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COMPLEX ENERGIES AND THE LAMBERT *W* FUNCTION

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We apply the Lambert W function in the context of complex energy values in statistical systems of fermions and bosons. We derive the condition for the transformation connecting fermions and bosons. We discuss the physical significance of these results and investigate the conditions under which bosonization effects take place. The fermion and boson statistical and structural distributions discussed in this work suggest the possibility of their extensions to generalized Planck distributions.

1. Introduction

The Lambert *W* function is one of the most ubiquitous functions that yields fruitful results in a wide variety of multifarious problems [1]–[3]. It has been remarkably useful in quantum statistical problems as well as in various mathematical problems involving exponential or logarithmic functions. Incidentally, it can also be used to solve various equations involving exponentials, as in the case of Bose–Einstein and Fermi—Dirac statistics. In an earlier study, Sidharth and Valluri [4] have studied and applied the Lambert *W* function to find the Cosmic Microwave Background temperature of the universe. Also, very recently [5], the authors of this paper used this versatile function to substantiate the minimal conductivity in case of graphene. In this paper, we utilize this function for a study of the Fermi–Dirac distribution to derive some new results.

In Sec. 2, the energy of a particle is extended to the complex domain. In Sec. 3, we delineate the physical significance of this complex energy by turning to the Schrödinger equation and confirming that the *resonance condition* remains unaltered. In Sec. 4, we consider the mathematical conditions where the phenomenon of *bosonization* can take place. In the final Sec. 5, we conclude and discuss further opportunities to explore questions involving complex energies.

2. Energy in the complex domain

An earlier work by Valluri et al. [6] has introduced the quantity M related to the Fermi–Dirac distribution as

$$M = A \int_0^\infty F(E) \, dE \tag{1}$$

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where

$$F(E) = \frac{E^{f(r)}}{e^{\beta E + \alpha} + 1}, \qquad \alpha = -\beta\mu, \tag{2}$$

with μ being the total chemical potential and the index function $f(r)$ real-valued and continuous such that $f \colon \mathbb{R} \mapsto \mathbb{R}_+ = \{x \geqslant 0\}$, and $A \in \mathbb{R}$. We noted that M is an arbitrary physical quantity and $f(r)$ is an index function that can be r and is to be specialized later as the logistic function. If E is considered a continuous real variable and α, r, and T are kept fixed, it has been shown in [6] that the extremum in E of $F(E)$ on the interval $(0, \infty)$ must satisfy

$$E = \frac{1}{\beta}[f(r) + W_j\{f(r)e^{-\alpha - f(r)}\}] \tag{3}$$

where $\beta = 1/kT$ (k being the Boltzmann constant) and the subscript j denotes the branches of the Lambert W function and

$$W_j\{f(r)e^{-\alpha - f(r)}\} = \beta E - f(r). \tag{4}$$

We intend to further study Eqs. (2) and (3) in the context of complex energies. First, in examining Eq. (3), we consider complex E; $f(E)$ is then complex and hence provides a whole new spectrum of distributions that represent a probabilistic interpretation of the parameters of the system under consideration. We consider r to be a continuous variable instead of a fixed parameter and find the extremum in r of Eq. (3). Differentiating Eq. (3) with respect to r, we have

$$\frac{dE}{dr} = \frac{1}{\beta}\left[\frac{df}{dr} + \frac{dz}{dr}\frac{d}{dz}W_j(z)\right], \qquad z = f(r)e^{-\alpha - f(r)}. \tag{5}$$

Now, let $f(r)$ be a standard *logistic function* in analogy with the Fermi–Dirac distribution

$$f(r) = \frac{1}{1 + e^{-r}} = \frac{e^r}{1 + e^r}, \tag{6a}$$

$$\frac{df}{dr} = f'(r) = f(r)[1 - f(r)], \qquad \frac{dz}{dr} = f(r)[1 - f(r)]^2 e^{-\alpha - f(r)}. \tag{6b}$$

We let $W(z)$ denote $W_j(z)$ unless otherwise indicated. We know that

$$\frac{dW(z)}{dz} = \frac{1}{z + e^{W(z)}}. \tag{7}$$

Using Eqs. (6) and (7) in Eq. (5), we have

$$\frac{dE}{dr} = \frac{1}{\beta}\left[f(r)\{1 - f(r)\} + \frac{f(r)\{1 - f(r)\}^2 e^{-\alpha - f(r)}}{f(r)e^{-\alpha - f(r)} + e^{W\{f(r)e^{-\alpha - f(r)}\}}}\right].$$

Setting $dE/dr = 0$, we have

$$1 + \frac{\{1 - f(r)\}e^{-\alpha - f(r)}}{f(r)e^{-\alpha - f(r)} + e^{W\{f(r)e^{-\alpha - f(r)}\}}} = 0.$$

Rearranging the terms, we arrive at

$$e^{W\{f(r)e^{-\alpha - f(r)}\} + \alpha + f(r)} + 1 = 0. \tag{8}$$

Now, we recall Euler's identity $e^{\pm i(2n+1)\pi} + 1 = 0$, where n is an integer. We can then write

$$W\{f(r)e^{-\alpha - f(r)}\} + \alpha + f(r) = \pm i(2n + 1)\pi,$$

and then using this relation in Eq. (3), we obtain

$$E = \frac{1}{\beta}[\pm i(2n+1)\pi - \alpha].\tag{9}$$

As we can see, E has been extended to the complex domain, and this energy is independent of the coordinate r.

We have maximized the energy in terms of r considering the condition for stationary points and derived the extremum condition $dE/dr = 0$. This provides the condition for resonances or bound states, i.e., singularities, to exist. This is because $f(r)$ is the logistic function introduced in Eq. (6a). It is worth mentioning that there might exist other index functions $f(r)$ that lead to complex energies similar to (9). We have merely laid out an example following the methodology of which several such functions might be chosen that correspond to complex energies independent of r.

There might be interesting applications of Eq. (9). Because α/β yields the chemical potential μ, the complex energy depends on the internal energy. Therefore, there is a number of new metastable states that arise from the analytic properties of energy. The first term inside the square bracket in (9) multiplied by $1/\beta$ yields a complex temperature (because $\beta = 1/kT$). That suggests a quantum statistical system, out of thermal equilibrium, which is undergoing a phase transition. Indeed, in the complex plane, this indicates a spontaneous magnetization for the metastable states due to the Curie–Weiss model [7] given by the relation

$$\chi = \frac{C}{T - T_{\mathrm{c}}}\tag{10}$$

where C is the Curie constant and T_{c} is the critical temperature below which spontaneous magnetization occurs. Another possibility is a connection with the Fisher zeros [8] that occur in determining the zeros of the partition function in the complex plane. Therefore, the Lambert W function can be applied to a diverse range of physical applications.

From Eq. (9),

$$E^* = -\frac{1}{\beta}[\mp i(2n+1)\pi + \alpha], \qquad |E|^2 = EE^* = \frac{1}{\beta^2}[\alpha^2 + (2n+1)^2\pi^2]$$

where $|E|$ denotes the modulus of E. This finally yields

$$|E| = \frac{1}{\beta}\sqrt{\alpha^2 + (2n+1)^2\pi^2}, \qquad \alpha = -\mu\beta.\tag{11}$$

This result can have new implications. The energy E has been extended to the complex domain. If E is treated as a holomorphic function in the complex plane, one can obtain similar results for the Bose–Einstein statistics discussed in Sec. 4. This could provide a benchmark for novel explorations in the fields of quantum mechanics and quantum statistics. In the next section, we discuss the physical significance of this complex energy considering the one-dimensional Schrödinger equation.

3. Physical significance of the complex energy

Previously, Sudarshan et al. [9], Bailey and Schieve [10], and Sidharth [11] have explored the possibilities of complex energy in quantum mechanics. Also, Bender et al. [12] studied non-Hermitian Hamiltonians that yield real spectra. In a similar context, Valluri and Romo [13] had obtained an analytic expression for the derivative of the phase shift with respect to angular momentum and momentum for complex potentials.

In the preceding section, we extended the energy to the complex domain. The energy in Eq. (9) is merely a consequence of maximizing the energy and using the Lambert-W function; the methodology suggests the

decay process of fermions in the complex domain. It is known that the Lee–Friedrichs model [14], [15] serves as an important mathematical model for decay–scattering processes in the field theory of elementary particles. Sudarshan et al. [9] have shown that complex-energy solutions of the Lee–Friedrichs model can yield consistent results in which the resonances contribute in the same way as bound states when the completeness property is used to expand the inner product.

To further illustrate the significance of complex energy, we consider the case of a simple one-dimensional finite square potential, such that the Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi, \qquad V(x) = \begin{cases} V_0, & 0 < x < L, \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

(where $\hbar = h/2\pi$ (h is Planck's constant) and the symbols have their usual meanings. Because complex energy indicates a non-Hermitian Hamiltonian, $V(x)$ itself is complex. This is because a complex potential in the complex plane, with a constant imaginary part, essentially implies complex energy. For real energy, the wave numbers in the potential-free region and within the potential are respectively given by

$$k_1 = \frac{\sqrt{2mE}}{\hbar}, \qquad k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}.$$

Here, the transmission coefficient (for $E > V_0$) is q

$$T = \frac{4E(E - V_0)}{4E(E - V_0) + V_0^2 \sin^2\left[\sqrt{2m(E - V_0)}L/\hbar\right]}.$$

In this context, we see that T attains its maximum value of 1 when

$$\sin^2\left[\sqrt{2m(E - V_0)}\frac{L}{\hbar}\right] = 0 \quad \text{or} \quad k_2 = \frac{n\pi}{L} \tag{13}$$

which is the resonance condition for k_2 real.

We investigate this problem when the energy is complex as in our case. Essentially, the wavenumber k_2 is then complex. We write $k_2 L = z$, where z is a complex number. Again, we know that $\sin z = (e^{iz} - e^{-iz})/2i$. Because the resonance condition requires $\sin z = 0$, the above relation yields $z = \pm n\pi$. Thus, we finally have $k_2 L = z = \pm n\pi$. The resonance condition implies singularities on the imaginary axis when the energy is extended to the complex domain. This means that interesting results can still be obtained while considering complex energies in the case of quantum systems.

Similarly, in the case $E < V_0$, where the energy is less than the barrier height, we have the transmission coefficient

$$T = \frac{4E(V_0 - E)}{4E(V_0 - E) + V_0^2 \sinh^2(\kappa L)}, \qquad \kappa = ik_2.$$

Interestingly, these novel complex-energy states can be regarded as new metastable states because they arise from our consideration of the Lambert W function; this is an interesting feature of our work. This is where the significance of the Lambert W function is manifest. These metastable states suggest new decay mechanisms or spontaneous disintegration with probabilistic features different from the already known ones. Essentially, this suggests new physical implications pertinent to decay mechanisms, new bound states of particles or other subatomic phenomena.

Incidentally, it is known that the problem of scattering by a square well is intimately related to the scattering by a potential barrier with $E > V_0$ [16]. For the quantum finite square well, Roberts and

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Valluri [17] discuss in detail the insights provided by the solutions in the complex plane in terms of the Lambert W function of the eigenvalue problem. They do not explicitly discuss or calculate the reflection and transmission coefficients, which are well known in the literature on quantum mechanics. With appropriate changes, the analysis for the quantum finite square well can be applied to the case of the potential barrier with $E > V_0$.

In the preceding section, we resorted to the fundamental properties of the Lambert W function to derive a complex form of the energy, which elucidates some interesting features of a physical system. This methodology essentially broadens the perspective of known physical systems to the complex domain. All this essentially points towards a new approach to an altogether new scenario concerning complex energies. Considering a standard plane wavefunction of the form $\psi(x,t) = e^{i(kx-\omega t)}$, we have the energy in the form

$$E = \hbar kc = E_0 + \frac{1}{2}i\Gamma, \tag{14}$$

where c is the speed of light. Here, the wave-vector k is complex. Hence, the wavefunction $\psi(x,t)$ has the exponentially decaying term $e^{-\alpha x}$, where $\alpha = \Gamma/2$; this explains the decay of the particle. Therefore, one can have physical applications in the complex domain for complex energies.

In a very closely related context, Sidharth and Das [18] have recently shown that considering Ito's stochastic methods in the *zitterbewegung* region, the Schrödinger equation becomes modified and exhibits a confinement-like nature as in the case of QCD. Interestingly, the *zitterbewegung* region below the Compton scale has several interesting modifications [19]. The introduction of complex energy in such a case might altogether bring about a new scenario. In the next section, we consider the phenomenon of bosonization.

4. Bosonization

In this section, we find a condition, in terms of the Lambert W function, for the bosonization effect to occur. Bosonization has been discussed over the previous decades by several authors [20]–[25]. Essentially, the interactions between electrons are modeled as bosonic interactions [20], [21]. Sidharth [26], [27] has considered this topic, although in a different approach.

We first consider the logistic function

$$g(\delta) = \frac{e^\delta}{e^\delta + 1}, \qquad \delta = \beta E + \alpha, \qquad \frac{d\delta}{dE} = \beta. \tag{15}$$

Using this relation, we can express (2) as

$$F(E) = \frac{g(\delta)}{e^\delta} E^{f(r)}.$$

Now, keeping the parameters α and T fixed, we differentiate the above relation with respect to E:

$$\frac{dF}{dE} = f(r)E^{(f(r)-1)}\frac{g(\delta)}{e^\delta} + E^{f(r)}\frac{1}{e^\delta}g'(\delta) - \frac{E^{f(r)}g(\delta)}{e^\delta}\beta. \tag{16}$$

Maximizing $F(E)$ with respect to the energy E (setting $dF/dE = 0$) and then simplifying the terms, we obtain

$$\frac{dg(\delta)}{dE} + g(\delta)\left[\frac{f(r)}{E} - \beta\right] = 0.$$

We easily integrate the equation to obtain

$$\ln g = \int_0^E \left[\beta - \frac{f(r)}{E}\right] dE.$$

Considering the function $f(r)$ to be independent of E, we derive $\ln g = \beta E - \ln E^{f(r)}$. More precisely, $g(\delta)E^{f(r)} = e^{\beta E}$. Hence, using the definition of the logistic function in (15) and simplifying, we obtain $E^{f(r)} - e^{-\alpha} = e^{\beta E}$. Multiplying both sides by βE and then exploiting the definition of the Lambert W function, we derive

$$W_{j\mathrm{F}}[(\beta E)(E^{f(r)} - e^{-\alpha})] = \beta E, \tag{17}$$

where $W_{j\mathrm{F}}$ is the Lambert W function for the Fermi–Dirac statistics.

Now, in a similar manner we can consider the logistic function

$$g(\delta) = \frac{e^{\delta}}{e^{\delta} - 1}, \qquad \delta = \beta E + \alpha, \qquad \frac{d\delta}{dE} = \beta, \tag{18}$$

Thereby, the Bose–Einstein statistics

$$B(E) = \frac{E^{b(r)}}{e^{\beta E + \alpha} - 1} \tag{19}$$

can be rewritten as

$$B(E) = \frac{g(\delta)}{e^{\delta}} E^{b(r)}. \tag{20}$$

The index function $b(r)$ is real-valued and continuous such that $b \colon \mathbb{R} \mapsto \mathbb{R}_+$, just as in case of the function $f(r)$. Now, following the same methodology and using (18) and (20), we can derive the following result for the Bose–Einstein statistics:

$$W_{j\mathrm{B}}[(\beta E)(E^{b(r)} + e^{-\alpha})] = \beta E. \tag{21}$$

Here, $W_{j\mathrm{B}}$ denotes the jth Lambert W function for the Bose–Einstein distribution.

Now, for a fermion–boson or boson–fermion transmutation, the key parameters are the energy E and the temperature T. Because the transmutation is essentially of a phase transition type, there has to be a critical temperature T_{c} and a critical energy E_{c}, which must manifestly be the threshold values. As in the general case for an arbitrary physical system of particles, beyond these critical values the transmutations cannot occur. Hence, these values must be fixed for most arbitrary statistical ensembles. Therefore, using this approach and considering (17) and (21), we must have

$$W_{j\mathrm{F}}[(\beta E)(E^{f(r)} - e^{-\alpha})] = W_{j\mathrm{B}}[(\beta E)(E^{b(r)} + e^{-\alpha})], \tag{22}$$

keeping in mind that the energy and temperature are at their critical values during the transmutation. Although the last relation (22) brings forth the statistical nature of fermions and bosons rather than their particle nature, it is worth mentioning that if we consider the system to be isolated and the potential energy to be translationally and rotationally invariant, then the angular momentum is conserved [28].

However, we can conclude from the above derivation that relation (22) must be satisfied for a fermion–boson transmutation. Essentially, in the study of bosonization, where interacting fermions are transformed into interacting or noninteracting bosons, condition (22) might provide a new perspective with respect to the corresponding Lambert W functions. This result could be used in the case of two-dimensional crystal structures like graphene to see if such a transmutation is possible.

The main results of this paper are given by Eqs. (9), (11), and (22). The first two represent the energy values in two different forms and may be useful for future research. Relation (22) is novel and interesting in the sense that it can delineate the affine connection between fermions and bosons.

5. Conclusions

In this work, we have found two different expressions for energy when it is considered a complex variable. The boson–fermion transmutation can be possible in the case of monoenergetic beams [26]. This transmutation is a consequence of the monoenergetic nature of the beam of particles that could occur in the case of nanotubes. We have derived a condition for bosonization effects to occur. It may open a new path for further investigations on this topic.

In a different context, it is worth mentioning that the complex domain into which the energy has been extended may be intricately related to the noncommutative nature of spacetime [29]–[32] (also see several references therein). Sidharth and Das have explored a variety of topics in the field of noncommutative geometry [5], [33]–[36], following Snyder's approach [37]. Recently, Sidharth and Das have shown that one could extract the rudimentary features of special relativity from the basic considerations of noncommutativity [38]. It would be interesting to see whether the noncommutative nature of spacetime plays any significant role in the complex domain, for the quantum statistical situations discussed in our work.

The property of bosonization of fermions leads to interesting applications. The work of Vega Monroy [39] on the Bose–Einstein condensation (BEC) of paired photons and dressed electrons in graphene shows that the condensate fraction N and the BCS-like critical temperature T_c can be usefully connected in terms of the multi-branched Lambert W function of the temperature, which displays the logarithmic structure dependence of N and T_c. Such a BEC is experimentally feasible. BEC and superfluidity of 2D quasiparticles in graphene in a strong magnetic field has been predicted [40].

The quantum Hall effect has been discussed in the context of noncommutative geometry in [41]. It is interesting to observe the geometric insights obtained on extremization of the Fermi–Dirac and Bose–Einstein distributions. Such insights suggest the possibility of extending to generalized Planck distributions and studying their structural properties [42]. The Lambert W function and its generalizations as well as the generalized Planck distributions will have many more interesting applications in diverse fields [43].

Conflicts of interest. The authors declare no conflicts of interest.

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