Stability Analysis of Dispersive-Dissipative Waves

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Introduction and Motivation

- We consider previous studies on **non-linear**, **dispersive Hamiltonian** systems
- Their dispersion equations can be written as polynomials where roots indicate stability
- **Korteweg-de-Vries Equation**:

$$u_t = \alpha u_{xxx} + \beta u_{5x} + \sigma(u^{p+1})_x \tag{1}$$

- **Kawahara Equation**:

$$u_t = \alpha u_{xxx} + \beta u_{5x} + \sigma(u^2)_x \tag{2}$$

Introduction and Motivation

What if we introduce **dissipation**?

- Physically: energy is not conserved
- Mathematically: introduction of a second order differential term

Dissipative Equation

Equation 1: Generalized KdV-Burgers equation

$$u_t + (\alpha + \beta u)uu_x + \gamma u_{xx} - \delta u_{xxx} = 0 \tag{3}$$

Equation 2:

$$u_t = -\alpha u^2 u_x - \beta u_{xx} - \gamma u_{xxx} + \tau(uu_x)_x + \delta u_{xxxx} \tag{4}$$

Equation 3: General Kawahara with 2nd order derivative term

$$u_t = \gamma u_{xx} + \alpha u_{xxx} + \beta u_{5x} + \sigma(u^2)_x \tag{5}$$

Method 1

1. Considering the stability of a travelling wave solution, we move to a frame of reference travelling at speed V

$$u(x,t) = \tilde{u}(x - Vt, t) \tag{6}$$

2. Consider the steady state solution in the travelling frame ($\tilde{u}_t = 0$)

$$Vu_x + (\alpha + \beta u)uu_x + \gamma u_{xx} - \delta u_{xxx} = 0 \tag{7}$$

1. Assuming a periodic wave with period 2π for simplicity, we can approximate $u \approx e^{ikx}$ and sub this into (7)
2. Assuming small-amplitude waves, we can neglect the non-linear term.

$$ikV + k^2\gamma - ik^3\delta = 0 \tag{8}$$

(we will henceforth set $V = V_0$)

Results

- In Fourier space:

$$\lambda_m^\mu = i(m+\mu)V_0 + (m+\mu)^2\gamma - i(m+\mu)^3\delta \tag{9}$$

- We can ensure a fixed eigenvalue by, for fixed γ, choosing

$$\delta = -i\gamma\frac{(m+\mu)((m+\mu)-1)+(n+\mu)(1-(n+\mu))}{(n+\mu)(1-(n+\mu)^2)+(m+\mu)((m+\mu)^2-1)} \tag{10}$$

New Dissipative Equations

Using a more general method:

- 7th order KdV with 2nd order term:

$$u_t + \delta u_{2x} + \alpha u_{3x} + \beta u_{5x} + \gamma u_{7x} = 0 \tag{11}$$

- 7th order KdV with 4th order term:

$$u_t + \delta u_{4x} + \alpha u_{3x} + \beta u_{5x} + \gamma u_{7x} = 0 \tag{12}$$

Method 2

1. We begin similarly - move to travelling frame, find steady state solution, neglect non-linearities and approximate $u \approx e^{ikx}$ giving us:

$$i\delta k - \alpha k^2 + \beta k^4 - \gamma k^6 + V = 0 \tag{13}$$

2. Assuming periodicity with period 2π, we obtain symmetry and can set k = 1

$$V_0 = -i\delta + \alpha - \beta + \gamma \tag{14}$$

1. In general, eigenvalue collisions for odd order polynomials have the form

$$p(\mu + n) - p(\mu) = 0 \tag{15}$$

2. For our equation:

$$\begin{aligned} p(\mu, n) = \gamma(\mu + n)^7 - \beta(\mu + n)^5 + \alpha(\mu + n)^3 - i\delta(\mu + n)^2 \\ -\gamma\mu^7 + \beta\mu^5 - \alpha\mu^3 + i\delta\mu^2 - (-i\delta + \alpha - \beta + \gamma)n \end{aligned} \tag{16}$$

1. Binomial theorem tells us:

$$(\mu + n)^N = \sum_{k=0}^{N} \binom{N}{k} \mu^{N-k} n^k \tag{17}$$

2. So we can use Pascal's Triangle to expand!

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1. Giving us:

$$\begin{aligned} p(\mu, n) = \gamma(7\mu^6 n + 21\mu^5 n^2 + 35\mu^4 n^3 + 35\mu^3 n^4 + 21\mu^2 n^5 + 7\mu n^6 + n^7) \\ -\beta(5\mu^4 n + 10\mu^3 n^2 + 10\mu^2 n^3 + 5\mu n^4 + n^5) \\ +\alpha(3\mu^2 n + 3\mu n^2 + n^3) \\ -i\delta(2\mu n + n^2) \\ -(-i\delta + \alpha - \beta + \gamma()n = 0. \end{aligned} \tag{18}$$

1. We now simplify by setting $s = \mu(\mu + n)$
2. Physically, this is related to the **signature**: contribution of eigenvalues to the Hamiltonian
3. Assumes equal contribution from both eigenvalues

1. Resulting polynomial:

$$\begin{aligned} q(s,n) = & -\gamma(n^6+7n^4s+14n^2s^2+7s^3) \\ & +\beta(n^4+5n^2s+5s^2)-\alpha(n^2+3s) \\ & +i\delta(n)-i\delta+\alpha-\beta+\gamma \end{aligned} \tag{19}$$

2. Set $q(s,n) = p_0(s)$ to create a Sturm chain

Sturm Chain

$$f(x)x^5 - 3x - 1 \tag{20}$$

1. Set $f(x) = f_0(x)$
2. $f_1(x) = f_0'(x)$
3. $f_2(x) = -Rem\frac{f_0(x)}{f_0'(x)}$
4. Input values in the domain to find number of real roots in the given interval of values

Numerical Results

$$s = \frac{1}{3}\left(1 - n^2\right)$$

Physical Interpretation

- We have impossible scenarios when we have imaginary velocities!

Further Work

Use ideas from pure math!

- **Galois Theory**
- **Cauchy Index**