maps

$$o = \begin{cases} X \longrightarrow \prod_{i\in I} V_i \\ (x_i)_{i\in I} \mapsto (o_i(x_i))_{i\in I} \end{cases} \qquad \text{and } t = \begin{cases} X \longrightarrow \prod_{i\in I} V_i \\ (x_i)_{i\in I} \mapsto (t_i(x_i))_{i\in I} \end{cases}$$

Observe that $o$ and $t$ map $X$ to $V$, since they are compatible with $q$ morphisms. They act as identity on $V$, since they act as identity component by component. The quadruplet $G = (X, V, o, t)$ is therefore a graph. It is a projective limit in the sense of the category of sets and so one can show that it is also a projective limit in sense of qmorphisms.

Now to define $\mathscr{C}$ as a preprofinite category, we take $F$ to be the forgetful functor on it. It is faithful and transforms projective limits of graphs into projective limits of sets.

Finally let $A, B, C$ be graphs, $u, v$ qmorphisms with $u$ surjective and $f$ a map from $A$ to $C$, such that:

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ u \uparrow & \nearrow_{v} & \\ B & & \end{array}$$

commutes. Let us show that $f$ is a qmorphism. Let $x \in A$ and $d \in \{o, t\}$. Then there exists $y \in B$, such that $u(y) = x$. Then

$$d(f(x)) = d(f(u(y))) = d(v(y)) = v(d(y)) = f(u(d(y))) = f(d(u(y))) = f(d(x))$$

We conclude that $f$ is indeed a qmorphism. □

As such we can define profinite graphs as follow:

**Definition 4.1.3** (Profinite graph)**.** *A profinite graph is a profinite object* $(\Gamma, (\phi_{i)})_{i\in I}$ *in the preprofinite category of abstract graphs.*

While seeing profinite graphs as limits of graphs is a valid definition, one can also see them as simply profinite spaces with a graph structure that is continuous for the graph topology. We have the characterization that follows:

**Theorem 4.1.4** (characterization of profinite graphs)**.** *Let $\Gamma$ be a compact Hausdorff topological space, together with continuous maps $o, t$ from $\Gamma$ to a closed subspace $V$, such that $(\Gamma, V, o, t)$ is a graph. Then $\Gamma$ is a profinite graph if and only if $\Gamma$ is totally disconnected as a topological space.*

*Proof.* One implication is simply consequence of 3.1.10.

To prove the other implication, by 3.1.12 it is enough to prove that if $\Gamma$ is compact Hausdorff, then $(\Gamma, V, o, t)$ is a profinite-compatible object. Let $\Omega$ be the set of open equivalence relations on $\Gamma$. First let us show that any $R \in \Omega$, contains an open $R'$ compatible with the graph structure.

For $R \in \Omega$, take

$$R' = \{(x, y) \in R | o(x)Ro(y) \text{ and } t(x)Rt(y)\} \subseteq R$$

which is clearly an equivalence relation. Let us show it is open. First by compactness of $G$, there are only finitely many equivalence classes for $R$. Let us call $\phi$ the natural projection on ${}^{G}\!/_{R}$. If then the finite set ${}^{\Gamma}\!/_{R}$ gets equipped with its discrete topology, $\phi$ is then continuous, since the preimage of a singleton by $\phi$ is an equivalence class and since $R$ is open, then this class is open. Now consider

$$\psi = \begin{cases} \Gamma \longrightarrow {{}^{\Gamma}\!/_{R}}^{3} \\ x \mapsto (\ \phi(x), \phi(o(x)), \phi(t(x))\ ) \end{cases}$$

This is a continuous map, since each component is continuous (since $o, t$ are continuous). For an $x \in \Gamma$, we get that

$$xR' = \psi^{-1}(\psi(x))$$

Indeed: if $xR'y$, then $xRy$, so $\phi(x) = \phi(y)$, $o(x)Ro(y)$,so $\phi(o(x)) = \phi(o(y))$ and similarly
$\phi(t(x)) = \phi(t(y))$. As such we get that $\psi(x) = \psi(y)$. On the other hand if $\psi(y) = \psi(x)$, we get that $\phi(x) = \phi(y)$, $\phi(o(x)) = \phi(o(y))$ and $\phi(t(x)) = \phi(t(y))$, therefore $xR'y$. We have shown that $xR'$ is a preimage of a singleton by a continuous function, therefore $xR'$ is an open subset, proving that $R'$ is a open relation. Finally let us prove that $R'$ is compatible with the graph structure. Take $d \in \{o, t\}$ and $xR'y$. In this case $d(x)Rd(y)$, $o(d(x)) = d(x)$ and $o(d(y)) = d(y)$, so $o(d(x))Ro(d(y))$ and by the same reasoning $t(d(x))Rt(d(y))$. As such, $d(x)R'd(y)$, proving that $R'$ is compatible with the graph structure.

This means that for every open equivalence relation $R$ in $\Omega$, there exists an open equivalence relation $R' \subseteq R$, such that ${}^{\Gamma}\!/_{R'}$ is a graph and the natural projection of $\Gamma$ on ${}^{\Gamma}\!/_{R'}$ is a morphism and therefore by 3.1.12 $\Gamma$ is a profinite graph. $\square$

Now we will define the group of automorphisms and an action of a profinite group.

### 4.1.1 Group of automorphisms

Let $\Gamma$ be a profinite graph. We denote $Aut(G)$ the group of continuous graph automorphisms of $\Gamma$. I.e $g \in Aut(\Gamma)$ if and only if: $g$ is bijective and $g$ and $g^{-1}$ send vertices to vertices and preserve the incidence maps $o$ and $t$. According to 3.1.1, we can equip the automorphisms of $\Gamma$ as a profinite space (without considering its graph structure) with a structure of a topological group with the compact open topology. Now the automorphisms of $\Gamma$ as automorphisms of graph are a closed subgroup of all the automorphisms of the profinite set $\Gamma$, so it is a topological group for the induced topology. Naturally we want $Aut(\Gamma)$ to be a profinite group, but unfortunately, we will see that it is not always the case. We have however the criterion that follows.

**Proposition 4.1.5** (profiniteness criterion)**.** *Let $\Gamma$ be a profinite graph and $G$ a closed subgroup of $Aut(\Gamma)$. Suppose that $\Gamma$ is written as the projective limit of graphs $(\Gamma_i)_{i\in I}$, together with transition maps $(\phi_{i,j})_{j\geq i}$ and with natural projections $p_i$. In that case, we get that $G$ is profinite if and only if for every $i \in I$, the set $A_i = \{p_i \circ g | g \in G\}$ is finite.*

*Proof.* Before we begin, notice that $G$ as a topological space is homeomorphic to the projective limit of $A_i$ with transition maps $(\psi_{i,j})_{j\geq i} = \begin{cases} A_j \longrightarrow A_i \\ p_j \circ f \mapsto p_i \circ f \end{cases}$ and equipped with product topology (each $A_i$ having the discrete topology). Indeed for every $i \in I$, we have a continuous map $\psi_i$ from $G$ to $A_i$ that to $f$ associates $A_i$. It is continuous with $A_i$ being equipped with discrete topology, since for any $f \in G$, $\psi_i^{-1}(\{p_i \circ f\}) = \{g \in G | p_i \circ g = p_i \circ f\}$, which is open by the definition of the topology on $Aut(\Gamma)$. The maps $\psi_i$ are clearly compatible with the transition maps $\psi_{i,j}$, therefore there exists a unique map $\Psi$ from $G$ to $\varprojlim_{i\in I} A_i$, such that $\forall i \in I, \pi_i \circ \Psi = \psi_i$, with $\pi_i$ being the natural projection of the limit on $A_i$. The map $\Psi$ is injective. Indeed if $f, g \in G$ are such that $\forall i \in I$, $p_i \circ f = p_i \circ g$, then clearly $f = g$. Now let us show that $\Psi$ is surjective. Let $(p_i \circ g_i)_{i\in I}$ be a collection in $\varprojlim_{i\in I} A_i$.

Define $g = \begin{cases} \Gamma \longrightarrow \Gamma \\ x \mapsto (\ p_i \circ g_i(x)\ )_{i\in I} \end{cases}$. The relation $g$ is a well defined function from $\Gamma$ to $\Gamma$. Let us prove that it is in $G$. First we start by proving that it is in $Aut(\Gamma)$. For that we take $x \in \Gamma$ and $d \in \{o, t\}$ one of the incidence maps. To prove that $d(g(x)) = g(d(x))$, it is enough to prove that for every $i \in I$, $p_i(d(g(x))) = p_i(g(d(x)))$. Now if we take $i \in I$, we get that

$$p_i(g(d(x))) = p_i g_i(d(x)) = d(p_i g_i(x)) = d(p_i(g(x))) = p_i(d(g(x)))$$

since $g_i$ and $p_i$ are morphisms of graphs. Finally $p_i(g(d(x))) = p_i(d(g(x)))$ as expected. The next step is to prove that $g$ is continuous. For that we take $x, y \in \Gamma$ and $i \in I$. Now let $i_0$ be such that for every $j \geq i_0$, if $p_j(x) = p_j(y)$, then $p_i(g_i(x)) = p_i(g_i(y))$(by continuity of the maps $p_i$ and $g_i$). Now if $j \geq i_0$, and $p_j(x) = p_j(y)$, we have that

$$p_i(g(x)) = p_i g_i(x) = p_i g_i(y) = p_i(g(y))$$

Now we need to prove that the inverse of $g$ is continuous as well. The map $g'$ associated to $(g_i^{-1})_{i\in I}$ is clearly a continuous inverse morphism of $g$, therefore $g$ is indeed in $Aut(\Gamma)$.

Now we need to show that $g$ is specifically in $G$. We will show that $g \in \overline{G}$ and therefore $g \in G$, since $G$ is closed. Let $V$ be a neighborhood of $g$. Then there exists $i \in I$, such that the set $\{g' \in Aut(\Gamma) | p_i \circ g = p_i \circ g'\} \subseteq V$. In that case we get that $g_i \in V$ and since $g_i \in G$, we get that $V \cap G \neq \emptyset$. Since this is true for all neighborhoods $V$ of g, it proves that $g \in \overline{G}$. The topological space $G$ is therefore homeomorphic to $\varprojlim_{i\in I} A_i$.

Now we can finally finish the proof: assume first that $A_i$ is finite for every $i$. The group $G$ is a projective limit of finite sets and so it is a profinite set, therefore a totally disconnected compact Hausdorff topological group and as such it is a profinite group by: 3.2.3.

On the other hand if we suppose that $G$ is profinite, then we have the following: Take $i \in I$. The set $\Psi_i(G) = A_i$ then is compact by the continuity of $\Psi_i$. Since it is equipped with the discrete topology, it has to be finite. □

*Remark.* This result is very similar to the Theorem 5.3 in the DDMS book [26], which establishes a sufficient condition for automorphisms of a profinite group to be profinite. For a description of the automorphism group of finitely generated profinite groups, one may also see [46]: in Theorem 1.3 John Smith shows that automorphism group of a finitely generated profinite group is compact and consequently profinite. Note that if we assume a profinite group $G$ to be finitely generated then an automorphism of $G$ is determined by its image on the generators and if we compose by a natural projection onto the quotient by an open normal subgroup, we end up with only finitely many options, hence from the point of view we adopted in the previous proposition applied to groups rather than graphs, we get indeed that $Aut(G)$ is profinite as stated in the theorem.

## 4.2 Connectedness

Since the correct arrows in the category of profinite graphs aren't morphisms, but qmorphisms, we will need to adapt our definition of a path slightly when compared to the one given in Chapter 2, so that paths are preserved by qmorphisms rather than just morphisms. We will give the following definition of a path:

**Definition 4.2.1.** *Let $(\Gamma, V(\Gamma), o, t)$ be an abstract graph. For an incidence map $d \in \{o, t\}$, denote $\overline{d} = \begin{cases} t \text{ if } d = o \\ o \text{ if } d = t \end{cases}$. We define a path as a finite sequence $(x_1, \cdots, x_n)$ of elements of $\Gamma$, such that there exist incidence maps $d_1, \cdots, d_n \in \{o, t\}$, such that for all $k < n$, $\overline{d_k}(x_k) = d_{k+1}(x_{k+1})$.*

Notice that since for every vertex $x \in V(\Gamma)$, $o(x) = t(x)$, this new definition of path allows paths to remain stationary at vertices between steps. That is the only difference with our previous definition of path in Chapter 2, so the notion of path-connected components will remain unaffected.

We define an equivalence relation on $\Gamma$, simply as: $x \sim y$ if and only if there exists a path $(x_1, \cdots, x_n)$, such that $x_1 = x$ and $x_n = y$. Note that this definition applies to vertices and edges alike. This relation is compatible with graph structure, so we can quotient a graph by it. Unfortunately it will not be compatible with the profinite structure, i.e the relation is not always open, or even closed for the profinite topology, which will lead us to define a more general notion of connectedness for profinite graphs.

**Proposition 4.2.2** (qmorphisms preserve paths)**.** *Let $\Gamma, \Gamma'$ be two abstract graphs, $f$ a qmorphism from $\Gamma$ to $\Gamma'$ and $(x_1, \cdots, x_n)$ a path in $\Gamma$. Then $(f(x_1), \cdots, f(x_n))$ is a path in $\Gamma'$.*

*Proof.* Since $(x_1, \cdots, x_n)$ is a path take $d_1, \cdots, d_n \in \{o, t\}$ such that for all $k < n$, $\overline{d_k}(x_k) = d_{k+1}(x_{k+1})$. Then using the fact that $f$ is a qmorphism we get that:

$$\overline{d_k}(f(x_k)) = f(\overline{d_k}(x_k)) = f(d_{k+1}(x_{k+1})) = d_{k+1}(f(x_{k+1}))$$

□

Now that we know that qmorphisms preserve paths, we will give a definition of connectedness that works for profinite graphs.

**Definition 4.2.3.** *A profinite graph $\Gamma$ is said to be connected if for every continuous qmorphism $f$ from $\Gamma$ into a finite graph, the image of $f$ is path-connected.*

While path-connected profinite graphs are connected (qmorphisms preserve paths), it is only a small part of connected profinite graphs. Ribes in Example 2.1.8 in [41] gives an example of a connected profinite graph that is connected, but has a vertex with no edges. We shall give a similar example that has two such vertices: one on each side.

**Example 4.2.4** (A graph that is connected, but not path-connected)**.** We define a graph $\Gamma$ with the set of vertices $\mathbb{Z} \cup \{-\infty, +\infty\}$ and edges $\{e_i | i \in \mathbb{Z}\}$. The origin map $o$ is then defined as: $o(e_i) = i$ and a terminus map defined as: $t(e_i) = i + 1$. We then get a graph that looks like:

$$-\infty \qquad \cdots \longrightarrow -2 \xrightarrow[e_{-2}]{} -1 \xrightarrow[e_{-1}]{} 0 \xrightarrow[e_0]{} 1 \xrightarrow[e_1]{} 2 \longrightarrow \cdots \qquad +\infty$$

We equip it with projections $p_n$ onto path from $-n$ to $n$, with:

$$p_n(x) = \begin{cases} x \text{ if } x \in \mathbb{Z} \text{ and } -n \leq i \leq n \\ n \text{ if } x \in \mathbb{Z} \text{ and } i \geq n \\ -n \text{ if } x \in \mathbb{Z} \text{ and } i \leq -n \\ (i, i+1) \text{ if } i \in \mathbb{Z}, -n \leq i \leq n \text{ and } x = e_i \\ -n \text{ if } i \in \mathbb{Z}, i < -n \text{ and } x = e_i \\ n \text{ if } i \in \mathbb{Z}, i > n \text{ and } x = e_i \end{cases}$$

Similarly to Ribes's example, one can show that these projections induce on the graph a profinite structure. As a side note, this graph is an example of a profinite graph whose group of automorphisms is not profinite. One can prove that it is isomorphic to $\mathbb{Z}$ with discrete topology. The automorphisms in this case are simply translations, with infinities being fixed.

The interesting property of this graph is that even though it is not connected, if we admit in some sense limits of paths, the graph would still be path-connected. To formally describe this situation, I came up with the definition that follows.

**Definition 4.2.5.** *Let $\Gamma$ be a profinite graph. We call a superpath a* finite sequence *$(C_1, \cdots, C_n)$ of* path-connected components *of $G$, such that $\forall i < n$, $\overline{C_i} \cap \overline{C_{i+1}} \neq \emptyset$, with for $A \subseteq \Gamma$, $\overline{A}$ being the topological closure of $A$ in $\Gamma$ for the profinite topology.*

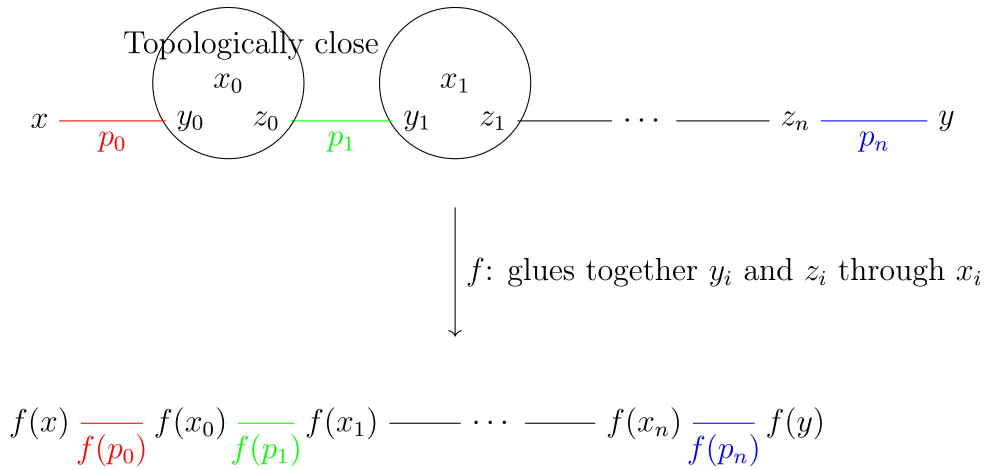
*We then call a profinite graph $\Gamma$ superpath-connected, if for every $x, y \in \Gamma$, there exists a superpath $(C_1, \cdots, C_n)$, such that $x \in C_1$ and $y \in C_n$.*

The path-connected components of a graph are disjoint, but not necessarily their closures. Requiring that $\overline{C_i} \cap \overline{C_{i+1}} \neq \emptyset$ means that any element in $C_i$ can approach any element in $C_{i+1}$ by paths, so even though it is not possible to get from one component to another in finitely many steps, it is possible to do it by taking limits.

An interesting question one can ask is whether this superpath connectivity is simply equivalent to connectivity or if it is still not enough. We will prove that if a profinite graph is superpath-connected, then it is connected, we will give one case where these notions are equivalent and finally we will show that they are not equivalent in general.

**Proposition 4.2.6.** *Let $\Gamma$ be a profinite graph. If $\Gamma$ is superpath-connected, then $\Gamma$ is connected.*

*Proof.* Assume that $\Gamma$ is superpath-connected. Let $f$ be a continuous qmorphism from $\Gamma$ to a finite graph. Let $x, y \in \Gamma$. To prove that $\Gamma$ is connected, we need to show that there exists a path from $f(x)$ to $f(y)$ in the graph $im(f)$, to prove that $\Gamma$ is connected. Since $\Gamma$ is superpath-connected, we take $(C_1, \cdots, C_n)$ a superpath from $x$ to $y$. Take $x_i \in \overline{C_i} \cap \overline{C_{i+1}}$. Now for $i$ between 0 and $n-1$, we take $y_i \in C_i$, such that $f(x_i) = f(y_i)$ using the continuity of $f$ and the fact that $x_i \in \overline{C_i}$. Furthermore for all $i$ between 0 and $n-1$, we take $z_i \in C_{i+1}$, such that $f(x_i) = f(z_i)$, which is possible for the same reasons. The set $f(C_i)$ is connected for all $i$, since $C_i$ is connected and $f$ is a qmorphism. We then take $p_0$ a path from $x$ to $y_0$, $p_n$ a path from $y_n$ to $y$ and for $i$ between 1 and $n-1$, we take $p_i$ a path from $f(z_{i-1})$ to $f(y_i)$, since they both belong to the connected subgraph $f(C_i)$. Notice that the terminal vertex of the path $f(p_i)$ is $f(y_i) = f(x_i) = f(z_i)$, which is the initial vertex of the path $f(p_{i+1})$. We can therefore concatenate the paths $f(p_0), \cdots, f(p_n)$ to obtain a path $p$ from $f(x)$ to $f(y)$. The image that follows illustrates the proof:




□

Now we will see one case, where being connected and superpath-connected is equivalent, but first we will prove a very useful lemma.

**Lemma 4.2.7.** *Let $\Gamma$ be a profinite graph. If it is a proper disjoint union of two open subgraphs, then it is not connected.*

*Proof.* Suppose that $\Gamma = \Gamma' \amalg \Gamma''$, with $\Gamma'$ and $\Gamma''$ open subgraphs. Let $f$ be a map from $\Gamma$ to $\{0, 1\}$ defined by

$$f(x) = \begin{cases} 0 \text{ if } x \in \Gamma' \\ 1 \text{ if } x \in \Gamma'' \end{cases}$$

If we consider $\{0, 1\}$ to be a graph with no edges and two vertices, we get that f is a continuous qmorphism. It is indeed continuous, since constant on the two open disjoint sets. Now let $d$ be either the terminus or the origin map. Let $x \in \Gamma$. Without loss of generality, we assume that $x \in \Gamma'$, and we show that $f(d(x)) = d(f(x))$. We have that $d(x) \in \Gamma'$, since $G'$ is a subgraph. Therefore we get that $f(d(x)) = 0 = d(f(x))$. We have found a continuous surjective qmorphism into a finite disconnected graph, therefore $G$ is not connected. □

**Proposition 4.2.8.** *Let $\Gamma$ be a profinite graph, with finitely many path-connected components. Then $\Gamma$ is connected if and only if it is superpath-connected.*

*Proof.* We already know that one implication is true in general case, so it is enough to prove that if a graph $\Gamma$ has finitely many path-connected components and is connected, then it is superpath-connected.

Since the relation "There exists a superpath between $x$ and $y$" is an equivalence relation, we can partition $\Gamma$ into superpath-connected components. We start by proving that there are only finitely many of these superpath-connected components.

Let $X$ denote the set of path-connected components. Consider the function $\Phi$ from the set of superpath-connected components, to the power set of $X$: $\mathscr{P}(X)$ that to a superpath-connected component $C$ associates the set

$$\Phi(C) = \{c \in X | c \subseteq C\}$$

i.e that set of all path-connected components included in $C$. Let us show that $\Phi$ is injective. Suppose that $\Phi(C) = \Phi(C')$. It is enough by symmetry of this relation to prove that $C \subseteq C'$ to conclude that $C = C'$. Let $x \in C$. Consider $c$ the path-connected component of $x$. Then $c \subseteq C$, by definition of a superpath-connected component. In that case, we also get that $c \subseteq C'$, since $\Phi(C) = \Phi(C')$, proving that $x \in C'$. Since $\Phi$ is an injection from the set of superpath-connected components into the finite set $\mathscr{P}(X)$, we get that the graph has indeed finitely many superpath-connected components.

The next step is to prove that superpath-connected components are all closed. Let $C$ be a superpath-connected component, then let us prove that $C = \bigcup_{c \in \Phi(C)} \overline{c}$, with $\Phi$ the injection from above. If $x \in C$, then if we take $c$ its path-connected component, then $c \subseteq C$, therefore $x \in \bigcup_{c \in \Phi(C)} \overline{c}$. On the other hand let us now take $x$ in some $\overline{c}$, with $c \subseteq C$. Let $c'$ be the connected component of $x$. Then by definition $x \in \overline{c} \cap \overline{c'}$. Then $(c', c)$ is a superpath from $x$ to any element of $c \subseteq C$, hence $x$ is in the superpath-connected component $C$. We then get that $C$ is therefore a finite union of closed sets and is therefore closed.

Now by contraposition, we will prove that if $\Gamma$ has finitely many path-connected components and is connected, then it is superpath-connected. Assume that $\Gamma$ is not superpath-connected. Take $C$ a superpath-connected component. The set $C$ is clopen, because it is closed and its complement is a union of finitely many closed sets (the other superpath-connected components). We will furthermore prove that $C$ is a subgraph. If $x \in C$, we take $c$ a path-connected component of $x$. Then $t(x) \in c$ and $o(x) \in c$. therefore $t(x), o(x) \in C$, confirming that $C$ is indeed an open subgraph of $\Gamma$. Its complement is a disjoint union of open subgraphs, therefore is itself an open subgraph.

The graph $\Gamma$ is then a proper union of two open subgraphs, therefore by the lemma 4.2.7, we get that $\Gamma$ is not connected. □

If we want to come up with an example of a profinite graph that is connected, but not superpath-connected, we will need to find a graph with infinitely many path-connected components. To construct such a graph, we will make use of limits of profinite graphs. We will first however need the proposition that follows:

**Proposition 4.2.9** (Limits of connected graphs are connected)**.** *Let* $((\Gamma_i)_{i \in I}, (f_{i,j})_{j \geq i \in I})$ *be a projective system of profinite connected graphs, then if the limit* $\Gamma$ *of this projective system has all of its natural projections surjective, then it is connected.*

*Proof.* Let $g$ be a surjective continuous qmorphism from $\Gamma$ to some finite graph $A$. We need to show that $A$ is path-connected. Write for $a \in A$, $U_a = g^{-1}(\{a\})$, which is a clopen set in $\Gamma$. We will proceed as follows:

- Prove that there exists $i \in I$, such that for every $a, a' \in A$, if $a \neq a'$, then $\phi_i(U_a)$ and $\phi_i(U_{a'})$ are disjoint.

- Prove then that the map $\tilde{g} = \begin{cases} \Gamma_i \longrightarrow A \\ x \mapsto g(y) \text{ if } \phi_i(y) = x \end{cases}$ is well defined and continuous.
- Prove that the map $\tilde{g}$ is a qmorphism.
- Conclude using $\tilde{g}$ that $A$ is path-connected.

Let us fix $a \in A$, and write $U = U_a$. Take $x \in U$. First notice that

$$\bigcap_{i \in I} f_i^{-1}(\{f_i(x)\}) \cap U^{\mathsf{c}} = \emptyset$$

Indeed if there was a $y \in U^{\mathsf{c}}$, such that $\forall i \in I, f_i(x) = f_i(y)$, then $x = y$, which is impossible since $U^{\mathsf{c}} \cap U = \emptyset$. The map $f_i$ is continuous, so $f_i^{-1}(\{f_i(x)\})$ is closed. By compactness of $U^{\mathsf{c}}$, there exists then an $i_x$, such that $f_{i_x}^{-1}(\{x\}) \cap U^{\mathsf{c}} = \emptyset$. Since $\Gamma_{i_x}$ is a profinite graph, if we denote $\Omega_x$ the set of clopen neighborhoods of $f_{i_x}(x)$ in $\Gamma_{i_x}$, we get that $\bigcap\limits_{V \in \Omega_x} V = \{f_{i_x}(x)\}$. As such, we end up with:

$$\bigcap_{V \in \Omega_x} f_{i_x}^{-1}(V) \cap U^{\mathsf{c}} = \emptyset$$

Since all the $V \in \Omega_x$ are closed, by compactness there exists a $V_x \in \Omega_x$, such that $f_{i_x}^{-1}(V_x) \cap U^{\mathsf{c}} = \emptyset$. The sets $f_{i_x}^{-1}(V_x)(x \in U)$ form an open cover of the compact set $U$, therefore there exist $x_1, \cdots, x_n \in U$, such that $\bigcup\limits_{k=1}^{n} f_{i_{x_k}}^{-1}(V_{x_k}) = U$. Now we take $i_a$ an upper bound of the $i_k$. Let us show that for $i \geq i_a$, $f_i(U) \cap f_i(U^{\mathsf{c}}) = \emptyset$. By contradiction, assume that there exists $x \in U$ and $y \in U^{\mathsf{c}}$, such that $f_i(x) = f_i(y)$. Since $x \in U$, then there exists $k$, such that $x \in f_{i_k}^{-1}(f_{i_k}(V_{x_k}))$. Using the transition map $f_{i,i_k}$, we deduce then that $f_{i_{x_k}}(x) = f_{i_{x_k}}(y)$, which proves that $y \in f_{x_k}^{-1}(V_{x_k})$ which is a contradiction. Finally just take $i$ an upper bound of all the $i_a$ and it is clear that

$$\forall a, a' \in A, a \neq a' \Rightarrow f_i(U_a) \cap f_i(U_{a'}) = \emptyset$$

Now define

$$\tilde{g} = \begin{cases} \Gamma_i \longrightarrow A \\ x \mapsto g(y) \text{ if } x = f_i(y) \end{cases}$$

Let us show that $\tilde{g}$ is well defined and is continuous.
First it is well defined, because if $f_i(y) = f_i(y')$, then $U_{g(y)} = U_{g(y')}$ and therefore $g(y) = g(y')$. Now to prove that $\tilde{g}$ is continuous, it is enough to show that the inverse image of singletons is closed, since $A$ is finite. We have $\tilde{g}^{-1}(\{a\}) = f_i(U_a)$. We know that $U_a$ is closed, $\Gamma$ is compact and $f_i$ is continuous, therefore $g^{-1}(\{a\})$ is closed and $\tilde{g}$ is continuous.

The next step is to prove that $\tilde{g}$ is a qmorphism. If we take $y \in \Gamma_i$, we take $x \in \Gamma$, such that $f_i(x) = y$. In that case if $d$ is an incidence map, we get that $f_i(d(x)) = d(y)$, since $f_i$ is a qmorphism. In this case, we get that $\tilde{g}(d(y)) = g(d(x)) = d(g(x))$, since

$g$ is a qmorphism. We therefore conclude that $\tilde{g}(d(y)) = d(\tilde{g}(y))$ and that $\tilde{g}$ is a qmorphism.

Now to conclude: we know that $\Gamma_i$ is connected and $\tilde{g}$ is trivially a surjective map, therefore $A$ is path-connected, proving that $\Gamma$ is connected. □

A corollary of this statement is that a profinite graph is connected if and only if it is a limit of a projective system of path-connected graphs with surjective transition maps.

**Example 4.2.10** (A connected graph that is not super-path-connected)**.** The rough idea behind this proof is taking the case that makes path-connectedness fail: i.e a limit of paths and adapt it to superpaths instead. Basically instead of taking limits of paths, we will take limits of superpaths and given that our category is stable by limits, it will give us a valid counterexample.

Let $P_n$ for $n \in \mathbb{N}$, be the path from $-n$ to $n$, i.e the graph with

$$V(P_n) = \{k \in \mathbb{Z} \,||k| \leq n\}$$

and

$$E(P_n) = \{(k, k+1) | k \in \mathbb{Z}, -n \leq k < n\}$$

Take $p_{m,n}$, for $n \geq m$ qmorphisms from $P_n$ to $P_m$ defined by the formula:

$$p_{m,n} = \begin{cases} P_n \longrightarrow P_m \\ x \mapsto \begin{cases} m \text{ if } o(x) \geq m \\ -m \text{ if } t(x) \leq -m \\ x \text{ else} \end{cases} \end{cases}$$

Take $Z$ to be the limit of the projective system $((P_n)_{n\in\mathbb{N}}, (p_{m,n})_{n\geq m})$. As a reminder, $Z$ looks like:

$$-\infty \qquad \cdots \to -2 \xrightarrow{e_{-2}} -1 \xrightarrow{e_{-1}} 0 \xrightarrow{e_0} 1 \xrightarrow{e_1} 2 \to \cdots \qquad +\infty$$

Now construct $Z_n$ as follows: Define $A_n = \coprod_{k=0}^{n} Z$. A finite disjoint union of profinite graphs is a profinite graph. Now take $R_n$ the relation on $A_n$ with

$$R_n = \{((+\infty, k), (-\infty, k+1)) | 0 \leq k < n\} \cup \{((-\infty, k+1), (+\infty, k)) | 0 \leq k < n\} \cup \Delta_n$$

where $\Delta_n$ is simply the diagonal of $A_n$. The relation $R_n$ basically glues the endpoints of paths together creating one big superpath. The relation $R_n$ is an equivalence relation on $Z_n$. It is closed, as a union of three closed subsets of $A_n^2$. It is compatible with the graph structure, because it only identifies the infinities, which are isolated vertices. We therefore have that $Z_n = {A_n}/{R_n}$ is a profinite graph.

$Z_n$ has finitely many path-connected components: the

$$C_{2i} = \{(a, i) | a \neq \pm\infty\}$$

with $i < n$,

$$C_{2i+1} = \{\overline{(+\infty, i)}^{R_n}\}$$

again with $i < n$ and

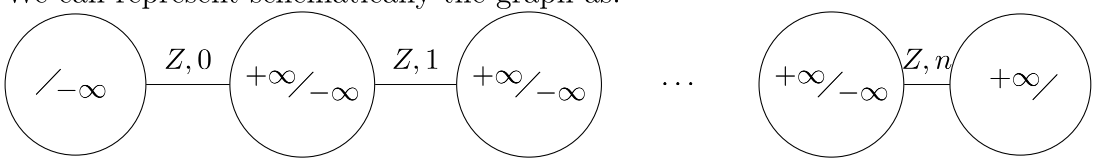
$$C_{-1} = \{(-\infty, 0)\}$$

and

$$C_{2n+1} = \{(+\infty, n)\}$$

The graph $Z_n$ is super-path-connected, since the closure of $C_{2i}$ and $C_{2i+1}$ have a non empty intersection ( they both contain $\overline{(+\infty, i)} = \overline{(-\infty, i+1)}$ ). Therefore we get that $Z_n$ is connected.

We can represent schematically the graph as:




With the circles being in the closure of the path-connected $Z, i$ ($i$-th copy of $Z$ in the disjoint union).

Now define for $n \geq m$ the maps:

$$f_{m,n} = \begin{cases} Z_n \longrightarrow Z_m \\ (x, k) \mapsto \begin{cases} (+\infty, k) \text{ if } k > m \\ (x, k) \text{ else} \end{cases} \end{cases}$$

The maps are well defined, since the $-\infty$ and $+\infty$ that get identified get sent by the maps to the same point. Let us prove that these maps are continuous qmorphisms. Consider $g_{m,n}$ a map from $A_n$ to $A_m$ defined with the same formula as $f_{n,m}$, i.e:

$$g_{m,n} = \begin{cases} A_n \longrightarrow A_m \\ (x, k) \mapsto \begin{cases} (x, k) \text{ if } k > m \\ (x, k) \text{ else} \end{cases} \end{cases}$$

$g_{m,n}$ is continuous for every $n \geq m$, because $A_n$ is a disjoint union of open subsets $Z$, upon which $g_{m,n}$ either behaves like identity or is constant. The map $g_{m,n}$ then composed with the natural projection $\pi_n$ on $Z_n$ is continuous as well. Finally $g_{m,n}$ is constant on the equivalence classes of the relation $R_n$ on $A_n$, so factors into a unique continuous map from $Z_n$ to $Z_m$ and that map is $f_{m,n}$. The map $f_{m,n}$ is therefore a morphism of profinite graphs.

Now consider $\hat{Z} = \varprojlim_{n \in N} Z_n$. It is a profinite connected graph as a limit of profinite connected graphs with surjective transition maps. Now define $\infty$ as the vertex $((+\infty, n))_{n\in\mathbb{N}}$. It has no edges coming into it or from it. Indeed take for a $d \in \{o, t\}$, a sequence $e = ((u_n, n))_{n\in\mathbb{N}}$, such that $d(e) = ((+\infty, n)_{n\in\mathbb{N}})$. In that case we get $\forall n \in \mathbb{N}, (d(u_n), n) = (+\infty, n)$. However the vertices $(+\infty, n)$ are all isolated, so that proves that $u_n = (+\infty, n)$, so $e = \infty$, proving that $\infty$ is isolated.

Now we will prove that there exists no superpath between $u = ((0, 0))_{n\in\mathbb{N}}$ and $\infty$.

We will prove it in the following steps:

- Prove that for every vertex $x \in \hat{Z}$, there exists $m \in \mathbb{N} \cup \{\infty\}$, such that

$$\forall n \geq m, x_n = x_m \text{ and that } \forall n < m, x_n = (\infty, m) \text{ and } x_m \in Z_m$$

- Prove that the path-connected components of the vertices

$$(u_{m,n})_{n\in\mathbb{N}} = \begin{cases} (+\infty, n) \text{ if } n < m \\ (0, m) \text{ else} \end{cases}$$

  are the vertices

$$u(k)_{n,m} = \begin{cases} (+\infty, n) \text{ if } n < m \\ (k, m) \text{ else} \end{cases}$$

  together with edges between these vertices.

- Prove that if we take $C_{2m}$ a path-connected component of $(u_{m,n})_{n\in\mathbb{N}}$, then the closure of $C_{2m}$ is the set $C_{2m} \cup \{(v_{m,n})_{n\in\mathbb{N}}, (v_{m+1,n})_{n\in\mathbb{N}}\}$, with

$$(v_{m,n})_{n\in\mathbb{N}} = \begin{cases} (+\infty, n) \text{ if } n < m \\ (-\infty, m) \text{ else} \end{cases}$$

- Prove that $(v_{m,n})_{n\in\mathbb{N}}$ isn't in the closure of any other path-connected component besides its own, $C_{2m}$ and $C_{2m+2}$.

- Conclude.

Suppose $((x_n, y_n))_{n\in\mathbb{N}}$ is a vertex in $\hat{Z}$. We shall prove that if $y$ is not a bounded sequence, then $x_n$ will be equal to infinity and $y_n$ to $n$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Since $y$ is not bounded, there exists $n' > n$, such that $y_{n'} > n \geq y_n$. In that case, we get that $f_{n',n}(x_{n'}, y_{n'}) = (+\infty, n) = (x_n, y_n)$. If that happens, we simply put $m = \infty$.

Therefore we may now assume that $y$ is bounded. In this case, it has a maximal value $m$. If we take now an $n \in \mathbb{N}$, such that $y_n = m$.

Observe that $f_{n,m}((x_n, y_n)) = \begin{cases} (+\infty, m) \text{ if n>m} \\ (y_m, m) \text{ if n=m} \end{cases}$

This proves that $y_m = m$. Let us observe that for all $n \geq m$, $y_n = m$ and $x_n = m$. Indeed if $n \geq m$, then $m \geq y_n$ and so we have that $f_{n,m}(x_n, y_n) = (x_n, y_n)$. As such we obtain that $y_n = m$. $m$ is also the least integer for which $y$ attains the value $m$.

Now we need to prove that for a vertex of the form $u_m$, the path-connected component of $u_m$ is exactly the set $\{u(k)_m | k \in \mathbb{Z}\}$ together with the edges connected to it. We simply denote

$u(k)_m = (u(k)_{m,n})_{n\in N}$ and $v_m = (v_{m,n})_{n\in\mathbb{N}}$.

First we start by proving that if $e \in \hat{Z}$ and $o(e) = u(k)_m$, then either $e = u(k)_m$, or $t(e) = u(k+1)_m$. Assume that $e \neq u(k)_m$. Then for all $n_0 \in \mathbb{N}$, there exists a $n > n_0$, such that $e_n \neq u(k)_{m,n}$. By contradiction now assume that there exists $n_0 \in \mathbb{N}$, such that $t(e_n) \neq u(k+1)_{m,n}$. Take $n > \max\{n_0, m\}$, such that $e_n \neq u(k)_{m,n}$. Since $o(e_n) = u(k)_{m,n} = (k, m)$, we know that $e_n$ is an edge in $Z$. There is however only one edge in $Z$ coming out of $k$ and that is the edge with terminus at $k+1$, hence $t(e_n) = (k+1, m) = u(k+1)_{m,n}$. Now if we apply the transition map $f_{n,n_0}$, we get that $t(e_{n_0}) = u(k+1)_{m,n_0}$ which is our contradiction. Using the same method, one can prove that if $e$ is such that $t(e) = u(k)_m$, then either $e = u(k)_m$, or $o(e) = u(k-1)_m$. From this it follows that $C_{2m} = \{x \in \hat{Z} | \exists k \in \mathbb{Z}, o(x) = u(k)_m\}$, i.e the set of all the $u(k)_m$ together with edges connected to $u(k)_m$.

Now we need to prove that $\overline{C_{2m}} = C_{2m} \cup \{v_m, v_{m+1}\}$. Let $\pi_n$ be the natural projection of $\hat{Z}$ onto $Z_n$. Let us show first that $C_{2m} \cup \{v_m, v_{m+1}\}$ is closed. Consider $F_n = \{(x, m) \in Z_n\}$. It is basically the set of elements of the m th component in $Z_n$ seen as the union $\coprod_{k=0}^{n} Z$. Let us show that $F_n$ is closed in $Z_n$. For that we write again $Z_n$ as the quotient ${A_n}/{R_n}$. One can see that the preimage of $F_n$ by the natural projection on the quotient is the closed component $m$-th $Z$ in $A_n$ together with the closed set $\{(+\infty, m-1), (-\infty, m+1)\}$. As such, since the preimage of $F_n$ is closed, $F_n$ itself is closed. Now let us show that $C_{2m} \cup \{v_m, v_n\} = \bigcap_{n \geq m} \pi_n^{-1}(F_n)$.

Let $x \in \bigcap_{n \geq m} \pi_n^{-1}(F_n)$. Then $x_n = (a_n, m)$. The sequence $a_n$ however has to be constant, so $x_n = (a, m)$ for all $n \geq m$. In that case we can observe that $x$ is either an edge connecting to some $u(k)_m$ or some $u(k)_m$ or one of the infinities. That concludes the proof that $C_{2m} \cup \{v_m, v_{m+1}\}$ is closed.

To prove now that $\overline{C_{2m}}$ is equal to that set, all we need to prove is that both $v_m$ and $v_{m+1}$ are in its closure. Notice that $\pi_m$ is a bijection between $C_{2m} \cup \{v_m, v_{m+1}\}$ and $F_m$.We already know that $\pi_m$ is surjective. The reason it is injective is simply, because if $m$ th components are equal, all components above will be simply by what we have proved earlier and the components below are always equal by applying the transition maps $f_{m,n}$. The map $\pi_m$ is therefore a homeomorphism between $\overline{C_{2m} \cup \{v_m, v_{m+1}\}}$ and $F_m$. Since $(-\infty, m)$ and $(+\infty, m)$ are in the closure of $\pi_m(C_{2m})$, then $v_m$ and $v_{m+1}$ will be in the closure of $C_{2m}$ in the closed subset $C_{2m} \cup \{v_m, v_{m+1}\}$. Since the subset is closed, we get indeed that $\overline{C_{2m}} = C_{2m} \cup \{v_m, v_{m+1}\}$

Now we will need to prove that the only path-connected components such that $v_m$ is in their closure are $C_{2m}, \{v_m\}, C_{2m+2}$. To prove this, we now define $C_{2m+1}$ as $\{v_m\}$ and we shall prove that $Z \setminus \infty = \bigcup_{n \in \mathbb{N}} C_n$. We start by proving that every vertex is in such a component. As we have proved earlier, there exists $m \in \mathbb{N}$, such that $\forall n \geq m$, $\pi_n(x) = \pi_m(x)$ and $\forall n < m$, $\pi_n(x) = (\infty, n)$. If we then write $x_m = (a, m)$, we get that either $x_m = u(a)_m$, if $a \in \mathbb{Z}$ or $x_m \in \{v_m, v_{m+1}\}$. This proves that $x_m \in C_{2m-1} \cup C_{2m} \cup C_{2m+1}$. Now we know that every vertex is in one of the components and since they are path-connected, every edge must be in one of them as well.

Now we can finally write the conclusion. By contradiction, assume that there exists a sequence of path-connected components $X_0, X_1, \cdots, X_l$, with $u_0 \in X_0$ and $\infty \in X_l$ and $\overline{X_k} \cap \overline{X_{k+1}} \neq \emptyset$. Without loss of generality, we may assume that this sequence is without repetition. In that case $X_{l-1} \neq \{\infty\}$ We then have to have that there exists $m \in \mathbb{N}$, such that $X_{l-1} = C_m$. However we know that $\infty \notin \overline{C_m}$, so we have a contradiction.

Now we have proven superpath-connectedness is not always enough to prove connectedness. There is however still a topological notion that explains what connectedness is. As seen in the 4.2.7, if there exists a non constant qmorphism into the discrete space $\{0, 1\}$: that seems close to one possible definition of connectedness in general topological spaces, where a space $X$ is connected if and only if every continuous map from $X$ to $\{0, 1\}$ is constant. This indicates there could be some topology for which a profinite graph is connected in the sense of profinite graphs if an only if it is connected in the sense of this topology. It cannot be the profinite topology since we know that profinite topologies are totally disconnected, which is the exact opposite of what we want. We will however show that there is a coarser topology on a profinite graph for which the connected graphs are exactly those connected for that topology.

**Proposition 4.2.11.** *Let $\Gamma$ be a profinite graph. We have that $\Gamma$ is connected if and only if every continuous qmorphism from $\Gamma$ into the discrete graph with vertices $\{0,1\}$ and no edges is constant.*

*Proof.* First suppose that $\Gamma$ is connected. Then if $f$ is a qmorphism into $\{0,1\}$ then its image has to be path-connected. As such, $f$ has to be constant, because if it were not its image would have been the whole $\{0,1\}$, which is not path-connected.

Now suppose on the other hand that $\Gamma$ is not connected. Let us construct a non constant continuous qmorphism from $\Gamma$ to $\{0,1\}$. Since $\Gamma$ is not connected, there exists a qmorphism $f$ from $\Gamma$ to some finite graph $X$, such that $im(f)$ is not path-connected. Let $C$ then be one of the path-connected component of $im(f)$. Let $u$ be a map from $im(f)$ to $\{0,1\}$ defined by

$$u(x) = \begin{cases} 0 \text{ if } x \in C \\ 1 \text{ else} \end{cases}$$

$u$ is qmorphism: indeed if we take $x \in C$, then $(o(x), x, t(x))$ is a path, so $o(x), t(x)$ and $x$ are all in $C$, therefore get sent to 0 by $u$. If $x$ isn't in $C$, then none of the $t(x), o(x)$ and $x$ can be in $C$, therefore $u(x) = u(o(x)) = u(t(x)) = 1$. It is automatically continuous, since for discrete graphs, all maps automatically are. Then $u \circ f$ is a non constant continuous qmorphism from $\Gamma$ to $\{0,1\}$. □

Again let us reiterate that this result is strongly analogous to a possible definition of connectedness. Now we will finally define the topology that is compatible with our notion of connectedness.

**Definition 4.2.12** (Subgraph)**.** *Let $(\Gamma, V, t, o)$ be an abstract graph. A subset $A$ of $\Gamma$ is called a subgraph, if $\forall d \in \{t, o\}$, $d(A) \subseteq A$.*

We observe that for a subgraph $A$, $(A, A \cap V, t_{|A}, o_{|A})$ is a graph.

**Proposition 4.2.13.** *Let $\Gamma$ be a profinite graph. The following statements are true:*

*i. The set $\tau$ of all open subgraphs of $\Gamma$ forms a topology on $\Gamma$.*

*ii. The functions $o, t$ and all continuous qmorphisms for the profinite topology are continuous for the topology given by $\tau$.*

*iii. $\Gamma$ is connected in the sense of profinite graphs if and only if it is connected for the topology given by $\tau$.*

*Proof.* i. Let $\Gamma_1$ and $\Gamma_2$ be two open subgraphs. The set $\Gamma_1 \cap \Gamma_2$ is an open set for the profinite topology. Now let us prove it is a subgraph. If $d \in \{o, t\}$ is one of the incidence maps and $x \in \Gamma_1 \cap \Gamma_2$, then since $\Gamma_1$ and $\Gamma_2$ are subgraphs, $d(x)$ has to be by definition in both of them.

Let $\Gamma_i$ be a collection of subgraphs. Let us define $\Gamma'$ to be their union. The set $\Gamma'$ is open in the profinite topology as a union of opens. To prove that $\Gamma'$ is a subgraph, we again take $d \in \{o, t\}$, Then if $x \in \Gamma'$, there exists $i$ such that $x \in \Gamma_i$, therefore $d(x) \in \Gamma_i \subseteq \Gamma'$.

The empty set is an open subgraph, since $\forall x \in \emptyset, d(x) \in \emptyset$ is vacuously true and $\Gamma$ is trivially an open of $\Gamma$. This concludes the proof that $\tau$ is a set of opens.

ii. Let $\Gamma'$ be an open subgraph of $\Gamma$ and $d \in \{o, t\}$. Let us prove that $d^{-1}(\Gamma')$ is a subgraph as well. If $x \in \Gamma$, such that $d(x) \in \Gamma'$, then we have $d(o(x)) = d(t(x)) = d(x)$, therefore both $o(x)$ and $t(x)$ are in $\Gamma'$, proving the continuity of the map $d$.

Now let $f$ be a continuous qmorphism from a profinite graph $\Gamma$ to a profinite graph $\Gamma'$. Let $U$ be an open subgraph of $\Gamma'$. Let us show that $f^{-1}(U)$ is an open subgraph of $\Gamma$. Let $d$ be again one of the two incidence maps. Let $x \in f^{-1}(U)$. Since $U$ is a subgraph of $\Gamma$, we have that $d(f(x)) \in U$. Since $f$ is a qmorphism, we get that $d(f(x)) = f(d(x))$, so $d(x) \in f^{-1}(U)$, concluding our proof that qmorphisms which are continuous for the profinite topology are continuous for the open-subgraph topology.

iii. Suppose first that $\Gamma$ is not connected for the open subgraph topology. Then $\Gamma$ is a disjoint union of open subgraphs, so by lemma 4.2.7, $\Gamma$ is not connected as a profinite graph. Now by contraposition suppose that $\Gamma$ is not connected in the profinite sense. In that case by the proposition 4.2.9 , there exists a non constant qmorphism from $\Gamma$ to $\{0, 1\}$. By continuity of $f$ in sense of the open subgraph topology, we conclude that $\Gamma$ is not connected for that topology, since we found a non constant continuous map from $\Gamma$ to the discrete space $\{0, 1\}$.

□

Let us now briefly revisit the example in Ribes [41](2.1.8). Now that we know in this light that the profinite connectedness is in fact a topological notion of connectedness, we can see that the compactification of the path $P$ in $\mathbb{N}$ is a close analogue to the famous example in $\mathbb{R}^2$: with the curve $(x, \sin(\frac{1}{x})), x \in (0, 1]$ in $\mathbb{R}$, The curve is path-connected, just like the path $P$ is, but its closure in a bigger space is not connected. The details for this counter example can for example be seen in the following topology textbook: [32] page 141.

## 4.3 Chain complex of profinite graphs

As we have seen in the section 2.4, a graph chain complex is useful for finding properties of a graph: mainly whether the graph is connected and whether is is a tree. We will see that it is the case for profinite graphs as well, but we need to change the notion of a complex to adapt it to the profinite structure. Instead of working with complexes over rings, we will work with complexes over profinite rings. Another difference to resolve is that in the case of abstract graphs, we take a sequence $R[E(\Gamma)] \to R[V(\Gamma)] \to R$. The problem with the case of profinite graphs is that $E(\Gamma)$ is not necessarily a profinite space. It is when $E(\Gamma)$ is closed in which case we can do the same theory, but with profinite topology added on top. If not what we do is by factoring by the set of vertices $V(\Gamma)$ (which is always closed for a profinite graph), we collapse all vertices onto a single point. That point has to be zero for the complex to make sense, which is why we have the definition that follows:

**Definition 4.3.1.** *Let $\Gamma$ be a profinite graph. Consider the pointed profinite space $({}^{\Gamma}\!/_{V(\Gamma)}, *)$, with $*$ being the class $V(\Gamma)$ in the quotient. Take $R$ to be a profinite ring. We take $\partial$ the continuous map from $({}^{\Gamma}\!/_{V(\Gamma)}, *)$ to $R[[V(\Gamma)]]$ the free profinite module over $V(\Gamma)$ that to $\overline{x} \in {}^{\Gamma}\!/_{V(\Gamma)}$ associates $t(x) - o(x)$. It sends $*$ onto $0$ and therefore can be extended to a linear continuous map from the free pointed module $R[[{}^{\Gamma}\!/_{V(\Gamma)}, *]]$ to $R[[V(\Gamma)]]$, which we shall call $\partial$ as well. Now consider $\epsilon$ the continuous map from $V(\Gamma)$ to $R$ constant and equal to $1$. It can be extended to a continuous $R$-linear map from the free module $R[[V(\Gamma)]]$ into $R$, which we shall call the augmentation map and denote it $\epsilon$ as well. We observe that $\epsilon \circ \partial = 0$ and therefore we can consider the complex:*

$$R[[{}^{\Gamma}\!/_{V(\Gamma)}, *]] \xrightarrow{\quad \partial \quad} R[[V(\Gamma)]] \xrightarrow{\epsilon} R \longrightarrow 0$$

*We call it the chain complex of $\Gamma$ with coefficients in $R$.*

*We call the zeroth homology the profinite $R$ module $H_0(\Gamma, R) = {}^{\ker(\epsilon)}\!/_{im(\partial)}$ and the first homology the profinite $R$-module: $H_1(\Gamma, R) = \ker(\partial)$.*

We can do a more classical homology without pointed spaces if we consider $E(\Gamma)$ closed we shall now prove that if that is the case the two give isomorphic chain complexes.

**Proposition 4.3.2.** *Let $\Gamma$ be a profinite graph. Let*

$$R[[{}^{\Gamma}\!/_{V(\Gamma)}, *]] \xrightarrow{\quad \partial \quad} R[[V(\Gamma)]] \xrightarrow{\epsilon} R \longrightarrow 0$$

*be its complex. Suppose that $E(\Gamma)$ is closed in $\Gamma$. Now consider the continuous map $\partial'$ from $E(\Gamma)$ to $V(\Gamma)$ that to $e$ associates $t(e) - o(e)$. We extend it into a continuous $R$-linear map from $R[[E(\Gamma)]]$ to $R[[V(\Gamma)]]$, which we shall still call $\partial'$. Then there exists an isomorphism $u$ from $R[[{}^{\Gamma}\!/_{V(\Gamma)}, *]]$ to $R[[E(\Gamma)]]$, such that the diagram:*

$$\begin{array}{ccccc}
R[[\Gamma/V(\Gamma), *]] & \xrightarrow{\partial} & R[[V(\Gamma)]] & \xrightarrow{\epsilon} & R \longrightarrow 0 \\
\downarrow u & & \downarrow id & & \downarrow id \\
R[[E(\Gamma)]] & \xrightarrow{\partial'} & R[[V(\Gamma)]] & \xrightarrow{\epsilon} & R \longrightarrow 0
\end{array}$$

*commutes. In particular the chain complexes are isomorphic.*

*Proof.* Consider the map $u$ from $\Gamma$ into $R[[E(\Gamma)]]$ that sends any element of $V(\Gamma)$ onto 0 and an element of $E(\Gamma)$ onto itself. The map $u$ is continuous on both $V(\Gamma)$ and $E(\Gamma)$, which are disjoint opens in $\Gamma$, therefore it is continuous. Furthermore it is constant on $V(\Gamma)$, therefore it can be factored into a continuous map which we still shall call $u$ from $\Gamma/V(\Gamma)$ to $R[[E(\Gamma)]]$. We have that $u(*) = 0$ and therefore $u$ can be extended to a unique $R$-linear continuous map from $R[[\Gamma/V(\Gamma), *]]$ to $R[[E(\Gamma)]]$. To prove that $u$ is an isomorphism of profinite $R$-modules, we shall find an explicit inverse. Simply consider $v$ the continuous map from $E(\Gamma)$ to $\Gamma/V(\Gamma)$ that sends an edge into its class in the quotient. It can be extended to a $R$-linear map from $R[[E(\Gamma)]]$ to $R[[\Gamma/V(\Gamma), *]]$. The maps $u$ and $v$ restricted to the bases are inverse to each other, therefore they are inverses to each other as maps of modules. Finally let us prove that $\partial' \circ u = \partial$. Let $x \in \Gamma/V(\Gamma)$. Then

$$\partial'(u(x)) = \partial'(0) = 0 = \partial(u(x)), \text{ if } x \text{ is the class } V(\Gamma)$$

If $x \in E(\Gamma)$, then $\partial'(u(x)) = t(x) - o(x) = \partial(x)$. Since $\partial' \circ u$ and $\partial$ are equal on the basis of $R[[\Gamma/V(\Gamma), *]]$, then they are equal everywhere which concludes the proof.

Since the chain complexes are isomorphic, so are the homologies. □

Now we will prove that the notion of connectedness is characterized by the homology.

**Lemma 4.3.3.** *Let $\alpha$ be a $q$ morphism of profinite graphs $\Gamma, \Gamma'$ and $R$ a profinite ring.*

*a) If $\alpha$ is surjective, then $H_0(\alpha, R)$, the induced morphism on the zeroth homology is surjective.*

*b) If $\alpha$ is injective, then $H_1(\alpha, R)$ is injective.*

*c) If $\Gamma$ is a projective limit of a projective system of profinite graphs $\Gamma_i$, then $H_0(\Gamma, R)$ is a projective limit of the $H_0(\Gamma_i, R)$ and $H_1(\Gamma, R)$ is the projective limit of the $H_1(\Gamma_i, R)$.*

*Proof.* a) Suppose that $\alpha$ from $\Gamma$ to $\Gamma'$ is surjective. By the lemma 3.3.3, if $\epsilon'$ is the augmentation map of $R[[V(\Gamma')]]$, then $R[V(\Gamma')] \cap \ker(\epsilon')$ is dense in $R[[V(\Gamma')]]$. The module $R[V(\Gamma')] \cap \ker(\epsilon')$ is generated by the $x - y$, with $x, y \in V(\Gamma')$. The linear map $\tilde{h}$ from $\ker(\epsilon)$ to $\ker(\epsilon')$ then is surjective, since $im(\tilde{h})$ is closed in $\ker(\epsilon')$ and contains all the $x - y$ , with $x, y \in V(\Gamma')$ by surjectivity of $\alpha$. We then get that $H_0(\alpha)$ is surjective.

b) Now suppose that $\alpha$ is injective. By 3.3.4, the linear map induced by $\alpha$ from $R[[{}^{\Gamma}\!/_{V(\Gamma)}, *]]$ to $R[[{}^{\Gamma'}\!/_{V(\Gamma')}, *]]$ is injective and therefore $H_1(\alpha)$ is injective as well.

c) Let $(p_i)_{i\in I}$ be the natural projections of $\Gamma$ onto $\Gamma_i$ with $\Gamma$ seen as a projective limit of the $\Gamma_i$. Let $\partial_i$ be the boundary map from $R[[{}^{\Gamma_i}\!/_{V(\Gamma_i)}, *]]$ to $R[[V(\Gamma_i)]]$ and $\epsilon_i$ the augmentation map on $R[[V(\Gamma_i)]]$. Furthermore let $\partial$ be the boundary map of $\Gamma$ and $\epsilon$ the augmentation map from $R[[V(\Gamma)]]$. One can check that:

$$\forall i \in I, p_i\partial = \partial_i p_i$$

and that:

$$\forall i \in I, \epsilon_i p_i = \epsilon$$

With $p_i$ denoting the appropriate induced linear map. Now consider the unique isomorphism $f$ from $R[[{}^{\Gamma}\!/_{V(\Gamma)}, *]]$ to the limit $L = \varprojlim_{i\in I} R[[{}^{\Gamma}\!/_{V(\Gamma_i)}, *]]$, such that $\pi_i \circ f = p_i$, with $\pi_i$ being the natural projection of $L$ onto $R[[{}^{\Gamma_i}\!/_{V(\Gamma_i)}, *]]$. Let us show that $f$ is an isomorphism from $\ker(\partial)$ to $\{(x_i)_{i\in I} \in L | \forall i \in I, \partial_i(x_i) = 0\} = \lim_{i\in I} \ker(\partial_i)$. The map $f$ is injective as a restriction of an injective map. To prove that $f$ is surjective, consider $(y_i)_{i\in I}$ a collection in $\lim_{i\in I} \ker(\partial_i)$. Since it is a collection in $L$, then there exists $x \in R[[{}^{\Gamma}\!/_{V(\Gamma_i)}, *]]$, such that $f(x) = (y_i)_{i\in I}$. Let us show that $\partial(x) = 0$. For that, it is enough to prove that $\forall i \in I$, $p_i(\partial(y)) = 0$. If we take $i \in I$, we obtain that $p_i(\partial(y)) = \partial_i p_i(y) = \partial_i(x_i) = 0$. Since $f$ is an isomorphism and $\ker(\partial_i)$ is by definition $H_1(\Gamma_i, R)$, then $H_1(\Gamma, R)$ and $\lim_{i\in I} H_1(\Gamma_i, R)$ are isomorphic.

Let us prove the same result for $H_0$. Consider $g$ the unique isomorphism of projective limits from $R[[V(\Gamma)]]$ to $L' = \varprojlim_{i\in I} R[V(\Gamma_i)]$. Now restrict $g$ onto $\ker(\epsilon)$ and compose it with natural projections into quotients to get a map $\tilde{g}$ from $\ker(\epsilon)$ to $\varprojlim_{i\in I} H_0(\Gamma_i, R)$. The map $\tilde{g}$ is surjective for the same reason the map $f$ was. Let us show that its kernel is exactly $im(\partial)$. To do that suppose that $g(y) = (\partial_i(p_i(x_i)))$.

Now by contradiction assume that there is no $x \in R[[{}^{\Gamma}\!/_{V(\Gamma)}]]$, such that $y = \partial(x)$. In that case:

$$\bigcap_{i\in I} (p_i \circ \partial)^{-1}(\{p_i(y)\}) = \emptyset$$

Indeed if there was an $x$ in that set, then for all $i \in I$, $p_i(\partial(x)) = p_i(y)$ and therefore $\partial(x) = y$. Using the continuity of the map $p_i \circ \partial$ and compactness, we then get that there exist $i_1, \cdots, i_n \in I$, such that:

$$\bigcap_{k=1}^{n} (p_{i_k} \circ \partial)^{-1}(\{p_{i_k}(y)\}) = \emptyset$$

Now consider $j$ an upper bound of $\{i_1, \cdots, i_n\}$ and $\phi_{i_k,j}$ the transition map from $\Gamma_j$ to $\Gamma_{i_k}$. In that case:

$$p_{i_k}\partial(x_j) = \phi_{i_k,j}(p_j(\partial(x_j))) = \phi_{i_k,j}(\partial_j(p_j(x_j))) = \phi_{i_k,j}(p_j(y)) = p_{i_k}(y)$$

Since this is true for an arbitrary $k$, we get that $x_j \in \bigcap_{k=1}^{n} (p_{i_k} \circ \delta)^{-1}(\{p_{i_k}(y)\})$, which is a contradiction. This proves that $y \in im(\partial)$, hence we conclude that $g$ is an isomorphism of $R$ modules and that

$$H_0(\Gamma, R) \cong \varprojlim_{i \in I} H_0(\Gamma_i, R)$$

□

**Proposition 4.3.4** (Connectivity criterion)**.** *A profinite graph $\Gamma$ is connected if and only if for every profinite ring $R$, $H_0(\Gamma, R) = \{0\}$.*

*Proof.* If we write $\Gamma$ as a projective limit of $\Gamma_i$ of finite graphs with the natural projections $p_i$ all surjective, then $\Gamma$ is connected if and only if all the $\Gamma_i$ are path-connected. Furthermore by the lemma if all the $p_i$ are surjective, then all the $H_0(p_i, R)$ are surjective. Hence $H_0(\Gamma, R)$ is equal to zero if and only if $H_0(\Gamma_i, R)$ is zero for all $i$. It is therefore enough to prove the theorem for the case $\Gamma$ finite.

Assume first then that $\Gamma$ is connected. We can complete $\Gamma$ into an undirected graph $\tilde{\Gamma}$(as defined in chapter 2), by formally adding inverses of the edges. The edges in $\Gamma$ will then form an orientation on $\tilde{\Gamma}$ and the homology computed in the sense of this chapter will be the same homology as defined in: 2.4. Furthermore since we assumed that $\Gamma$ is connected, then $H_0(\Gamma, \mathbb{Z}) = \{0\}$. Then by the universal coefficient theorem: if $R$ is a ring, we get an exact sequence of the form:

$$0 \longrightarrow H_0(\Gamma, \mathbb{Z}) \otimes R \longrightarrow H_0(\Gamma, R) \longrightarrow Tor_1(H_{-1}(\Gamma, \mathbb{Z}), R) \longrightarrow 0$$

We have that $H_0(\Gamma, \mathbb{Z}) = \{0\}$ and $H_{-1}(\Gamma, \mathbb{Z}) = \{0\}$, therefore $H_0(\Gamma, R) = \{0\}$. This being true independently of the ring $R$, it is true for any profinite ring as well.

Now suppose on the other hand that $H_0(\Gamma, R) = \{0\}$ is true for any profinite ring $R$. Let us show that $\Gamma$ is connected. We simply take $R = \mathbb{F}_2$, the finite field with two elements. Let $x, y \in V(\Gamma)$. Let us show that there exists a path from $x$ to $y$. Since $y - x \in im(\partial)$, there exist $e_1, \cdots, e_n \in E(\Gamma)$, such that $x - y = \partial(e_1) + \cdots + \partial(e_n)$. Just like in the proof of 2.4.3, we can prove by induction on $n$ that we can rearrange the edges into a path from $x$ to $y$. It is in fact even a little easier, since thanks to the coefficients in $\mathbb{F}_2$, we do not need to worry about signs. The graph $\Gamma$ is therefore connected and that concludes the proof. □

## 4.4 Profinite Trees

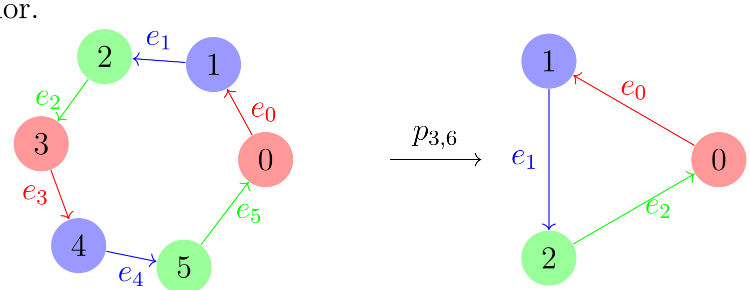
So far the approach we took for generalizing a notion from graph theory to profinite graphs, we proceeded by taking limits of finite graphs that verified the notion we wanted to generalize. Following this approach, we could define a profinite tree as a limit of finite trees. Such a definition however defies an intuition one usually has of trees: that is connected graphs with no cycles.

Take for example the following profinite graph:

For $n \in \mathbb{N} \setminus \{0\}$, we define $C_n$ to be the cycle of length $n$: i.e a tree whose vertices are: elements of $\mathbb{Z}/_{n\mathbb{Z}}$ and whose edges are $\{e_0, \cdots, e_{n-1}\}$ such that for all $k \in \mathbb{Z}/_{n\mathbb{Z}}$, $o(e_k) = k$ and $t(e_k) = k+1$. For $m$ dividing $n$, the transition map from $C_n$ to $C_m$ is the

$$p_{m,n} = \begin{cases} C_n \longrightarrow C_m \\ x \mapsto \begin{cases} \overline{x}^m \text{ if } x \in \mathbb{Z}/_{n\mathbb{Z}} \\ e_{\overline{k}^m} \text{ if } x = e_k, k \in \mathbb{Z}/_{n\mathbb{Z}} \end{cases} \end{cases}$$

The image below illustrates the transition map for the case $n = 6$ and $m = 3$. The edges and vertices of a certain color are sent to the edges or vertices of the same color.




The limit graph has no cycles, because any loop for smaller $n$ will eventually get lifted into a path without a loop for a high enough multiple of $m$. When looking for trees then, the right notion seems then to be cycles, rather than just plain limits of trees. The tool for detecting cycles in graphs is $H_1$, but we cannot simply pick it with coefficients in $\mathbb{Z}$, because that would disregard the profinite structure, so we take the coefficients in the profinite completion of $\mathbb{Z}$: $\hat{\mathbb{Z}}$ and end up with the definition that follows:

**Definition 4.4.1.** *Let $\Gamma$ be a profinite graph. We say that $\Gamma$ is a tree, if $\Gamma$ is connected and $H^1(\Gamma, \hat{\mathbb{Z}}) = \{0\}$.*

We will show that the example that we gave above is a tree in the sense of this definition. What we can observe is that profinite trees are connected and without cycles, but we will also show that not all connected profinite graphs without cycles are profinite trees.

*Remark.* There is a more precise notion of a tree. Sometimes instead of studying actions of all profinite groups, we want to restrict ourselves to only pro-$p$ groups or more generally groups with only certain primes involved. In that case we would consider $H_1$ with coefficients of certain products of $p$- completions of $\mathbb{Z}$: $\mathbb{Z}_p$. If we would denote the set primes in the product $\pi$, then the definition of a $\pi$-tree would be a connected profinite graph with $H^1(\Gamma, \prod_{p\in\pi} \mathbb{Z}_p)$. However we won't study this case in here. It is covered in more details in Ribes's book [41] section 2.4.

**Proposition 4.4.2** (properties of profinite trees)**.**

*a) A finite tree is a profinite tree.*

*b) Limits of profinite trees with surjective natural projections are profinite trees.*

*c) A profinite tree has no cycles: i.e for any path $(x, e_1, \cdots, e_n, x)$, with $e_1, \cdots, e_n \in E(G)$, there exist distinct $i, j$, such that $e_i = e_j$.*

*Proof.* a) Let $T$ be a finite tree. The graph $T$ is then by definition path-connected, so it is connected. Now consider $\partial$ to be the boundary map from $\hat{\mathbb{Z}}[[E(T)]]$ to $\hat{\mathbb{Z}}[[V(T)]]$. Since $E(T)$ is finite, we have that $\hat{\mathbb{Z}}[[E(T)]]$ is equal to the free $\hat{\mathbb{Z}}$-module $\hat{\mathbb{Z}}[E(T)]$ Therefore the homology $H_1$ is just a regular homology, without the profinite structure and we can compute it using the universal coefficient theorem. $T$ is a tree, therefore we get that $H_1(T, \mathbb{Z}) = \{0\}$. Using the universal coefficient theorem, we get the following exact sequence:

$$0 \to H_1(T, \mathbb{Z}) \otimes \hat{\mathbb{Z}} \longrightarrow H_1(T, \hat{\mathbb{Z}}) \longrightarrow Tor^1(H_0(T, \mathbb{Z}), \hat{\mathbb{Z}}) \longrightarrow 0$$

Since $T$ is a tree, we get $H_1(T, \mathbb{Z})$ and $H_0(T, \mathbb{Z})$ that are zero which leads to the conclusion that $H_1(T, \hat{\mathbb{Z}})$ is zero and therefore $T$ is a profinite tree.

b) Suppose that $T$ is a limit of trees $(T_i)_{i\in I}$ with surjective natural projections, then $T$ is connected as a limit of connected profinite graphs. Furthermore we have by the lemma 4.3.3 that $H_0(T, \hat{\mathbb{Z}})$ is isomorphic to $\lim_{i\in I} H_0(T_i, \hat{\mathbb{Z}})$, which are all zero since $T_i$s are assumed to be trees.

In particular a limit of finite trees is a profinite tree.

c) Let $p = (x, e_1, \cdots, e_n, x)$ be a path. There exist then incidence maps $d_1, \cdots, d_n \in \{o, t\}$, such that $d_1(e_1) = x$, $\overline{d_n}(e_n) = x$ and such that for all $k < n$, $\overline{d_k}(e_k) = d_{k+}(e_{k+1})$ We write for $k$ between 1 and $n$

$$\epsilon_k = \begin{cases} 1 \text{ if } d_k = o \\ -1 \text{ if } d_k = t \end{cases}$$

Since $p$ is a path from $x$ to itself, we get that $\partial(\sum_{k=1}^{n} \epsilon_k e_k) = 0$. Since $\partial$ is injective map (by assumption $T$ is a profinite tree), we get that $\sum_{k=1}^{n} \epsilon_k e_k = 0$, which implies that there exists some $i, j$ distinct such that $\epsilon_i e_i = \epsilon_j e_j$ and $e_i = e_j$. The graph $T$ is therefore without cycles.

□

A natural question is whether a connected profinite graph without cycles is a profinite tree: we will now show that it is not the case.

**Example 4.4.3** (A connected profinite graph with no cycles that is not a tree)**.** Take $p$ some prime number. For $n \in \mathbb{N}$, consider $C_n$ the cycle of length $p^n$, i.e a graph whose vertices are $\mathbb{Z}/p^n\mathbb{Z}$ and edges $e_0^n, \cdots, e_{p^n-1}^n$, with $o(e_k^n) = k$ and $t(e_k^n) = k+1$. We consider for $n \geq m$, the transition map from $C_n$ to $C_m$:

$$\pi_{m,n} = \begin{cases} C_n \longrightarrow C_m \\ x \mapsto \begin{cases} \overline{x}^{p^m} \text{ if } x \in \mathbb{Z}/p^n\mathbb{Z} \\ \overline{k}^{p^m} \text{ if } x = e_k, k \in \mathbb{Z}/p^n\mathbb{Z} \end{cases} \end{cases}$$

and take $G$ the limit of the $C_n$. The graph $G$ is then connected as a limit of connected graphs. Let us prove that $G$ has no cycles. By contradiction, assume that there is a cycle $(x, e_1, \cdots, e_a, x)$ with $e_1, \cdots, e_a$ distinct edges in $G$. Since $e_1, \cdots, e_a$ are distinct, there exists a number $m \in \mathbb{N}$, such that $\pi_m(e_1), \cdots, \pi_m(e_a)$ are all distinct (with $\pi_m$ the natural projection of $G$ on $C_m$). Now let $n$ be a natural number, such that $p^n > a$ and $n \geq m$. In that case $(\pi_n(x), \pi_n(e_1), \cdots, \pi_n(e_a), \pi_n(x))$ is a cycle in $C_n$ of length $a < p^n$, which is impossible, since the length of all cycles in $C_n$ has to be a multiple of $p^n$.

Now that we know that $G$ is without cycles, let us prove that is nevertheless not a profinite tree. In order to do that, we need to find $a \in \hat{Z}[[G/V(G), *]]$, such that $\partial(a) = 0$, but $a \neq 0$.

We know that $\hat{Z}$ can be represented as the infinite product $\prod_{q \text{ prime}} \mathbb{Z}_q$. Take $q$ a prime that is coprime with $p$. Then $q^n$ for all $n$ is coprime with $p$ and therefore $p$ is invertible modulo $q^n$. Using the axiom of choice, we can construct a sequence $(u_n)_{n\in\mathbb{N}}$, such that $pu_n \equiv 1[q^n]$ and $u_n \equiv u_{n-1}[q^{n-1}]$. This sequence gives us a number $v$ in $\mathbb{Z}_q$ and therefore in $\hat{\mathbb{Z}}$. The number $v$ has the property that $pv$ is equal to $\tilde{1}^q$, which we define as the number in $\hat{\mathbb{Z}}$ associated to the number 1 in $\mathbb{Z}_q$. Now define a sequence $v_n$ by $v_0 = \tilde{1}^q$ and $v_n = v^n$ for $n \neq 0$. Then take

$$(a_n)_{n\in\mathbb{N}} = (v_n \sum_{k=0}^{p^n-1} e_k^n)$$

Due to the property $pv_n = v_{n-1}$, we get that this sequence is compatible with transition maps from $\hat{\mathbb{Z}}[E(G_n)]$ to $\hat{\mathbb{Z}}[E(G_m)]$ for $n \geq m$ and it therefore defines a non zero element in $\hat{\mathbb{Z}}[[G/V(G), *]]$, which we shall call $a$. Observe that $\partial(a)$ is zero, because component by component, $a$ is a multiple of a cycle. However $a$ is not zero, proving that $G$ is indeed not a profinite tree.

**Example 4.4.4** (A profinite tree that is not a limit of finite trees)**.** This time consider the graph $G$ from 4.4 that is the limit of cycles $C_n$, with transition maps $p_{m,n}$ for $m$ dividing $n$. Let us show that such a graph is a profinite tree. Consider $u \in \ker(\partial)$ an element on the augmentation map. By contradiction, assume it is non zero, then there exists $n \in \mathbb{N}$, such that $\pi_n(a) \neq 0$, with $\pi_n$ being the natural projection of $\hat{Z}[[{}^{G}\!/\!_{V(G)}]]$ to $\hat{Z}[E(C_n)]$. The element $\pi_n(u)$ is then a cycle in $\hat{Z}[E(C_n)]$. Using the universal coefficient theorem, we can observe that cycles in $C_n$ are generated by $e_0^n + \cdots + e_{n-1}^n$ and therefore there exists $u \in \hat{\mathbb{Z}}$, $u \neq 0$, such that $\pi_n(u) = a(e_0^n + \cdots + e_{n-1}^n)$. Since $a$ is not zero, there exists an integer $n_1$, such that $p_{n_1}(a)$ is a non zero element in ${}^{\mathbb{Z}}\!/\!_{n\mathbb{Z}}$, with $p_{n_1}$ the natural projection of $\hat{\mathbb{Z}}$ on ${}^{\mathbb{Z}}\!/\!_{n_1\mathbb{Z}}$. Now let $m = n_1 n$. We have $\pi_m(u) = b \sum_{k=1}^{m} e_{k-1}^m$, with $b \in \hat{\mathbb{Z}}$. Then by applying the transition map from $\hat{\mathbb{Z}}[E(G_m)]$ to $\hat{\mathbb{Z}}[E(G_n)]$, we get that $n_1 b \sum_{k=1}^{n} e_{k-1}^n = a \sum_{k=1}^{n} e_{k-1}^n$ and therefore $n_1 b = a$. This is however impossible, since $p_{n_1}(n_1 b) = 0$ and $p_{n_1}(a) \neq 0$, which concludes the proof that $G$ is indeed a profinite tree.

## 4.5 Profinite covering graphs

In this section, we will generalize the notions we have seen in chapter 2. We will focus mainly on normal covering graphs, which in case of profinite graphs will be called Galois coverings. We will give a definition using a free action of a profinite group, we will show that in the finite case it's equivalent to the classical definition given in chapter 2 and then we will prove that every Galois covering can be seen as a profinite limit of finite covering graphs. This section is based on the work of Amrita Acharyya , Jon M Corson and Bikash Das in [1]. Note that it uses undirected profinite graphs, while up until now we have been working with directed ones. Furthermore just like in [1], we will assume that our profinite graphs have a closed set of edges i.e that they are limits in the category of graphs with morphisms.

### 4.5.1 Undirected profinite graphs

First note that the category of undirected graphs together with morphisms form a pre-profinite category as defined in 3.1.4. We can then define undirected profinite graphs as follows:

**Definition 4.5.1** (Undirected profinite graphs)**.** *We define an undirected profinite graph as a profinite object $(\Gamma, (\phi_{i\in I})_{i\in I})$ in the category of undirected abstract graphs.*

Just like in 4.1.4, we have the following characterization of undirected profinite graphs:

**Theorem 4.5.2** (characterization of undirected profinite graphs)**.** *Let $\Gamma$ be a compact Hausdorff topological space, together with continuous maps $o, t$ from $\Gamma$ to a clopen subspace $V$, such that $(\Gamma, V, o, t)$ is an undirected graph with a continuous inversion map. Then $\Gamma$ is a profinite undirected graph if and only if $\Gamma$ is totally disconnected as a topological space.*

*Proof.* We will take a similar approach as in 4.1.4. We know that one implication is a simple consequence of 3.1.10. To prove the other implication, we take $\Omega$ the set of clopen equivalence relations on $\Gamma$. Now let $R \in \Omega$. Since $E(\Gamma)$ and $V(\Gamma)$ form an open partition of $\Gamma$, we have that $R \cap (E(\Gamma)^2 \amalg V(\Gamma)^2)$ is an open equivalence relation that has the property that edges can only be equivalent to edges. Now define

$$R' = \{(x,y) \in R \cap (E(\Gamma)^2 \amalg V(\Gamma)^2), o(x)Ro(y), t(x)Rt(y) \text{ and if } x \in E(\Gamma), \text{ then } \overline{x}R\overline{y}\}$$

Similarly to 4.1.4, we can show that this equivalence relation is compatible with the graph structure and is open. Thus we conclude that $\Gamma$ is indeed an undirected profinite graph.

□

### 4.5.2 Covering graphs form a preprofinite category

As a reminder in section 2.2.7, for an abstract undirected graph $\Gamma$ and $x$ a vertex in $V(\Gamma)$, we define a star at a vertex x: $St(\Gamma, x) := \{e \in G | o(e) = x \text{ or } t(e) = x\}$. Now we define the following category: The objects are triplets $(\Gamma, \Delta, \xi)$, with $\Gamma$ and $\Delta$ abstract undirected graphs and $\xi$ a morphism of graphs, such that for every $x \in \Gamma$, $\xi$ is a bijection from $St(\Gamma, x)$ to $St(\Delta, \xi(x))$. The objects are then called coverings. A morphism of coverings $(\Gamma, \Delta, \xi)$ and $(\Gamma', \Delta', \xi')$ is a pair of morphisms of graphs $(u, v)$, such that the diagram:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\xi} & \Delta \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ \Gamma' & \xrightarrow[\xi']{} & \Delta' \end{array}$$

commutes. One has to be a little careful here as we do not assume $\xi$ to be surjective, which is typically the case in the theory of covering graphs. Let us show that this category is preprofinite in the sense of the definition in 3.1.4. First, we define the forgetful functor $F$ in the following way: To an object $O = (\Gamma, \Delta, \xi)$, we associate $F(O) = \Gamma \amalg \Delta$ and to a morphism $(u, v)$ of objects $(\Gamma, \Delta, \xi)$ and $(\Gamma', \Delta', \xi')$, we associate the map

$$F((u,v)) = \begin{cases} \Gamma \amalg \Delta \longrightarrow \Gamma' \amalg \Delta' \\ x \mapsto \begin{cases} u(x) \text{ if } x \in \Gamma \\ v(x) \text{ if } x \in \Delta \end{cases} \end{cases}$$

The functor $F$ is faithful.

Now we will prove that projective limits exist in this category and that $F$ commutes with them.

Let $((O_i)_{i \in I}, (\phi_{i,j}, \psi_{i,j})_{j \geq i})$ be a projective system in the category of covering graphs, such that for all $i \in I$, $F(O_i)$ is a finite set. We write for $i \in I$, $O_i = (\Gamma_i, \Delta_i, \xi_i)$. We then define $(\Gamma, (p_i)_{i \in I})$ as the limit of the system $((\Gamma_i)_{i \in I}, (\phi_{j,i})_{j \geq i})$ in the category of graphs with morphisms and $(\Delta, p'_i)$ to be the limit of $((\Delta_i)_{i \in I}, (\psi_{i,j})_{j \geq i})$ in the same category. The morphisms $(\xi_i \circ p_i)_{i \in I}$ are compatible with the transition maps $(\psi_{i,j})_{j\geq i}$ and therefore by definition of a projective limit, there exists a unique morphism $\xi$ from $\Gamma$ to $\Delta$ , such that the diagram:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\xi} & \Delta \\ p_i \downarrow & & \downarrow p'_i \\ \Gamma_i & \xrightarrow[\xi_i]{} & \Delta_i \end{array}$$

commutes for every $i \in I$. Let us show that $(\Gamma, \Delta, \xi)$ is a projective limit. We need to prove that $\xi$ is locally bijective. For that, take $x \in V(\Gamma)$.

For every $i \in I$, we denote $\eta_i$ the inverse map of $\xi_i$ from $St(\Delta, \xi_i(x_i))$ to $St(\Gamma_i, x_i)$. Now take $(e_i)_{i\in I} \in St(\Delta, \xi(x))$. We have $\forall i \in I, e_i \in St(\Delta, x_i)$. Indeed we have that $d((e_i)_{i\in I}) = x$, with $d$ being either $o$ or $t$ and so $d(e_i) = x_i$ for all $i \in I$, so $e_i \in St(\Delta_i, x_i)$ for all $i \in I$. The collection $(\eta_i(e_i))_{i\in I}$ is well defined. Now let us prove that it is compatible with the transition maps $\phi_{i,j}$. For that take $i \in I$ and $j \in I$, such that $j \geq i$. We have that:

$$\xi_i(\phi_{i,j}(\eta_j(e_j))) = \psi_{i,j}(\xi_j(\eta_j)(e_j)) = \psi_{i,j}(e_j) = e_i$$

Since

$$\xi_i(\phi_{i,j}(\eta_j(e_j))) = e_i$$

Then by applying $\eta_i$, we get:

$$\phi_{i,j}(\eta_j(e_j)) = \eta_i(e_i)$$

proving the compatibility of the collection with the transition maps. We conclude that:
$(\eta_i(e_i))_{i\in I} \in \Gamma$. More precisely we have that $(\eta_i(e_i))_{i\in I} \in St(\Gamma, x)$. We can then define the map:

$$\begin{cases} St(\Delta, \xi(x)) \longrightarrow St(\Gamma, x) \\ (e_i)_{i\in I} \mapsto (\eta_i(e_i))_{i\in I} \end{cases}$$

This map is the local inverse of $\xi$, since their both compositions equal identity component by component.

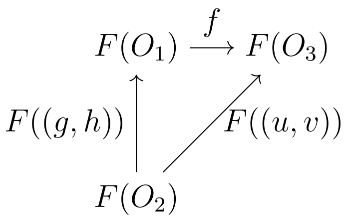
We have proven therefore that $(\Gamma, \Delta, \xi)$ is a cover, now we need to prove that it is a projective limit, but that comes directly from the fact that both $\Gamma$ and $\Delta$ are projective limits.

Now we need to prove that $\Gamma \amalg \Delta$ is also a projective limit of the sets $\Gamma_i \amalg \Delta_i$. Write for $j \geq i$

$$u_{i,j} = \begin{cases} \Gamma_j \amalg \Delta_j \longrightarrow \Gamma_i \amalg \Delta_i \\ x \mapsto \begin{cases} \phi_{i,j}(x) \text{ if } x \in \Gamma_j \\ \psi_{i,j}(x) \text{ if } x \in \Delta_j \end{cases} \end{cases}$$

And denote $A$ the limit of the projective system $((\Gamma_i \amalg \Delta_i)_{i\in I}, (u_{i,j})_{j\geq i})$. Note that if $(a_i)_{i\in I} \in A$, then either for all $i$, $a_i$ is in $\Gamma_i$, or for all $i$, $a_i$ is in $\Delta_i$. Thanks to that we obtain a straightforward isomorphism between $\Gamma \amalg \Delta$ and $A$, proving that $F((\Gamma, \Delta, \xi))$ is isomorphic to the projective limit of the $F((\Gamma_i, \Delta_i, \xi_i))$.

There is one last property of $F$ to prove.
Let $O_1 = (\Gamma_1, \Delta_1, \xi_1), O_2 = (\Gamma_2, \Delta_2, \xi_2)$ and $O_3 = (\Gamma_3, \Delta_3, \xi_3)$ be three objects in the category of covering graphs. Let $(u, v)$ be a morphism from $O_2$ to $O_3$ and $(g, h)$ a morphism from $O_2$ to $O_1$, such that $F((g, h))$ is surjective and finally let $f$ be a map from $F(O_1)$ to $F(O_3)$, such that the diagram:




commutes. Let us show that there exists a morphism $(w, z)$ from $F(O_1)$ to $F(O_3)$, such that $F((w, z)) = f$.

First let us prove that $f(\Gamma_1) \subseteq \Gamma_3$ and $f(\Delta_1) \subseteq \Delta_3$. We take $x \in \Gamma_1$. Since $F((u, v))$ is surjective, and $x \in \Gamma_1$, there exists $y \in \Gamma_2$, such that $u(y) = x$. Then

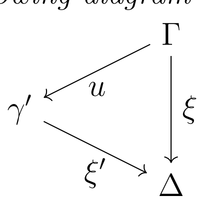
$$f(x) = F((g, h))(y) = g(y) \in \Gamma_3$$

The proof that $f(\Delta_1) \subseteq \Delta_3$ follows exactly the same logic. In that case denote $f_1$, $f$ restricted to $\Gamma_1$ and $f_2$, $f$ restricted to $\Delta_2$. The fact that $f_1$ and $f_2$ are morphisms follows simply from that fact that graphs are a preprofinite category. We therefore conclude that $f = F((f_1, f_2))$.

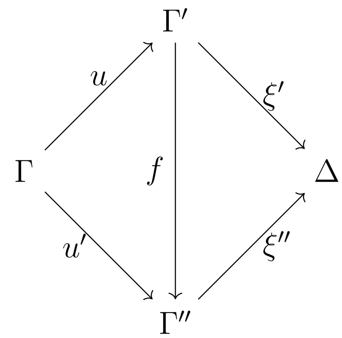
As such, we can therefore define a category of profinite graph coverings as profinite limits of finite coverings of finite graphs.

Now we want to establish a Galois theory of profinite covering graphs to generalize what we have seen in chapter 2. In order to do that, we will need to define the notion of a subcovering graph and a notion of a normal cover. For normal coverings, we will use the definition coming from Ribes [41] in Chapter 2: on Galois coverings.

**Definition 4.5.3.** *Let $C = (\Gamma, \Delta, \xi)$ be a covering graph, we call a subcovering graph of $C$ a triplet $(\Gamma', \xi', u)$ with $\xi'$ and $u$ morphisms of graphs, such that $(\Gamma', \Delta, \xi')$ is a covering graph of $\Delta$ and $(\Gamma, \Gamma', u)$ is a covering graph of $\Gamma'$ and such that the following diagram commutes:*




*A morphism of subcovering graphs of $(\Gamma, \Delta, \xi)$: $(\Gamma', \xi', u)$ and $(\Gamma'', \xi'', u')$, Is a continuous morphism of graphs $f$ from $\Gamma'$ to $\Gamma''$ such that the following diagram commutes:*




**Definition 4.5.4** (Group action on a graph)**.** *Let $G$ be an abstract group and $\Gamma$ an abstract graph. An action of $G$ on $\Gamma$, is a morphism $\phi$ from $G$ to $Aut(\Gamma)$. We often denote $\phi(g)(x)$ as $g \cdot x$. We say that $G$ acts without edge inversions if $\forall e \in E(\Gamma), g \cdot e \neq \overline{e}$.*

Not that of a group acts we can without edge inversions, the quotient graph of an undirected graph is also undirected.

**Definition 4.5.5** (Galois coverings)**.** *Let $G$ be a profinite group acting freely (and continuously) without edge inversions on an undirected profinite graph $\Gamma$. Let $\Delta$ be the profinite graph ${}^{\Gamma}\!/_{G}$, i.e $\Gamma$, quotiented by the closed relation: $x \sim y \Leftrightarrow \exists g \in G, x = g \cdot y$. Finally let $\xi$ be the natural projection of $\Gamma$ on $\Delta$. We call the triple $(\Gamma, \Delta, \xi)$ a Galois covering.*

We can now prove that if $(\Gamma, \Delta, \xi)$ is a Galois covering, then is a covering.

Let $x \in V(\Gamma)$. First let us show that $\xi$ is injective on $St(\Gamma, x)$. Let $e, e' \in St(\Gamma, x)$ be two edges, such that $\xi(e) = \xi(e')$. In that case there exists $g \in G$, such that $e' = g \cdot e$. We now distinguish two cases:

- Case 1: $o(e) = x$ and $t(e') = x$.
  In that case $t(g \cdot e) = x = o(\overline{e'})$, thus $g \cdot x = x$ and since $G$ acts freely, that implies that $g = 1_G$ and $e = e'$.
- Case 2: $o(e) = x$ and $o(e') = x$. Then $g \cdot x = x$ and so $e = e'$.

Now let us prove that $\xi$ is surjective. Let $[e] \in St(\Delta, [x])$, with $[x]$ denoting the class of $x$ in the quotient. In that case, there exists $g \in G$, such that $o(e) = g \cdot x$. Thus we have that $g^{-1} \cdot e \in St(\Gamma, x)$ and $\xi(g^{-1} \cdot e) = [e]$. We can therefore conclude that $(\Gamma, \Delta, \xi)$ is indeed a covering. We will now show more specifically that it is a profinite covering.

**Proposition 4.5.6.** *Let $(\Gamma, \Delta, \xi)$ be a Galois covering, with $G$ acting without edge inversions, then $(\Gamma, \Delta, \xi)$ is a profinite covering.*

*Proof.* We will adapt the proof in Ribes [41], Proposition 3.1.3.

Let $\mathcal{N}$ denote the set of all open subgroups of $G$. For $N \in \mathcal{N}$, we denote $\pi_N$ the natural projection of $\Gamma$, onto the quotient graph $\Gamma_N = {}^{\Gamma}\!/_{N}$. The group ${}^{G}\!/_{N}$ is then a finite group and acts freely on $\Gamma_N$, therefore there exists a directed set $I'_N$, such that $\Gamma_N$ is decomposed into $\varprojlim_{i\in I'_N} \Gamma_{N,i}$, with $\Gamma_{N,i}$ finite undirected graph and ${}^{G}\!/\!_{N}$ acting freely on $\Gamma_{N,i}$. Denote $\phi_{N,i}$ the natural projection of $\Gamma_N$ onto $\Gamma_{N,i}$. Now for $g \in {}^{G}\!/\!_{N}$, consider

$$F_{g,i} = \{e \in E(\Gamma_N), \phi_{N,i}(g\cdot e) = \phi_{N,i}(\overline{e})\}$$

Since ${}^{G}\!/\!_{N}$ acts without inversions of edges, we get that $\bigcap_{i\in I'_N} F_{g,i} = \emptyset$. Indeed if it were nonempty and contained an $e \in E(\Gamma)$, then $g \cdot e = \overline{e}$, which is a contradiction. By compactness, we can therefore find an $i_g \in I'_N$, such that $F_{g,i_g} = \emptyset$. Then denote $i_N$ an element of $I'_N$ that is greater than all the $i_g$. (Possible since ${}^{G}\!/\!_{N}$ is finite). Then denote $I_N = \{i \in I'_N | i \geq i_N\}$. One can then prove that $\Gamma_N$ is a projective limit $\varprojlim_{i\in I_N} \Gamma_{N,i}$ and for all $i \in I_N$ ${}^{G}\!/\!_{N}$ acts without edge inversions. The collection $(\Gamma_{N,i}, \xi_{N.i}, \Delta_{N.i})$, with $\xi_{N.i}$ the natural projection of $\Gamma_{N.i}$ on $\Delta_{N,i} = {}^{\Gamma_{N,i}}\!/\!_{{}^{G}\!/\!_{N}}$ is therefore a finite covering. One can prove that $\Gamma$ is a limit of the $\Gamma_{N,i}$ and therefore it is a profinite covering. □

This result justifies the definition that follows:

**Definition 4.5.7.** *A profinite covering* $(\Gamma, \Delta, \xi)$ *is called normal, if it is isomorphic to a Galois covering.*

Alternalitevely if we take $\Gamma$ and $\Delta$ be two graphs. Let $\xi$ be a surjective morphism from $\Gamma$ to $\Delta$, we can say that $(\Gamma, \xi)$ is a Galois covering of $\Delta$ if there exists $G$ a profinite group acting freely on $\Gamma$, such that

$$\forall x, y \in \Gamma, \xi(x) = \xi(y) \Leftrightarrow \exists g \in G, g \cdot x = y$$

As shown in [41],section 3.2 of Chapter 3, a Galois covering can be seen as a subgroup of automorphisms of the graph. We will show more precisely that in case of a connected profinite graph it corresponds to the automorphisms fixing $\xi$, which in terms of chapter 2 would correspond to deck transformations.

**Proposition 4.5.8.** *Let* $(\Gamma, \xi)$ *be a Galois covering of* $\Delta$ *with the profinite group* $G$*. Suppose that* $\Gamma$ *is connected: then* $G$ *is isomorphic to the automorphisms of* $\Gamma$*, fixing* $\xi$*, i.e the subgroup*

$$Aut_\xi(\Gamma) = \{g \in Aut(\Gamma) | \xi \circ g = \xi\}$$

*Proof.* By adapting the Proposition 3.1.3 in [41] to undirected graphs, $(\Gamma, \xi)$ can be decomposed into finite Galois coverings with surjective natural projections: $(\Gamma_i, \xi_i)_{i\in I}$. We then denote for an $i \in I$ $p_i$ the natural projection of $\Gamma$ onto $\Gamma_i$ and $p'_i$ the natural projection of $\Delta$ onto $\Delta_i$ with:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\xi} & \Delta \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p'_i} \\ \Gamma_i & \xrightarrow[\xi_i]{} & \Delta_i \end{array}$$

Furthermore take for $j \geq i$, $\phi_{i,j}$ the transition map from $\Gamma_j$ to $\Gamma_i$. Now let $f \in Aut_\xi(G)$. Since $\xi \circ f = \xi$, we pick $a \in \Gamma$ and then there exists by definition a $g \in G$, such that:

$$f(a) = g \cdot a$$

Now for every $i \in I$ take $N_i$ the open normal subgroup of $G$, such that ${}^{G}\!/_{N}$ together with $(\Gamma_i, \xi_i)$ is a Galois covering of $\Delta_i$. We then get

$$\forall g \in G, \forall x \in \Gamma, p_i(g \cdot x) = \overline{g}^{N_i} \cdot p_i(x)$$

Take $i \in I$. By continuity of $f$, there exists $j \geq i$, such that if $p_j(x) = p_j(y)$, then $p_i(f(x)) = p_i(f(y))$.

Since $\Gamma$ is connected, then $\Gamma_j$ is connected as well. We can then do an induction: Consider

$$A = \{u \in \Gamma_j | \forall x \in \Gamma, p_j(x) = u \Rightarrow p_i(f(x)) = \overline{g}^{N_i} \cdot \phi_{i,j}(u)\}$$

We will now prove that $A$ is non empty and has the two following inductive properties:

$$\forall u \in A, \forall e \in \Gamma_j, o(e) = u \text{ or } t(e) = u \Rightarrow e \in A$$

$$\forall u \in A, o(u) \in A \text{ and } t(u) \in A$$

Basically we require that if $u$ is in $A$, then all of its neighbors are in $A$ as well, which coupled with the connectedness of $\Gamma$ will let us conclude that $A$ is equal to $\Gamma_j$.

First of all $A$ is non empty, since $p_j(a) \in A$. Now suppose that $u \in A$ and take $e' \in \Gamma_j$, such that $d(e') = u$, with $d$ an incidence map (origin or terminus). Let $e \in \Gamma$, such that $p_j(e) = e'$. Since $\xi(f(e)) = e$, there exists $g' \in G$, such that $f(e) = g' \cdot e$. In that case $p_i(f(e)) = \overline{g'}^{N_i} \cdot p_i(e)$. Since $p_i$ and $f$ are morphisms, we get that $d(p_i(f(e))) = p_i(f(d(e)))$. Since $p_j(d(e)) = u$, then by definition of the set $A$, we get: $d(p_i(f(e))) = \overline{g}^{N_i} \cdot \phi_{i,j}(u)$. Now on the other hand we also have that $f(e) = g' \cdot e$ and so $p_i(f(e)) = \overline{g'}^{N_i} \cdot \phi_{i,j}(u)$. Since the action of ${}^{G}\!/_{N_i}$ on $\Gamma_i$ is free we get that $\overline{g'}^{N_i} = \overline{g}^{N_i}$ and as such, $p_i(f(e)) = \overline{g}^{N_i} \cdot e$. This being true for all $e$, such that $p_j(e) = e'$ we get that $e' \in A$. The proof that if $u \in A$, then $o(u)$ and $t(u)$ are in $A$ is very similar, so we skip it.

We conclude therefore that $A = \Gamma_j$. Now let us show that this implies that for every $x \in \Gamma, f(x) = g \cdot x$. Let $x \in \Gamma$, then: $p_j(x) \in A$. Therefore $p_i(f(x)) = \overline{g}^{N_i} \cdot p_i(x) = p_i(g \cdot x)$. That being true for every $i \in I$ and every $x \in \Gamma$, we conclude that $f = g$. □

Now we prove more generally that for connected profinite coverings, the group of deck transformations is a profinite group.

**Proposition 4.5.9.** *Let* $(\Gamma, \Delta, \xi)$ *be a connected profinite covering graph (*$\Gamma$ *and* $\Delta$ *are connected ), then together with its open compact topology:* $Aut_\xi(\Gamma)$ *is a profinite group.*

*Proof.* Since $(\Gamma, \Delta, \xi)$ is profinite, it is a limit of $(\Gamma_i, \Delta_i, \xi_i)_{i\in I}$ finite covering graphs. Furthermore by corestricting the natural projections $(p_i, p_i')$ to their image we can assume that $p_i, p_i'$ are surjective for all $i \in I$. In that case, since $\Gamma$ and $\Delta$ are connected, we get that for every $i \in I$, $\Gamma_i$ and $\Delta_i$ are connected.

Now to prove that $Aut_\xi(\Gamma)$ is profinite, we make first the simple observation that $Aut_\xi(\Gamma)$ is closed in $Aut(\Gamma)$, so it inherits its topological group structure. Now using, the result from the proposition 4.1.1, all we need to prove is that for all $i \in I$ , the set $A_i = \{p_i \circ g | g \in Aut_\xi(\Gamma)\}$ is finite. Take $a \in V(\Gamma)$. Let us show that the map

$$\Phi = \begin{cases} A_i \longrightarrow \Delta_i \\ p_i \circ g \mapsto p_i(g(a)) \end{cases}$$

is injective. Let $g, h \in Aut_\xi(\Gamma)$, such that $p_i(g(a)) = p_i(h(a))$. By continuity of $g$ and $h$, there exists $j \in I$, such that

$$\forall x, y \in \Gamma, p_j(x) = p_j(y) \Rightarrow p_i(g(x)) = p_i(g(y)) \text{ and } p_i(h(x)) = p_i(h(y))$$

Now consider

$$X = \{u \in \Gamma_j | \forall x \in \Gamma, p_j(x) = u \Rightarrow p_i(g(x)) = p_i(h(x))\}$$

Since $\Gamma_j$ is connected, we can do the same induction as in the previous proposition. We need to prove the statements:

$$\forall u \in X, \forall e \in \Gamma_j, o(e) = u \text{ or } t(e) = u \Rightarrow e \in X$$

$$\forall u \in X, o(u) \in A \text{ and } t(u) \in X$$

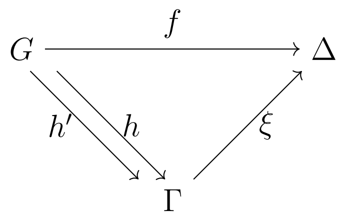
Assume that $u \in X$ and let $e \in \Gamma_j$, such that $d(e) = u$, with $d$ being an incidence map (terminus or origin) and let $e' \in \Gamma$, such that $p_j(e') = e$. In that case $p_j(d(e')) = u$ and by definition of $X$: $p_i(d(g(e))) = p_i(d(h(e)))$. Then we get that $p_i(g(e))$ and $p_i(h(e))$ are both edges in $St_{\Gamma_i}(g(e))$. Furthermore, since $\xi_i(p_i(g(e))) = p_i(\xi(g(e))) = p_i(\xi(e)) = \xi_i(p_i(h(e)))$, we get by injectivity of $\xi_i$ at the star of $p_i(d(e))$ that $p_i(g(e)) = p_i(h(e))$ and therefore $e' \in X$ as expected. Just like in the previous proposition, the proof of the second statement needed for the induction is very similar and will be skipped.

We now can conclude, using that $\Gamma_j$ is connected that $X = \Gamma_j$. Now if we take $x \in \Gamma$, then $p_j(x) \in \Gamma_j$ and therefore $p_i(g(x)) = p_i(h(x))$. Since this is true for all $x \in \Gamma$, then $p_i \circ g = p_i \circ h$, proving that the map $\Phi$ we started with is injective. Since $\Delta_i$ is a finite set and $\Phi$ is an injection, we get that $A_i$ is a finite set as well.

We can therefore conclude that $Aut_\xi(\Gamma)$ is indeed profinite. □

*Remark.* A similar proof would work even under the assumption that $\Gamma$ has a finite number of connected components, but I suspect that the statement can be false if we get infinitely many connected components.

**Proposition 4.5.10** (Uniqueness of liftings)**.** *Let* $(\Gamma, \Delta, \xi)$ *be a profinite covering graph. Let* $G$ *be a profinite connected graph and* $f$ *a continuous morphism from* $G$ *to* $\Delta$*. Let* $u \in \Gamma$ *and* $v \in \Delta$*, such that* $f(u) = \xi(v)$*. Let* $h, h'$ *be two continuous morphisms, such that the diagram*



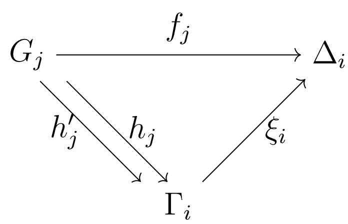

*commutes and such that* $h(u) = h'(u) = v$*, then* $h = h'$*.*

*Proof.* We start by writing $(\Gamma, \Delta, f)$ as a limit of $((\Gamma_i, \Delta_i, f_i))_{i \in I}$ finite covers. We denote $p_i$ the natural projection of $\Gamma$ on $\Gamma_i$ and $q_i$ the natural projection of $\Delta$ on $\Delta_i$. We also write $G$ as a limit of $(G_j)_{j \in J}$ and denote $\pi_j$ the natural projection of $G$ onto $G_j$. We assume that all the $\pi_j$ are surjective and $G_j$ are therefore connected.

To prove that $h' = h$, we show that $\pi_i \circ h = \pi_i \circ h'$ for all $i \in I$. Let $j \in J$ such that

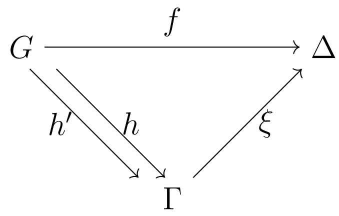
$$\forall j' \geq j, \forall x, y \in G, \pi_{j'}(x) = \pi_{j'}(y) \Rightarrow$$
$$(\ p_i(h(x)) = p_i(h(y)), p_i(h'(x)) = p_i(h'(y)) \text{ and } q_i(f(x)) = q_i(f(y))\ )$$

Let $h_j, h'_j$ be the induced morphisms from $G_j$ to $\Gamma_j$ and $f_j$ the induced q-morphism from $G_j$ to $\Delta_i$. In that case we get that:




Now assuming that we have proved the proposition in the case of finite graphs, we can conclude that $h_j = h'_j$. By contradiction now assume that $p_i \circ h \neq p_i \circ h'$. In that case there exists $x \in G$, such that $p_i(h(x)) \neq p_i(h'(x))$. In that case $h_j(\pi_j(x)) \neq h'_j(\pi_j(x))$, which is a contradiction.

Now we shall prove that the result is true in the finite case. From now on we assume that $(\Gamma, \xi)$ is a finite cover of the finite graph $\Delta$ and $G$ is a finite connected graph with a morphism $f$. Furthermore assume that $u \in G$ and $v \in \Gamma$, such that $f(u) = \xi(v)$. Assume that $h, h'$ are two morphisms from $G$ to $\Gamma$, such that the following diagram commutes:




Finally assume that $h(u) = h'(u) = v$. Let us prove that under these conditions, $h = h'$. To prove it, we will use the connectedness of $\Gamma$ and use induction. Let

$$A = \{x \in G | h(x) = h'(x)\}$$

To prove that $A = G$, we us induction and we shall prove that

$$\forall x \in G, o(x) \in A \text{ or } t(x) \in A \Rightarrow x \in A$$

and

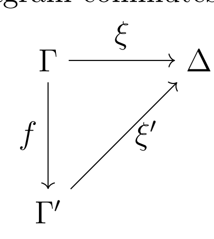
$$\forall x \in A, o(x) \in A \text{ and } t(x) \in A$$

Let $d \in \{o, t\}$ be an incidence map. Assume that $d(x) \in A$. Then $h(d(x)) = h'(d(x))$ and $h(x), h'(x)$ are therefore in the star of $d(h(x))$. Furthermore $\xi(h(x)) = f(x) = \xi(h'(x))$ and by injectivity of $\xi$ on stars we get that $h(x) = h'(x)$ and therefore $x \in A$. Now suppose that $x \in A$. We get that $h(x) = h'(x)$ and therefore $d(h(x)) = d(h'(x))$. Since $h, h'$ are morphisms, we get that $h(d(x)) = h'(d(x))$ and therefore $d(x) \in A$. We conclude that $A = G$, hence the lifting is indeed unique. □

Now we are ready to state the fundamental theorem of Galois theory of profinite covering graphs.

**Theorem 4.5.11** (Fundamental theorem of Galois theory of profinite covering graphs). *Let* $(\Gamma, \Delta, \xi)$ *be a profinite normal connected covering graph. Then there is an inclusion reversing bijection* $\Phi$ *between the closed subgroups of* $Aut_\xi(\Gamma)$ *and subcovering connected graphs* $(\Gamma', \Delta, \xi')$ *(up to isomorphism) of* $(\Gamma, \Delta, \xi)$.

*Proof.* Let $H$ be a closed subgroup of $Aut_\xi$. The group $H$ then acts continuously and freely on $\Gamma$. Now let $f$ be the natural projection of $\Gamma$ onto $\Gamma' = \Gamma/H$ the graph of orbits under the action of $H$. We have that for every $h \in H$ and every $x \in \Gamma$, $\xi(h \cdot x) = \xi(x)$ and therefore by the universal property of the quotient, there exists a unique continuous morphism of graphs $\xi'$ from $\Gamma'$ to $\Delta$, such that the following diagram commutes




$(\Gamma, \xi', f)$ is a subcovering, but we need to show that it is profinite and connected.

We denote $G = Aut_\xi(\Gamma)$ and we decompose $\Gamma$ into $(\Gamma_i, \Delta_i, \xi_i)_{i \in I}$ Galois coverings with $(N_i)_{i \in I}$ being a basis of open normal subgroups of $G$ and $G/N_i$ acting freely and without edge inversions on $(\Gamma_i, \Delta_i, \xi_i)$. For every $i \in I$, $N_i \cap H$ is an open normal subgroup of $H$. The group ${}^{H}\!/_{H \cap N_i}$ injects itself naturally into ${}^{G}\!/_{N_i}$. Consider then

$$\Gamma'_i = {}^{\Gamma_i}\!/_{\left({}^{H}\!/_{H \cap N_i}\right)}$$

If we denote $f_i$ the natural projection of $\Gamma_i$ onto $\Gamma'_i$, there exists $\xi'_i$ morphism such that following diagram commutes:

$$\begin{array}{ccc} \Gamma_i & \xrightarrow{\ \xi_i\ } & \Delta_i \\ {\scriptstyle f_i}\downarrow & \nearrow{\scriptstyle \xi'_i} & \\ \Gamma'_i & & \end{array}$$

All the $(\Gamma'_i, \Delta_i, \xi')$ are covering graphs. Now if we denote $(\phi_{i,j}, \psi_{i,j})_{j \geq i}$ the transition maps for $(\Gamma_i, \Delta_i, \xi_i)_{i \in I}$, we have using the universal property of quotients that there exist transition maps $\phi'_{i,j}$, such that for every $j \geq i$, the diagram:

$$\begin{array}{ccc} \Gamma_j & \xrightarrow{\ f_j\ } & \Gamma'_j \\ {\scriptstyle \phi_{i,j}}\downarrow & & \downarrow{\scriptstyle \phi'_{i,j}} \\ \Gamma_i & \xrightarrow[\ f_i\ ]{} & \Gamma'_i \end{array}$$

commutes. Now let us show that $(\Gamma', \Delta, \xi')$ is a projective limit of $(\Gamma'_i, \Delta_i, \xi'_i)$ for the transition maps $\phi'_{j,i}$. If $p_i$ is the natural projection of $\Gamma$ onto $\Gamma_i$, we denote $p'_i$, the natural map from $\Gamma'$ to $\Gamma'_i$, such that the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ f_i \circ p_i\ } & \Gamma'_i \\ {\scriptstyle f}\downarrow & \nearrow{\scriptstyle p'_i} & \\ \Gamma' & & \end{array}$$

commutes. To prove that $(\Gamma', \Delta, \xi')$ together with the maps $(p'_i, q_i)_{i \in I}$ ($q_i$ being the natural projection of $\Delta$ onto $\Delta_i$) is a projective limit of the $(\Gamma'_i, \Delta_i, \xi_i)_{i \in I}$ together with the transition maps $(\phi'_{i,j}, \psi_{i,j})_{j \geq i}$, we need to prove that if $(x_i)_{i \in I}$ is a collection in $\prod_{i \in I} \Gamma'_i$ compatible with $\phi'_{i,j}$, then there exists a unique $x$ in $\Gamma'$, such that for all $i \in I$, $p'_i(x) = x_i$.

<u>Uniqueness</u> Suppose that for all $i \in I$, $p'_i(x) = p'_i(y)$. Let us write $x = f(x')$ and $y = f(y')$. Then for every $i \in I$, there exists $\overline{h_i}^{N_i \cap H} \in {}^{H}\!/_{N_i \cap H}$, such that $p_i(x') = \overline{h_i}^{N_i} \cdot p_i(y')$. If we take $j \geq i$, we have that $\phi_{i,j}(p_j(x')) = \overline{h_i}^{N_i} p_i(y)$, but also that $\phi_{i,j}(p_j(x')) = \overline{h_i}^{N_j} p_j(y')$. Since ${}^{G}\!/_{N_i}$ acts freely on $\Gamma_i$, we get that $\overline{h_j}^{N_i} = \overline{h_i}^{N_i}$. The collection $(h_i)_{i \in I}$ is compatible with the transition maps in the profinite group $G$ and $(N_i)_{i \in I}$ form a neighborhood basis of $1_G$ and $H$ is a closed subgroup of $G$,

therefore there exists $h \in H$, such that $\forall i \in I, \overline{h}^{N_i} = \overline{h_i}^{N_i}$. In that case we get that for every $i \in I$, $p_i(x') = p_i(h \cdot y')$ and therefore $x' = h \cdot y'$. As such since $f$ is fixed by the action of $G$, we get that $f(x') = f(y')$, so $x = y$.

Existence

Let $(x_i)_{i \in I} \in \prod_{i \in I} \Gamma'_i$ be a collection compatible with the transition maps. By contradiction, assume that there is no $x \in \Gamma$, such that $\forall i \in I, f_i(p_i(x)) = x_i$. In that case by compactness and compatibility of the collection with the transition maps, there exists an $i \in I$, such that for all $x \in \Gamma$, $f_i(p_i(x)) \neq x_i$. That is however a contradiction, since both $p_i$ and $f_i$ are surjective maps. In that case let $x \in \Gamma$, such that $\forall i \in I, f_i(p_i(x)) = x_i$. In that case for all $i$ in $I$, $p'_i(f(x)) = x_i$, which concludes the proof of existence.

Note that $\Gamma'_i$ are all path-connected, since $\Gamma_i$ are and the projection $f_i$ is surjective.

We now know that $(\Gamma', \Delta, \xi')$ is a limit of finite path-connected covers, therefore is a connected profinite cover. We conclude that $(\Gamma', \xi', f)$ is then a profinite subcover of $\Gamma$. We shall denote it $\Phi(H)$. Furthermore, we denote $\xi_H = \xi'$ and $f_H = f'$.

Let us now show that $H$ and $H'$ are equal if and only if $\Phi(H)$ is isomorphic to $\Phi(H')$.

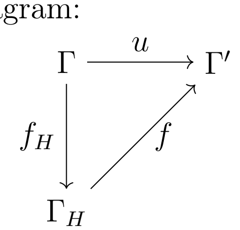
Suppose that $\Phi(H)$ and $\Phi(H')$ are isomorphic and let us show that $H$ and $H'$ are equal.

Let $u$ be the isomorphism between $\Phi(H)$ and $\Phi(H')$. Then simply take $h \in H$ and let us show that $h \in H'$. Let us take $a$ a vertex in $\Gamma$. We have that

$$f_{H'}(h \cdot a) = u(f_h(h \cdot a)) = u(f_H(a)) = f_{H'}(a)$$

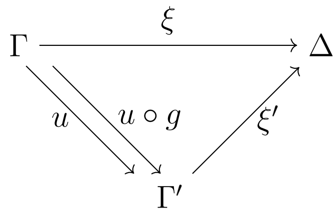
Therefore there exists by definition of $f_{H'}$ an $h' \in H$, such that $h' \cdot a = h \cdot a$. Since $G$ acts freely, we get that $h = h'$, from which we conclude that $h \in H'$. Since $H \subseteq H'$ and $H, H'$ play symmetric roles, we get that the two subgroups are equal.

This proves that $\Phi$ is an injective map. Now we need to prove that $\Phi$ is surjective on subcovering graphs that is: if $(\Gamma', \xi', u)$ is a profinite subcovering graph of $(\Gamma, \Delta, \xi)$, then there exists a closed subgroup $H$ of $G$, such that $\Phi(H)$ and $(\Gamma', \Delta, \xi')$ are isomorphic. Let $H = Aut_u(\Gamma)$, then $H$ is a closed subgroup of $G$. Let us prove that $\Phi(H)$ and $(\Gamma, \Gamma', \xi')$ are isomorphic. The map $u$ is by definition invariant by the action of $H$, therefore there exists a natural map $f$ from $\Gamma_H$ to $\Gamma'$, such that the diagram:



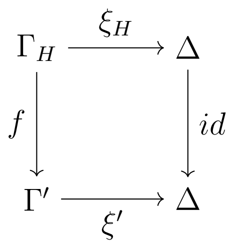

commutes. Now let us prove that $f$ is bijective. The map $f$ is surjective because $u$ is (due to the fact that $\Gamma'$ and $\Gamma$ are connected and $u$ is a continuous covering morphism). Now let $\overline{x}^H, \overline{y}^H \in \Gamma_H$, such that $f(\overline{x}^H) = f(\overline{y}^H)$. In that case: $u(x) =$

$u(y)$ and therefore $\xi'(u(x)) = \xi'(u(y))$, thus $\xi(x) = \xi(y)$. Therefore there exists $g \in G$, such that $g(x) = y$ and as such, $u(g(x)) = u(x)$. Using the fact that $\Gamma$ is connected and the previous proposition about uniqueness of lifts, we get that $u \circ g = u$. Indeed: we have that the following diagram commutes:




commutes that $u \circ g(x) = u(x)$ and that $(\Gamma', \Delta, \xi')$ is a profinite covering, so by the uniqueness of the lift given in the previous proposition, we get that $u \circ g = u$. Therefore $g \in Aut_u(\Gamma) = H$ and as such $f_H(x) = f_H(y)$ and so $f$ is injective.

Now let us prove that the following diagram commutes:



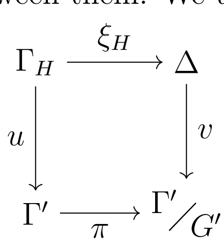

Let $\overline{x}^H \in \Gamma_H$. In that case:

$$\xi'(f(\overline{x})) = \xi'(u(x)) = \xi(x) = \xi_H(\overline{x})$$

proving that the diagram indeed commutes. We therefore get that $\Gamma_H$ and $\Gamma'$ are isomorphic, showing that $\Phi$ is indeed a bijection between closed subgroups of $G$ and isomorphism classes of connected profinite subcovering graphs of $(\Gamma, \Delta, \xi)$.

The next step is to show that $\Phi$ is inclusion reversing: i.e if $H$ is a subgroup of $H'$, then $\Phi(H')$ is a subcovering graph of $\Phi(H)$. That is however straightforward: since there exists a natural surjection from $\Gamma_{H'}$ to $\Gamma_H$ which to a class $\overline{x}^{H'}$ associates the class $\overline{x}^H$.

The last step is to show that $\Phi(H)$ is normal if and only if $H$ is a normal subgroup of $G$. Suppose first that $\Phi(H)$ is normal. In that case there exists a profinite group $G'$ and a graph $\Gamma'$, such that $\Phi(H)$ is isomorphic to the covering $(\Gamma', {}^{\Gamma'}\!/_{G'}, \pi)$, with $\pi$ being the natural projection of $\Gamma'$ onto ${}^{\Gamma'}\!/_{G'}$. We then denote $(u, v)$ an isomorphism between them. We then have the following commutative diagram:



Now take $g \in G$ and $h \in H$. Let us prove that $ghg^{-1} \in H$. Let $a \in \Gamma_H$. Then there exists $g' \in G'$, such that $u(\overline{g \cdot x}) = g' \cdot u(\overline{a})$. Using the fact that $\Gamma$ is connected and that $G'$ acts freely on $\Gamma'$, we can then show that for every $x \in \Gamma$, $u(\overline{g \cdot x}) = g' \cdot u(\overline{x}^H)$. Using the same result for $g^{-1}$, we get that there exists $g'' \in G'$, such that $u(\overline{g^{-1} \cdot x}^H) = g'' \cdot u(\overline{x}^H)$ for every $x \in \Gamma$. We then get that $u(\overline{a}^H) = g'g''u(\overline{a}^H)$ and using the fact that $G'$ acts freely on $\Gamma'$, we get that $g' = g''^{-1}$. Furthermore we get that $u(\overline{ghg^{-1} \cdot a}^H) = g' \cdot u(\overline{hg^{-1} \cdot a}^H) = g' \cdot u(\overline{g^{-1} \cdot a}^H) = g'g'' \cdot u(\overline{a}^H) = u(\overline{a}^H)$. Since $u$ is a bijection, we get that $\overline{ghg^{-1} \cdot a}^H = \overline{a}^H$ and therefore $ghg^{-1} \in H$ as announced. The group $H$ is therefore normal, which ends the first part of the proof.

If on the other hand $H$ is normal, we get that the profinite group ${}^{G}\!/_{H}$ acts continuously and freely on $\Gamma_H$. Now we need to prove that $\Delta$ is isomorphic to $\Gamma_H /_{{}^{G}\!/_{H}}$. The map $\xi_H$ is constant on the equivalence classes of ${}^{G}\!/_{H}$, since if $g \in G$: $\xi_H(\overline{g}^H \cdot \overline{x}^H) = \xi_H(\overline{g \cdot x}) = \xi(g \cdot x) = \xi(x) = \xi_H(\overline{x}^H)$. $\xi_H$ is surjective, since $\xi$ is. Finally, suppose that $\xi_H(\overline{x}) = \xi_H(\overline{y})$. Then $\xi(x) = \xi(y)$ and so $\exists g \in G, y = g \cdot x$, therefore $\overline{y}^H = \overline{g}^H \cdot \overline{x}^H$ and so we can conclude that $\Gamma_H$ is indeed normal. □

To conclude this section we will generalize the notion of the universal covering graph to profinite coverings.

**Definition 4.5.12** (Universal covering graph)**.** *Let $\Delta$ be a connected profinite graph. A connected profinite covering graph $(\tilde{\Delta}, \Delta, p)$ is called universal if every connected covering graph $(\Gamma, \Delta, f)$ is isomorphic to a subcovering $(\tilde{\Delta}, \Delta, p)$.*

*Remark.* We could have also worked by choosing a distinguished point for a covering, in which case we would have had uniqueness of the isomorphism map $h$ such that $(\Gamma, f, h)$ is a subcovering of $(\tilde{\Delta}, \Delta, f)$ with the assumption that it sends a distinguished point $c$ of $\tilde{\Delta}$ onto a distinguished point $a \in \Gamma$ , such that $f(a) = p(c)$ as in the definition 32 of [1].

Finally we state the theorem of existence of universal coverings.

**Theorem 4.5.13.** *Let $\Delta$ be a connected profinite graph. Then the universal profinite covering graph of $\Delta$ exists, is normal and is unique up to isomorphism of coverings of graphs.*

*Proof.* We will prove the uniqueness here. The proof of existence can be found in [1] section 3.5. Let $(\Gamma, \Delta, p)$ and $(\Gamma', \Delta, p')$ be two universal coverings of $\Delta$. Choose a distinguished point $c \in \Gamma$. Then there exists a covering morphism $u$ from $\Gamma$ to $\Gamma'$, such that $p' \circ u = p$. Also there exists a morphism $v$, such that $p \circ v = p'$. Let us observe that for any $g \in Aut_p(\Gamma)$, $g \circ v$ is also a covering morphism, such that $p \circ g \circ v = p'$. We will use this to correct $v$ in order so that it becomes an inverse of $u$. We have $p \circ v \circ u = p' \circ u = p$. Therefore there exists $g \in Aut_p(\Gamma)$, such that $v \circ u(c) = g(c)$. In that case $g^{-1} \circ v \circ u(c) = c$ and using the connectedness, we can then conclude that $g^{-1} \circ v \circ u = id_\Gamma$. We also have $u(g^{-1}(v(u(c)))) = u(g^{-1}(g(c))) = u(c)$ and again using the connectedness, we get that $u \circ g^{-1} \circ v = id$ and hence $(\Gamma, \Delta, p)$ and $(\Gamma', \Delta, p')$ are isomorphic. □

## 4.6 Colors and color substitutions

In this section, we want to generalize the results in Chapter 1 to profinite graphs. We will construct a group action Cayley graph for finite sets. In order to do that, we will need to define a notion of color (or label) on a profinite graph. In the end, we will show that the Lovász construction that is used for dropping colors in abstract graphs works for profinite graphs as well for finitely many colors. A reader interested in exploring labeling on finite graphs may read for example [9].

Now to define a notion of color, we need some definition that is compatible with the profinite structure of a graph. Hence the set of colors has to have a topology to ensure such compatibility. I came therefore with the definition that follows:

**Definition 4.6.1** (Edge colored profinite graphs)**.** *An edge-colored profinite graph is a triplet $(G, c, C)$, where $G$ is a profinite graph, $C$ a topological set, called the set of colors and $c$ a continuous map from $E(G)$ to $C$.*

*A morphism of edge-colored graphs $(G, c, C)$ to $(G', c', C')$ is a morphism $f$ from $G$ to $G'$, such that $\forall x, y \in E(G)$, $c(x) = c(y) \Rightarrow c'(f(x)) = c'(f(y))$.*

*For an edge-colored profinite graph $(G, c, C)$, we define $Aut_c(G)$ as the group of automorphisms preserving colors, i.e: $Aut_c(G) = \{g \in Aut(G) | c \circ g = c\}$.*

Since the only colorings we will be interested in are those on edges, we will simply call edge-colored graphs as 'colored graphs' from now on. Naturally we want to give the set $Aut_c(G)$ a structure of a topological group, coming from $Aut(G)$. For that, we would like to have $Aut_c(G)$ closed inside of $Aut(G)$. Luckily there is a very loose sufficient condition on $C$ for that to happen.

**Proposition 4.6.2.** *Let $(G, c, C)$ be a colored profinite graph. Then if $C$ is a $T_1$ separated space (all singletons are closed) then $Aut_c(G)$ is closed in $Aut(G)$ for the compact open topology.*

*Proof.* Suppose that $g \in \overline{Aut_c(G)}$. Let us show that $g$ preserves colors. Let $x \in E(G)$. To show that $c(g(x)) = c(x)$, we will show that $c(x) \in \overline{\{c(g(x))\}}$. Let $p_i$ be the natural projections onto $G_i$ with $G$ seen as a projective limit of finite graphs $G_i$: $\varprojlim_{i \in I} G_i$. Let $V$ be a neighborhood of $c(g(x))$. By continuity of $c$, there exists $i_0 \in I$, such that $\forall i \geq i_0, \forall y \in E(G), p_i(y) = p_i(g(x)) \Rightarrow c(y) \in V$. Since $g \in \overline{Aut_c(G)}$, there exists $g' \in Aut_c(G)$ such that $p_{i_0} \circ g = p_{i_0} \circ g'$. In that case: $p_{i_0}(g(x)) = p_{i_0}(g'(x))$ and therefore by continuity: $c(x) = c(g'(x)) \in V$. Since this is true for every neighborhood $V$ of $c(g(x))$, we get that $c(x) \in \overline{\{c(g(x))\}} = \{c(g(x))\}$ and so we conclude that $g \in Aut_c(G)$. □

Now we will define a profinite group action Cayley graph. The basic idea behind this construction is to take the Cayley graph as defined in [41], Chapter 1 and extend it with the group action structure that I constructed in the discrete case.

**Definition 4.6.3** (Colors and color substitutions)**.** *Let $G$ be a profinite group generated topologically by a closed subset $S$ and $X$ a profinite set. Suppose that $G$ acts* *continuously on $X$. We take $\tilde{S} = S \cup \{1_G\}$ and define $F = G \times X$, which we equip with its product topology (it is then a profinite space). Then we put*

$$\Gamma = G \times \tilde{S} \amalg F \amalg X$$

*We equip it with the disjoint topology. Now we define $V(\Gamma) = G \times \{1\} \amalg X$, which is a closed subset of $\Gamma$. Let us define the origin and terminus maps, to make $\Gamma$ into a graph. We write:*

$$o = \begin{cases} \Gamma \longrightarrow V(\Gamma) \\ x \mapsto \begin{cases} x \text{ if } x \in X \\ (g,1) \text{ if } x = (g,s) \in G \times \tilde{S} \\ (g,1) \text{ if } x = (g,u) \in G \times X \end{cases} \end{cases}$$

*We also define:*

$$t = \begin{cases} \Gamma \longrightarrow V(\Gamma) \\ x \mapsto \begin{cases} x \text{ if } x \in X \\ (gs,1) \text{ if } x = (g,s) \in G \times \tilde{S} \\ u \text{ if } x = (g,u) \in G \times X \end{cases} \end{cases}$$

*These two maps are continuous on each of the disjoint components and therefore are continuous for the disjoint topology. Their restriction to $V(\Gamma)$ is equal to identity and therefore they define a profinite graph. Now we need to put colors on that graph.*

*We define $C = \tilde{S} \amalg X \amalg$. We equip $C$ with disjoint topology, ($\tilde{S}$ and $X$ having their respective profinite topologies). We define*

$$c = \begin{cases} E(\Gamma) \longrightarrow C \\ x \mapsto \begin{cases} s \text{ if } x = (g,s) \in G \times \tilde{S} \\ g^{-1} \cdot u \text{ if } x = (g,u) \in G \times X \end{cases} \end{cases}$$

*The map $c$ is continuous, because the action of $G$ on $X$ is continuous. As such this defines a colored graph that we shall call the Profinite group action Cayley graph and denote it $Cay(G,S,X)$*

We have the following theorem:

**Theorem 4.6.4.** *Let $G$ be a profinite group topologically generated by a closed subset $S$ and acting on a profinite set $X$. Let $Cay(G,S,X) = (\Gamma, c, C)$ be the associated group action Cayley graph. Then there exists $\Phi$ an isomorphism of topological groups from $G$ to $Aut_c(\Gamma)$, such that $Aut_c(\Gamma)$ $\forall x \in X, g \cdot x = \Phi(g)(x)$.*

*Proof.* Let

$$\Phi = \begin{cases} G \longrightarrow Aut_c(\Gamma) \\ g \mapsto \begin{cases} \Gamma \longrightarrow \Gamma \\ x \mapsto \begin{cases} (gg',s) \text{ if } x = (g',s) \in G \times \tilde{S} \\ (gg', g \cdot u) \text{ if } x = (g',u) \in G \times X \\ (g \cdot x) \text{ if } x \in X \end{cases} \end{cases} \end{cases}$$

$\Phi(g)$ is a continuous morphism on $\Gamma$, since $G$ is a topological group and the action of $G$ on $X$ is continuous. The map $\Phi(g)$ preserves colors. To prove it, take $x \in \Gamma$ and we distinguish two cases.

- Case 1: $x = (g', s) \in G \times \tilde{S}$

$$c(x) = s = c(gg', s) = c(\Phi(g)(x))$$

- Case 2: $x = (g', u) \in G \times X$.

$$c(x) = g'^{-1} \cdot u = (g'^{-1}g^{-1}) \cdot g \cdot u = (gg')^{-1} \cdot (g \cdot u) = c(\Phi(g)(x))$$

The map $\Phi$ is injective, since $\Phi(g)(1,1) = (g,1)$. Now let us prove that $\Phi$ is surjective. Let $u \in Aut_c(\Gamma)$. Then $u$ has to send the vertex $(1,1)$ onto some $(g,1)$ with $g \in G$. Now let us prove that $\forall g' \in G$, $u(g') = \Phi(g)(g')$. Using the standard proof for Cayley graphs as done in 1.5.1, we can prove that for every $g' \in < S >$ and for every $s \in \tilde{S}$, $u((g',s)) = \Phi(g)(g',s)$. Since $< S >$ is dense in $G$ and $u, \Phi(g)$ are continuous, then they are equal on $G \times S$. Now we need to prove that they are equal on $G \times X$. Let $(g',x) \in G \times X$. Since $u$ preserves the colors, we get that $u((g',x)) = (g'',y)$, with $g'^{-1} \cdot x = g''^{-1} \cdot y$.
We get $(g'',1) = o(u(g',x)) = u((g',1))$ and therefore $g'' = gg'$, proving that

$$\Phi(g)((g',x)) = u((g',x))$$

Finally if $x \in X$, we get that

$$u(x) = u(t((1,x))) = t(u(1,x)) = t(\Phi(g)(u(1,x))) = \Phi(g)(x)$$

We therefore conclude that $u = \Phi(g)$, proving that $\Phi$ is surjective.

Now we need to show that $\Phi$ and $\Phi^{-1}$ are continuous.

For that we write $G$ as a limit of $(G_i)_{i \in I}$ with natural projections $\pi_i$ and $X$ a limit of $(X_j)_{j \in J}$, with natural projection $\psi_j$. Then for $i \in I$ and $j \in J$ take

$$u_{i,j} = \begin{cases} \Gamma \longrightarrow G_i \times \tilde{S}_i \amalg G_i \times X_i \amalg X_i \\ x \mapsto \begin{cases} (\phi_i(g), \phi_i(s)) \text{ if } x = (g,s) \in G \times \tilde{S} \\ (\phi_i(g), \psi_j(x)) \text{ if } x = (g,x) \in G \times X \\ (\psi_j(x)) \text{ if } x \in X \end{cases} \end{cases}$$

The profinite set $\Gamma$ together with the transition maps $u_{i,j}$ is a projective limit of $(G_i \times \tilde{S}_i \amalg G_i \times X_j \amalg X_j)_{(i,j) \in I \times J}$, with the partial order on $I \times J$ defined component by component. We will use these maps $u_{i,j}$ to prove continuity of $\Phi$ and the fact that $\Phi$ is open. Let $g \in G$ , $i \in I$ and $j \in J$. The action of $G$ on $X$ is continuous, so there exists $i_0 \in I$, such that

$$\forall i' \geq i_0, \forall g' \in G, \forall x \in X, \pi_{i'}(g') = \pi_{i'}(g) \Rightarrow \psi_j(g \cdot x) = \psi_j(g' \cdot x)$$

Now take $i_1$ greater than both $i$ and $i_0$ and let $i' \geq i_1$. If $\pi_{i'}(g) = \pi_{i'}(g')$, then $\pi_i(g) = \pi_i(g')$ and $\forall x \in X, \psi_j(g \cdot x) = \psi_j(g' \cdot x)$, which proves that $u_{i,j} \circ \Phi(g) = u_{i,j} \circ \Phi(g')$, therefore $\Phi$ is continuous. Since $G$ is compact and $\Phi$ is injective, $\Phi$ is a homeomorphism to its image, so $\Phi^{-1}$ is continuous as well.

This concludes the proof that $G$ and $Aut_c(\Gamma)$ are isomorphic as topological groups. □

Just like in the case of finite graphs, an interesting question is whether is possible to drop the colors, while preserving the automorphism group. I have shown that it is possible in case that both $X$ and $S$ are finite sets by generalizing the Lovász construction given in 1.3.3. I have also shown that it is possible even without the assumption that $X$ and $S$ are finite, but in that case the set of edges will no longer be closed. We will show that such constructions also preserve connectedness, but for that we need the proposition that follows.

**Proposition 4.6.5.** *Let $G$ be a connected profinite graph, $G'$ a profinite graph and $h$ an injective continuous map from $V(G)$ to $V(G')$, such that*

$$\forall e \in E(G), h(o(e)) \text{ and } h(t(e)) \text{ are in the same path-connected component in } G'$$

*Finally suppose that every $g' \in V(G')$ is in a path-connected component of some $h(g)$ with $g \in V(G)$. Then we have that $G'$ is connected.*

*Proof.* To show that $G'$ is connected, we will show that if $R$ is a profinite ring, then $H_0(G', R) = \{0\}$. The map $h$ is a continuous map from $V(G)$ to $V(G')$. It therefore induces a unique continuous linear map of modules $\tilde{h}$: from $R[[V(G)]]$ to $R[[V(G')]]$. Let $\epsilon$ be the augmentation map from $R[[V(G)]]$ to $R$, let $\epsilon'$ be the augmentation map from $R[[V(G')]]$ to $R$.
Let $\partial_1$ be the boundary map from $R[[{}^{G}\!/_{V(G)}, *]]$ to $R[[V(G)]]$ and let $\partial_1'$ be the boundary map from $R[[{}^{G'}\!/_{V(G')}, *]]$ to $R[[V(G')]]$.

We observe that $\epsilon' \circ \tilde{h} = \epsilon$, since it is true for $R[V(G)]$, which is dense in $R[[V(G)]]$. Then we get that $\tilde{h}(\ker(\epsilon)) \subseteq \ker(\epsilon')$, and hence $\tilde{h}$ sends cycles to cycles. Now we need to prove that $\tilde{h}(im(\partial_1)) \subseteq im(\partial_1')$, i.e that it sends boundaries to boundaries. The set $im(\partial_1')$ is closed in $\ker(\epsilon')$ by compactness. We also get that $\partial_1(R[[{}^{G}\!/_{V(G)}, *]]) = \overline{\partial_1(R[{}^{G}\!/_{V(G)}])}$, since $R[{}^{G}\!/_{V(G)}, *]$ is dense in $R[[{}^{G}\!/_{V(G)}, *]]$. Therefore if we prove that $\tilde{h}(\partial_1(R[{}^{G}\!/_{V(G)}, *])) \subseteq im(\partial_1')$, then $\tilde{h}(im(\partial_1)) \subseteq im(\partial_1')$ Now if we take $u \in G$, then $h(o(u))$ and $h(t(u))$ are in the same path component, therefore there exists $p = (u_1, \cdots, u_n)$ a path in $G'$ from $h(o(u))$ to $h(t(u))$. Now take $\tilde{p}$ the corresponding element in ${}^{G'}\!/_{V(G')}$. We then get that $h(\partial_1(\overline{u})) = h(t(u)) - h(o(u)) = \partial_1'(\tilde{p})$, hence $\tilde{h}(R[{}^{G}\!/_{V(G)}, *]) \subseteq im(\partial_1')$ and so $\tilde{h}$ sends boundary to boundary.The induced map $\tilde{h}$ therefore factors into a unique continuous map $H_0(h)$ from $H_0(G)$ into $H_0(G')$. Now let us show that $H_0(h)$ is surjective.

Using the lemma 3.3.3, we know that $\ker(\epsilon')$ is topologically generated by $\overline{u - v}$, with $u, v \in V(G')$. If we therefore prove that all these generators are in the image of $H_0(h)$, then we will know that $H_0(h)$ is surjective. Now let $u, v \in V(G')$. By assumption on $h$, we know that there exist $x, y \in V(G)$ and paths $p$ from $u$ to $h(x)$ and $p'$ from $v$ to $h(y)$. Let $\tilde{p}$ and $\tilde{p}'$ be the corresponding elements in $R[[{}^{G'}\!/\!_{V(G')}]]$. We then get that $u - v = h(x) - \partial_1'(p) + \partial_1'(p') - h(y)$ and therefore in the homology $H_0(h)(\overline{x-y}) = \overline{u-v}$. The map $H_0(h)$ is therefore surjective. Since $G$ is connected, we know that $H_0(G, R) = \{0\}$ and by surjectivity of $H_0(h)$, we get that $H_0(G', R) = \{0\}$. This being true for every profinite ring $R$, the graph $G'$ is connected as well. □

Now we are ready to state the theorem for Lovász construction.

**Theorem 4.6.6** (Finite color substitution)**.** *Let $(G, f, X)$ be an edge colored profinite graph such that $E(G)$ is closed in $G$ and $X$ is a finite set together with its discrete topology. Then there exists a profinite graph $G'$ with a closed set of edges, a continuous injective map $h$ from $V(G)$ to $V(G')$ and an isomorphism of topological groups $\Phi$ from $Aut_f(G)$ to $Aut_f(G')$ such that $\forall g \in Aut_f(G), \Phi(g) \circ h = h \circ g$.*

*Furthermore if we assume $G$ to be connected, we may assume $G'$ to be connected as well.*

*Also if we assume $G$ to be superpath-connected, we may assume $G'$ to be superpath-connected as well.*

*Proof.* Up to relabeling, we can assume without loss of generality that $X$ is the set $\{1, \cdots, n\}$ with some $n \in \mathbb{N}$. Let us write

$$A(G) = \{(e, k, f(e)) | e \in E(G), k \leq f(x) + 2\}$$

and

$$V(G') = V(G) \amalg A(G)$$

We equip $A(G)$ with the following topology: For $e \in E(G)$ and $V$ a neighborhood of $e$ in $E(G)$ contained in $f^{-1}(\{f(e)\})$ and $k \leq f(e) + 2$, define:

$$N_{e,k,V} = \{(e', k, f(e')) | e' \in V\}$$

Then we say that $U$ is open in $A(G)$ if and only if for all $(e, k, f(e)) \in U$, there exists $N$ neighborhood of $e$ contained in $f^{-1}(\{f(e)\})$, such that $N_{e,k,V} \subseteq U$. Let us prove that we define in such a way a topology. We observe that the empty set is open.

The set $A(G)$ is open: indeed if we take $e \in E(G)$ and $k \leq f(e) + 2$, $V = f^{-1}(\{f(e)\})$ is a neighborhood of $e$ by continuity of $f$ and we have $N_{e,k,f^{-1}(\{f(V)\})} \subseteq A(G)$. We observe that a union of open sets is open. Finally let $U, U'$ be two open sets. Let us show that $U \cap U'$ is open. Let $(e, k, f(e)) \in U \cap U'$. Then there exist $V, V' \subseteq f^{-1}(\{f(x)\})$, such that $N_{e,k,V} \subseteq U$ and $N_{e,k,V'} \subseteq V'$. We then have that $V \cap V' \subseteq f^{-1}(\{f(x)\})$ and $N_{e,k,V} \cap N_{e,k',V'} = N_{e,k,V \cap V'}$, therefore $U \cap U'$ is open. We have finished the proof that $A(G)$ is indeed a topological space.

Let us now show that $A(G)$ defined in this way is compact.

Consider the compact set

$$E(G) \times \{0, \cdots, n+2\}$$

and the map $F$ from this set to $A(G)$, defined by:

$$F((e,k)) = \begin{cases} (e, k, f(e)) \text{ if } k \leq f(e) + 2 \\ (e, f(e)+2, f(e)) \text{ else} \end{cases}$$

This map is surjective. Let us show that it is continuous. Let $e \in E(G)$ and let $N_{e,k,V}$ be a neighborhood of $F((e,k))$. Then $F(V \times \{k\}) \subseteq N_{e,k,V}$, if $k \leq f(e)+2$, so $F$ is continuous at $(e,k,V)$. If on the other hand $k \geq f(e)+2$, we get that $F(V \times \{k\}) \subseteq N_{e,f(e)+2,V}$, so $F$ is continuous at $(e,k)$ as well. The map $F$ being continuous at every point, we conclude that $A(G)$ is compact. We define $V(G') = V(G) \amalg A(G)$ together with its disjoint topology. It is compact as a disjoint union of two compacts. For $e \in E(G)$ and $k \leq f(e)+2$, we now write $u(e,k) = (e,k,f(e))$.

Now define

$$\begin{gathered} E(G') = \{(o(e), u(e,0))|\ e \in E(G)\} \cup \\ \{u(e,1), u(e,0)|\ e \in E(G)\} \cup \\ \{(u(e,f(e)), t(e))|\ e \in E(G)\} \cup \\ \{(u(e,k), u(e,k+1))|\ e \in E(G), 1 \leq k \leq f(e)+1\} \end{gathered}$$

We equip it with the topology induced by the product topology on the compact set $V(G')^2$.

Let us now show that $E(G')$ is compact. As it is a union of four components, we can simply show individually that each of these components is compact.

- For the first component, consider the map $F$ from $E(G)$ to $\{(t(e), u(0,e))|e \in E(G)\}$ that to $e$ associates $F(e) = (o(e), u(0,e))$. Let us show that $F$ is continuous. Let $e \in E(G)$. Consider $V$ a neighborhood of $t(e)$ in $V(G)$ and $V'$ a neighborhood of $e$ in $E(G)$ contained in $f^{-1}(\{f(e)\})$. Let us show that there exists a neighborhood of $e$ that gets sent by $F$ to $V \times N_{e,o,V'}$. Since $o$ is continuous, there exists $U \subseteq E(G)$ neighborhood of $e$, such that $o(U) \subseteq V$. If we write $W = U \cap V'$, we get that $F(W) \subseteq V \times N_{e,0,V'}$, proving the continuity of $F$. Given that $F$ is continuous and surjective and $E(G)$ is compact, we get that $\{(t(e), u(0,e))|e \in E(G)\}$ is compact.

- The proof that $\{(u(e,f(e)), t(e))\}$ is compact is very similar to the previous proof, except it uses the continuity of the terminus map, rather than the origin map.

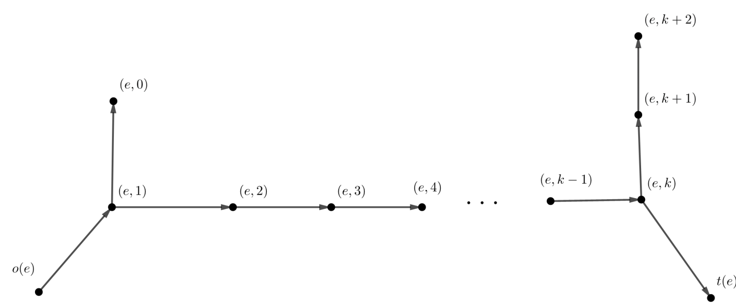
- Consider $F$ the map from $A(G)$ to $\{(u(e,k), u(e,k+1))|e \in E(G), k \leq f(e)+2\}$ that to $u(e,k)$ associates $(u(e,k), u(e,k+1))$, if $0 < k < f(e)+2$.

  The map $F$ is surjective. Let us prove it is continuous. Let $e \in E(G)$ and $k < f(e)+2$ Let $V, V'$ be two neighborhoods of $e$ contained in $f^{-1}(\{f(e)\})$, then we get that $F(N_{e,k,V \cap V'}) \subseteq N_{e,k,V} \times N_{e,k+1,V'}$, proving that $f$ is continuous at $(e,k,f(e))$. From the surjectivity of $F$, the continuity of $F$ and the compactness of $A(G)$, we get that the set $\{(u(e,k), u(e,k+1))\}$ is compact.

- Similarly we can prove that $\{(u(e,1),u(e,0))|e \in E(G)\}$ is compact.

We conclude that $E(G')$ is compact. Now let us define incidence and origin maps on $G' = V(G') \amalg E(G')$ equipped with its disjoint topology. On $V(G')$ we define them as identities. On $E(G')$ we define them as projection on the first and the second component, which makes them continuous, since the topology of $E(G')$ is induced by that of $V(G')^2$.

Then if we consider $(G', V(G'), o, t)$ it is a graph with a compact topology. We observe that this topology is Hausdorff. To obtain $G' = V(G') \amalg E(G')$, we essentially substitute in $G$ an edge of color $k$ with the following graph:




Now we need to prove that $G'$ is profinite.

For that it is sufficient to prove that there exists a set $\Omega$ of open equivalence relations on $G'$, whose intersection is the diagonal. First take $\Omega'$ a set of open relations on $G$ compatible with the incidence maps, whose intersection is the diagonal. We also require that only vertices can be equivalent to vertices, which is possible, since $E(G)$ is closed in $G$. Now for $R \in \Omega$, we define an equivalence relation $\tilde{R}$ on each disjoint component of $G$ separately.

- For two elements in $x, y \in V(G)$, we say that $x\tilde{R}y$ if and only if $xRy$.
- For two elements $u(e,k)$ and $u(e',k')$ in $A(G)$, we say that $u(e,k)\tilde{R}u(e',k')$ if and only if $f(e) = f(e')$, $k = k'$ and $eRe'$.
- Two elements $(o(e), u(e,0))$ and $(o(e'), u(e',0))$ are equivalent if and only if $eRe'$ and $f(e) = f(e')$.
- Two elements $(u(e, f(e)), t(e))$ and $(u(e', f(e')), t(e'))$ are equivalent if and only if $eRe'$ and $f(e) = f(e')$.
- Two elements $(u(e,k), u(e,k+1))$ and $(u(e',k), u(e',k+1))$ are equivalent if and only if $eRe'$ and $f(e) = f(e')$.
- Two elements $(u(e,1), u(e,0))$ and $(u(e',1), u(e',0))$ are equivalent if and only if $eRe'$ and $f(e) = f(e')$.

And no two other elements are equivalent. The relation $\tilde{R}$ is an equivalence relation on each of these disjoint components. No two elements from two distinct components are equivalent and hence $R'$ is an equivalence relation on $G'$.

Then we define $\Omega = \{\tilde{R} | R \in \Omega'\}$. Let us show first that all the relations in $\Omega$ are open. Let $x \in G'$: we distinguish several cases:

- $x \in V(G)$:

  In that case: $x\tilde{R} = xR \cap V(G)$, which is an open subset of $V(G)$, therefore of $G'$.

- $x = u(e, k)$.

  Let us show that $x\tilde{R}$ is open. Let $x'\tilde{R}x$. In that case $x' = u(e', k)$ with $e'Re$ and $f(e') = f(e)$. Let then $V$ be an open neighborhood of $x'$ contained both in $eR$ and $f^{-1}(\{e\})$. We get that $N(e', k, V) \subseteq xR$.The set $xR$ is therefore by definition open in $A(G)$, hence in $G'$.

- $x = (o(e), u(e, 0))$.

  We write $F = \{(o(e'), u(e', 0)) | e' \in E(G)\}$, which is open in $E(G')$, since its complement $\{u(e', f(e')), t(e') | e' \in E(G)\} \cup \{(u(e, k), u(e, k+1)) | k < f(e)+2\}$ is closed. To prove that $x\tilde{R} = x\tilde{R} \cap F$ is open, we will prove that it is a neighborhood of each of its points. Let $x'Rx$. We write $x' = (o(e'), u(e', 0))$. We then get that $eRe'$ and $f(e) = f(e')$. Now consider $V$ a neighborhood of $e$ contained in $eR$ and in $f^{-1}(\{f(e)\})$. We then have that $F \cap V \times N(e', 0) \subseteq x\tilde{R}$. Indeed if $x'' \in F \cap V$, then $x'' = (o(e''), u(e'', 0))$ and

  $e''Re$, $f(e'') = f(e)$, so $x''Rx$. The set $xR$ is therefore open in $F$ and such it is open in $E(G')$.

- The last three cases are proven similarly.

The intersection of all open relations $R$ on $G$ is by assumption reduced to the diagonal. We therefore have that the intersection of all $\tilde{R}$ is reduced to the diagonal as well. We therefore conclude that $G'$ is indeed a profinite graph. Furthermore the set of $E(G')$ is closed in $G'$.

Now let us prove that the group of automorphisms fixing colors of $G$ is isomorphic to the automorphism group of $G'$.

To a continuous automorphism $g$, such that $f \circ g = f$, we associate the automorphism

$$\Phi(g) = \begin{cases} G' \longrightarrow G' \\ x \mapsto \begin{cases} g(x) \text{ if } x \in V(G) \\ u(g(e), k) \text{ if } x = u(e, k) \\ (o(g(e)), u(g(e), 0)) \text{if } x = (o(e), u(e, 0)) \\ (u(g(e), k), u(g(e), k+1)) \text{ if } x = (e, k) \text{ and } k < f(e) + 2 \\ (u(g(e), 1), u(g(e), 0)) \text{ if } x = (u(e, 1), u(e, 0)) \\ (u(g(e), f(e)), g(t(e))) \text{ if } x = (u(e, f(e)), t(e)) \end{cases} \end{cases}$$

This function is well defined, because $g$ preserves colors and is continuous, simply because $g$ is. It is an automorphism with the inverse $\Phi(g^{-1})$ and we have $\Phi(gg') =$

$\Phi(g)\Phi(g')$. Given its formulas $\Phi(g)$ is a morphism as it sends edges on vertices and preserves the origin and terminus maps. The map $\Phi$ is therefore a morphism of groups. We define then the map $h$ to be the natural injection of $V(G)$ into $V(G')$ and we have that

$$\forall g \in Aut_f(G)\Phi(g) \circ h = h \circ g$$

We define a continuous qmorphism $a$ from $G'$ to $G$ as follows:

$$a = \begin{cases} G' \longrightarrow G \\ x \mapsto \begin{cases} x \text{ if } x \in V(G) \\ o(e) \text{ if } x = u(e,k) \\ e \text{ if } x = (o(e), u(e,0)) \\ e \text{ if } x = (u(e,f(e)), t(e)) \\ o(e) \text{ if } x = (u(e,k), u(e,k+1)) \text{ and } k < f(e)+2 \\ o(e) \text{ if } x = (u(e,1), u(e,0)) \end{cases} \end{cases}$$

Note that $a \circ \Phi(g) = g \circ a$ and that $a$ is surjective.

$\Phi$ is injective, because of the formula $a \circ \Phi(g) = g \circ a$.

Now to prove the surjectivity of $\Phi$, we take $g$ a continuous automorphism of $G'$. The map $g$ then has to send a vertex of the type $u(e, f(e)+2)$ on a vertex of in-degree 1 connected to a vertex of in-degree 1 and out-degree 2. The only such vertices are the vertices of the type $u(e', f(e')+2)$. The path $(u(e,f(0)), \cdots u(e,f(e)))$ then has to become $(u(e',0), \cdots, u(e', f(e')))$ by similar arguments as used in 1.3.2. The vertices in $V(G)$ can then only be sent on vertices in $V(G)$ by bijectivity. Then write

$$g' = \begin{cases} G \longrightarrow G \\ x \mapsto \begin{cases} g(x) \text{ if } x \in V(G) \\ a \circ g(\ ((u(x,f(x)), t(x))\ )\ ) \text{ if } x \in E(G) \end{cases} \end{cases}$$

The map $g'$ is continuous given that the maps $a$, $e \mapsto (u(e,f(e)), t(e))$ and $g$ are continuous and $V(E)$ and $E(G)$ are two open disjoint subsets of $G$. The map $g'$ preserves the colors of the edges, since as we have shown, $g(u(e,f(e))) = u(e, f(e'))$, with $f(e) = f(e')$. We therefore conclude that $g' \in Aut_f$ and we have $\Phi(g') = g$. The map $\Phi$ is therefore an isomorphism of groups. Now we need to show that both $\Phi$ and $\Phi^{-1}$ are continuous.

Let $g \in Aut_f(G)$ and $R'$ an open relation on $G'$ We know that there exists $R$ an open relation on $G$, such that $\tilde{R} \subseteq R'$. Now if $g' \in Aut_f(G)$, such that $\forall x \in G$, $g(x)Rg'(x)$, then $\forall x \in G', \Phi(g)(x)\tilde{R}\Phi(g')(x)$, so $\forall x \in G', \Phi(g)(x)R'\Phi(g)(x)$, proving the continuity of $\Phi$.

Now on the other hand take $g \in Aut(G')$ and $R$ a relation on $G$. If we then take a $g' \in Aut(G')$, such that $\forall x \in G', g(x)\tilde{R}g(x')$, then for every $x \in G$, $\Phi^{-1}(g)(x)R\Phi^{-1}(g')(x)$, proving that $\Phi^{-1}$ is continuous as well.

This concludes the proof that $\Phi$ is an isomorphism of topological groups $Aut_f(G)$ and $Aut(G')$ and as we have shown earlier $\forall g \in Aut_f(G), \Phi(g) \circ h = h \circ g$ with $h$ the natural injection of $V(G)$ into $V(G')$.

Now assume that $G$ is connected. We will prove that $G'$ is connected. We will do it using the proposition 4.6.5.

Consider the natural injection $h$ from $V(G)$ into $V(G')$. If we take $e \in E(G)$, we have a path from $o(e)$ to $t(e)$ given by the sequence of vertices

$$( \ o(e), u_0, \cdots, u_{f(e)}, t(e) \ )$$

with $u_i$ being the vertex $u(e,i)$ It is also clear that there is a path from any element of $G'$ into the image of $h$ and therefore by the proposition 4.6.5 $G'$ is connected.

Assume now instead that $G$ is superpath-connected. Let us show that $G'$ is connected. Since every element in $G' \setminus V(G)$ is in the path-connected component of some vertex in $V(G)$, it is enough to prove that all the vertices in $V(G)$ are in the same superpath-connected component. Let $x, y \in V(G)$. Since by assumption $G$ is superpath-connected, there exist $C_1, \cdots, C_n$ path-connected components in $G$ and a sequence $x_i \in \overline{C_i} \cap \overline{C_{i+1}}$, such that $x \in C_1$ and $y \in C_n$. Now consider $C_i' = a^{-1}C_i$. Note that all the vertices in $C_i \subseteq C_i'$ are in the same path-connected component of $G'$ and since every element of $C_i' \setminus C_i$ is in the path-connected component of some element in $C_i$, we get that $C_i'$ is path-connected. Now let us prove that $\overline{C_i'} \cap \overline{C_{i+1}'} \neq \emptyset$ for every $i < n$. Let $y_i$ be such that $a(y_i) = x_i$. Let us prove that $y_i \in \overline{C_i'} \cap \overline{C_{i+1}'}$. If $y_i \in V(G)$, then $y_i = x_i \in \overline{C_i} \subseteq \overline{C_i}$, since $V(G)$ is closed in $G'$. Otherwise we may assume that $y_i = (u(e, f(e), t(e))$ with $x_i = e \in E(G)$. Now let $V$ be a neighborhood of $y_i$. In that case there exists $V'$ a neighborhood of $e$ contained in $f^{-1}(f(e))$ and $U$ a neighborhood of $t(e)$, such that $N_{V',e,f(e)} \times U \subseteq V$. By continuity of the terminus map take $U' \subseteq V'$, such that $t(U') \subseteq U$. Since $e \in \overline{C_i}$, we get that there exists $e' \in U' \cap C_i$. In that case we get that $(u(e', f(e'), t(e'))) \in N_{V',e,f(e)} \subseteq V$. Furthermore $a( \ (u(e', f(e')), t(e')) \ ) = e' \in C_i$, thus $(u(e', f(e')), t(e')) \in V \cap C_i'$, proving that $y_i \in \overline{C_{i+1}}$. We then get that $x \in C_1'$, $y \in C_n'$ and for all $i$, $\overline{C_i'} \cap \overline{C_{i+1}'} \neq \emptyset$ with $C_i'$ connected for all $i$, which means that $x$ and $y$ are in the same superpath-connected component. □

We can also prove a profinite version of a generalization of the theorem of Frucht, which was proved by Sadibussi in 1960. The approach here remains essentially the same: we replace colors by non isomorphic graphs, which have no automorphisms. However we have to be careful, because these graphs will not be profinite, which is where I had the idea to do a certain profinite completion that still makes the graphs non isomorphic and have no additional automorphisms.

**Lemma 4.6.7** (Sabidussi (1960))**.** *For every ordinal $\alpha$, there exists an abstract undirected loopless graph $T_\alpha$, such that $T_\alpha$ has no automorphisms and its cardinal is greater than that of $\alpha$. Furthermore if $\beta$ is an ordinal distinct from $\alpha$, then $T_\beta$ and $T_\alpha$ are non isomorphic as abstract graphs.*

**Theorem 4.6.8** (Infinite color substitution)**.** *Let $(\Gamma, c)$ be a profinite edge colored loopless graph, with $C$ a Hausdorff set of colors, such that $E(\Gamma)$ is closed in $\Gamma$. There exists a profinite graph $\Gamma'$ without colors and maps $\iota, \Phi$ such that:*

*$\iota$ is a continuous injection from $V(\Gamma)$ to $V(\Gamma')$ and $\Phi$ is an isomorphism of topological groups $Aut_c(\Gamma)$ and $Aut_c(\Gamma')$ with $\forall g \in Aut_c(\Gamma), \Phi(g) \circ \iota = \iota \circ g$.*

*If furthermore $\Gamma$ is connected, then we can choose $\Gamma'$ to be connected as well. If $\Gamma$ is superpath-connected, then we can choose $\Gamma'$ to be superpath-connected as well.*

*Proof.* First up to taking a bijection between the set of colors $C$ and a subset of the proper class of infinite ordinals, we can using the Sabidussi's lemma associate to each color $x \in C$ an abstract graph $T_x$ with no automorphisms and such that $T_x$ and $T_y$ are isomorphic as abstract graphs if and only if $x = y$. Furthermore without loss of generality, we may assume that the graphs are pairwise disjoint.

Now denote

$$T = \{(o(x), t(x), e) | e \in E(\Gamma), x \in T_{c(e)}\} \cup \{(x, \infty, e) | e \in E(\Gamma), x \in V(T_{c(e)})\} \cup \{(\infty, y, e) | e \in E(\Gamma), y \in V(T_{c(e)})\} \cup \{(\infty, \infty, e) | e \in E(\Gamma)\}$$

with $\infty$ a point that is in none of the $T_a$'s nor in $\Gamma$.

For $e \in E(\Gamma)$, denote $T_e$ the corresponding copy of $T_{c(e)}$ in $T$, i.e the set

$$T_e = \{(o(x), t(x), e) | x \in T_{c(e)}\}$$

Now we write $\Gamma$ as a projective limit of some finite graphs $(\Gamma_i)_{i \in I}$, with surjective natural projections $p_i$. Furthermore let $J$ be the directed set of finite subsets of $\bigcup_{a \in C} T_a$ for the order of inclusion. We equip the set $I \times J \times J$ with the product order. For each $i \in I$ and $X, Y \in J$, we define

$$\begin{aligned} A_{i,X,Y} = & \{(x, y, p_i(e)) | e \in E(\Gamma), x \in X, y \in Y \text{ and } \exists u \in T_{c(e)}, o(u) = x \text{ and } t(u) = y\} \cup \\ & \{(x, \infty, p_i(e)) | e \in E(\Gamma), x \in V(T_{c(e)}) \cap X\} \cup \\ & \{(\infty, y, p_i(e)) | e \in E(\Gamma), y \in V(T_{c(e)}) \cap Y\} \cup \\ & \{(\infty, \infty, p_i(e)) | e \in E(\Gamma)\}. \end{aligned}$$

These sets are finite, since $X, Y$ is finite and so is $\{p_i(e) | e \in E(\Gamma)\}$ for all $i \in I$. Now define for $X \in J$ and $x \in \bigcup_{e \in E(\Gamma)} V(T_{c(e)}) \cup \{\infty\}$:

$$h_X(x) = \begin{cases} x \text{ if } x \in X \\ \infty \text{ else} \end{cases}$$

Now we will define the transition map from $A_{j,X,Y}$ to $A_{i,X',Y'}$ for $(i, X, Y) \leq (j, X', Y')$. Write:

$$\phi_{(i,X,Y),(j,X',Y')} = \begin{cases} A_{j,X',Y'} \longrightarrow A_{i,X,Y} \\ (x, y, p_j(e)) \mapsto (h_X(x), h_Y(y), p_i(e)) \end{cases}$$

This map is well defined, because the projections $p_i$ are compatible with the transition maps of the graph $\Gamma$. It is a transition map on each component, hence it is a transition map for the product order. Now we will define the natural projections of the set $T$ onto $A_{i,X,Y}$. Write:

$$p_{i,X,Y} = \begin{cases} T \longrightarrow A_{i,X,Y} \\ (x, y, e) \mapsto (h_X(x), h_Y(y), p_i(e)) \end{cases}$$

We shall now prove that $T$ together with these projections is the limit of the projective system:

$$((A_{i,X,Y})_{i\in I,X,Y\in J},(\phi_{(i,X,Y),(j,X',Y')})_{(i,X,Y)\leq(j,X',Y')})$$

Observe that the projections are compatible with transition maps. Let us to prove that if $(u_{i,X,Y})_{(i,X,Y)\in I\times J\times J}$ is a collection compatible with the transition maps, then there exists $u\in T$, such that $\forall(i,X,Y)\in I\times J\times J$, $p_{(i,X,Y)}(u)=u_{(i,X,Y)}$.

For that purpose we distinguish four cases.

- Case 1: For all $(i,X,Y)\in I\times J\times J, u_{i,X,Y}=(\infty,\infty,p_i(e_{i,X,Y}))$.

  Write $e_{i,\emptyset,\emptyset}=e_i$. The collection $p_i(e_i)$ is compatible with the transition maps of $\Gamma$ and $E(\Gamma)$ is closed in $\Gamma$, hence there exists an $e\in E(\Gamma)$ such that $\forall i\in I, p_i(e)=p_i(e_i)$. By compatibility with the transition maps, we get that for all $(i,X,Y)\in I\times J\times J$:

  $$u_{i,X,Y}=(\infty,\infty,p_i(e))$$

- Case 2: There exists $(i_0,X_0,Y_0)\in I\times J\times J$, such that $u_{i_0,X_0,Y_0}=(x,y,p_{i_0}(e'))$ with $x\in T_a, y\in T_a$ and $a=c(e')$.

  For $i\geq i_0$, we take $e_i\in E(\Gamma)$, such that $u_{i,X_0,Y_0}=(x,y,p_i(e_i))$. The elements $p_i(e_i)$ are compatible with transition maps of the graph $\Gamma$ and are all in $E(\Gamma)\cap c^{-1}(\{a\})$, which is closed in $E(\Gamma)$, therefore there exists $e\in E(\Gamma)$, such that $c(e)=a$ and $\forall i\geq i_0, p_i(e)=p_i(e_i)$. Using then the compatibility with the transition maps, we can prove that

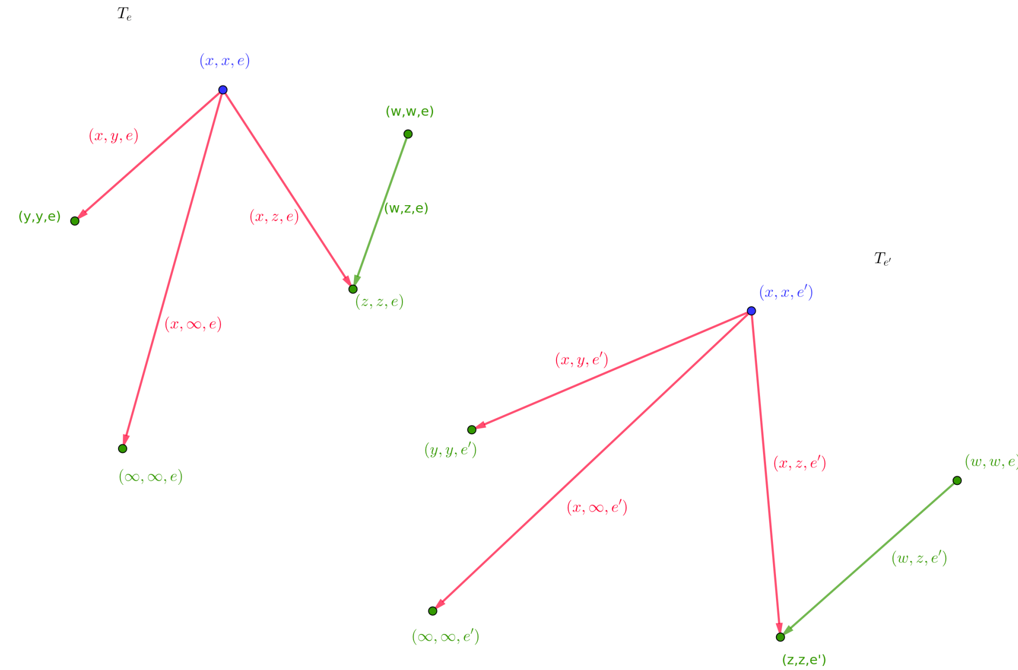
  $$\forall(i,X,Y)\in I\times J\times J, u_{i,X,Y}=p_{i,X,Y}((x,y,e))$$

  The two remaining are proven very similarly to the case two, so we will just mention them for the sake of completeness.

- Case 3: There exists $(i_0,X_0,Y_0)\in I\times J\times J$, such that $u_{i_0,X_0,Y_0}=(x,\infty,p_i(e'))$ with $x\in T_{c(e')}$ and such that for all $I\times J\times J$ the second component of $u_{i,X,Y}$ is $\infty$.

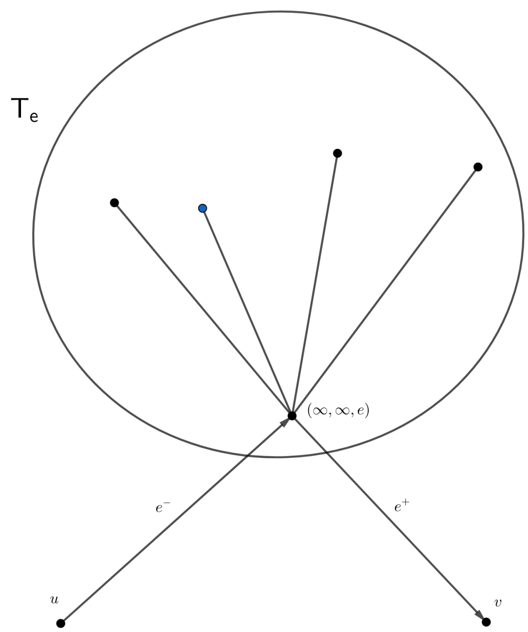
- Case 4: There exists $(i_0,X_0,Y_0)\in I\times J\times J$, such that $u_{i_0,X_0,Y_0}=(\infty,y,p_{i_0}(e'))$ with $y\in T_{c(e')}$ and such that for all $(i,X,Y)\in I\times J\times J$ the first component of $u_{i,X,Y}$ is $\infty$.

Let us take a step back and examine this construction. We start with a disjoint union of the graphs $T_{c(e)}$ and we have to complete it in order to make it a profinite graph. The most naive completion would be a single point compactification. The problem with this approach is that the origin and terminus maps will not necessarily be continuous, unless the graph is locally finite. Instead we identify the graphs $T_e$ with the product of vertices $V(T_e)\times V(T_e)$ with the diagonal corresponding to vertices and its complement to the edges. This identification works, because none of the $T_e$ has neither loops nor multiple edges between two vertices. We compactify these graphs with point at infinity and edges connecting all points to infinity and vice versa. We illustrate the resulting graph below. Choose $i \in I$ and two edges $e, e'$, such that $p_i(e) = p_i(e')$ and such that $c(e) = c(e')$. Then pick a $x \in T_{c(e)}$ and write $X = Y = \{x\}$. We illustrate the elements $u, v \in T$, such that $p_{i,X,Y}(u) = p_{i,X,Y}(v)$ with the same colors.




Now that we have constructed the graph $T$, we will use it to create the graph $\Gamma'$ with the desired properties. The construction of $\Gamma'$ proceeds as follows:

Take $e \in E(\Gamma)$ an edge from $u$ to $v$. Replace it with the edges $e^+$ and $e^-$, with $e^-$ going from $u$ to $(\infty, \infty, e)$ and $e^+$ from $(\infty, \infty, e)$ to $v$, as in the following picture:




The resulting graph will have the desired properties, because by construction, the graphs $T_e$ and $T_{e'}$ can't be isomorphic if $e$ and $e'$ have distinct colors.

Define $E^+ = \{e^+ | e \in E(\Gamma)\}$ and $E^- = \{e^- | e \in E(\Gamma)\}$ two disjoint copies of $E(\Gamma)$. We write:

$$\Gamma' = V(\Gamma) \amalg E^- \amalg E^+ \amalg T$$

and equip it with the disjoint topology. Since all the individual topologies on the disjoint components are profinite, the overall topology will be profinite as well. To make it into a graph, we need to define a closed subset of vertices and the continuous maps origin and terminus.

We write:

$$V(\Gamma') = V(\Gamma) \cup \{(x, x, e) | e \in E(\Gamma), x \in V(T_{c(e)})\}$$

as the set of vertices. To prove that is closed, we will show that its complement is a neighborhood of each of its points. Take $q$ in the complement of $V(\Gamma')$. We differentiate several cases.

- Case 1: $q \in E^+ \cup E^-$

  By the definition of disjoint topology $E^+ \cup E^-$ then is a neighborhood of $q$ and it does not intersect $V(\Gamma')$.

- Case 2: $q = (\infty, y, e)$, with $y \in V(T_{c(e)})$.

  Let $i$ be any element of $I$, $X = \{y\}$ and $Y = \{y\}$. Now take $q' = (x', y', e') \in T$, such that $p_{i,X,Y}(q') = p_{i,X,Y}(q) = (\infty, y, p_i(e))$. That means that $h_X(x') = \infty$ and therefore $x' \notin \{y\}$, so $x' \neq y$. Furthermore $h_Y(y') = y$ and therefore $y' = y$. As a conclusion $q' = (x', y, e')$, with $x' \neq y$, therefore $q'$ cannot be in $V(\Gamma')$. We conclude that the complement of $V(\Gamma')$ is indeed a neighborhood of $q$.

- Case 3: $q = (x, \infty, e)$ with $x \in V(T_{c(e)})$.

  This case is essentially the same as the previous one and will therefore be omitted.

- Case 4: $q = (x, y, e)$ wtih $(x, y)$ being an edge in $T_{c(e)}$.

  Take again $i$ any element of $I$, $X = \{x\}$ and $Y = \{y\}$. Now assume that $q' = (x', y', e') \in T$ is such that $p_{i,X,Y}(q') = p_{i,X,Y}(q) = (x, y, p_i(e))$. Then we get that $x = x'$ and $y = y'$. We conclude that $q'$ is not a vertex. We have again that the complement of the set of vertices is a neighborhood of $q$.

From this we can indeed conclude that $V(\Gamma)$ is closed in $\Gamma$. Now is the time to define the origin and terminus maps and prove that they are continuous.

Write

$$o = \begin{cases} \Gamma' \longrightarrow V(\Gamma') \\ q \mapsto \begin{cases} q \text{ if } q \in V(\Gamma) \\ o(e) \text{ if } q = e^- \in E^- \\ (\infty, \infty, e) \text{ if } q = e^+ \\ (x, x, e) \text{ if } q = (x, y, e) \in T \end{cases} \end{cases}$$

and

$$t = \begin{cases} \Gamma' \longrightarrow V(\Gamma') \\ q \mapsto q \text{ if } q \in V(\Gamma) \\ (\infty, \infty, e) \text{ if } q = e^- \\ t(e) \text{ if } q = e^+ \\ (y, y, e) \text{ if } q = (x, y, e) \in T \end{cases}$$

Since both cases are very similar, we will only prove that $o$ is continuous. To do that, we will prove that $o$ is continuous at every point. Take $q \in V(\Gamma')$. We differentiate the following cases:

- Case 1: $q \in V(\Gamma)$.

  This part is straightforward, because of the structure of $\Gamma$ and continuity of its own origin map.

- Case 2: $q \in E^-$:

  Same reasoning as before.

- Case 3: $q = e^+ \in E^+$.

  Let $i \in I, X \in J$ and $Y \in J$. If $e'^+ \in E^+$ is such that $p_i(e') = p_i(e)$, then

  $$p_{i,X,Y}(o(e')) = p_{i,X,Y}(\infty, \infty, e') = (\infty, \infty, p_i(e')) = p_{i,X,Y}(o(e^+))$$

  This proves that $o$ is continuous at $q$.

- Case 4: $q = (x, y, e) \in T$, with $x$ and $y$ vertices in $T_{c(e)}$.

  Let $i \in I$, $X \in J$ and $Y \in J$. We write $X' = \{x\}$ and $Y' = \emptyset$. Now suppose that $q' = (x', y', e') \in T$, such that

  $$p_{i,X',Y'}(q') = p_{i,X',Y'}(x, y, e) = (x, \infty, p_i(e))$$

  Then $x = x'$ and therefore $o(q') = (x, x, e')$. We get that

  $$p_{i,X,Y}(o(q')) = (h_X(x), h_Y(x), p_i(e')) = (h_X(x), h_Y(x), p_i(e)) = p_{i,X,Y}(o(q))$$

  This proves the continuity of $o$ at $q$.

- Case 5: $q = (x, \infty, e) \in T$, with $x$ a vertex in $T_e$.

  Let again $i \in I$, $X \in J$ and $Y \in J$. Write $X' = \{x\}$ and $Y' = \emptyset$. Suppose that $q' = (x', y', e')$ is such that

  $$p_{i,X',Y'}(q') = p_{i,X',Y'}(q) = (x, \infty, p_i(e))$$

  We then get that $x' = x$ and as such

  $$p_{i,X,Y}(o(q')) = (h_X(x), h_Y(x), p_i(e')) = (h_X(x), h_x(Y), p_i(e)) = p_{i,X,Y}(o(q))$$

- Case 6: $q = (\infty, y, e) \in T$. Let $i \in I$ and $X, Y \in J$.

  Let $X' = X \cup Y$ and $Y' = \emptyset$. Suppose that $q' = (x', y', e') \in T$, such that:

  $$p_{i,X',Y'}(q') = (h_{X'}(x'), \infty, p_i(e')) = h_{i,X,Y}(q) = (\infty, \infty, p_i(e))$$

  Since $h_{X'}(x') = \infty$, we deduce that $x'$ is neither in $X$ nor $Y$. As such we get:

  $$p_{i,X,Y}(o(q')) = (h_X(x'), h_Y(x'), p_i(e')) = (\infty, \infty, p_i(e)) = h_{i,X,Y}(q)$$

- Case 7: $q = (\infty, \infty, e)$. Let $i \in I$ and $X, Y \in J$. Write $X' = X \cup Y$ and $Y' = \emptyset$. Suppose that $q' = (x', y', e' \in E) \in T$, such that

  $$p_{i,X',Y'}(q') = (h_{X'}(x'), h_{Y'}(y'), p_i(e')) = p_{i,X',Y'}(\infty, \infty, e) = (\infty, \infty, p_i(e))$$

  Then since $h_{X'}(x') = \infty$, we can conclude that $x' \notin X'$ and so $x'$ is neither in $X$ nor in $Y$. From this we get that

  $$p_{i,X,Y}(o(x', y', e')) = (h_X(x'), h_Y(x'), p_i(e')) = (\infty, \infty, p_i(e))$$

Note that while the projections $p_{i,X,Y}$ define a profinite topology, they are not q-morphisms. Now that we proved that $\Gamma'$ is a profinite graph, we will prove that it has the properties of the proposition. The injection $\iota$ from $V(\Gamma)$ to $V(\Gamma')$ is simply the natural inclusion, which is continuous.

Now define:

$$\Phi = \begin{cases} Aut_c(\Gamma) \longrightarrow Aut(\Gamma') \\ g \mapsto \begin{cases} \Gamma' \longrightarrow \Gamma' \\ q \mapsto \begin{cases} g(q) \text{ if } q \in V(\Gamma) \\ g(e)^- \text{ if } q = e^- \\ g(e)^+ \text{ if } q = e^+ \\ (x, y, g(e)) \text{ if } q = (x, y, e) \in T \end{cases} \end{cases} \end{cases}$$

This map is well defined, because $g$ preserves colors, so if $x$ or $y$ or both valid vertices are in $T_e$, then they are the same vertices in $T_{g(e)}$, which is the copy of the same set $T_{c(e)} = T_{c(g(e))}$. We can check at each point that $\Phi(g)$ is continuous.

Now let us show that $\Phi$ is continuous.
Let $(i, X, Y) \in I \times J \times J$. If we take $g, g' \in Aut_c(G)$, such that $p_i \circ g = p_i \circ g'$, then for every $q$ in the complement of $T$, $p_i \circ \phi(g)(q) = p_i \circ \Phi(g')(q)$ and for every $q = (x, y, e)$ in $T$, we get that

$$\pi_{i,X,Y} \circ \Phi(g)(q) = (h_X(x), h_Y(y), p_i(g(e))) = (h_X(x), h_Y(y), p_i(g'(e))) = \pi_{i,X,Y} \circ \Phi(g')(q)$$

which concludes the proof of the continuity of $\Phi$. All that remains to do if to find the inverse and prove that it is continuous as well.

Now let us take $g \in Aut(\Gamma')$. The only one sided edges in this graph are those in $E^+$ and $E^-$, so $g(E^-) \subseteq E^+ \cup E^-$. Now let us show that $g(E^-) = E^-$ and therefore that $g(E^+) = g(E^+)$. If we take $e^- \in E^-$, then $o(e^-) \in V(\Gamma)$. If by contradiction $g(e^-) \in E^+$, then $o(e^-)$ gets sent by $g$ to some $(\infty, \infty, e')$. However $(\infty, \infty, e')$ has some two sided edges connected to it and $o(e^-) = o(e)$ does not, which contradicts the fact that $g$ is an automorphism of a graph. As such, we get that $g(E^-) = E^-$.

Since $g$ is an automorphism sending infinity on infinity, we get that for every $e \in E(\Gamma)$, there exists a unique $u_g(e) \in E(\Gamma)$, such that $g(\infty, \infty, e) = (\infty, \infty, u_g(e))$. Let us show that $u_g$ preserves colors and is continuous. The fact that $u_g$ preserves colors is simply because if we restrict $g$ to $T_e$, we get that $g(T_e) \subseteq T_{u(e)}$ and $g(T_{u(e)}) \subseteq T_e$, so $g$ is an isomorphism between $T_e$ and $T_{u(e)}$. By construction such an isomorphism is only possible if $c(u_g(e)) = c(e)$.

Now for the continuity of $u_g$. Let $i \in I$. By continuity of $g$, there exists $(j, X, Y) \in I \times J \times J$, such that if $p_{j,X,Y}(x, y, e') = p_{j,X,Y}(\infty, \infty, e)$, then

$$p_{i,\emptyset,\emptyset}(g(x, y, e')) = p_{i,\emptyset,\emptyset}(g(\infty, \infty, e)) = (\infty, \infty, p_i(u(e)))$$

Now if we take $e'$, such that $p_j(e') = p_j(e)$, then

$$p_{i,\emptyset,\emptyset}(g(\infty, \infty, e')) = p_{i,\emptyset,\emptyset}(g(\infty, \infty, e))$$

and so $p_i(u(e')) = p_i(u(e))$, proving the continuity of $u$.

From what precedes we can see that $g$ sends $V(\Gamma)$ onto $V(\Gamma)$ and thus we can define:

$$\Psi = \begin{cases} Aut(\Gamma') \longrightarrow Aut_c(\Gamma) \\ g \mapsto \begin{cases} \Gamma \longrightarrow \Gamma \\ q \mapsto \begin{cases} g(q) \text{ if } q \in V(\Gamma) \\ u_g(q) \text{ if } q \in E(\Gamma) \end{cases} \end{cases} \end{cases}$$

$\Psi$ is then an inverse to $\Phi$.

To prove the continuity of $\Psi$, we take an $i \in I$. Then we take $g, g' \in Aut(\Gamma')$, such that for every $x \in T$, $p_{i,\emptyset,\emptyset}(g(x)) = p_{i,\emptyset,\emptyset}(g'(x))$ and for every $x \notin T$, $p_i(g(x)) = p_i(g'(x))$. In that case if we take $a \in V(\Gamma)$:

$$p_i \circ \Psi(g)(a) = p_i \circ g(a) = p_i \circ g'(a) = p_i \circ \Psi(g)(a)$$

If on the other hand $a \in E(\Gamma)$:

$$p_i \circ \Psi(g)(e) = p_i(u_g(e)) = p_i(u_{g'}(e)) = p_i \circ \Psi(g')(e)$$

We can therefore conclude that $\Psi$ is indeed continuous, which concludes that $Aut_c(\Gamma)$ and $Aut(\Gamma')$ are isomorphic.

Finally assume that $\Gamma$ is connected and let us prove that $\Gamma'$ is connected. Take $\iota$ the natural injection of $V(\Gamma)$ to $V(\Gamma')$. We have for every $e \in E(\Gamma)$ that $\iota(o(e))$ and $\iota(t(e))$ are in the same component in $\Gamma'$, since $(o(e), e^-, e^+, t(e))$ forms a path from $o(e)$ to $t(e)$. Every element of $\Gamma'$ is in the path component of some $\iota(u)$ with $u \in V(\Gamma)$, so by 4.6.5, $\Gamma'$ is connected.

Similarly to 4.6.6 we can prove that if we $\Gamma$ is superpath-connected, then $\Gamma'$ is superpath-connected as well. □

## 4.7 Application to the topology of profinite groups

In this section we will prove that every profinite group is a group of autohomeomorphisms of a compact connected Hausdorff space together with its open compact topology.

It is a generalization of a result published by Karl Hofmann and Sidney Morris in [25] and [24] proving that every profinite group with one generator is a group of automorphisms of a Hausdorff compact space. The approach is the following: we start with a profinite graph and we replace each edge with a specially constructed curve called de Groot space, which is compact connected and has no local autohomeomorphisms. These curves were constructed by de Groot in [20] and [19]. Once the edges are replaced in such a way, we will show that the autohomeomorphism group is isomorphic to the automorphism group of the graph. Then using the profinite analogue of the theorem of Sabidussi, we will construct a graph with a given profinite group as a group of automorphisms and closed set of edges.

Due to certain difficulties related to whether the set of edges of a profinite graph is closed or not, I originally proved this theorem only for finitely generated profinite groups as opposed to all profinite groups. It is important to mention that such a result has already been established in a more general context by Paul Gartside and

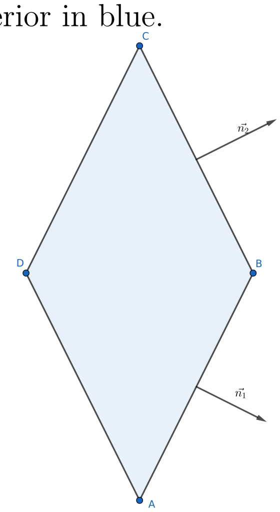
Anerin Glyn in [18], who showed that every metric profinite group is a group of autohomeomorphisms of a connected compact Hausdorff space. Their methodology was however different from the one that I and Morris with Hofmann used.

Before proving the Morris and Hofmann conjecture, we will shed more light on the de Groot construction as given in [20] as in its original form it is not very detailed. While the original proof uses undefined curves called propellers, in our version we will use rhombi which have the same topological properties needed in the proof and will enable us to do the proof in greater detail, while making use of convex geometry. We start by properly defining a rhombus.

**Definition 4.7.1** (Rhombus)**.** *Let $\mathbb{R}^2$ be the two dimensional affine space equipped with its standard dot product $\cdot$ giving a Euclidean metric $d$ and the standard orientation. A rhombus is a set of four points $A, B, C, D$ in $\mathbb{R}^2$, such that $d(A, B) = d(B, C) = d(C, D) = d(D, A) > 0$ and such that the oriented angles $\widehat{DAB}$, $\widehat{ABC}$, $\widehat{BCD}$ and $\widehat{CDA}$ all have a measure between $0$ and $\pi$. We call the segments $[AB]$,$[BC]$, $[CD]$ and $[DA]$ the faces of the rhombus.*

*Now take $\vec{n_1}, \vec{n_2}$ the unique unit vectors, such that the angles $(\widehat{\vec{AB}, \vec{n_1}})$ and $(\widehat{\vec{BC}, \vec{n_2}})$ have a measure $\frac{\pi}{2}$. We say that a point $X$ is below a face $f$, if $\vec{AX} \cdot \vec{n_1} < 0$ and $f = [AB]$,or $\vec{BX} \cdot n_2 < 0$ and $f = [BC]$ or $\vec{CX} \cdot n_1 > 0$ and $f = [CD]$ or $\vec{DA} \cdot \vec{n_2} > 0$ and $f = [AD]$. We say that the point is above the face $f$, if the inequalities above are in the other way. We call the interior of the rhombus $ABCD$ the set of points which are below all its faces.*

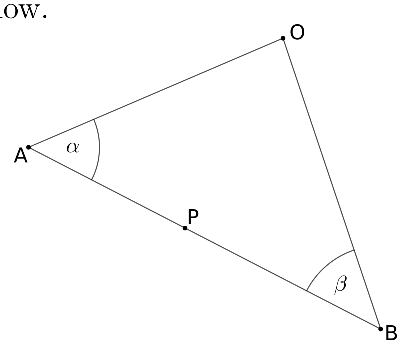
Here is an illustration of a rhombus together with the vectors $\vec{n_1}$ and $\vec{n_2}$ and its interior in blue.




Now we will prove a property of rhombus that we will use.

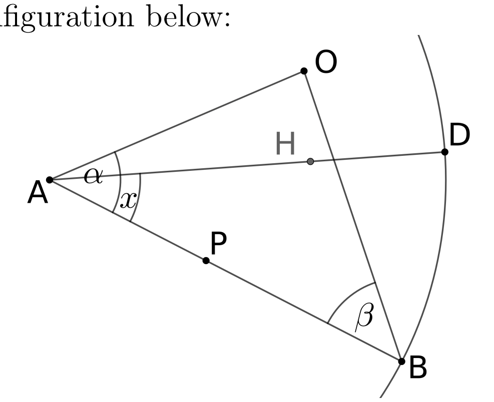
**Lemma 4.7.2.** *Let $R$ be a rhombus and $O$ a point that is neither in the interior of $R$ nor on the faces of $R$. Let $P$ be the closest point of the faces to $O$, then $O$ is above at least one of the faces containing $P$.*

*Proof.* We have two distinct possibilities: either $P$ is in the interior of one of the faces or $P$ is on the intersection of two faces. Let us begin by examining the case where $P$ is in the interior of a face $f = [AB]$. We denote the measure of the angle $\widehat{OAB}$ in $[-\pi, \pi]$ and $\beta$ the measure of the angle $\widehat{ABO}$ in $[-\pi, \pi]$. Up to a choice of orientation we may assume that $\alpha$ and $\beta$ are positive. The situation is illustrated below.




Now let $D$ and $C$ be the points such that $R$ is the rhombus $ABCD$. We denote $x$ the measure of the angle $\widehat{BAD}$. To prove that $O$ is above the face $[AB]$, we shall prove that $x$ is negative and therefore the orientation we chose does not conflict with the definitions above and below for the rhombus $ABCD$.

By contradiction, assume that $x > 0$. We will deal with the case where $x < 2\alpha$. We take $H$ the orthogonal projection of $O$ onto the line $(AD)$. We illustrate this configuration below:




Now assume that $d(A, O) \leq d(A, B)$. We will prove that $H$ belongs to the segment $[AD]$ and that the distance $d(H, O)$ is smaller than $d(P, O)$, which will give a contradiction. We have $\vec{AH} \cdot \vec{AD} = \vec{AO} \cdot \vec{AD} = d(A, D)d(A, O)cos(x - \alpha)$. Since $0 < x < 2\alpha$, we get that $-\alpha < x - \alpha < \alpha$. Now $\alpha$ has to be less than $\frac{\pi}{2}$, otherwise $P$ wouldn't be on the segment $[AB]$. From this we deduce that $cos(\alpha - x)$ is positive and hence the angle between $\vec{AH}$ and $\vec{AD}$ is 0. Furthermore $d(A, H) = cos(x - \alpha)d(A, O) < d(A, D)$ and thus $H$ belongs to $[AD]$. In this case $d(O, H) = |sin(x - \alpha)|d(A, O) < sin(\alpha)d(A, O) = d(O, P)$, since $0 < x < 2\alpha$. Now assume instead that $d(A, O) > d(A, B)$. If $H$ belongs to $[AD]$, the same reasoning still works and we find a point on one of the faces of $R$ that is closer to $O$ than $P$. If on the other hand $H$ does not belong to $[AD]$, then we get that

$$d(O, D)^2 = d(A, O)^2 + d(A, D)^2 - 2d(A, D)d(A, O)cos(x - \alpha)$$

by the law of cosines.

Furthermore we have that $cos(x-\alpha)d(A,O) = d(A,H)$ and hence

$$cos(x-\alpha)AO > AD$$

From this we obtain that

$$d(O,D)^2 < d(A,O)^2 - d(A,D)^2$$

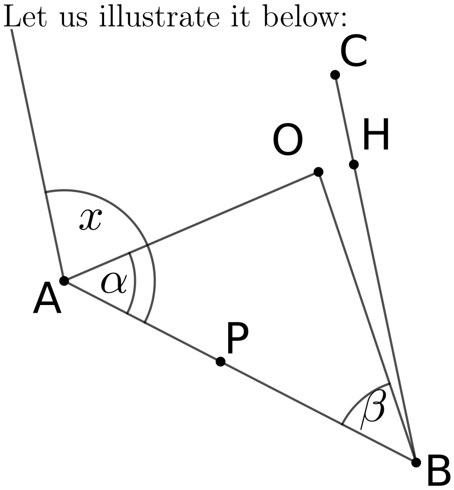
By the theorem of Pythagoras, we also get that

$$d(O,P)^2 = d(A,O)^2 - d(A,P)^2 > d(A,O)^2 - d(A,B)^2$$

From this we conclude that $d(O,D)^2 < d(O,P)^2$, which is again a contradiction.

Now that we have dealt with the case $0 < x < 2\alpha$, we will work on the case $\pi > x > \pi - 2\beta$ and this time we take $H$ to be the projection of $O$ onto the line $(BC)$.

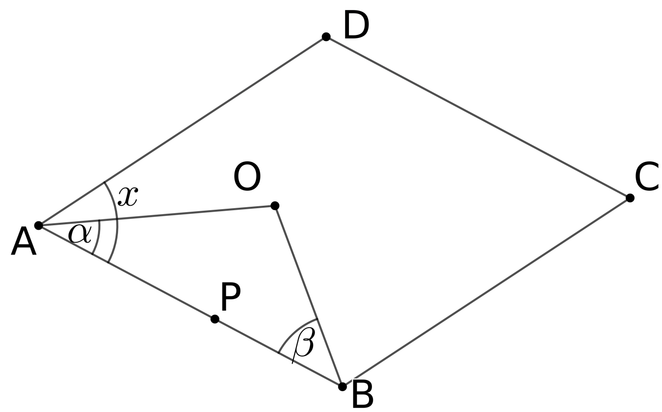
Let us illustrate it below:




The angles $\widehat{ABC}$ and $\widehat{DAB}$ are complementary, since $ABCD$ is a rhombus and thus if $\widehat{DAB}$ is between $\pi-2\beta$ and $\pi$, then $\widehat{ABC}$ will be between 0 and $2\beta$ and thus we can do the same reasoning to prove that either $H$ is on a face of the rhombus and is closer to $O$ than $P$ or $C$ is closer to $O$ than $P$ and get a contradiction.

We have so far examined two cases: $x$ in the open interval $(0, 2\alpha)$ and $x$ in the open interval $(\pi - 2\beta, \pi)$. If these two intervals overlap each other that is to say $2\alpha > \pi - 2\beta$, which is equivalent to $\alpha + \beta > \frac{\pi}{2}$, then we are done. Suppose then instead that $\alpha + \beta \leq \frac{\pi}{2}$. We will then show that for $\alpha \leq x \leq \pi - \beta$, $O$ is on the rhombus $R$ (either inside or on a face).

Let us illustrate it below:

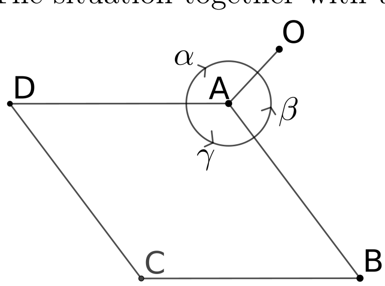



Since the angle $\widehat{OAB}$ is $\alpha$ and is non negative, $O$ is below or on the face $[AB]$. The point $O$ is below or on the line $(AD)$, since the angle $\widehat{OAD}$ is $x - \alpha \geq 0$. The point $O$ is below or on the line $(BC)$, since the angle $\widehat{OBC}$ is $\pi - x - \beta \geq 0$. Finally we need to prove that $O$ is below or on the line $(CD)$. Now let $\vec{n}$ be the normalization of the vector $\vec{PO}$ and $\vec{m}$ it's complement into a direct orthonormal basis. Since the line $(CD)$ is orthogonal to $\vec{n}$, to prove that $O$ is below the face $[CD]$ or on the line $(CD)$, it is enough to prove that that the height of $D$ in the coordinate system defined by $A, \vec{n}, \vec{m}$ is at least that of $O$. The height of $O$ is $d(P,O) = d(A,O)sin(\alpha) = d(B,O)sin(\beta)$. The height of $D$ is $d(A,B)sin(x)$. It is therefore increasing on the interval $[\alpha, \frac{\pi}{2}]$ and since we get that for $x = \alpha$ it is $d(A,B)sin(\alpha) \geq d(A,O)sin(\alpha)$ (the angle $\widehat{AOB}$ is obtuse), we get that for every $x$ between $\alpha$ and $\frac{\pi}{2}$, the point $O$ is below the face $[CD]$. Now if we take $x$ in $[\frac{\pi}{2}, \pi - \beta]$, we recenter our coordinate system in $B$. Since we shift by a vector orthogonal to $\vec{n}$, the height won't change and we can do the same reasoning. The height of $C$ will then be $sin(\pi - x)d(A,B) = sin(x)d(A,B)$. In that case the height is decreasing on the interval $[\frac{\pi}{2}, \pi - \beta]$. At the value $\beta$ it is greater than that of $O$, so we have our conclusion.

Since we have shown that it is impossible for the angle $x$ to be positive, we now know that it is negative and hence $O$ is above the face $[AB]$ as expected.

Now we need to deal with the special case where the projected point $P$ is equal to one of the vertices of the rhombus. Suppose without loss of generality that $P = A$.

The situation together with the labels of important angles is illustrated below:



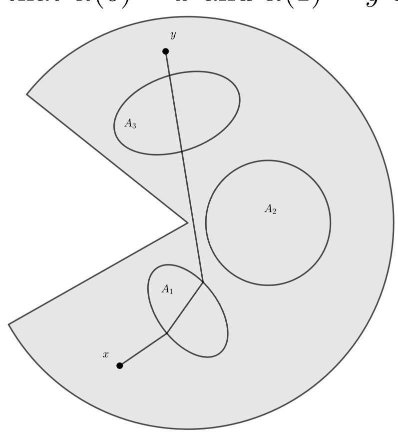

Note that we have both the angles $(\widehat{\vec{AD}, \vec{AO}})$ and $(\widehat{\vec{AB}, \vec{AO}})$ are obtuse, since the orthogonal projection of $O$ on respectively $(AD)$ and $(AB)$ is on respectively the semi lines $[DA)$ and $[BA)$.

Now let $\alpha$ be the measure between $-\pi$ and $\pi$ of the angle $(\widehat{\vec{AD}, \vec{AO}})$ and $\beta$ the measure between $-\pi$ and $\pi$ of the angle $(\widehat{\vec{AB}, \vec{AO}})$. Now we shall prove that $O$ is above the face $[AD]$ or the face $[BA]$. To do that it is enough to prove that $\alpha$ is less than 0 or $\beta$ is greater than 0. Suppose that $\beta < 0$. Then since the angle $(\widehat{\vec{AB}, \vec{AO}})$ is obtuse, we get that $-\pi < \beta \leq -\frac{\pi}{2}$. Now let $\gamma$ be the measure between $-\pi$ and $\pi$ of the angle $(\widehat{\vec{AD}, \vec{AB}})$. We have that $\alpha \equiv \beta + \gamma[2\pi]$. Furthermore $-\pi < \beta + \gamma < -\frac{\pi}{2}$. From that we conclude that $-\pi < \alpha < \frac{\pi}{2}$. However we know that the angle $\alpha$ is obtuse, hence $-\pi < \alpha, \leq -\frac{\pi}{2}$ and therefore $O$ is above the face $[AD]$.

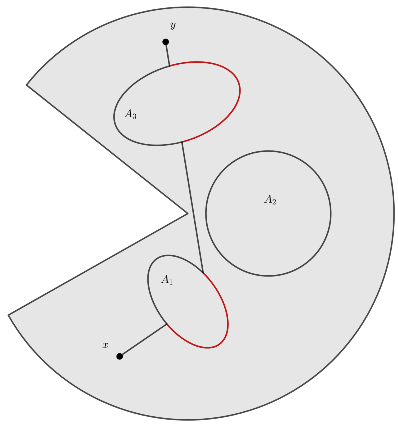
We therefore conclude that in all cases there exists a face containing the closest point to $O$, such that $O$ is above that face. □

**Lemma 4.7.3.** *Let $C \subseteq \mathbb{R}^2$ be a set that is broken line connected. Let $A_1, \cdots, A_n$ be open and pairwise disjoint subsets of $C$, such that for all $k$ the boundary $\partial A_k$ is path-connected and if a line intersects $A_k$, then it intersects $\partial A_K$ at finitely many points. We then have that $C \setminus (A_1 \cup \cdots \cup A_n)$ is path-connected.*

*Proof.* Take $x, y \in C \setminus (A_1 \cup \cdots \cup A_n)$. Now take $\alpha : [0,1] \to C$ a broken line in $C$, such that $\alpha(0) = x$ and $\alpha(1) = y$ as illustrated below




This broken line can of course intersect some of the $A_k$, however whenever it enters an open $A_k$ at a point $P$, it has to exit it at a point $P'$. By assumption, the boundary is path-connected, hence we replace the portion of the broken line inside $A_k$ between $P$ and $P'$ by a path on the boundary from $P$ to $P'$ as illustrated in red below:




By assumption each segment in the broken line can intersect $A_k$ only finitely many of times, hence the total number of replacements will be finite and the resulting path will connect $x$ and $y$ in $C \setminus (A_1 \cup \cdots \cup A_n)$. □

**Lemma 4.7.4.** *Let $X$ be a compact Hausdorff topological space and $\Omega$ a non empty totally ordered set for inclusion of closed connected subspaces. Then $\bigcap_{A\in\Omega} A$ is a connected topological space.*

*Proof.* Write $Y = \bigcap_{A\in\Omega} A$. Let $F_1$ and $F_2$ be two closed subsets of $X$, such that $Y$ is a disjoint union of $F_1$ and $F_2$. Since $X$ is compact Hausdorff, it is normal, i.e. there exist two $U_1$ and $U_2$ two disjoint opens in $X$, such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$. Now $\bigcap_{A\in\Omega} A \setminus (U_1 \cup U_2) \subseteq Y \setminus (F_1 \cup F_2) = \emptyset$ is a decreasing intersection of closed sets, therefore by compactness for some $A \in \Omega$, $A \setminus (U_1 \cup U_2) = \emptyset$. We then have $A \subseteq U_1 \cup U_2$ with $A \cap U_1$ and $A \cap U_2$ being disjoint since $U_1$ and $U_2$ are disjoint. The set $A$ is by assumption connected, so without loss of generality, we may assume that $U_2 \cap A = \emptyset$. In that case $F_2 \subseteq Y \cap U_2 \subseteq U_2 \cap A = \emptyset$ and hence $Y$ is connected. □

**Lemma 4.7.5.** *Let $D$ be a closed disc of positive radius and let $A_1, \cdots, A_n$ be a sequence of distinct points on $D$, such that $A_1$ and $A_n$ are on the boundary of $D$. Suppose furthermore that for every $k$ between 1 and $n-1$ and every $k', k''$ distinct from $k$ and $k+1$, the points $A_{k'}$ and $A_{k''}$ are on the same open half space delimited by the line $(A_k A_{k+1})$. Let $\ell$ be the broken line $(A_1 \cdots A_n)$. Then there exists an open convex subset $U \subseteq D$, whose boundary is $\ell$ and such that $D \setminus U$ is path-connected. We call $U$ the inside of $\ell$ and $D \setminus (U \cup \ell)$ the outside*

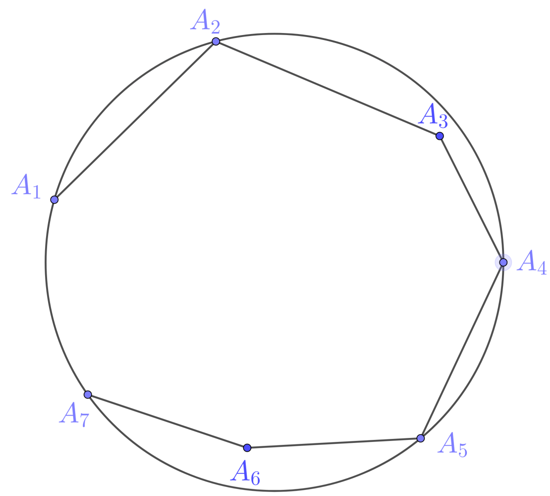
*Furthermore suppose that $B_1, \cdots, B_m$ is another such sequence with broken line $\ell'$ associated to it. Suppose that $\ell' \subseteq U \cup \ell$ and the the outside of $\ell$ and the outside of $\ell'$ have a non empty intersection. Furthermore assume that $\ell$ and $\ell'$ have at most one point of intersection $P$ and that $P \in \{A_1, \cdots, A_n\} \cap \{B_1, \cdots, B_m\}$. Finally assume one of the following holds true for $P$:*

- *$P = A_1$ or $P = A_n$*
- *$P \neq B_1$ and $P \neq B_m$*

- *$P = B_1$, $P = A_k$ and $A_{k-1}$ and $A_{k+1}$ are on the same side of the face $[B_1B_2]$.*
- *$P = B_m$, $P = A_k$ and $A_{k-1}$ and $A_{k+1}$ are on the same side of the face $[B_{m-1}B_m]$*

*Then the inside of $\ell'$ is included in the inside of $\ell$.*

*Proof.* Here is an illustration for the situation with $n = 7$.




In case $n = 2$, the result is trivial. If $n > 2$, for each $k < n$, define $P_k$ as the intersection of $D$ and the open half plane delimited by $(A_kA_{k+1})$ and containing all the other $A$s. Define then $U = \bigcap_{k=1}^{n-1} P_k$. The set $U$ has an empty intersection with the broken line $\ell$.The set $U$ is convex as an intersection of convex spaces. Let us show that the boundary of $U$ is $l$. First note that if $x \in \overline{U}$, then $x$ has to be on the intersection of closed half planes delimited by $(A_k, A_{k+1})$. If $x$ is not on either of these lines, then $x$ is inside of $U$. If on the other hand $x$ is on a line $(A_k, A_{k+1})$: we deal with the following cases:

- Case 1: $x$ is on the line $(A_1, A_2)$.

  Since $A_1$ is on the boundary of $D$ and $A_2$ is in $D$, then the semi line $A_1 + t\vec{A_1A_2}$ with $t < 0$ is outside of $D$, hence $x$ cannot be on that semi line. The point $x$ is therefore on the semi line $[A_1A_2)$. Since $A_1$ is on the open half space $P_2$, then the semi line $A_2 + t\vec{A_2A_1}$ $(t < 0)$ is on the opposite open half plane and $x$ cannot belong to it. The point $x$ is therefore on the semi line $[A_2A_1)$ and hence $x$ in on the segment $[A_1A_2]$ and therefore on the broken line $\ell$.

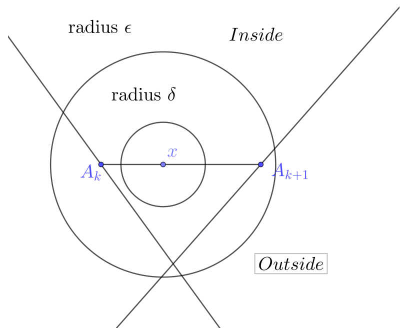
- Case 2: $x$ is on a line $(A_kA_{k+1})$ with $k$ between 2 and $n-2$.

  $x$ is simultaneously on the closed half planes $\overline{P_{k-1}}$ and $\overline{P_k}$, hence $x$ has to be on the segment $[A_kA_{k+1}]$ using a similar reasoning to the previous case.

- Case 3: $x$ is on a line $(A_{n-1}A_n)$.

  This case is symmetric to the case 1, hence we can prove that $x$ is on the segment $[A_{n-1}A_n]$

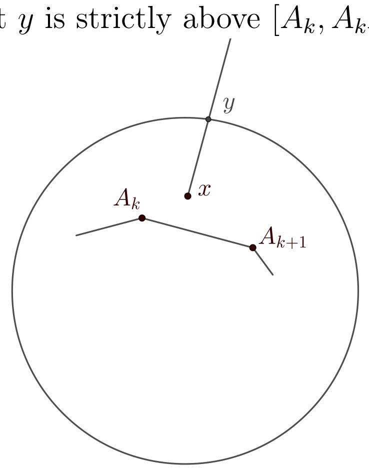
We have now proved that $\overline{U} \subseteq U \cup \ell$. Now let us prove that $\ell \subseteq \overline{U}$. Take $x$ on $\ell$ and an $\epsilon > 0$. Since $x \in l$, there exists $k$, such that $x$ belongs to the segment $[A_k, A_{k+1}]$. Now suppose that $x$ is not an extreme point, then since $x$ is strictly below all the other faces, then there exists $\delta < \epsilon$, such that the circle centered at $x$ and of radius $\delta$ is below all these faces as illustrated below:




We then simply take a point in the smaller disc that is below the face $[A_k A_{k+1}]$ and prove that $x \in \overline{U}$. If $x$ is one of the vertices $A_k$ the proof is essentially the same, except we take a radius $\delta$, such that $x$ is below all the faces $[A_{k'} A_{k'+1}]$ with $k'$ distinct from $k-1$ and $k+1$ and pay attention to the angles when picking a point inside $U$.

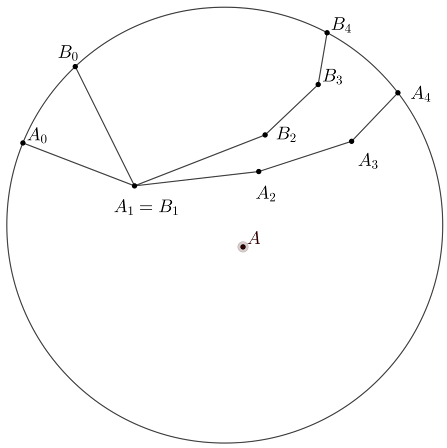
Since $U$ is open and $\overline{U} = U \cup l$ and $\ell \cap U = \emptyset$, then $\partial U = \ell$ as expected.

Now we need to prove that $D \setminus (U \cup l)$ is path-connected. First we show that all the points in $D \setminus (U \cup l)$ are path-connected to the boundary of $D$. Let $x \in D \setminus (U \cup l)$. Then $x$ is strictly above some face $[A_k A_{k+1}]$. Now just draw a semi line starting at $x$, orthogonal to the face $[A_k A_{k+1}]$ and moving away from it. Since the semi line is unbounded and $x$ is in $D$, it has to intersect the boundary of $D$ on a point $y$. The point $y$ is strictly above $[A_k, A_{k+1}]$ and hence is on the exterior of $l$.

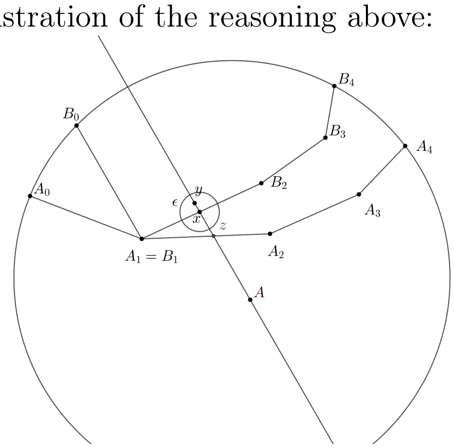



Now we need to prove that the boundary of $D$ intersected with the complement of $\overline{U}$ is connected. To prove that it is connected it is enough to prove that its complement in the circle is. The set $\mathscr{C}$ cannot be the whole circle and therefore is included inside the circle without a point, which is homeomorphic to the open interval $]0, 1[$. We have therefore the property that path-connected subspaces are intervals. Furthermore $\mathscr{C}$ is the intersection of the circle and all the subsets that are above the faces $[A_k, A_{k+1}]$, which itself are connected. That intersection contains the point $A_0$ in common hence is a non empty interval and therefore path-connected. We can therefore conclude indeed that $D \setminus \overline{U}$ is connected.

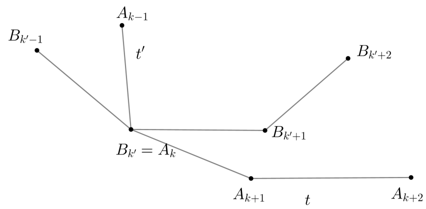
Now we need to prove the second part of our statement: we take $B_0, \cdots, B_m$ another sequence of points, $\ell'$ the broken line $(B_0, \cdots, B_m)$. Assume that $\ell'$ is included in $\overline{U}$ and that $\ell$ and $\ell'$ have only one point of intersection and such that there exists a point $A$ outside of $\ell$ and $\ell'$. A picture of the situation is shown below:




Now pick $[B_k B_{k+1}]$ a face, such that $A$ is strictly above it. Furthermore, we pick a point $x$ strictly between $B_k$ and $B_{k+1}$: then $x$ is strictly inside the curve $\ell$. Then there exists an $\epsilon > 0$, such that the open ball around $x$ is inside $\ell$. Now pick $y$ a point on the semi line $[Ax)$ that is inside the ball of radius $\epsilon$ and inside of $\ell'$. The semi line $[yx)$ can only intersect $\ell'$ in one point, since the interior of $\ell'$ is convex. Furthermore $y$ is inside $\ell$, while $A$ is outside therefore it intersects $\ell$ at a unique point $z$. This point $z$ is on the semi line $[xA)$, since the segment $[yx]$ is included in the ball of center $x$ and radius $\epsilon$ and therefore $y$ is outside of $\ell'$. Here is the illustration of the reasoning above:




Now if $\ell$ and $\ell'$ have no common points of intersection, all points on $\ell$ then have to be on the outside of $\ell'$. Now assume that they have one common point of intersection: say $A_k$. If $k$ is zero or $n$, then the point is extremal and one can prove that $\ell$ is on the outside of $\ell'$. Suppose now that $A_k$ is not an extremal point. Take $t$ to be the connected component of $z$ (the point on $\ell$ exterior to $\ell'$) on $\ell \setminus \{A_k\}$. The whole $t$ is then on the exterior of $\ell'$. Now if $t'$ is the other connected component of $\ell \setminus \{A_k\}$, then $t'$ is either completely on the outside of $\ell'$ or completely on the inside. By contradiction, assume it is on the inside. First assume that $A_k = B_{k'}$, with $B_{k'}$ an internal point. Let us draw that situation:



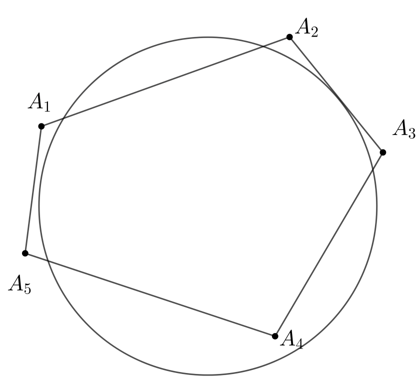

Now choose an orientation, such that the angle $B_{k'+1}\widehat{B_{k'}}B_{k'-1}$ is between 0 and $\pi$. Then since $A_{k-1}$ and $B_{k'-1}$ are on the same side of the face $[B_{k'}B_{k'+1}]$, then the angle $B_{k'+1}\widehat{B_{k'}}A_{k-1}$ is between 0 and $\pi$. The angle $A_{k-1}\widehat{A_k}B_{k'+1}$ is then between $-\pi$ and 0. The angle $B_{k'-1}\widehat{B_{k'}}B_{k'+1}$ is between $-\pi$ and 0. Since $A_{k-1}$ is on the same side of the face $[B_{k'-1}B_{k'}]$ as $B_{k'+1}$, then the angle $B_{k'-1}\widehat{B_{k'}}A_{k-1}$ is between $-\pi$ and 0 as well. That means that the angle $B_{k'-1}\widehat{A_k}A_{k-1}$ is between 0 and $\pi$, while $B_{k'+1}\widehat{A_k}A_{k-1}$ is between $-\pi$ and 0, contradicting the fact that $B_{k'-1}$ and $B_{k'+1}$ are on the same side of the face $[A_{k-1}A_k]$.

Now assume instead that $B_{k'}$ is an extremal point, say $k' = 0$ then by assumption $A_{k-1}$ and $A_{k+1}$ are on the same side of the face $[B_0B_1]$ and hence the curve $\ell$ stays on the outside of $\ell'$. Now let us prove that $U'$ the inside of $\ell'$ is included in $U$: the inside of $\ell$ Let $x \in U'$. Let us pick $y \in \ell'$, such that $y \in U$. Since $\overline{U'}$ is convex the segment $[xy]$ is included in $\overline{U'}$ and $y$ is the only of its points $\ell'$. Furthermore, $[xy]$ does not intersect $\ell'$, since $\ell'$ is on the outside of $\ell$ and $y$ isn't a point that $\ell$ and $\ell'$ have in common. The points $x$ and $y$ are therefore in the same path-connected component of $D \setminus \ell$. Since $y \in U$, this implies that $x \in U$ as expected.

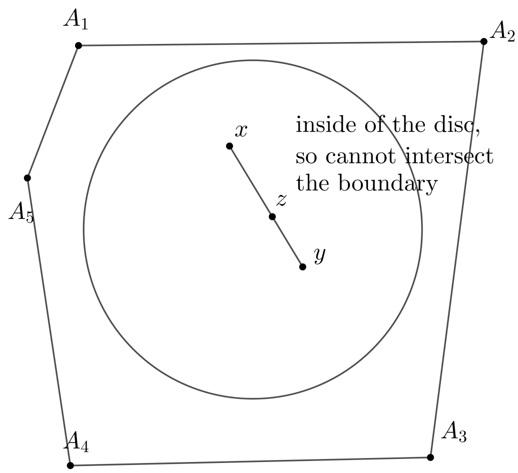
□

**Lemma 4.7.6.** *Let $B$ be an open disc in $\mathbb{R}^2$ of a positive radius and $P = (A_1 \cdots A_n)$ a convex polygon. Let $U$ be the interior of $P$. Then $U \setminus B$ can be written as a finite disjoint union of path-connected opens, whose boundary in $\mathbb{R}^2 \setminus B$ is path-connected.*

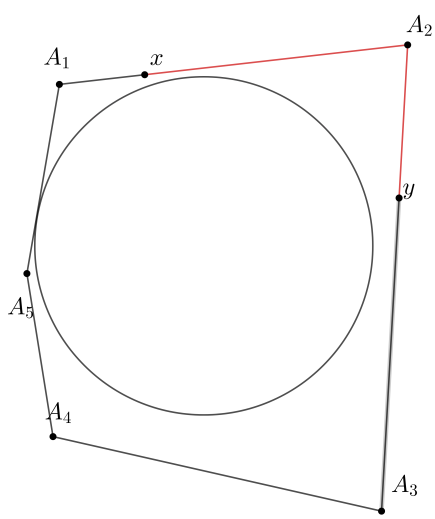
*Proof.* A drawing of a generic situation for $n = 5$ goes as follows:



Now let $\mathscr{C}$ be the boundary of $B$. First let us treat the case when $\partial P$ does not intersect $\mathscr{C}$. If all points of $\partial P$ are in $B$, then by convexity the whole interior of the polygon is included in $B$ and so $U \setminus B = \emptyset$. Now let us assume that there is a point on $\partial P$ outside of $B$. If $U \cap B = \emptyset$, then the result is trivial. If not take $x_0 \in U \cap B$. Let us show that $U \setminus B$ is connected. By the lemma 4.7.3, it is enough to show that $B$ is included in $U$. Let $y$ be another point in $B$. By contradiction, assume that $y \notin U$, then the segment $[xy]$ crosses the boundary of $U$: $\partial P$ at some point $z$. We however assumed that $\partial P \cap B = \emptyset$, hence we obtain a contradiction. A picture illustrating this proof is below:



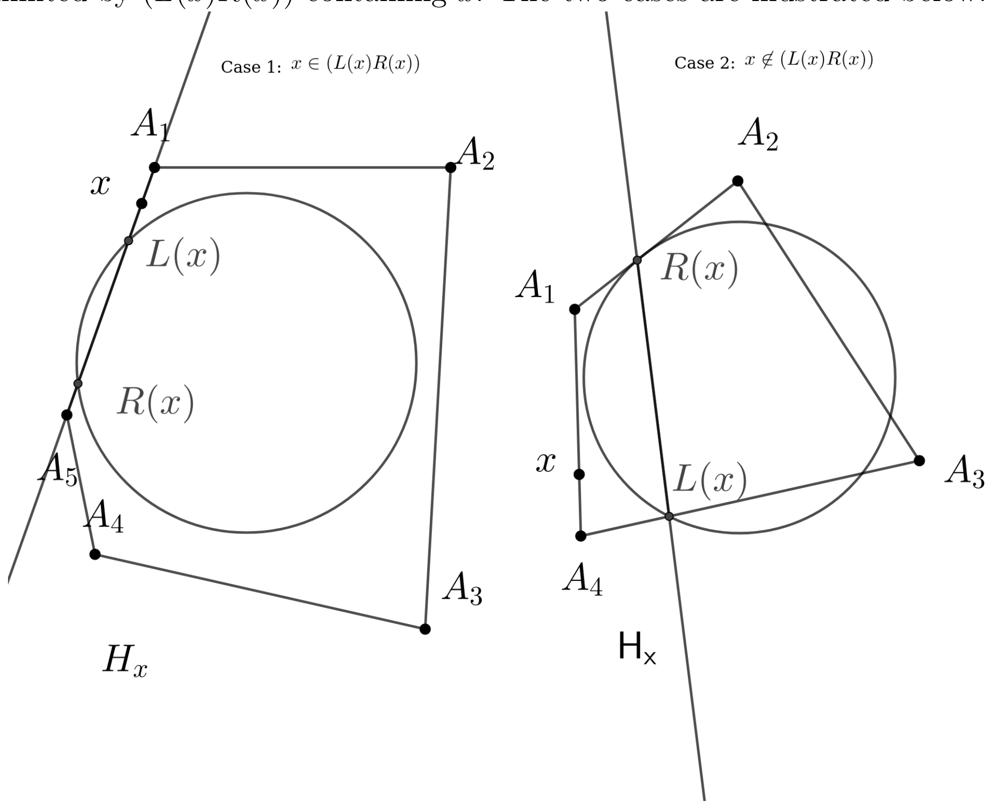

Now let us assume that $\partial P$ intersects $\mathscr{C}$ at exactly one point. If all other points of $\partial P$ are inside the open disc $B$, then by convexity $U \subseteq B$. Now assume that there is at least on point outside of $B$: $y$. Let us show that then $\partial P$ does not intersect $B$. By contradiction, assume there exists a point of intersection $y \in U$. Then there exists a path on $\partial P$ from $x$ to $y$ that doesn't go trough the unique point of intersection between $\mathscr{C}$ and $\partial P$ as illustrated below:




That path would have to then intersect $\mathscr{C}$ at some point, which is a contradiction.

Then we use again the lemma 4.7.3 to conclude.

Now assume that there are two or more points of intersection. We parameterize $\partial P$ by a continuous function $\gamma$ from $S^1$ to $\partial P$. Now for each $x$ on $\partial P$ outside of $U$, we associate $L(x)$ the first point on $\partial P \cap \mathscr{C}$ on the left of $x$ and $R(x)$ the first point on $\partial P \cap \mathscr{C}$ on the right of $x$. Now if $x$ belongs to the line $(L(x)R(x))$, we can find a point on $\partial P$ that is outside of $U$ and not on $(L(x)R(x))$. In that case define $H_x$ as the half open plane delimited by $(L(x)R(x))$ containing that point. If on the other hand $x$ does not belong to $(L(x)R(x))$, then we define $H_x$ as the open plane delimited by $(L(x)R(x))$ containing $x$. The two cases are illustrated below:



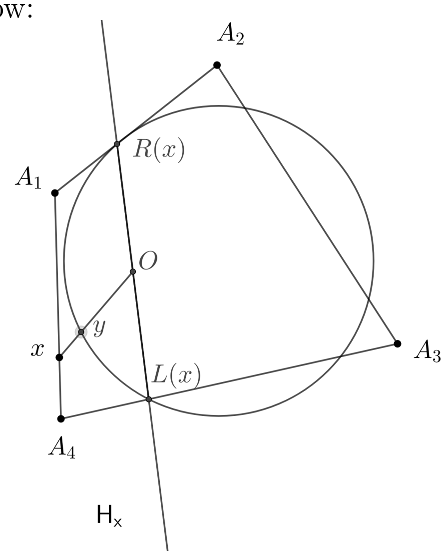

Now we define $C_x = H_x \cap [U \setminus B]$. We will prove that $C_x$ is path-connected. We will do it in the following steps:

- Prove that $H_x$ intersected with $\partial P$ is the curve between $L(x)$ and $R(x)$ going from left to right with our chosen parameterization $\gamma$ if $x \notin (L(x)R(x))$.
- Prove that $\mathscr{C} \cap H_x$ doesn't intersect $\partial P$.
- Prove that $\mathscr{C} \cap H_x \subseteq C_x$.
- Prove that every $y \in C_x$ is connected by a path to $\mathscr{C} \cap H_x$.

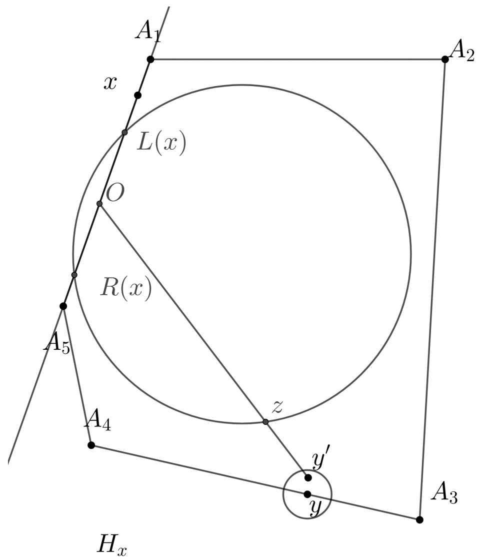
We start by proving that $H_x \cap \partial P$ is the curve between $L(x)$ and $R(x)$ if $x$ is not on $(L(x)R(x))$. Now take $t_0, t_1, t_2 \in S_1$. such that $L(x) = \gamma(t_0)$, $x = \gamma(t_1)$ and $R(x) = \gamma(t_2)$. Let us take $t \in S^1$ between $t_1$ and $t_3$. By contradiction, assume that $\gamma(t) \notin H_x$. In that case by continuity, there exists $t'$ between $t_0$ and $t_1$ or between $t_1$ and $t_2$, such that $\gamma(t') \in (L(x)R(x))$, which is impossible, since $P$ is a convex polygon, so a line that is not a face can intersect its boundary at at most two points. By a similar reasoning we can prove that points whose parameter is not between $t_0$ and $t_2$ is on the opposite half plane to $H_x$.

Now we need to prove that $\mathscr{C} \cap H_x$ doesn't intersect $\partial P$. First assume that $x$ is not on $(L(x)R(x))$. If then $\mathscr{C} \cap H_x$ intersected $\partial P$, it would intersect $\partial P$ on a point between $L(x)$ and $P(x)$, which is a contradiction. If on the other hand $x \in (L(x)R(x))$, then it cannot intersect another face besides the one $x$ is on, because we otherwise would find a $t$ between $t_0$ and $t_2$, such that $\gamma(t)$ is on $\mathscr{C}$. It also cannot intersect the face of $x$ more than at two points and both $R(x)$ and $L(x)$ are outside of $H_x$.

Next step is to prove that $\mathscr{C} \cap H_x \subseteq C_x$. The arc $\mathscr{C} \cap H_x$ doesn't intersect $\partial P$, therefore it is either entirely in $U$ or entirely outside. Let us prove that there exists at least one point inside. Now let us assume that $x \notin (L(x)R(x))$ Take $O$ the middle of the segment $[L(X)R(x)]$. Then $O$ is inside $U$ as well as inside $B$. The segment $[Ox]$ with the point $x$ excluded is then therefore inside $U$. It has to intersect $\mathscr{C}$ at some point $y$, since $x$ is outside $B$ and hence $y \in U$. Furthermore the segment $[Ox]$ with $O$ excluded is inside $H_x$, hence $y \in H_x \cap \mathscr{C}$. We illustrate this part of the proof below:




Now instead assume that $x \in (L(x)R(x))$. Take again $O$ in the middle of the segment $[L(x)R(x)]$. Furthermore take $y \in H_x \cap \partial P$. The point $y$ is outside of $\overline{B}$ and in $H_x$, so we take an open ball small enough centered around $y$ such that the ball is inside $H_x \setminus \overline{B}$. Then take $y' \in U$ that is inside that ball. The segment $[y'O]$ without $O$ is in $H_x \cap U$ and it has to intersect $\mathscr{C}$ at a point $z \in H_x \cap U$. We illustrate this part of the proof as well:




Finally we shall prove that every $y \in C_x$ is connected by a path to $\mathscr{C} \cap H_x$. We write again $O$ to be the center of $[L(x)R(x)]$. The segment $[yO]$ has to intersect $\mathscr{C}$ at a point $z$ by the same reasoning as above. Furthermore $[yz]$ is included in $H_x \cap U$ and is outside $B$, proving that $y$ is connected by a path to $H_x \cap \mathscr{C}$.

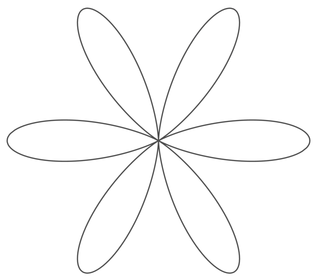
The arc $H_x \cap \mathscr{C}$ is path-connected proving therefore that the whole $C_x$ is path-connected.

$C_x$ is furthermore open in $\mathbb{R}^2 \setminus B$ as an intersection of two open sets: $U \setminus B$ and $H_x \setminus B$. Now we shall prove that every $y \in U \setminus B$ is in some $C_x$. Let $y \in U \setminus B$. Then the semi-line coming from $y$ and going in the opposite direction from the center of $B$ has to intersect at some point $x$ with the boundary $\partial P$, since $P$ is bounded. Let us prove that $y \in C_x$. By contradiction, assume that $y$ is on the lower plane of $H_x$. It cannot be on the segment $[R(x)H(x)]$, since either $y$ would be on a face which is impossible or $y$ would be inside of the disc $B$. The segment $[yx]$ would then have to intersect $[R(x)L(x)]$, which is impossible, since we chose it in such a way that it cannot enter in the disc $B$, hence $y \in H_x$ and therefore $y \in C_x$ as expected.

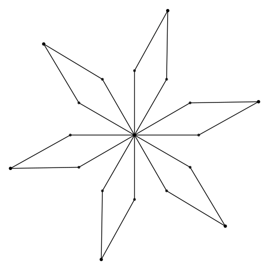
Finally using a similar reasoning to the one above, we can prove that the open sets in the set $\{C_x | x \in \partial P \setminus \overline{B}\}$ are disjoint, which concludes the proof. □

**Theorem 4.7.7** (de Groot)**.** *There exists a connected Hausdorff compact and locally connected space $H$ and $a, b \in H$, two distinct points in $H$ such that: $H \setminus \{a, b\}$ is connected and if $U$ is an open in $H$ and $f$ a continuous injective and open map from $U$ to $H$, then $\forall x \in U, f(x) = x$.*

*Proof.* We will detail the proof done by de Groot in [20]. While de Groot originally used unspecified curves that are bouquets of propellers, which looked as follows:

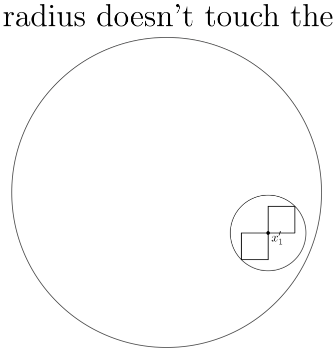


We will replace these propellers by rhombi, so that the proof can be reduced to convex geometry. The bouquets will then look as follows:

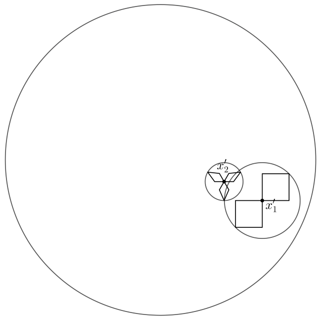


We can construct a flower of rhombi centered at a point $X$ of radius $r$ and of $n$ petals $R(X, r, n)$ as follows. Take $D(X, r)$ the closed disc of radius $r$ centered at $X$ and $D(X, \frac{r}{2cos(\frac{\pi}{2n})})$ a closed disc of radius $\frac{r}{2cos(\frac{\pi}{2n})}$. Split the smaller disc into $2n$ equal parts by taking the points that correspond to $2n$-th roots of unity. On the exterior disc take the $n$th roots of unity shifted by the angle $\frac{\pi}{2n}$. Then connect appropriately to get $n$ rhombi whose only common point is $X$ as shown in the illustration above.

Now we start the construction of $H$ by picking $D$ a disc of radius 1 in $\mathbb{R}^2$ and choose $(x_n)_{n \in \mathbb{N} \setminus \{0\}}$ a sequence in the interior of $D$ and dense in $D$. Now we construct a sequence $D_n$ by induction. We start with $D_0 = D$. We construct $D_1$, by taking $x'_1 = x_1$ and constructing around it a flower $R(x'_1, r_1, 2)$ with two petals, such that its radius doesn't touch the edges of $D$ as illustrated below.



Then we obtain $D_1$, simply by removing the interior of the two rhombi. We construct $D_2$ in the following way: choose $x'_2$ as the first of the $x_n$ that is not on a rhombus in $D_1$ (neither in the interior nor on the edges). It exists because the complement of a rhombus is open and the sequence is dense in the disc $D$. Now choose a disc centered at $x'_2$ of a radius $r_2$ that is less than $\frac{1}{2}$, such that the disc is disjoint with the flower centered at $x'_1$ as well as with the edges of $D$. Finally we take a flower with three petals centered at $x'_2$ and of radius of the given disc. We illustrate one such case below:



We then obtain $D_2$ by removing the interior of all rhombi of the flower centered at $x'_2$ from $D_1$.

Now suppose by induction that we have constructed a $D_n$ and $x'_1, \cdots, x'_n$ with the following properties:

- For every $k$, $x'_k$ is a center of a flower with $k+1$ rhombi and of a radius $r_k \leq \frac{1}{2^k}$.
- For every $k > 1$ $x'_k$ the first $x_m$ that is in none of the flowers centered at $x_1, \cdots x_{k-1}$.
- For every $k > 1$ the disc of radius $r_k$ centered on $x'_k$ has an empty intersection with the flowers centered respectively at $x'_1, \cdots x'_{k-1}$ and an empty intersection with the boundary of $D$.
- $D_n$ is the complement of the interiors of all the flowers centered at $x'_1, \cdots, x'_n$ in $D$.

We construct $D_{n+1}$ as follows: Take $x'_{n+1}$ to be the first $x_m$, such that $x_m$ is in none of the flowers so far constructed. Then take $r_{n+1}$ a radius smaller than $\frac{1}{2^{n+1}}$, such that the disc centered at $x'_{n+1}$ has an empty intersection with all the flowers so far constructed as well as the boundary of $D$. Construct a flower of radius $r_{n+1}$ centered at $x'_{n+1}$ and remove its interior to obtain $D_{n+1}$.

Now define

$$H = \bigcap_{n \in \mathbb{N}} D_n$$

We shall prove that $H$ has the following properties:

- $H$ is compact and connected.
- For every $n \in \mathbb{N} \setminus \{0\}$, if there is no $k \in \mathbb{N} \setminus \{0\}$, such that $x_n = x'_k$, then $x_n$ belongs to one of the flowers.
- The sequence $(x'_n)_{n \in \mathbb{N} \setminus \{0\}}$ is dense in $H$.
- For every $x \in H$ distinct from all the $x'_k$ and every $U$ a neighborhood of $x$, there exists $C \subseteq U$ a neighborhood of $x$, such that: $C$ and $C \setminus \{x\}$ are connected.

- For every $n \geq 1$ and every $U$ a neighborhood of $x'_n$, there exists $C \subseteq U$ a neighborhood of $x_n$, such that $C$ is connected and $C \setminus \{x'_n\}$ has $n+1$ connected components dense in $C$.

From these properties, we will then be able to prove that $H$ is a de Groot space.

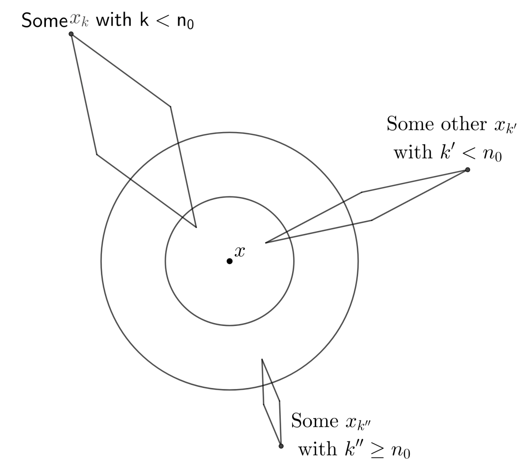
To prove that $H$ is compact and connected, we first prove that all the $D_n$ are. They are compact, because closed in $D$, which is compact. To prove they are connected, we notice that all the interiors of the rhombi in $D_n$ are disjoint convex sets with a path-connected boundary. By the lemma 4.7.3, $D_n$ is then path-connected, hence connected. $H$ is a decreasing intersection of compact connected subsets and is therefore itself compact connected.

Now we need to prove that all the $x_n$ that are not of the form $x'_k$ are on the flowers (including their interiors). By contradiction take $n$ the smallest positive integer such that $\forall k \in \mathbb{N} \setminus \{0\}$ $x_n \neq x'_k$ and $x_n$ is on none of the flowers. Then for every $n' \leq n$, $x_{n'}$ is either equal to some $x'_k$ or is on some flower. Now take $k_0$ the largest $k$, such that

$$\exists n' < n, x'_{k_0} = x_{n'}$$

The number $k_0+1$ is the smallest $n'$, such that $x_{n'}$ is on none of the flowers centered at $x'_{k'}$ for $k' \leq k_0$. For all $n' \leq n$, $x_{n'}$ is on one of the flowers with indexes up to $k_0$ because of how the sequence $x'$ is constructed and therefore $x'_{k_0+1} = x_n$, which is a contradiction.

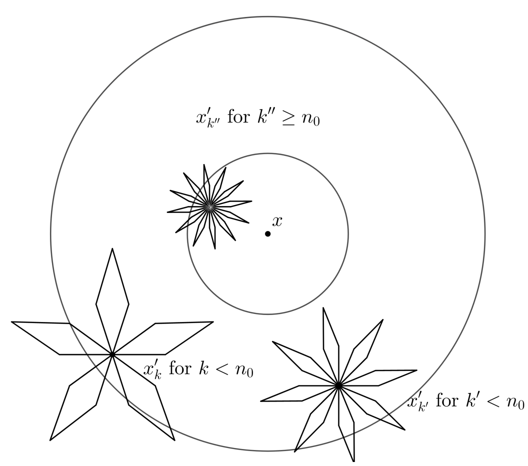
The next step is to establish that the sequence $(x'_n)_{n \in \mathbb{N} \setminus \{0\}}$ is dense in $H$. By contradiction, assume that it is not. Then there exists $x \in H$ and a radius $r > 0$, such that for all integers $n$, $x'_n$ does not belong to the disc (closed ball) centered at $x$ of radius $r$: $D(x, r)$. Now let $n_0$ be such that the radius of the flower centered at $x_{n_0}$: $r_{n_0}$ is less than $\frac{r}{2}$. Denote $C$ the open ball centered at $x$ of radius $\frac{r}{2}$ with all the flowers up to rank $n_0$ removed. The ball $C$ cannot be empty, since then we could write $B(x, \frac{r}{2})$ as a finite disjoint union of open sets, which is impossible since $B(x, \frac{r}{2})$ is connected. We then get that $C$ with interiors of all flowers removed is equal to $C$, since all the flowers from the rank $n_0$ are centered outside of the open ball $B(x, r)$ and of radius $\frac{r}{2}$, so don't intersect $C$ at all. Furthermore $C$ is open in $D$, therefore there exists $n \in \mathbb{N}$, such that $x_n$ is in $C$ by density of the sequence. The point $x_n$ is distinct from all the $x'_k$ and therefore has to be on of one of the flowers, but that is impossible, since all the flowers are by construction disjoint with $C$, hence the contradiction. The illustration for the proof is shown below:




The next step in the proof is to show that for every $x \in H$ distinct from the $x'_k$ and for every $U$ a neighborhood of $x$, there exists $C$ a neighborhood of $x$, such that $C \setminus \{x\}$ and $C$ are connected. We take $x \in H$ distinct from the $x'_k$s. We will show that there exists $A_n$ an increasing sequence of connected subsets of $U$, such that $x$ is in none of the $A_n$ and such that $\bigcup_{n \in \mathbb{N}} A_n \cup \{x\}$ is a neighborhood of $x$. To do that we will distinguish three cases:

Case 1: $x$ is neither an element of the boundary of $D$ nor on any of the rhombi.

Since $U$ is a neighborhood of $x$, there exists a $r > 0$, such that the closed disc of radius $r$ intersected with $H$: $D(x,r) \cap H$ is a subset of $U$. Furthermore we can choose $r > 0$ small enough such that $D(x,r)$ doesn't intersect the boundary of $D$. Let $n_0$ be in $\mathbb{N}$, such that the flower centered on $x'_{n_0}$ is of radius less than $\frac{r}{3}$. Let $r' > 0$ be a radius strictly smaller than $\frac{r}{3}$, such that none of the flowers centered at $x'_1, \cdots, x'_{n_0}$ intersect the open ball centered at $x$. Here is an illustration of the configuration:

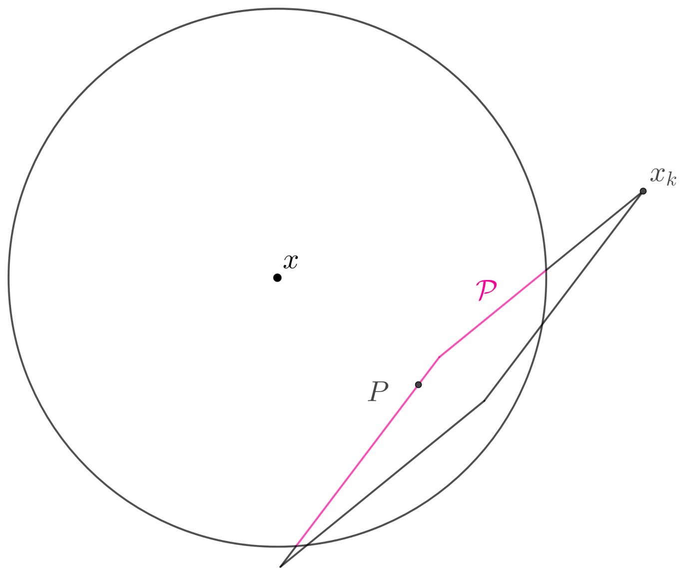



Now let's take $r'_n$ a sequence of radii converging monotonely to zero with $r_0 = r'$. Consider $R_n = D(x,r) \backslash B(x, r'_n)$ a ring centered at $x$. ($B(x, r'_n)$ being the open ball of radius $r'_n$ centered at $x$). Note that none of the flowers intersect both the outer edge of the ring and the inner edge. To prove it: first take $x'_n$ with $n < n_0$. Then the flower centered at $x'_n$ does not intersect the inner edge by definition. If on the other hand

$n \geq n_0$,then suppose that the flower centered at $x'_n$ intersects the inner edge. Take $x'$ to be the point of intersection. We get that $d(x, x'_n) \leq d(x, x') + d(x', x) < \frac{r}{3} + \frac{r}{3}$. The circle centered at $x'_n$ and of radius $r' < \frac{r}{3}$ is therefore included in the interior of $D(x, r)$ and hence does not intersect the outer edge. The flower centered at $x'_n$ cannot therefore intersect the edge either.

Now let $m$ be an integer greater than 1 and $\mathscr{R}$ a rhombus that is part of a flower centered at some $x'_k$, $k \leq m$. We will define an open $\Upsilon(\mathscr{R})$ as follows:

- If $\mathscr{R}$ intersects the outer edge, we take $P$ the closest point on $R$ to $x$. Now take $\mathscr{P}$ the path-connected component on $\partial\mathscr{R} \cap D(x, r)$ of $P$ as illustrated below.




In the case of the illustration that would be two faces.

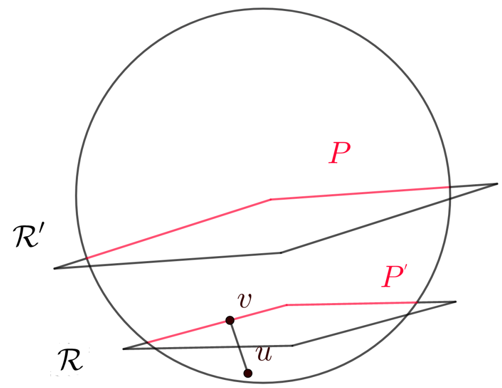
Define then $\Upsilon(\mathscr{R})$ as follows: if $\mathscr{R}$ is entirely included in $D(x, r)$, then $\Upsilon(\mathscr{R})$ is simply the interior of $\mathscr{R}$. If it is not entirely included, then $\mathscr{P}$ is a convex broken line starting and ending on the boundary of $D(x, r)$, hence by the lemma 4.7.5 it splits $B(x, r)$ into two path-connected sets and we take $\Upsilon(\mathscr{R})$ to be the region not containing $x$.

First note that $\Upsilon(\mathscr{R})$ is included in the ring $R_n$. It is by definition included in $D(x, r)$.

Now let us show that it doesn't intersect with $D(x, r')$. Note that by the lemma 4.7.2 $x$ is above some face containing $P$ and therefore is outside of $\Upsilon(\mathscr{R})$. Furthermore if there was a point $y \in D(x, r') \cap \Upsilon(\mathscr{R})$, then the segment $[xy]$, would have to intersect the boundary of the set $\Upsilon(\mathscr{R})$, which is impossible, since then $D(x, r')$ and the rhombus $\mathscr{R}$ would have a non empty intersection.

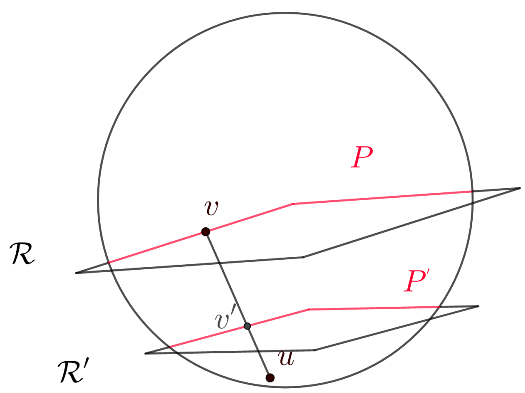
Furthermore since rhombus is a convex polygon, we have that the interior of $\mathscr{R}$ is a subset of $\Upsilon(\mathscr{R})$. Now let us show that if $\mathscr{R}'$ is another rhombus that intersects the outer edge such that $\Upsilon(\mathscr{R}') \cap \Upsilon(\mathscr{R}) \neq \emptyset$, then either $\Upsilon(\mathscr{R}') \subseteq \Upsilon(\mathscr{R})$ or vice versa.

Take $u \in \Upsilon(\mathscr{R}') \cap \Upsilon(\mathscr{R})$. Now let us take $v$ a point on $\partial\Upsilon(\mathscr{R})$ By convexity of $\Upsilon(\mathscr{R})$ the segment $[uv]$ is included in $\overline{\Upsilon(\mathscr{R})}$ with $v$ being the only point of the segment on the boundary. Suppose first that this segment doesn't intersect the boundary of $\Upsilon(\mathscr{R}')$ as shown below:




In the case the segment is wholly included into $\Upsilon(\mathscr{R}')$ and so $v$ is in $\Upsilon(\mathscr{R}')$.

Now suppose that it does intersect at some point:



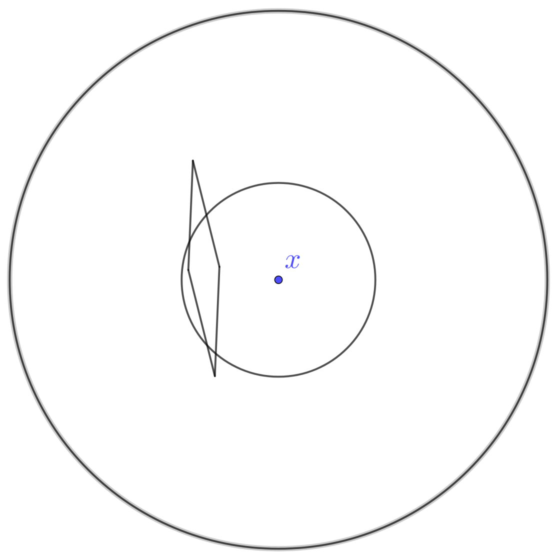

we simply exchange the roles of $\mathscr{R}$ and $\mathscr{R}'$ and we obtain a point $v'$ on the boundary of $\Upsilon(\mathscr{R}')$ that is inside of $\Upsilon(\mathscr{R})$. Hence we can suppose after potentially relabeling that $\Upsilon(\mathscr{R}')$ contains a point on the boundary of $\Upsilon(\mathscr{R})$

Now if $\mathscr{R}$ is entirely included into $B(x,r)$, then its interior is in $\Upsilon(\mathscr{R}')$ by convexity. Indeed the boundary of $\Upsilon(\mathscr{R}')$ and the polygon convex polygon $\mathscr{R}'$ have at most one point in common, $\Upsilon(\mathscr{R}')$ contains a point of the boundary of $\mathscr{R}'$ and $\Upsilon(\mathscr{R}')$ is a convex region. In case that both $\mathscr{R}$ and $\mathscr{R}'$ are included inside of $B(x,r)$, then $\Upsilon(\mathscr{R})$ and $\Upsilon(\mathscr{R}')$ are disjoint by construction, which cannot happen, since we assume that there is at least a point in common. In case that $\mathscr{R}'$ is entirely included $B(x,r)$, then $\mathscr{R}$ would have a point on the boundary of $\mathscr{R}'$ in its interior, which would contradict the construction of the flowers. Finally in case that neither $\mathscr{R}$ nor $\mathscr{R}'$ are fully included inside $B(x, r)$, we apply the lemma 4.7.5 to prove that $\Upsilon(\mathscr{R}) \subseteq \Upsilon(\mathscr{R}')$

- If $\mathscr{R}$ does not intersect inner or outer edge and is fully included in the ring, then we simply take $\Upsilon(\mathscr{R})$ to be equal to $\mathscr{R}$. By the same reasoning as previously, we can prove that it doesn't intersect any other $\Upsilon(\mathscr{R})$ as previously defined or is entirely included in it.

- If $\mathscr{R}$ intersects the inner edge and therefore not the outer edge, we take $\Upsilon(\mathscr{R}) = \mathscr{R} \setminus B(x, r'_n)$ Note that $\Upsilon(\mathscr{R})$ cannot intersect with any other opens $\Upsilon(\mathscr{R}')$.

  Now we will show that $\Upsilon(\mathscr{R})$ can be written as a disjoint union of finite opens with path-connected boundaries.




  By the lemma 4.7.6 the interior of $\mathscr{R}$ splits into opens whose boundary is path-connected. Since $\mathscr{R}$ is entirely included in $D(x, r)$ its intersection with $D(x, r)$ stays path-connected as well.

Now that for every rhombus $\mathscr{R}$, we defined $\Upsilon(\mathscr{R})$, we define $A_{m,n}$, as the ring $R_n$ without all the $\Upsilon(\mathscr{R})$ with $\mathscr{R}$ being a rhombus centered at $x'_k$, $k' \leq m$. Note that using the lemma 4.7.3 $A_{m,n}$ is path-connected. It is also closed in $\mathbb{R}^2$, so compact. Then $A_n = \bigcap_{m \geq 1} A_{m,n}$ is compact and path-connected by 4.7.4. Furthermore $A_n$ is a subset of $R_n \cap H$, since for every rhombus $\mathscr{R}$, the interior of $\mathscr{R} \cap R_n$ is included in $\Upsilon(\mathscr{R})$.

Now write

$$C = \bigcup_{n \geq 1} A_n$$

$C$ is an increasing union of connected spaces, so is itself connected. It is a subset of $H \setminus \{x\}$ as it doesn't contain $x$ and each $A_n$ is contained in $H$. Now let us show that $C \cup \{x\}$ is a neighborhood of $x$ in $H$.

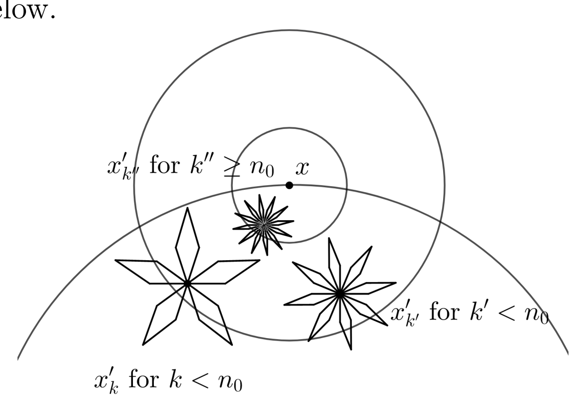
We will show that

$$B(x, r') \cap H \subseteq C \cup \{x\}$$

Take $y \in B(x, r') \cap H$. Take $n \in \mathbb{N}$ sufficiently large so that $y \notin B(x, r_n)$. Let us show that $y \in A_n$. To do that, we need to show that $y$ is not in any $\Upsilon(\mathscr{R})$ for every rhombus. By contradiction assume that there exists a rhombus $\mathscr{R}$, such that $y \in \Upsilon(\mathscr{R})$. In that case the open $\Upsilon(\mathscr{R})$ intersects the inner disc $D(x, r')$ and therefore $\mathscr{R}$ does not intersect the outer disc $D(x, r)$, in which case we know that $\Upsilon(\mathscr{R})$ is equal to the interior of $\mathscr{R}$ intersected with the ring $R_n$. This implies that $y$ is in the interior of $\mathscr{R}$, which is a contradiction, since $y \in H$. We have shown that $C \cup \{x\}$ contains a non empty open ball centered at $x$, therefore it is a neighborhood of $x$. Finally let us show that $C \cup \{x\}$ is connected. The set $C$ contains all the $x'_n$ belonging to the open ball $B(x, r') \cap H$. Furthermore $x'_n$ is a dense sequence in $H$ so it is dense in an open of $H$, therefore $x$ is in the topological closure of $C$ into $H$ and therefore $C \cup \{x\}$ is connected.

Case 2: $x$ is on one the boundary of $D$.

The approach is the same as in the previous case. First we take $r > 0$ small enough so that $D(x, r) \cap H$ is included in the neighborhood $U$. The next step is just like previously to take $n_0$, such that for every $n \geq n_0$, the radius $r_n$ of the flower centered at $x_{n'}$ is less than $\frac{r}{3}$. Then take $r'$ a radius small enough such that none of the flowers up to rank $n_0$ intersects $B(x, r')$. The illustration for the situation is below.

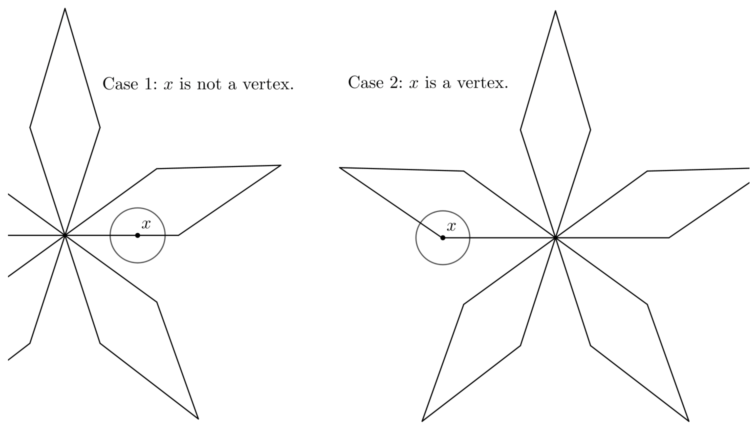



Now we take $r'_n$ a sequence of radii converging to 0 and for a rhombus $\mathscr{R}$, we define $\Upsilon(\mathscr{R})$ either the interior of $\mathscr{R}$ if the rhombus does not intersect the edges for $D(x, r)$ or we take the connected component of $\partial R$ closes to $x$ and then take the interior of that curve just like in the lemma 4.7.5. We can then proceed just like in case 1: the only difference is: we have to show that $\Upsilon(\mathscr{R})$ is a subset of $D$, which is not completely obvious if $\mathscr{R}$ intersects the edges of $B(x, r)$. We take $y$ a point in $B(x, r)$ that is not in $D$. Then the segment $[xy]$ doesn't intersect the boundary of $\Upsilon(\mathscr{R})$, which is included in $D$. Since $x$ is not in $\Upsilon(\mathscr{R})$ and hence the whole segment is outside of $\Upsilon(\mathscr{R})$ and $y \notin \Upsilon(\mathscr{R})$.

We can therefore just like in the previous case construct a sequence of connected closed subsets $A_n$ of $U$, such that $\bigcup_{n \geq 1} A_n \cup \{x\}$ is a connected neighborhood of $x$.

Case 3: $x$ is on one of the rhombi say $\mathscr{R}_0$, without being any of the centers.

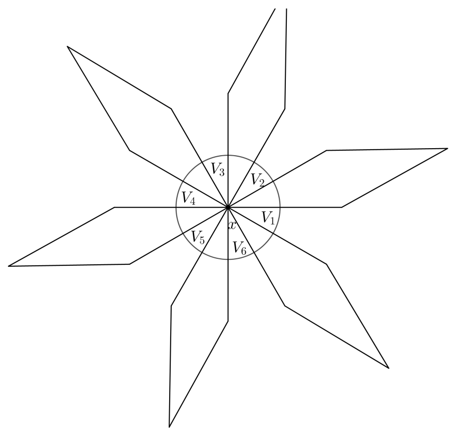
Take $r$ a radius small enough such that $D(x,r) \cap H$ is contained in $U$ such that this disc intersects the boundary of the corresponding flower at two points as shown below:



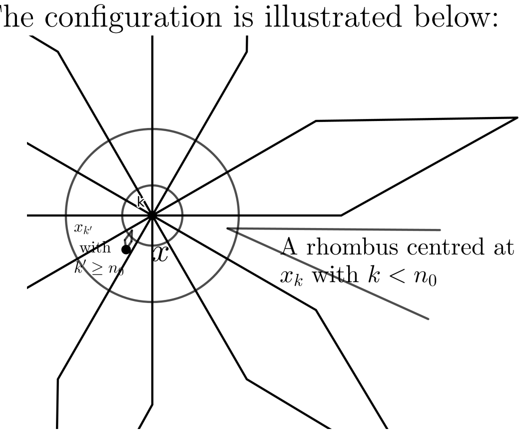

We then define $V$ as $D(x,r)$ without the interior of $\mathscr{R}_0$. Now just like in the two previous cases, take $n_0$ large enough, so that for all $n \geq n_0$, the radius $r_n$ is less that $\frac{r}{3}$. Now none of the flowers with ranks between 1 and $n_0$ except potentially some $n_1$ contain $x$, hence we take $r' < \frac{r}{3}$ small enough so that for all $k \in \{1, \cdots, n_0\} \setminus \{n_1\}$, none of the rhombi with center at $x'_k$ intersects $B(x,r')$. We define for a rhombus $\mathscr{R}$ distinct from $\mathscr{R}_0$, $\Upsilon(\mathscr{R})$ just as previously. If it intersects the outer circle $\mathscr{C}(x,r)$, we take $P$ a point on $\mathscr{R}$ closest to $x$, $\ell$ its connected component in $D(x,r)$ and we take $\Upsilon(\mathscr{R})$ the region delimited by the broken line $\ell$ not containing $x$. Otherwise define $\Upsilon(\mathscr{R})$ as $D(x,r) \setminus B(x,r')$ intersected with the interior of $\mathscr{R}$. To show that we are again in the same case, we just need to prove that $\Upsilon(\mathscr{R})$ is included in $V$ for every $\mathscr{R}$ distinct from $\mathscr{R}_0$. First if $\mathscr{R}$ doesn't intersect the outer circle, then the interior of $\mathscr{R}$ and $\mathscr{R}_0$ have an empty intersection, hence $\Upsilon(\mathscr{R}) \subseteq V$. If it intersects the outer circle,just take $y$ inside $\mathscr{R}_0 \cap D(x,r)$. The segment $[xy]$ doesn't intersect the boundary of $\Upsilon(\mathscr{R})$ and $x$ is on the outside of $\Upsilon(\mathscr{R})$, hence $y$ is on the outside of $\Upsilon(\mathscr{R})$ as well.

We then construct again a sequence $A_n \subseteq H$ of closed connected subsets of $H$, such that $\bigcup_{n\in\mathbb{N}} A_n \cup \{x\}$ is a connected neighborhood of $x$.

Now take $x = x'_m$ and $U$ a neighborhood of $x$. Pick $r > 0$ small enough so that $B(x,r) \subseteq U$ and such that $D(x,r)$ doesn't contain any of the upper faces of the rhombi as shown below:



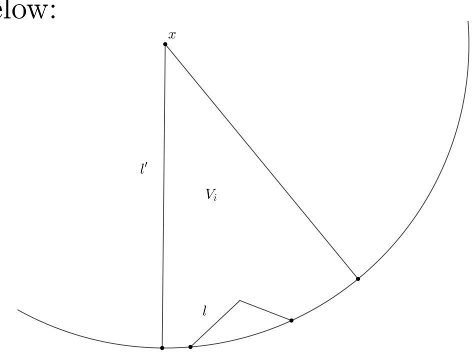

We also denote $V_i$ for $i$ between 1 and $m$ as the $m$ distinct connected components of $D(x,r)\setminus\{x\}$ without the interior of the flower centered at $x$. The next steps are similar to what we did in the previous part of the proof: we take $n_0$, such that for all $n \geq n_0$,the radius $r_n$ is less than $\frac{r}{3}$. Then we take $r'$ a radius small enough such that none of the flowers centered at $x'_k$ with $k \in \{1,\cdots,n_0\}\setminus\{m\}$ intersects $B(x,r')$. The configuration is illustrated below:




Note that again none of the flowers except those centered at $x$ can intersect both the circle centered at $x$ and of radius $r$ and that of radius $r'$. Just like previously, if a rhombus $\mathscr{R}$ centered at some $x_k$ with $k \neq m$ intersects the outer circle $\mathscr{C}(x,r)$, we take $\ell$ are the connected component in the boundary of $\mathscr{R}\setminus D(x,r)$ of the closest point to $x$ and then we take $\Upsilon(\mathscr{R})$ the region in $D(x,r)$ delimited by $\ell$ not containing $x$. If $\mathscr{R}$ doesn't intersect the outer circle, or intersects it only at one point, we take $\Upsilon(\mathscr{R})$ to simply be the interior of $\mathscr{R}$. We shall show that $\Upsilon(\mathscr{R})$ intersects at most one $V_i$.

First we deal with the case when $\Upsilon(\mathscr{R})$ is simply the interior of $\mathscr{R}$. Then if $\Upsilon(\mathscr{R})$ intersected $V_i$ and $V_j$, with $i \neq j$. take $y$ in $V_i \cap \Upsilon(\mathscr{R})$ and $y' \in V_j \cap \mathscr{R}$. The segment $[yy']$ is included in $\Upsilon(\mathscr{R})$ by convexity and therefore has to intersect one of the rhombi $\mathscr{R}'$ centered at $x$, since $V_i$ and $V_j$ are two different path components of $D(x,r)\setminus\{x\}$ without the interior of the flower centered at $x$. That implies that $\mathscr{R}$ and $\mathscr{R}'$ have a non trivial intersection, which is a contradiction.

If on the other hand we suppose that $\Upsilon(\mathscr{R})$ intersects $\mathscr{C}(x,r)$ at more than one point, let us show again that $\Upsilon(\mathscr{R})$ intersects at most one $V_i$. First assume that there exists $y \in \Upsilon(\mathscr{R}) \cap V_i$ for some $i$. Let us show that then $\Upsilon(\mathscr{R}) \subseteq V_i$. Let $\ell'$ be the boundary of $V_i$ it is then a broken line passing by $x$ with extremities intersecting $\mathscr{C}(x, r)$ and $V_i$ is a convex region delimited by $\ell'$. We have the situation illustrated below:




We know that $x$ is on the exterior of $\Upsilon(\mathscr{R})$, therefore the segment $[xy]$ must intersect the boundary of $\Upsilon(\mathscr{R})$ at some point $z$. By convexity, $z$ belongs to $V_i$. Since the boundary of $\Upsilon(\mathscr{R})$: $\ell$ and $\ell'$ don't intersect each other, the whole $\ell$ is inside $V_i$. The point $x$ is on the exterior of both $\ell$ and $\ell'$, hence by the lemma 4.7.5 $\Upsilon(\mathscr{R}')$ is included in $V_i$.

Now take $r'_n$ a sequence of radii monotonely converging to 0 with $r'_0 = r'$. Define $A_n$ as $D(x, r) \setminus B(x, r_n)$ without all the $\Upsilon(\mathscr{R})$ with $\mathscr{R}$ all rhombi not containing $x$ and without the interior of the flower centered at $x$. Just like previously one can show that $A_n \subseteq H \setminus \{x\}$ and that $\bigcup_{n\in\mathbb{N}} A_n \cup \{x\}$ is a neighborhood of $x$. Let us show that $V_i \cap \bigcup_{n\in\mathbb{N}} A_n$ are the connected components of $\bigcup_{n\mathbb{N}} A_n$. Write for every $i$ between 1 and $m$, $B_{n,i} = A_n \cap V_m$. Using the fact that $\Upsilon(\mathscr{R})$ is included inside $V_m$ if it has a non empty intersection, we can prove using the lemma 4.7.3 that $B_{n,i}$ is connected for every $n$. We therefore get that $B_i = \bigcup_{n\in\mathbb{N}} B_{n,i}$ is connected as an increasing union of connected spaces. Now to prove that it is a connected component, let us show that it is a clopen. It is closed as an intersection of the closed subset $V_i$ with $\bigcup_{n\in\mathbb{N}} A_n$. Now $V_i \setminus \{x\}$ is open in $H \setminus \{x\}$, which contains $\bigcup_{n\in\mathbb{N}} A_n$ therefore $V_i \cap \bigcup_{n\in\mathbb{N}} A_n$ is open in $\bigcup_{n\in\mathbb{N}} A_n$. Now simply define $C = \bigcup_{n\in\mathbb{N}} A_n \cup \{x\}$. $C$ is a neighborhood of $x$. It is also a union of connected spaces that have the point $x$ in common therefore is itself connected. Finally $C \setminus \{x\}$ has $m$ connected components which all have the point $x$ in their closure in $C$.

We have finally proven all the properties of $H$ we need to show that it is a de Groot space. Take now $a, b \in H$ two opposite points on $D$. As a reminder we need to prove that

- $H$ is compact Hausdorff and locally connected.
- $H \setminus \{a, b\}$ is connected.
- If $U \subset H$ is an open and $f : U \to H$ is a continuous injective open map, then $\forall x \in U, f(x) = x$.

We already know that $H$ is compact Hausdorff and locally connected. Now let us prove that $H \setminus \{a, b\}$ is connected. Since $H$ is compact Hausdorff, there exists $U' \subseteq H$ and open neighborhood of $a$ and $V'$ an open neighborhood of $b$, such that $U \cap V = \emptyset$. Now take $C \subseteq U'$ connected neighborhood of $a$, such that $C \setminus \{a\}$ is connected and $C' \subseteq V'$ a connected neighborhood of $b$. Now take $U \subseteq C$ an open neighborhood of $a$ and $V \subseteq C'$ an open neighborhood of $b$. For every $x \in H \setminus \{a, b\}$ take $C_x \subseteq H \setminus \{a, b\}$ a connected neighborhood of $x$ containing an open neighborhood of $U_x$ of $x$. The set $\{U_x | x \in H \setminus \{a, b\}\} \cup U \cup V$ is an open covering of $H$, therefore by compactness it has to have a finite subcovering $U_{x_1}, \cdots, U_{x_n}, U, V$. We then write

$$U_0 = U,\ U_{n+1} = V \text{ and } U_i = U_{x_1} \text{ for } i \in \{1, \cdots, n\}$$
$$C_0 = C,\ C_{n+1} = C' \text{ and } C_i = C_{x_i} \text{ for } i \in \{1, \cdots, n\}$$

Now let $W$ be the connected component of $H \setminus \{a, b\}$ containing $C_1$. Let us show that $W = H \setminus \{a, b\}$. Let $m \leq n + 1$ be maximal, such that up to permutation $U \cup \bigcup_{k=1}^{n} U_{x_k} \subseteq W$. By contradiction, assume that $m < n + 1$. Then $U_0 \cup \cdots \cup U_m$ is an open in $H$ and not all $U_i$ with $i > m$ can be disjoint with it, otherwise $H$ could be written as a disjoint union of two opens. Up to permutation we may therefore assume that $U_{m+1} \cap (U_1 \cup \cdots \cup U_m) \neq \emptyset$. We therefore have that there exists an $i \leq m$, such that $U_i \cap U_{m+1} \neq \emptyset$. The set $W$ contains therefore an element of $C_{m+1}$, which is connected. Furthermore $W$ is connected hence $W$ contains $C_{m+1}$ and in particular it contains $U_{m+1}$. Finally by the same reasoning we can prove that $W$ contains $U_0 \setminus \{a\}$ and $U_{n+1} \setminus \{b\}$.

Now the final step is to prove that if $f : U \longrightarrow H$ is an injective continuous and open map for some $U$ open in $H$, then $\forall x \in U, f(x) = x$. To prove it we shall first prove that if $x'_m \in U$, then $f(x'_m) = x'_m$ and then conclude by density of the sequence $(x'_n)_{n \geq 1}$. Consider $y = f(x'_m)$. We have that $f(U)$ is an open neighborhood of $y$ since $f$ is an open map, therefore there exists $C'$ a neighborhood of $y$ contained in $f(U)$, such that $C' \setminus \{y\}$ has $n$ connected components with the property that for every $A \subseteq C' \setminus \{y\}$ connected component of $C \setminus \{y\}$, $y \in \overline{A}$. Now $f^{-1}(C')$ is an open neighborhood of $x_m$, therefore there exists $C \subseteq C'$ a connected neighborhood of $x_m$, such that $C \setminus \{x_m\}$ has $m + 1$ connected components. Let us prove that $m + 1 \leq n$. To prove it we shall show that there exists a continuous map from $C \setminus \{x'_m\}$ to $\mathbb{Z}$ that has at least $n$ values. Take $u : C' \setminus \{y\} \longrightarrow \mathbb{Z}$ having a different value on each connected component. This map is continuous. Now consider the map $u \circ f : C \longrightarrow \mathbb{Z}$. It is continuous as a composition of two continuous maps. Furthermore it has at least $n$ values. Indeed if we pick $A$ a connected component of $C' \setminus \{y\}$, we know that $f(C)$ is a neighborhood of $y$, hence $f(C) \cap A \neq \emptyset$. We then pick $a \in C$, such that $f(a) \in A$. In that case $u(f(a))$ will have the same value as $u$ does on $A$. This implies in particular that $n \geq 2$, since $m + 1 \geq 2$. We therefore get that $f(x'_m)$ has to get sent on a $x'_{m'}$, with $m' \geq m$. But by the same reasoning $f^{-1}$ has to send $x'_{m'}$ onto $x_{m''}$, with $m'' \geq m'$ and therefore $m = m'$. We therefore conclude that $f$ fixes all the $x'_n$ that are in $U$. Now the sequence $(x'_n)$ is dense in $H$, therefore it is dense in $U$ as well. We can therefore conclude by continuity that $f$ fixes all the elements of $U$. $\square$

Now we are almost ready to prove the Morris and Hofmann conjecture, but before we will need the following lemma:

**Lemma 4.7.8.** *Let $\Gamma$ be a profinite graph and $d \in \{o, t\}$ an incidence map. Let $x \in V(\Gamma)$ and let $U$ be a neighborhood of $x$ in $\Gamma$ containing the set $d^{-1}(\{x\})$. There exists then $N$ a clopen neighborhood of $x$ in $V(\Gamma)$, such that $d^{-1}(N) \subseteq U$.*

*Proof.* Since $V(\Gamma)$ is a profinite set, we can find $\mathcal{N}$ be a system of clopen neighborhoods of $x$ in $V(\Gamma)$, such that $\bigcap\limits_{N\in\mathcal{N}} N = \{x\}$. In that case we get that $\bigcap\limits_{N\in\mathcal{N}} d^{-1}(N) = d^{-1}(\{x\})$. Since $d^{-1}(\{x\}) \subseteq U$, we then get that $\bigcap\limits_{N\in\mathcal{N}} d^{-1}(N) \setminus U$ is an empty intersection of closed sets in $\Gamma$. By compactness of $\Gamma$, there exists then an $N \in \mathcal{N}$, such that $d^{-1}(N) \setminus U = \emptyset$ and therefore $d^{-1}(N) \subseteq U$. □

**Theorem 4.7.9.** *Let $\Gamma$ be a profinite graph. There exists a compact Hausdorff space $\tilde{\Gamma}$, such that the group of automorphisms of $\Gamma$ is isomorphic to the group of autohomeomorphisms of $\tilde{\Gamma}$. Furthermore if we assume $\Gamma$ to be connected, then $\tilde{\Gamma}$ is connected as a topological space.*

Originally, I proved this theorem with the assumption that the set of edges in $\Gamma$ is closed. That was incompatible with the theorem 4.6.8, since the resulting profinite graph with no colors doesn't have a closed set of edges since the compactifying points at infinity are in the topological closure of the set of edges. The main difficulty in proving the general version

*Proof.* Take $(H, a, b)$ a doubly pointed de Groot space. Take $\sim$ an equivalence relation on $\Gamma \times H$, such that the equivalence classes are as follows:

$$\overline{(u,x)} = \begin{cases} \{(u,x)\} \text{ if } x \neq a, b \text{ and } u \in E(\Gamma) \\ \{(u',a)|o(u') = o(u)\} \cup \{(u',b)|t(u') = o(u)\} \cup \{(o(u),x')|x' \in H\} \text{ if } x = a \\ \{(u',a)|o(u') = t(u)\} \cup \{(u',b)|t(u') = t(u)\} \cup \{(t(u),x')|x' \in H\} \text{ if } x = b \\ \{(u',a)|o(u') = u\} \cup \{(u',b)|t(u') = u\} \cup \{(u,x')|x' \in H\} \text{ if } u \in V(\Gamma) \end{cases}$$

Now define $\tilde{\Gamma} = {}^{\Gamma \times H}\!\big/_{\sim}$. We need to show that $\tilde{\Gamma}$ has the desired properties. We will now define here all the important notations that we will use throughout the proof:

- $\pi$ the natural projection of $\Gamma \times H$ onto $\tilde{\Gamma}$.
- $V(\tilde{\Gamma}) = \pi(\Gamma \times \{a, b\})$ and $E(\tilde{\Gamma}) = \tilde{\Gamma} \setminus V(\tilde{\Gamma})$.
- $H_1$, $H_2$ two disjoint connected opens of $H$, such that $a \in H_1$ and $b \in H_2$.

The idea of the proof is that we replace the edges by "intervals", which is called the topological realization of the graph. However what we require out of these intervals is that they have no local homeomorphisms besides identity, in that way we do not create additional homeomorphisms that would simply act on the intervals themselves. That means instead of taking intervals in $\mathbb{R}$ like it is traditionally done with topological graphs, we will take the de Groot space $H$. Furthermore we require that a homeomorphism of the new space $\tilde{\Gamma}$, does not jump between intervals and stays on the same one. That will be guaranteed by the fact that our graphs are profinite, hence totally disconnected, thus it is impossible to jump continuously from edge to edge, since our interval $H$ is connected.

The proof proceeds in the following steps:

- Step 1: Show that $V(\tilde{\Gamma})$ and $V(\Gamma)$ are homeomorphic by a homeomorphism $\alpha$. That way the extremities of the edges correspond to vertices of $V(\Gamma)$.
- Step 2: Show that every homeomorphism $g$ sends $V(\tilde{\Gamma})$ onto $V(\tilde{\Gamma})$
- Step 3: Show that every homeomorphism $g$ sends an open interval $\pi(\{e\} \times H)$ onto an open interval $\pi(\{e_g \times H\})$ for all $e \in E(\Gamma)$
- Step 4: Using the fact that $H$ has no non trivial local homeomorphisms, show that for every homeomorphism $g$ of $\tilde{\Gamma}$, every edge $e \in E(\Gamma)$ and every $x \in H \setminus \{a, b\}$ $g(\overline{(e,x)}) = \overline{(e_g, x)}$.
- Step 5: Using the continuity of $g$, conclude that for every $x$ this time in $H$, allowing the extremities, we get $g(\overline{(e,x)}) = \overline{(e_g, x)}$. That sufficiently determines the structure of the automorphisms to proceed with the rest of the proof.
- Step 6: Define the isomorphism from $Aut(\Gamma)$ to $Aut(\tilde{\Gamma})$ and its inverse.
- Step 7: Prove that if $\Gamma$ is connected, then $\tilde{\Gamma}$ is connected.

We have $\pi^{-1}(V(\tilde{\Gamma})) = \Gamma \times \{a, b\} \cup V(\Gamma) \times H$, therefore $V(\tilde{\Gamma})$ is closed in $\tilde{\Gamma}$ and therefore compact. Now define

$$\alpha = \begin{cases} V(\tilde{\Gamma}) \longrightarrow V(\Gamma) \\ u \mapsto \begin{cases} o(e) \text{ if } u = \overline{(e,a)} \text{ and } e \in E(\Gamma) \\ t(e) \text{ if } u = \overline{(e,b)} \text{ and } e \in E(\Gamma) \\ v \text{ if } u = \overline{(v,x)} \in \pi(V(\Gamma) \times H) \end{cases} \end{cases}$$

Let us show that $\alpha$ is a well defined continuous and bijective function. The fact that it is well defined is because of how the equivalence relation works. Now let us prove the continuity of $\alpha$. Let $u \in V(\tilde{\Gamma})$ and $U$ and open neighborhood of $\alpha(u)$ in $V(\Gamma)$. Now let $U_1 = o^{-1}(U)$ and $U_2 = t^{-1}(U)$, which are two opens in $\Gamma$ by continuity of the origin and terminus maps. Now let $U' = \pi(U_1 \times H_1 \cup U_2 \times H_2 \cup (U_1 \cap U_2) \times H)$. To show that $U'$ is open, we will show that $\pi^{-1}(U') = U_1 \times H_1 \cup U_2 \times H_2 \cup (H_1 \cap H_2) \times H$. Let $(u, x)$ be such that there exists $(u', x') \in U_1 \times H_1 \cup U_2 \times H_2 \cup (U_1 \cap U_2) \times H$ such that $\overline{(u,x)} = \overline{(u', x')}$. Let us differentiate all the possible cases:

- Case 1 $u \in V(\Gamma)$: We have that: $o(u) = t(u) = u \in U$ ,therefore $u \in (U_1 \cap U_2) \times H$. For the rest of the cases, we are going to assume that $u \in E(\Gamma)$.

- Case 2: $x' \neq a, b$.

  In that case we simply get $u = u'$ and $x = x'$ and $(u', x')$ is in $U_1 \times H_1 \cup U_2 \times H_2 \cup (U_1 \cap U_2) \times H$, then so is $(u, x)$.

- Case 3: $x = a$ , $x' = b$

  Then we get that $t(u') = o(u)$. Now $t(u') \in U$ and therefore $o(u) \in U$ and so $u \in U_1 = o^{-1}(U)$ and therefore $(u, x) \in U_1 \times H_1$. To prove that $(u, x) \in U_1 \times H_1 \cup U_2 \times H_2$ for the remaining cases, the approach is very similar, so we will simply list them here for the sake of completeness.

- Case 4: $x' = b$ and $x = a$.

- Case 5: $x' = a$ and $x = a$.

- Case 6: $x' = b$ and $x = b$.

Now that we know that $U'$ is open, we get that $U' \cap V(\tilde{\Gamma})$ is an open neighborhood of $u$. Now let us prove that $\alpha(U' \cap V(\tilde{\Gamma})) \subseteq U$. Let $v \in U' \cap V(\tilde{\Gamma})$. If $v = \overline{(e, a)}$ with $e \in E(\Gamma)$, then $e \in o^{-1}(U)$ and so $\alpha(v) = o(e) \in U$. If $v = \overline{(e, b)}$ with $e \in E(\Gamma)$, then $\alpha(v) = t(e) \in U$. Finally if $v = (v', x)$ with $v' \in V(\Gamma)$, then $v' = o(v') = t(v')$ and $v' \in U$. That concludes the proof of continuity of $\alpha$.

Now let us prove that $\alpha$ is injective. Suppose that $\alpha(v) = \alpha(v')$. Write for example $u = \overline{(e, a)}$ and $u' = \overline{(e', b)}$ with $e, e' \in E(\Gamma)$ (all the other cases are similar). In that case $o(e) = t(e)$ and by definition of the equivalence relation $u = u'$. The surjectivity of $\alpha$ simply comes from the fact that $\alpha(\overline{(u, a)}) = u$ for every $u \in V(\Gamma)$. Note that since $\alpha$ is a bijective continuous map between two compact spaces, it is a homeomorphism.

Now that we have properly defined the map $\alpha$, we shall investigate the structure of the autohomeomorphisms of $\tilde{\Gamma}$. We shall prove that for every $g \in \overline{Aut(\Gamma)}$ and for every $e \in E(\Gamma)$, there exists $e_g \in E(\Gamma)$, such that $\forall x \in H$, $g(\overline{(e, x)}) = \overline{(e_g, x)}$.

First let us start by proving that $g(V(\tilde{\Gamma})) \subseteq V(\tilde{\Gamma})$. By contradiction assume that there exists a $\overline{(u, x)} \in V(\tilde{\Gamma})$ with $u \in V(\Gamma)$, $e' \in E(\Gamma)$ and $x' \in H \setminus \{a, b\}$ such that $g(\overline{(u, a)}) = \overline{(e', x')}$. Then by the continuity of $g$, there exists $U$ a neighborhood of $\overline{(u, a)}$, such that $g(U) \subseteq \pi(E(\Gamma) \times H \setminus \{a, b\})$, since $\pi(E(\Gamma) \times H \setminus \{a, b\})$ is an open neighborhood of $(e', x')$. We will distinguish three cases:

- Case 1: The vertex $u$ is isolated. Then for every $h \in H$, $\pi^{-1}(U)$ is a neighborhood of $(u, h)$, therefore there exists $U_h$ an open neighborhood of $u$ and $I_h$ an open neighborhood of $h$, such that $\pi(U_h \times I_h) \subseteq U$. The set $H$ is compact, therefore we can find $h_1, \cdots, h_n \in H$, such that $H = I_{h_1} \cup \cdots \cup I_{h_n}$. Now write

  $$U' = \bigcap_{k=1}^{n} U_{h_k}$$

  Which is an open neighborhood of $u$. Since $u$ is isolated, we have $d^{-1}(\{u\}) = \{u\}$ with $d$ being the origin or the terminus map and thus $o^{-1}(\{u\}) \cup t^{-1}(\{u\}) \subseteq$

$U'$. By the lemma 4.7.8, there exists $V$ an open neighborhood of $u$, such that $o^{-1}(V) \cup t^{-1}(V) \subseteq U'$.

If $(o^{-1}(V) \cup t^{-1}(V)) \cap E(\Gamma) = \emptyset$, then the set $\pi((o^{-1}(V) \cup t^{-1}(V)) \times H)$ is an open neighborhood of $\overline{(u,a)}$. By continuity of $g^{-1} \circ \pi$, there exists $H'$ an neighborhood of $x'$, such that

$$\forall x \in H', g^{-1}(\overline{(e',x)}) \in \pi((o^{-1}(V) \cup t^{-1}(V)) \times H) \subseteq V(\tilde{\Gamma})$$

By local connectedness of $H$, there exists a $C \subseteq H'$ a connected neighborhood of $x'$. Now define

$$\phi = \begin{cases} C \longrightarrow V(\Gamma) \\ x \mapsto \alpha(g^{-1}(\overline{(e',x)})) \end{cases}$$

It is a continuous map from a connected set $C$ into a totally disconnected set $V(\Gamma)$, therefore it is constant. However $\phi$ has to also be injective, which is a contradiction.

Now if we assume instead that $(o^{-1}(V) \cup t^{-1}(V)) \cap E(\Gamma)$ is non empty, then take $e \in E(\Gamma)$, such $d(e) \in U$ with $d$ either the origin or the terminus map. The vertex $d(e)$ is not isolated and we have $g(\overline{d(e),a}) \in \pi(E(\Gamma) \times H \setminus \{a,b\})$, therefore $g(\overline{d(e),a}) \notin V(\tilde{\Gamma})$ and we may defer to the cases that follow to obtain a contradiction.

- Case 2: There exists $e \in E(\Gamma)$, such that $o(e) = u$. Remember that the set $U$ is open in $\tilde{\Gamma}$ and that $g(U) \subseteq \pi(E(\Gamma) \times H \setminus \{a,b\})$.

  There exists then $N$ an open neighborhood of $e$ and $I$ a connected neighborhood of $a$ not containing $b$, such that $\pi(N \times I) \subseteq U$.

  Now define a map $\phi$ as:

$$\phi = \begin{cases} I \longrightarrow E(\Gamma) \\ x \mapsto e'' \mid \overline{(e'',x'')} = g(\overline{(e,x)}) \end{cases}$$

  The map $\phi$ is well defined since the projection $\pi$ is injective on $E(\Gamma) \times H \setminus \{a,b\}$. It is a continuous map from a connected set $I$ to the totally disconnected set $\Gamma$, hence it is constant, thus equal to $e'$ at all points.

  Now let $I' \subseteq I$ be an open neighborhood of $a$. Let us define a map $\psi$ by

$$\psi = \begin{cases} I' \longrightarrow H \\ x \mapsto x'' \mid g(\overline{(e,x)}) = \overline{(e',x'')} \end{cases}$$

  It is an injective continuous and open map since $g$ is a homeomorphism. By the property of rigidity of $H$, we have that $\forall x \in I, \psi(x) = x$: in particular $\psi(a) = a$, which is the desired contradiction, since $\psi$ can only take values in $H \setminus \{a,b\}$.

- Case 3: There exists $e \in E(\tilde{\Gamma})$, such that $t(e) = u$. This case is essentially the same as the case 2, therefore will be omitted.

We conclude therefore that $g(V(\tilde{\Gamma})) \subseteq V(\tilde{\Gamma})$ and thus $g$ sends "vertices" on "vertices". By bijectivity of $g$, we then get that $g(E(\tilde{\Gamma})) \subseteq E(\tilde{\Gamma})$.

The next step is to show that

$$\forall e \in E(\Gamma), \exists e' \in E(\Gamma), \forall x \in H \setminus \{a,b\}, \exists y \in H \setminus \{a,b\}, g(\overline{(e,x)}) = \overline{(e',y)}$$

Consider

$$\phi = \begin{cases} H \setminus \{a,b\} \longrightarrow E(\Gamma) \\ x \mapsto e' \mid \exists y \in H \setminus \{a,b\}, g(\overline{(e,x)}) = \overline{(e',y)} \end{cases}$$

Let us show that $\phi$ is well defined and continuous. The map $\phi$ is well defined simply because

$$g(\pi(\{e\} \times H \setminus \{a,b\})) \subseteq \pi(E(\Gamma) \times H \setminus \{a,b\})$$

Now to show that $\phi$ is continuous, suppose that $g(\overline{(e,x)}) = \overline{(e',y)}$ and let $V$ be an open neighborhood of $e'$. Then $\pi(V \times H \setminus \{a,b\})$ is an open neighborhood of $\overline{(e',y)}$ in $\tilde{\Gamma}$. By continuity of $g$, there exists $U$ an open neighborhood of $\overline{(e,x)}$, such that $g(U) \subseteq \pi(V \times H \setminus \{a,b\})$. By continuity of $\pi$ at $(e,x)$, there exists $W$ a neighborhood of $x$ not containing $a, b$, such that $\pi(\{e\} \times W) \subseteq U$. Let us show that $\phi(W) \subseteq V$. Let $x' \in W$. Then $g(\overline{(e,x')}) \in \pi(V \times H \setminus \{a,b\})$ and so there exists $e'' \in V$ and $y' \in H \setminus \{a,b\}$, such that $g(\overline{(e,x')}) = \overline{(e'',y')}$. We therefore get that $\phi(x') = e'' \in V$ as expected. The map $\phi$ is a continuous map from the connected set $H \setminus \{a,b\}$ to $E(\Gamma)$, which is totally disconnected and therefore is constant. We denote $e_g$ its value.

Now we will show that $\forall x \in H \setminus \{a,b\}, g(\overline{(e,x)}) = \overline{(e_g,x)}$. Define

$$\psi = \begin{cases} H \setminus \{a,b\} \longrightarrow H \setminus \{a,b\} \\ x \mapsto y \mid g(\overline{(e,x)}) = \overline{(e_g,y)} \end{cases}$$

The relation $\psi$ is a well defined function. It is injective, because $g$ is injective and it is open and continuous due to $g$ being a homeomorphism. Then we get $\forall x \in H \setminus \{a,b\}, \psi(x) = x$. As such we indeed have $\forall x \in H \setminus \{a,b\}, g(\overline{(e,x)}) = \overline{(e_g,x)}$.

Now we shall prove that $g(\overline{(e,a)}) = \overline{(e_g,a)}$ and that $g(\overline{(e,b)}) = \overline{(e_g,b)}$. Since the proofs of these two statements are similar, we will only do one of them.

Write $u = \alpha(g(\overline{(e,a)}))$. By contradiction assume that $u \neq o(e_g)$. Then there exists $U$ a neighborhood of $u$, such that $o(e_g) \notin U$. By continuity of $g$, there exists $V$ a neighborhood of $\overline{(e,a)}$, such that

$$\forall q \in V, g(q) \in \pi(\ o^{-1}(U) \times H_1 \cup t^{-1}(U) \times H_2 \cup (o^{-1}(U) \cap t^{-1}(U) \times H\ )$$

By continuity of $\pi$, there exists $W$ a neighborhood of $a$ contained in $H_1$, such that

$$\forall x \in W, \overline{(e,x)} \in V$$

Now pick an $x \in W$ distinct from $a$. We then have

$$g(\overline{(e,x)}) = \overline{(e_g, x)} \in \pi(o^{-1}(U) \times H_1 \cup t^{-1}(U) \times H_2 \cup (o^{-1}(U) \cap t^{-1}(U)) \times H)$$

Since $x \notin H_2$, we get that $(e_g, x) \in o^{-1}(U) \times H_1 \cup (o^{-1}(U) \cap t^{-1}(U)) \times H$. Hence $o(e_g) \in U$, which is a contradiction. As such, we conclude that

$$\forall e \in E(\Gamma), \exists e_g \in E(\Gamma), \forall x \in H, g(\overline{(e,x)}) = \overline{(e_g, x)}$$

Now that we have a description of the autohomeomorphisms of $\tilde{\Gamma}$, we can prove that $Aut(\Gamma)$ is isomorphic to $Aut(\tilde{\Gamma})$ as a topological group.

Define

$$\Phi = \begin{cases} Aut(\Gamma) \longrightarrow Aut(\tilde{\Gamma}) \\ g \mapsto \begin{cases} \tilde{\Gamma} \longrightarrow \tilde{\Gamma} \\ \overline{(u,x)} \mapsto \overline{(g(u),x)} \end{cases} \end{cases}$$

First we need to prove that $\Phi$ is well defined. Take $g \in Aut(\Gamma)$. Now consider $\tilde{g}$ the map from $\Gamma \times H$ to $\tilde{\Gamma}$, given by $\tilde{g}(u,x) = \overline{(g(u),x)}$. The map $\tilde{g}$ is continuous as a composition of a continuous map on $\Gamma \times H$ with the natural projection $\pi$. Because $g$ is an automorphism of a graph, $\tilde{g}$ is compatible with the equivalence relation $\sim$ and therefore factors into a continuous map from $\tilde{\Gamma}$ to $\tilde{\Gamma}$ , $\Phi(g)$. The map $\Phi$ is a morphism of groups. Now we need to prove that $\Phi$ is continuous for the open compact topology.

For that we take $K$ a compact and $U$ an open in $\tilde{\Gamma}$, such that $\Phi(g)(K) \subseteq U$. Now we write $\Gamma$ as a projective limit of some $\Gamma_i$ indexed by a directed set $I$ and $p_i$ the natural projections. For $i \in I$ and $u \in \Gamma$, we denote $P_i(u) = \{u' \in \Gamma | p_i(u') = p_i(u)\}$ Now if $(u,x) \in \pi^{-1}(\Phi(g)(K))$, then $\overline{(u,x)} \in U$ and therefore by the continuity of $\pi$, there exists $i_{e,x} \in I$ and $H_{e,x} \subseteq H$, such that $\pi(P_{i_{e,x}}(e) \times H_{e,x}) \subseteq U$. The $P_{i_{e,x}} \times H_{e,x}$ cover $\pi^{-1}(\Phi(g)(K))$, so by compactness there exists a finite family $P_{i_{e_1,x_1}} \times H_{e_1,x_1}, \cdots, P_{i_n,x_n} \times H_{e_n,x_n}$ covering $\pi^{-1}(g(K))$ as well. For simplicity we will denote $P_k = P_{i_{e_k,x_k}}$ and $H_k = H_{e_k,x_k}$. Now let $i$ be an upper bound of $\{i_1, \cdots, i_n\}$ and let $g' \in Aut(\Gamma)$, such that $p_i \circ g = p_i \circ g'$. Let us prove that $\phi(g')(K) \subseteq U$. Let $\overline{(u,x)} \in K$. Then $\overline{(g(u),x)} \in \Psi(g)(K)$, therefore there exists an $m$, such that $(g(u),x) \in P_m \times H_m$. Now $p_i \circ g = p_i \circ g'$ and therefore $p_m(g'(u)) = p_m(g(u))$, from which we get that $(g'(u),x) \in P_m \times H_m$ and thus $\Phi(g')(\overline{(u,x)}) = \overline{(g'(u),x)} \in U$. We therefore conclude that $\Phi$ is a continuous map. To finish the proof that $Aut(\Gamma)$ and $Aut(\tilde{\Gamma})$ are isomorphic, we will define a continuous inverse of $\Phi$.

To define the inverse, we take a $x_0 \in H \setminus \{a,b\}$. We define a map

$$\beta = \begin{cases} \pi(E(\Gamma) \times \{x_0\}) \longrightarrow E(\Gamma) \\ \overline{(e,x_0)} \mapsto e \end{cases}$$

This map is a homeomorphism. Now write

$$\Psi = \begin{cases} Aut(\tilde{\Gamma}) \longrightarrow Aut(\Gamma) \\ g \mapsto \begin{cases} \Gamma \longrightarrow \Gamma \\ u \mapsto \begin{cases} \alpha \circ g \circ \alpha^{-1}(u) \text{ if } u \in V(\Gamma) \\ \beta \circ g \circ \beta^{-1}(u) \text{ if } u \in E(\Gamma) \end{cases} \end{cases} \end{cases}$$

This is where the main difference between assuming that the set of edges $E(\Gamma)$ is closed in $\Gamma$ or not lies. The difficulty here is that while neighborhoods of edges are essentially the same between $\tilde{\Gamma}$ and $\Gamma$, it is not the case for the neighborhoods of vertices. In case of vertices, every edge that is incident to the vertex has to be in its neighborhood. If we assume the set of edges to be closed, we can circumvent this problem by dealing with vertices and edges separately and using the fact that $\alpha$ and $\beta$ are isomorphisms. If on the other hand, we do not make this assumption, we will have to add the edges incident to a vertex and their neighborhoods and use them when proving continuity.

Let us now prove $\Psi(g)$ is continuous for all $g$. Let $u \in \Gamma$. To prove that $\Psi(g)$ is continuous at $u$, we distinguish two cases.

- Case 1: $u \in E(\Gamma)$. Let $V \subseteq E(\Gamma)$ be a neighborhood of $\Psi(g)(u)$. By continuity of $\beta \circ g \circ \beta^{-1}$, there exists $U$ a neighborhood of $u$, such that $\Psi(g)(U) \subseteq V$.

- Case 2: $u \in V(\Gamma)$. Let $U$ be a clopen neighborhood of $\Psi(g)(u)$.

  For every edge $e \in (o^{-1}(\{u\}) \cup t^{-1}(\{u\}))$, take $V_e \subseteq E(\Gamma)$ a clopen neighborhood of $e$. The family $(V_e)_{e\in D}$ with $D = (o^{-1}(\{u\}) \cup t^{-1}(\{u\})) \cap E(\Gamma)$ together with $U$ covers $o^{-1}(\{u\}) \cup t^{-1}(\{u\})$, therefore by compactness there exist $e_1, \cdots, e_n \in D$, such that $o^{-1}(\{u\}) \cup t^{-1}(\{u\}) \subseteq U \cup V_{e_1} \cup \cdots \cup V_{e_n}$. By the lemma 4.7.8, let then $U'$ be a neighborhood of $\Psi(g)(u)$, such that

  $$o^{-1}(U') \subseteq U \cup V_{e_1} \cup \cdots \cup V_{e_n}$$

  and

  $$t^{-1}(U') \subseteq U \cup V_{e_1} \cup \cdots \cup V_{e_n}$$

  Now define $U_1 = o^{-1}(U')$ and $U_2 = t^{-1}(U')$. Let $V$ be the set

  $$V = \pi(U_1 \times H_1 \cup U_2 \times H_2 \cup (U_1 \cap U_2) \times H)$$

  The set $V$ is then open in $\tilde{\Gamma}$ and therefore by continuity of $g$, there exists $N$ a neighborhood of $\overline{(u, x_0)}$, such that $g(N) \subseteq V$. Furthermore the set $\pi\big((\bigcup_{k=1}^{n} V_{e_k}) \times \{x_0\}\big)$ is closed in $\tilde{\Gamma}$, thus by continuity of $g$, the set $g^{-1}(\tilde{\Gamma} \setminus \pi(\bigcup_{k=1}^{n} V_{e_k} \times \{x_0\}))$ is an open neighborhood of $\overline{(u, x_0)}$. Write then

  $$N' = N \setminus g^{-1}\big(\pi(\bigcup_{k=1}^{n} V_{e_k} \times \{x_0\})\big)$$

  By continuity of the natural projection, there exists then $A$ a neighborhood of $u$ and $I$ a neighborhood of $x_0$, such that $\pi(A \times I) \subseteq N'$. Observe that

  $$\psi(g)(A) \subseteq U_1 \cup U_2 \subseteq U \cup V_{e_1} \cup \cdots \cup V_{e_n}$$

Furthermore if $\overline{(e,x_0)} \in N'$ with $e \in E(\Gamma)$, then by definition

$$g(\overline{(e,x_0)}) \notin \pi(\bigcup_{k=1}^{n} V_{e_k} \times \{x_0\})$$

and therefore $\Psi(g)(e) \notin \bigcup_{k=1}^{n} V_{e_k}$. This proves that

$$\forall u' \in A, \Psi(g)(u') \notin \bigcup_{k=1}^{n} V_{e_k}$$

We then get that

$$\Psi(g)(A) \subseteq U' \setminus (V_{e_1} \cup \cdots \cup V_{e_n}) \subseteq U$$

We conclude therefore that $\Psi(g)$ is continuous at $u$.

We now need to prove that for all $g$, $\Psi(g)$ is compatible with the origin and terminus map. We will only prove it for origin as the proof for terminus is essentially the same. Take $e \in E(\Gamma)$. We have

$$\Psi(g)(o(e)) = \alpha(g\alpha^{-1}(o(e))) = \alpha(g(\overline{(e,a)})) = \alpha(\overline{(\Psi(g)(e),a)}) = o(\Psi(g)(e))$$

Finally to conclude that $Aut(\tilde{\Gamma})$ and $Aut(\Gamma)$ are isomorphic, we need to prove that $\Psi$ is a continuous map. For that let $K$ be a compact in $\Gamma$ and $U$ an open in $\Gamma$, such that $\Psi(g)(K) \subseteq U$. For each $u \in K$, we construct a clopen set $N_u \subseteq \Gamma$ as follows:

First if $u \in E(\Gamma)$, we take $N_u \subseteq E(\Gamma)$ a clopen subset of $\Gamma$, such that $\Psi(g)(N_u) \subseteq U \cap E(\Gamma)$.

If $u \in V(\Gamma)$ denote $u' = \Psi(g)(u)$, by what we have seen when proving the continuity of $\Psi(g)$, we can construct a clopen $A_u \subseteq E(\Gamma)$ and an open neighborhood of $u'$ in $V(\Gamma)$ $B_u$, such that

$$o^{-1}(B_u) \cup t^{-1}(B_u) \subseteq U \cup A_u$$

Observe that $U \setminus A_u$ is a neighborhood of $u'$ and take $N_u$ to be a clopen neighborhood of $u$, such that $\Psi(g)(N_u) \subseteq U \setminus A_u$. Since $N_u$ form an open cover of $K$, therefore by compactness of $K$, there exist $u_1, \cdots, u_n \in \Gamma$, such that

$$K \subseteq N_{u_1} \cup \cdots \cup N_{u_n}$$

Now let $i$ be an integer between 1 and $n$: we will distinguish two cases:

- Case 1: $u_i$ is an edge.

  Then we have

$$g(\pi(N_{u_i} \times \{x_0\})) \subseteq (U \cap E(\Gamma)) \times H \setminus \{a,b\}$$

- Case 2: $u_i$ is a vertex.

  Then we have

  $$g(\pi(N_{u_i} \times \{x_0\})) \subseteq S(B_{u_i}) \setminus \pi(A_{u_i} \times \{x_0\}) \cup \pi((U \cap E(\Gamma)) \times H \setminus \{a, b\})$$

  With

  $$S(B_{u_i}) = \pi(o^{-1}(B_{u_i}) \times H_1 \cup t^{-1}(B_{u_i}) \times H_2 \cup (o^{-1}(B_{u_i}) \cap t^{-1}(B_{u_i}) \times H))$$

These inclusions define an open neighborhood of $g$ in the open compact topology which we shall call $\mathscr{V}$. Let us now show that $\forall g' \in \mathscr{V}, \Psi(g')(K) \subseteq U$. Take $g' \in \mathscr{V}$ and let $u \in K$. Then there exists $i$, such that $u \in N_{u_i}$.

If $u_i$ is an edge, then by definition: $g'(\overline{(u, x_0)}) \in (U \cap E(\Gamma)) \times H \setminus \{a, b\}$ and thus $\Psi(g')(u) \in U$.

If on the other hand $u_i$ is a vertex, then we have two possibilities.

We can have $g'(\overline{(u, x_0)}) \in S(B_{u_i}) \setminus \pi(A_{u_i} \times \{x_0\})$, in which case we conclude that either $t(\Psi(g')(u))$ or $o(\Psi(g')(u))$ is in $B_u$. If that is the case we get that $\psi(g)(u)$ is in $U \setminus A_{u_i}$. The other possibility is that $g'(\overline{(u, x_0)}) \in (U \cap E(\Gamma) \times H \setminus \{a, b\})$, in which case $\Psi(g')(u) \in U$.

We therefore conclude that $\Psi$ is a continuous map and we have that $Aut(\Gamma)$ and the autohomeomorphisms of $\tilde{\Gamma}$ are isomorphic as topological groups.

Finally let us prove that $\tilde{\Gamma}$ is connected if $\Gamma$ is connected. To prove it, we take $f$ a continuous map from $\tilde{\Gamma}$ to $\{0, 1\}$. Since $H$ is a connected set and $f$ a continuous map, we have that

$$\forall u \in \Gamma, \forall x, x' \in H, f\left(\overline{(u, x)}\right) = f\left(\overline{(u, x')}\right)$$

Define

$$\phi = \begin{cases} \Gamma \longrightarrow \{0, 1\} \\ u \mapsto f(\overline{(u, x_0)}) \end{cases}$$

Let us show that $\phi$ is a continuous qmorphism from $\Gamma$ to the discrete graph $\{0, 1\}$. To prove that it is a qmorphism, we just observe that

$$\forall e \in E(\Gamma), f\left(\overline{(e, a)}\right) = f\left(\overline{(e, x_0)}\right) = f\left(\overline{(e, b)}\right)$$

To prove the continuity of $\phi$, we take a $u \in \Gamma$ and show that $\phi$ is constant on a neighborhood of $u$. By the continuity of $f$, there exists $U$ a neighborhood of $\overline{(u, x_0)}$, such that $f$ is constant on $U$. Now by continuity of the natural projection $\pi$, there exists $U'$ a neighborhood of $u$ in $\Gamma$, such that $\pi(U') \subseteq U$. We then have that $\phi$ is constant on $U'$.

Since $\phi$ is a continuous qmorphism from $\Gamma$ to $\{0, 1\}$, we have that $\phi$ is constant. Given that for every $u \in \Gamma$, $f$ is constant on $\pi(\{u\} \times H)$, we can conclude that $f$ is a constant function and thus $\tilde{\Gamma}$ is connected. □

As a corollary of this result, we have the theorem:

**Theorem 4.7.10.** *Let $G$ be a profinite group. Then there exists a compact connected Hausdorff set $X$, such that $G$ is isomorphic to the group of autohomeomorphisms of $X$ with the open compact topology.*

*Proof.* In case $G$ is the singleton $\{1_G\}$, just pick $X$ to be the de Groot space $H$ constructed earlier.

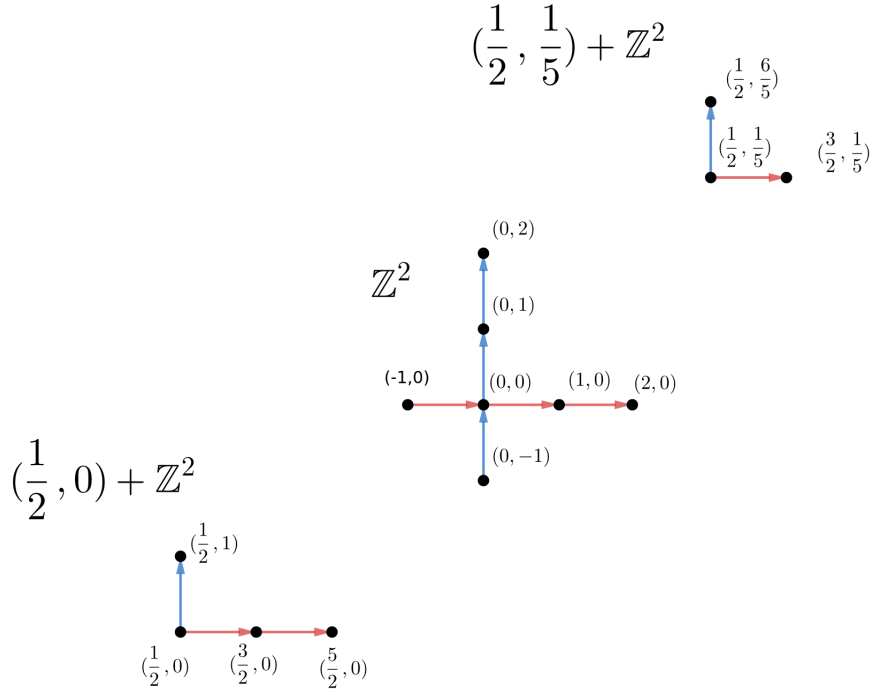
In case $G$ is a finite group that is not a singleton, take $S$ to be the set $G \setminus \{1_G\}$ and $\Gamma$ to be the Cayley graph $Cay(G, S)$ together with its standard colors. Next take $\Gamma'$ to be the colorless graph whose group of automorphisms is that of $\Gamma$ i.e. $G$. Finally by 4.7.9 let $X$ be a compact Hausdorff connected space whose group of autohomeomorphisms is $Aut(\Gamma') = G$.

Finally in case $G$ is an infinite group, pick $N$ an open proper normal subgroup of $G$. Such a group exists, since otherwise $G$ would have to be finite. Furthermore let $x_1, \cdots, x_n$ be representatives of the classes in ${}^{G}\!/\!{}_{N}$,with $x_1 \in N$ and $x_1 \neq 1_G$. Consider then $S = x_2N \cup \{x_1, \cdots, x_n\}$. The set $S$ is closed in $G$ as a union of a closed and a finite set. Furthermore it generates $G$ as an abstract group, so the graph $\Gamma = Cay(G, S)$ is not only connected as a profinite graph, but in fact path-connected. Now by the theorem 4.6.8 take $\Gamma'$ to be a colorless connected graph, such that $G = Aut_c(\Gamma) \cong Aut(\Gamma')$. We then take by the theorem 4.7.9 $X$ to be the connected compact Hausdorff space, whose group of autohomeomorphisms is isomorphic to $Aut(\Gamma') \cong G$.

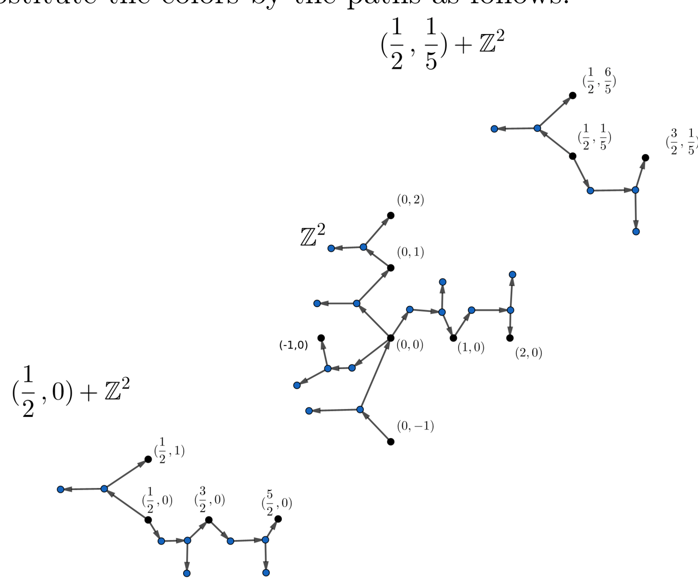
□

As mentioned earlier, I proved originally this result for only finitely generated profinite groups. It is worth mentioning that in case of finitely generated groups the theorem does not extend an already proven result due to Gartside and Glyn in [18]. They proved it for the case of metric profinite groups and as we will show in the next section finitely generated profinite groups are in fact metric. Our approach followed rather the ideas of Hoffmann and Morris in [25] and by adopting the language of profinite graphs and introducing the notion of colors and color substitution, I expanded on those ideas and proven the general case. Finally we give an example of the construction of the space $X$ for the group $\mathbb{Z}_3 \times \mathbb{Z}_3$, with $\mathbb{Z}_3$ being the three-adic numbers.

**Example 4.7.11.** The illustration of the Cayley graph of $\mathbb{Z}_3 \times \mathbb{Z}_3$ is below with blue corresponding to translation by $(0,1)$ and red by $(1,0)$:



It is composed of all the disjoint components corresponding to the cosets. While this graph is not path-connected, with the topology of a profinite graph, we can approach any element of it by a sequence in the component corresponding to $\mathbb{Z}^2$. For example, $(\frac{1}{2},0)$, can be approached by the sequence $((1+\sum_{k=0}^{n} 3^k, 0))_{n\in\mathbb{N}}$. We then substitute the colors by the paths as follows:



Note that the color substitution here is slightly simplified compared to the general one in 4.6.6, but in this context it still works and is slightly clearer.

After the color substitution is done, we will get that the group of continuous automorphisms of the new profinite graph is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. The final step is to replace each oriented edge by a doubly pointed deGroot curve and the group of autohomeomorphisms of the resulting space will be isomorphic to the group $\mathbb{Z}_3 \times \mathbb{Z}_3$ as a topological group.

## 4.8 Finitely generated profinite groups are metric

To establish this result, we will prove that the open subgroups of a finitely generated profinite group are countable and then we will use it to construct a valuation on such profinite groups.

**Proposition 4.8.1.** *For every finitely generated profinite group $G$ and every index $n \in \mathbb{N}$, the set of open subgroups of $G$ of index $n$ is finite.*

*Proof.* This proof can be found in [26] Proposition 1.6. Let $m$ be the number of generators. Then we can inject the continuous functions from $G$ to $\mathcal{S}_n$(the symmetric group with discrete topology) into $\mathcal{S}_n^m$. Now by axiom of choice, for each $H$ open subgroup of $G$ of index $n$, choose $f_H$ a bijection between ${}^{G}\!/_{H}$ and $\{1, \cdots, m\}$, such that $f_H(H) = 1$.

Furthermore for an open subgroup $H$ and each $g \in G$, we can define a permutation on ${}^{G}\!/_{H}$, given by the formula: $\overline{g'} \cdot g^{-1} = \overline{g' g^{-1}}$. Such a permutation is well defined: indeed if $\overline{g'} = \overline{g''}$, we get that $(g'g^{-1})(g''g^{-1})^{-1} = g'g''^{-1} \in H$ and so $\overline{g'g^{-1}} = \overline{gg''^{-1}}$. Now for every $H$, we define

$$\Phi(H) = \begin{cases} G \longrightarrow \mathcal{S}_n \\ g \mapsto \begin{cases} \{1, \cdots n\} \longrightarrow \{1, \cdots, n\} \\ k \mapsto f_H(f_H^{-1}(k) \cdot g^{-1}) \end{cases} \end{cases}$$

Let us show that $\Phi(H)$ is a continuous morphism. We have for $k \in \{1, \cdots, n\}$ and $g, g' \in G$ that $\Phi(H)(gg') = f_H(f_H^{-1}(k)(gg')^{-1}) = f_H(f_H^{-1}(k)g'^{-1}g^{-1}) = \Phi(H)(g)(\Phi(H)(g')(k))$ and so $\Phi(H)$ is indeed a morphism. Now let us show that it is continuous by proving its kernel is the open $H$. We get that $\Phi(H)(g) = id$ if and only if for every $k$ between 1 and $n$, $f_H(f_H^{-1}(k) \cdot g^{-1}) = k$, which since $f_H$ is a bijection is equivalent to saying that for every $g' \in G$, $\overline{g'g^{-1}} = \overline{g'}$, which is equivalent to $g$ being in $H$.

Finally let us prove that $\Phi$ is an injective function. Suppose that $\Phi(H) = \Phi(H')$. Let us prove that $H \subseteq H'$. Let $h \in H$. Since $\Phi(H)(h) = \Phi(H')(h)$, we get that $\Phi(H')(h)(1) = 1$. And therefore $f_H(H' \cdot h^{-1}) = 1$, thus $H' \cdot h^{-1} = H'$, which proves that $h \in H'$. Since $H, H'$ have symmetric roles we conclude that $H = H'$ and that $\Phi$ is an injective function.

Since $Hom_c(G, \mathcal{S}_n)$ is finite and $\Phi$ is an injection from $X_n$ to $Hom_c(G, \mathcal{S}_n)$, we conclude that $X_n$ is finite. □

From this follows immediately the:

**Corollary 4.8.1.1.** *Let $G$ be a finitely generated profinite group. Then the set of opens in $G$ is countable.*

Now we can conclude that a finitely generated profinite group is metrizable by the Urysohn metrization theorem stating that every regular Hausdorff space such that every point has a countable basis of neighborhoods is metrizable. We can however provide a more straightforward proof written below.

**Proposition 4.8.2.** *Let $G$ be a finitely generated profinite group, then $G$ is metrizable.*

*Proof.* We take $(N_n)_{n\in\mathbb{N}}$ a sequence of all open normal subgroups, which is possible since open subgroups of $G$ are countable. Now we will define a valuation on the group $G$, which will induce our metric.

We define $U_n = \bigcap\limits_{k=0}^{n} N_k$, which is an open normal subgroup of $G$. Now for $g \in G$, we define $v(g) = min\{n \in \mathbb{N} \cup \{\infty\} | g \notin U_n\}$. Let us prove that $v$ is a valuation, i.e:

- $\forall g, g' \in G, v(gg'^{-1}) \geq min(v(g), v(g'))$
- $\forall g \in G, v(g) = \infty \Leftrightarrow g = 1_G$

For more details about valuations on groups see [43].

Let $n = v(g)$ and $m = v(g')$. Assume by contradiction that $v(gg') < n$ and $v(gg') < m$. Without loss of generality, we may assume that $n \leq m$. In that case $gg'^{-1} \notin U_{v(gg'^{-1})}$ and $g, g'\ inU_{n-1}$. Since $U_{n-1}$ is a subgroup of $U_{v(gg'^{-1})}$, we get that $gg'^{-1} \in U_{v(gg'^{-1})}$, which is a contradiction. Furthermore we have that $v(1_G) = \infty$, since as subgroups, all $U_n$ have to contain $1_G$. Finally we know that since $G$ is profinite $\bigcap\limits_{n\in\mathbb{N}} N_n = \{1_G\}$, hence if $v(g) = \infty$, then $g = 1_G$.

Now let us show that $v$ defines a metric. Write

$$d(g, g') = \begin{cases} 2^{-v(gg'^{-1})} \text{ if } v(gg'^{-1}) \neq \infty \\ 0 \text{ else} \end{cases}$$

Let us check that $d$ verifies the three axioms of an ultrametric.

- Let $g, g' \in G$. We have that $d(g, g') = 0 \Leftrightarrow v(gg'^{-1}) = \infty \Leftrightarrow gg'^{-1} = 1_G \Leftrightarrow g = g'$.
- We have $v(gg'^{-1}) = v(g'g^{-1})$ for all $g, g' \in G$, so $d(g, g') = d(g', g)$.
- Let $g, g', g'' \in G$. We have $v(gg''^{-1}) = v((gg'^{-1})(g''g'^{-1})^{-1}) \geq min(v(gg'^{-1}), v(g''g'^{-1}))$. We therefore get that $d(g, g') \leq max(d(g, g'), d(g', g''))$, since $x \mapsto 2^{-x}$ is a decreasing function.

The valuation $v$ defines therefore a metric and more precisely an ultrametric on $G$. Now we just need to prove that the topology induced by $v$ is the profinite topology on $G$. To prove that the topologies are the same, we will now show that the neighborhoods for both topologies coincide. Let then $V$ be a neighborhood of $g_0 \in G$ for the profinite topology. Then there exists $n \in \mathbb{N}$, such that $g_0 \cdot N_n \subseteq V$ and so $g_0 \cdot U_n \subseteq V$. Now

$$U_n = \{g \in G | v(g) > n\}$$

so $g_0 U_n$ is an open neighborhood of $g_0$ for the topology induced by the valuation and thus $V$ is an open neighborhood of $g_0$ for the valuation topology. Now on the other hand let us suppose that $V$ is a neighborhood of $g_0$ for the valuation topology. Then there exists $\epsilon > 0$, such that $\{g \in G | v(g_0 g^{-1}) > log_2(-\epsilon)\} \subseteq V$. Now take $n \in \mathbb{N}$, such that $n > log_2(-\epsilon)$. Now let us show that $U_n \cdot g_0 \subseteq V$. Take $g \in U_n$. Now $v(g_0(gg_0)^{-1}) = v(g^{-1}) = n > log_2(-\epsilon)$. Therefore $d(g_0, gg_0) < \epsilon$ and $U_n g_0 \subseteq V$ and since $U_n g_0$ is open for the profinite topology, we get that $V$ is a neighborhood of $g_0$ for the profinite topology. We therefore conclude that $G$ is metrizable. $\square$

# Conclusion

The original motivation of this thesis for studying graphs was Galois theory. The main motivation was to find connections between Galois groups: permutations on roots of polynomials fixing algebraic equations and automorphisms of graphs: permutations on vertices preserving edges. In the first chapter we saw an example of how graphs can be used to solve a Galois theory question: by the theorem of János Kollár and Ervin Fried, proving that every finite group is a group of automorphisms of some finite extension of $\mathbb{Q}$. This theorem uses an old theorem known as the theorem of Frucht that establishes that every finite group is a group of automorphisms of a finite graph. It is proved by starting with a Cayley graph and by substituting edges of certain colors by graphs. In Galois theory not only Galois groups but their actions on roots as well are of interest, hence we extended Cayley graphs to not only represent the group itself, but as well the action of a group on a set: we called such an extension the group action Cayley graphs.

In the second chapter we saw another way of creating links between graph theory and Galois theory using a tool coming from algebraic topology called covering graphs. We have shown how covering graphs are analogous to field extensions, normal covering graphs to normal field extensions and deck transformations to automorphisms of field extensions.

Since this thesis uses many different kind of profinite structures such as: profinite sets, profinite groups, profinite rings and profinite modules, in the third chapter we grouped their common properties into one categorical notion called a profinite structure.

Their profinite topology is an important property in the study of infinite Galois groups. Hence in order to generalize the links between Galois theory and Graph theory that we explored in the first two chapters, we needed to equip the graphs with a profinite topology as well in order to obtain the profinite graphs studied in the fourth chapter. We explored the notions of Galois coverings for profinite graphs as well as generalizations of notions such as homology and connectedness to profinite graphs. We constructed the profinite group action Cayley graphs, which can represent the action of a profinite group on a profinite set. We then defined a notion of a color on a the edges of a profinite graph and we have seen two color substitution theorems on profinite graphs with a closed set of edges. The first theorem deals with graphs with a closed set of edges and finitely many colors. After substituting the edges in such a graph with finite graphs, we obtain a graph without colors whose set of edges is still closed. The second theorem deals with graphs with a closed set of edges and infinitely many colors. After substituting the edges in such a graph with infinite and compactified graphs, we obtain a graph without colors, but whose set of edges is no longer closed. Those two theorems make it possible to for instance drop the colors in the profinite group action Cayley graphs and represent an action of a profinite group on a profinite set by a profinite graph without colors. As seen in the Chapter 3, since etale algebras can be represented as actions of the absolute Galois group on a profinite set, we can then represent any etale algebra by a profinite graph. Last application of the first color substitution theorem we gave is to prove the Morris-Hofmann conjecture stating that every profinite group is a group of autohomeomorphisms of a Hausdorff connected compact space. We it using the profinite Cayley graphs and the color substitutions theorems.

Let us finally mention some possible areas of research.

The first observation to be made is that the infinite substitution theorem does not preserve the property of having closed set of edges: it would be interesting to find a substitution that can preserve such a property. Furthermore the construction we gave is non canonical, increases in size very quickly and doesn't use the fact that the set of colors is profinite. One could possibly find a more suitable family of profinite graphs to substitute the colored edges with than just compactified Sabidussi graphs.

Another possible research suggested to me by Professor Ribes would be to investigate which graphs have an automorphism group with the property to be pro-$p$ and I believe a possible way to start investigating this question would be to start with finite graphs and work out if there is a relation between sizes of orbits of the natural action of the autmorphism group and the relation of being a $p$-group.

Trough the substitution of colors we have seen one example where we start with classical theorems on graphs and we generalize them to the case of profinite graphs. It would be very interesting to see what other notions from graph theory could be generalized to profinite graphs. One such potential example of study would be the graph spectrum. We know that profinite groups are limit of finite graphs. Could there be for instance in the case of profinite metric graphs be a way to study how the spectrum of these finite graphs evolve as the precision increases? Another line of research that could be explored would be to study the Galois theory of etale algebras using profinite graphs. The starting point for such an approach would be the representation of an etale algebra by a profinite graph we saw in the Chapter 4. In this way a number of ideas from both graph theory, Galois theory, algebraic geometry, topology and combinatorics of covering spaces can be promoted for their mutual benefit.

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