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# Polynomial Identities of Algebras with Actions: A Unified Combinatorial Approach

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# Abstract

Suppose $R$ is an associative algebra with $1_R$ and $A$ is an associative or Lie algebra equipped with an $R$-module action with the property that the algebra of endomorphisms on $A$ defined by the $R$-action is of finite dimension. In this thesis, we establish a series of conditions that ensure that $A$ is a PI-algebra. This work extends a collection of results in associative and Lie PI-theory.

First, we use the added structure of an $R$-action on the associative algebra $A$ to extend the classical notion of polynomial identity to so called $R$-identities. We then ask: is the existence of an $R$-identity sufficient to ensure that $A$ is a PI-algebra? In general, the answer is negative; nonetheless, we prove that if the $R$-action is 'compatible' with the multiplicative structure of $A$, a suitable condition on the $R$-identity yields a positive result. With the aim of extending this result, we then consider associative algebras $A$ satisfying the following property: for all $a_1, \ldots, a_d$ in $A$, the product $a_1 \cdots a_d$ is a linear combination of elements of the form $(R \cdot a_{\sigma(1)}) \cdots (R \cdot a_{\sigma(d)})$, where $\sigma$ is a non-identity permutation in $S_d$; we call these algebras $R$-rewritable. We conclude that if the $R$-action is 'compatible' with the multiplicative structure of an $R$-rewritable algebra $A$, then $A$ is a PI-algebra; moreover, we give an explicit polynomial identity for $A$. To obtain these results, we associate to each algebra a numerical sequence, denoted $\pi_n(A)$, which shares some important properties of the codimension sequence of $A$. In particular, we prove that $A$ is a PI-algebra if and only if $\pi_n(A) < n!$, for some positive integer $n$, thereby providing a new combinatorial characterization of PI-algebras. Lastly, we prove that analogous results hold when $A$ is a Lie algebra.



**Keywords:** Polynomial identities, PI-algebras, identical relations with actions, group-graded algebras, Hopf algebra actions, automorphisms, anti-automorphisms, involutions, derivations

# Summary for Lay Audience

An algebra is a set of objects together with operations of multiplication, addition, and scalar multiplication by elements of a field (such as the real numbers) such that these operations behave nicely with one another. Suppose that we can find a non-zero polynomial $f(x_1, \ldots, x_n)$ in non-commuting indeterminates $x_1, \ldots, x_n$ that vanishes when evaluated at arbitrary elements of a given algebra $A$ over a field; in this case, we say that $A$ is a PI-algebra satisfying the polynomial identity $f \equiv 0$. Satisfying a polynomial identity has far reaching consequences on the structure of the algebra in question. Thus, it is interesting to provide criteria for when a given algebra is a PI-algebra. This is the general goal of this thesis.

# Co-Authorship Statement

Chapters 4, 5, and 6 of this thesis incorporate material which results from joint research with Professor David M. Riley and is based on the papers:

[CMR1] M. Cárdenas Montoya and D.M. Riley. On the identical relations of associative and Lie algebras equipped with an action, submitted (2021).

[CMR2] M. Cárdenas Montoya and D.M. Riley. Combinatorial criteria for an algebra endowed with an action to satisfy a polynomial identity, in preparation.

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# Contents

## List of Abbreviations, Symbols, and Nomenclature

| | |
|---|---|
| $k$ | field |
| $A^{(-)}$ | Lie algebra of an associative algebra $A$ |
| $X$ | countably infinite set of non-commuting indeterminates |
| $k\langle X\rangle$ | free associative algebra on $X$ over $k$ |
| $[x, y]$ | Lie commutator of $x$ and $y$ |
| $UT_n(k)$ | algebra of $n \times n$ upper triangular matrices with entries in $k$ |
| $S_n$ | symmetric group on $\{1, \ldots, n\}$ |
| $M_n(k)$ | algebra of $n \times n$ matrices with entries in $k$ |
| $E$ | Grassmann (or exterior) algebra of a countably infinite dimensional vector space |
| $s_n(x_1, \ldots, x_n)$ | standard polynomial of degree n |
| $\mathrm{Id}(A)$ | set of polynomial identities of the algebra $A$ |
| $\langle S\rangle_T$ | T-ideal generated by $S$ |
| $k\langle X\rangle^{\mathrm{gr}}$ | $G$-graded algebra of countable rank over $k$ |
| $\mathrm{Aut}(A)^*$ | group of automorphisms and anti-automorphisms of $A$ |
| $k\langle X \mid G\rangle$ | free algebra on $X$ with $G$-action |
| $a_d(n)$ | number of $d$-indecomposable permutations in $S_n$ |
| $\kappa(d, m)$ | see Lemma 2.25 |
| $k[G]$ | group algebra of a group $G$ |
| $k[G]^*$ | dual of the group algebra |
| $U(\mathfrak{g})$ | universal enveloping algebra of a Lie algebra $\mathfrak{g}$ |
| $k\langle X \mid H\rangle$ | free $H$-algebra |
| $\mathrm{End}_k(A)$ | algebra of $k$-linear maps on $A$ |
| $R$ | associative algebra with 1 |
| $\mathcal{B}$ | basis for $R$ containing $1_R$ |

| | |
|---|---|
| $k\langle X \mid R \rangle$ | free associative algebra generated by $x^b$, with $x \in X$, $b \in \mathcal{B}$ |
| $P_n$ | space of multilinear polynomials in $x_1, \dots, x_n$ |
| $c_n(A)$ | $n$-th codimension of $A$ |
| $\pi_d(A)$ | see Definition 4.12 |
| $P_n^R$ | space of multilinear polynomials in $x_1^{r_1}, \dots, x_n^{r_n}$, with $r_i \in R$ |
| $c_n^R(A)$ | $n$-th $R$-codimension of $A$ |
| $\mathrm{Id}(A \mid R)$ | set of $R$-identities of $A$ |
| $\lambda_A$ | see Definition 4.5 |
| $L$ | Lie algebra |
| $\mathrm{Der}_k(A)$ | Lie algebra of derivations of $A$ |
| $F\langle X \rangle$ | free non-associative algebra on $X$ over $k$ |
| $\mathscr{L}\langle X \rangle$ | free Lie algebra on $X$ over $k$ |
| $[x_{i_1}, \dots, x_{i_n}]$ | left-normed commutator |
| $\mathrm{Id}(L)$ | set of Lie polynomial identities of the algebra $L$ |
| $Q_n$ | space of multilinear Lie polynomials in $x_0, \dots, x_n$ |
| $c_n(L)$ | $n$-th codimension of $L$ |
| $\pi_n(L)$ | see Definition 5.21 |
| $\mathscr{L}\langle X \mid R \rangle$ | free Lie algebra generated by $x^b$, with $x \in X$, $b \in \mathcal{B}$ |
| $\mathrm{Id}(L \mid R)$ | set of $R$-identities of $L$ |
| $c_n^R(L)$ | $n$-th $R$-codimension of $L$ |
| $\lambda_L$ | see Definition 5.16 |

# Chapter 1

# Introduction

Associative and Lie PI-algebras are the central characters of this story. To introduce them properly, we first require some notation. Denote by $k\langle X\rangle$ the free associative algebra on a countably infinite set $X$ of non-commuting indeterminates over a field $k$, and fix $f(x_1, \ldots, x_n) \in k\langle X\rangle$. We shall say that an associative algebra $A$ satisfies the polynomial identity $f(x_1, \ldots, x_n) \equiv 0$ if, for all $a_1, \ldots, a_n \in A$, $f(a_1, \ldots, a_n) = 0$. Similarly, let $\mathscr{L}\langle X\rangle$ denote the free Lie algebra on $X$ over the field $k$, and fix $g(x_1, \ldots, x_n) \in \mathscr{L}\langle X\rangle$. We shall say that a Lie algebra $L$ satisfies the polynomial identity $g(x_1, \ldots, x_n) \equiv 0$ if, for all $a_1, \ldots, a_n \in L$, $g(a_1, \ldots, a_n) = 0$. An algebra satisfying a (non-trivial) polynomial identity is called a *PI-algebra*.

Satisfying a (non-trivial) polynomial identity has significant impact on the structure of an associative algebra over a field. For example, the famous Kurosh Problem:

*Is every finitely generated nil algebra nilpotent?*

has a positive solution in the class of all PI-algebras, as first proved by Kaplansky in [14]. It follows from Kaplansky's theorem that Köthe's Conjecture:

*The sum of two left-sided nil ideals in a ring is nil.*

also has a positive solution in the class of PI-algebras. Kaplansky's result is regarded by many as one of the deepest results in ring theory. More recently, Zelmanov proved the remarkable fact that the Lie-theoretic analogue of the Kurosh Problem:

*If every element in a finitely generated Lie algebra is ad-nilpotent, does it follow that the Lie algebra is nilpotent?*

also has a positive solution in the class of all Lie PI-algebras (see [23], for example). Counterexamples constructed by Golod and Shafarevich in [13] show that Kurosh's problems for associative algebras and Lie algebras each have negative solutions, in general.

In light of the above, it is interesting to provide criteria for an associative or Lie algebra to satisfy a polynomial identity. This is the general goal of this thesis.

Given an algebra endowed with additional structure (for example, a group action, or a group grading) it is often convenient to extend our classical notion of polynomial identity by taking into account the added structure. One then examines the identities that arise in this form and their connection to the classical polynomial identities. For instance, suppose that an associative algebra $A$ admits a $k$-linear involution (that is, a $k$-linear anti-automorphism of $A$ of order 2). Consider the free associative algebra $k\langle X^*\rangle$ on the set $X^* = \{x, x^* : x \in X\}$. An element $f \in k\langle X^*\rangle$ is known as a $*$-*polynomial*; moreover $f \equiv 0$ is called a $*$-*identity* of $A$ if $f$ vanishes under all evaluations in $A$, where the involution is used in the evaluation. A celebrated theorem proved by Amitsur in [2] asserts that, whenever $A$ satisfies a (non-trivial) $*$-identity, $A$ satisfies a classical polynomial identity. Another interesting example along the same lines is due to Bergen and Cohen. In [10], they proved that if the identity component of a $G$-graded algebra $A$ is a PI-algebra, then $A$ itself must be a PI-algebra; here, $G$ denotes a finite group.

An associative algebra $A$ admitting a $k$-linear involution $*$ defines a $k[G]$-module action on $A$, where $G = \{1, *\}$, while an associative algebra graded by a finite group $G$ can be regarded as a $k[G]^*$-module algebra (see Definition 2.37 and Proposition 2.42). Thus, motivated by the results of Amitsur, Bergen, and Cohen (and similar results in the literature) we centre our research around the study of algebras $A$ equipped with a (left unitary $k$-linear) $R$-module action,

$R \to \mathrm{End}_k(A)$, where $R$ denotes a given unitary associative algebra, and $\mathrm{End}_k(A)$ denotes the algebra of $k$-linear maps on $A$ with the usual operations. We seek sufficient conditions for $A$ to be a PI-algebra. Chapter 2 contains the necessary background material; additionally, some of the results that motivate our work are discussed in more detail. In Chapter 3, we introduce our general framework, which encompasses each of the aforementioned results and more. The main work is carried out in Chapters 4 and 5, and applications of our main results are discussed in Chapter 6. Unlike the proofs of many classical results in the literature, including Amitsur's, Bergen's, and Cohen's work mentioned above, the proofs of our main results are constructive in the sense that they provide a concrete bound on the minimal degree of a polynomial identity for $A$; in fact, some of our main results include explicit polynomial identities for $A$.

Because our algebras are equipped with the additional structure of an $R$-action, the notion of an $R$-identity arises naturally, and with it the question:

*Is the presence of a non-trivial R-identity a sufficient condition for an algebra to be a PI-algebra?*

By imposing suitable conditions on the $R$-identities, we obtain a positive result; in this regard, Theorem 4.8 is the main result in the associative case, while Theorem 5.20 is the main result in the Lie case; the proofs of these theorems are carried out in Sections 4.1 and 5.2 respectively. Next, we study general conditions on the $R$-action which ensure that the algebra in question is a PI-algebra. The main result is Theorem 4.24 in the associative case (see Section 4.4), and Theorem 5.26 in the Lie case (see Section 5.4). To prove these results, we associate to each algebra a numerical sequence which turns out to be quite interesting on its own; indeed, this sequence provides a delightful combinatorial characterization for PI-algebras; it is introduced in Section 4.3 for associative algebras, and Section 5.3 for Lie algebras. Lastly, we discuss some applications of our main results in Chapter 6. We focus primarily on applications pertaining to algebras equipped with Hopf actions. Additionally, we recover several previously known results. In this sense, this thesis forms an umbrella over a collection of results from the

PI-theory literature.

# Chapter 2

# Background and motivation

## 2.1 Basic definitions and examples

Throughout this work, $k$ will denote the common base field of our algebras, vector spaces, and tensor products. Recall that a *$k$-algebra* is a vector space $A$ equipped with a binary operation $\cdot : A \times A \to A$, called multiplication, such that, for all $a_1, a_2, a_3 \in A$ and $\alpha \in k$, each of the following conditions hold:

1. $(a_1 + a_2) \cdot a_3 = a_1 \cdot a_3 + a_2 \cdot a_3$

2. $a_1 \cdot (a_2 + a_3) = a_1 \cdot a_2 + a_1 \cdot a_3$

3. $\alpha(a_1 \cdot a_2) = (\alpha a_1) \cdot a_2 = a_1 \cdot (\alpha a_2)$

As is usual, we will write $a_1 a_2$ in place of $a_1 \cdot a_2$. We will say that $A$ is *associative* if, for all $a_1, a_2, a_3 \in A$, $(a_1 a_2) a_3 = a_1(a_2 a_3)$, and we will say it is *unitary* if there exists an element $1_A \in A$ (called the identity) such that, for all $a \in A$, $1_A a = a 1_A = a$. An algebra that is not necessarily associative will be called a *non-associative* algebra.

A (non-associative) algebra $A$ will be called a *Lie algebra* if it satifies both the *anti-commutative law* and the *Jacobi identity*; that is, for all $a_1, a_2, a_3 \in A$,

1. $a_1^2 = 0$ (anti-commutative law)

2. $(a_1a_2)a_3 + (a_2a_3)a_1 + (a_3a_1)a_2 = 0$ (Jacobi identity)

Note that every associative algebra $A$ is a Lie algebra with respect to the new multiplication

$$[a_1, a_2] = a_1a_2 - a_2a_1, \text{ for all } a_1, a_2 \in A.$$

The resulting Lie algebra will be denoted $A^{(-)}$.

The *free associative algebra* of a countably infinite set $X = \{x_1, x_2, \dots\}$ over $k$, denoted $k\langle X\rangle$, is the algebra of polynomials in the non-commuting indeterminates $x_i \in X$; on occasion, we will use $y, z, y_i, z_i$, etc., to denote elements of $X$. A basis for $k\langle X\rangle$ is given by the set of all words in the alphabet $X$; the empty word is denoted by 1, and the product of two words is defined by concatenation.

**Proposition 2.1.** *Given an associative algebra $A$, any set theoretic map $\varphi\colon X \to A$ can be uniquely extended to a homomorphism of algebras $\bar{\varphi}\colon k\langle X\rangle \to A$.*

If $f \in k\langle X\rangle$, we write $f(x_1, \dots, x_n)$ to indicate that the only indeterminates occurring in $f$ are precisely $x_1, \dots, x_n$. Given $a_1, \dots, a_n \in A$, the *evaluation* $f(a_1, \dots, a_n)$ corresponds to the image of $f$ under $\bar{\varphi}$ (see Proposition 2.1), where $\varphi\colon X \to A$ is any map satisfying $\varphi(x_i) = a_i$, for all $1 \le i \le n$.

A non-zero scalar multiple of a word is called a *monomial*. The *degree of $x_i$* in a monomial $m \in k\langle X\rangle$, denoted $\deg_i(m)$, corresponds to the number of occurrences of $x_i$ in $m$. The *degree of $m$*, denoted $\deg(m)$, is defined as the sum of all $\deg_i(m)$. The *degree of a polynomial $f \in k\langle X\rangle$*, denoted $\deg(f)$, is the maximum amongst all values of $\deg(m)$ as $m$ varies amongst all monomials of $f$.

### 2.1.1 PI-algebras

**Definition 2.2.** Fix an associative algebra $A$ and a polynomial $f \in k\langle X\rangle$. We shall say that $f \equiv 0$ is a *polynomial identity* for $A$ if $\bar{\varphi}(f) = 0$, for every map $\varphi\colon X \to A$ (see Proposition 2.1). Equivalently, $f(x_1, \ldots, x_n) \equiv 0$ is a polynomial identity for $A$ if, for all $a_1, \ldots, a_n \in A$, $f(a_1, \ldots, a_n) = 0$.

**Example 2.3.** Important algebraic properties can be expressed in the language of polynomial identities. Fix an associative algebra $A$.

1. $A$ is *commutative* if and only if it satisfies the polynomial identity $[x_1, x_2] \equiv 0$; here, $[x_1, x_2] = x_1x_2 - x_2x_1$ denotes the Lie commutator of $x_1$ and $x_2$.
2. $A$ is *nilpotent* of index $n \geq 1$ if and only if $x_1 \cdots x_n \equiv 0$ is a polynomial identity for $A$.
3. $A$ is *nil of bounded index* if and only if there exists an integer $n \geq 1$ such that $x_1^n \equiv 0$ is a polynomial identity for $A$.

**Definition 2.4.** An associative algebra $A$ is called a *PI-algebra* if it satisfies a non-trivial (i.e., non-zero) polynomial identity $f \equiv 0$.

**Example 2.5.** Let $UT_n(k)$ denote the *algebra of $n\times n$ upper triangular matrices* with entries in $k$. Observe that, for all $A_1, A_2 \in UT_n(k)$, $[A_1, A_2]$ is a strictly upper triangular matrix. It follows easily that $UT_n(k)$ is a PI-algebra satisfying the polynomial identity

$$[x_1, x_2] \cdots [x_{2n-1}, x_{2n}] \equiv 0.$$

**Example 2.6.** Let $M_n(k)$ denote the *algebra of $n \times n$ matrices* with entries in $k$. By a theorem of Amitsur and Levitzki (see [3]), $M_n(k)$ satisfies the *standard polynomial identity* of degree $2n$:

$$s_{2n}(x_1, \ldots, x_{2n}) = \sum_{\sigma\in S_{2n}} \operatorname{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(2n)} \equiv 0,$$

where $S_{2n}$ denotes the symmetric group on $\{1, \ldots, 2n\}$. Hence, every matrix algebra is a PI-algebra.

**Example 2.7.** To conclude this round of examples, consider the *Grassmann algebra* over a field $k$ of characteristic different from 2. Suppose that $V$ is a vector space with basis $\{e_n : n \in \mathbb{Z}^+\}$. The Grassmann (or exterior) algebra of $V$, denoted by $E$, is the associative algebra generated by $\{e_n : n \in \mathbb{Z}^+\}$ with defining relations

$$e_ie_j + e_je_i = 0, \text{ for all } i, j \in \mathbb{Z}^+.$$

A basis for $E$ is given by $\mathcal{B} = \{1, e_{i_1} \cdots e_{i_k} : i_1 < \cdots < i_k,\ k \geq 1\}$. Note that monomials in the $e_i$'s of even length lie in the centre of $E$. From this simple observation, it is easy to see that

$$[[x_1, x_2], x_3] \equiv 0$$

is a polynomial identity for $E$. Indeed, any commutator of two elements of $E$ is a linear combination of monomials in the $e_i$'s of even length. Hence, $E$ is a PI-algebra.

### 2.1.2 T-ideals and multilinear identities

A polynomial $f \in k\langle X\rangle$ is *linear in* $x_i$ if $\deg_i(m) = 1$, for every monomial $m$ of $f$. The polynomial $f(x_1, \ldots, x_n)$ is called *multilinear* if it is linear in $x_i$, for all $1 \leq i \leq n$. For instance, the (multilinear) polynomial

$$s_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)}$$

is known as the *standard polynomial of degree n*.

Multilinear polynomials play an important role in PI-theory. First, observe that one can easily determine whether a multilinear polynomial $f(x_1, \ldots, x_n) \in k\langle X\rangle$ is an identity for a given algebra $A$. Indeed, it is enough to check whether $f(x_1, \ldots, x_n)$ vanishes on basis elements.

Moreover, every PI-algebra satisfies some multilinear polynomial identity.

**Theorem 2.8.** *If an associative algebra A satisfies a polynomial identity of degree n, then it satisfies a multilinear identity of degree n.*

*Proof.* This result follows from the well-known multilinearization process. See Theorem 1.3.7 in [12] for details. □

**Definition 2.9.** Given an algebra $A$, we will write $\mathrm{Id}(A)$ to denote the set of polynomial identities of $A$:

$$\mathrm{Id}(A) = \{f \in k\langle X\rangle \colon f \equiv 0 \text{ on } A\}.$$

Clearly, $\mathrm{Id}(A)$ is a two-sided ideal of $k\langle X\rangle$. Moreover, if $f(x_1, \ldots, x_n) \in \mathrm{Id}(A)$, then given any homomorphism of algebras $\psi\colon k\langle X\rangle \to k\langle X\rangle$, $\psi(f) \in \mathrm{Id}(A)$. Indeed, if $\psi(x_i) = g_i$, for each $1 \leq i \leq n$, then $\psi(f) = f(g_1, \ldots, g_n) \in \mathrm{Id}(A)$. Two-sided ideals with this property are called *T-ideals*.

**Definition 2.10.** We will say that a two-sided ideal $I$ of $k\langle X\rangle$ is a *T-ideal* if $\psi(I) \subseteq I$, for all endomorphisms $\psi\colon k\langle X\rangle \to k\langle X\rangle$. Fix a set of polynomials $S = \left\{f_j(x_1, \ldots, x_{n_j}) \in k\langle X\rangle \colon j \in J\right\}$.

1. The *T-ideal generated by S* is the smallest T-ideal of $k\langle X\rangle$ containing $S$, it will be denoted by $\langle S\rangle_T$; in other words,

$$\langle S\rangle_T = \mathrm{span}_k \left\{u_j f_j(g_1, \ldots, g_{n_j})v_j \in k\langle X\rangle \colon u_j, g_1, \ldots, g_{n_j}, v_j \in k\langle X\rangle, j \in J\right\}.$$

2. We will refer to the elements of $\langle S\rangle_T$ as *consequences* of the polynomials $f_j \in S$.

**Example 2.11.** Consider the polynomial $f(x) = x^2$. Then,

$$f(x_1 + x_2) - f(x_1) - f(x_2) = x_1x_2 + x_2x_1$$

is a (multilinear) consequence of $f$.

**Remark 2.12.** The Specht problem asks whether every set of polynomial identities of an algebra is a consequence of a finite number of identities; Kemer's positive solution to the Specht problem for associative algebras over a field of characteristic 0 is one of the greatest achievements of PI-theory (see [15]). However, given an algebra $A$, it is very difficult to find an explicit set $S \subseteq k\langle X\rangle$ such that $\langle S\rangle_T = \mathrm{Id}(A)$. In fact, to this day, finding such a set remains an open problem even for the matrix algebra $M_3(\mathbb{Q})$!

## 2.2 Background results

Next, we present a selection of results in PI-theory which serve as motivation for our work. Our intention is twofold. First, to introduce the reader to more "general" notions of polynomial identities which arise naturally when the algebra in question is equipped with some additional structure. Second, to illustrate that the existence of such "general" identities can be used as criteria to determine whether a given algebra is a PI-algebra. Throughout this section, $A$ will denote an associative algebra.

### 2.2.1 Graded identities

When the associative algebra $A$ is equipped with a group grading, one may speak of more "general" polynomial identities of $A$. In this section, we introduce *graded identities* and present a well-known result due to Bergen and Cohen which relates graded identities and ordinary polynomial identities.

Fix a finite multiplicative group $G$. Recall that a vector space decomposition

$$A = \bigoplus_{g\in G} A^{(g)}$$

is a $G$-grading of $A$ provided $A^{(g_1)}A^{(g_2)} \subseteq A^{(g_1g_2)}$, for all $g_1, g_2 \in G$. The subspaces $A^{(g)}$ are called the *homogeneous components* of $A$. An element $a \in A$ is called *homogeneous of degree $g$* if $a \in A^{(g)}$.

Throughout this subsection, let $X$ denote the disjoint union of the sets $X^{(g)} = \{x_1^{(g)}, x_2^{(g)}, \dots\}$, with $g \in G$. We shall say that a monomial $x_{i_1}^{(g_1)} \cdots x_{i_n}^{(g_n)} \in k\langle X \rangle$ has *homogeneous degree $g$* if $g_1 \cdots g_n = g$. We will write $k\langle X \rangle^{(g)}$ to denote the vector space spanned by all monomials having homogeneous degree $g$. We may now equip the free associative algebra on $X$ with a $G$-grading as follows:

$$k\langle X \rangle = \bigoplus_{g \in G} k\langle X \rangle^{(g)}.$$

We will write $k\langle X \rangle^{\mathrm{gr}}$ to emphasize that we are referring to the free associative algebra on $X$ with this particular grading. The algebra $k\langle X \rangle^{\mathrm{gr}}$ is known as the *free $G$-graded algebra of countable rank over $k$*. Elements of $k\langle X \rangle^{\mathrm{gr}}$ are called *graded polynomials*.

**Proposition 2.13.** *Given any $G$-graded algebra $A$, any set theoretic map $\varphi\colon X \to A$ satisfying $\varphi\left(X^{(g)}\right) \subseteq A^{(g)}$, for all $g \in G$, can be uniquely extended to a homomorphism $\bar{\varphi}\colon k\langle X \rangle^{gr} \to A$ of $G$-graded algebras.*

**Definition 2.14.** Fix a $G$-graded algebra $A$. A graded polynomial $f \in k\langle X \rangle^{\mathrm{gr}}$ is a *graded identity* of $A$ if

$$f \in \bigcap \ker \bar{\varphi},$$

where the intersection runs over all maps $\varphi\colon X \to A$ satisfying $\varphi\left(X^{(g)}\right) \subseteq A^{(g)}$, for all $g \in G$, and $\bar{\varphi}$ is defined as in Proposition 2.13.

Notice that satisfying a graded identity is a much weaker condition than satisfying an ordinary polynomial identity. For example, we may equip any algebra $A$ with a trivial $G$-grading, where

$$A^{(g)} = \begin{cases} A \text{ if } g = 1_G \\ 0 \text{ otherwise} \end{cases}$$

In this case, any graded polynomial $f\left(x_{i_1}^{(g_1)}, \dots, x_{i_n}^{(g_n)}\right)$ is a graded identity for $A$ provided $g_i \neq 1_G$, for all $1 \leq i \leq n$. Thus, even when $A$ satisfies a non-trivial graded identity, it may not be a PI-algebra. Nonetheless, by imposing suitable conditions on the graded identity, we can obtain a positive result. Indeed, as a consequence of Theorem 2.15, any $G$-graded algebra $A$ satisfying a graded identity of the form

$$\sum_{\sigma \in S_d} \alpha_\sigma x_{\sigma(1)}^{(1)} \cdots x_{\sigma(d)}^{(1)} \equiv 0,$$

is a PI-algebra.

**Theorem 2.15** (Bergen, Cohen)**.** *Suppose an algebra $A$ is graded by a group $G$ such that $|G| = n$. If $A^{(1)}$ satisfies a polynomial identity of degree $d$, then $A$ satisfies an ordinary polynomial identity of the form $s_{nd}^m(x_1, \dots, x_{nd}) \equiv 0$, for some positive integer $m$.*

*Proof.* See [10]. □

In [4], Bahturin, Giambruno, and Riley recovered an explicit bound for the degree of the polynomial identity in Theorem 2.15 in terms of $d$ and $|G|$. Let $e = 2.71\dots$ denote the base of the natural logarithm.

**Theorem 2.16** (Bahturin, Giambruno, Riley)**.** *Suppose $A = \bigoplus_{g \in G} A^{(g)}$ is a $G$-graded algebra, with $G$ a finite group. If $A^{(1)}$ satisfies an identity of degree $d$, then $A$ satisfies a polynomial identity of degree $n$, for all*

$$n > e|G|\left(d|G| - 1\right)^2.$$

### 2.2.2 $G$-identities

When a finite group $G$ acts on the associative algebra $A$ by automorphisms and anti-automorphisms, it is once again possible to speak of more "general" polynomial identities of $A$. In this section, we introduce $G$-identities and $*$-identities. Throughout, $\mathrm{Aut}^*(A)$ will denote the group of automorphisms and anti-automorphisms of $A$ (under composition of functions) and $G$ will denote a finite subgroup of $\mathrm{Aut}^*(A)$. Additionally, we will write $H$ to denote the subgroup $G \cap \mathrm{Aut}(A)$.

Reverting back to our standard notation, $X$ will denote a countably infinite set of indeterminates $x_1, x_2, \ldots$. We will write $k\langle X \mid G\rangle$ to denote the free associative algebra on the set $\{x^g \colon x \in X, g \in G\}$. We can define a group action on $k\langle X \mid G\rangle$ as follows. First, let $(x^{g_1})^{g_2} = x^{g_2 g_1}$, for all $x \in X$ and $g_1, g_2 \in G$. Now, if $v, w$ are monomials and $g \in G$, set

$$(vw)^g = \begin{cases} v^g w^g \text{ if } g \in H \\ w^g v^g \text{ if } g \in G\backslash H \end{cases}.$$

Finally, extend this action to all of $k\langle X \mid G\rangle$ by linearity. The algebra $k\langle X \mid G\rangle$ is known as the *free algebra on X with G-action*. Elements of $k\langle X \mid G\rangle$ are called *G-polynomials*.

**Proposition 2.17.** *Any set theoretic map $\varphi \colon X \to A$ extends uniquely to a homomorphism $\bar{\varphi} \colon k\langle X \mid G\rangle \to A$ such that, for all $x \in X$ and $g \in G$, $\bar{\varphi}(x^g) = \bar{\varphi}(x)^g$.*

**Definition 2.18.** We say that a $G$-polynomial $f\left(x_{i_1}^{g_1}, \ldots, x_{i_n}^{g_n}\right) \in k\langle X \mid G\rangle$ is a *$G$-identity* for $A$ if

$$f \in \bigcap \ker \bar{\varphi},$$

where the intersection runs over all maps $\varphi \colon X \to A$, and $\bar{\varphi}$ is defined as in Proposition 2.17.

Observe that if $A$ is a PI-algebra satisfying a non-trivial polynomial identity $f(x_1, \ldots, x_n) \equiv 0$, then it satisfies a non-trivial $G$-identity $f(x_1^1, \ldots, x_n^1) \equiv 0$. As the next example illustrates, the converse is not true.

**Example 2.19.** Suppose $k$ is a field of characteristic different from 2. Let $B = \mathrm{span}_k\{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\}$, and consider the algebra $A = k\langle X\rangle \oplus B$ with component-wise multiplication:

$$(a_1 + b_1)(a_2 + b_2) = a_1 a_2 + b_1 b_2, \text{ for all } a_1, a_2 \in k\langle X\rangle \text{ and } b_1, b_2 \in B.$$

Now consider the automorphism $\varphi \in \mathrm{Aut}^*(A)$ given by $\varphi(a + b) = a - b$, for all $a \in k\langle X\rangle$ and

$b \in B$. Additionally, let $G = \{1, \varphi\}$. The algebra $A$ satisfies the $G$-identity

$$\left(x_1 - x_1^{\varphi}\right)\left(x_2 - x_2^{\varphi}\right) \equiv 0.$$

However, A is not a PI-algebra.

Nonetheless, the converse is true if we impose suitable conditions on the $G$-identity. Indeed, in [5], Bahturin, Giambruno, and Zaicev established the following connection between the existence of a special type of $G$-identity, which they called *essential*, and the existence of an ordinary polynomial identity of $A$.

**Theorem 2.20** (Bahturin, Giambruno, Zaicev)**.** *Let $A$ be an associative algebra and $G$ a finite subgroup of $Aut^*(A)$. Suppose $A$ satisfies an essential $G$-identity of degree $d$:*

$$x_1^1 \cdots x_d^1 + \sum_{\sigma \in S_d, \sigma \neq 1} \sum_{g=(g_1,\dots,g_d) \in G^d} \alpha_{\sigma,g} x_{\sigma(1)}^{g_1} \cdots x_{\sigma(d)}^{g_d} \equiv 0.$$

*Then $A$ satisfies a non-zero polynomial identity, whose degree is bounded by $\kappa(d, |G|)$.*

We leave the intricate definition of the function $\kappa(d, |G|)$ for Subsection 2.2.3.

Now, suppose that $A$ admits an involution $*$ (that is, a $k$-linear anti-automorphism of $A$ of order 2) and consider the group $G = \{1, *\}$. In this case, $G$-polynomials and $G$-identities are called $*$-polynomials and $*$-identities, respectively. In [2], Amitsur proved that if $A$ satisfies a non-trivial $*$-identity, then it satisfies an ordinary polynomial identity (no additional conditions on the $*$-identity required!). The following quantitative version of Amitsur's result follows from Theorem 2.20.

**Theorem 2.21.** *Let $A$ be an algebra with involution $*$ satisfying a non-trivial $*$-identity of degree $d$. Then, $A$ satisfies a non-trivial polynomial identity whose degree is bounded by the function $f(2d, 2)$.*

*Proof.* See Theorem 10.3.3 in [12]. ◻

### 2.2.3 Combinatorics of words

The combinatorial properties of the permutation group $S_n$ are among the main tools used to prove Theorems 2.16 and 2.20. In this subsection, we introduce these properties; they will be used extensively in order to prove the main results in this thesis. The reader can safely skip this subsection and refer back as the need arises.

In the sequel, permutations in $S_n$ are ordered lexicographically; that is, given two permutations $\alpha, \beta \in S_n$, we will write $\alpha < \beta$ if, for some $k \geq 0$,

$$\alpha(1) = \beta(1), \ldots, \alpha(k) = \beta(k) \text{ and } \alpha(k+1) < \beta(k+1).$$

**Definition 2.22.** Let $d \leq n$ be positive integers. A permutation $\sigma \in S_n$ is called *d-bad* if there exists a sequence of integers $1 \leq h_1 < \cdots < h_d \leq n$ such that $\sigma(h_1) > \cdots > \sigma(h_d)$; otherwise, $\sigma$ is called *d-good*.

For example, consider the following permutation in $S_5$:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix}.$$

This permutation is 3-good but 2-bad; for instance, $2 < 4$, yet $\sigma(2) > \sigma(4)$.

By extension, a monomial of the form $x_{\sigma(1)} \cdots x_{\sigma(n)} \in k\langle X \rangle$ will be called $d$-good (respectively, $d$-bad) if the corresponding permutation $\sigma \in S_n$ is $d$-good (respectively, $d$-bad). In the next chapters, we will need an upper bound for the number of $d$-good words in $k\langle X \rangle$.

**Lemma 2.23.** *For all positive integers $d \leq n$, the number of $d$-good permutations in $S_n$ does not exceed $\frac{(d-1)^{2n}}{(d-1)!}$.*

*Proof.* See Theorem 1.8 in [20]. □

Next, we introduce the notion of *d-indecomposable* permutations; this is a generalization of the notion of $d$-good permutations.

**Definition 2.24.** Let $d \leq n$ be positive integers. A permutation $\sigma \in S_n$ is called *d-decomposable* if there exists a sequence of integers

$$1 \leq h_1 \leq t_1 < h_2 \leq t_2 < \cdots < h_d \leq t_d \leq n, \tag{2.1}$$

such that the following conditions are satisfied:

1. $\sigma(h_1) > \sigma(h_2) > \cdots > \sigma(h_d)$;
2. $t_i = h_{i+1} - 1$, for each $1 \leq i \leq d-1$; and,
3. $\sigma(h_i) > \sigma(k)$, for all $h_i < k \leq t_i$ and $1 \leq i \leq d$.

If no such $d$-decomposition exists, $\sigma$ is called *d-indecomposable*. We shall write $a_d(n)$ to denote the number of $d$-indecomposable permutations in $S_n$.

Consider the following permutation in $S_6$:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 4 & 1 & 2 \end{pmatrix}.$$

This permutation is 3-decomposable; for instance, we can take $h_1 = 2, t_1 = 3, h_2 = 4, t_2 = 5$, and $h_3 = t_3 = 6$.

A monomial of the form $x_{\sigma(1)} \cdots x_{\sigma(n)} \in k\langle X \rangle$ will be called $d$-indecomposable (respectively, $d$-decomposable) if the corresponding permutation $\sigma \in S_n$ is $d$-indecomposable (respectively, $d$-decomposable). Notice that a $d$-decomposable word $w = x_{\sigma(1)} \cdots x_{\sigma(n)}$ can be written in the form $w = w_0 w_1 \cdots w_d w_{d+1}$, where, for each $1 \leq i \leq d$, $w_i = x_{h_i} \cdots x_{t_i}$ is a non-empty monomial. Moreover, given any non-trivial permutation $\tau \in S_d$, $w_0 w_{\tau(1)} \cdots w_{\tau(d)} w_{d+1} < w_0 w_1 \cdots w_d w_{d+1}$.

We will also require an upper bound for the number of $d$-indecomposable permutations in $S_n$.

In [5], Bahturin, Giambruno and Zaicev gave the following estimate for $a_d(n)$.

**Lemma 2.25.** *Fix positive integers $d \leq n$ and $m$. Set $k = d + \lfloor log_2 m \rfloor$, $N = 2^{k2^{k+1}}$, $p_2 = 2^{k2^k}$, and define $p_j$, for each $j > 2$, to be the integer for which*

$$\underbrace{log_N \ldots log_N}_{j-2} p_j = p_2.$$

*Set $\kappa(d,m) = log_2 p_d$. If $n \geq \kappa(d,m)$, then $a_d(n) < \left(\frac{1}{m}\right)^n n!$.*

**Remark 2.26.** Henceforth, we reserve the notation $\kappa(d,m)$ for the map defined in Lemma 2.25.

The following elementary computation will simplify some of our proofs later on.

**Lemma 2.27.** *Fix positive integers $d \geq 2$ and $m$. If $t = \kappa(d, m+1)$, then $m^{t+1} a_d(t) < t!$.*

*Proof.* Note that, for all $d \geq 2$, we have

$$t = \kappa(d, m+1) \geq \kappa(2, m+1) = \log_2 p_2 = k2^k = (2 + \lfloor \log_2 (m+1) \rfloor)\, 2^{\left(2+\lfloor \log_2 (m+1) \rfloor\right)} > m \log_2 m.$$

Then,

$$t > m \log_2 m \geq \frac{\log_2 m}{\log_2 \left(\frac{m+1}{m}\right)}; \tag{2.2}$$

indeed, for all positive integers $m$:

$$\log_2 \left(\frac{m+1}{m}\right)^m > 1.$$

It follows from Equation (2.2) that

$$\left(\frac{m}{m+1}\right)^t \leq \frac{1}{m},$$

and thus, by Lemma 2.25,

$$m^{t+1} a_d(t) < m^{t+1} \left(\frac{1}{m+1}\right)^t t! \leq t!,$$

as required. □

## 2.3 Hopf algebras and their actions

In this section, we reformulate some of the previous results using the language of Hopf actions. We restrict ourselves to introducing only the most basic notions about Hopf algebras which will be needed in the sequel. Most importantly, we introduce three classical examples: the group algebra $k[G]$, the linear dual of the group algebra $k[G]^*$, and the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. Additionally, we introduce a useful convention known as Sweedler's notation, which will be used throughout this work. For a complete account on Hopf algebras and their actions, we refer the reader to [18] or [22].

### 2.3.1 Algebras, coalgebras, and bialgebras

An associative algebra with 1 can be regarded as a triple $(A, m, u)$ consisting of a vector space $A$, a linear map $m\colon A \otimes A \to A$ called *multiplication*, and a linear map $u\colon k \to A$ called the *unit map*, such that the following diagrams commute:

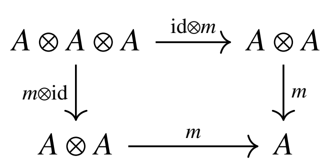



Figure 2.1: Associative property

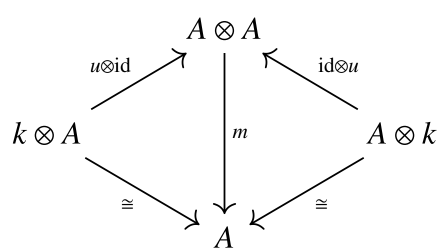



Figure 2.2: Unitary property

The advantage of this approach is that it leads naturally to the notion of a *coalgebra* by "dualizing" or "turning all arrows around".

**Definition 2.28.** A *coalgebra* $C$ over $k$ is a triple $(C, \Delta, u)$ consisting of a vector space $C$, a linear map $\Delta\colon C \to C \otimes C$ called *comultiplication*, and a linear map $\epsilon\colon C \to k$ called the *counit* *map*, such that the following diagrams commute:

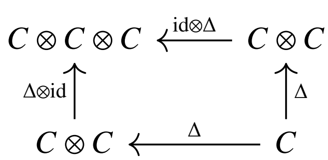



Figure 2.3: Coassociative property

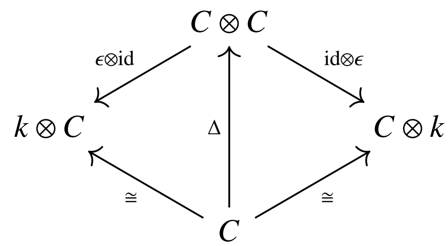



Figure 2.4: Counitary property

**Remark 2.29** (Sweedler's notation)**.** Given a coalgebra $(C, \Delta, u)$ and an element $c \in C$, we can write $\Delta(c)$ as a sum of pure tensors:

$$\Delta(c) = \sum_j c_{1j} \otimes c_{2j}.$$

Henceforth, we drop the index $j$ and express $\Delta(c)$ symbolically as

$$\Delta(c) = \sum_{(c)} c_1 \otimes c_2.$$

This convention is known as *Sweedler's notation*.

**Definition 2.30.** If $A_1$ and $A_2$ are algebras, a linear map $f\colon A_1 \to A_2$ is an *algebra map* if the following diagrams commute:

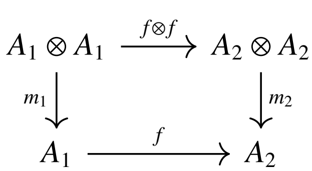



Figure 2.5: Multiplicative

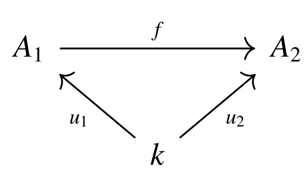



Figure 2.6: Unit preserving

**Definition 2.31.** If $C_1$ and $C_2$ are coalgebras, a linear map $g\colon C_1 \to C_2$ is a *coalgebra map* if the following diagrams commute:

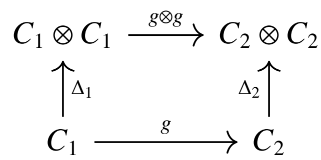


Figure 2.7: Comultiplicative

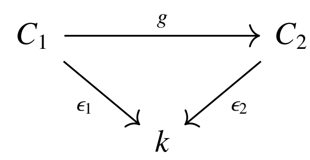


Figure 2.8: Counit preserving

Let $(A_1, m_1, u_1)$ and $(A_1, m_1, u_1)$ be algebras. The tensor product $A_1 \otimes A_2$ has a natural algebra structure. Define $m_{A_1 \otimes A_2}$ to be the composite:

$$A_1 \otimes A_2 \otimes A_1 \otimes A_2 \xrightarrow{\mathrm{id} \otimes t \otimes \mathrm{id}} A_1 \otimes A_1 \otimes A_2 \otimes A_2 \xrightarrow{m_1 \otimes m_2} A_1 \otimes A_2,$$

where $t \colon A_2 \otimes A_1 \to A_1 \otimes A_2$ denotes the twist map $a_2 \otimes a_1 \mapsto a_1 \otimes a_2$. Additionally, let $u_{A_1 \otimes A_2}$ denote the composite

$$k \xrightarrow{\cong} k \otimes k \xrightarrow{u_1 \otimes u_2} A_1 \otimes A_2.$$

Then $(A_1 \otimes A_2, m_{A_1 \otimes A_2}, u_{A_1 \otimes A_2})$ is an algebra. Similarly, if $(C_1, \Delta_1, \epsilon_1)$ and $(C_2, \Delta_2, \epsilon_2)$ are coalgebras, the tensor product $C_1 \otimes C_2$ can be made into a coalgebra in a natural way. Define $\Delta_{C_1 \otimes C_2}$ to be the composite:

$$C_1 \otimes C_2 \xrightarrow{\Delta_1 \otimes \Delta_2} C_1 \otimes C_1 \otimes C_2 \otimes C_2 \xrightarrow{\mathrm{id} \otimes t \otimes \mathrm{id}} C_1 \otimes C_2 \otimes C_1 \otimes C_2,$$

where $t \colon C_1 \otimes C_2 \to C_2 \otimes C_1$ denotes the twist map $c_1 \otimes c_2 \mapsto c_2 \otimes c_1$. Additionally, let $\epsilon_{C_1 \otimes C_2}$ denote the composite

$$C_1 \otimes C_2 \xrightarrow{\epsilon_1 \otimes \epsilon_2} k \otimes k \xrightarrow{\cong} k.$$

Then $(C_1 \otimes C_2, \Delta_{C_1 \otimes C_2}, \epsilon_{C_1 \otimes C_2})$ is a coalgebra.

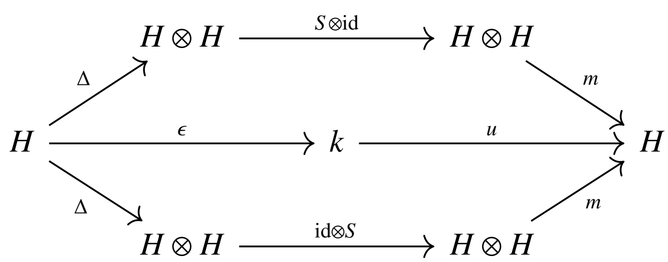
**Definition 2.32.** Let $(H, m, u)$ be an algebra and let $(H, \Delta, \epsilon)$ be a coalgebra. We call $(H, m, u, \Delta, \epsilon)$ a *bialgebra* if either of the following (equivalent) conditions holds:

1. $m$ and $u$ are coalgebra maps.

2. $\Delta$ and $\epsilon$ are algebra maps.

### 2.3.2 Hopf algebras

**Definition 2.33.** A *Hopf algebra* is a bialgebra $H$ with a linear map $S : H \to H$, called the *antipode*, such that the following diagram commutes.




**Example 2.34** (The group algebra)**.** Let $G$ be a multiplicative group. The *group algebra* $k[G]$ consists of all formal finite sums of the form

$$a = \sum_{g \in G} \alpha_g \cdot g$$

with $\alpha_g \in k$. We identify $g \in G$ with $1 \cdot g \in k[G]$ and regard $k[G]$ as a vector space over $k$ with elements of $G$ as a basis. The group algebra $k[G]$ has a natural Hopf algebra structure. Indeed, given $g_1, g_2 \in G$, let

$$m(g_1 \otimes g_2) = g_1 g_2$$

$$u(1_k) = 1_G$$

$$\Delta(g_1) = g_1 \otimes g_1$$

$$\epsilon(g_1) = 1_k$$

$$S(g_1) = g_1^{-1}$$

Extend these maps linearly to obtain a Hopf algebra structure $(k[G], m, u, \Delta, \epsilon, S)$ on $k[G]$.

Given a vector space $V$, we will write $V^* = \mathrm{Hom}_k(V, k)$ to denote its linear dual.

**Example 2.35** (The linear dual of the group algebra)**.** Let $G$ denote a finite multiplicative group and let $\{\rho_g : g \in G\}$ denote the standard dual basis of $k[G]^*$:

$$\rho_g(h) = \begin{cases} 1 \text{ if } g = h \\ 0 \text{ otherwise} \end{cases}$$

Given $g_1, g_2 \in G$, define

$$m(\rho_{g_1} \otimes \rho_{g_2}) = \begin{cases} \rho_{g_1} \text{ if } g_1 = g_2 \\ 0 \text{ otherwise} \end{cases}$$

and

$$u(1_k) = 1_{k[G]^*} = \sum_{g \in G} \rho_g.$$

Additionally, for each $g \in G$, let

$$\Delta\left(\rho_g\right) = \sum_{h \in G} \rho_{gh^{-1}} \otimes \rho_h,$$

$$\epsilon\left(\rho_g\right) = \begin{cases} 1_k \text{ if } g = 1_G \\ 0 \text{ otherwise} \end{cases}$$

and $S(\rho_g) = \rho_{g^{-1}}$. Extend these maps linearly to obtain a Hopf algebra $(k[G]^*, m, u, \Delta, \epsilon, S)$.

**Example 2.36** (Universal enveloping algebra)**.** Fix a Lie algebra $\mathfrak{g}$. A (unitary) associative algebra $A$ is called the *universal enveloping algebra of* $\mathfrak{g}$ if

1. $\mathfrak{g}$ is isomorphic to a subalgebra of $A^{(-)}$.

2. Given a (unitary) associative algebra $B$ and a homomorphism of Lie algebras $\varphi\colon \mathfrak{g} \to B^{(-)}$, there exists a unique homomorphism of associative algebras $\bar{\varphi}\colon A \to B$ which

extends $\varphi$, i.e., $\bar{\varphi}$ is equal to $\varphi$ on $\mathfrak{g}$.

The universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is unique up to isomorphism. Henceforth, it shall be denoted as $U(\mathfrak{g})$. A basis for $U(\mathfrak{g})$ can be chosen in the form

$$e_1 \cdots e_n,\ e_1 \leq \cdots \leq e_n,\ n = 0, 1, 2, \ldots$$

where each $e_i$ is an element of an ordered basis $E$ for $\mathfrak{g}$. We can equip $U(\mathfrak{g})$ with a coalgebra structure; define $\Delta\colon U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ via $\Delta(1) = 1 \otimes 1$ and $\Delta(x) = x \otimes 1 + 1 \otimes x$, for all $x \in \mathfrak{g}$. Additionally, define $\epsilon\colon U(\mathfrak{g}) \to k$ via $\epsilon(1) = 1_k$ and $\epsilon(x) = 0$, for all $x \in \mathfrak{g}$. Extend both maps linearly and multiplicatively to obtain a coalgebra $(U(\mathfrak{g}), \Delta, \epsilon)$.

### 2.3.3 Hopf algebra actions

**Definition 2.37.** Fix an algebra $A$ and a Hopf algebra $(H, m, u, \Delta, \epsilon, S)$. The algebra $A$ is called an *$H$-module algebra*, or *$H$-algebra* for short, if $A$ is an $H$-module, and, for each $h \in H$ and $a_1, a_2 \in A$,

$$h \cdot (a_1 a_2) = \sum_{(h)} (h_1 \cdot a_1)(h_2 \cdot a_2)^1. \tag{2.3}$$

Moreover, if $A$ is associative and unitary, we require $h \cdot 1_A = \epsilon(h) 1_A$, for all $h \in H$.

**Remark 2.38.** In [9], Berele extended this definition by requiring, in place of (2.3), that, for each $h \in H$, there exists $h_{(1)}, h_{(2)}, h'_{(1)}$, and $h'_{(2)}$ in $H$ (not necessarily coming from the coproduct) such that, for all $a_1, a_2 \in A$,

$$h \cdot (a_1 a_2) = \sum (h_{(1)} \cdot a_1)(h_{(2)} \cdot a_2) + \left(h'_{(1)} \cdot a_2\right)\left(h'_{(2)} \cdot a_1\right).$$

This will motivate one of our principal definitions in Chapter 3 (see Definition 3.5).

Given a Hopf algebra $(H, m, u, \Delta, \epsilon, S)$ with a fixed basis $B$ containing $1_H$, denote by $k\langle X \,|\, H\rangle$

[1] note Sweedler's notation in action!

the free associative algebra generated by the set of indeterminates $\left\{x^b : x \in X, b \in B\right\}$. Elements in $k\langle X \mid H\rangle$ will be called *H-polynomials*. Given $x \in X$, we identify $x = x^{1_H}$ and

$$x^h = \sum_{i=1}^{n} \alpha_i x^{b_i},$$

for each linear combination of basis elements $h = \alpha_1 b_1 + \cdots + \alpha_n b_n \in H$.

**Remark 2.39.** $k\langle X \mid H\rangle$ is naturally an $H$-algebra; indeed, for each $h \in H$,

$$h \cdot \left(x_{i_1}^{b_1} \cdots x_{i_n}^{b_n}\right) = \sum_{(h)} x_{i_1}^{h_1 b_1} \cdots x_{i_n}^{h_n b_n},$$

where $\Delta^{n-1}(h) = \sum_{(h)} h_1 \otimes \cdots \otimes h_n$. See [6] for details.

**Proposition 2.40.** *Suppose that $A$ is an $H$-algebra. Given any map $\varphi\colon X \to A$, there is a unique algebra homomorphism extension $\bar{\varphi}\colon k\langle X \mid H\rangle \to A$ such that $\bar{\varphi}\left(x^h\right) = h \cdot \varphi(x)$, for all $x \in X$ and $h \in H$.*

**Definition 2.41.** We shall say that an $H$-polynomial $f \in k\langle X \mid H\rangle$ is an *H-identity* for an $H$-algebra $A$ if,

$$f \in \bigcap \ker \bar{\varphi},$$

where the intersection runs over all maps $\varphi\colon X \to A$, and $\bar{\varphi}$ is defined as in Proposition 2.40.

We have seen that when an associative algebra $A$ is equipped with a group grading, an action by automorphisms and anti-automorphisms, or an involution, the presence of a certain type of graded identity, $G$-identity, or $*$-identity is sufficient to conclude that $A$ is a PI-algebra. We now ask:

*When does the existence of an $H$-identity force the existence of an ordinary polynomial identity for $A$?*

We can draw some positive results from Theorems 2.15 and 2.20. Fix an associative unitary algebra $A$ and a finite multiplicative group $G$. First, observe that a group grading on $A$ corresponds to a particular Hopf algebra action.

**Proposition 2.42.** *If $A$ is graded by $G$,*

$$A = \bigoplus_{g \in G} A_g,$$

*then $A$ is a $k[G]^*$-module algebra, where $\rho_g \cdot (\sum_{h \in G} a_h) = a_g$, for each $\rho_g$ in the standard dual basis for $k[G]^*$. Conversely, if $A$ is a $k[G]^*$-module algebra, then $A$ is graded by $G$, where $A_g = \rho_g \cdot A$, for each $g \in G$.*

*Proof.* See Proposition 1.3 in [11]. □

Therefore, we can rephrase Theorem 2.15 in the language of Hopf actions and $H$-identities as follows.

**Corollary 2.43.** *Fix a unitary associative algebra $A$ and let $H = k[G]^*$. If $A$ is an $H$-algebra satisfying an $H$-identity of the form*

$$\sum_{\sigma \in S_d} \alpha_\sigma x_{\sigma(1)}^{\rho_{1_G}} \cdots x_{\sigma(d)}^{\rho_{1_G}} \equiv 0,$$

*then it is a PI-algebra.*

Along the same lines, we have the following result in the case $A$ is equipped with a group action via a group homomorphism $G \to \mathrm{Aut}(A)$.

**Proposition 2.44.** *Any group action $G \to Aut(A)$ makes $A$ into a $k[G]$-module algebra. Conversely, if $A$ is a $k[G]$-module algebra, this arises from a group action $G \to Aut(A)$.*

*Proof.* See Proposition 1.2 in [11]. □

However, observe that a group action $G \to \mathrm{Aut}^*(A)$ makes $A$ into a $k[G]$-module, but not into a $k[G]$-algebra in the sense of Definition 2.37. Due to this obstruction, we cannot reformulate Theorem 2.20 using Hopf actions. Nonetheless, the following special case follows from Proposition 2.44.

**Corollary 2.45.** *If A is a $k[G]$-algebra satisfying a $k[G]$-identity of the form:*

$$x_1^1 \cdots x_d^1 + \sum_{\sigma \in S_d, \sigma \neq 1} \sum_{g=(g_1,\ldots,g_d) \in G^d} \alpha_{\sigma,g} x_{\sigma(1)}^{g_1} \cdots x_{\sigma(d)}^{g_d} \equiv 0,$$

*then A is a PI-algebra.*

Of course, the question of when the existence of an $H$-identity on $A$ ensures that $A$ satisfies an ordinary polynomial identity remains open for arbitrary Hopf algebras. In order to address this, we propose a general framework that will allow us to consider graded identities, $G$-identities, $*$-identities, $H$-identities and other "general" polynomial identities of interest simultaneously.

# Chapter 3

# A unified approach

Our research will focus on algebras $A$ which are endowed with a (left unitary $k$-linear) $R$-module action, where $R$ denotes a given unitary associative algebra. By an $R$-module action on $A$, we mean a homomorphism of unitary $k$-algebras, $R \to \mathrm{End}_k(A)$, where $\mathrm{End}_k(A)$ denotes the algebra of linear maps on $A$ with the usual operations. First, we consider the case when $A$ is an associative algebra.

Let us fix a basis $\mathcal{B}$ of $R$ containing $1_R$ and denote by $k\langle X \,|\, R\rangle$ the free associative algebra generated by the set of indeterminates $\{x^b\colon\, x \in X, b \in \mathcal{B}\}$. Elements in $k\langle X \,|\, R\rangle$ will be called *$R$-polynomials*. For each $x \in X$ and each $k$-linear combination of basis elements

$$r = \alpha_1 b_1 + \cdots + \alpha_n b_n \in R,$$

we will identify

$$x^r = \alpha_1 x^{b_1} + \cdots + \alpha_n x^{b_n} \text{ and } x^{1_R} = x.$$

**Remark 3.1.** For an arbitrary $R$, the free associative algebra $k\langle X \,|\, R\rangle$ need not have a natural $R$-action. Compare this with $k\langle X\rangle^{\mathrm{gr}}$ which is equipped with a $G$-grading (see Subsection 2.2.1), $k\langle X \,|\, G\rangle$ which is equipped with a $G$-action (see Subsection 2.2.2), or $k\langle X \,|\, H\rangle$ which is equipped with an $H$-action (see Subsection 2.3.3).

**Proposition 3.2.** *If an associative algebra $A$ is endowed with an $R$-module action, then any map $\varphi\colon X \to A$ has a unique $k$-algebra homomorphic extension $\bar{\varphi}\colon k\langle X \mid R\rangle \to A$ such that, for all $x \in X$ and $b \in \mathcal{B}$, $\bar{\varphi}\left(x^b\right) = b \cdot \varphi(x)$.*

**Definition 3.3.** Suppose that $A$ is an associative algebra endowed with an $R$-module action. We shall say that an $R$-polynomial $f \in k\langle X \mid R\rangle$ is an *$R$-identity* for $A$ if,

$$f \in \bigcap \ker \bar{\varphi},$$

where the intersection runs over all maps $\varphi\colon X \to A$, and $\bar{\varphi}$ is defined as in Proposition 3.2.

For example, a graded identity is an $R$-identity, where $R = k[G]^*$; a $G$-identity is an $R$-identity, where $R = k[G]$; and, clearly, an $H$-identity is an $R$-identity, where $R = H$.

**Remark 3.4.** If $f \in k\langle X \mid R\rangle$, we will write $f(x_1, \ldots, x_n)$ to indicate that the only indeterminates *possibly* occurring in $f$ are those of the form $x_1^b, \ldots, x_n^b$, with $b \in \mathcal{B}$.

Previous examples show that, in general, $A$ could satisfy a nontrivial $R$-identity and yet might not be a PI-algebra. Our primary goal in this thesis is to provide conditions which ensure that $A$ satisfies a classical polynomial identity. Most critically, we will require that the $R$-action on $A$ is compatible with the multiplicative structure of $A$. In order to formulate this 'compatibility' condition precisely, we draw some inspiration from Remark 2.38 and propose the following definition.

**Definition 3.5.** Let $A$ be a not necessarily associative algebra equipped with an $R$-module action.

1. We shall say that $A$ is a *generalized $R$-algebra* if, for each $r \in R$, there exists finitely

many pairs $(l_i^+, r_i^+)$ and $(l_i^-, r_i^-)$ in $R^2$ (with $i$ ranging over a finite set $I$) such that

$$r \cdot (a_1 a_2) = \sum_{i \in I} (l_i^+ \cdot a_1)(r_i^+ \cdot a_2) + \sum_{i \in I} (l_i^- \cdot a_2)(r_i^- \cdot a_1),$$

for all $a_1, a_2 \in A$.

2. We shall say that $A$ is a *positive generalized R-algebra* if, for each $r \in R$, there exists finitely many pairs $(l_i^+, r_i^+)$ in $R^2$ (with $i$ ranging over a finite set $I$) such that

$$r \cdot (a_1 a_2) = \sum_{i \in I} (l_i^+ \cdot a_1)(r_i^+ \cdot a_2),$$

for all $a_1, a_2 \in A$.

There are many classes of such actions in 'Nature'. For example, if $H$ is a Hopf algebra and $A$ is an $H$-algebra, then $A$ is a positive generalized $H$-algebra. Perhaps the most important example of a generalized $R$-algebra action that is not positive is the case when $A$ admits an involution; more generally, if $G$ is any group acting as automorphisms and anti-automorphisms on $A$, then $A$ is a generalized $R$-algebra, where $R = k[G]$. In the next chapters we investigate:

*When is a generalized R-algebra a PI-algebra?*

We address this question in Chapters 4 and 5 for associative and Lie algebras, respectively.

# Chapter 4

# Polynomial identities of associative algebras with actions

Our main objective in this chapter is to establish a series of combinatorial conditions that ensure that a given associative algebra $A$ is a PI-algebra. Initially, we focus on the $R$-identities of an algebra $A$ equipped with an $R$-module action; more concretely, we examine what type of $R$-identities force the existence of a classical identity for $A$. This work is carried out in Section 4.1. We reformulate the main results of Section 4.1 in the language of generalized polynomial identities (GPI's) in Section 4.2. With the aim of extending our initial results, in Section 4.3 we introduce one of our most important tools: the sequence $\pi_n(A)$. We prove the following characterization: $A$ is a PI-algebra if and only if $\pi_n(A) < n!$, for some positive integer $n$. We then exploit the tools developed in Section 4.3 to study some general conditions on the $R$-action which ensure that $A$ is a PI-algebra. This is done in Section 4.4.

## 4.1 Conditions on the $R$-identities

Throughout this section we shall follow the notation introduced in Chapter 3; in particular, we assume $A$ is endowed with a (left unitary $k$-linear) $R$-module action, where $R$ denotes a given unitary associative algebra. We begin this chapter by examining the following question:

*When can the existence of an $R$-identity for $A$ be used as criteria to establish that $A$ is a PI-algebra?*

To provide an answer, it will be necessary to introduce some additional terminology and tools. We remark that, in general, $A$ could satisfy a non-trivial $R$-identity and yet might not be a PI-algebra (for instance, see Example 2.19). Hence, we will require a special type of $R$-identity.

**Definition 4.1.** For each positive integer $n$, let

$$P_n^R = \operatorname{span}_k \left\{ x_{\sigma(1)}^{r_1} \cdots x_{\sigma(n)}^{r_n} \in k\langle X \,|\, R\rangle \colon \sigma \in S_n, r_1, \ldots, r_n \in R \right\}.$$

Inspired by the work of Bahturin, Giambruno, and Zaicev in [5], we propose the following definition.

**Definition 4.2.** We shall say that an $R$-identity of $A$ of the form

$$x_1 \cdots x_d - \sum x_{\sigma(1)}^{r_1} \cdots x_{\sigma(d)}^{r_d} \equiv 0$$

*rewrites* $A$ if $\sum x_{\sigma(1)}^{r_1} \cdots x_{\sigma(d)}^{r_d}$ is an element of $P_d^R$ without any terms of the form $x_1^{r_1} \cdots x_d^{r_d}$.

**Definition 4.3.** We will write $\operatorname{Id}(A \,|\, R)$ to denote the set of $R$-identities of $A$. We define the *$n$-th $R$-codimension of $A$*, denoted by $c_n^R(A)$, as

$$c_n^R(A) = \dim_k \left( \frac{P_n^R}{P_n^R \cap \operatorname{Id}(A \,|\, R)} \right).$$

The following lemma is analogous to Lemma 10.1.2 in [12].

**Lemma 4.4.** *For all positive integers $n$,*

$$c_n(A) = \dim_k \left( \frac{P_n + I}{I} \right) \leq c_n^R(A),$$

*where* $I = P_n^R \cap \mathrm{Id}(A \,|\, R)$.

*Proof.* Under the identifications made in Chapter 3, we have $P_n \subseteq P_n^R$; hence,

$$P_n \cap \mathrm{Id}(A) = P_n \cap \left(P_n^R \cap \mathrm{Id}(A \,|\, R)\right).$$

It follows that

$$\frac{P_n}{P_n \cap \mathrm{Id}(A)} \cong \frac{P_n + \left(P_n^R \cap \mathrm{Id}(A \,|\, R)\right)}{P_n^R \cap \mathrm{Id}(A \,|\, R)} \leq \frac{P_n^R}{P_n^R \cap \mathrm{Id}(A \,|\, R)},$$

as vector spaces. □

Suppose now that $A$ is a generalized $R$-algebra. The algebraic structure of $\mathrm{Id}(A \,|\, R)$ is not as rich as that of $\mathrm{Id}(A)$; recall that $\mathrm{Id}(A)$ is a T-ideal of $k\langle X\rangle$. In comparison, while $\mathrm{Id}(A \,|\, R)$ is a two-sided ideal of $k\langle X \,|\, R\rangle$, it is not invariant under endomorphisms of $k\langle X \,|\, R\rangle$. Hence, even when $f(x_1, \ldots, x_n) \in \mathrm{Id}(A \,|\, R)$, given $g_1, \ldots, g_n \in k\langle X \,|\, R\rangle$, $f(g_1, \ldots, g_n)$ need not be an $R$-identity of $A$. In fact, because there is no natural $R$-action on $k\langle X \,|\, R\rangle$ (see Remark 3.1), the expression $f(g_1, \ldots, g_n)$ may not be a well defined $R$-polynomial in the first place. We take advantage of the generalized $R$-algebra structure of $A$ to rectify this situation. Recall that $\mathcal{B}$ denotes a fixed basis of $R$ containing $1_R$.

**Definition 4.5.** Given a generalized $R$-algebra $A$ and $b \in \mathcal{B}$, fix a choice of finitely many elements $b_\sigma = (b_{\sigma,1}, \ldots, b_{\sigma,n}) \in R^n$, for each $\sigma \in S_n$, such that, for all $a_1, \ldots, a_n \in A$,

$$b \cdot (a_1 \cdots a_n) = \sum_{\sigma \in S_n} \sum_{b_\sigma} (b_{\sigma,1} \cdot a_{\sigma(1)}) \cdots (b_{\sigma,n} \cdot a_{\sigma(n)}).$$

Define

$$b \cdot \left(x_{i_1}^{e_1} \cdots x_{i_n}^{e_n}\right) = \sum_{\sigma \in S_n} \sum_{b_\sigma} x_{i_{\sigma(1)}}^{b_{\sigma,1} e_{\sigma(1)}} \cdots x_{i_{\sigma(n)}}^{b_{\sigma,n} e_{\sigma(n)}} \in k\langle X \,|\, R\rangle,$$

for each $x_{i_1}, \ldots, x_{i_n} \in X$ and $e_1, \ldots, e_n \in \mathcal{B}$.

1. These basis assignments induce a well-defined linear map which will be denoted by

$\lambda_A \colon R \to \mathrm{End}_k\,(k\,\langle X\,|\,R\rangle)$.

2. For each $r \in R$ and $f \in k\,\langle X\,|\,R\rangle$, we will write $f^r$ for $\lambda_A(r)\,(f)$.

Henceforth, every associative algebra $A$ equipped with a generalized $R$-action will be implicitly equipped with a fixed linear map $\lambda_A \colon R \to \mathrm{End}_k(k\,\langle X\,|\,R\rangle)$. We remark that, in general, $\lambda_A$ need not be unique nor an algebra homomorphism; nevertheless, as indicated in Definition 4.6, if $f(x_1, \ldots, x_n) \in k\,\langle X\,|\,R\rangle$ and $g_1, \ldots, g_n \in k\,\langle X\,|\,R\rangle$, the application of $\lambda_A$ does allow us to regard $f(g_1, \ldots, g_n)$ as a well-defined $R$-polynomial in $k\,\langle X\,|\,R\rangle$.

**Definition 4.6.** Given $g_1, \ldots, g_n \in k\,\langle X\,|\,R\rangle$, fix a map $\varphi\colon X \to k\,\langle X\,|\,R\rangle$ such that $\varphi(x_1) = g_1, \ldots, \varphi(x_n) = g_n$. This map extends uniquely to a homomorphism of algebras $\bar{\varphi}\colon k\,\langle X\,|\,R\rangle \to k\,\langle X\,|\,R\rangle$ with the property that, for each $x \in X$ and $b \in \mathcal{B}$, $\bar{\varphi}(x^b) = \lambda_A(b)(\varphi(x))$. For each $f(x_1, \ldots, x_n) \in k\,\langle X\,|\,R\rangle$, we define $f(g_1, \ldots, g_n) = \bar{\varphi}(f)$.

Furthermore, we have:

**Lemma 4.7.** *If $f(x_1, \ldots, x_n) \equiv 0$ is an $R$-identity for $A$ and $g_1, \ldots, g_n \in k\,\langle X\,|\,R\rangle$, then $f(g_1, \ldots, g_n) \equiv 0$ is also an $R$-identity for $A$.*

Finally, recall that permutations in $S_n$ are ordered lexicographically; that is, given two permutations $\alpha, \beta \in S_n$, we will write $\alpha < \beta$ if, for some $k \geq 0$,

$$\alpha(1) = \beta(1), \ldots, \alpha(k) = \beta(k) \text{ and } \alpha(k+1) < \beta(k+1).$$

Furthermore, we shall define $x^{r_1}_{\alpha(1)} \cdots x^{r_n}_{\alpha(n)} < x^{s_1}_{\beta(1)} \cdots x^{s_n}_{\beta(n)}$ in $P^R_n$ if $\alpha < \beta$. We are ready to prove the main result of this section.

**Theorem 4.8.** *Let $A$ be an associative algebra equipped with a generalized $R$-action corresponding to $\rho\colon R \to End_k(A)$ with the property that $m = \dim_k \rho(R)$ is finite. If there exists an $R$ identity of degree $d$ rewriting $A$:*

$$f = x_1 \cdots x_d - \sum x^{r_1}_{\sigma(1)} \cdots x^{r_d}_{\sigma(d)} \equiv 0,$$

*then $A$ is a PI-algebra satisfying an ordinary polynomial identity of degree $n = \kappa(d, m)$ (for the definition of $\kappa(d, m)$, see Lemma 2.25). If moreover the action is a positive generalized $R$-action, we may take $n = \lceil em(d-1)^2 \rceil$ ($e$ denotes the base of the natural logarithm).*

*Proof.* We may replace $R$ by $\rho(R)$ to assume that $\dim_k R = m$.

1. Suppose $A$ is a generalized $R$-algebra and let $n = \kappa(d, m)$. We will prove that $c_n(A) < n!$. By Lemma 4.4, it suffices to show that $\dim_k(P_n + I)/I < n!$, where $I = P_n^R \cap \mathrm{Id}(A \mid R)$. To this end, consider

$$W = \mathrm{span}_k \left\{ x_{\upsilon(1)}^{r_1} \cdots x_{\upsilon(n)}^{r_n} : \upsilon \in S_n \text{ is } d\text{-indecomposable and } r_1, \ldots, r_n \in R \right\}.$$

If we can show that $P_n \leq W + I$, then, by Lemma 2.25, we would obtain

$$\dim_k(P_n + I)/I \leq \dim_k(W + I)/I \leq m^n a_d(n) < m^n \left(\frac{1}{m}\right)^n n! = n!;$$

consequently, we would be able to conclude that $A$ satisfies an ordinary polynomial identity of degree $n$, as required.

In order to prove $P_n \leq W + I$, we argue by contradiction and fix the smallest permutation $\tau \in S_n$ for which $x_{\tau(1)} \cdots x_{\tau(n)} \notin W + I$. Since $\tau$ must be $d$-decomposable, there exists a sequence of integers

$$1 \leq h_1 \leq t_1 < \cdots < h_d \leq t_d \leq n$$

that determine a $d$-decomposition for $w = x_{\tau(1)} \cdots x_{\tau(n)}$; we partition $w$ accordingly:

$$w = \underbrace{\left(x_{\tau(1)} \cdots\right)}_{w_0} \underbrace{\left(x_{\tau(h_1)} \cdots x_{\tau(t_1)}\right)}_{w_1} \underbrace{\left(x_{\tau(h_2)} \cdots x_{\tau(t_2)}\right)}_{w_2} \cdots \underbrace{\left(x_{\tau(h_d)} \cdots x_{\tau(t_d)}\right)}_{w_d} \underbrace{\left(\cdots x_{\tau(n)}\right)}_{w_{d+1}}.$$

Because $f \equiv 0$ is an $R$-identity for $A$, then, by Lemma 4.7, so is

$$w_0 f(w_1,\dots,w_d)w_{d+1} = w - \sum_{1\neq\sigma\in S_d}\sum_{r} w_0 w^{r_1}_{\sigma(1)}\cdots w^{r_d}_{\sigma(d)} w_{d+1} \equiv 0. \tag{4.1}$$

Furthermore, when we expand each term in the sum on the right into a linear combination of basis monomials in $P^R_n$, the $d$-decomposition of $w$ and the generalized $R$-algebra action together force each of these basis monomials to be smaller than $w$. Consider $\upsilon < \tau$. Then $x_{\upsilon(1)}\cdots x_{\upsilon(n)} \in W + I$ by the minimality of $\tau$. Since $g(x^{s_1}_1,\dots,x^{s_n}_n) \equiv 0$ is an $R$-identity of $A$, for all $s_1,\dots,s_n \in R$, whenever $g(x_1,\dots,x_n) \equiv 0$ is an $R$-identity of $A$, it follows that

$$x^{s_1}_{\upsilon(1)}\cdots x^{s_n}_{\upsilon(n)} \in W + I, \text{ for all } s_1,\dots,s_n \in R.$$

Consequently, $w \in W + I$ by (4.1). This contradiction completes the proof.

2. Suppose $A$ is a positive generalized $R$-algebra and let $n = \lceil em(d-1)^2\rceil$. Consider the subspace

$$W = \operatorname{span}_k\left\{x^{r_1}_{\upsilon(1)}\cdots x^{r_n}_{\upsilon(n)} : \upsilon \in S_n \text{ is } d\text{-good and } r_1,\dots,r_n \in R\right\}.$$

Once again, we have $P_n \leq W+I$. Indeed, suppose that this were false, and fix the smallest permutation $\tau \in S_n$ for which

$$x_{\tau(1)}\cdots x_{\tau(n)} \notin W + I.$$

Since $\tau$ must be $d$-bad, there is a sequence of integers $1 \leq h_1 < \cdots < h_d \leq n$ with the property that $\tau(h_1) > \cdots > \tau(h_d)$. Partition $w = x_{\tau(1)}\cdots x_{\tau(n)}$ accordingly:

$$w = \underbrace{(x_{\tau(1)}\cdots)}_{u}\underbrace{(x_{\tau(h_1)}\cdots)}_{w_1}\underbrace{(x_{\tau(h_2)}\cdots)}_{w_2}\cdots\underbrace{(x_{\tau(h_d)}\cdots x_{\tau(n)})}_{w_d}.$$

By Lemma 4.7, because $f \equiv 0$ is an $R$-identity of $A$, so is

$$uf(w_1, \ldots, w_d) = w - \sum_{1 \neq \sigma \in S_d} \sum_r u w_{\sigma(1)}^{r_1} \cdots w_{\sigma(d)}^{r_d} \equiv 0. \tag{4.2}$$

Moreover, when we expand each term in the sum on the right into a linear combination of basis monomials in $P_n^R$, the partition of $w$, followed by the positivity of the generalized $R$-algebra action, forces each of these basis monomials to be smaller than $w$. Arguing exactly as before shows that

$$x_{\upsilon(1)}^{s_1} \cdots x_{\upsilon(n)}^{s_n} \in W + I,$$

for all $\upsilon < \tau$ and $s_1, \ldots, s_n \in R$. Thus, we have found our contradiction: $w \in W + I$.

Because Lemma 2.23 assures us that number of $d$-good permutations in $S_n$ is at most $\frac{(d-1)^{2n}}{(d-1)!}$, it follows that

$$\dim_k(P_n + I)/I \leq \dim_k(W + I)/I \leq m^n \frac{(d-1)^{2n}}{(d-1)!}.$$

Therefore, substituting $n = \lceil em(d-1)^2 \rceil$ into the inequality $(\frac{n}{e})^n < n!$ yields

$$\dim_k(P_n + I)/I \leq \frac{m^n (d-1)^{2n}}{(d-1)!} < \frac{n!}{(d-1)!}.$$

Thus, by Lemma 4.4, $c_n(A) < n!$, and so $A$ satisfies a polynomial identity of degree $n$.

□

## 4.2 Combining multiple actions: *R*-GPIs

Suppose that $R_1$ and $R_2$ are unitary subalgebras of $\mathrm{End}_k(A)$ such that $A$ is a (positive) generalized $R_i$-algebra, for $i = 1, 2$. Then, $A$ is a (positive) generalized $R$-algebra, where $R$ is the subalgebra of $\mathrm{End}_k(A)$ generated by $R_1$ and $R_2$. In this way, we can combine multiple types of generalized $R$-algebra actions into one.

In this section, we shall use the above-mentioned observation to reformulate Theorem 4.8 in the language of generalized polynomial identities for a given unitary associative algebra $A$. We remind the reader that $f$ is a *generalized polynomial of degree* $d$ if

$$f(x_1, \dots, x_d) = \sum_{\sigma \in S_d} f^{\sigma}(x_1, \dots, x_d),$$

where, for each $\sigma \in S_d$,

$$f^{\sigma}(x_1, \dots, x_d) = \sum_{j=1}^{\alpha_\sigma} \alpha_{0,\sigma,j} x_{\sigma(1)} \alpha_{1,\sigma,j} x_{\sigma(2)} \cdots \alpha_{d-1,\sigma,j} x_{\sigma(d)} \alpha_{d,\sigma,j};$$

in this equation, $\alpha_\sigma$ denotes a fixed positive integer and each $\alpha_{i,\sigma,j} \in A$. A generalized polynomial $f$ with the property that $f(a_1, \dots, a_n) = 0$, for all $a_1, \dots, a_n \in A$, is known as a *generalized polynomial identity* of $A$, (or GPI for short).

Now, suppose $A$ is equipped with a generalized $R$-module action $\rho\colon R \to \mathrm{End}_k(A)$. We shall call an identical relation of the form

$$f = x_1 \cdots x_d - \sum_{1 \neq \sigma \in S_d} \sum_{\alpha} \sum_{r} \alpha_0 x_{\sigma(1)}^{r_1} \alpha_1 x_{\sigma(2)}^{r_2} \cdots \alpha_{d-1} x_{\sigma(d)}^{r_d} \alpha_d \equiv 0$$

an *R-GPI rewriting A* provided $f(a_1, \dots, a_d) = 0$, for all $a_1, \dots, a_d \in A$, where the $R$-action is applied before multiplication. Let $\mu\colon A \to \mathrm{End}_k(A)$ denote the action of $A$ by left multiplication, and write $E(R, f)$ to denote the unitary subalgebra of $\mathrm{End}_k(A)$ generated by $\rho(R)$ and the elements $\mu(\alpha_i)$, for each $\alpha_i \in A$ appearing as entries in between the indeterminates in $f$. By our opening remark, if $A$ satisfies the $R$-GPI $f \equiv 0$, we can view $A$ as a generalized $E(R, f)$-algebra and $f(x_1, \dots, x_d)x_{d+1} \equiv 0$ as an $E(R, f)$-identity of degree $d + 1$ rewriting $A$.

By the preceding discussion, we can reformulate Theorem 4.8 as follows.

**Corollary 4.9.** *Let A be a unitary associative algebra.*

1. *Suppose that R is a unitary subalgebra of* $\mathrm{End}_k(A)$ *such that A is a positive generalized R-algebra. This occurs, for example, whenever R is generated by algebra endomorphisms and derivations. If A satisfies an R-GPI* $f \equiv 0$ *of degree d rewriting A such that* $m = \dim_k E(R, f)$ *is finite, then A satisfies an ordinary polynomial identity of degree* $\lceil emd^2 \rceil$.

2. *Suppose that R is a unitary subalgebra of* $\mathrm{End}_k(A)$ *such that A is a generalized R-algebra. This occurs, for example, whenever R is generated by algebra endomorphisms, algebra anti-endomorphisms and derivations. If A satisfies an R-GPI* $f \equiv 0$ *of degree d rewriting A such that* $m = \dim_k E(R, f)$ *is finite, then A satisfies an ordinary polynomial identity of degree* $\kappa(d+1, m)$.

Similarly, for a GPI rewriting $A$ without $R$-actions,

$$f = x_1 \cdots x_d - \sum_{1 \neq \sigma \in S_d} \sum_{\alpha} \alpha_0 x_{\sigma(1)} \alpha_1 x_{\sigma(2)} \cdots \alpha_{d-1} x_{\sigma(d)} \alpha_d \equiv 0,$$

we will denote by $E(f)$ the subalgebra of $\mathrm{End}_k(A)$ generated by the elements $\mu(\alpha_i)$, for each $\alpha_i \in A$ appearing in $f$. In this case, $A$ can be viewed as a positive generalized $E(f)$-algebra, and $f(x_1, \ldots, x_d)x_{d+1} \equiv 0$ can be viewed as an $E(f)$-identity of degree $d+1$ rewriting $A$.

**Corollary 4.10.** *If A satisfies a GPI* $f \equiv 0$ *of degree d rewriting A such that* $m = \dim_k E(f)$ *is finite, then A satisfies an ordinary polynomial identity of degree* $\lceil emd^2 \rceil$.

A generalized polynomial $f$ is called an *essential* GPI of $A$ if the two-sided ideal generated by the values $f^\sigma(a_1, \ldots, a_n)$, with $a_1, \ldots, a_n \in A$, contains $1_A$. In [21], Rowen showed that if $A$ satisfies an essential GPI, then $A$ is a PI-algebra in the classical sense. Rowen's theorem was extended by Kharchenko to include GPIs with actions by automorphisms (from a possibly infinite group) and the composition of derivations (in the case when the characteristic is zero). See Theorems 2.6.4 and 2.6.8 in [16] for further details. The approach used by Rowen and Kharchenko does not produce bounds on the degree of the polynomial identity satisfied by $A$. In contrast, observe that our work does yield quantitative results along these same lines.

Next, we work towards extending Theorem 4.8. The main tool that will allow us to accomplish this is the sequence $\pi_n(A)$, which we now introduce.

## 4.3 The sequence $\pi_n(A)$ for an associative algebra $A$

To every associative algebra $A$ we can associate a numerical sequence known as the *codimension sequence of* $A$.

**Definition 4.11.** Fix a positive integer $n$.

1. We will write $P_n$ to denote the following subspace of $k\langle X\rangle$:

$$P_n = \operatorname{span}_k \left\{x_{\sigma(1)} \cdots x_{\sigma(n)} \in k\langle X\rangle : \sigma \in S_n\right\}.$$

2. The integer
$$c_n(A) = \dim \frac{P_n}{P_n \cap \operatorname{Id}(A)}$$
is called the *n-th codimension* of the algebra $A$.

We highlight the following simple yet useful characterization: an associative algebra $A$ is PI-algebra if and only if $c_n(A) < n!$, for some positive integer $n$. Indeed, if $A$ satisfies a polynomial identity of degree $n$, then $A$ satisfies a multilinear identity of degree $n$; this is precisely equivalent to the condition $c_n(A) < n!$.

While the codimension sequence is an important tool used to prove existence theorems in PI-theory, given an algebra $A$, it is generally quite difficult to compute $c_n(A)$ explicitly. In fact, explicit values of $c_n(A)$ are known only for very few algebras. In this section, we introduce a new numerical sequence that might serve as a more tractable alternative to $c_n(A)$ for some purposes. Besides being interesting on its own, this new sequence will be key to prove some of our main results.

**Definition 4.12.** Fix a positive integer $n$. For every $a_1, \dots, a_n \in A$, let

$$\pi_n(a_1, \dots, a_n) = \dim_k \left(\operatorname{span}_k \{a_{\sigma(1)} \cdots a_{\sigma(n)} : \sigma \in S_n\}\right);$$

we define

$$\pi_n(A) = \max \{\pi_n(a_1, \dots, a_n) : a_1, \dots, a_n \in A\}.$$

**Proposition 4.13.** *If $\pi_d(A) < d!$, then $A$ is a PI-algebra satisfying a classical polynomial identity of degree $d!d$.*

*Proof.* Let $w_1, \dots, w_{d!}$ be a complete list of all multilinear monomials in $k\langle X\rangle$ of the form $x_{\sigma(1)} \cdots x_{\sigma(d)}$, with $\sigma \in S_d$. Now let $a_1, \dots, a_d \in A$. Then, by assumption, if we evaluate $x_i \mapsto a_i$, for each $1 \le i \le d$, the corresponding images of $w_1, \dots, w_{d!}$ in $A$ are $k$-linearly dependent. Next, consider the standard polynomial of degree $d!$:

$$s_{d!}(x_1, \dots, x_{d!}) = \sum_{\tau \in S_{d!}} \operatorname{sgn}(\tau) x_{\tau(1)} \cdots x_{\tau(d!)}.$$

Observe that, if we specialize $x_j$ to $x_i$, for any $1 \le i < j \le d!$, then

$$s_{d!}(x_1, \dots, x_i, \dots, x_i, \dots, x_{d!}) = 0.$$

It follows that $A$ satisfies the following (non-trivial) identity:

$$\sum_{\tau \in S_{d!}} \operatorname{sgn}(\tau) w_{\tau(1)} \cdots w_{\tau(d!)} \equiv 0.$$

□

We now explore the relationship between $c_n(A)$ and $\pi_n(A)$.

**Proposition 4.14.** *Let $A$ be any associative algebra. Then $\pi_n(A) \le c_n(A)$, for each positive* *integer n.*

*Proof.* Notice that the $k$-space $P_n/P_n \cap \mathrm{Id}(A)$ has a basis consisting of elements of the form $x_{\sigma(1)} \cdots x_{\sigma(n)} + P_n \cap \mathrm{Id}(A)$, where each $\sigma$ lies in some fixed subset of $S_n$. Now consider any $a_1, \ldots, a_n \in A$. Because the subalgebra of $A$ generated by $a_1, \ldots, a_n$ is the homomorphic image of the relatively free algebra modulo $\mathrm{Id}(A)$ under $x_i + \mathrm{Id}(A) \mapsto a_i$, for $1 \leq i \leq n$, and $x_i + \mathrm{Id}(A) \mapsto 0$, otherwise, it follows that the assignments $x_{\sigma(1)} \cdots x_{\sigma(n)} + P_n \cap \mathrm{Id}(A) \mapsto a_{\sigma(1)} \cdots a_{\sigma(n)}$ induce a well-defined linear transformation from $P_n/P_n \cap \mathrm{Id}(A)$ onto $\mathrm{span}_k \{a_{\sigma(1)} \cdots a_{\sigma(n)} : \sigma \in S_n\}$. Thus, $\pi_n(a_1, \ldots, a_n) \leq c_n(A)$, as required. □

We have seen that an associative algebra $A$ is PI-algebra if and only if $c_n(A) < n!$, for some positive integer $n$. It follows immediately from Propositions 4.13 and 4.14 that the same is true if we consider $\pi_n(A)$ in place of $c_n(A)$.

**Lemma 4.15.** *If $A$ satisfies a polynomial identity of degree $n$, then $\pi_n(A) \leq c_n(A) < n!$. Conversely, if $\pi_n(A) < n!$, then $A$ satisfies a polynomial identity of degree $n!n$.*

This new characterization of PI-algebras in terms of the sequence $\pi_n(A)$ will be key to prove our main result in Section 4.4.

### 4.3.1 Computing $\pi_n(A)$

In this subsection, we compute the values of $\pi_n(A)$ for the $2 \times 2$ matrix algebra, $M_2(k)$, and the Grassmann algebra of a countably infinite-dimensional vector space.

**Example 4.16.** Let $M_2(k)$ denote the algebra of all $2 \times 2$ matrices with entries from $k$. The codimension sequence of $M_2(k)$ in characteristic zero was shown by Procesi in [19] to be:

$$c_n(M_2(k)) = \frac{1}{n+2}\binom{2n+2}{n+1} - \binom{n}{3} + 1 - 2^n, \text{ for all } n \geq 3.$$

In contrast, one can quickly check that $\pi_n(M_2(k)) = 4$, for all $n \geq 3$, by hand, by considering the matrices

$$a_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \; a_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \; a_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Further computations carried out in SageMath motivate the following conjecture.

**Conjecture 4.17.** For all positive integers $n, k$,

$$\pi_n\left(M_k(\mathbb{Q})\right) = \min\{k^2, n!\}.$$

**Proposition 4.18.** *Let $E$ be the Grassmann algebra of an infinite-dimensional vector space $V$ over a field $k$ of characteristic different from 2. Then $\pi_n(E) = c_n(E) = 2^{n-1}$.*

*Proof.* Because the $n$-th codimension of $E$ is known to equal $2^{n-1}$ by a theorem of Krakowski and Regev (see [17]), it suffices by Proposition 4.14 to show that $2^{n-1} \le \pi_n(E)$. To see why this inequality holds, fix an ordered basis $\mathcal{E} = \{e_1, e_2, \dots\}$ for $V$. Recall that the ordered products of distinct basis elements from $\mathcal{E}$ form a basis $\mathcal{W}$ of $E$; the ordered products of even length form a basis $\mathcal{W}_0$ for the subspace denoted by $E_0$, while the ordered products of odd length form a basis $\mathcal{W}_1$ for the subspace denoted by $E_1$. In this way, $E = E_0 \oplus E_1$ is a $\mathbb{Z}_2$-grading of $E$, where $E_0$ is central in $E$ and elements $b_1, b_2$ in $E_1$ anticommute: $b_1 b_2 = -b_2 b_1$.

For each positive integer $n$, let $y_n = e_{3n-2}e_{3n-1}$, $z_n = e_{3n}$, and set $x_n = y_n + z_n$. We will show that

$$\pi_n(x_1, \dots, x_n) = 2^{n-1};$$

it suffices to exhibit a linearly independent set $X_n$ consisting of $2^{n-1}$ elements of the form $x_{\sigma(1)} \cdots x_{\sigma(n)}$, with $\sigma \in S_n$. To simplify the notation, for each $\omega = x_{\sigma(1)} \cdots x_{\sigma(n)}$, let $\overline{\omega}$ denote the word obtained from $\omega$ by permuting the last two factors:

$$\overline{\omega} = x_{\sigma(1)} \cdots x_{\sigma(n-2)} x_{\sigma(n)} x_{\sigma(n-1)}.$$

Let $X_1 = \{x_1\}$ and, for each positive integer $n \geq 2$, let

$$X_n = \{\omega_1 x_n < \cdots < \omega_{2^{n-2}} x_n < \overline{\omega_1 x_n} < \cdots < \overline{\omega_{2^{n-2}} x_n}\},$$

where $X_{n-1} = \{\omega_1 < \cdots < \omega_{2^{n-2}}\}$. We can easily verify that $X_2$ is linearly independent. Indeed, one can easily check that the coordinate matrix

$$M_2 = \left[ \left[ x_1 x_2 \right]_{\mathcal{W}} \mid \left[ \overline{x_1 x_2} \right]_{\mathcal{W}} \right]$$

in the standard basis $\mathcal{W}$ has rank 2:

| | $x_1x_2$ | $\overline{x_1x_2}$ |
|---|---|---|
| $y_1y_2$ | 1 | 1 |
| $z_1y_2$ | 1 | 1 |
| $y_1z_2$ | 1 | 1 |
| $z_1z_2$ | 1 | −1 |

Similarly, if $n = 3$, the coordinate matrix

| | $x_1x_2x_3$ | $x_2x_1x_3$ | $\overline{x_1x_2x_3}$ | $\overline{x_2x_1x_3}$ |
|---|---|---|---|---|
| $y_1y_2y_3$ | 1 | 1 | 1 | 1 |
| $z_1y_2y_3$ | 1 | 1 | 1 | 1 |
| $y_1z_2y_3$ | 1 | 1 | 1 | 1 |
| $z_1z_2y_3$ | 1 | −1 | 1 | −1 |
| | | | | |
| $y_1y_2z_3$ | 1 | 1 | 1 | 1 |
| $z_1y_2z_3$ | 1 | 1 | 1 | −1 |
| $y_1z_2z_3$ | 1 | 1 | −1 | 1 |
| $z_1z_2z_3$ | 1 | −1 | −1 | 1 |

corresponding to

$$M_3 = \left[ \left[ x_1x_2x_3 \right]_{\mathcal{W}} \mid \left[ x_2x_1x_3 \right]_{\mathcal{W}} \mid \left[ \overline{x_1x_2x_3} \right]_{\mathcal{W}} \mid \left[ \overline{x_2x_1x_3} \right]_{\mathcal{W}} \right]$$

has rank 4. Note that $M_3$ has the form:

$$\left[ \begin{array}{c|c} M_2 & M_2 \\ \hline M_2 & * \end{array} \right]$$

Next, observe that each element in $X_n$ is a linear combination of $2^n$ basis elements in $\mathcal{W}$; the linear ordering $y_1 < \cdots < y_n < z_1 < \cdots < z_n$ induces a right-to-left lexicographic ordering on these basis elements. Now, consider the coordinate matrix $M_n$ of $X_n$ with respect to this ordered basis, where columns 1 through $2^{n-2}$ correspond to elements $\omega_1 x_n, \ldots, \omega_{2^{n-2}} x_n$, with $\omega_i \in X_{n-1}$, while columns $2^{n-2}+1$ through $2^{n-1}$ correspond to elements of the form $\overline{\omega_1 x_n}, \ldots, \overline{\omega_{2^{n-2}} x_n}$, with $\omega_i \in X_{n-1}$. Then, $M_n$ has the form:

$$\left[ \begin{array}{c|c} M_{n-1} & M_{n-1} \\ \hline M_{n-1} & * \end{array} \right]$$

where the lower right matrix is of the form

$$\left[\begin{array}{c:c} M_{n-2} & \begin{array}{c:c} M_{n-3} & \begin{array}{c:c} M_{n-4} & \begin{array}{c} \begin{array}{c:c} M_2 & M \\ \hdashline -M_2 & * \end{array} \\ \iddots \end{array} \\ \hdashline -M_{n-4} & * \end{array} \\ \hdashline -M_{n-3} & * \end{array} \\ \hdashline -M_{n-2} & * \end{array}\right]$$

with

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}$$

Using the fact that

$$M_{n-1} = \left[\begin{array}{c:c} M_{n-2} & M_{n-2} \\ \hdashline M_{n-2} & * \end{array}\right], M_{n-2} = \left[\begin{array}{c:c} M_{n-3} & M_{n-3} \\ \hdashline M_{n-3} & * \end{array}\right], \ldots, M_3 = \left[\begin{array}{c:c} M_2 & M_2 \\ \hdashline M_2 & * \end{array}\right],$$

we can perform elementary row and column operations to transform $M_n$ into a matrix of the form

$$\left[\begin{array}{c:c} M_{n-1} & 0 \\ \hdashline 0 & * \end{array}\right]$$

Indeed, write $R_i$ and $C_j$ to denote the $i$-th row and $j$-th column of $M_n$, respectively. Perform the following elementary column operations

$$\begin{gathered} C_{2^{n-2}+1} - C_1 \to C_{2^{n-2}+1} \\ \vdots \\ C_{2^{n-1}} - C_{2^{n-2}} \to C_{2^{n-1}} \end{gathered}$$

followed by the following elementary row operations

$$\begin{gathered} R_{2^{n-1}+1} - R_1 \to R_{2^{n-1}+1} \\ \vdots \\ R_{2^n} - R_{2^{n-1}} \to R_{2^n} \end{gathered}$$

In this way, we can obtain a matrix of the form indicated above, where the lower right matrix corresponds to

$$
\left[\begin{array}{c:c}
0 & \begin{array}{c:c}
0 & \begin{array}{c:c}
0 & \begin{array}{c} \begin{array}{c:c} 0 & M' \\ \hdashline -2M_2 & * \end{array} \\ \iddots \end{array} \\ \hdashline
-2M_{n-4} & *
\end{array} \\ \hdashline
-2M_{n-3} & *
\end{array} \\ \hdashline
-2M_{n-2} & *
\end{array}\right]
$$

with

$$
M' = \begin{bmatrix} 0 & 0 \\ 0 & -2 \\ -2 & 0 \\ -2 & 2 \end{bmatrix}
$$

Thus, it is clear that

$$
\operatorname{rank}(M_n) = 2 + \sum_{k=2}^{n-1} \operatorname{rank}(M_k).
$$

Using strong induction, we can assume $\mathrm{rank}(M_k) = 2^{k-1}$, for all $2 \leq k \leq n-1$. Then,

$$\sum_{k=2}^{n-1} \mathrm{rank}(M_k) = \sum_{k=2}^{n-1} 2^{k-1} = 2^{n-1} - 2,$$

and $\mathrm{rank}(M_n) = 2^{n-1}$, as required. □

### 4.3.2 A new 'generic' polynomial for PI-algebras

To conclude this section, we present an interesting application. By a well-known theorem of Amitsur ([1]), every PI-algebra satisfies some power

$$s_n(x_1, \dots, x_n)^d \equiv 0$$

of the standard identity, where

$$s_n(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)};$$

thus, every PI-algebra satisfies a polynomial identity of a 'generic' type. Using a different approach, Regev was able to provide a quantitative version of Amitsur's theorem in [20]. The proof of Proposition 4.13 provides us with another such generic polynomial identity:

**Corollary 4.19.** *If an algebra $A$ satisfies an arbitrary polynomial identity of degree $d$, then $A$ satisfies the following specific polynomial identity of degree $d!d$:*

$$s_{d!}\left(x_{\sigma_1(1)} \cdots x_{\sigma_1(d)}, \dots, x_{\sigma_{d!}(1)} \cdots x_{\sigma_{d!}(d)}\right) \equiv 0,$$

*where $S_d = \{\sigma_i \colon 1 \leq i \leq d!\}$.*

*Proof.* As a consequence of the multilinearization process, if $A$ satisfies a polynomial identity of degree $d$, then it satisfies a multilinear identity of degree $d$. Thus, $\pi_d(A) < d!$. It remains to examine the proof of Proposition 4.13. □

## 4.4 Conditions on the $R$-action

In this section, we will exploit the tools developed in Section 4.3 to study the implications of the following condition on a generalized $R$-algebra $A$.

**Definition 4.20.** Let $A$ be an associative algebra equipped with an $R$-module action. We shall say $A$ is *$R$-rewritable of degree $d$* if, for every $a_1, \ldots, a_d \in A$,

$$a_1 \cdots a_d \in \operatorname{span}_k \left\{ \left(R \cdot a_{\sigma(1)}\right) \cdots \left(R \cdot a_{\sigma(d)}\right) : 1 \neq \sigma \in S_d \right\}.$$

Moreover, we shall say that $A$ is *$R$-rewritable* if $A$ is $R$-rewritable of degree $d$, for some $d$.

Clearly, if $A$ satisfies an $R$-identity of degree $d$ rewriting $A$, then $A$ is $R$-rewritable of degree $d$. Our goal in this section is to extend Theorem 4.8 to $R$-rewritable algebras. Our results will be valid for algebras equipped with the following type of $R$-module actions.

**Definition 4.21.** Let $A$ be a not necessarily associative algebra equipped with an $R$-module action.

1. We shall say that $A$ is a *hypomorphic $R$-algebra* if, for all $a_1, a_2 \in A$,

$$R \cdot (a_1 a_2) \subseteq \operatorname{span}_k \left\{ (R \cdot a_1)(R \cdot a_2) + (R \cdot a_2)(R \cdot a_1) \right\}.$$

2. We shall say that $A$ is a *positive hypomorphic $R$-algebra* if, for all $a, b \in A$,

$$R \cdot (a_1 a_2) \subseteq \operatorname{span}_k \left\{ (R \cdot a_1)(R \cdot a_2) \right\}.$$

Notice that, while all (positive) generalized $R$-algebras are clearly (positive) hypomorphic $R$-algebras, there is no reason to expect that the converse should hold.

**Theorem 4.22.** *Let $A$ be an associative algebra equipped with a hypomorphic $R$-action $\rho\colon R \to \mathrm{End}_k(A)$ with the property that $m = \dim_k \rho(R)$ is finite. If $A$ is $R$-rewritable of degree $d$, then $\pi_t(A) < t!$, where $t = \kappa(d, m)$.*

*Proof.* Fix elements $a_1, \ldots, a_t \in A$. We will show that $\pi_t(a_1, \ldots, a_t) < t!$. Consider the vector space

$$W = \mathrm{span}_k \left\{ a_{\sigma(1)}^{r_1} \cdots a_{\sigma(t)}^{r_t} \colon \sigma \in S_t \text{ is } d\text{-indecomposable}, r_1, \ldots, r_t \in R \right\}.$$

By Lemma 2.25, for each $n \geq \kappa(d, m)$, the number of $d$-indecomposable permutations in $S_n$ is strictly smaller than $\left(\frac{1}{m}\right)^n n!$. Hence, because $t = \kappa(d, m)$ and $m = \dim_k \rho(R)$, we have

$$\dim_k W < m^t \left(\frac{1}{m}\right)^t t! = t!$$

Thus, it suffices to prove that

$$\mathrm{span}_k \{a_{\sigma(1)} \cdots a_{\sigma(t)} \colon \sigma \in S_t\} \subseteq W.$$

Suppose, to the contrary, that there exists some $\sigma \in S_t$ for which $a_{\sigma(1)} \cdots a_{\sigma(t)} \notin W$, and fix the smallest permutation $\tau \in S_t$ for which it is possible to find $s_1, \ldots, s_t \in R$ such that $a_{\tau(1)}^{s_1} \cdots a_{\tau(t)}^{s_t} \notin W$. Then $\tau$ has a $d$-decomposition

$$1 \leq h_1 \leq t_1 < h_2 \leq t_2 < \cdots < h_d \leq t_d \leq t$$

and we can partition $a = a_{\tau(1)}^{s_1} \cdots a_{\tau(t)}^{s_t}$ accordingly:

$$a = \underbrace{\left(a_{\tau(1)}^{s_1} \cdots\right)}_{u} \underbrace{\left(a_{\tau(h_1)}^{s_{h_1}} \cdots a_{\tau(t_1)}^{s_{t_1}}\right)}_{w_1} \cdots \underbrace{\left(a_{\tau(h_d)}^{s_{h_d}} \cdots a_{\tau(t_d)}^{s_{t_d}}\right)}_{w_d} \underbrace{\left(\cdots a_{\tau(t)}^{s_t}\right)}_{v}.$$

Since $A$ is $R$-rewritable of degree $d$, we have

$$w_1 \cdots w_d \in \operatorname{span}_k \{(R \cdot w_{\sigma(1)}) \cdots (R \cdot w_{\sigma(d)}) : 1 \neq \sigma \in S_d\}.$$

Consequently, because $A$ is a hypomorphic $R$-algebra by hypothesis, the defining conditions of the $d$-decomposition of $\tau$ force $a$ to be a $k$-linear combination of elements of the form $a_{\sigma(1)}^{r_1} \cdots a_{\sigma(t)}^{r_t}$, where $\sigma < \tau$ and $r_1, \ldots, r_t \in R$. But every element of this form lies in $W$ (by the minimality of $\tau$); hence, so does $a$. This contradiction proves that, for every $\sigma \in S_t$, $a_{\sigma(1)} \cdots a_{\sigma(t)} \in W$; therefore, we have $\pi_t(a_1, \ldots, a_t) \leq \dim_k W < t!$, as desired. □

If $A$ is equipped with a positive $R$-hypomorphic action, we can improve the bound on the degree in Theorem 4.22 as follows:

**Theorem 4.23.** *Let $A$ be an associative algebra equipped with a positive hypomorphic $R$-action $\rho\colon R \to End_k(A)$ with the property that $m = \dim_k \rho(R)$ is finite. If $A$ is an $R$-rewritable algebra of degree $d$, then $\pi_t(A) < t!$, where $t = \lceil em(d-1)^2 \rceil$ ($e$ denotes the base of the natural logarithm).*

Our proof of Theorem 4.23 is very similar to that of Theorem 4.22 but uses the simpler notion of $d$-good permutations in place of $d$-indecomposable permutations.

*Proof.* Fix elements $a_1, \ldots, a_t \in A$. We will show that $\pi_t(a_1, \ldots, a_t) < t!$. Consider the vector space

$$W = \operatorname{span}_k \left\{ a_{\sigma(1)}^{r_1} \cdots a_{\sigma(t)}^{r_t} : \sigma \in S_t \text{ is } d\text{-good}, r_1, \ldots, r_t \in R \right\}.$$

By Lemma 2.23, the number of $d$-good permutations in $S_t$ does not exceed $\frac{(d-1)^{2t}}{(d-1)!}$. Hence,

$$\dim_k W \leq m^t \frac{(d-1)^{2t}}{(d-1)!}$$

It is easy to see that, for every positive integer $n$, $\left(\frac{n}{e}\right)^n < n!$. From this inequality, we deduce that

$$m^t(d-1)^{2t} \le \left(\frac{t}{e}\right)^t < t!$$

Thus, our proof will be complete once we show that

$$\operatorname{span}_k \{a_{\sigma(1)} \cdots a_{\sigma(t)} : \sigma \in S_t\} \subseteq W.$$

Suppose, to the contrary, that there exists some $\sigma \in S_t$ for which $a_{\sigma(1)} \cdots a_{\sigma(t)} \notin W$, and fix the smallest permutation $\tau \in S_t$ for which it is possible to find $s_1, \ldots, s_t \in R$ such that $a^{s_1}_{\tau(1)} \cdots a^{s_t}_{\tau(t)} \notin W$. Because $\tau$ is $d$-bad, we can find a sequence of integers $1 \le h_1 < \cdots < h_d \le t$ such that $\tau(h_1) > \cdots > \tau(h_d)$. Partition $a = a^{s_1}_{\tau(1)} \cdots a^{s_t}_{\tau(t)}$ accordingly:

$$a = \underbrace{\left(a^{s_1}_{\tau(1)} \cdots\right)}_{u}\underbrace{\left(a^{s_{h_1}}_{\tau(h_1)} \cdots\right)}_{w_1}\underbrace{\left(a^{s_{h_2}}_{\tau(h_2)} \cdots\right)}_{w_2} \cdots \underbrace{\left(a^{s_{h_d}}_{\tau(h_d)} \cdots a^{s_t}_{\tau(t)}\right)}_{w_d}.$$

Since $A$ is $R$-rewritable of degree $d$, we have

$$w_1 \cdots w_d \in \operatorname{span}_k \left\{(R \cdot w_{\sigma(1)}) \cdots (R \cdot w_{\sigma(d)}) : 1 \ne \sigma \in S_d\right\}.$$

Consequently, because $A$ is a positive hypomorphic $R$-algebra by hypothesis, the defining conditions of the $d$-bad permutation $\tau$ force $a$ to be a $k$-linear combination of elements of the form $a^{r_1}_{\sigma(1)} \cdots a^{r_t}_{\sigma(t)}$, where $\sigma < \tau$ and $r_1, \ldots, r_t \in R$. But every element of this form lies in $W$ (by the minimality of $\tau$); hence, so does $a$. This contradiction proves that, for every $\sigma \in S_t$, $a_{\sigma(1)} \cdots a_{\sigma(t)} \in W$; therefore, we have $\pi_t(a_1, \ldots, a_t) \le \dim_k W < t!$, as desired. □

The next theorem is the main result of this chapter. It ties together the results in Sections 4.3 and 4.4.

**Theorem 4.24.** *Let $A$ be an associative hypomorphic $R$-algebra such that the algebra of endomorphisms on $A$ defined by the $R$-action is $m$-dimensional. Denote by $\sigma_1 < \cdots < \sigma_{t!}$ the distinct permutations in $S_t$, listed in the standard lexicographical order. If $A$ is $R$-rewritable of* *degree $d$, then $A$ satisfies the following classical polynomial identity of degree $t!t$:*

$$\sum_{\tau \in S_{t!}} \operatorname{sgn}(\tau) \left( x_{\sigma_{\tau(1)}(1)} \cdots x_{\sigma_{\tau(1)}(t)} \right) \cdots \left( x_{\sigma_{\tau(t!)}(1)} \cdots x_{\sigma_{\tau(t!)}(t)} \right) \equiv 0,$$

*where $t = \kappa(d, m)$. If the action is positive, we may take $t = \lceil em(d-1)^2 \rceil$.*

# Chapter 5

# Polynomial identities of Lie algebras with actions

Given a Lie algebra equipped with an action, what conditions ensure that it is a Lie PI-algebra? In this chapter, we investigate this question; our research will lead us to the natural Lie-theoretic analogues of Theorems 4.8 and 4.24. Before we can formulate our main results, we formally introduce the basic notions and notation.

## 5.1 Lie PI-algebras

We remind the reader that a (non-associative) algebra $L$ is called a *Lie algebra* if it satisfies both the *anticommutative law* and the *Jacobi identity*; that is, for all $a_1, a_2, a_3 \in L$,

1. $a_1^2 = 0$ (anticommutative law)

2. $(a_1 a_2)a_3 + (a_2 a_3)a_1 + (a_3 a_1)a_2 = 0$ (Jacobi identity)

**Example 5.1.** Let $A$ be an associative algebra. Denote by $A^{(-)}$ the vector space $A$ together with the operation

$$[a_1, a_2] = a_1 a_2 - a_2 a_1, \text{ for all } a_1, a_2 \in A.$$

It is easy to see that $A^{(-)}$ is a Lie algebra.

**Example 5.2.** Suppose $A$ is a (non-associative) algebra. A linear map $\delta\colon A \to A$ is called a *derivation* of $A$ if

$$\delta(a_1a_2) = \delta(a_1)a_2 + a_1\delta(a_2), \text{ for all } a_1, a_2 \in A.$$

Let $\mathrm{Der}_k(A)$ denote the set of all derivations of $A$. Given $\delta_1, \delta_2 \in \mathrm{Der}_k(A)$, the linear map $\delta_1\delta_2$ is generally not a derivation. However,

$$[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$$

is easily seen to be a derivation. Hence, $\mathrm{Der}_k(A)$ is a Lie subalgebra of $\mathrm{End}_k(A)^{(-)}$.

In order to define a polynomial identity for $L$, we introduce the free Lie algebra on the ordered set of non-commutative indeterminates $X = \{y < x_0 < x_1 < \cdots\}$. Let $F\langle X\rangle$ denote the free non-associative algebra on $X$, and let $I$ denote the smallest ideal of $F\langle X\rangle$ containing all elements of the form $f^2$ and $(fg)h + (hf)g + (gh)f$, where $f, g, h \in F\langle X\rangle$. The quotient algebra $\mathscr{L}\langle X\rangle = F\langle X\rangle / I$ is called *the free Lie algebra generated by $X$*.

**Proposition 5.3.** *Given a Lie algebra $L$, any mapping $\varphi\colon X \to L$ extends uniquely to a homomorphism of Lie algebras $\overline{\varphi}\colon \mathscr{L}\langle X\rangle \to L$ such that $\overline{\varphi}(x) = \varphi(x)$, for all $x \in X$.*

**Remark 5.4.** By Proposition 5.3, the identity map $\iota\colon X \to X$ extends to a homomorphism of Lie algebras $\bar{\iota}\colon \mathscr{L}\langle X\rangle \to k\langle X\rangle^{(-)}$. It can be seen that $\bar{\iota}$ is injective.

Henceforth, we use Remark 5.4 to identify $\mathscr{L}\langle X\rangle$ with the Lie subalgebra of $k\langle X\rangle^{(-)}$ generated by $X$. Elements of $\mathscr{L}\langle X\rangle$ are called *Lie polynomials*. A commutator of elements of $X$ is called a *Lie monomial*. For all $n \geq 3$, the left-normed commutator is defined inductively as

$$[x_{i_1}, \ldots, x_{i_n}] = [[x_{i_1}, \ldots, x_{i_{n-1}}], x_{i_n}].$$

**Definition 5.5.** Given a polynomial $f \in \mathscr{L}\langle X\rangle$, we shall say that $f \equiv 0$ is an identity for a Lie algebra $L$ if

$$f \in \bigcap \ker \overline{\varphi},$$

where the intersection runs over all maps $\varphi\colon X \to L$; the maps $\overline{\varphi}\colon \mathscr{L}\langle X\rangle \to L$ are defined as in Proposition 5.3.

The set of polynomial identities of $L$, denoted $\mathrm{Id}(L)$, forms a two-sided ideal of $\mathscr{L}\langle X\rangle$ which is invariant under all endomorphisms of $\mathscr{L}\langle X\rangle$.

**Definition 5.6.** We shall say that a Lie algebra $L$ is a PI-algebra if $\mathrm{Id}(L) \neq 0$.

**Example 5.7.** If $L$ is a finite dimensional Lie algebra with the property that $\dim_k(L) < n$, then $L$ satisfies the *standard Lie identity* of degree $n+1$:

$$\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\,[x_0, x_{\sigma(1)}, \ldots, x_{\sigma(n)}] \equiv 0.$$

Because we can identify every Lie polynomial $f \in \mathscr{L}\langle X\rangle$ with an associative polynomial in $k\langle X\rangle$, we can define $\deg(f)$ and speak of *multilinear Lie polynomials* as in the associative case.

**Definition 5.8.** The space of multilinear Lie polynomials in $x_0, x_1, \ldots, x_n$ will be denoted by $Q_n$.

The contents of the following proposition are well known (see Proposition 12.2.6 in [12]).

**Proposition 5.9.**

1. $\dim_k Q_n = n!$

2. *Any multilinear Lie polynomial $f$ in $x_0, x_1 \ldots, x_n$ is a linear combination of the monomials*

$$[x_0, x_{\sigma(1)}, \ldots, x_{\sigma(n)}],\ \sigma \in S_n. \tag{5.1}$$

3. *The elements* (5.1) *are linearly independent over $k$.*

Observe that there may be dependence relations between different Lie monomials. For instance,

$$[[x_1, x_2], [x_3, x_4]] - [x_1, [x_2, [x_3, x_4]]] + [x_2, [x_1, [x_3, x_4]]] = 0.$$

**Definition 5.10.** For each positive integer $n$, we will write $c_n(L)$ to denote the $n$-th codimension of $L$; that is,

$$c_n(L) = \dim_k \left( \frac{Q_n}{Q_n \cap \mathrm{Id}(L)} \right).$$

Notice that $L$ satisfies a Lie polynomial identity of degree $n+1$ whenever $c_n(L) < n!$.

## 5.2 Conditions on the $R$-identities

In this section, we provide a Lie-theoretic analogue of Theorem 4.8. We will denote by $\mathscr{L}\langle X \,|\, R\rangle$ the free Lie algebra on the set of indeterminates $\{x^b \colon x \in X, b \in \mathcal{B}\}$. Elements of $\mathscr{L}\langle X \,|\, R\rangle$ will be called *Lie $R$-polynomials*. As usual, for each $x \in X$ and each $k$-linear combination of basis elements

$$r = \alpha_1 b_1 + \cdots + \alpha_n b_n \in R,$$

we will identify

$$x^r = \alpha_1 x^{b_1} + \cdots + \alpha_n x^{b_n} \text{ and } x^{1_R} = x.$$

Now, suppose we are given a Lie algebra $L$ equipped with an $R$-action.

**Proposition 5.11.** *Any map $\varphi \colon X \to L$ has a unique Lie $k$-algebra homomorphic extension $\bar{\varphi} \colon \mathscr{L}\langle X \,|\, R\rangle \to L$ such that, for all $x \in X$ and $b \in \mathcal{B}$, $\bar{\varphi}\left(x^b\right) = b \cdot \varphi(x)$.*

**Definition 5.12.** An $R$-polynomial $f \in \mathscr{L}\langle X \,|\, R\rangle$ will be called an *$R$-identity* for $L$ if, for all $\varphi \colon X \to L$, $\bar{\varphi}(f) = 0$; here, $\bar{\varphi}$ is defined as in Proposition 5.11. The set of $R$-identities of $L$ will be denoted $\mathrm{Id}(L \,|\, R)$.

**Definition 5.13.** For each positive integer $n$, let

$$Q_n^R = \operatorname{span}_k \left\{ \left[ x_0^{r_0}, x_{\sigma(1)}^{r_1}, \ldots, x_{\sigma(n)}^{r_n} \right] : r_0, r_1, \ldots, r_n \in R, \sigma \in S_n \right\}.$$

The *n-th R-codimension of L*, denoted $c_n^R(L)$, is given by

$$c_n^R(L) = \dim_k \frac{Q_n^R}{Q_n^R \cap \operatorname{Id}(L \mid R)}.$$

**Lemma 5.14.** *For all positive integers n,*

$$c_n(L) = \dim_k \frac{Q_n + Q_n^R \cap \operatorname{Id}(L \mid R)}{Q_n^R \cap \operatorname{Id}(L \mid R)} \leq c_n^R(L).$$

*Proof.* We have $Q_n \subseteq Q_n^R$; hence,

$$Q_n \cap \operatorname{Id}(L) = Q_n \cap \left( Q_n^R \cap \operatorname{Id}(L|R) \right).$$

It follows that

$$\frac{Q_n}{Q_n \cap \operatorname{Id}(L)} \cong \frac{Q_n + \left( Q_n^R \cap \operatorname{Id}(L|R) \right)}{Q_n^R \cap \operatorname{Id}(L|R)} \leq \frac{Q_n^R}{Q_n^R \cap \operatorname{Id}(L|R)},$$

as vector spaces. □

**Definition 5.15.** We will say that an $R$-identity of $L$ of the form

$$[x_0, x_1, \ldots, x_d] - \sum \left[ x_0^{r_0}, x_{\sigma(1)}^{r_1}, \ldots, x_{\sigma(d)}^{r_d} \right] \equiv 0,$$

*rewrites L* if

$$\sum \left[ x_0^{r_0}, x_{\sigma(1)}^{r_1}, \ldots, x_{\sigma(d)}^{r_d} \right]$$

is an element of $Q_d^R$ without any terms of the form $\left[ x_0^{r_0}, x_1^{r_1}, \ldots, x_d^{r_d} \right]$.

Suppose now that $L$ is a generalized $R$-algebra (see Definition 3.5); observe that because we are dealing with a Lie algebra $L$, this is equivalent to the assumption that for each $r \in R$, there exists finitely many elements $(r_1, r_2) \in R^2$ such that, for all $a_1, a_2 \in L$,

$$r \cdot [a_1, a_2] = \sum_{(r_1, r_2)} [r_1 \cdot a_1, r_2 \cdot a_2] .$$

Indeed, observe that the anti-commutativity property of $L$ implies that, for each $a_1, a_2 \in L$, $a_1 a_2 = -a_2 a_1$:

$$a_1 a_2 + a_2 a_1 = a_1^2 + a_1 a_2 + a_2 a_1 + a_2^2 = (a_1 + a_2)^2 = 0.$$

Thus, there is no distinction between generalized $R$-algebras and positive generalized $R$-algebras.

We shall see that if the algebra of endomorphisms on $L$ defined by a generalized $R$-action is finite dimensional, and $L$ satisfies an $R$-identity of degree $d + 1$ rewriting $L$, then $L$ satisfies an ordinary polynomial identity. We will need to work over $\mathscr{L} \langle X \,|\, R \rangle$; in particular, it will be necessary to make sense of expressions of the form

$$f(g_1, \ldots, g_n), \text{ where } f, g_1, \ldots, g_n \in \mathscr{L} \langle X \,|\, R \rangle.$$

In principle, such expression may not be defined because there may not be a natural $R$-action on $\mathscr{L} \langle X \,|\, R \rangle$; nonetheless, as in the associative case, we can take advantage of the generalized $R$-algebra structure of $L$ to fix a linear map $\lambda \colon R \to \operatorname{End}_k (\mathscr{L} \langle X \,|\, R \rangle)$ which can then be used to give a precise meaning to these expressions.

**Definition 5.16.** Fix a generalized $R$-algebra $L$. For each $b \in \mathcal{B}$ and $n \geq 2$, fix a choice of finitely many elements $(b_1, \ldots, b_n)$ in $R^n$ satisfying

$$b \cdot [a_1, \ldots, a_n] = \sum_{(b_1, \ldots, b_n)} [b_1 \cdot a_1, \ldots, b_n \cdot a_n] ,$$

for all $a_1, \ldots, a_n \in L$; then, for each $\left[x_{i_1}^{e_1}, \ldots, x_{i_n}^{e_n}\right]$ in a basis of left-justifed Lie monomials in

$\mathscr{L}\langle X \mid R\rangle$, define

$$b \cdot \left[x_{i_1}^{e_1}, \ldots, x_{i_n}^{e_n}\right] = \sum_{(b_1,\ldots,b_n)} \left[x_{i_1}^{b_1 e_1}, \ldots, x_{i_n}^{b_n e_n}\right].$$

1. These assignments induce a well-defined linear map; denote it by $\lambda_L \colon R \to \mathrm{End}_k(\mathscr{L}\langle X \mid R\rangle)$.

2. For each $r \in R$ and $f \in \mathscr{L}\langle X \mid R\rangle$, we shall write $f^r$ for $\lambda_L(r)(f)$.

Henceforth, every Lie algebra $L$ equipped with a generalized $R$-action will be implicitly equipped with a fixed linear map $\lambda_L \colon R \to \mathrm{End}_k(\mathscr{L}\langle X \mid R\rangle)$. We may regard expressions of the form $f(g_1, \ldots, g_n)$ as well-defined $R$-polynomials in $\mathscr{L}\langle X \mid R\rangle$ using $\lambda_L$ as follows.

**Definition 5.17.** Given $g_1, \ldots, g_n \in \mathscr{L}\langle X \mid R\rangle$, fix a map $\varphi \colon X \to \mathscr{L}\langle X \mid R\rangle$ with the property that $\varphi(x_1) = g_1, \ldots, \varphi(x_n) = g_n$. This map extends uniquely to a homomorphism of Lie algebras $\bar{\varphi} \colon \mathscr{L}\langle X \mid R\rangle \to \mathscr{L}\langle X \mid R\rangle$ such that, for each $x \in X$ and $b \in \mathcal{B}$, $\bar{\varphi}\left(x^b\right) = \lambda_L(b)(\varphi(x))$. For each $f(x_1, \ldots, x_n) \in \mathscr{L}\langle X \mid R\rangle$, we define $f(g_1, \ldots, g_n) = \bar{\varphi}(f)$.

Moreover, we have:

**Lemma 5.18.** *Suppose $f(x_1, \ldots, x_n) \equiv 0$ is an $R$-identity for a generalized $R$-algebra $L$. Then, for any given $g_1, \ldots, g_n \in \mathscr{L}\langle X \mid R\rangle$, $f(g_1, \ldots, g_n) \equiv 0$ is also an $R$-identity for $L$.*

Lastly, we present the following technical lemma. It follows by expanding $w'$ into a linear combination of left-justified Lie monomials using the Jacobi identity.

**Lemma 5.19.** *Fix elements $a_0, a_1, \ldots, a_n$ in a Lie algebra $L$ and a permutation $\sigma \in S_n$. Consider the sequence of integers*

$$1 \le h_1 \le t_1 < h_2 \le t_2 < \cdots < h_d \le t_d \le n,$$

*where $t_1 = h_2 - 1, \ldots, t_{d-1} = h_d - 1$. Let $w = \left[a_0, a_{\sigma(1)}, \ldots, a_{\sigma(n)}\right]$, $w_0 = \left[a_0, a_{\sigma(1)}, \ldots, a_{\sigma(h_1-1)}\right]$,*

$$w_1 = \left[a_{\sigma(h_1)}, \ldots, a_{\sigma(t_1)}\right], \ldots, w_d = \left[a_{\sigma(h_d)}, \ldots, a_{\sigma(t_d)}\right],$$

*and* $w' = [w_0, w_1, \dots, w_d, \dots, a_{\sigma(n)}]$. *If, for all* $h_i < k \le t_i$ *and* $1 \le i \le d$, *we have* $\sigma(h_i) > \sigma(k)$, *then*

$$w - w' \in \operatorname{span}_k \{[a_0, a_{\sigma'(1)}, \dots, a_{\sigma'(n)}] : \sigma' < \sigma\}.$$

We are now ready to prove the main result of this section.

**Theorem 5.20.** *Let $L$ be a Lie algebra equipped with a generalized $R$-algebra action given by* $\rho\colon R \to End_k(L)$ *with the property that* $m = \dim \rho(R)$ *is finite. If*

$$[x_0, x_1, \dots, x_d] - \sum_{1 \ne \sigma \in S_d} \sum_r \left[x_0^{r_0}, x_{\sigma(1)}^{r_{\sigma(1)}}, \dots, x_{\sigma(d)}^{r_{\sigma(d)}}\right] \equiv 0 \tag{5.2}$$

*is an $R$-identity of degree* $d + 1$ *rewriting $L$, then $L$ satisfies an ordinary polynomial identity of degree* $n = \kappa(d, m + 1)$.

*Proof.* We can replace $R$ by $\rho(R)$ to assume $R$ is $m$-dimensional. Let $I = Q_n^R \cap \mathrm{Id}(L \,|\, R)$, and let

$$W = \operatorname{span}_k \left\{\left[x_0^{r_0}, x_{\upsilon(1)}^{r_1}, \dots, x_{\upsilon(n)}^{r_n}\right] : r_0, r_1, \dots, r_n \in R, \upsilon \in S_n \text{ is } d\text{-indecomposable}\right\}.$$

We claim that $Q_n \le W + I$. Suppose that this were not the case, and let $\tau \in S_n$ denote the least permutation for which $[x_0, x_{\tau(1)}, \dots, x_{\tau(n)}] \notin W + I$. Then $\tau$ is $d$-decomposable; so, we can fix a $d$-decomposition $1 \le h_1 \le t_1 < \cdots < h_d \le t_d \le n$ of $\tau$ and set

$$w = \Big[\underbrace{[x_0, x_{\tau(1)}, \dots, x_{\tau(h_1-1)}]}_{w_0}, \underbrace{[x_{\tau(h_1)}, \dots, x_{\tau(t_1)}]}_{w_1}, \dots, \underbrace{[x_{\tau(h_d)}, \dots, x_{\tau(t_d)}]}_{w_d}, x_{\tau(t_d+1)}, \dots, x_{\tau(n)}\Big].$$

In light of Lemma 5.19, we can find scalars $\alpha_\sigma \in k$ such that

$$[x_0, x_{\tau(1)}, \dots, x_{\tau(n)}] = w + \sum_{\sigma < \tau} \alpha_\sigma [x_0, x_{\sigma(1)}, \dots, x_{\sigma(n)}].$$

It follows by the minimality of $\tau$ that the linear sum on the right lies in $W + I$; thus, we will obtain the required contradiction if we can show that $w$ also lies in $W + I$. To this end, notice that, because Equation (5.2) is an $R$-identity for $L$, then, by Lemma 5.18, so is

$$w - \sum_{1\neq\sigma\in S_d} \sum_{r} \left[ w_0^{r_0}, w_{\sigma(1)}^{r_{\sigma(1)}}, \dots, w_{\sigma(d)}^{r_{\sigma(d)}}, x_{\tau(t_d+1)}, \dots, x_{\tau(n)} \right] \equiv 0. \tag{5.3}$$

Thus, to show $w \in W + I$, it suffices for us to prove that each summand

$$y_{\sigma,r} = \left[ w_0^{r_0}, w_{\sigma(1)}^{r_{\sigma(1)}}, \dots, w_{\sigma(d)}^{r_{\sigma(d)}}, x_{\tau(t_d+1)}, \dots, x_{\tau(n)} \right]$$

in (5.3) lies in $W + I$. Suppose we could show that

$$z_\sigma = [w_0, w_{\sigma(1)}, \dots, w_{\sigma(d)}, x_{\tau(t_d+1)}, \dots, x_{\tau(n)}]$$

lies in $W + I$. Then, by Lemma 5.18, every evaluation of $z_\sigma$ of the form $x_i \mapsto x_i^{s_i}$, $s_i \in R$, would also lie in $W + I$; since $L$ is a generalized $R$-algebra, it would follow that $y_{\sigma,r} \in W + I$. So, let $[x_0, x_{\upsilon(1)}, \dots, x_{\upsilon(n)}]$ be the left-justified Lie monomial corresponding to $z_\sigma$ in Lemma 5.19. Because $\sigma \neq 1$, the $d$-decomposition of $\tau$ forces $\upsilon < \tau$. Hence, by Lemma 5.19, $z_\sigma \in W + I$. It follows that $w \in W + I$, proving the claim.

Using the claim, we can bound $c_n(L)$ as follows:

$$c_n(L) = \dim_k \left( \frac{Q_n + I}{I} \right) \leq \dim_k \left( \frac{W + I}{I} \right) \leq m^{n+1} a_d(n).$$

Consequently, Lemma 2.27 applied to the assumption $n = \kappa(d, m + 1)$ allows us to conclude that $c_n(L) < n!$, as required. □

## 5.3 The sequence $\pi_n(L)$ for a Lie algebra $L$

The numerical sequence $\pi_n(A)$ was key to prove Theorem 4.24. In order to prove its Lie-theoretic analogue, we introduce the numerical sequence $\pi_n(L)$, for a given Lie algebra $L$.

**Definition 5.21.** For each $a_0, a_1, \ldots, a_n \in L$, let

$$\pi_n(a_0, a_1, \ldots, a_n) = \dim_k\left(\operatorname{span}_k\{[a_0, a_{\sigma(1)}, \ldots, a_{\sigma(n)}] \in L\colon \sigma \in S_n\}\right);$$

we define

$$\pi_n(L) = \max\{\pi_n(a_0, a_1, \ldots, a_n)\colon a_0, a_1, \ldots, a_n \in L\}.$$

**Proposition 5.22.** *For each positive integer $n$, $\pi_n(L) \leq c_n(L)$.*

*Proof.* Let $I_n = Q_n \cap \mathrm{Id}(L)$ and $c = c_n(L)$. Fix a basis

$$\left\{f_i(x_0, x_1, \ldots, x_n) + I_n, i = 1, \ldots, c\right\}$$

of $Q_n/I_n$. Then, for every permutation $\sigma \in S_n$, we can find $\alpha_{\sigma,i} \in k$ such that

$$[x_0, x_{\sigma(1)}, \ldots, x_{\sigma(n)}] - \sum_{i=1}^{c} \alpha_{\sigma,i}\, f_i(x_0, x_1, \ldots, x_n) \equiv 0.$$

Hence, for every choice of elements $a_1, \ldots, a_n \in L$, we have

$$\operatorname{span}_k\{[a_0, a_{\sigma(1)}, \ldots, a_{\sigma(n)}] : \sigma \in S_n\} \subseteq \operatorname{span}_k\{f_i(a_1, \ldots, a_n)\colon 1 \leq i \leq c\}$$

and our claim follows. □

**Proposition 5.23.** *Fix a positive integer $d$. If $L$ satisfies a polynomial identity of degree $d+1$, then $\pi_d(L) < d!$. Conversely, if $\pi_d(L) < d!$, then $L$ satisfies a polynomial identity of degree $(d+1)!+1$.*

*Proof.* If $L$ satisfies a polynomial identity of degree $d+1$, then it satisfies a multilinear identity $f(x_0, x_1, \ldots, x_d)$ of degree $d+1$. Hence, $f \in Q_d \cap \mathrm{Id}(L)$ forcing $c_d(L) < \dim_k Q_d = d!$. In this way, the first implication follows from Proposition 5.22.

To prove the converse, denote by $\sigma_1 < \cdots < \sigma_{d!}$ the distinct permutations in $S_d$, listed in the standard lexicographical order. Now, observe that $L$ satisfies the polynomial identity of degree $(d+1)! + 1$ given by

$$f(y, x_0, \ldots, x_d) = \sum_{\tau \in S_{d!}} \operatorname{sgn}(\tau)[y, w_{\tau(1)}, \ldots, w_{\tau(d!)}] \equiv 0,$$

where $w_i = [x_0, x_{\sigma_i(1)}, \ldots, x_{\sigma_i(d)}]$, for each $1 \le i \le d!$. Indeed, it is easy to see that $L$ vanishes on $f$ using the same sort of reasoning as in the proof of Proposition 4.13. It remains to verify that $f$ is nontrivial. For each $1 \le i \le d!$, let $\bar{w}_i = x_0 x_{\sigma_i(1)} \cdots x_{\sigma_i(d)}$; in other words, $\bar{w}_i$ is the associative monomial in $k\langle X\rangle$ obtained from $w_i$ by replacing the Lie brackets with associative products. Because $y\bar{w}_1 \cdots \bar{w}_{d!}$ is the unique smallest monomial appearing in the associative expansion of $f$ in $k\langle X\rangle$, $f$ is nontrivial. □

## 5.4 Conditions on the *R*-action

We now investigate general conditions on the $R$-action which ensure that $L$ is a Lie PI-algebra. Our main result, Theorem 5.26, is the Lie-theoretic analogue of Theorem 4.24.

Once again, it will be crucial for the $R$-action to be compatible with the multiplicative structure of $L$. Thus, we will focus on hypomorphic Lie $R$-algebras (see Definition 4.21). By the anti-commutativity property of $L$, there is no reason for us to distinguish between hypomorphic Lie $R$-algebras and positive hypomorphic Lie $R$-algebras.

**Definition 5.24.** We shall say that the Lie algebra $L$ is *$R$-rewritable of degree $d$* if, for all $a_0, a_1 \ldots, a_d \in L$,

$$[a_0, a_1, \ldots, a_d] \in \operatorname{span}_k\Big\{ [R \cdot a_0, R \cdot a_{\sigma(1)}, \ldots, R \cdot a_{\sigma(d)}] : 1 \neq \sigma \in S_d \Big\}.$$

We shall see that if the algebra of endomorphisms on $L$ defined by a hypomorphic $R$-action is finite dimensional, then knowing that $L$ is $R$-rewritable ensures that it is a Lie PI-algebra.

**Theorem 5.25.** *Let $L$ be a Lie hypomorphic $R$-algebra such that the algebra of endomorphisms on $L$ defined by the $R$-action is $m$-dimensional. If $L$ is $R$-rewritable of degree $d$, then $\pi_t(L) < t!$, where $t = \kappa(d, m+1)$.*

*Proof.* In order to prove that $\pi_t(L) < t!$, fix elements $a_0, a_1, \ldots, a_t \in L$; it suffices to show that $\pi_t(a_0, \ldots, a_t) < t!$. First, we prove that $\pi_t(a_0, \ldots, a_t)$ is bounded above by $\dim_k W$, where

$$W = \operatorname{span}_k \left\{ \left[ a_0^{r_0}, a_{\sigma(1)}^{r_1}, \ldots, a_{\sigma(t)}^{r_t} \right] : r_0, r_1, \ldots, r_t \in R, \sigma \in S_t \text{ is } d\text{-indecomposable} \right\}.$$

We claim that, for all $\sigma \in S_t$, $[a_0, a_{\sigma(1)}, \ldots, a_{\sigma(t)}] \in W$. Suppose that this is not the case. Let $\tau$ denote the least element in $S_t$ for which there exist $s_0, s_1, \ldots, s_t \in R$ such that $a = \left[ a_0^{s_0}, a_{\tau(1)}^{s_1}, \ldots, a_{\tau(t)}^{s_t} \right] \notin W$. Observe that $\tau$ is $d$-decomposable; fix a $d$-decomposition

$$1 \le h_1 \le t_1 < h_2 \le t_2 < \cdots < h_d \le t_d \le t$$

of $\tau$ and set

$$w_0 = \left[ a_0^{s_0}, \ldots, a_{\tau(h_1-1)}^{s_{h_1-1}} \right], w_1 = \left[ a_{\tau(h_1)}^{s_{h_1}}, \ldots, a_{\tau(t_1)}^{s_{t_1}} \right], \ldots, w_d = \left[ a_{\tau(h_d)}^{s_{h_d}}, \ldots, a_{\tau(t_d)}^{s_{t_d}} \right].$$

Let $a' = \left[ w_0, w_1, \ldots, w_d, \ldots, a_{\tau(t)}^{s_t} \right]$ and

$$V = \operatorname{span}_k \left\{ \left[ a_0^{r_0}, a_{\sigma(1)}^{r_1}, \ldots, a_{\sigma(t)}^{r_t} \right] : r_0, r_1, \ldots, r_t \in R, \sigma < \tau \right\} \subseteq W.$$

In light of Lemma 5.19, $a - a' \in V$. On the other hand, because $L$ is $R$-rewritable of degree $d$,

$$[w_0, w_1, \ldots, w_d] \in \operatorname{span}_k \left\{ [R \cdot w_0, R \cdot w_{\sigma(1)}, \ldots, R \cdot w_{\sigma(d)}] : 1 \ne \sigma \in S_d \right\}.$$

Moreover, because $L$ is an $R$-hypomorphic algebra, each element of the form $\left[ w_0^{r_0}, w_{\sigma(1)}^{r_1}, \ldots, w_{\sigma(d)}^{r_d} \right]$, with $r_0, \ldots, r_d \in R$, is a linear combination of elements of the form

$$\left[\left[a_0^{e_0}, \ldots, a_{\tau(h_1-1)}^{e_{h_1-1}}\right], \left[a_{\tau(h_{\sigma(1)})}^{e_{h_{\sigma(1)}}}, \ldots, a_{\tau(t_{\sigma(1)})}^{e_{t_{\sigma(1)}}}\right], \ldots, \left[a_{\tau(h_{\sigma(d)})}^{e_{h_{\sigma(d)}}}, \ldots, a_{\tau(t_{\sigma(d)})}^{e_{t_{\sigma(d)}}}\right]\right],$$

where each $e_j \in R$. It follows that $a'$ is a linear combination of elements of the form

$$\left[a_0^{e_0}, \ldots, a_{\tau(h_1-1)}^{e_{h_1-1}}, \left[a_{\tau(h_{\sigma(1)})}^{e_{h_{\sigma(1)}}}, \ldots, a_{\tau(t_{\sigma(1)})}^{e_{t_{\sigma(1)}}}\right], \ldots \right.$$
$$\left. \ldots, \left[a_{\tau(h_{\sigma(d)})}^{e_{h_{\sigma(d)}}}, \ldots, a_{\tau(t_{\sigma(d)})}^{e_{t_{\sigma(d)}}}\right], a_{\tau(t_d+1)}^{s_{t_d+1}}, \ldots, a_{\tau(t)}^{s_t}\right],$$

with $\sigma \neq 1$. Seeing that the sequence of integers $1 \leq h_1 \leq t_1 < h_2 \leq t_2 < \cdots < h_d \leq t_d \leq t$ is a $d$-decomposition for $\tau$, it is easy to see that every element of this form is contained in $V$ using Lemma 5.19. But then, so is $a'$ (and hence $a$), yielding the desired contradiction. This proves that $\pi_t(a_0, a_1, \ldots, a_t) \leq \dim_k W$.

Now, recall that $a_d(t)$ denotes the number of $d$-indecomposable permutations in $S_t$. Hence:

$$\dim_k W \leq m^{t+1} a_d(t).$$

Finally, Lemma 2.27 applied to the assumption $t = \kappa(d, m+1)$ allows us to conclude that $m^{t+1} a_d(t) < t!$, thus proving that $\pi_t(a_0, a_1, \cdots, a_t) < t!$, as required. □

Proposition 5.23 and Theorem 5.25 together yield the main result of this chapter.

**Theorem 5.26.** *Let $L$ be a Lie hypomorphic $R$-algebra such that the algebra of endomorphisms on $L$ defined by the $R$-action is $m$-dimensional. Denote by $\sigma_1 < \cdots < \sigma_{t!}$ the distinct permutations in $S_t$, listed in the standard lexicographical order. If $L$ is $R$-rewritable of degree $d$, then $L$ satisfies the following ordinary polynomial identity of degree $(t+1)! + 1$:*

$$\sum_{\tau \in S_{t!}} \operatorname{sgn}(\tau) \left[y, \left[x_0, x_{\sigma_{\tau(1)}(1)}, \ldots, x_{\sigma_{\tau(1)}(t)}\right], \ldots, \left[x_0, x_{\sigma_{\tau(t!)}(1)}, \ldots, x_{\sigma_{\tau(t!)}(t)}\right]\right] \equiv 0,$$

*where* $t = \kappa(d, m + 1)$.

# Chapter 6

# Applications

In this chapter we discuss some applications of our main results. Throughout, $A$ will denote an associative algebra while $L$ will denote a Lie algebra. In our first corollary, we present some direct consequences of Theorem 4.8 and Theorem 4.24.

**Corollary 6.1.**

1. *Suppose that $M$ is a monoid of order $m$ acting as algebra endomorphisms on $A$, and let $k[M]$ denote the semigroup algebra of $M$ over $k$.*

   *(a) If $A$ is $k[M]$-rewritable of degree $d$, then $A$ satisfies an ordinary polynomial identity of degree bounded by $t!t$, where $t = \lceil em(d-1)^2 \rceil$.*

   *(b) If there exists a $k[M]$-identity of degree $d$ rewriting $A$, then $A$ satisfies an ordinary polynomial identity of degree bounded by $t = \lceil em(d-1)^2 \rceil$.*

2. *Suppose that $M$ is a monoid of finite order $m$ acting as algebra endomorphisms and anti-endomorphisms on $A$.*

   *(a) If $A$ is $k[M]$-rewritable of degree $d$, then $A$ satisfies an ordinary polynomial identity of degree $t!t$, where $t = \kappa(d, m)$.*

(b) *If there exists a $k[M]$-identity of degree $d$ rewriting $A$, then $A$ satisfies an ordinary polynomial identity of degree $t = \kappa(d, m)$.*

3. *If $A$ admits a $k$-linear involution $*$ and $A$ satisfies any $*$-identity of degree $d$, then $A$ satisfies an ordinary polynomial identity of degree $t = \kappa(2d, 2)$.*

*In all cases, $A$ satisfies the following classical polynomial identity of degree $t!t$:*

$$\sum_{\tau \in S_{t!}} \operatorname{sgn}(\tau) \left( x_{\sigma_{\tau(1)}(1)} \cdots x_{\sigma_{\tau(1)}(t)} \right) \cdots \left( x_{\sigma_{\tau(t!)}(1)} \cdots x_{\sigma_{\tau(t!)}(t)} \right) \equiv 0,$$

*where $\sigma_1 < \cdots < \sigma_{t!}$ denote the distinct permutations in $S_t$, listed in the standard lexicographical order.*

Observe that in the case when $M$ is a group, Part 2.b in Corollary 6.1 is precisely Theorem 1 in [5], by Bahturin, Giambruno and Zaicev. Part 3 is a quantitative form of Amitsur's Theorem 1 in [2]; it follows from Part 2 as shown in Theorem 10.3.3 in [12]. We also remark that Parts 1 and 2 also hold under the weaker assumption that the homomorphic image of $k[M]$ in $\operatorname{End}_k(A)$ is $m$-dimensional.

Similarly, we have the following result for Lie algebras, which follows directly from Theorem 5.20 and Theorem 5.26.

**Corollary 6.2.** *Suppose that $M$ is a monoid of order $m$ acting as algebra endomorphisms and anti-endomorphisms on $L$.*

1. *If $L$ is $k[M]$-rewritable of degree $d$, then $L$ satisfies a polynomial identity of degree bounded by $(t + 1)! + 1$, where $t = \kappa(d, m + 1)$.*

2. *If there exists a $k[M]$-identity of degree $d + 1$ rewriting $L$, then $L$ satisfies a polynomial identity of degree bounded by the function $t = \kappa(d, m + 1)$.*

*In both cases, L satisfies the following ordinary polynomial identity of degree* $(t+1)!+1$*:*

$$\sum_{\tau\in S_{t!}} \operatorname{sgn}(\tau)\left[y, \left[x_0, x_{\sigma_{\tau(1)}(1)}, \dots, x_{\sigma_{\tau(1)}(t)}\right], \dots, \left[x_0, x_{\sigma_{\tau(t!)}(1)}, \dots, x_{\sigma_{\tau(t!)}(t)}\right]\right] \equiv 0,$$

*where* $\sigma_1 < \cdots < \sigma_{t!}$ *denote the distinct permutations in* $S_t$*, listed in the standard lexicographical order.*

We point out that, in the case when $M$ is a group, Part (2) of Corollary 6.2 follows from a theorem of Bahturin, Zaicev and Sehgal proved in [8].

## 6.1 Associative algebras equipped with Hopf actions

Next, we provide an answer to the question posed at the end of Subsection 2.3.3: given an $H$-algebra, when does the existence of an $H$-identity force the existence of a classical polynomial identity?

**Proposition 6.3.** *Let H be a Hopf algebra, and suppose that A is an H-algebra (in the sense of Definition 2.37) such that the corresponding action* $\rho\colon H \to End_k(A)$ *has the property that* $m = \dim_k \rho(H)$ *is finite.*

1. *If A is H-rewritable of degree d, then A is a PI-algebra satisfying a polynomial identity of degree* $t!t$*, where* $t = \lceil em(d-1)^2 \rceil$.

2. *If there exists an H-identity of degree d rewriting A,*

$$x_1 \cdots x_d - \sum x^{h_1}_{\sigma(1)} \cdots x^{h_d}_{\sigma(d)} \equiv 0,$$

*then A is a PI-algebra satisfying a polynomial identity of degree* $t = \lceil em(d-1)^2 \rceil$.

*In both cases, A satisfies the following classical polynomial identity of degree $t!t$:*

$$\sum_{\tau \in S_{t!}} \operatorname{sgn}(\tau) \left( x_{\sigma_{\tau(1)}(1)} \cdots x_{\sigma_{\tau(1)}(t)} \right) \cdots \left( x_{\sigma_{\tau(t!)}(1)} \cdots x_{\sigma_{\tau(t!)}(t)} \right) \equiv 0,$$

*where $\sigma_1 < \cdots < \sigma_{t!}$ denote the distinct permutations in $S_t$, listed in the standard lexicographical order.*

We highlight two special instances of this proposition in the following corollaries.

**Corollary 6.4.** *Let $\mathfrak{g}$ be a Lie algebra acting on an associative algebra $A$ via a Lie algebra homomorphism $\rho : \mathfrak{g} \to Der_k(A)$. Then $\rho$ extends to an associative algebra homomorphism $\bar{\rho} : H \to End_k(A)$, where $H = U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. Suppose that $m = \dim_k \bar{\rho}(H)$ is finite.*

1. *If $A$ is $H$-rewritable of degree $d$, then $A$ is a PI-algebra satisfying a polynomial identity of degree $t!t$, where $t = \lceil em(d-1)^2 \rceil$.*

2. *If $A$ satisfies an $H$-identity of degree $d$ rewriting $A$, then $A$ satisfies an ordinary polynomial identity of degree $\lceil em(d-1)^2 \rceil$.*

**Corollary 6.5.** *Suppose $k$ is a field of positive characteristic $p$, and let $\mathfrak{g}$ be a restricted Lie algebra acting on an associative algebra $A$ via a restricted Lie algebra representation $\rho : \mathfrak{g} \to Der_k(A)$. Then $\rho$ extends to an associative algebra homomorphism $\bar{\rho} : H \to End_k(A)$, where $H = u(\mathfrak{g})$ is the restricted universal enveloping algebra of $\mathfrak{g}$. Suppose $m = \dim_k \mathfrak{g}$ is finite (so that $\dim_k H = p^m$).*

1. *If $A$ is $H$-rewritable of degree $d$, then $A$ is a PI-algebra satisfying a polynomial identity of degree $t!t$, where $t = \lceil ep^m(d-1)^2 \rceil$.*

2. *If $A$ satisfies an $H$-identity of degree $d$ rewriting $A$, then $A$ is a PI-algebra satisfying a polynomial identity of degree $\lceil ep^m(d-1)^2 \rceil$.*

Now, let $G$ be a finite group of order $m$. Recall that a vector space decomposition

$$A = \bigoplus_{g \in G} A_g$$

is a $G$-grading of $A$ provided $A_g A_h \subseteq A_{gh}$, for all $g, h \in G$. We saw in Chapter 2 that, by a theorem of Bahturin, Giambruno, and Riley ([4]), whenever the identity component $A_1$ satisfies a polynomial identity of degree $d$, the entire algebra $A$ satisfies a polynomial identity of degree $\lceil em(dm-1)^2 \rceil$. We now recover this result. First, we prove an intermediate result.

Recall that, if $H$ is a Hopf algebra and $A$ is an $H$-algebra, then the subspace of *$H$-invariants* is given by

$$A^H = \{a \in A : \ h \cdot a = \epsilon(h)a, \text{ for every } h \in H\}.$$

**Theorem 6.6.** *Let $H$ be an $m$-dimensional semisimple commutative Hopf algebra, and let $A$ be an associative $H$-algebra. If there exists an $H$-identity of degree $d$ rewriting $A^H$, then there exists an $H$-identity of degree $dm$ rewriting $A$.*

*Proof.* The arguments used in Sections 1 and 2 of [10] allow us to extend scalars in order to assume that $H$ splits over $k$. Thus, by Lemma 4 in [10], $H$ is isomorphic, as Hopf algebras, to $(k[G])^*$, the dual of the group algebra $k[G]$, where $G$ is the group of all algebra maps from $H$ to $k$ under the convolution operation. Hence, we shall assume that $H = (k[G])^*$. Let $\mathcal{B} = \{\rho_g \mid g \in G\}$ denote the standard dual basis of $(k[G])^*$.

Because $\sum_{g \in G} \rho_g = 1_H$, we may express $x_1^{1_H} \cdots x_{dm}^{1_H}$ as a sum of elements of the form $x_1^{e_1} \cdots x_{dm}^{e_{dm}}$, where each $e_i \in \mathcal{B}$. Fix $e_1, \ldots, e_{dm} \in \mathcal{B}$. Our proof will be complete once we show that $x_1^{e_1} \cdots x_{dm}^{e_{dm}} \in W + \mathrm{Id}\,(A \mid H)$, where

$$W = \mathrm{span}_k \left\{ x_{\tau(1)}^{h_1} \cdots x_{\tau(dm)}^{h_{dm}} \mid 1 \neq \tau \in S_{dm}, h_1, \ldots, h_{dm} \in H \right\}.$$

According to Lemma 4.1 in [4], for any word of length $dm = d|G|$ in $G$, there exists a string of $d$ consecutive subwords, each with trivial evaluation. Hence, we may write $x_1^{e_1}\cdots x_{dm}^{e_{dm}}$ as $uy_1\cdots y_dv$, where each submonomial $y_i$ is of the form $x_{i_1}^{\rho_{g_1}}\cdots x_{i_l}^{\rho_{g_l}}$ with $g_1\cdots g_l = 1$. Since $A$ is a $(k[G])^*$-algebra, $A$ is $G$-graded with $A_g = \rho_g \cdot A$, for each $g \in G$ (see Proposition 1.3 in [11]). Thus, each $y_i$ evaluates into $A_1 = \rho_1 \cdot A = A^H$,

Now suppose that

$$f(x_1,\dots,x_d) = x_1\cdots x_d - \sum_{1\neq\sigma\in S_d}\sum_h x_{\sigma(1)}^{h_1}\cdots x_{\sigma(d)}^{h_d} \equiv 0$$

is an $H$-identity for $A^H$. Then, since each $y_i$ evaluates into $A^H$,

$$uf(y_1,\dots,y_d)v = x_1^{e_1}\cdots x_{dm}^{e_{dm}} - \sum_{1\neq\sigma\in S_d}\sum_h uy_{\sigma(1)}^{h_1}\cdots y_{\sigma(d)}^{h_d}v \equiv 0$$

is an $H$-identity on the whole of $A$. Because the $H$-algebra action on $k\langle X\,|\,H\rangle$ is positive, $uy_{\sigma(1)}^{h_1}\cdots y_{\sigma(d)}^{h_d}v$ lies in $W$, for each $\sigma \neq 1$ and $h$ in the sum. It follows that $x_1^{e_1}\cdots x_{dm}^{e_{dm}} \in W + \mathrm{Id}\,(A\,|\,H)$, as required. □

Combining Proposition 6.3 with Theorem 6.6 now yields the following results from [4]. In order to deduce Part (2) from Part (1), recall that $A$ is $G$-graded precisely when $A$ is a $(k[G])^*$-algebra with $A_1 = A^{(k[G])^*}$ (see [11], for example).

**Corollary 6.7.**

1. *Let $H$ be an $m$-dimensional semisimple commutative Hopf algebra, and suppose $A$ is an associative $H$-algebra. If $A^H$ satisfies a polynomial identity of degree $d$, then $A$ is a PI-algebra satisfying a polynomial identity of degree $\lceil em(dm-1)^2\rceil$.*

2. *If $G$ is a group with finite order $m$ and $A$ is a $G$-graded associative algebra whose identity component satisfies a polynomial identity of degree $d$, then $A$ is a PI-algebra satisfying a polynomial identity of degree $\lceil em(dm-1)^2\rceil$.*

In [6], Bahturin and Linchenko studied the properties of Hopf algebras $H$ necessary to guarantee the existence of a polynomial identity on $H$-algebras $A$ whenever $A^H$ satisfies a polynomial identity. Their Proposition 6 states that, if $H$ is finite-dimensional but not semisimple, then there exists an associative $H$-algebra $A$ such that $A^H$ is a PI-algebra and yet $A$ is not. We can strengthen this result as follows:

**Corollary 6.8.** *Let $H$ be a finite-dimensional Hopf algebra that is not semisimple. Then there exists an associative $H$-algebra $A$ such that $A^H$ is a PI-algebra but $A$ does not satisfy a nonzero $H$-identity rewriting $A$.*

## 6.2 Lie algebras equipped with Hopf actions

The results discussed in Section 6.1 come with natural Lie-theoretic analogues which we now present. Our first proposition follows directly from Theorem 5.26 and Theorem 5.20.

**Proposition 6.9.** *Let $H$ be a Hopf algebra, and suppose that $L$ is an $H$-algebra (in the sense of Definition 2.37) such that the corresponding action $\rho\colon H \to End_k(L)$ has the property that $m = \dim_k \rho(H)$ is finite.*

1. *If $L$ is $H$-rewritable of degree $d$, then $L$ satisfies an ordinary polynomial identity of degree $(t+1)!+1$, where $t = \kappa(d, m+1)$.*
2. *If $L$ satisfies an $H$-identity of degree $d+1$ rewriting $L$, then $L$ satisfies an ordinary polynomial identity of degree $t = \kappa(d, m+1)$.*

We point out that in both cases of Proposition 6.9, $L$ satisfies the following ordinary polynomial identity of degree $(t+1)!+1$:

$$\sum_{\tau \in S_{t!}} \operatorname{sgn}(\tau) \left[y, \left[x_0, x_{\sigma_{\tau(1)}(1)}, \ldots, x_{\sigma_{\tau(1)}(t)}\right], \ldots, \left[x_0, x_{\sigma_{\tau(t!)}(1)}, \ldots, x_{\sigma_{\tau(t!)}(t)}\right]\right] \equiv 0,$$

where $\sigma_1 < \cdots < \sigma_{t!}$ denote the distinct permutations in $S_t$, listed in the standard lexicographical order.

**Corollary 6.10.** *Let $\mathfrak{g}$ be a Lie algebra acting on $L$ by derivations and antiderivations via a Lie algebra representation $\rho\colon \mathfrak{g} \to End_k(L)$. Then $\rho$ extends to an associative algebra homomorphism $\bar{\rho} : H \to End_k(L)$, where $H = U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. Suppose that $m = \dim_k \bar{\rho}(H)$ is finite.*

1. *If $L$ is $H$-rewritable of degree $d$, then $L$ satisfies an ordinary polynomial identity of degree $(t+1)! + 1$, where $t = \kappa(d, m+1)$.*
2. *If there exists an $H$-identity of degree $d+1$ rewriting $L$, then $L$ satisfies an ordinary polynomial identity of degree $\kappa(d, m+1)$.*

**Corollary 6.11.** *Let the characteristic of $k$ be $p > 0$, and let $\mathfrak{g}$ be a restricted Lie algebra acting on $L$ by derivations and antiderivations via a restricted Lie algebra representation $\rho : \mathfrak{g} \to End_k(L)$. Then $\rho$ extends to an associative algebra homomorphism $\bar{\rho} : H \to End_k(L)$, where $H = u(\mathfrak{g})$ is the restricted universal enveloping algebra of $\mathfrak{g}$. Suppose $m = \dim_k \mathfrak{g}$ is finite (so that $\dim_k H = p^m$).*

1. *If $L$ is $H$-rewritable of degree $d$, the $L$ satisfies an ordinary polynomial identity of degree $(t+1)! + 1$, where $t = \kappa(d, p^m + 1)$*
2. *If there exists an $H$-identity of degree $d+1$ rewriting $L$, then $L$ satisfies an ordinary polynomial identity of degree $\kappa(d, p^m + 1)$.*

Next we present a Lie-theoretic analogue of Theorem 6.6.

**Theorem 6.12.** *Let $H$ be an $m$-dimensional semisimple commutative Hopf algebra, and let $L$ be a Lie $H$-algebra. If there exists an $H$-identity of degree $d+1$ rewriting $L^H$, then there exists an $H$-identity of degree $(d+1)m$ rewriting $L$.*

*Proof.* As explained in the proof of Theorem 6.6, we may assume, without loss of generality, that $H = (k[G])^*$, where $G$ is the group of all algebra maps from $H$ to $k$ under the convolution operation. Suppose

$$f(x_0, x_1, \ldots, x_d) = [x_0, x_1, \ldots, x_d] - \sum_{1 \neq \sigma \in S_d} \sum_h \left[x_0^{h_0}, x_{\sigma(1)}^{h_{\sigma(1)}}, \ldots, x_{\sigma(d)}^{h_{\sigma(d)}}\right] \equiv 0$$

is an $H$-identity rewriting $L^H$, and let $k = dm + m - 1$. Using the fact that $\sum_{g \in G} \rho_g = 1_H$, we may write $[x_0, x_1, \ldots, x_k]$ as a sum of elements of the form $\left[x_0^{e_0}, x_1^{e_1}, \ldots, x_k^{e_k}\right]$ with $e_i \in \mathcal{B} = \{\rho_g \mid g \in G\}$. Let

$$W = \operatorname{span}_k \left\{\left[x_0^{h_0}, x_{\upsilon(1)}^{h_1}, \ldots, x_{\upsilon(k)}^{h_k}\right] \mid 1 \neq \upsilon \in S_k, h_0, \ldots, h_k \in H\right\}.$$

Our proof will be complete once we show that each $\left[x_0^{e_0}, x_1^{e_1}, \ldots, x_k^{e_k}\right] \in W + \operatorname{Id}(L \mid H)$. To this end, fix $e_0, e_1, \ldots, e_k \in \mathcal{B}$. By Lemma 4.1 in [4], for any word of length $(d+1)m = (d+1)|G|$ in $G$, there exists a string of $d+1$ consecutive subwords each with trivial evaluation. Hence, we may bracket $x_0^{e_0}, x_1^{e_1}, \ldots, x_k^{e_k}$ into a product of the form

$$\Bigg[u, \underbrace{\left[x_{i_0}^{\rho_{g_{i_0}}}, x_{i_0+1}^{\rho_{g_{i_0+1}}}, \ldots, x_{i_1-1}^{\rho_{g_{i_1-1}}}\right]}_{w_0}, \ldots, \underbrace{\left[x_{i_d}^{\rho_{g_{i_d}}}, x_{i_d+1}^{\rho_{g_{i_d+1}}}, \ldots, x_l^{\rho_{g_l}}\right]}_{w_d}, v\Bigg],$$

where $g_{i_0} g_{i_0+1} \cdots g_{i_1-1} = \cdots = g_{i_d} g_{i_d+1} \cdots g_l = 1$. Because $L$ is a $(k[G])^*$-algebra, it is $G$-graded with $L_g = \rho_g \cdot L$, for each $g \in G$. Thus, each $w_0, w_1, \ldots, w_d$ evaluates into $L_1 = L^H$, so that $[u, f(w_0, w_1, \ldots, w_d), v] \equiv 0$ is an $H$-identity on all of $L$. In other words,

$$[u, [w_0, w_1, \ldots, w_d], v] - \sum_{1 \neq \sigma \in S_d} \sum_h \left[u, \left[w_0^{h_0}, w_{\sigma(1)}^{h_{\sigma(1)}}, \ldots, w_{\sigma(d)}^{h_{\sigma(d)}}\right], v\right] \equiv 0 \tag{6.1}$$

is an $H$-identity on $L$. Using the Jacobi identity to open the Lie brackets, it is easy to see that $\left[x_0^{e_0}, x_1^{e_1}, \ldots, x_k^{e_k}\right] - [u, [w_0, w_1, \ldots, w_d], v] \in W$. Similarly, because the $H$-algebra action on $k\langle X \mid H\rangle$ is positive, we also have

$$\left[u, \left[w_0^{h_0}, w_{\sigma(1)}^{h_{\sigma(1)}}, \ldots, w_{\sigma(d)}^{h_{\sigma(d)}}\right], v\right] \in W,$$

for each $\sigma \neq 1$ and $h$ in the sum. Finally, because (6.1) is an $H$-identity on $L$, it follows that $\left[x_0^{e_0}, x_1^{e_1}, \ldots, x_k^{e_k}\right] \in W + \mathrm{Id}\,(L\,|\,H)$, as required. □

As a consequence of Corollary 6.10 and Theorem 6.12, we obtain the following result.

**Corollary 6.13.**

1. *Let $H$ be an $m$-dimensional semisimple commutative Hopf algebra, and suppose $L$ is a Lie $H$-algebra. If $L^H$ satisfies a polynomial identity of degree $d + 1$, then $L$ satisfies a polynomial identity of degree $\kappa((d + 1)m - 1, m + 1)$.*

2. *If $G$ is a finite group of order $m$ and $L$ is a $G$-graded Lie algebra whose identity component satisfies a polynomial identity of degree $d + 1$, then the algebra $L$ itself satisfies a polynomial identity of degree $\kappa((d + 1)m - 1, m + 1)$.*

Part (2) compares to Theorem 1 in [7] (see also [8]).

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| | B.Sc. Mathematics, *Summa cum laude*<br>National University of Colombia<br>2015 |
| | Academic Excellence Scholarship<br>National University of Colombia<br>2011 - 2015 |
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**Publications:**

[CMR1] M. Cárdenas Montoya and D.M. Riley. On the identical relations of associative and Lie algebras equipped with an action, submitted (2021).

[CMR2] M. Cárdenas Montoya and D.M. Riley. Combinatorial criteria for an algebra endowed with an action to satisfy a polynomial identity, in preparation.