Stability Analysis of Dispersive-Dissipative Waves

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Summer 2021
We consider previous studies on non-linear, dispersive Hamiltonian systems.

Their dispersion equations can be written as polynomials where roots indicate stability.

**Korteweg-de-Vries Equation:**

\[
  u_t = \alpha u_{xxx} + \beta u_{5x} + \sigma (u^{p+1})_x 
\]  

**Kawahara Equation:**

\[
  u_t = \alpha u_{xxx} + \beta u_{5x} + \sigma (u^2)_x 
\]
What if we introduce **dissipation**?

- Physically: energy is not conserved
- Mathematically: introduction of a second order differential term
Equation 1: Generalized KdV-Burgers equation

\[ u_t + (\alpha + \beta u)uu_x + \gamma u_{xx} - \delta u_{xxx} = 0 \]  

Equation 2:

\[ u_t = -\alpha u^2 u_x - \beta u_{xx} - \gamma u_{xxx} + \tau (uu_x)_x + \delta u_{xxxx} \]  

Equation 3: General Kawahara with 2nd order derivative term

\[ u_t = \gamma u_{xx} + \alpha u_{xxx} + \beta u_{5x} + \sigma (u^2)_x \]
Considering the stability of a travelling wave solution, we move to a frame of reference travelling at speed $V$

$$u(x, t) = \tilde{u}(x - Vt, t)$$  \hspace{1cm} (6)

Consider the steady state solution in the travelling frame ($\tilde{u}_t = 0$)

$$Vu_x + (\alpha + \beta u)uu_x + \gamma u_{xx} - \delta u_{xxx} = 0$$  \hspace{1cm} (7)
1. Assuming a periodic wave with period $2\pi$ for simplicity, we can approximate $u \approx e^{ikx}$ and sub this into (7)

2. Assuming small-amplitude waves, we can neglect the non-linear term.

\[ ikV + k^2\gamma - ik^3\delta = 0 \] (8)

(we will henceforth set $V = V_0$)
In Fourier space:

\[ \lambda_m^\mu = i(m + \mu)V_0 + (m + \mu)^2\gamma - i(m + \mu)^3\delta \quad (9) \]

We can ensure a fixed eigenvalue by, for fixed \(\gamma\), choosing

\[ \delta = -i\gamma\frac{(m + \mu)((m + \mu) - 1) + (n + \mu)(1 - (n + \mu))}{(n + \mu)(1 - (n + \mu)^2) + (m + \mu)((m + \mu)^2 - 1)} \quad (10) \]
New Dissipative Equations

Using a more general method:

- 7th order KdV with 2nd order term:

\[
 u_t + \delta u_{2x} + \alpha u_{3x} + \beta u_{5x} + \gamma u_{7x} = 0 \tag{11}
\]

- 7th order KdV with 4th order term:

\[
 u_t + \delta u_{4x} + \alpha u_{3x} + \beta u_{5x} + \gamma u_{7x} = 0 \tag{12}
\]
We begin similarly - move to travelling frame, find steady state solution, neglect non-linearities and approximate $u \approx e^{ikx}$ giving us:

$$i\delta k - \alpha k^2 + \beta k^4 - \gamma k^6 + V = 0 \quad (13)$$

Assuming periodicity with period $2\pi$, we obtain symmetry and can set $k = 1$

$$V_0 = -i\delta + \alpha - \beta + \gamma \quad (14)$$
In general, eigenvalue collisions for odd order polynomials have the form

\[ p(\mu + n) - p(\mu) = 0 \quad (15) \]

For our equation:

\[
p(\mu, n) = \gamma (\mu + n)^7 - \beta (\mu + n)^5 + \alpha (\mu + n)^3 - i\delta (\mu + n)^2 \\
- \gamma \mu^7 + \beta \mu^5 - \alpha \mu^3 + i\delta \mu^2 - (-i\delta + \alpha - \beta + \gamma) n
\]  

(16)
Binomial theorem tells us:

\[(\mu + n)^N = \sum_{k=0}^{N} \binom{N}{k} \mu^{N-k} n^k\]  \hspace{1cm} (17)

So we can use Pascal’s Triangle to expand!
```
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
```
Giving us:

\[ p(\mu, n) = \gamma (7\mu^6 n + 21\mu^5 n^2 + 35\mu^4 n^3 + 35\mu^3 n^4 + 21\mu^2 n^5 + 7\mu n^6 + n^7) \]
\[ - \beta (5\mu^4 n + 10\mu^3 n^2 + 10\mu^2 n^3 + 5\mu n^4 + n^5) \]
\[ + \alpha (3\mu^2 n + 3\mu n^2 + n^3) \]
\[ - i\delta (2\mu n + n^2) \]
\[ - (-i\delta + \alpha - \beta + \gamma) n = 0. \]

(18)
1. We now simplify by setting $s = \mu(\mu + n)$

2. Physically, this is related to the **signature**: contribution of eigenvalues to the Hamiltonian

3. Assumes equal contribution from both eigenvalues
1 Resulting polynomial:

\[ q(s, n) = -\gamma(n^6 + 7n^4s + 14n^2s^2 + 7s^3) + \beta(n^4 + 5n^2s + 5s^2) - \alpha(n^2 + 3s) + i\delta(n) - i\delta + \alpha - \beta + \gamma \]  

(19)

2 Set \( q(s, n) = p_0(s) \) to create a Sturm chain
\( f(x)x^5 - 3x - 1 \) \hspace{1cm} (20)

1. Set \( f(x) = f_0(x) \)
2. \( f_1(x) = f_0'(x) \)
3. \( f_2(x) = -\text{Rem} \frac{f_0(x)}{f_0'(x)} \)
4. Input values in the domain to find number of real roots in the given interval of values
Numerical Results

\[ s = \frac{1}{3} (1 - n^2) \]
We have impossible scenarios when we have imaginary velocities!
Further Work

Use ideas from pure math!

- Galois Theory
- Cauchy Index