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Essays on the All-Pay Auction

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A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Economics
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Abstract

Three all-pay auction models are examined. The first is a symmetric two-player binary-signal all-pay auction with correlated signals and interdependent valuations. The first chapter provides a complete characterization of each form of equilibrium and gives conditions for their existence. The main finding is that there generically exists a unique equilibrium. The unique equilibrium can only be one of four forms of equilibria. I apply my all-pay auction model to elections, where a candidate that receives good news from the polls behaves in a rationally overconfident manner and reduces her equilibrium effort. Consequently, the other candidate can win the election in an upset.

The second chapter extends Chapter 1's model to N signals. In comparison, the binary model allows for a guess-verify approach. However, the number of possible guesses increases rapidly when N increases. Hence such an approach is infeasible. Chapter 2's approach is centered around linear algebra techniques and a novel notion of a weakly monotone equilibrium. In a weakly monotone equilibrium the bid supports are ordered by the strong set order but not necessarily separated like the traditional monotone equilibrium. I classify these weakly monotone equilibria into four primary forms. I characterize each form and find sufficient conditions for their existence. Furthermore, for the model used in Rentschler and Turocy (2016), I provide a novel necessary and sufficient condition for the existence of a traditional monotone equilibrium.

The third chapter considers a two-stage game: a negotiation stage followed by a conflict stage in case the negotiations break down. In a setting with multi-dimensional correlated types, two players compete over a good that is of uncertain but common value. Conflict is modeled as an all-pay auction, which endogenizes the cost of conflict. In the literature, which assumes independent private values or costs, a peaceful equilibrium, in which war occurs with zero probability need not exist. I find that in my correlated pure common-value model, a peaceful equilibrium always exists and is essentially unique. Further, I show that adding private values to this model worsens the prospect of peace, and conflict might occur.

Summary for Lay Audience

Every bidder pays their bid in the all-pay auction, but only the highest bidder wins the prize. The fact that everybody pays their bid, a sunk cost, makes the all-pay auction a very appealing theoretical framework to study. There are various real-world competitions where there is this kind of cost. For example, the effort and time in applying for a job are sunk, but only one person gets the job. Countries spend money on their military, but only one country can win the war. All political candidates in an election spend time and money on their campaigns, but only one candidate wins the election.

Because all-pay processes are widespread, it is crucial to solve the mathematical all-pay auction model under reasonable assumptions. The goal of this thesis is to solve the all-pay auction in a general model such that it is applicable to real-world examples.

I use the all-pay auction model to study election campaigns in my thesis. I show that the 2016 US presidential election results might have been caused by rational overconfidence. Clinton, whom the pollsters favored, exhibited a form of complacency which gave her opponent Trump a chance to win the election.

Further, I apply the all-pay auction to military conflict. I provide conditions to avoid war by proposing a peaceful solution.

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Contents

Introduction

Every bidder pays their bid in the all-pay auction, but only the highest bidder wins the prize. The sunk-cost feature makes the all-pay auction a very appealing theoretical framework to study various real-world applications. For example, the all-pay auction can be used to study election campaigns (Chapter 1) and military conflicts between countries (Chapter 3). However, the set of parameter values usable in the all-pay auction had to be extended to study these applications and others.

For example, to realistically model elections, one needs to allow for a strong correlation between the private information of the political candidates. Further, to model military conflict, the value of the prize needs to depend on both players' private information (interdependent valuation). The all-pay auction had generally remained unsolved for both correlation and interdependent valuation.

This thesis' three chapters characterize the different equilibria of the two-player all-pay auction, allowing for both correlation and interdependent valuation. Chapters 1 and 2 consider a static symmetric all-pay auction model. Static means that there is no time element in the game. At the same time, symmetry implies that the players are identical before realizing their private information. Finally, Chapter 3 considers a dynamic (multi-stage) asymmetric all-pay auction model.

Chapter 1 confines attention to a binary-type model. Each player receives a private type that can be high or low. The binary type restriction implies that there generically exists a unique equilibrium. This unique equilibrium can take four possible forms, which are fully characterized. Allowing for correlation and interdependence can change the predictions of the all-pay auction; it is now possible that a low type beats a high type in an upset. Finally, the binary model is applied to elections, where a candidate who receives good news from the polls behaves in a rationally overconfident manner and reduces her equilibrium effort. Consequently, the other candidate can win the election in an upset.

Chapter 2 extends Chapter 1's model to N types. Relaxing the binary structure implies that we may no longer have a unique equilibrium. Furthermore, the binary type space allows for a guess-verify approach. Since the amount of possible guesses increases rapidly when N increases, such an approach is infeasible for larger type spaces. This chapter provides a new methodology to solve the all-pay auction. In this methodology, linear algebra techniques are used to characterize the bid-distribution slopes and derive conditions for all four equilibria from the binary type model in the general N-type model. These four forms are called the primary forms. Further, "mixed-form" equilibria can exist, which are combinations of two or more primary forms.

Chapter 3 considers a two-stage game: a negotiation stage followed by a conflict stage in case the negotiations break down. In this game, two countries negotiate over a good of an uncertain but common value. Conflict is modeled as an all-pay auction, which endogenizes the cost of conflict.

In the literature, which assumes independent private values or costs, a peaceful equilibrium, in which war occurs with zero probability, need not exist. First, in a common-value model, a peaceful equilibrium always exists if two countries are equally powerful. Furthermore, this peaceful equilibrium is essentially unique. Second, adding uncertainty and differences in power to this model worsens the prospect of peace, and conflict might occur. Third, correlation between private information might lead to a scenario where a weak type rejects a peace deal that a stronger type may accept.

Chapter 1

Upsets in the All-Pay Auction

1.1 Introduction

In recent elections, the front-runner in the polls loses out to an underdog.[1] These surprising outcomes, which I call upsets, can trivially occur if polls fail to capture the true preferences of the electorate. All-pay auction models can offer an alternative explanation so that even if polls represent the true preferences, an upset can occur due to rational overconfidence. However, to realistically model applications such as elections, the standard assumptions of the all-pay auction need to be relaxed.

I study a symmetric two-player binary signal all-pay auction model with arbitrarily correlated signals and interdependent valuations.[2] I fully characterize all the equilibria and find conditions for their existence. Furthermore, I find that a unique equilibrium exists under a generic condition on the primitives. In fact, it is unique for all but one configuration of the primitives. Further, I prove that the unique equilibrium satisfies some important properties that I describe below.

The all-pay auction is an auction in which each participant pays their bid, but only the highest bidder wins the auction. It is effective as a theoretical framework for understanding conflict with irreversible investments. There are many other applications besides elections where the all-pay auction has been used as a theoretical framework; R&D investment decisions by competing firms (Siegel, 2014), military spending by countries (Zheng, 2019), and legal expenses in litigation (Baye et al, 2005). For all these

[1] A prominent recent example of such an upset is the 2016 US presidential election. Another example is Brexit, where "remain" was favored by telephone pollsters before the referendum. The 2020 US presidential election, which did not end in an upset, was much closer than polls anticipated, as for the 2014 Scottish independence referendum.

[2] The importance of two-player all-pay auctions is well established in the introduction of Siegel (2014). Likewise, the particular relevance of binary signal models is well-motivated in Chi et al.(2019).

applications, relaxing the assumptions of the all-pay auction might also be necessary.

However, solving for the equilibria of the all-pay auction game is in general difficult. The literature often imposes restrictive assumptions, and even then, generally restricts attention to a class of monotone equilibria. A *monotone equilibrium* in an auction is an equilibrium in which a higher bid is associated with a higher signal; thus, it is an ordered separating equilibrium. This monotonicity imposes structure on the equilibrium, which is then used to solve for the equilibrium. For the all-pay auction, Krishna and Morgan (1997) find the sufficient condition on the primitives for the existence of a monotone equilibrium in a symmetric model where signals are affiliated and continuously distributed. Siegel (2014) finds the counterpart for this condition for an asymmetric finite signal model. However, the Krishna, Morgan, and Siegel (henceforth KMS) condition can be violated in reasonable situations. When it is not satisfied, it is generally not known what kind of equilibria exist.

In particular, the KMS condition is violated when the players' signals exhibit strong dependence. This type of violation happens in an election campaign. The political candidates receive signals on the voting population's preferences. If the signals are derived from unbiased samples of the voting population, then there is a strong dependence between the signals of the different candidates. In this case, I show that a monotone equilibrium does not exist.

In order to move beyond the restrictions of the monotone equilibrium, I introduce the novel notion of a *weakly monotone equilibrium*. A weakly monotone equilibrium is an equilibrium in which the bid supports of each player are ordered by the strong set order.[3] When the equilibrium bid supports are disjoint as well, this is a monotone equilibrium. Further, when an equilibrium is both symmetric and weakly monotone, the probability of winning the auction increases in the player's signal.

The recent literature has characterized some equilibria that they consider non-monotone, for example, Liu and Chen (2016) and Chi et al. (2019). This paper classifies the non-monotone equilibria in both papers as weakly monotone. In fact, any equilibrium in the binary signal all-pay auction is weakly monotone if the generic condition is satisfied. In this sense, the generic condition weakly restores the monotonicity lost when departing from the KMS condition.

The main theorem of the paper states that, given a generic condition, a unique equilibrium exists. The unique equilibrium can only be one of four forms of equilibria, all

[3] If there are two signals, low and high, then each player has two different bid strategies depending on what signal is realized. In this setting, the strong set order is that for any two bids, one belonging to the low signal's bid-support and one belonging to the high signals bid-support, the minimum of the two bids belongs to the low signal's bid-support and the maximum belongs to the high signal's bid-support.

of which are weakly monotone and symmetric. Before this paper, it was known that a unique symmetric equilibrium exists for specific all-pay auction models. I strengthen these results to show that generally, there exists no asymmetric equilibrium.[4]

The generic condition states that the payoff for the two signals cannot be the same for all possible strategies the opponent can play. This condition holds for almost all sets of primitives. Further, it is satisfied for most relevant economic environments, e.g., private value models, common value models, and models in which the signals are correlated with the state of the world.

I show that in almost every symmetric all-pay auction, every equilibrium is symmetric.[5] This is perhaps surprising as there are plenty of symmetric games with asymmetric equilibria. Symmetry implies that the bid support for each different signal is an interval. A necessary condition of a weakly monotone equilibrium is that the bid supports are intervals. Hence these two properties are linked. Relaxing symmetry assumption yields the possibility of non-monotonicity, and disconnected bid supports in equilibrium. In the conclusion, I provide an asymmetric example that satisfies the generic condition in which a non-monotone equilibrium exists.[6]

The unique equilibrium can be one of four different forms, all of which are weakly monotone. One of the four weakly monotone equilibria is the traditional monotone equilibrium, Siegel (2014) provides the unique construction of such an equilibrium. This paper offers the unique construction and full characterization for the other three kinds of equilibria in the most general setting given binary signals and symmetry.

The first of three weakly monotone equilibrium is a completely overlapping equilibrium. An equilibrium is completely overlapping if the supports of bid strategies coincide regardless of the signal. The second weakly monotone equilibrium is dubbed the highly

[4]Rentschler and Turocy (2016) show that there exists a symmetric equilibrium in a symmetric n-signal model. They also provide an algorithm that can find a symmetric equilibrium. However, they do not characterize this equilibrium. My contribution is that in the binary model, I fully characterize the equilibrium their algorithm would produce and determine the unique equilibrium. Chi et al.(2019) use a model with an unknown world state in which the players receive affiliated signals while having log super modular valuation functions. In this model, they show that the all-pay auction has a unique symmetric equilibrium. I compliment their result by showing that it is the unique equilibrium, at least for the two-player case see corollary 2.4.16.

[5]When the generic condition is satisfied, then every equilibrium is symmetric. However, when the generic condition fails infinitely many equilibria exist. Also, infinitely many asymmetric equilibria exist.

[6]Any equilibrium that is not weakly monotone is non-monotone. Any equilibrium where the supports are not intervals is non-monotone. In general, an asymmetric model is more chaotic due to the existence of mass points at the zero bid. The form of equilibrium depends on which player has a mass point at zero, and it is ex-ante hard to know which player has a mass point.

competitive equilibrium and has been documented in the recent literature.[7] In the highly competitive equilibrium, profits are zero for both players, but a high signal player has a strictly higher probability of winning the auction.

I call the third and final weakly monotone equilibrium the uncompetitive equilibrium.[8] The uncompetitive equilibrium can arise because of negative dependence. If player 1 gets a high signal, it is more likely that player 2 has a low signal. Intuitively, if player 2 has a low signal and player 1 has a high signal, and payoffs are increasing in the signal, player 1 should win the auction had they played a monotone equilibrium provided that it exists. Nevertheless, in the uncompetitive equilibrium, there can be an upset, and player 2 wins.[9]

In the uncompetitive equilibrium, an upset happens because of rational overconfidence. Player 1 in the previous scenario is fairly certain that he is favored, and in knowing this, trades some probability of winning for a larger payoff if he wins. The smaller the uncertainty, the more he dares to lower his bid, with the asymmetric complete-information case as the limit. In doing so, the favored player allows the unfavored player to win with a higher probability.

A player with a high signal has a positive expected payoff in the uncompetitive equilibrium as in a monotone equilibrium. A player with a low signal has zero expected payoff. Further, the probability of winning with a high signal is higher than the probability of winning with a low signal.

I apply my all-pay auction model to explain election upsets. In this application, there are two unobserved states of the world and two candidates. The state represents the electoral preferences over the two candidates. In one state, candidate 1 is favored, and in the other state, candidate 2 is favored. The candidates observe a noisy public signal (the polls) which is correlated with the state. I connect the likelihood of an upset to the level of accuracy of the polls and explain that being favored by the polls can lead to rational overconfidence.

An upset is an event in which a low signal player beats a high signal player. The

[7] Rentschler and Turocy (2016) call the environment in which this equilibrium exists the highly competitive environment, hence the equilibrium is highly competitive. Liu and Chen (2016), Chi (2018), and Chi et al. (2019) characterize this equilibrium in their respective models. I generalize their characterization to interdependent valuations and arbitrary correlations.

[8] Liu and Chen (2016) characterize this uncompetitive equilibrium in a private value model with binary correlated signals. I generalize this characterization to interdependent valuations. They show existence by guess and verify. I provide a different approach as I show that the bid supports have to be connected intervals and weakly monotone in the strong set order. This implies that only four types of equilibria can exist. Furthermore, I show that given the correct assumptions on the primitives, the uncompetitive equilibrium exists and is the unique equilibrium of the game.

[9] An upset can also occur in the highly competitive equilibrium and the fully overlapping equilibrium.

two other models that can generate upsets are a complete information asymmetric all-pay auction and an asymmetric independent private value all-pay auction model. For the former model, one cannot investigate the informational quality of polls and their relationship to an upset. While for the latter, independence and private values rule out any form of public polls. Furthermore, the private realized value cannot contain any information about the true state of the world since this would imply a correlation between the candidates' values. Finally, since both models require asymmetry, they also require that one candidate is ex-ante favored, while the whole point of polling before an election is that this is uncertain.

Concerning the 2016 US presidential election, the model offers some insights. First, as expected, if the public polls were inaccurate, the model predicts a higher probability of an upset.[10] Second, and more interestingly, even if the polls were accurate, the model predicts Hilary Clinton to act in a complacent manner, and this allows an upset to occur. There is some evidence for this complacency as Trump held 14 campaign events in the swing state of Pennsylvania compared to only 10 for Clinton. Furthermore, Clinton's pattern of fewer campaign events is observable in all swing states that Trump won.[11]

The chapter is outlined as follows. I discuss the model in Section 2. Section 3 characterizes the different equilibria and leads to Theorem 1 that shows the uniqueness of the equilibrium and how the primitives of the model partition the equilibrium space. Section 4 discusses the payoff properties of each equilibrium. Section 5 provides the election upset application. Finally, in Section 6, I conclude. All omitted proofs are in the Appendix.

[10] After the 2016 presidential election, in response to the critique that election polls are unreliable. The American Association for Public Opinion Research (AAPOR) released a report. This report offered various theories to explain the inaccuracy of the polls. The leading theory states that voters make their decision late. Polls only take the early decision-makers into account, leading to a noisy signal. When this late group grows in size, the signal becomes noisier.

[11] The website FairVote.org has tracked all campaign visits for both candidates in 2016. Interestingly, in all 11 swing states, including Pennsylvania, Trump has more campaign visits than Clinton.

1.2 Binary-Signal Model

Consider a symmetric all-pay auction game with binary signals. A single indivisible object is auctioned to one of two risk-neutral players. Prior to bidding both players receive a private signal $t_i \in \{L, H\}$. In the all-pay auction, both players place a sealed bid $b_i \in \mathbb{R}_+$ at the same time. The player with the higher bid wins the object, but both players pay their bid. In the case of a tie, both players win with equal probability.

I define the non-negative function $f : \{L, H\} \times \{L, H\} \to [0, 1]$, where f is the known symmetric joint prior distribution over the signals. With slight abuse of notation, denote the marginal distribution as $f(t_i) := \sum_{\{L,H\}} f(t_i, t_{-i})$. I allow for any level of correlation and make no further assumptions on f, opposed to the full support assumption on the joint distribution that is generally used in the literature. For convenience, I do assume that $f(t_i) > 0$, such that, $f(t_{-i}|t_i) = \frac{f(t_i,t_{-i})}{f(t_i)}$ is still well defined.[12] The value of the prize is strictly positive and allows for interdependent valuation. Let $v_i : \{L, H\} \times \{L, H\} \to \mathbb{R}_{++}$ denote the value of the prize for player $i = 1, 2$. By symmetry $v_i(t_i, t_j) = v(t_i, t_j)$.

Throughout this text, I interchangeably use the word type and signal. That is, a low type has a low signal, and a high type has a high signal. A very useful object in the analysis of the all-pay auction is defined in equation (1.1). For $k \in \{L, H\}$ and $j \in \{L, H\}$ let

$$V^k(j) := f(j|k)v(k, j). \tag{1.1}$$

I call these objects "primal" because they depend only on the primitives of the model (f, v). These primal objects contain all the information necessary to completely characterize any equilibrium. $V^k(j)$ can be loosely interpreted as the contribution to the expected payoff for type k given that his opponent is of type j. Together these objects form the expected value of the prize for type k in equation (1.2). I make the following final assumption on the types that $H > L$ such that, $\mathbb{E}[v|\ H] \geq \mathbb{E}[v|\ L]$.[13]

$$\mathbb{E}[v|\ k] = V^k(L) + V^k(H). \tag{1.2}$$

A strategy $G_i = \left(G_i^L, G_i^H\right)$ for player i is a pair of cumulative distribution functions, i.e. $G_i^k(b) := \Pr(b_i \leq b|k)$. Define $\hat{G}_i^k(b) := \left(\Pr(b_i < b|k) + \frac{1}{2}\Pr(b_i = b|k)\right)$, then conditional on G_{-i}, the expected payoff for i's type k bidding b is

[12] This assumption is not necessary, if $f(t_i) = 0$, then the model coincides with a known common-value model.

[13] If you allow for $\mathbb{E}[v|\ H] < \mathbb{E}[v|\ L]$, then all the analysis is the same but with H and L switched. For example, the interpretation of a monotone equilibrium would then be a monotone decreasing equilibrium.

$$U_i(b, k \mid G_{-i}) = V^k(L)\hat{G}^L_{-i}(b) + V^k(H)\hat{G}^H_{-i}(b) - b. \tag{1.3}$$

I'm interested in the Bayes Nash equilibria of this game. A strategy profile $G := (G_1, G_2)$ is an equilibrium if for each player $i = 1, 2$ and type $k \in \{L, H\}$, G_i^k assigns measure 1 to $\text{argmax}_b U_i(b, k \mid G_{-i})$.

Throughout this paper, I denote the support of G_i^k as supp $\left[G_i^k\right]$. Since the support of a distribution is the smallest closed set with full measure and in any equilibrium, all bids are bounded there exists a maximum and a minimum element for supp $\left[G_i^k\right]$.

$$\begin{aligned} \bar{B}_i^{\,k} &= \max \text{ supp } \left[G_i^k\right] \\ \underline{B}_i^{\,k} &= \min \text{ supp } \left[G_i^k\right] \end{aligned} \tag{1.4}$$

In this paper, I distinguish three classes of equilibria: weakly monotone, monotone, and non-monotone. All three classes are defined in terms of the strong set order.

Strong Set Order, $\leq_s$ Consider two compact subsets of the real line, $A, B \subset \mathbb{R}$, define the binary relation $\leq_s$ as follows: $A \leq_s B$ if for all $b \in A$ and for all $b' \in B$; $\max\{b, b'\} \in B$ and $\min\{b, b'\} \in A$.[14] Further, denote $A <_s B$, if $\max A \leq \min B$.

Definition An equilibrium G is weakly monotone if for both i, supp $\left[G_i^L\right] \leq_s$ supp $\left[G_i^H\right]$.

Definition An equilibrium G is monotone if for both i, supp $\left[G_i^L\right] <_s$ supp $\left[G_i^H\right]$.[15]

Notice that monotone implies weakly monotone but the converse is generally not true. The difference between the two classes of equilibria is that in a weakly monotone equilibrium an interval of bids may belong to support of both types, while this is impossible in a monotone equilibrium. When a bid belongs to the interior (denoted by int) of the bid support of both types of a player, I call this an *overlap*.

Definition A strategy G_i has an overlap if int $\left(\text{supp}\left[G_i^L\right]\right) \cap$ int $\left(\text{supp}\left[G_i^H\right]\right) \neq \emptyset$.

The final class of equilibrium is the *non-monotone equilibrium*, an equilibrium is non-monotone if it is not weakly monotone.

[14] This is the definition of the strong set order when restricted to the real line. The general definition was first introduced in Veinott (1989) and popularized in Milgrom and Shannon (1994).

[15] This definition coincides with the standard definition in Siegel (2014). An equilibrium G is monotone if for both $i, b \in$ supp $\left[G_i^L\right]$ and $b' \in$ supp $\left[G_i^H\right]$ implies that $b' \geq b$.

1.3 Characterizing Equilibrium

In characterizing the equilibrium, the most effort is geared towards establishing that any equilibrium is symmetric (Proposition 1.3.2). First, some necessary conditions for equilibrium need to be established (Lemma 1.3.1). Second, the generic condition is introduced, with various implications for the First-order conditions (Appendix B.1). Finally, it is shown that every equilibrium is symmetric.

The rest of the results follow as a consequence of symmetry and the binary signal structure. Binary signals and symmetry imply that each bid-support is an interval, which implies that each equilibrium is weakly monotone (Proposition 1.3.3). Once at this point, one can use exhaustion and the First-order conditions (Henceforth, FOCs) to characterize each candidate for equilibrium fully. Theorem 1.3.8 provides the uniqueness of the equilibrium and the partition of the space of the primitives.

The following notation is used for the highest bid of i when playing a strategy G_i,

$$x_i^G := \sup\left(\text{supp}\left[G_i^L\right] \cup \text{supp}\left[G_i^H\right]\right). \tag{1.5}$$

Lemma 1.3.1 shows that in any equilibrium at least one type of each player has 0 in its support and both players share the same bid supremum, i.e. $x_1^G = x_2^G = x^G$. Further, the union of the supports of all types of a player forms a closed interval $\left[0, x^G\right]$, and on this interval the equilibrium bid distribution of each type for both players is continuous. Lastly, since this paper relaxes the full-support assumption for f, both players may have probability mass (an atom) for some type at the zero bid. This double atom at zero occurs for example in the election application in which both low types have an atom at zero.

However, if the full-support assumption is satisfied at most one player has an atom at zero. This is because at least one of the players has an incentive to bid ϵ above zero. Without full support, if both players have an atom at zero in equilibrium for some signal, then it is a zero probability event that both players have that signal at the same time. For example, in the election application, this is due to a perfect negative correlation, in which it is impossible that both players get a low signal simultaneously. Therefore, even if player 1 receives signal L, player 2 receives signal H, and therefore both players can have an atom at zero for the signal L, and neither player has an incentive to deviate.

Lemma 1.3.1. *In any equilibrium G,*

1. $\text{supp}\left[G_i^L\right] \cup \text{supp}\left[G_i^H\right] = \left[0, x^G\right]$.
2. *For both $k \in \{L, H\}$, G_i^k is continuous on $\left[0, x^G\right]$.*

3. *If* $V^k(j) > 0$ *and if* $G_i^k(0) > 0$ *then* $G_{-i}^j(0) = 0$ *and if* $G_{-i}^j(0) > 0$ *then* $G_i^k(0) = 0$.

Proof to the claims in lemma 1.3.1 can be found in appendix A.1. Lemma 1.3.1 implies that the union of the supports is an interval, but it does not imply that supp $\left[G_i^k\right]$ is. It could be that supp $\left[G_i^k\right]$ has a gap, but if G is weakly monotone then supp $\left[G_i^k\right]$ is an interval.[16]

To illustrate the difference between non-monotone and weakly monotone equilibria, consider figure 1. In figure 1 the primal objects are as follows: $V^L(L) = V^H(L) = \frac{1}{3}$, $V^H(H) = V^L(H) = \frac{2}{3}$. The figure illustrates the bid supports of 2 equilibria, $G^* = (G_1, G_2^*)$ and $G = (G_1, G_2)$.[17]

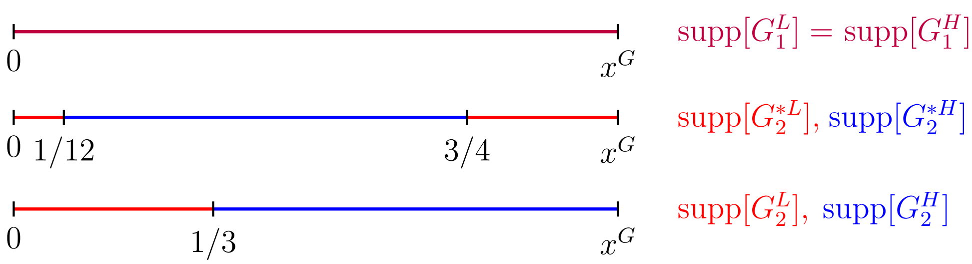

Figure 1.1: G_2^* is a non-monotone strategy since supp$[G_2^{*L}]$ has a gap between 1/12 and 3/4, which implies the supports can't be ordered by the strong set order. G_1 is a weakly monotone stragegy since supp $\left[G_1^L\right] \leq_s$ supp $\left[G_1^H\right]$ and it has an overlap. G_2 is a monotone strategy, since supp $\left[G_2^L\right] \leq_s$ supp $\left[G_2^H\right]$ and the interior of the supports are disjoint.

Notice that both $G^* = (G_1, G_2^*)$ and $G = (G_1, G_2)$ are asymmetric equilibria, where

[16]If the support has a gap it means that the support is not connected, i.e. there is an open interval of bids that belongs to the convex hull of the support but that does not belong to the support. A formal definition can be found in the appendix, definition A.1.

[17]The strategies for figure 1:

$$G_1^L(b) = G_1^H(b) = \begin{cases} b & b \leq 1 \\ 1 & b > 1 \end{cases}. \tag{1.6}$$

$$G_2^{*L}(b) = \begin{cases} 3b & b \leq \frac{1}{12} \\ \frac{1}{2} & \frac{1}{12} \leq b \leq \frac{3}{4} \\ 3b-2 & \frac{3}{4} \leq b \leq 1 \end{cases},\ G_2^{*H}(b) = \begin{cases} 0 & b \leq \frac{1}{12} \\ \frac{3}{2}b - \frac{1}{8} & \frac{1}{12} \leq b \leq \frac{3}{4} \\ 1 & b \geq \frac{3}{4} \end{cases}. \tag{1.7}$$

$$G_2^L(b) = \begin{cases} 0 & b \leq 0 \\ 3b & 0 \leq b \leq \frac{1}{3} \\ 1 & b \geq \frac{1}{3} \end{cases},\ G_2^H(b) = \begin{cases} 0 & b \leq \frac{1}{3} \\ \frac{3}{2}b - \frac{1}{2} & \frac{1}{3} \leq b \leq 1 \\ 1 & b \geq 1 \end{cases}. \tag{1.8}$$

G^* is non-monotone and G is weakly monotone.[18] In G^*, the probability of winning is decreasing in type for player 2, while the probability of winning is increasing in type in G and in both equilibria, the expected payoff for each player is zero. This example shows that there can be a multiplicity of equilibria, asymmetry, and non-monotonicity.

However, it turns out that these three things can only happen if the following knife-edge condition is met, that is, when $V^L(L) = V^H(L)$ and $V^H(H) = V^L(H)$. If the objects value strays but a little from these equalities such that $V^H(H) > V^L(H)$ or $V^L(L) < V^H(L)$, then there exists a unique equilibrium, which is symmetric and weakly monotone. Hence, I consider the condition that $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$ to be a *generic condition*, because it satisfied for almost all sets of primitives and the set where this condition is satisfied is dense in $\mathbb{R}^2$.[19]

When the generic condition is satisfied, any equilibrium is symmetric. The proof follows from the FOCs, which imply that if a player's bid belongs to their own overlap, this bid also belongs to the other player's overlap. Hence, any overlap is a mutual overlap. Further, on a non-overlapping subset $S \subset \left[0, x^G\right]$ of bids, the same type for each player must bid. For example, if $b \in \text{supp}\left[G_1^H\right]$ and $b \notin \text{supp}\left[G_1^L\right]$, then $b \in \text{supp}\left[G_2^H\right]$ and $b \notin \text{supp}\left[G_2^L\right]$.

Lastly, the bid-distribution densities are uniquely determined by the primitives, and the primitives are symmetric; hence these densities are the same. Since there are no mass points anywhere, the bid supports are symmetric, and both players' bid-distribution densities are the same, the bid distributions must be identical for both players. Hence the equilibrium is symmetric.

Proposition 1.3.2. *If $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$, then any equilibrium G is symmetric.*

Formal proof can be found in the appendix. Since all equilibria are symmetric, the i subscript is dropped. Symmetry allows me to show that $\text{supp}\left[G^k\right]$ is an interval. The interval property restricts the number of forms an equilibrium can take, and because of

[18] To show that G^* and G are both equilibria first notice that neither player ever bids above $x^G = 1$, since this gives a negative payoff. Second, player 2 is best responding in both equilibria because for for all $b \in [0,1]$, $U_2(b, H \mid G_1) = U_2(b, L \mid G_1) = V^L(L)G_1^L(b) + V^L(H)G_1^H(b) - b = \frac{1}{3}b + \frac{2}{3}b - b = 0$. Player 1 is best responding to G_2^*, since for all $b \in [0, 1/12]$ $U_1(b, H \mid G_2^*) = U_1(b, L \mid G_2^*) = V^L(L)G_2^{*L}(b) + V^L(H)G_2^{*H}(b) - b = \frac{1}{3}(3b) - b = 0$. For all $b \in [1/12, 3/4]$, $U_1(b, H \mid G_2^*) = U_1(b, L \mid G_2^*) = V^L(L)G_2^{*L}(b) + V^L(H)G_2^{*H}(b) - b = 1/12 + \frac{2}{3}(\frac{3}{2}b - \frac{1}{8}) - b = 0$. Similar substitutions can be done to show that $U_1(b, H \mid G_2^*) = U_1(b, L \mid G_2^*) = 0$ for all $b \in [3/4, 1]$ and that $U_1(b, H \mid G_2) = U_1(b, L \mid G_2) = 0$ for all $b \in [0, 1]$.

[19] That is, the conditions can be written as two difference, which are points in $\mathbb{R}^2$: $x_1 = V^H(H) - V^L(H)$ and $x_2 = V^H(L) - V^L(L)$. The set of points where these differences satisfy the generic condition is dense in $\mathbb{R}^2$. Only the origin of $\mathbb{R}^2$ is not in this set, see figure 1.3.

the binary signal structure, only two forms are non-monotone. In these non-monotone forms, the support of one type is completely contained within the other. For example, the high type's support is contained in the low type's support: $\underline{B}^L = 0$, $\underline{B}^H > 0$ and $\bar{B}^H < \bar{B}^L = x^G$.[20] In Proposition 1.3.3, I show that an equilibrium of such a non-monotone form cannot exist.

Proposition 1.3.3. *If $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$, then any symmetric equilibrium G is weakly monotone.*

Proof of Proposition 1.3.3, can be found in the appendix. Thus, if $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$ then any equilibrium G is symmetric and weakly monotone. Because of the binary structure only the following 4 forms can be an equilibrium, proof can be found in appendix, Lemma B.4.1. Figure 1.2 illustrates the equilibrium bid distributions for each of the 4 forms.

- **a)** Weakly monotone and completely overlapping: $\underline{B}^L = \underline{B}^H = 0$ & $\bar{B}^L = \bar{B}^H$.
- **b)** Monotone: $\underline{B}^L = 0$ and $\bar{B}^L = \underline{B}^H < \bar{B}^H$.
- **c)** Weakly monotone and semi-overlapping: $\underline{B}^L = \underline{B}^H = 0$ and $\bar{B}^L < \bar{B}^H$.
- **d)** Weakly monotone and semi-overlapping: $\underline{B}^H > \underline{B}^L = 0$ and $\bar{B}^L = \bar{B}^H$.

Form (b): The Monotone Equilibrium. The traditional monotone equilibrium has been thoroughly studied by the literature. For example, Krishna and Morgan (1997) and Siegel (2014) find the sufficient KMS condition for existence, and Lu and Parreiras (2017) give a necessary and sufficient condition for a continuous type model and Schouten (2021) provides a necessary and sufficient condition for any finite N type model. Due to the strong properties of this form, and its type revealing properties ex-post, this equilibrium is of interest. Lemma 1.3.4 provides the necessary and sufficient condition for the existence of a monotone equilibrium in this model.[21]

Lemma 1.3.4 (KMS)**.** $V^H(j) \geq V^L(j)\ \forall j \in \{L, H\} \iff$ *existence of an equilibrium of form **(b)**.*

Assuming KMS to hold is restrictive and therefore it's important to know the equilibrium behavior of the players for other values of the primitives. This is particularly

[20] Alternatively, L's is support is completely contained in H's support in this same way. That is, $\underline{B}^H = 0$, $\underline{B}^L > 0$ and $\bar{B}^L < \bar{B}^H = x^G$.

[21] In the binary model the KMS condition from Siegel (2014) and the necessary and sufficient condition from Schouten (2021) coincide. This implies that KMS is also necessary for the binary model, however, this is not generally true for any larger type spaces.

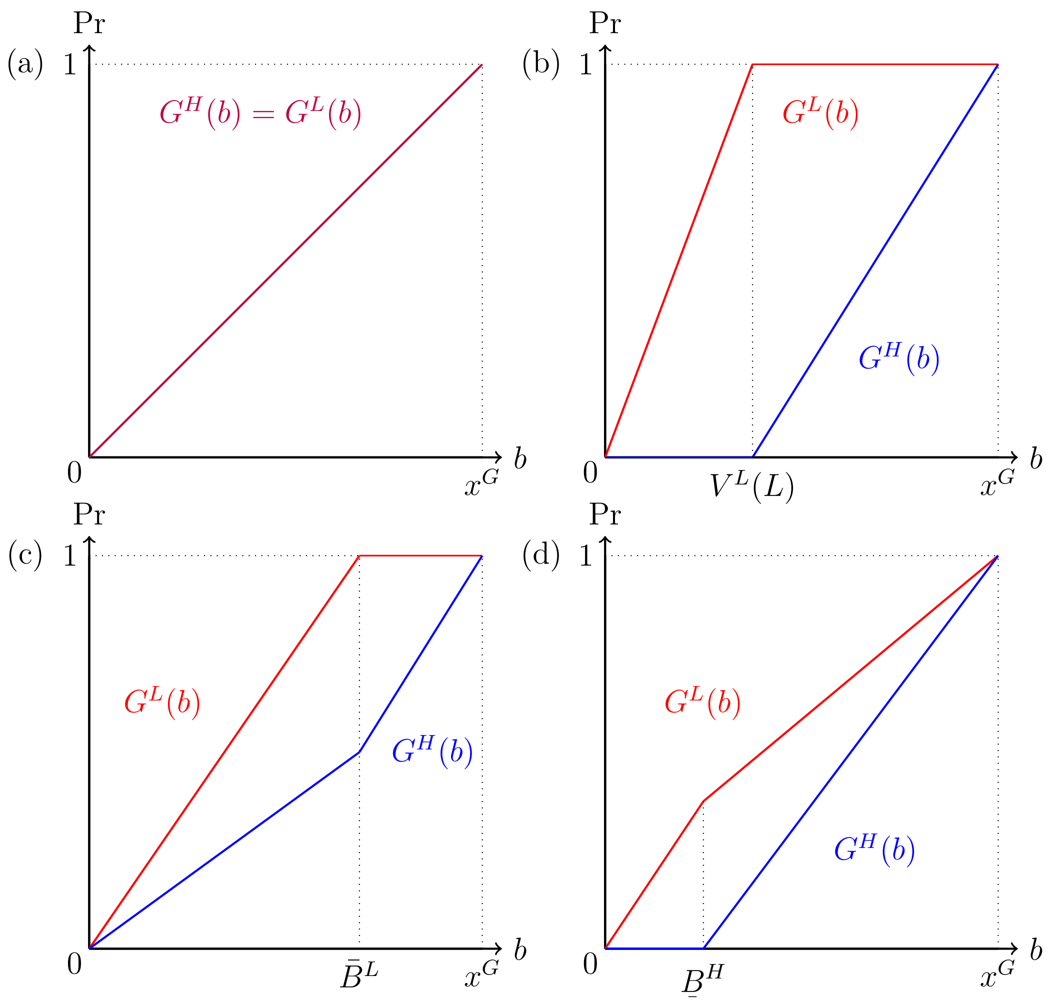

Figure 1.2: The four graphs are equilibrium bid distributions, with one distribution for each type. Each graph is labeled by the letter of the form. The y-axis, is the probabilities, while the x-axis is the bids. Thus, for each bid b, the y-axis provides the probability that a type bids b or below. Recall that the symbol x^G denotes the highest bid in equilibrium when bidding x^G the probability that type L or H bids lower is 1.

important if you want to be agnostic about exact values of the primitives and rather make qualitative assumptions. For example, $v(t_i, t_{-i})$ is increasing in t_i and t_1 and t_2 are negatively correlated, that is, $f(H|L) > f(H|H)$ and $f(L|H) > f(L|L)$. If the correlation is strong enough s.t $V^L(H) = f(H|L)v(L,H) > f(H|H)v(H,H) = V^H(H)$, there exists no monotone equilibrium.

Bid Densities on the Overlap. When KMS cannot be satisfied the equilibrium has an overlap. On the overlap a low signal player could upset the high signal player and win the auction. Further, if KMS doesn't hold and $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$ then either (1.9) or (1.10) is true.

$$V^H(H) > V^L(H) \text{ and } V^L(L) > V^H(L), \tag{1.9}$$

$$V^H(H) < V^L(H) \text{ and } V^L(L) < V^H(L). \tag{1.10}$$

Unlike when KMS holds, if either (1.9) or (1.10) is true, it is no longer always better to be the high type. A consequence of this is that now the low type can win the auction against a high type on the overlap. The slopes on the overlap are uniquely determined in equilibrium by the FOCs and for any bid b on the overlap, $\frac{\partial G^L(b)}{\partial b} = x_L$ and $\frac{\partial G^H(b)}{\partial b} = x_H$. Proof and more details on these claims can be found in the appendix.

$$x_L := \frac{V^H(H) - V^L(H)}{V^L(L)V^H(H) - V^L(H)V^H(L)}, \tag{1.11}$$

$$x_H := \frac{V^L(L) - V^H(L)}{V^L(L)V^H(H) - V^L(H)V^H(L)}. \tag{1.12}$$

Form (a): Fully Overlapping Weakly Monotone equilibrium. The fully overlapping equilibrium is straightforward and can only exist if the expected value of the prize is equal for both types. For any fully overlapping equilibrium it must be that $\frac{\partial G^H(b)}{\partial b} = \frac{\partial G^L(b)}{\partial b}$ for all b, and $x^G = \frac{1}{\frac{\partial G^L(b)}{\partial b}}$. This follows from the observation that bids supports of both types are the same and no type has an atom at zero. The proof of Lemma 1.3.5 can be found in the appendix.

Lemma 1.3.5. $\mathbb{E}[v|\ H] = \mathbb{E}[v|\ L] \iff$ *existence of an equilibrium of form (a).*

Form (c): The Highly Competitive Equilibrium. Form (c) has been characterized in Liu and Chen (2016), Chi (2018), and Chi et al (2019) in their respective models. They consider this form non-monotone since it could be that a low type beats a high type. But the bid-supports are ordered by the strong set order, and therefore I consider it weakly monotone. Furthermore, the probability to win the auction is increasing in type.

In this form, neither type has a positive expected payoff. This payoff is zero since neither player has an atom at the zero bid, which means that the payoff for bidding zero is zero. Every type has zero in its support. In equilibrium, each type is indifferent between all bids in its support. Hence, the expected payoff for each type is zero.

Form (c) is an equilibrium in what Rentschler and Turocy (2016) call the highly competitive environment. In terms of the primal objects, this means that (1.9) is satisfied.

The reason it is called "highly competitive" is because this kind of order can be born from dependence. If player 1 has a high signal it is more likely that player 2 has a high signal, likewise with a low signal. Thus, if a firm has a high signal about the demand of a market it is more likely that other firms have a high signal as well. Therefore, the high signal firms can expect greater competition. This dampens the profitability and therefore in this form a high signal player extracts no information-rent, unlike the monotone equilibrium.

The following construction generates a symmetric strategy profile consistent with form (c), and fully characterizes an equilibrium of this form. This construction can only be an equilibrium if $V^L(L) > 0$ and $V^H(H) > 0$.[22] Thus, $G^H(b)$ is well-defined in equilibrium.

Construction (c):

1. $x^G = \mathbb{E}\left[v|\ H\right] = V^H(L) + V^H(H)$

2. $\bar{B}^L = \frac{1}{x_L}$

3. $G^L(b) = \begin{cases} 0 & b < 0 \\ bx_L & b \in [0, \bar{B}^L] \\ 1 & b > \bar{B}^L \end{cases}$

4. $G^H(b) = \begin{cases} 0 & b < 0 \\ bx_H & b \in [0, \bar{B}^L] \\ \frac{b - V^H(L)}{V^H(H)} & b \in [\bar{B}^L, x^G] \\ 1 & b > x^G \end{cases}$

Lemma 1.3.6. *If $V^H(j) \neq V^L(j)$ for all $j \in \{L, H\}$ then construction (c) produces the unique candidate for an equilibrium of form (c).*

Form (d): The Uncompetitive Equilibrium. This form has received the least amount of attention in the literature.[23] Which is somewhat surprising as it has just as much economic meaning as form (c). In an equilibrium of this form, the players have a

[22]Since (1.9) is satisfied in equilibrium, then $V^H(L) \geq 0 \implies V^L(L) > 0$. Likewise, $V^L(H) \implies V^H(H) > 0$.

[23]The only published paper I'm aware of that mentions this form is Liu and Chen (2016). They characterize it in their binary correlated private value model. There is also a working paper from Chen (2019) that discusses it as a weak competition equilibrium. The published version Chen (2021) removed any mention of it.

positive expected payoff, although it is not monotone. Specifically, a high signal player has a positive expected payoff while the low signal player has zero expected payoff.[24]

When the primal objects are such that (1.10) is satisfied, this form is an equilibrium. When this form is an equilibrium, I call it uncompetitive, because the ordering of the primal objects can arise due to negative dependence, that is, it is more likely if player 1 has a high signal that player 2 has a low signal and vice versa. Because, a high signal player is more likely to face a low signal player, competition is lessened and the high signal player retains a profit. An application of the uncompetitive equilibrium can be found in section 1.5. The following construction generates a symmetric strategy profile consistent with form (d).

Construction (d):

1. $x^G = \mathbb{E}[v|\ L] = V^L(L) + V^L(H)$.
2. $\underline{B}^H = x^G - \frac{1}{x_H}$
3. $G^H(b) = \begin{cases} 0 & b \leq \underline{B}^H \\ x_H(b - \underline{B}^H) & b \in [\underline{B}^H, x^G] \\ 1 & b > x^G \end{cases}$.
4. If $V^L(L) > 0$, $G^L(b) = \begin{cases} 0 & b < 0 \\ \frac{b}{V^L(L)} & b \in [0, \underline{B}^H] \\ \frac{\underline{B}^H}{V^L(L)} + x_L\left(b - \underline{B}^H\right) & b \in [\underline{B}^H, x^G] \\ 1 & b > x^G \end{cases}$
5. If $V^L(L) = 0$, $G^L(b) = \begin{cases} 0 & b < 0 \\ \left(1 - \frac{x_L}{x_H}\right) & b = 0 \\ \left(1 - \frac{x_L}{x_H}\right) + x_L b & b \in [0, \frac{1}{x_H}] \\ 1 & b > \frac{1}{x_H} \end{cases}$

This equilibrium is interesting since it is an equilibrium in which both players can have an atom at zero. A double atom at zero occurs if $f(L, L) = 0$, which can happen if we have a perfect negative correlation. This happens in the Election-Upset application.

[24] You can observe that the payoff for H is positive because either $\underline{B}^H > 0$ when $V^L(L) > 0$ or $\underline{B}^H = 0$ when $V^L(L) = 0$ and L has an atom at zero. When the opponent has an atom at zero, you can bid zero and get a strictly positive payoff hence in equilibrium all bids must give a strictly positive payoff. When $\underline{B}^H > 0$, it must be that on the interval of bids $\left(0, \underline{B}^H\right)$ the low type bids. When $V^H(L) > V^L(L)$, H makes an incremental gain on this interval hence at $\underline{B}^H$ the payoff is strictly positive.

Lemma 1.3.7. *If $V^H(j) \neq V^L(j)$ for all $j \in \{L, H\}$ then construction (d) produces the unique candidate for an equilibrium of form (d).*

So far I have shown that if $V^H(j) \neq V^L(j)$ for at least $j \in \{L, H\}$, that any equilibrium G is monotone and symmetric, that it can take four possible forms and the forms are uniquely constructed. What remains to be shown is uniqueness and sufficiency which can be found in the appendix.

Theorem 1.3.8 (Partition Theorem)**.** *If $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$ then a unique equilibrium G exists, which is of form:*

a) *if* $\mathbb{E}[v|\ H] = \mathbb{E}[v|\ L]$

b) *if $V^H(j) \geq V^L(j)$ for all $j \in \{L, H\}$.*[25]

c) *If $V^H(H) > V^L(H)$ and $V^L(L) > V^H(L)$.*

d) *If $V^H(H) < V^L(H)$ and $V^L(L) < V^H(L)$.*

For a graphical representation of Theorem 1.3.8 see Figure 1.3. Theorem 1.3.8 implies that if $V^L(L)V^H(H) - V^L(H)V^H(L) \neq 0$, a unique weakly-monotone equilibrium exists. This is because $V^L(L)V^H(H) - V^L(H)V^H(L) \neq 0 \implies V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$. The converse is not necessarily true, if $V^H(H) = V^H(L) > V^L(L) = V^L(H)$ then $V^L(L)V^H(H) - V^L(H)V^H(L) = 0$, but a unique equilibrium exists, which is monotone.

Corollary 1.3.9. *If $V^L(L)V^H(H) - V^L(H)V^H(L) \neq 0$ then there exist a unique equilibrium.*

A standard model of private values with arbitrary correlation between the players' signals has a unique equilibrium. In this model each player's valuation depends only on their own private signal and the valuation increases with the signal. In such a model the generic condition is always satisfied and hence a unique equilibrium exists.

Corollary 1.3.10. *A correlated private value model in which the value is increasing with the signal has a unique equilibrium.*

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[25]Recall this is the KMS condition, and the generic condition requires one of inequalities to be strict. Because I assume that $\mathbb{E}[v|\ H] \geq \mathbb{E}[v|\ L] \iff V^H(H) + V^H(L) \geq V^L(H) + V^L(L)$, these 4 forms partition the set of primitives, since form (b) included all equilibria where one inequality is an equality.

Proof. WLOG let $v(t_i, t_{-i}) = t_i$. Need to show that $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$. Suppose not, then

$$\begin{aligned} V^H(L) = V^L(L) &\Longleftrightarrow f(H,L)f(L)H = f(L,L)f(H)L, \\ V^H(H) = V^L(H) &\Longleftrightarrow f(H,H)f(L)H = f(L,H)f(H)L. \end{aligned} \tag{1.13}$$

Substitution of one of the equations in (1.13) into the other yields that:

$$\frac{f(L,L)}{f(H,L)} = \frac{f(L,H)}{f(H,H)}. \tag{1.14}$$

By symmetry, $f(H,L) = f(L,H) \implies f(H,H) = f(L,L) = \frac{1}{4}$. But then (1.13) implies that $H = L$, which contradicts that $L < H$. □

In a model where there is an uncertain common value of the prize and in which each player gets an affiliated signal over this common value, a unique equilibrium exists. A version of such a model is used in Chi et al (2019).

Corollary 1.3.11. *An affiliated common-value model a la Chi et al (2019) has a unique equilibrium.*

Proof. Since, $V^H(H)V^L(L) - V^L(H)V^H(L) > 0$. □

1.4 Payoffs and Upsets

The expected payoff is zero for both players in forms (a) and (c). In forms (b) and (d), the expected payoff can be greater than zero. More precisely the expected payoff for the low type is still zero but the expected payoff for the high type is positive. When $V^L(L) > 0$ then for any $b \in \text{supp}\left[G^H\right]$ the expected payoff for the high type is:

$$U(b, H|\ G) = \left(\frac{V^H(L)}{V^L(L)} - 1\right)\underline{B}^H \geq 0. \tag{1.15}$$

If $V^L(L) = 0$ then for any $b \in \text{supp}\left[G^H\right]$ the expected payoff is given by (1.16).[26]

$$U(b, H|\ G) = V^H(L)G^L(0) > 0 \tag{1.16}$$

[26]If $V^L(L) = 0$ then both low types have an atom at zero in either form (b) or (d). For form (b) if $V^L(L) = 0$, then the low types of both players have an atom at zero, and the size of the atom is 1.

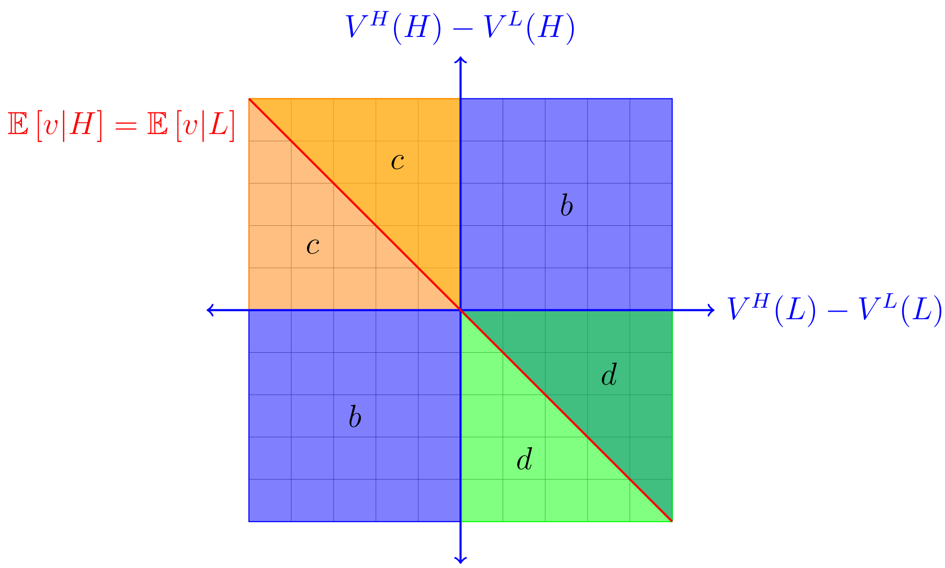

Figure 1.3: This figure is a graphical representation of Theorem 1, and each point in the picture maps to a unique equilibrium. The horizontal axis is the difference $V^H(L)-V^L(L)$ and the vertical axis is the difference $V^H(H) - V^L(H)$. On the red diagonal line, the unique equilibrium is the fully overlapping equilibrium. To the right of the red line are the weakly monotone increasing equilibria, and to the left of the red line are the weakly monotone decreasing equilibria. Decreasing equilibria are ruled out by the assumption that expected payoffs are increasing in type. If this assumption is relaxed, these equilibria exist. The axes are blue because the unique equilibrium is monotone on the axes. The remaining labels in the figure correspond to the labels given in the statement of Theorem 1. The generic condition only rules out the origin.

Clearly if the difference $V^H(L) - V^L(L)$ increases, (1.15) increases. In a monotone equilibrium, $\underline{B}^H = V^L(L)$, then the difference $V^H(L) - V^L(L)$ is the sole determinant of the expected payoff. In the weakly monotone equilibrium, the expected payoff depends on all the primal objects.

The probability of winning the auction as a low type against a high type (an upset) is zero in form (b). But the probability for an upset in the equilibrium of form (d) is greater than zero but less than a half. Denote the probability that a low type beats a high type as $Prob\{win = L|lose = H\}$. The calculations for the following propositions can be found in the appendix.

Proposition 1.4.1. *In an equilibrium of form (d):*

1. $U(b, H|\ G) = \mathbb{E}[v|\ H] - \mathbb{E}[v|\ L] > 0$ *for all* $b \in \operatorname{supp}[G^H]$.

2. $Prob\{win = L\,|\,lose = H\} = \frac{1}{2}\left[\frac{V^L(H)-V^H(H)}{V^H(L)-V^L(L)}\right] < \frac{1}{2}$.

Forms (a) and (c) also have a positive probability for an upset. Naturally, the probability for an upset in form (a) is a half. Since the expected payoff for both types in that equilibrium is the same it might be unfair to regard it as an upset. However, form (c) does have an upset probability in the true sense. A lower signal implies a lower expected value of the price, and still, the low signal player can win from a higher signal player, just like form (d).

Proposition 1.4.2. *In an equilibrium of form (c):*

1. $Prob\{win = L \,|\, lose = H\} = \frac{1}{2}\left[\frac{V^L(L)-V^H(L)}{V^H(H)-V^L(H)}\right] < \frac{1}{2}$.

1.5 Application: Election Upset

Suppose there is a blue candidate (b) and a red candidate (r) that compete in an election. I model this as an all-pay auction, in which the candidates simultaneously choose their effort to try to mobilize voters to come out and vote for them. The winner of the election is the candidate that chooses the highest effort.

Let there be two unordered unobserved states of world $\Omega = \{\omega_b, \omega_r\}$; that is, either the state is blue or red. The state represents the (majority) of the voting population preferences; they either prefer the blue or the red candidate. In the case of a referendum, it represents the populations preference over a single issue.[27]

If the candidate wins the election, they get a payoff of y^k, such that $k \in \{L, H\}$. When the state of the world matches the winning candidate's color, the winner gets $y^H > y^L$. Otherwise, the winner gets y^L. A candidate's payoff depends on the state of the world because they care to some extent about the well-being of the voter. If a candidate gets elected in the wrong state, they are happy about the win. However, they impose the "wrong" policy in the eyes of the population, and this backlash results in a lower payoff. However, for an utterly cynical candidate, the populations' preferences may not enter their payoffs directly. Regardless, it most likely does enter indirectly through reputation effects. The "wrong" policy affects this cynical candidate's legacy, which in turn may impact future elections performances and or other future work-related prospects.[28]

[27] Even in a general election; the preference of the population may be caputured by a single issue. For example, Nationalism versus Globalism, Public Health versus Economic performance, i.e lock downs or no lock downs.

[28] The analysis is more interesting if the payoffs of a candidate incorporate the state of the world. Without this assumption, both candidates are equally favored to win regardless of the state since they have no incentive to act on the state realizations. This implies they play the same strategy regardless of the state, which leads to an equilibrium of form (a).

An implicit assumption of this model is that it is a close election; either candidate can win the election regardless of the true state of the world. Thus, a blue (red) candidate can win in a red (blue) state. Without this implicit assumption, the winner is be pre-determined by the state of the world and neither candidate would put any effort.[29]

Information Structure of the Candidates. Neither candidate can observe the state of the world but they each get a signal about the state of the world $t_i \in T = \{B, R\}$. The state of the world is correlated with the signal by means of the known prior distribution f, in which $\alpha > \frac{1}{2}$ dictates the quality of the signal i.e. $f_i(\omega_b|B) = f_i(\omega_r|R) = \alpha$. That is, if $\alpha = 1$, both players are completely informed about the state of the world, if $\alpha = \frac{1}{2}$, they are completely uninformed. After the 2016 United States presidential election, the quality of the polls was under siege. The model would capture this with a low α (i.e. poor signal quality).

Since candidate i does not know the state of the world, he forms an expectation conditional on the realized signals of both candidates $\boldsymbol{t} = (t_1, t_2)$.

$$v_i(\boldsymbol{t}) := f_i(\omega_i|\boldsymbol{t})y^H + f_i(\omega_{-i}|\boldsymbol{t})y^L. \tag{1.17}$$

Then candidate i's expected payoff for exerting e_i effort with signal t_i is given by:

$$U_i(e_i, t_i \,|\, e_{-i}) = \sum_{t_{-i}\in\{R,B\}} \Pr\left(t_{-i}|t_i\right) v_i(\boldsymbol{t}) Pr\left[e_i > e_{-i}\right] - e_i. \tag{1.18}$$

Consider the mapping from this model to the reduced form all-pay auction model. Notice that for $i = b$: $H = B$ and $L = R$. While for $i = r$: $H = R$ and $L = B$. This means that for $i = b$, $\Pr\left(B|B\right) = f\left(L|H\right)$ and for $i = r$, $\Pr\left(B|B\right) = f\left(H|L\right)$.[30] Recall that $V^k(j) = f(j|k)v(k,j)$. Then the expected payoff is:

$$U_i(e_i, k|\, G_{-i}) = \sum_{j\in\{L,H\}} V^k(j) G^j_{-i}(e_i) - e_i. \tag{1.19}$$

The final assumption is that the two candidates use the same public signal to make their decisions. A public signal here is that the signal each player receives coincides. That is, either both players get a red signal or both get a blue signal.

[29] This implicit assumption is in line with observations in the real world. In 2016, Clinton spent more than 95% of her campaign visits in the 11 swing states and Trump spent more than 90% of his campaign visits in the 11 swing states according to non-profit organization FairVote. Suggesting that indeed neither candidate puts very much effort into the remaining 39 non-swing states.

[30] Further, for $i = b$: $\Pr\left(B|R\right) = f\left(L|L\right)$ and $\Pr\left(R|B\right) = f\left(H|H\right)$, vice versa for $i = r$.

- **Assumption 1 (Public Signal)** : $Pr\{B, R|\omega\} = Pr\{R, B|\omega\} = 0, Pr\{B, B|\omega_b\} = Pr\{R, R|\omega_r\} = \alpha$ and $Pr\{B, B|\omega_r\} = Pr\{R, R|\omega_b\} = 1 - \alpha$.

This kind of correlation structure is justified if both candidates use the same sample of voters for inference or follow the same news sources and polls to infer the state of the world. In Section 1.5.2, I change Assumption 1 and allow for signals to be private information.

1.5.1 Public Signal

In an environment with a public signal, the unique equilibrium is the uncompetitive equilibrium. Thus, there is a strictly positive probability for an *upset*. This means that if the public signal is blue, the red candidate has a positive probability of winning the election. This upset occurs because of rational overconfidence and costly effort.

In equilibrium, rational overconfidence occurs because the poll-favored candidate is more likely to have the majority of the popular support. Therefore, the favored candidate lowers her minimum effort to get a higher payoff if she wins. The behavior of the favored candidate is rational because she maximizes her expected payoff. The behavior is overconfident because she can win with certainty if she expends more effort.[31] In reaction to this lower effort, the unfavored candidate increases his maximum effort, which allows for an upset.

Lemma 1.5.1. *The unique equilibrium of this game exists is the uncompetitive equilibrium of form d.*

Proof. The prior joint distribution of signals with abuse of notation is $f(H, H) = f(L, L) = 0$ and $f(H, L) = f(L, H) = \frac{1}{2}$.[32] Then to find the primal objects,

$$\begin{aligned} V^H(H) =& V^L(L) = 0, \\ V^H(L) =& f(L|H)v(H) = \alpha y^H + (1-\alpha)y^L, \\ V^L(H) =& f(H|L)v(L) = (1-\alpha)y^H + \alpha y^L. \end{aligned} \tag{1.20}$$

Thus, $V^H(L) > V^L(H)$ since $\alpha > \frac{1}{2}$, this implies that $\mathbb{E}[v|H] > \mathbb{E}[v|L]$. Because $V^H(L) > V^L(L)$ and $V^L(H) > V^H(H)$, by theorem 2.4.4 the unique equilibrium is the uncompetitive equilibrium. □

[31] This uncompetitive equilibrium is reminiscent of the fable "The Tortoise and the Hare" in which the fast hare loses a race to the tortoise by taking a nap.

[32] Calculations are: $f(H, L) = f(\omega_B)f(B, B|\omega_b) + f(\omega_R)f(B, B|\omega_r) = \frac{1}{2} * \alpha + \frac{1}{2} * (1-\alpha) = \frac{1}{2}$, the other is similar.

An *upset* is the event in which the public signal indicates that the state is blue but the red candidate wins and vice versa. Let $t = B$ ($t = R$) be the event that both candidates receive a blue (red) signal. The probability of an upset is given by:

$$\Pr[\text{upset}] := \Pr\left[e_b < e_r | t = B\right] = \Pr\left[e_r < e_b | t = R\right]. \tag{1.21}$$

Using proposition 1.4.1, (1.22) provides a closed form solution for this probability,

$$\Pr[\text{upset}] = \frac{(1-\alpha)y^H + \alpha y^L}{2(\alpha y^H + (1-\alpha)y^L)}. \tag{1.22}$$

Equation (1.22) implies the probability of an upset is decreasing in α. If $\alpha = 1$, $Pr[\text{upset}] = \frac{y^L}{2y^H} < \frac{1}{2}$. As $\alpha \to \frac{1}{2}$, $Pr[\text{upset}] \to \frac{1}{2}$. When the signal becomes less informative, the probability of an upset increases. This result is intuitive, since when the polls have little predictive power the probability that they are wrong increases.

In this public signal setting, α is a measure of fundamental uncertainty. A low α means that the signal contains little information about the state of the world and the expected payoffs. When α increases, the expected payoff increases for the favored candidate, and the expected payoff decreases for the unfavored candidate, which discourages the unfavored candidate to put in the effort.[33]

An upset in my model occurs because of rational overconfidence. Concerning the 2016 US presidential election, Trump's upset in Swing states likes Pennsylvania may have been exacerbated by inaccuracy in polling. Since a low α implies a higher probability for an upset. The 2016 AAPOR post-election report list various reasons why polls might have been inaccurate. The central leading hypothesis of the report is the late decision-making theory. The late decision-making theory states that there is a group of voters that make their decisions very late or even in the voting booth and therefore polling is ineffective. The report gives empirical evidence of this theory in 4 swing states: Florida, Michigan, Pennsylvania, and Wisconsin. In these states, between 11-15 percent of the voters decided in the last week of the election.[34]

If candidates have private signals instead of a public signal then lower α no longer leads to a larger probability for an upset. This is because a low α implies that the equilibrium is monotone, in which no upset can occur. This means that any upset occurs

[33] Discouragement means that the unfavored candidate increases his probability mass at the zero bid, which reduces the overlap and reduces the probability of an upset.

[34] A discussion on 2016 election outcomes and late decision-making theory can be found in the full report at https://www.aapor.org/Education-Resources/Reports/. In particular, the report claims that major events before an election can impact late decision-makers. In 2016, the Clinton emails probe investigation was announced just prior to election day.

solely because of rational overconfidence.

1.5.2 Private Signals

When there is strategic uncertainty because each player has their own private predictions about the state of the world, the analysis changes. I replace Assumption 1 with Assumption 2, in which the signals are private and conditional on the state of the world the signals have no dependence. This implies that any correlation between the signals received by the candidates is through the state of the world. If both candidates do in-house polling based on their own collected samples, the results of the polls are strongly correlated if the samples are unbiased. Therefore, α not only dictates the accuracy of polls but also the level of correlation between the two polls.

Assumption 2 (Private Signals) : $Pr\{t_1, t_2|\omega\} = Pr\{t_1|\omega\} * Pr\{t_2|\omega\}$.
When signals are private there still is a unique equilibrium, but now it is either an uncompetitive equilibrium (form d) or a monotone equilibrium (form b). Proposition 1.5.2 shows that if α is sufficiently high, then the unique equilibrium is uncompetitive. Proposition 1.5.3 shows that the probability of an upset is increasing in α.

Proposition 1.5.2. *If* $\frac{y^H}{y^L} < \frac{\alpha}{(1-\alpha)}$ *then the unique equilibrium is an uncompetitive equilibrium. Otherwise the unique equilibrium is monotone.*

Proof. The calculations of the primal objects can be found in appendix.

$$\begin{aligned} V^H(H) &= \alpha(1-\alpha)\left(y^H + y^L\right) \\ V^L(L) &= \alpha(1-\alpha)\left(y^H + y^L\right) \\ V^H(L) &= \alpha^2 y^H + (1-\alpha)^2 y^L \\ V^L(H) &= (1-\alpha)^2 y^H + \alpha^2 y^L \end{aligned} \tag{1.23}$$

$$\begin{aligned} &V^H(H) - V^L(H) < 0 \\ &\qquad \iff \left[\alpha(1-\alpha) - (1-\alpha)^2\right] y^H + \left[\alpha(1-\alpha) - \alpha^2\right] y^L < 0 \\ &\qquad \iff \left((1-\alpha)y^H - \alpha y^L\right)[2\alpha - 1] \\ &\qquad \iff \frac{y^H}{y^L} < \frac{\alpha}{1-\alpha} \end{aligned} \tag{1.24}$$

Also notice that, $V^H(L) > V^L(L)$ and that $V^H(H) + V^H(L) - V^L(H) - V^L(L) = \mathbb{E}[v \mid H] - \mathbb{E}[v \mid L] > 0$ for all $\alpha > \frac{1}{2}$. Hence by theorem 2.4.4 the unique equilibrium is of form (d) when $\frac{y^H}{y^L} < \frac{\alpha}{1-\alpha}$, otherwise it is of form (b).

□

Proposition 1.5.3. *If $\frac{y^H}{y^L} < \frac{\alpha}{(1-\alpha)}$ then the probability of an upset is increasing in α.*

Proof. From proposition 1.4.1 follows that:

$$\begin{aligned}\Pr[\text{upset}] =& \frac{1}{2}\left[\frac{V^L(H) - V^H(H)}{V^H(L) - V^L(L)}\right] \\ =& \frac{\left(\alpha y^L - (1-\alpha)y^H\right)}{2\left(\alpha y^H - (1-\alpha)\, y^L\right)}\end{aligned} \tag{1.25}$$

$\frac{d\,Pr[\text{upset}]}{d\,\alpha} = \frac{(y^H)^2 - (y^L)^2}{2(\alpha y^H - (1-\alpha)y^L)^2} > 0$, since $y^H > y^L$. □

In this setting α affects the probability for an upset in two ways. The first effect is the same in the benchmark; high α implies low fundamental uncertainty, which reduces the probability of an upset. The second effect, that high α reduces strategic uncertainty, is absent in the benchmark. If α is high and a candidate receives a favorable signal then it becomes increasingly likely that the opponent has an unfavorable signal. This leads the favored candidate to be more rationally overconfident and lower their minimum bid, thereby increasing the probability of an upset. In equilibrium, the effect of strategic uncertainty outweighs the effect of fundamental uncertainty. Hence, the probability for an upset increases when α increases.

If $\frac{y^H}{y^L} \geq \frac{\alpha}{(1-\alpha)}$, then α is sufficiently low such that the unique equilibrium is the traditional monotone equilibrium. In this equilibrium, there can be no upsets. This is because when α is low, strategic uncertainty is high, which implies there is no rational overconfidence. The lack of rational overconfidence means a higher minimum bid for the favored candidate, which causes the equilibrium to separate. If $\frac{y^H}{y^L} < \frac{\alpha}{(1-\alpha)}$, strategic uncertainty is sufficiently low such that a high signal candidate dares to lower their minimum bid. In reaction to this, a low signal candidate raises their maximum bid allowing an upset to occur.

For the 2016 presidential election. If poll quality is bad, that is, α is low such that $\frac{y^H}{y^L} \geq \frac{\alpha}{(1-\alpha)}$, then a high signal candidate beats a low signal candidate with certainty. Therefore, in the private signal setting it cannot be that α is low and the Trump victory was an upset.[35]

Alternatively, If $\frac{y^H}{y^L} < \frac{\alpha}{(1-\alpha)}$, Trump's victory in the swing states can be explained as an upset. In this case, Clinton lost the election due to rational overconfidence. The narrative that Clinton supporters did not go out to vote could be correct. Within the model,

[35] An upset is a low signal candidate beating a high signal candidate, this cannot happen in a monotone equilibrium.

this is explained due to the lack of effort in the swing states by Hillary Clinton. There is evidence for this lack of effort. In all swing states, Clinton had fewer campaign visits than Trump and a higher percentage of Democrats didn't vote compared to Republicans.[36]

1.6 Conclusion

This chapter maps out the entire relevant set of equilibria in the all-pay auction. Even though it confines attention to a binary model, it motivates the use of a new class of monotone equilibria, those that are only weakly monotone. The use of a binary model does not have to be restrictive as Chi et al. (2019) mention in their introduction, since there are plenty of environments that can be modeled with just two signals. Furthermore, weakly monotone equilibria have economic relevance and have a useful structure that allows for monotone comparative statics.

That the equilibrium of the all-pay auction is unique is a useful feature, especially in any application where the desire is to make strong predictions. That the unique equilibrium is symmetric and weakly monotone allows this model to be easily applicable.

Future work can include extensions to any number of signals or perhaps to a continuum of signals. Other directions include asymmetric primitives and many bidders. In general, many bidders may be difficult as Siegel (2014) indicates in his conclusion, but Chi et al. (2019) show that within the binary setting there are tools to venture forth.

Extending to an asymmetric setting is non-trivial because in an asymmetric equilibrium there can be atoms at zero for only 1 player. My approach can't determine which player has the atoms at zero ex-ante. As a result of this atom, the equilibrium might not be weakly monotone as the following example shows.

Non-monotone Equilibrium in an Asymmetric Setting. Let the primal objects be such that $V_1^H(H) = 2 > V_1^L(H) = 1, V_1^H(L) = 2 > V_1^L(L) = 1, V_2^H(H) = 2 > V_2^L(H) = 1$ and $V_2^H(L) = 1 < V_2^L(L) = 2$. In a symmetric setting with player 1's primitives, the equilibrium would be monotone of the form (b) and in the symmetric setting with player 2's primitives, it would be a weakly monotone equilibrium of form

[36] Effort is costly in this model and the real world. This effort could be thought of as the number of campaign visits in a state. The non-profit organization FairVote collected data on the campaign visits for Hillary Clinton and Donald Trump in the 2016 election. It turns out that in all Swing States; Colorado, Florida, Iowa, Michigan, New Hampshire, Nevada, North Carolina, Ohio, Pennsylvania, Virginia, and Wisconsin, Trump had more campaign visits. Furthermore, Survey Monkey presented a survey after the 2016 election about registered voters that didn't vote. They found that a higher percentage of self-identified Democrats didn't vote compared to self-identified Republicans.

(a). Nevertheless, there exists a non-monotone equilibrium, in which player 1 plays a monotone strategy and player 2 plays a non-monotone strategy.

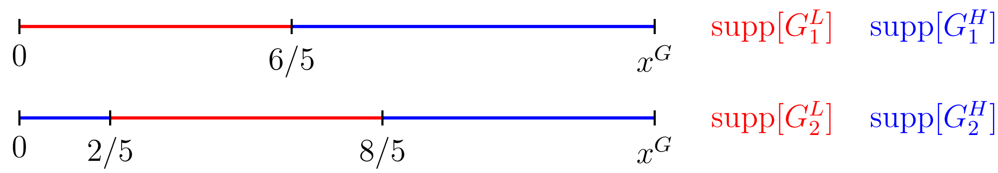

Figure 1.4: Player 1's low type has an atom at zero of size 0.2. Player 2's High type has a gap in the bid support between (2/5,6/5). More details and proof of equilibrium can be found in appendix B.

Chapter 2

The Symmetric Weakly Monotone Equilibrium of the All-Pay Auction

2.1 Introduction

The two-player all-pay auction is an excellent framework to study conflict with sunk costs.[1] This chapter is concerned with a general methodology for solving the all-pay auction. In particular, I use my approach to solve for a general all-pay auction model with N types, arbitrary level of dependence between private information, and interdependent valuation.

Many real-world applications can benefit from this chapter. Recall from Chapter 1 that in an election, there is a strong dependence between the private information of the two players through polling. Furthermore, the payoff of each candidate depends on the opponent's type (interdependent valuation). In a general N-type model, it is not known what kind of equilibria exist with such assumptions. While at the same time, having more than two types is often a more reasonable assumption.[2]

Traditionally, the incomplete information all-pay auction literature has been concerned with the study of monotone equilibria. A monotone equilibrium in an auction is an equilibrium in which a higher bid is associated with a higher type. In addition, a monotone equilibrium has economic relevance in that the allocation of the prize is efficient.

The original starting point was an all-pay model with independent private values

[1]There is extensive literature on two-player all-pay auctions, in which their importance has been established. For example in Siegel (2014), Lu and Parreiras (2017), and Chen (2021), to name a few.

[2]For example, a poll may predict that candidate a or b is favored. However, it is also possible that a poll predicts a tight race. This implies that we require at least three types.

(IPV).[3] In such an IPV model, the unique equilibrium is monotone. However, if valuations are interdependent or information is correlated, a monotone equilibrium may not exist.[4] In a setting with interdependence, Krishna and Morgan (1997) find the sufficient condition on the primitives for the existence of a monotone equilibrium in a symmetric model where types are affiliated and continuously distributed. Siegel (2014) finds the counterpart for this condition for an asymmetric finite type model.

One objective of this chapter is to replace The Krishna, Morgan, and Siegel (henceforth KMS) with a weaker condition, a condition that is not just sufficient but also necessary. I provide this with Condition B, which is a necessary and sufficient condition for the existence of a monotone equilibrium. Condition B is significant considering that there exist relevant economic environments where KMS is never satisfied, but a monotone equilibrium does exist. An example of this is the model of Chen (2021). In the appendix, I show that KMS is not satisfied for the model in Chen (2021) but Condition B is.[5]

The Condition B is novel in my environment of finite types with type dependence and interdependent valuations. However, with a continuum of types, Lu and Parreiras (2017) have provided necessary and sufficient conditions for the existence of a monotone equilibrium. Lu and Parreiras (2017) rely on the properties of their so-called tying function to define their conditions. The benefit of a symmetric model is that Condition B can be characterized fully in terms of the primitives.If I define Condition B for an asymmetric environment. I face the same problem as Lu and Parreiras. That is, Condition B is only partially characterized in terms of the primitives.

Even though the finite type and infinite type all-pay auction models appear to be closely related, the literature of both strands has been mostly separate. So far, very few attempts have been made to reconcile the two. One exception is Zheng (2019). In this paper, Zheng captures both models utilizing his distributional approach in an IPV setting.[6]

Chapter 1 has moved past the traditional monotone equilibrium in a binary type model. However, it doesn't provide existence conditions or characterizations for these equilibria in a general model. This chapter moves beyond the monotone equilibrium in

[3]Some examples of papers with IPV models include Hillman and Riley (1989), Amann and Leininger (1996), and Zheng (2019).

[4]See Chapter 1 for this non-existence of a monotone equilibrium in the election example.

[5]Proposition 1 of Chen (2021) proves that a monotone equilibrium exists, Condition B replicates his result as is shown in lemma C.3.1. The exercise in the appendix shows the ease of use of the condition and its applicability. The ease and applicability are particularly useful when the goal is revenue analysis for different economic environments and do not want to spend time on existence for each particular environment.

[6]An interesting project would be to see if it is possible to extend his distributional approach to a wider set of interdependence and correlation.

a general model.

First, I focus my attention on my novel notion of a weakly monotone equilibrium. A weakly monotone equilibrium is an equilibrium where the type's bid-supports are ordered by strong set order. This class of equilibria is of interest for a few reasons: the probability of winning the auction increases in type, the bid-supports are intervals, and in many economically relevant environments, every equilibrium is weakly monotone.[7]

Second, this Chapter characterizes four different "primary" forms of weakly monotone equilibria. These are "primary" because they exist for any general N type model. Further, if $N = 2$, they are the only form of equilibrium that can exist.[8] This paper is the first paper to provide different existence conditions for all primary forms in a general N-type model. Furthermore, I show that when $N > 2$ that there exist "mixed" forms. These mixed-form equilibria are combinations of two or more primary forms.

The model in this chapter is a slightly more general version of the model in Rentschler and Turocy (2016).[9] Their paper proves that a symmetric equilibrium exists and provides an algorithm that can find a symmetric equilibrium without characterizing the equilibrium. In this chapter, I characterize the economically relevant equilibria and find conditions for their existence.

Rentschler and Turocy's algorithm can compute the equilibrium by observing that the expected payoffs are a system of linear equations. Using a linear algebra approach, this chapter uses the same observation to provide existence conditions for different equilibria. The main contribution of this chapter is this new methodology for solving the all-pay auction. For example, by assuming the matrix of coefficients(the primitives of the model) to be invertible: i.e., the determinant is non-zero, the slopes of the bid-distributions are unique. When the slope is unique, certain forms of the multiplicity of equilibria are ruled out.[10]

This chapter outlines as follows. First, I establish the model and my linear algebra approach in sections 2 and 3. In section 4, I characterize equilibria's primary forms, provide existence conditions, and comment on mixed forms. I conclude in section 5 and

[7]First note that traditional monotone equilibrium is weakly monotone, but not vice versa. The interval property is useful for tractability when finding the equilibrium. Many economically relevant environments include; Correlated private values, binary type space models, and complete information models.

[8]Chapter 1 shows there exists a unique equilibrium of the binary model, which takes one of the primary forms.

[9]I allow for an arbitrary level of dependence, while Rentschler and Turocy (2016) assume that prior probability for both players having the same type is strictly positive. Their assumption rules out the election application model in the previous Chapter.

[10]In the language of Rentschler and Turocy (2016), the family of supporting solutions for an active set is one dimensional.

provide more details on examples and proofs in the appendix.

2.2 The Preliminaries

2.2.1 N-Type Model

Consider a symmetric all-pay auction game with two competing players. A single indivisible object is auctioned to one of the two risk-neutral players. In the all-pay auction, both players place a sealed bid $b_i \in \mathbb{R}_+$ simultaneously. The player with the higher bid wins the object, but both players pay their bid. In the case of a tie, both players win with equal probability.

Prior to bidding both players realize their private type $t_i \in T := \{1, 2, \cdots, N\}$. The types are distributed according to the symmetric joint prior distribution f. With slight abuse of notation let $f(t_i) := \sum_{j=1}^{N} f(t_i, j)$ denote the probability of a player being type t_i. I assume that $f(t_i) > 0$ such that $f(t_{-i}|t_i) = \frac{f(t_i, t_{-i})}{f(t_i)}$ is well defined.[11]

Let v_i denote the value of the prize for player $i = 1, 2$. The function $v_i(t_i, t_{-i})$ is strictly positive, real-valued and may depend on both players' types. For any two types $k, j \in T$ symmetry implies that $v_1(k, j) = v_2(k, j) = v(k, j)$. Because the value of the prize may depend on the opponent's type, define the *conditional payoff* for a player with type $t \in T$ when the opponent has type $j \in T$ as in (2.1).

$$V^t(j) := f(j|t)v(t, j) \tag{2.1}$$

If a player has type t, the conditional payoff is the probability of the opponent having type j multiplied by the value of the prize when the player's type is t and the opponent has type j. The conditional payoff objects form the *expected value* of the auctioned object for type t.

$$\mathbb{E}\left[v \mid t\right] = \sum_{j=1}^{N} V^t(j). \tag{2.2}$$

The expected value is the best guess of the value of the prize, given that I observe my type but not the opponent's type. I assume the expected value of the prize is (weakly) increasing in type.

[11] It is common to assume that f has full support: $f(t_1, t_2) > 0$ for all $(t_1, t_2) \in T^2$. I relax this assumption, and this allows for perfect positive and perfect negative dependence.

$$\mathbb{E}\left[v|\ t\right] \text{is increasing in } t. \tag{2.3}$$

2.2.2 Strategies and Weak Monotonicity

A strategy $G_i = (G_i^1, \cdots, G_i^N)$ for player i is N cumulative distribution functions, i.e. $G_i^t(b) := Pr(b_i \leq b|t)$. That is, if player 1 bids b, $G_2^t(b)$ dictates the probability that player 1 bids higher than type t of player 2. However, if there is a tie, the tie breaking rule implies that player 1 wins half the time. Define $\hat{G}_i^k(b) := \left(Pr(b_i < b|t) + \frac{1}{2}Pr(b_i = b|t)\right)$, then conditional on G_{-i}, the expected payoff for i's type $t \in T$ bidding b is

$$U_i(b, t\,|\,G_{-i}) = \sum_{j=1}^{N} V^t(j)\hat{G}_{-i}^j(b) - b. \tag{2.4}$$

I'm interested in the Bayes Nash equilibria of this game. A strategy profile $G := (G_1, G_2)$ is an equilibrium if for each player $i = 1, 2$ and each type $t \in T$, G_i^t assigns measure 1 to $\text{argmax}_b U_i(b, t\,|\,G_{-i})$.

Since the support (henceforth denoted by supp) of a distribution is the smallest closed set with full measure and in any equilibrium, all bids are bounded there exists a maximum, and a minimum element for supp $[G_i^t]$.

$$\begin{aligned} \max \text{ supp}\left[G_i^t\right] &= \bar{B}_i{}^t \\ \min \text{ supp}\left[G_i^t\right] &= \underline{B}_i{}^t \end{aligned} \tag{2.5}$$

Like Chapter 1, I distinguish three classes of Bayesian Nash equilibria: weakly monotone, monotone, and non-monotone. Recall, the strong set order ($\leq_s$): definition 1.2.

Definition An equilibrium G is weakly monotone if for both players, and for all $t < t'$: supp $[G_i^t] \leq_s$ supp $\left[G_i^{t'}\right]$. An equilibrium is monotone if for all $t < t'$: supp $[G_i^t] <_s$ supp $\left[G_i^{t'}\right]$. An equilibrium is non-monotone if the equilibrium is not weakly monotone.

2.3 Linear Algebra Approach

2.3.1 An Example

In order to motivate the linear algebra approach used in this chapter, consider the following example depicted in figure 2.1. Section 2.3.2 formalizes the approach described in the example. The example shows how the linear approach finds sufficient conditions for the existence of the equilibrium depicted in Figure 2.1.

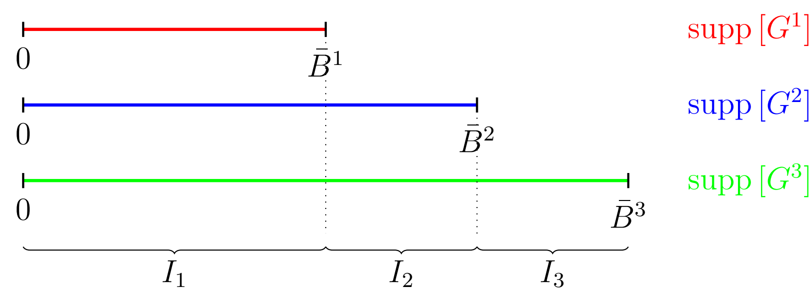

Figure 2.1: Let S denote the subset of types that bid on a set of bids. On the first interval I_1 all types bid, hence $S = T$. On the second interval I_2, only the two highest types bid, which means that $S = \{2, 3\}$. On third and last interval I_3, $S = \{3\}$.

If the construction in figure 2.1 is to be an equilibrium, then on each interval, each type has to be indifferent. For example, for any two bids $b, b' \in I_2$ the following must be true in equilibrium.

$$\begin{aligned} U_i(b', 3 \,|\, G_{-i}) - U_i(b, 3 \,|\, G_{-i}) &= \\ V^3(3)(G^3(b') - G^3(b)) + V^3(2)(G^2(b') - G^2(b)) - b' + b &= 0 \\ \iff V^3(3)\frac{G^3(b') - G^3(b)}{b' - b} + V^3(2)\frac{G^2(b') - G^2(b)}{b' - b} &= 1 \\ \iff V^3(3)\frac{dG^3}{db} + V^3(2)\frac{dG^2}{db} &= 1 \end{aligned} \tag{2.6}$$

$$\begin{aligned} U_i(b', 2 \,|\, G_{-i}) - U_i(b, 2 \,|\, G_{-i}) &= \\ V^2(3)(G^3(b') - G^3(b)) + V^2(2)(G^2(b') - G^2(b)) - b' + b &= 0 \\ \iff V^2(3)\frac{dG^3}{db} + V^2(2)\frac{dG^2}{db} &= 1 \end{aligned} \tag{2.7}$$

In other words, the first-order conditions need to be satisfied for types 2 and 3 on I_2. Equations (2.6) and (2.7) are linear system of equations which can be rewritten in matrix notation.

$$\begin{pmatrix} V^3(3) & V^3(2) \\ V^2(3) & V^2(2) \end{pmatrix} \cdot \begin{pmatrix} \frac{dG^3}{db} \\ \frac{dG^2}{db} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.8}$$

If $V^3(3)V^2(2) - V^3(2)V^2(3) \neq 0$, $\frac{dG^3}{db}$ and $\frac{dG^2}{db}$ are uniquely determined by the system (2.8). Equation (2.9) provides the slope of types 2 and 3 on I_2. This approach can be used to find the slopes on all three intervals.

$$\begin{aligned} \frac{dG^3}{db} &= \frac{V^2(2) - V^3(2)}{V^3(3)V^2(2) - V^3(2)V^2(3)} \\ \frac{dG^2}{db} &= \frac{V^3(3) - V^2(3)}{V^3(3)V^2(2) - V^3(2)V^2(3)} \end{aligned} \tag{2.9}$$

A necessary condition for equilibrium in figure 2.1 is that the slopes are such that they satisfy the linear system generated by the types that bid on it. However, this is not sufficient. In order for the bid-supports in figure 2.1 to be equilibrium, each type must not have a profitable deviation outside their bid-supports.

For example, consider type 1, which does not bid on I_2. Then, in equilibrium, it has to be the case that bidding on I_2 is not profitable for type 1. That is, any bid $b \in I_2$ should not give a higher payoff than $\bar{B}^1$.

$$\begin{aligned} U_i(b, 1 \,|\, G_{-i}) - U_i(\bar{B}^1, 1 \,|\, G_{-i}) &= \\ V^1(3)(G^3(b) - G^3(\bar{B}^1)) + V^1(2)(G^2(b) - G^2(\bar{B}^1)) - b + \bar{B}^1 &\leq 0 \\ \iff V^1(3)\frac{dG^3}{db} + V^1(2)\frac{dG^2}{db} &\leq 1 \end{aligned} \tag{2.10}$$

Since (2.8) is invertible, $V^1(3)$ and $V^1(2)$ can be written as unique linear combinations of the coefficients in (2.8).

$$\begin{aligned} V^3(3)\alpha_3 + V^2(3)\alpha_2 &= V^1(3) \\ V^3(2)\alpha_3 + V^2(2)\alpha_2 &= V^1(2) \end{aligned} \tag{2.11}$$

Plug (2.11) in (2.10) and see in (2.12) that if the coefficients are such that $\alpha_3 + \alpha_2 < 1$, type 1 has no incentive to deviate to any bid on I_2. The last equivalence follows from (2.6) and (2.7). One can use Cramer's rule to solve for the alphas in terms of the primitives. Hence, we found a sufficient condition for type 1 not to bid on I_2. This approach can be replicated at every interval. The intuition described here is used to derive Condition C and Condition D.

$$
\begin{aligned}
V^1(3)\frac{dG^3}{db} + V^1(2)\frac{dG^2}{db} &\leq 1 \\
\iff \left(V^3(3)\alpha_3 + V^2(3)\alpha_2\right)\frac{dG^3}{db} + \left(V^3(2)\alpha_3 + V^2(2)\alpha_2\right)\frac{dG^2}{db} &\leq 1 \\
\iff \alpha_3\left(V^3(3)\frac{dG^3}{db} + V^3(2)\frac{dG^2}{db}\right) + \alpha_2\left(V^2(3)\frac{dG^3}{db} + V^2(2)\frac{dG^2}{db}\right) &\leq 1 \\
\iff \alpha_3 + \alpha_2 &\leq 1
\end{aligned}
\tag{2.12}
$$

2.3.2 First Order Conditions and the Matrix of Coefficients

The bid supremum of a player's strategy is defined as follows.

$$
x_i^G := \sup\left(\cup_{j\in T}\operatorname{supp}\left[G_i^j\right]\right). \tag{2.13}
$$

Lemma 2.3.1. *In any equilibrium G,*

1. $\cup_{k\in T}\operatorname{supp}\left[G_i^k\right] = \left[0, x^G\right]$.
2. *For all $k \in T$, G_i^k is continuous on $\left[0, x^G\right]$.*
3. *For $k, j \in T$, if $V^k(j) > 0$ and if $G_i^k(0) > 0$ then $G_{-i}^j(0) = 0$ and if $G_{-i}^j(0) > 0$ then $G_i^k(0) = 0$.*

The results of lemma 2.3.1 have been discussed in Chapter 1. Essentially it states that on the bid space $[0, x^G]$, the bid-distributions for each type is continuous. Furthermore, the bid distributions are also differentiable almost everywhere on the bid space (proposition 2.3.2). Therefore, the first-order conditions need to be satisfied for almost every bid in equilibrium.

Proposition 2.3.2. *There is a open set $\mathbb{A} \subseteq \left[0, x^G\right]$ s.t. $\mathbb{A}$ has full Lebesgue measure in $\left[0, x^G\right]$ and for both i and all k, $G_i^k(\cdot)$ and $U_i(\cdot, k|G_{-i})$ are differentiable for all $b \in \mathbb{A}$.*

Proof of proposition 2.3.2 can be found in the appendix, it follows from Lebesgue's Theorem. Consider any subset of types $S \subseteq T$. Then let $\beta(S)$ be the subset of the differentiable points on which only the types S bid.

$$
\beta(S) := \mathbb{A} \cap \left\{b \in \left[0, x^G\right] \mid b \in \cap_{t\in S}\operatorname{supp}\left[G^t\right] \;\&\; b \notin \cup_{t\in T\setminus\{S\}}\operatorname{supp}\left[G^t\right]\right\} \tag{2.14}
$$

In any symmetric equilibrium, the following FOCs have to be satisfied for all $b \in \beta(S)$.

$$\forall t \in S : \sum_{j \in S} V^t(j) \frac{dG^j}{db} = 1 \tag{2.15}$$

Equation (2.15) is a linear system of equations used to solve for the slopes of the bid-distribution: $\frac{dG^j}{db}$. This is compactly written in matrix notation (2.16). Let $M(S)$ denote the *matrix of coefficients* (the coefficients are conditional payoff objects).[12] Let $\boldsymbol{x}$ be the vector of first derivatives, and let $\mathbf{1}$ denote a vector of ones. If $M(S)$ is invertible then in equilibrium $\beta(S)$ is non-empty only if equation (2.16) is satisfied.

$$\boldsymbol{x} = M(S)^{-1}\mathbf{1} : \boldsymbol{x} >> 0. \tag{2.16}$$

That $M(S)$ is invertible turns out to be a key assumption. In fact, if $T = \{L, H\}$ and $\det[M(T)] \neq 0$, there is a unique equilibrium which is symmetric and weakly monotone. In the general model, this assumption is not sufficient for such a strong result. However, it does lead to tractability, which allows for further results.

Generic Condition The Generic Condition is satisfied if for any subset $S \subseteq T$ s.t $|S| > 1$, if either one of the following conditions is true.

1. $\det[M(S)] \neq 0$.
2. $\det[M(S)] = 0$ but there are different two types $k, n \in S$: $V^k(j) > V^n(j)$ for all $j \in S$.

The first point implies that the slopes of the bid distributions are uniquely determined by (2.16) almost everywhere on the bid space $[0, x^G]$. The second condition is to allow for one type's payoff to be a strictly higher multiple of another type. For example, this second condition applies in an independent private value model.[13] That the condition is generic; is easily observed from the binary model where this condition is satisfied for almost every set of primitives.

[12] For example, let $T = \{1, 2, 3, 4, 5\}$ and if $S = \{1, 3, 5\}$, then

$$M(S) = \begin{pmatrix} V^5(5) & V^5(3) & V^5(1) \\ V^3(5) & V^3(3) & V^3(1) \\ V^1(5) & V^1(3) & V^1(1) \end{pmatrix}$$

The entries of the first row of $M(S)$ are the primal object for the highest type in S. The second row is for the second-highest type in S etc. The first column of $M(S)$ are the conditional payoff objects conditional on the opponent having the highest type in S. The second column conditional on the second-highest type in S, etc.

[13] In an independent private value model, $\det[M(S)] = 0$. However, since $v(k+1) > v(k)$: $V^{k+1}(j) = f(j)v(k+1) > f(j)v(k) = V^k(j)$ for all $k \in T$. This means that 2.18 is satisfied and a unique equilibrium exists which is monotone (Siegel, 2014).

Binary Model Generic Condition Let $T = \{L, H\}$, then $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$.

I leave it to the reader to check that the two conditions are equivalent in the binary setting. When the generic condition is satisfied, the slopes of the equilibrium bid distribution are uniquely determined by the first-order conditions.[14]

2.4 Characterizing Equilibrium

In order to characterize symmetric equilibria with type spaces this large, I confine attention to weakly monotone equilibria. Furthermore, I use specific equilibrium forms from Chapter 1 to put structure on the problem, allowing me to solve them. Chapter 1 shows that only four forms of equilibria can exist if the generic condition is satisfied in a binary type space. I call these four equilibria the primary forms, and they also exist in the general N-type model.

This chapter provides necessary and sufficient conditions for the existence of primary form (a) and primary form (b). Primary form (b) coincides with the notion of a traditional monotone equilibrium. These two forms have very simple bid-distribution slopes and constructions. I can use these two features to prove directly that my conditions A and B are necessary and sufficient for form a (Lemma 2.4.1) and form b (Theorem 2.4.4).

Forms (c) and (d) are tricky to characterize fully in a model where $N > 3$. It is tricky because the bid-distribution slope can become quite complex. Essentially, as the linear problem grows in dimension with N, the slopes are no longer closed form and can only be characterized by algorithms. This prevents a brute force approach used for forms (a) and (b).

The method I used to find the sufficient conditions for forms (c) and (d) are as follows. First, the generic condition and condition 2 imply that if a bid does not belong to the bid-support of a type, the FOCs are not satisfied at that bid (Lemma 1.3.6). This rules out some forms of non-monotone equilibria, but more importantly, it simplifies equilibrium verification since no type is ever indifferent outside its support.

Further restrictions imposed on Condition 2 results in Conditions C and D. In Condition C, not only are the FOCs not satisfied outside the bid support, but the type also makes an incremental loss outside his support. Condition D is the opposite in that now

[14]With generic condition satisfied for $S \subseteq T$ and if (1.11) is satisfied, then S is an active admissible set in the words of Rentschler and Turocy (2016). Furthermore, the generic condition implies that the family of supporting solutions for an active set is one dimensional.

the type makes incremental gain outside his support. These conditions imply several necessary conditions for the equilibrium to be weakly monotone (Lemma 2.4.8 and Lemma 2.4.14). This allows me to formulate sufficient conditions for the existence of form c (Corollary 2.4.12) and form d (Corollary 2.4.16), the proof for both is constructive.[15]

Throughout this paper, I also provide some evidence of mixed form equilibria (example 2.4.3). These mixes are combinations of different primary forms. I also provide an example in which no weakly monotone equilibria exist (example 2.4.2).

All the primal forms and their mixes are weakly monotone equilibria. Weak monotonicity implies that the bid-support of each type is an interval, further the probability to win the auction is increasing in type. However, it is not generally true that expected payoff is increasing in type (even if the expected value of the prize is strictly increasing in the type). Regardless, even if the equilibrium was strongly monotone (definition 1.2) instead of weakly monotone (definition 1.2), the expected payoff does not have to be increasing in type.[16]

2.4.1 Primary Form (a) and Primary Form (b)

Form (a): Fully Overlapping Weakly Monotone equilibrium.

An equilibrium of form (a) is the most simple; it requires that each type has the same bid distribution. The construction of form (a) follows these four simple steps.

Form (a):

1. $x^G := \mathbb{E}\left[v \mid N\right]$.
2. For all $t \in T$, $\operatorname{supp}\left[G^t\right] = \left[0, x^G\right]$.
3. For all $t \in T$, $G^t(0) = 0$.
4. Forall $t \in T$, and all $b \in \left[0, x^G\right]$, $\frac{dG^t}{db} = \frac{1}{\mathbb{E}[v \mid N]}$.

The necessary and sufficient conditions for the existence of such an equilibrium is given by (2.17). Like in Chapter 1, condition (2.17) states that the expected value of the prize needs to same for all types.

$$\textbf{Condition A} : \mathbb{E}\left[v \mid N\right] = \mathbb{E}\left[v \mid t\right] \;\; \forall t \in T. \tag{2.17}$$

[15]Conditions C and D are not in terms of the primitives, although they could be if a linear algebra problem in the appendix was solved.

[16]For example, let $T = \{1, 2, 3\}$, and $V^1(j) = 1$ and $V^2(j) = 2$ for all $j \in T$. Furthermore, let $V^3(1) = 1 + \epsilon$, $V^3(2) = 2 + \epsilon$ and $V^3(3) = 4$, s.t. $0 < \epsilon < \frac{1}{2}$. Condition B is satisfied for this example hence there exists a monotone equilibrium (form b). However, in equilibrium $U_i(\cdot, 2 \mid G_{-i}) = 1 > 2\epsilon = U_i(\cdot, 3 \mid G_{-i})$.

Lemma 2.4.1. *If* $\mathbb{E}[v \mid N] = \mathbb{E}[v \mid k]$ *for all* $k \in T \iff$ *form (a) is an equilibrium.*

Proof. ($\implies$) Notice that by point 3 of form a, that there are no atoms at zero. Hence, the expected payoff (2.4) at the zero bid is zero. Further, every type is indifferent between all bids in the bid space. Thus, the expected payoff for all bids on the bids space is zero. That is, for all $b \in \left[0, x^G\right] : U_i(b, k \mid G_{-i}) = \sum_{j=1}^{N} V^k(j)\hat{G}^j_{-i}(b) - b = \frac{\sum_{j=1}^{N} V^k(j)}{\mathbb{E}[v \mid t_N]} b - b = 0$. This follows from $\sum_{j=1}^{N} V^k(j) = \mathbb{E}[v \mid k] = \mathbb{E}[v \mid N]$. Further, it never a best response for any type to bid above $x^G = \mathbb{E}[v \mid t]$. So, no type has an incentive to deviate from form (a).

($\impliedby$) By means of contradiction suppose not, then there exists a $k \in T$ s.t. $\mathbb{E}[v \mid k] \neq \mathbb{E}[v \mid N]$. If $\mathbb{E}[v \mid k] < \mathbb{E}[v \mid N]$, then all bids in the open interval $\left(\mathbb{E}[v \mid k], x^G\right)$ are not best responses for type k. If $\mathbb{E}[v \mid k] > \mathbb{E}[v \mid N]$, then $U_i(x^G, k \mid G_{-i}) > U_i(b, k \mid G_{-i})$ for all $b \in \left[0, x^G\right)$, this implies that all bids on the open interval $\left(0, x^G\right)$ are not best responses for type k. An open interval has positive measure hence form (a) is not an equilibrium.

□

Form (b): The Monotone Equilibrium.

An equilibrium of form (b) is an equilibrium that is monotone in the sense of definition 1.2. Siegel (2014) provides sufficient conditions for the existence of a monotone equilibrium. This KMS condition is given by (2.18). Intuitively, KMS states that the expected payoff is increasing in a type independent of the opponent's type.

$$[\boldsymbol{KMS}]: \; V^t(j) \geq V^{t-1}(j) \;\; \forall j \in T \tag{2.18}$$

Condition 2.18 is also necessary if the type space is binary, but beyond two types, it is no longer necessary, as the example shows.

Example 1. Let $T = \{1, 2, 3\}$ and consider the following matrix of coefficients for the entire type space.

$$M(T) = \begin{pmatrix} V^3(3) & V^3(2) & V^3(1) \\ V^2(3) & V^2(2) & V^2(1) \\ V^1(3) & V^1(2) & V^1(1) \end{pmatrix} = \begin{pmatrix} 3 & 2.5 & 0.5 \\ 2 & 2 & 2 \\ 2.5 & 1 & 1 \end{pmatrix}$$

First note that the generic condition (2.3.2) is satisfied for this example.[17] Clearly, KMS (2.18) is not satisfied. Because, $V^3(1) = 0.5 < 1 = V^1(1)$ and $V^2(3) = 2 < 2.5 = V^1(3)$. But there exists a monotone equilibrium, which is illustrated in figure 2.2.

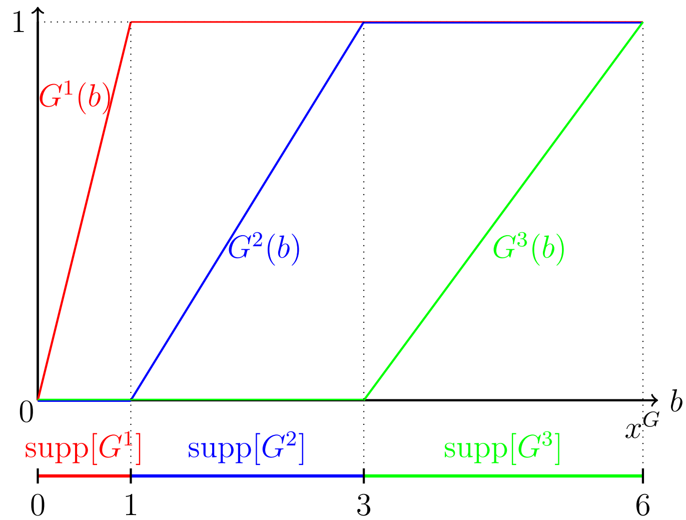

Figure 2.2: In this example observe that $\frac{dG^t}{db} = \frac{1}{t}$ on the support of each type. Notice that on the support of type 1, type 3 makes an incremental loss. That is, for $b \in [0, 1]$: $\frac{d\,U_i(b,3\,|\,G_{-i})}{d\,b} = V^3(1)\frac{dG^1}{db} - 1 = -\frac{1}{2}$. Further, on the support of type 2, type 3 makes an incremental gain. That is, for $b \in [1, 3]$: $\frac{d\,U_i(b,3\,|\,G_{-i})}{d\,b} = V^3(2)\frac{dG^2}{db} - 1 = \frac{1}{4}$. Since the length of the support of type 1 is one and length of the support of type 2 is two, the entire loss on the first interval is exactly offset by the entire gain on the second interval. Thus, type 3 is indifferent between bidding $b = 0$ and $b = 3$, and strictly worse off for all bids in between. By construction each type is indifferent on its own support i.e. $[3, 6]$, so type 3 has no incentive deviate. Similar calculations can be done to show that no type has an incentive to deviate.

Now consider the following necessary and sufficient condition for the existence of a monotone equilibrium.

Condition B For all $k \in T$, and for all $J_L \leq k - 1$

$$\sum_{j=1}^{J_L} \left[V^k(k-j) - V^{k-j}(k-j) \right] \geq 0 \tag{2.19}$$

[17]For the subsets $S = T, S = \{2, 3\}$ and $S = \{1, 3\}$, $\det M(S) \neq 0$. For $S = \{1, 2\}$, $\det M(S) = 0$, but type 2's primal objects are a strictly higher multiple of type 1's conditional on these subset of types (The entries of the row of type 2 of this matrix of coefficients is exactly twice that of the entries of the type 1's row). Thus, the second clause of (2.3.2) is satisfied.

and for all $J_H \leq N-k$

$$\sum_{j=1}^{J_H} \left[V^k(k+j) - V^{k+j}(k+j)\right] \leq 0 \tag{2.20}$$

Lemma 2.4.2. *Example 1 satisfies Condition B.*

Proof. Condition b for $N=3$ is satisfied if the inequalities in (2.21), (2.22) and (2.23) are satisfied. This is the case, where all inequalities are strict except for $J_L = 2$ in (2.21).

For $k=3$, $J_L \leq 2$ and $J_H \leq 0 \implies$

$$\begin{aligned} J_L = 1: \ V^3(2) &\geq V^2(2) \\ J_L = 2: \ V^3(2) + V^3(1) &\geq V^2(2) + V^1(1) \end{aligned} \tag{2.21}$$

For $k=2$, $J_L \leq 1$ and $J_H \leq 1 \implies$

$$\begin{aligned} J_L = 1: \ V^2(1) &\geq V^1(1) \\ J_H = 1: \ V^2(3) &\leq V^3(3) \end{aligned} \tag{2.22}$$

For $k=1$, $J_L \leq 0$ and $J_H \leq 2 \implies$

$$\begin{aligned} J_H = 1: \ V^1(2) &\leq V^2(2) \\ J_H = 2: \ V^1(2) + V^1(3) &\leq V^2(2) + V^3(3) \end{aligned} \tag{2.23}$$

□

Now compare the KMS condition with Condition B. The KMS condition requires $V^k(j)$ to be increasing in k for every j. Condition B does not require this, and in fact for $k > j$ it allows $V^k(j) < V^j(j)$ as long as $\sum_{i=j}^{k-1} V^i(i) \leq \sum_{i=j}^{k-1} V^k(i)$. In a symmetric monotone construction, each type t keeps the opponent type t indifferent on her support. Hence, if Condition B is satisfied, the sum of the incremental changes in utility is positive when type t reaches his own bid support from below, preventing a profitable deviation below his own bid support. Likewise, summing the incremental changes on all bid-supports above t's leads to a loss in utility.

Thus, Condition B guarantees against a global deviation. This is the reason why in example 1, $V^3(1) < V^1(1)$ is not enough to undo the monotone equilibrium. Because $V^3(1)+V^3(2) = V^1(1)+V^2(2)$ implies that the expected payoff for type 3 is non-negative when reaching his own bid-support.

In example 1, $V^1(3) - V^2(3) > 0$ also violates KMS, but again this does not violate Condition B. The reason for this is that the particular ordering of those primal objects plays no role in the construction of the symmetric monotone equilibrium. That is, in a monotone equilibrium, the opponent's type 3 never keeps type 2 indifferent on her support, and therefore type 1 cannot profit from the fact that $V^1(3) - V^2(3) > 0$.

In Condition B, if I restrict attention to $J_L = J_H = 1$, a necessary condition for the existence of a monotone equilibrium arises. That is, KMS has to be satisfied for all adjacent types only conditional on the opponent's type t. This is stated in lemma 2.4.3, from this lemma then follows that in a binary model KMS is necessary.

Lemma 2.4.3. *An equilibrium of form (b) exists only if* $V^{t-1}(t) \leq V^t(t) \leq V^{t+1}(t)$.

The proof of lemma 2.4.3 follows directly from theorem 2.4.4.

Theorem 2.4.4. *Condition B* $\Longleftrightarrow$ *the existence of an equilibrium of form (b).*

Proof. For the entirety of this proof assume that both players are playing G, the strategy prescribed by Siegel's construction.[18]

($\Longleftarrow$) (contrapositive) Suppose Condition B does not hold for some $k \in T$. This implies there exists a $J_L \leq k - 1$ such that:

$$\sum_{j=1}^{J_L} \Big[V^k(k-j) - V^{k-j}(k-j)\Big] < 0 \tag{2.24}$$

or a $J_H \leq N - k$ s.t.

$$\sum_{j=1}^{J_H} \Big[V^k(k+j) - V^{k+j}(k+j)\Big] > 0 \tag{2.25}$$

If (2.24) is true then k is making a loss on the J_L supports before his support, which implies he has a profitable deviation bidding the infimum of union of these interval. This follows from Siegel's construction, since for all $b \in \operatorname{supp}\big[G^k\big]$, $U_i(b, k \,|\, G_{-i}) = \pi_k$. When k bids $\underline{B}^k = \inf \operatorname{supp}\big[G^k\big] = \sum_{j=1}^{k-1} V^j(j)$, then $\sum_{j=1}^{N} V^k(j)\hat{G}^j_{-i}(\underline{B}^k) = \sum_{j=1}^{k-1} V^k(j)$, since $\hat{G}^j_{-i}(\underline{B}^k) = 1$ for all $j \leq k-1$ and $\hat{G}^j_{-i}(\underline{B}^k) = 0$ for all $j > k - 1$.

$$U_i(\underline{B}^k, k \,|\, G_{-i}) = \sum_{j=1}^{k-1} V^k(j) - \sum_{j=1}^{k-1} V^j(j) = \sum_{j=1}^{k-1} \Big(V^k(j) - V^j(j)\Big) = \pi_k. \tag{2.26}$$

[18] Siegels construction for a symmetric model implies $\operatorname{supp}\big[G_i^k\big] = \Big[\sum_{j=1}^{k-1} V^j(j), \sum_{j=1}^{k} V^j(j)\Big]$ and that $\frac{\partial G_i^k(b)}{\partial b} = \frac{1}{V^k(k)}$ $\forall b \in \operatorname{int}\big(\operatorname{supp}\big[G_i^k\big]\big)$. Siegel (2014) shows that this is the unique construction for a monotone equilibrium. Hence, if this construction does not yield an equilibrium, there is no monotone equilibrium.

But, k is better of bidding $\underline{B}^{k-J_L} = \sum_{j=1}^{k-1-J_L} V^j(j)$ instead, because of equation (2.24),

$$U_i(\underline{B}^k, k \,|\, G_{-i}) - U_i(\underline{B}^{k-J_L}, k \,|\, G_{-i}) = \sum_{j=k-J_L}^{k-1} \left[V^k(j) - V^j(j)\right] = \sum_{j=1}^{J_L} \left[V^k(k-j) - V^{k-j}(k-j)\right] < 0. \tag{2.27}$$

Likewise, if (2.25) is true, then k could make a profitable deviation on the J_H intervals above his support. Thus, Siegel's construction is not an equilibrium, Siegel (2014) implies that there exists no monotone equilibrium.

($\Longrightarrow$) Suppose Condition B is satisfied. Now k is contemplating a deviation to any $b \notin \text{supp}\left[G^k\right]$. If there exists a $b < \underline{B}^k$ (i.e. $k \neq 1$ and $\underline{B}^k \neq 0$), then there exists a $J_L \leq k-1$ s.t. $b \in \text{supp}\left[G^{k-J_L}\right] = \left[\sum_{j=1}^{k-J_L-1} V^j(j), \sum_{j=1}^{k-J_L} V^j(j)\right]$. Now either $V^{k-J_L}(k-J_L) \geq V^k(k-J_L)$ or $V^{k-J_L}(k-J_L) \leq V^k(k-J_L)$, if the former is true then k makes an incremental loss on supp $\left[G^{J_L}\right]$ and so WLOG let $b = \underline{B}^{k-J_L}$. But k has no incentive to deviate to $\underline{B}^{k-J_L}$ because (2.19) implies that:

$$U(\underline{B}^k, k|G_{-i}) - U(\underline{B}^{k-J_L}, k|G_{-i}) = \sum_{j=1}^{J_L} \left[V^k(k-j) - V^{k-j}(k-j)\right] \geq 0 \tag{2.28}$$

Likewise if $V^{k-J_L}(k-J_L) \leq V^k(k-J_L)$ then k makes an incremental gain on supp $\left[G^{k-J_L}\right]$, WLOG let $b = \bar{B}^{k-J_L} = \underline{B}^{k+1-J_L}$. Since (2.19) is satisfied for all J_L it is also satisfied for $J_L - 1$.[19] Hence,

$$U(\underline{B}^k, k|G_{-i}) - U(\underline{B}^{k+1-J_L}, k|G_{-i}) = \sum_{j=1}^{J_L-1} \left[V^k(k-j) - V^{k-j}(k-j)\right] \geq 0. \tag{2.29}$$

Thus, k has no incentive to deviate below his support. A similar proof can be used to show that he would not want to bid above his support. Therefore, as required, Siegel's construction produces an equilibrium.

□

[19] Notice that if $J_L = 1$ then the difference is zero, since $\bar{B}^{J_L} = \underline{B}^k$, and no deviation on $b \in \text{supp}\left[G^{J_L}\right]$ is profitable. If $J_L = 2$, then $U(\underline{B}^k, k|G_{-i}) - U(\bar{B}^{k-2}, k|G_{-i}) = V^k(k-1) - V^{k-1}(k-1) > 0$, etc.

2.4.2 Primary Form (c) and Primary Form (d)

The Generic Condition ensures a weak monotone equilibrium in the binary model. The Generic Condition is not enough to guarantee weak-monotonicity in the general model. An arbitrary form of non-monotonicity can arise when the set of best responses for a type is a strict superset of the bid-support. The type is indifferent on this complement region of the support, and a gap can be constructed. A gap implies a non-monotone equilibrium.[20]

Condition 2 ensures that if a bid does not belong to the bid support, the first-order conditions are not satisfied at that bid. This implies that the set of best responses in equilibrium is the same as the bid support almost everywhere. Hence in equilibrium, it is impossible to have a strategy with a gap, such that for all bids belonging to the gap, the player is indifferent. Example 3 shows that if Condition 2 is not satisfied, how such a gap can occur.

Condition 2 is satisfied if for all $k \in T$ and any subset $S \subseteq T \backslash \{k\}$:

$$V^k(j) = \sum_{t \in S} \alpha_t V^t(j) : \ \forall j \in S. \tag{2.30}$$

implies that $\sum_{t \in S} \alpha_t \neq 1$.[21]

In the binary type model from Chapter 1, Condition 2 is satisfied for almost all parameter values. Furthermore, because of the binary-type structure, even when Condition 2 is not satisfied, the unique equilibrium is weakly monotone. As Example 3 shows, when $N \geq 3$, the latter is no longer the case.

Calculating Condition 2 in terms of the primitives becomes increasingly costly as N increases. When $N = 2$, the condition is equivalent to 2 parameter assumptions (2.33). When $N = 3$ this increases to 9 parameter assumptions (2.34, 2.35, and 2.36). For general N the amount of parameter assumptions are provided by (2.32). Proof of Lemma 2.4.5 is relegated to Appendix C.1.

[20] Connected bids supports are a necessary condition for a weak monotone equilibrium, see Lemma A.1.8.

[21] The condition (2.30) can also be written in matrix notation(2.31). Where $V^k(S)$ is the vector of $V^k(j)$ for all $j \in S$. Let $M(S)^T$ denote the transpose of M(S), and a_t is the vector of linear coefficients. By the generic condition $M(S)^T$ is Invertible, hence a_t is uniquely determined. Now if all the elements in a_t don't sum to 1, condition 2 is satisfied.

$$M(S)^T a_t = V^k(S) \tag{2.31}$$

$$N\left[\sum_{k=1}^{N-1}\frac{(N-1)!}{k!(N-1-k)!}\right] \tag{2.32}$$

Lemma 2.4.5. *When $N=2$, Condition 2 is equivalent to:*

$$V^1(j)\neq V^2(j)\ \forall j\in\{1,2\}. \tag{2.33}$$

When $N=3$, Condition 2 is equivalent to following 9 conditions (3 per type):

$$\begin{gathered}V^3(2)\neq V^2(2)\ \&\ V^3(1)\neq V^1(1)\\ \&\ \Big(V^1(1)-V^2(1)\Big)\Big(V^3(2)-V^2(2)\Big)\neq\Big(V^3(1)-V^2(1)\Big)\Big(V^1(2)-V^2(2)\Big)\end{gathered} \tag{2.34}$$

$$\begin{gathered}V^2(3)\neq V^3(3)\ \&\ V^2(1)\neq V^1(1)\\ \&\ \Big(V^3(1)-V^2(1)\Big)\Big(V^1(3)-V^2(3)\Big)\neq\Big(V^1(1)-V^2(1)\Big)\Big(V^3(3)-V^2(3)\Big)\end{gathered} \tag{2.35}$$

$$\begin{gathered}V^1(3)\neq V^3(3)\ \&\ V^1(2)\neq V^2(2)\\ \&\ \Big(V^1(2)-V^2(2)\Big)\Big(V^3(3)-V^2(3)\Big)\neq\Big(V^2(2)-V^3(2)\Big)\Big(V^2(3)-V^1(3)\Big)\end{gathered} \tag{2.36}$$

Example 2. Let $T=\{1,2,3\}$ and consider the following matrix of coefficients for the entire type space.

$$M(T)=\begin{pmatrix}V^3(3) & V^3(2) & V^3(1)\\ V^2(3) & V^2(2) & V^2(1)\\ V^1(3) & V^1(2) & V^1(1)\end{pmatrix}=\begin{pmatrix}4 & 3 & 1\\ 2 & 3 & 1.5\\ 1 & 2.5 & 2\end{pmatrix}$$

Note that the generic condition is satisfied.[22] There exists a non-monotone equilibrium depicted in figure 2.3. This equilibrium is non-monotone because there is a gap in support of type 3. Since $V^3(2)=V^2(2)=3$, it is clear by equation (2.34) that Condition 2 is not satisfied. That $V^3(2)=V^2(2)$ implies that conditional on only facing an opponent with type 2, the payoff for a type 3 and 2 is the same.

On the gap region, only type 2 bids. Since the Generic Condition is satisfied, the bid density is uniquely determined on the gap. By the FOC when $S=\{2\}$ (see 2.15), the equilibrium bid density is $\frac{dG^2}{db}=\frac{1}{V^2(2)}$. Now because $V^3(2)=V^2(2)=3$, type 3 is indifferent between all bids on the gap, i.e. the FOC are satisfied.

[22] $M(T)$ is a positive definite matrix, which implies that all principal minors are positive. So what remain is check that $M(S)$ is non-singular for $S=\{1,2\}$ and $S=\{1,3\}$, which they both are.

For any bid $b \in (3.5, 5)$:

$$\frac{d\,U_i(b, 3 \,|\, G_{-i})}{d\,b} = V^3(2) * \frac{d\,G^2}{d\,b} - 1 = 0 \tag{2.37}$$

If instead Condition 2 was satisfied, for example if $V^3(2) = V^2(2) + \epsilon$ with $\epsilon > 0$. Then equation (2.37) implies that type 3 makes an incremental gain on the gap region, which implies that any bid below $b = 5$ cannot be a best response.

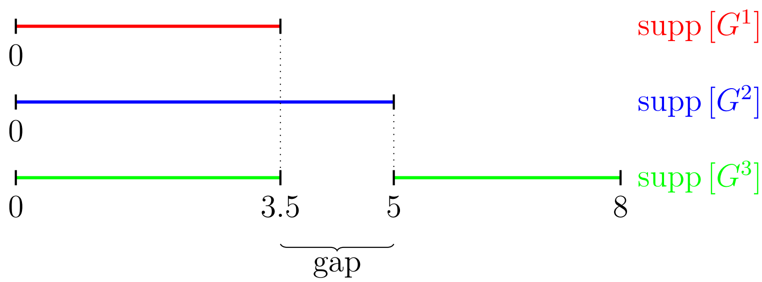

Figure 2.3: The following non-monotone construction is an equilibrium of Example 3. On the first interval, which has a length of $\frac{7}{2}$, all types bid and type 1 exhaust her bid probability. Then on the second interval only type 2 bids, she exhausts her remaining probability on the interval of length $\frac{3}{2}$. Finally, on the last interval of length 3, the highest type exhausts her remaining bid probability. Hence, Type 3 has a gap in her support.

Lemma 2.4.6. *If the Generic Condition is satisfied and Condition 2 is satisfied then in equilibrium if $b \notin supp[G^k]$ then $\frac{d\,U(b,k|\,G_{-i})}{d\,b} \neq 0$.*

Proof. Let $x_j = \frac{d\,G^j(b)}{d\,b}$ and consider any $k \in T$ and $S \subseteq T \backslash \{k\}$. For any $b \in \beta(S)$ the first order conditions imply:

$$\forall t \in S : \sum_{j \in S} V^t(j) x_j = 1 \tag{2.38}$$

Then since $M(S)$ spans its space by the Generic Condition. There exist a unique linear combination of the rows in $M(S)$, such that for all $j \in S$:

$$V^k(j) = \sum_{t \in S} \alpha_t V^t(j) \tag{2.39}$$

But by Condition 2, this combination is not affine hence $\sum_{t \in S} \alpha_t \neq 1$. Then some substitution yields the required result.

$$\begin{aligned}\frac{dU(b,k|\,G_{-i})}{db} &= \sum_{j\in S} V^k(j)x_j - 1\\ &= \sum_{j\in S}\sum_{t\in S}\alpha_t V^t(j)x_j - 1\\ &= \sum_{t\in S}\alpha_t\sum_{j\in S} V^t(j)x_j - 1\\ &= \sum_{t\in S}\alpha_t - 1 \neq 0\end{aligned} \tag{2.40}$$

□

Essentially, Condition 2 regulates the slope of expected payoff for a type k outside of k's bid-support. When Condition 2 is not satisfied for some subset $S \subseteq T\backslash\{k\}$, then on $\beta(S)$ type k is indifferent.

At first glance, Condition 2 may seem similar to the Generic Condition, but there are a few important differences. The generic condition requires that the row of type k in a system that includes k is not a linear combination of any of the other rows in this system. While Condition 2 states that for any system $M(S)$ (s.t. $k \notin S$). The row vector $\boldsymbol{y} = [V^k(\max S), \cdots, V^k(\min S)]$ is not a affine combination of any of the rows in $M(S)$.

Any vector $\boldsymbol{y}$ can be uniquely written as a linear combination of rows in $M(S)$, if $M(S)$ spans the space $\mathbb{R}^{|S|}$.[23] However, this linear combination might not be affine. If this row vector $\boldsymbol{y}$ is an affine combination, then in any equilibrium where $\beta(S) \neq 0$, type k is indifferent.

If Condition 2 is satisfied no gap can arise because one type is indifferent on any measurable subset outside of her bid-support. However, a gap can also arise if the gap can be partitioned into at least 2 parts. On the first part of the gap, the type with a gap makes an incremental loss, and on the second part of the gap, the type with a gap makes an incremental gain, s.t. the loss on the first and the gain on the second part exactly offset each other.[24] Condition C and Condition D deal with these kinds of gaps.

[23] A linear combination places no restrictions on its coefficients, while an affine combination is a linear combination in which the sum of the coefficient is one. A convex combination is an affine combination in which the coefficients are all positive. The differences matter, assuming convex-combinations are unnecessarily restrictive. In contrast, affine combinations lead to the result in Lemma 2.4.6.

[24] Naturally the gap can have more parts as long as in the end, the gap type is indifferent between the two endpoints, and it makes an incremental loss at the beginning and an incremental gain at the start.

Form (c): The Highly Competitive Equilibrium

An equilibrium of form (c) is a weakly monotone equilibrium. This form is called the highly competitive equilibrium because it can exist when there is a strong positive dependence. When there is a strong positive dependence, it is more likely that a player faces an opponent of the same type. This increases competition and is reflected by the expected payoff for the players, which is always zero in this equilibrium.

Chapter 1 provides necessary and sufficient conditions for the existence of an equilibrium of form (c) in the binary-type space. Further, for a binary-type space this form has received attention in Chen (2016) and Chi et al. (2019). However, no work has been done to prove existence of this form in a general N-type model.

Form (c):

1. For all $t \in T$, $0 \in \text{supp}[G^t]$ and $G^t(0) = 0$.
2. $\text{supp}[G^t]$ is an interval.
3. $\bar{B}^t$ is strictly increasing in t.
4. $\text{supp}[G^t] = [0, x^G]$

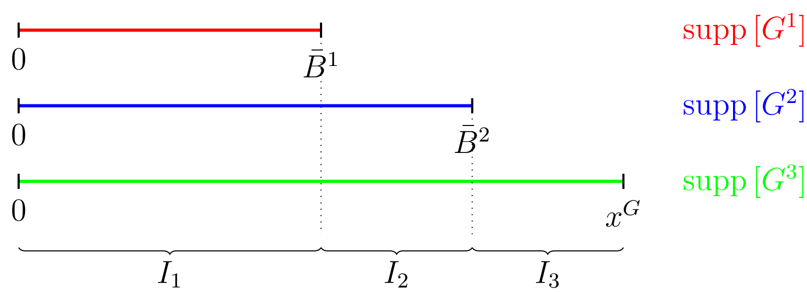

Figure 2.4: In a model with N types, an equilibrium of form (c) has N intervals that partition the bid space. On the first interval I_1 all types bid, and on every interval after, the lowest type drops out and does not bid there such that on the last interval I_N only the highest type N bids.

From Chapter 1, if $N = 2$ and the Generic Condition is satisfied, a necessary and sufficient condition for the existence of an equilibrium of form (c) is given by (2.41).

$$\text{For } j \neq k : V^k(k) > V^j(k). \tag{2.41}$$

The intuition behind (2.41) is that conditional on my opponent's type, it is more valuable (or more likely) to match that type. The main implication of (2.41) in the binary model is that if type j were to bid outside her own bid-support, this leads to an incremental loss. This intuition is key for the Condition C. Which ensures that each type makes an incremental loss on any subset of bids outside of its support: Lemma 2.4.7.

Condition C is satisfied for a type $k \in T$ if for all subsets $S \subseteq T \backslash \{k\}$;

$$V^k(j) = \sum_{t \in S} \alpha_t V^t(j) : \ \forall j \in S. \tag{2.42}$$

Where $\alpha_t \in \mathbb{R}$ such that $\sum_{t \in S} \alpha_t < 1$.

Lemma 2.4.7. *If Condition C is satisfied for type $k \in T$ then for all $b \in [0, x^G] \backslash \operatorname{supp}\left[G^k\right]$: $\frac{dU(b,k|\, G_{-i})}{db} < 0$.*

The proof of lemma 2.4.7 is almost the verbatim as the proof of lemma 2.4.6.

Hence, It is necessary that (2.41) is holds for Condition C to be satisfied.

Lemma 2.4.8. *Condition C is satisfied for all $k \in T$ only if (2.41) is satisfied and $V^k(k) > 0$ for all $k \in T$.*

Proof. Take any singleton set $S = \{j\}$, then Condition C states for any $k \neq j$ that $V^k(j) < V^j(j)$. This implies that $V^j(j) > 0$ since $V^k(j) \geq 0$. □

Like Condition 2, it is possible to derive conditions in terms of the primitives. The proof of Lemma 2.4.9 is the same as for 2.4.5 with the exception that every $\neq$ is $<$ instead. Notice that if $N = 2$, Condition C is equivalent to the necessary and sufficient conditions for the existence of form (c) from Chapter 1 (Theorem 1.3.8).

Lemma 2.4.9. *When $N = 2$, Condition C is equivalent to:*

$$V^1(2) < V^2(2) \ \& \ V^2(1) < V^1(1) \tag{2.43}$$

When $N = 3$, Condition C is equivalent to following 9 conditions (3 per type):

$$\begin{gathered} V^3(2) < V^2(2) \ \& \ V^3(1) < V^1(1) \\ \& \ \left(V^1(1) - V^2(1)\right)\left(V^3(2) - V^2(2)\right) < \left(V^3(1) - V^2(1)\right)\left(V^1(2) - V^2(2)\right) \end{gathered} \tag{2.44}$$

$$\begin{gathered} V^2(3) < V^3(3) \ \& \ V^2(1) < V^1(1) \\ \& \ \left(V^3(1) - V^2(1)\right)\left(V^1(3) - V^2(3)\right) < \left(V^1(1) - V^2(1)\right)\left(V^3(3) - V^2(3)\right) \end{gathered} \tag{2.45}$$

$$\begin{gathered} V^1(3) < V^3(3) \ \& \ V^1(2) < V^2(2) \\ \& \ \left(V^1(2) - V^2(2)\right)\left(V^3(3) - V^2(3)\right) < \left(V^2(2) - V^3(2)\right)\left(V^2(3) - V^1(3)\right) \end{gathered} \tag{2.46}$$

An example of when Condition C is satisfied can be found the correlated private value example (2.4.3). Lemma 2.4.7 implies that in equilibrium, the bid-support is an interval and that zero is in the bid-support for any type.

Lemma 2.4.10. *If Condition C is satisfied for all type $t \in T$ then in any symmetric equilibrium:*

1. $0 \in \operatorname{supp}[G^t]$.
2. $G^t(0) = 0$
3. $\operatorname{supp}[G^t]$ *is an interval.*
4. $x^G = \mathbb{E}[v|N]$

Proof.
1. By lemma 2.4.7, any $\underset{\sim}{B}^k > 0 \implies U(\underset{\sim}{B}^k, k|\, G_{-i}) - U(0, k|\, G_{-i}) < 0$.
2. By lemma 2.4.8: $V^t(t) > 0$. Then, Lemma 2.3.1 implies that for type t at most for one player can have an atom in equilibrium. Because the equilibrium is symmetric neither player has an atom. Hence, $G^t(0) = 0$ (see A.1 for the formal definition of an atom).
3. Suppose not and β is a gap of $\operatorname{supp}[G^t]$, see A.1 for the definition of a gap. The infimum and supremum of β belong to $\operatorname{supp}[G^t]$. Lemma 1.3.7 implies that $U(\inf\beta, k|\, G_{-i}) - U(\sup\beta, k|\, G_{-i}) > 0$. A contradiction of equilibrium.
4. By point 1. and 2., zero is in support of type N and there are no atoms at zero for any player or type, which means the expected payoff at $b = 0$ is zero for type N. Then if $x^G < \mathbb{E}[v|N]$, $b = 0$ is not a best response for type N, since when bidding $b = x^G$ the expected payoff is strictly greater than zero (since you always win the auction and pay less than $\mathbb{E}[v|N]$). Naturally, it can't be that $x^G > \mathbb{E}[v|N]$, since has x^G be the a best response for at least one type.

$\square$

If Condition C is satisfied a unique symmetric equilibrium exists, in which the bid-supports are ordered by the strong set order for some order. This means that there exists some possible different order than the natural order $\preceq$ on T such that for $t \preceq t'$ *supp* $[G^t] \leq_s$ *supp* $\left[G^{t'}\right]$.

Theorem 2.4.11. *Let the Generic Condition be satisfied. If Condition C is satisfied for all types $k \in T$, then there is a unique symmetric equilibrium. Further, in the unique equilibrium, the supports are ordered by the strong set order for some order on T.*

Proof. The proof is constructive. Since all types have zero in their support by Lemma 2.4.10, it must be that on the first interval I_1 of the bid space, all types have a strictly positive bid-distribution slope. By the Generic Condition, these slopes are uniquely determined. By Lemma 2.4.10, the bid-support is an interval for every type. Hence, there must be at least one type $k \in T$ that exhaust her bid probability on I_1. This constitutes step 1 of the construction. In step 2, on interval I_2, a proper subset $T^1 \subset T$ bids, which all have strictly positive bid-distribution slope , which are uniquely determined. Like in step 1, at least one type expends her remaining bid probability on I_2. For step 3, a proper subset $T^2 \subset T^1$ bids on the next interval I_3. The construction proceeds till finally type N expends their bid probability, and this concludes the construction. Since the slopes are uniquely determined at each step; the construction is unique. Hence there is a unique symmetric equilibrium.

□

What remains to be shown is that if Condition C is satisfied that $\bar{B}^t$ is increasing in t. This is difficult to show and it remains an open question. One alternative approach, is to pair Condition C with a condition that regulates the slopes on the bid space. That is, for each $T^n := \{n, \cdots, N\}$ and for all $b \in \beta(T^n)$: $\frac{d\,G^k(b)}{d\,b}$ is strictly decreasing in k. This, additional requirement is sufficient but not necessary for the existence of an equilibrium of form (c).[25]

Corollary 2.4.12. *Let the Generic Condition be satisfied. If Condition C is satisfied for all types. If $\mathbb{E}[v \mid t]$ is strictly increasing in t. If for all $n < N$ and all $b \in \beta(T^n)$ $\frac{d\,G^t(b)}{d\,b}$ is strictly decreasing in t, then the unique symmetric equilibrium is of form (c).*

Proof. First, theorem 2.4.11 applies, so form (c) is the unique symmetric equilibrium if it exists. That form (c) exists follows from the construction in the proof in theorem 2.4.11.

[25] Consider the example; let

$$M(T) = \begin{pmatrix} 3 & 1.5 & 0 \\ 1.5 & 2 & 0.5 \\ 0 & 1.2 & 1.5 \end{pmatrix}$$

By Lemma 2.4.9, condition C is satisfied, and $\mathbb{E}[v \mid k]$ is strictly increasing, which means there is a unique symmetric equilibrium by Theorem 2.4.11. One can show that this unique symmetric equilibrium is of form (c). However, on the first interval $\frac{d\,G^3(b)}{d\,b} > \frac{d\,G^2(b)}{d\,b}$. This means that on the first interval, the higher type 3 expends more bid probability than type 2. But on the second interval, type 2's slope is so much steeper than type 3 that type 2 exhausts her remaining bid probability before type 3.

Since the lowest type left at any step has the steepest slope, this type always exhausts the bid probability at that step of the construction. This means that construction in the proof of theorem 2.4.11 coincides with form (c). By the FOCs, the types are all indifferent on their bid supports. Further, in this construction, every type has zero in their bid support, and the bid supports are intervals, so we only need to check upward deviations. Since Condition C is satisfied, no type has an incentive to bid above their support by lemma 1.3.7; hence this construction is an equilibrium. □

The following conditions on the primitives are sufficient for an equilibrium of form (c).

1. If $V^k(k)$ is strictly increasing in k and $V^k(j) = \epsilon < V^1(1)$ for all $j \neq k$. This environment includes perfect correlation when $\epsilon = 0$. (- diagonal matrices)

   Proof. It is obvious that Condition C is satisfied, so I just have to show that the slope is decreasing in type. For any $k' > k$, the first order conditions for both types imply that:

   $$\begin{aligned} &\sum_{j=1}^{N}(V^{k'}(j) - V^k(j))\frac{d\,G^j(b)}{d\,b} = 0 \\ \iff (V^{k'}(k') - \epsilon)\frac{d\,G^{k'}(b)}{d\,b} &= (V^k(k) - \epsilon)\frac{d\,G^k(b)}{d\,b} \\ \iff \frac{\frac{d\,G^{k'}(b)}{d\,b}}{\frac{d\,G^k(b)}{d\,b}} &= \frac{(V^k(k) - \epsilon)}{(V^{k'}(k') - \epsilon)} < 0 \end{aligned} \tag{2.47}$$

   □

2. If for the environment above ϵ is a function of the type k, and $\epsilon(k)$ is increasing in k the slopes are ordered still (and the proof is the same except now the equality on the second line is a less than or equal to). But now condition c might not be satisfied. Condition c is however satisfied if $\epsilon(N) \leq \frac{1}{N-1}(\sum_{j=2}^{N-1}\epsilon(j) + V^1(1))$.

Form (d): The Uncompetitive Equilibrium.

This form is called the uncompetitive equilibrium because it can exist when there is a strong negative dependence. When there is a strong negative dependence, it is more likely that players have different types. This decreases competition and is reflected by the expected payoff for the players, which is greater than zero for all types except the lowest type.

Form (d) can be thought of as the another side of the coin of form (c). Many of the intuitions gathered for form (c) can be used here in reverse. Even though this equilibrium form has had little attention in the literature so far, many economic environments lend themselves to it.[26]

Form (d):

1. For all $t \in T$, $x^G \in \text{supp}[G^t]$.
2. $\text{supp}[G^t]$ is an interval.
3. $\underline{B}^t$ is strictly increasing in t.

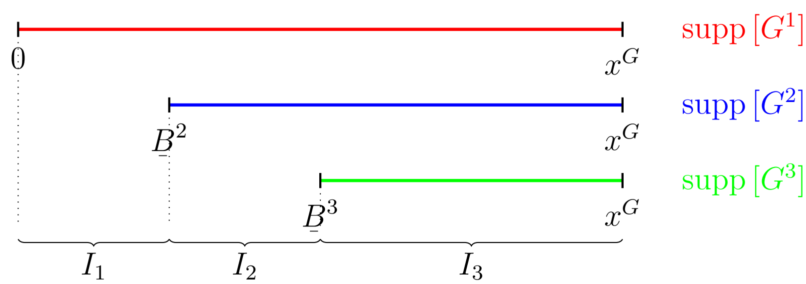

Figure 2.5: In a model with N types and the full-support assumption on the prior an equilibrium of form (d) has N intervals that partition the bid space. On every interval I_n, the first n types bid, such that on the last interval I_N all types bid.

Condition C implies slope of the expected payoff outside the bid-support is strictly Negative. While Condition D insures that the slope of the expected payoff outside the bid-support is strictly positive. Hence, Conditions C and D are mutually exclusive.

Condition D is satisfied for a type $k \in T$ if for all subsets $S \subseteq T \backslash \{k\}$;

$$V^k(j) = \sum_{t \in S} \alpha_t V^t(j) : \ \forall j \in S. \tag{2.48}$$

Where $\alpha_t \in \mathbb{R}$ such that $\sum_{t \in S} \alpha_t > 1$.

Just as Condition C, I derive condition Condition D in terms of the primitives. The proof is the same; just take care that when deriving the 3th condition for each type

[26]Chapter 1 uses this form to explain election upsets, in which one candidate is favored which implies the other candidate is unfavored. This form can also be used to model market shares for competing firms; if one firm's market share increases, the other share of the others must decrease.

when $N = 3$, the signs switches because it the equation is multiplied by the negative determinant.

Lemma 2.4.13. *When $N = 2$, Condition D is equivalent to:*

$$V^1(2) > V^2(2) \ \&\ V^2(1) > V^1(1) \tag{2.49}$$

When $N = 3$, Condition D is equivalent to following 9 conditions (3 per type):

$$\begin{gathered} V^3(2) > V^2(2) \ \&\ V^3(1) > V^1(1) \\ \&\ \Big(V^1(1) - V^2(1)\Big)\Big(V^3(2) - V^2(2)\Big) < \Big(V^3(1) - V^2(1)\Big)\Big(V^1(2) - V^2(2)\Big) \end{gathered} \tag{2.50}$$

$$\begin{gathered} V^2(3) > V^3(3) \ \&\ V^2(1) > V^1(1) \\ \&\ \Big(V^3(1) - V^2(1)\Big)\Big(V^1(3) - V^2(3)\Big) < \Big(V^1(1) - V^2(1)\Big)\Big(V^3(3) - V^2(3)\Big) \end{gathered} \tag{2.51}$$

$$\begin{gathered} V^1(3) > V^3(3) \ \&\ V^1(2) > V^2(2) \\ \&\ \Big(V^1(2) - V^2(2)\Big)\Big(V^3(3) - V^2(3)\Big) < \Big(V^2(2) - V^3(2)\Big)\Big(V^2(3) - V^1(3)\Big) \end{gathered} \tag{2.52}$$

Example 3. Let $T = \{1, 2, 3\}$ and consider the following matrix of coefficients for the entire type space.

$$M(T) = \begin{pmatrix} V^3(3) & V^3(2) & V^3(1) \\ V^2(3) & V^2(2) & V^2(1) \\ V^1(3) & V^1(2) & V^1(1) \end{pmatrix} = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 0.5 & 2 \\ 2.5 & 1.5 & 0.25 \end{pmatrix}$$

Note that the generic condition is satisfied: the matrix of coefficients is a definite negative matrix. Further, Condition D is satisfied because (2.50), (2.51), and (2.52) are satisfied. This kind of matrix can be generated if there is negative dependence; that is more likely my opponent has a different type than my own. Or if the value of the prize has a property where the value of the prize is higher if the two types don't match. The slopes are ordered on each interval, so by Corollary 2.4.16, an equilibrium of form (d) is the unique symmetric equilibrium.[27]

Lemma 2.4.14 list all the implications of Condition D. The proof is essentially the

[27] The easiest way to check that the slopes are ordered is to go to an online Cramer's rule calculator: https://matrix.reshish.com/cramer.php. For the third interval and use $M(T)x = 1$, and see that $x_1 > x_2 > x_3$, where x_1 is the slope for type 3 and x_3 is the slope for type 1. For the second interval, use $M(S)x = 1$ s.t. S={1,2}, $x_1 > x_2$, where x_1 is type 2's slope. Only the lowest type remains on the last interval, which expends her remaining bid probability.

same as for Lemmas 2.4.7 and 2.4.10.

Lemma 2.4.14. *If Condition D is satisfied for a type $k \in T$ then:*

1. *For all $b \in [0, x^G] \setminus \operatorname{supp}\left[G^k\right]$: $\frac{dU(b,k|G_{-i})}{db} > 0$.*
2. *$x^G \in \operatorname{supp}\left[G^k\right]$.*
3. *$\operatorname{Supp}\left[G^k\right]$ is an interval.*
4. *Further, if condition d is satisfied for all type's $k \in T$, then $V^k(k) < V^j(k)$ for all $j \neq k$.*[28]

Theorem 2.4.15. *Assume that $\mathbb{E}\left[v \,|\, t\right] < \mathbb{E}\left[v \,|\, t'\right]$ for $t < t'$, that the Generic Condition is satisfied and f has full support. If Condition D is satisfied for all types $k \in T$ then there is a unique symmetric equilibrium. Further, in the unique equilibrium the supports are ordered by the strong set order for some order on T.*

Proof. First, I establish that $0 \in \operatorname{supp}[G^1]$. Suppose not, then by Lemma 2.3.1, there is some other type $k > 1$ s.t. $0 \in \operatorname{supp}\left[G^k\right]$. Further, Lemma 2.3.1 implies there are no atoms at zero since f has full-support by assumption. Since zero is in k's support, and there are no atoms at zero, the expected payoff at the zero bid is zero for this type. This implies that in equilibrium $x^G = \mathbb{E}\left[v \,|\, k\right]$. But since $\mathbb{E}\left[v \,|\, k\right] > \mathbb{E}\left[v \,|\, 1\right]$ then $U(x^G, 1 \,|\, G_{-i}) < 0$, which contradicts Lemma 2.4.14.

By the logic of the previous paragraph, $0 \in \operatorname{supp}[G^1] \implies x^G = \mathbb{E}[v|1]$. For all types $k > 1$, $\underline{B}^k > 0$. Because, $U(x^G, k \,|\, G_{-i}) > 0 \implies 0 \notin \operatorname{supp}\left[G^k\right]$, since $U(0, k \,|\, G_{-i}) = 0$.

The rest of the proof is constructive. Start at the last interval of the bid-space. Since all types have x^G in their bid-support, it must be that on this interval, all types have strictly positive slopes. By the generic condition, these slopes are uniquely determined. Since all bid-supports are intervals, it must be that at least one type k expends all her bid probability on this last interval: $\left(\underline{B}^k, x^G\right)$. This constitutes step 1 of the construction. In step 2, a proper subset of types $T^1 \subset T$ has not expended all their bid probability but has $\underline{B}^k$ in their support. On the interval before $\left(\underline{B}^k, x^G\right)$, each of the remaining types bid with uniquely determined bid-distribution slopes. Again, at least one type must expend her probability on that interval, which ends step 2. This continues process continues till only type 1 is left. Type 1 expends her remaining bid probability on the first interval

[28] $V^k(k) < V^j(k)$ is necessary for the existence of an equilibrium of form (d) in the binary model. While, $\mathbb{E}\left[v \,|\, 1\right] < \mathbb{E}\left[v \,|\, 2\right]$ and $V^k(k) < V^j(k)$ is necessary and sufficient.

that has zero as the infimum. This ends the construction. The bid-distribution slopes are unique at each subset of bids, and by properties in Lemma 2.4.14 in any symmetric equilibrium, there is no other way to configure these subsets. Hence, this construction is a unique symmetric equilibrium.

□

Corollary 2.4.16. *Assume that $\mathbb{E}[v \mid t] < \mathbb{E}[v \mid t']$ for $t < t'$, that the Generic Condition is satisfied and f has full support. If Condition D is satisfied for all types $k \in T$ and If for all $n < N$ and all $b \in \beta(T^n)$ $\frac{dG^t(b)}{db}$ is strictly increasing in t, then the unique symmetric equilibrium is of form (d).*

Proof. First, theorem 2.4.15 applies, so form (d) is the unique symmetric equilibrium if it exists. That form (d) exists follows from the construction in the proof in theorem 2.4.15. Since the highest type left at any step has the steepest slope, this type is the only type that exhausts their bid probability at that step of the construction. This means that construction in the proof of theorem 2.4.15 coincides with form (d). By the FOCs, the types are all indifferent on their supports. Further, in this construction, every type has x^G in their bid support, and the bid supports are intervals, so we only need to check downward deviations. Since Condition D is satisfied, no type has an incentive to bid below their support by Lemma 2.4.14.1; hence this construction is an equilibrium. □

Perfect Negative Correlation.

Define perfect negative correlation as definition 2.4.2.

Perfect Negative Correlation For all $j < N$, f is such that $f(N-j|1+j) = 1$ and $f(\cdot|1+j) = 0$ otherwise.

Loosely speaking whenever there is strong positive correlation an equilibrium of form (c) exists. The limit of this is perfect positive correlation and then the unique symmetric equilibrium is of form (c). Likewise, if there is strong negative correlation an equilibrium of form (d) exists. However, if there is perfect negative correlation, the equilibrium is not of form (d). The reason for this is that if $V^t(t) = 0$ then both player's can have an atom at zero for that type t, see lemma 2.3.1. Furthermore, if $N \geq 3$ the equilibrium that exists with perfect negative correlation is not even weakly-monotone and Condition D is not satisfied. However, the probability to win the game is increasing in type.[29] This is illustrated in the figure below.

[29] In figure 2.6, prob$\{win|t=4\} = \frac{2V^4(1)-V^1(4)}{2V^4(1)}$, prob$\{win|t=3\} = \frac{2V^3(2)-V^2(3)}{2V^3(2)}$, prob$\{win|t=2\} = \frac{V^2(3)}{2V^3(2)}$ and prob$\{win|t=1\} = \frac{V^1(4)}{2V^4(1)}$. Clearly, $\frac{V^1(4)}{2V^4(1)} < \frac{V^2(3)}{2V^3(2)} \iff V^1(4)V^3(2) < V^4(1)V^2(3)$. This for $N = 4$, but it is true for any N.

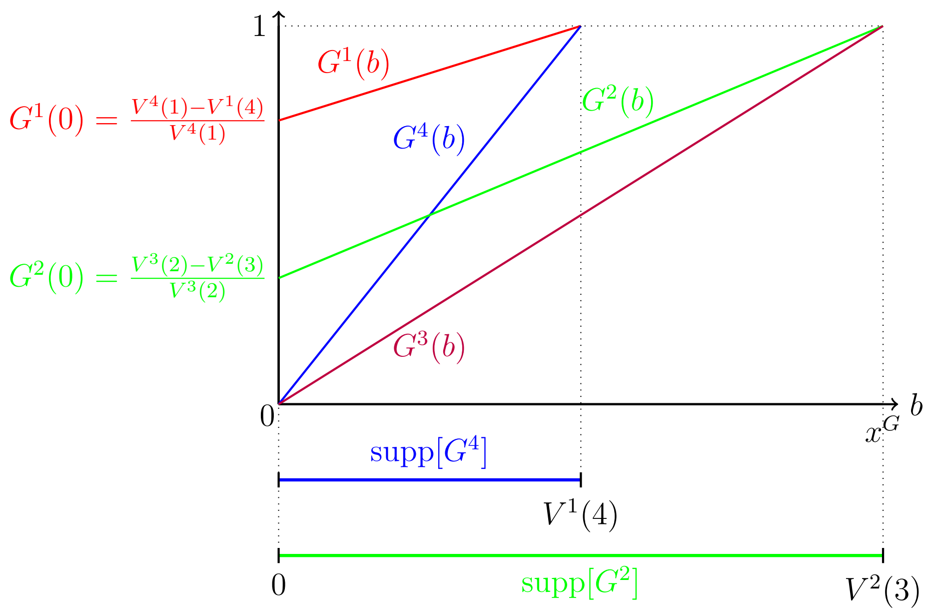

Figure 2.6: The symmetric equilibrium for perfect negative correlation when $N = 4$.

2.4.3 Weakly Monotone Mixed Forms

A mixed form equilibrium is a weakly monotone equilibrium that combines multiple primary forms. For example, in figure 2.7, the equilibrium is a mix between form (b) and (c).[30]

Correlated Private Value's. In the binary signal model, an equilibrium of form (c) exists if the positive correlation between the signals outweighs the increase in the prize value when obtaining a higher signal. Alternatively, if the correlation is weak, the equilibrium is of form (b). In the general model, the same intuition can be found. However, in the general model, if the value of the prize increases relative to the strength of the correlation, the equilibrium does not have to have to separate immediately: now mixed forms can arise.

This mixed-form equilibrium arises in the 3-type version of a correlated private value model. In this model, the conditional probability of facing the same type as your own is at least twice as high as facing any other type. At the same time, the payoff of being a higher type is at least $\frac{1}{\theta}$ times higher than any type below it. When the benefit is

[30]Rentschler and Turocy (2016) find another slightly different mixed form between form b and c in their example 3. In that example, they have a common-value prize and the player's types are correlated with this common value. In that example, a mixed form equilibrium exists is when the correlation is not too strong and not too weak. Chen (2021) finds a mix between form (a),form (c) and form (d), which exists when the correlation is strong, while if the correlation is weak, a form (b) equilibrium exists.

increased (When θ decreases), the equilibrium moves from an equilibrium of form (a) into (c) into a (b-c)-mixed and finally into a fully separating equilibrium of form (b).

Proposition 2.4.17. *Let $T = \{1, 2, 3\}$. Let the private value of winning be $v(3) = 4$, $v(2) = \theta 4$ and $v(1) = \theta^2 4$ with $\theta \in (0, 1]$. Furthermore, conditional on receiving the type t, it is two times more likely that the opponent's type is t than any adjacent type and it is four times more likely than types that are twice removed. The unique symmetric equilibrium is of form (a) if $\theta = 1$, of form (c) if $\theta > \frac{4}{7}$, of the (b-c) mixed-form if $\frac{7}{16} < \theta < \frac{4}{7}$ and of form (b) if $\theta \leq \frac{7}{16}$.*

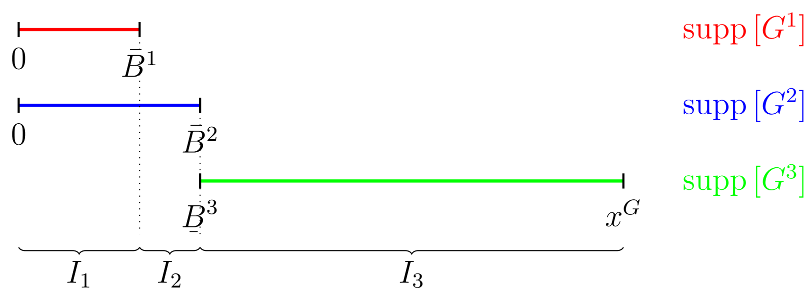

Figure 2.7: When $\frac{7}{16} < \theta < \frac{4}{7}$ there exists a (b-c) mixed form. In this form type 1 and 2 play a strategies consistent with form (c) and type 3 plays a strategy consistent with form (b).

The proof and more details on the example can be found in the appendix.

2.5 Conclusion

This paper investigated a symmetric two-player all-pay auction with interdependent valuations and arbitrary dependence. It provides novel necessary and sufficient conditions for the existence of a monotone equilibrium. Furthermore, this paper is the first to characterize the set of equilibria that are not monotone in a general finite type framework.

In order to move beyond the traditional monotone equilibrium, this paper focuses on a weaker notation of a monotone equilibrium. In the weakly monotone equilibrium, the bid-supports of the types are still ordered by the strong set order, which allows for monotone comparative statics in the sense of Milgrom and Shannon (1994).

This paper focuses on three new forms of weakly monotone equilibria, all of which have strong economic properties. Form (a) is the equilibrium that exists if the types value the prize the same. Form (c) exists when the players' payoffs are log supermodular or/and if the two players or if the signals have strong positive dependence. Form (d) exists when the payoffs are log submodular or/and if there is substantial negative dependence.

For forms (c) and (d), sufficient conditions are provided. However, if the open question in Appendix C.4 is answered, these conditions could be sharper in terms of the primitives. Condition 2 is used as a regularity condition in this paper, but I conjecture it could also be a sufficient condition for the uniqueness of a symmetric equilibrium.

A non-trivial extension to this project is to extend it to asymmetric primitives. One exception is Condition B, the necessary and sufficient condition for the existence of a monotone equilibrium. This condition can easily be extended to an asymmetric environment, but unfortunately, it can only be partially characterized. In a symmetric model, the construction of a monotone equilibrium is known, which allows the condition to be fully characterized. However, ex-ante, in the asymmetric setting, the construction is not known. Condition B is formulated given the construction. Hence one has to execute Siegel's algorithm first, and only then can one formulate the condition in terms of the primitives.

Another interesting extension is to see if these weakly monotone equilibria have counterparts in other auctions forms like first or second-price auctions and the static in the war of attrition. I think the first-price auction has to most promise since its payment rule is close to that of an all-pay auction. For the second-price auction, there is a known multiplicity of equilibria. Further, there exist pure strategy equilibria, even if the type space is finite. Hence weak monotonicity may have no bite.

Chapter 3

War and Peace, Conflict over a Prize with Common Value

3.1 Introduction

Conflict is inherent to the interaction of individuals and has been prevalent throughout history. Recently, we have seen large-scale conflicts arise in Afghanistan and Ukraine. Resolving and preventing these conflicts is essential. Therefore, it is necessary to understand the factors contributing to conflict escalation. For example, how does military power determine if a peaceful outcome can be achieved? Moreover, how does the value of the prize determine which peace deals are acceptable?

This chapter shows that peace can be guaranteed if two players are known to be equally strong. Furthermore, any peaceful split of the prize can be implemented. However, if one player is known to be stronger than the other, the strong prey upon the weak, and war may occur. In this scenario, war can be avoided if the peace deal is sufficiently lob-sided in favor of the strong. On the other hand, war can sometimes not be avoided if there is uncertainty over the players' military power. Surprisingly, when the common value of the prize is uncertain, a weak player may prefer a war. If this same player had been strong, they would have chosen peace.

When conflict arises over a prize, the value is often uncertain. Further, the value a player assigns to the prize might depend on the valuations of the other players. For example, the contested oil fields near the Arctic are believed to hold a quarter of the world's petroleum reserves. The disagreeing parties each have some signal about the amount of oil, but they cannot be sure unless they start extracting. Once either party obtains the oil, it is valued at the world's market price for crude oil. This example implies that ex-post, each player values the prize the same, i.e., there is a common value.[1]

A setting like the Arctic example, in which the objective of the war has an uncertain but common value, has to my knowledge not been studied in the conflict mediation literature.[2] The rest of the literature focuses on private value models, where the value of winning is known to the player. The private value or cost is often interpreted as private information on power or relative power. They suggest that the prospect of peace increases when players are more asymmetric in power.[3]

This chapter suggests that symmetry in power leads to peace. In a pure common-value model, where players differ only by their private information over the value of the prize, an essentially unique equilibrium exists.[4] In this equilibrium, war occurs with zero probability. War does not happen because any peaceful prize split is better than a costly war against an equally strong opponent. This is true even if the players have different expectations of the value of the prize.

When the players can differ in military power. Which is modeled as adding a private value element to the common value. Then, a war might happen with positive probability. Furthermore, any acceptable fixed peace deal is no longer acceptable if the relative differences in military power are sufficiently increased. Essentially, differences in military power reduce the perceived costliness of war for one player. This player has an incentive to start a war.

In this chapter, I consider a conflict game with two stages, a negotiation stage followed by a possible conflict stage. The conflict mediation literature that uses this type of game can be divided into two main branches. The first branch consists of models that have an exogenous conflict stage and the second branch has models that make conflict

[1]The common value in the example is the realized amount of oil sold at the market price of crude oil. However, because the amount of oil is unobserved, each player produces an estimate. In the interim, their expected valuation of the oil is a function of their estimate and the estimate of the other players.

[2]One exception is Sorokin and Winter (2020), since this paper allows for interdependent valuations. However, they study a very different conflict game. Their paper studies when a player should to back out of an ongoing costly conflict, and does not allow for a peaceful resolution prior to conflict.

[3]For example, Bester and Wrneryd (2006) and Zheng (2019) find that stochastically increasing the level of power of a player never hurts the existence of a peaceful equilibrium.

[4]Essentially unique means that it is the only equilibrium besides the war equilibrium. In a war equilibrium, both players play war with certainty. This war equilibrium always exists as a feature of the model; since war is a unilateral decision. Hence, if a player believes their opponent plays war certainty, the player choice between peace and war is inconsequential. However, the war equilibrium is unstable if any player's strategy is perturbed to play peace with *probability*, the other player does not play war with certainty anymore.

endogenous.[5] This paper belongs to the latter. When conflict is endogenized, one does not need to make assumptions on the cost of conflict as it is determined in the model. A player's outside option depends on how he thinks his opponent behaves if conflict were to happen. In turn, this allows the players to learn from the negotiations, and backward induction affects their behaviors in the negotiation stage.

This Chapter is closest to Zheng (2019), which also models the conflict stage as an all-pay auction. In an independent private value setting, Zheng (2019) finds the necessary and sufficient conditions on the primitives of the game to implement peace. Peace is implementable if and only if a Perfect Bayesian Equilibrium (PBE) exists, in which conflict occurs with zero probability. I allow for correlated two-dimensional types, where one dimension dictates the common value and the other the private value. I find that peace is always implementable in the "pure" common-value model and it might not be in a model with private value elements. Like Zheng (2019), I provide the necessary and sufficient conditions on the primitives of the game to implement peace.[6]

Intuitively, when the value of the prize is interdependent, good news for one player might also be good news for the other player.[7] In the all-pay auction, this difference between the models is demonstrated by probability mass at zero in the bid distributions. I show that in the common-value model of the all-pay auction, neither player has probability mass at the zero bid, which means neither player is completely discouraged to participate in a war. When players differ in a known private value element, which is interpreted as power here, I show that the "weaker" player has a mass at zero. The probability mass increases the benefit of war for the "stronger" player, which makes war more appealing for that player.

Uncertainty may lead to war and my findings corroborate this to some extent. When there is uncertainty over the private element, then greater uncertainty threatens peace.

[5] I endogenize conflict, as in Balzer and Schneider (2018), Zheng (2019), and Kamranzadeh and Zheng (2019). Bester and Wrneryd (2006), Fey and Ramsay (2011), and Hörner Morelli and Squintani (2015) belong to the recent exogenous conflict literature. Since conflict is exogenous in their models, they must make assumptions about its cost. They find that peace can be achieved only if war is sufficiently costly.

[6] Zheng (2019) has one dimension of private information, which is power. In the all-pay auction, this model is equivalent to a private value model, where your power level is the value of the prize. Zheng's model allows for both continuous and discrete type spaces. Thus, our models overlap when considering independently distributed discrete types with private values, and as such the condition to implement peace is the same for that case.

[7] If the value of the prize depends on both players' private information like in the common-value model, then if my private "signal" over the value increases, it increases the valuation of the object, but this increases the value of the object for my opponent as well. While in the private value model this is not the case.

This finding is consistent with the literature.[8] The uncertainty over the common-value element leads to a novel finding, in which there can be non-monotone behavior in the first stage of the game. In a model with a common-value element, it can be that a player with a low signal of the prize rejects a peace deal where a player with a higher signal may not. This cannot happen in an independent-private-value model, in which the player with the highest signal is always the most aggressive.

A common assumption in the conflict literature is that the private information of each player is not correlated with the other player's private information.[9] This independence assumption is sometimes technical. For example, it can guarantee the existence of a monotone equilibrium in the all-pay auction.[10] When there is one dimension of private information which is interpreted as power, independence might intuitively be fine. Since power is not necessarily correlated to the other player's power. Consider a common-value setting instead, where two players get information on the same prize, it must be that this information is positively correlated. In the Arctic example, if it turns out there is a lot of oil beneath the ice, then it should be more likely that each player's private information indicates this. It also means that, if one player has a high "signal" about the value of the prize, he knows his opponent is more likely to have received a high "signal" as well.

I allow private information to be correlated. I replace the independence condition with a novel condition called Preserve Monotonicity (PM). This condition, like independence, is a sufficient condition for the existence of a monotone equilibrium in the second stage of the all-pay auction game. Further, when PM is assumed, it is the unique equilibrium of that game. PM is a condition on the valuation function and distribution of private information. It rules out perfect negative and perfect positive correlation, but naturally, it allows for independence. Condition PM allows for the case of affiliated random variables, which was required to analyze the Arctic example.[11] PM is a multi-stage extension to a condition found in Siegel (2014) and Krishna and Morgan (1997). [12]

I characterize all the possible equilibria for the pure common-value model. I find that besides a default war equilibrium only one other equilibrium exists, which is completely

[8]For example, Sorokin and Winter (2020) find that increase in accuracy i.e. higher levels of certainty increases the probability of peace.

[9]For example in, Bester and Wrneryd (2006), Fey and Ramsay (2011), Hörner, Morelli and Squintani (2015), Balzer and Schneider (2018), Zheng (2019) and Kamranzadeh and Zheng (2019).

[10]Zheng (2019) and Kamranzadeh and Zheng (2019) rely on independence for existence.

[11]Affiliated random variables are positively correlated random variables. This terminology is often used in auctions. PM also allows negative correlation (Reverse affiliation) although that seems less of a fit in the rhetoric of a common value. Formal definitions can be found in the appendix.

[12]Their condition is sufficient for the existence of a monotone equilibrium in the static all-pay auction.

peaceful.[13] Further, I find that any peace deal is implementable, even when one player gets the whole prize and the other gets nothing. The common-value model is peaceful because if a player infers what type the opponent is, he knows the value of the prize, which is the same for both players. In conflict, this forces the opponent's expected payoff to zero. Thus, any strategy that signals information to the opponent is extremely costly. This "informational cost" can only occur in models that endogenize conflict. In exogenous conflict, signaling is costless since the game ends afterward.

In a private value model, the cost of signaling a player's private information is still relevant, but the difference in valuation between the players can outweigh this cost. Suppose that player 2 knows player 1's private signal (type) and realizes that player 1 values the object more than him. Player 2 never expends more resources than what the prize is worth to him. In the all-pay auction, this means player 2 does not bid above his valuation. This implies that player 1 has a positive expected payoff in conflict. Therefore player 1 might still play war and signal his "type" and war can occur on-path.

More formally, a player's payoff for deviating to war depends on the other player's probability mass at zero in the all-pay auction. I add to the all-pay auction literature by providing proof of absolute continuity for the bid distributions. I show this for a broad class of interdependent value all-pay auction games with correlated types. This allows me to conclude that there can never be any atom in the common-value model.[14] For my general model, it allows me to determine the atoms at zero in the case where a player makes a unilateral deviation to war.

Bester and Wrneryd (2006) find evidence that asymmetry in power leads to peace, in other words, deterrence of strength leads to peace. They base this on the finding that stochastically increasing the level of power of a player never hurts the existence of a peaceful equilibrium. This paper corroborates that finding as does Zheng (2019). Regardless, I argue that symmetry leads to peace.

First, the equilibrium is peaceful in the pure common-value model, where players are symmetric in power. Second, if the power level becomes more asymmetric between the players, an acceptable peace deal may no longer exist. For example, if an equal split is just acceptable for two symmetric players and the maximum power of one player is increased, then a peace equilibrium no longer exists. This means that the increase in the profitability of war for the stronger player outweighs the decrease for the weaker player.

Third and finally, I provide a different interpretation of the stochastic increase in

[13] The war equilibrium is a feature of this model since peace cannot be triggered unilaterally. When a player chooses war with certainty, the other player is indifferent between war and peace.

[14] Atom is a probability mass. No atom at zero means no probability mass at the zero bid. In the common-value auction, there is no atom for any bid.

power argument. In a private value model, the highest type is the most aggressive. In order to determine if war is profitable, this highest type calculates the probability he faces a weak opponent. When this is likely, war is profitable. Thus, this profitability decreases when the other player gets a stochastic increase. The reason for this decrease is that the high type is more likely to face an opponent as strong as him, in which war becomes costly. Therefore, peace is threatened by differences in power. This difference might be either realized ex-ante, in terms of differences in the level of power, or in the interim, when a player realizes he is strong, and the opponent might not be.

This chapter is outlined as follows. Section 2 establishes the model and notation. Section 3 discusses the pure common-value model. Section 4 discusses the combination of a private and common value model. In section 4.1, known differences in private value are added and in section 4.2 the differences are uncertain. Section 5 concludes.

3.2 Model

3.2.1 Notation and Assumptions

There are two players denoted by $i = 1, 2$. There is one prize that needs to be allocated to one of the two players. Player i has a type $t_i \in T_i$. Let $T_i := S_i \times P_i \subset \mathbb{R}_+ \times \mathbb{R}_{++}$ denote a two-dimensional set. The elements $s_i \in S_i$ contain information on the value of the prize for player i, henceforth called the signal of player i. The elements $p_i \in P_i$ contain information on the level of power of agent i, henceforth called the power of player i. The type space is assumed to be finite let $|T_i| = n_i$.

The common value of this prize is given by the function $V(s_1, s_2)$, which depends on both players' signals. I assume that $V(s_1, s_2)$ is strictly increasing in both signals and $0 < V(s_1, s_2) < \infty$. Let the types be distributed according to a prior joint distribution f, where $f(t) = f(t_i, t_{-i})$ denotes the probability of the event t. Let $f(t_i) = \sum_{j=1}^{n_j} f(t_i, t_j)$ denote the probability of t_i. Let $f(t_{-i}|t_i) = \frac{f(t_i,t_{-i})}{f(t_i)}$ be the conditional probability of t_{-i} given t_i. I further assume that f has full support, i.e. $f(t) > 0 \; \forall \; t \in T_1 \times T_2$.

3.2.2 The Game

The two agents compete over a prize of uncertain but common valuation. The uncertainty over the value arises because you can't observe the signal realisation of the opponent. The timing of the game is as follows:

1. Each player receives their type $t_i = (s_i, p_i)$ known only to themselves.
2. Each player simultaneously, independently and publicly announces to either accept peace or go to war.
3. If both players pick peace, the game ends and the prize goes to player 1 with υ probability and to player 2 with probability $(1 - \upsilon)$.
4. If either or both agents pick war, the next stage of the game is played, which is an all-pay auction.

3.2.3 The All-Pay Auction

If peace is not achieved, there is a war, in the form of an all-pay auction. In the all-pay auction, both players place a sealed bid at the same time. The player with the higher bid wins the prize, but both players pay their bid. In the case of a tie, both players win the prize with equal probability.

Given $t \in T_i \times T_{-i}$, a pair of bids (b_i, b_{-i}), a player with type t_i receives:

$$u_i(t, b_i) = PR_i(b_i \geq b_{-i})\, V(s_i, s_{-i}) - \frac{b_i}{p_i}. \tag{3.1}$$

Where $\Pr_i(b_i \geq b_{-i})$ is the probability of winning the auction.[15] In this game p_i modifies the costliness of bidding b. That is, it is less costly to bid b if you are stronger, which is the standard way to model military power in the literature.[16]

3.2.4 Strategies and Beliefs

Let $\sigma := (\sigma_1, \sigma_2)$ denote a strategy profile. A behavioral strategy σ_i is a probability distribution over the actions in each stage for each type and for all possible histories. H is the set of all possible histories of announcements by the players. For each possible announcement $h \in H$, a *strategy component* for t_i specifies the likelihood of an action. For example, for the strategy σ_i, the strategy component $\sigma_{i,h}(a|t_i)$, is the probability that player i with type t_i plays action a given that he observed announcement h.

[15] $Pr_i(b_i, b_{-i}) = 1$ if $b_i > b_{-i}$, $Pr_i(b_i, b_{-i}) = \frac{1}{2}$ if $b_i = b_{-i}$ and it is zero otherwise.

[16] Zheng (2019) and Kamranzadeh and Zheng (2022) have the same payoff function as (3.1), where power modifies the marginal cost of bidding b, but they differ since they assume a normalized known common value of $V(s_i, s_{-i}) = 1$.

In stage one, $h = \emptyset$, each t_i chooses between war and peace. That is, $\sigma_{i,\emptyset}(war|t_i) + \sigma_{i,\emptyset}(peace|t_i) = 1$. For stage two, the action chosen belongs to the entire positive real line and must be contingent on the action announced in stage 1. Given the announcement $h \in H$, a strategy component in stage 2 is a cumulative distribution function (CDF). With slight abuse of notation $\sigma_{i,h}(b|t_i)$ denotes the probability of t_i bidding at least b.

The posterior joint distribution is denoted by μ and is derived using Bayes's Rule whenever possible. When the players abide by σ and Bayes's rule applies, their actions are *on path*. Notice that besides depending on f, the on-path distribution over the types depends on the strategies σ and the announcement h. Hence, the on-path distribution is denoted as $\mu_{h,\sigma}$. Depending on μ, it is entirely possible that only a subset $T_i(\mu) \subset T_i$ remains if the all-pay auction is reached.[17]

Definition The type support of t_i given some belief μ is:

$$T_i(\mu) := \{t_i \in T_i|\ \exists\ t_{-i} \in T_{-i}\ s.t.\ \mu(t_i, t_{-i}) > 0\}\,. \tag{3.2}$$

3.3 Equilibrium

I study the Perfect Bayesian Equilibrium (PBE) of the before-mentioned game. Hence, if (σ, μ) is an equilibrium, then for all $h \in H$, σ_h is a Bayesian Nash equilibrium (BNE). Where σ_h is a strategy profile component at announcement h in the game.

Since war can be triggered unilaterally, three out of the four announcements trigger the all-pay auction.[18] The strategy profile σ and announcement h induce an all-pay auction game G.

$$G := (2, (\mathbb{R}_+)_{i\in N}, (T_i(\mu))_{i\in N}, (\mu), (u_i)_{i\in N}) \tag{3.3}$$

The first element of G is the number of players and the second element is the action space, the third element is the type space given μ, the fourth element is μ, and the fifth element is the payoff function (3.1).

Given $h \in H$ and σ, let $U_{i,h,\sigma}(t_i, b)$ denote the interim expected payoff for t_i in G. Fixing the opponent's bidding strategy $\sigma_{-i,h}$, a bid x belongs to the best response set $BR_i(t_i|\sigma, h)$, if for all other bids $y \neq x$, $U_{i,h,\sigma}(t_i, x) \geq U_{i,h,\sigma}(t_i, y)$. The strategy profile

[17] For example, if player 1 has two types: $T = \{t_1^1, t_1^2\}$. Let t_1^1 always play peace and t_1^2 always play war. On path, if player 1 announces peace, and the all-pay auction is reached $T(\mu_{(peace,a_2),\sigma}) = t_1^1 \subset T_1$. Likewise, if player 1 announces war, $T(\mu_{(war,a_2),\sigma}) = t_1^2 \subset T_1$.

[18] War is triggered when both players announce war, when player 1 announces war and player 2 announces peace, and vise versa. Only if both players announce peace does the war not occur.

component σ_h is an equilibrium of G if for both i and all their types t_i, $\sigma_{i,h}(b|t_i)$ assigns measure one to $BR_i(t_i|\sigma, h)$.

3.3.1 Preserving Monotonicity Condition (PM)

Unfortunately, the results of Chapters 1 and 2 do not apply to G because the players are potentially asymmetric. Because of this, it is required to confine attention to monotone equilibria, and the KMS condition needs to be satisfied to guarantee their existence.

However, even if KMS is assumed before the game starts, it can fail after the first stage because of increased correlation due to the actions played. Therefore, I make the Preserve Monotonicity assumption instead. PM ensures that a unique equilibrium exists in all the possible on-path all-pay auction stage games, which will be monotone. This existence is independent of whatever actions are played in the game's first stage. This result comes from Schouten (2020), but the appendix provides proof for this paper's particular setting.

Before I introduce PM, I first claim that the following game G^* is fully equivalent in the Myerson (1991) sense to G.

$$G^* := (2, (\mathbb{R}_+)_{i\in N}, (T_i(\mu))_{i\in N}, (\mu), (w_i)_{i\in N}). \tag{3.4}$$

The only difference is the payoff function w_i.

$$w_i(t, b) = PR_i(b)\, p_i V(s_i, s_{-i}) - b \tag{3.5}$$[19]

Given this equivalent game, it is convenient to rewrite the value of the prize as follows.

$$V_i(t_i, t_{-i}) := p_i V(s_i, s_{-i}) \tag{3.6}$$

I assume $V_i(t_i, t_{-i})$ to be increasing in t_i. Further, I assume T_i to be completely ordered, such that $t_i^1 \prec t_i^2 \prec \cdots \prec t_i^{n_i}$. An example of a total order in this setting is the following lexiographic order: $t \prec t' \iff s < s'$ or $s = s'$ and $p < p'$. If $T_i = S_i$ or $T_i = P_i$ then $\prec$ is $<$.

Preserving Monotonicity Condition (PM) is satisfied if for both players the following condition is satisfied. For any two consecutive signals $t_i' \prec t_i''$ of agent i and any

[19] Some authors call this strategically equivalent. The equivalence follows from Myerson (1991) page 72 and 73. The payoff is a linear transformation of the first, and will have the same solution. This other formulation does implies that continuation payoff are p_i times higher, which must be taken into account.

two signals of agent $-i$, $t'_{-i} \prec t''_{-i}$. The following 2 inequalities hold:

$$\frac{V_i(t'_i, t''_{-i})}{V_i(t''_i, t''_{-i})} \leq \frac{f(t'_i, t'_{-i})f(t''_i, t''_{-i})}{f(t''_i, t'_{-i})f(t'_i, t''_{-i})} \leq \frac{V_i(t''_i, t'_{-i})}{V_i(t'_i, t'_{-i})} \tag{3.7}$$

Essentially, PM bounds the correlation between the players' types. For example, PM rules out a perfect correlation between types, and independent variables always satisfy this condition. PM also implies that the payoffs exhibit increasing differences in the all-pay auction and that payoffs increase in type for all bids.

In the appendix, proof of Proposition 3.3.1 can be found. For the rest of this paper, I assume PM to hold.

Proposition 3.3.1. *If assumption PM is satisfied, a monotone equilibrium exists in any of the all-pay auction stage games subject to Bayes's rule, and it is the unique equilibrium of that stage game. Further, $U_{i,h,\sigma}(t_i, b)$ is strictly increasing in t_i for all $b > 0$.*

3.3.2 Existence of a War Equilibrium

Define an equilibrium in which both players choose war with certainty as a *war equilibrium*. Likewise, define an equilibrium where both players choose peace with certainty as a *peace equilibrium*. Since war can be triggered unilaterally, a war equilibrium exists. Another modeling option is to make peace a unilateral choice instead, then the war equilibrium no longer exists, and a peace equilibrium always exists. In this work, war is unilateral, since "wars begin when you will, but they do not end when you please".[20]

Proposition 3.3.2. *A war equilibrium exists.*

The intuition as to why a war equilibrium exists is straightforward. Since the opponent chooses war with certainty, my unilateral deviation alone can never lead to peace. If I choose peace regardless, the opponent's beliefs over my type are arbitrary. Therefore, choosing peace can at most lead to an all-pay auction game in which the opponent's posterior beliefs are different from the prior. Choosing peace is not the best response to a wide range of my opponent's beliefs, and in this case, there exists a war equilibrium.

For example, if *passive beliefs* are assumed a war equilibrium exists. Passive beliefs imply that $f = \mu$ if Bayes's rule does not apply. Due to passive beliefs, even if $-i$ observes a deviation, she cannot infer anything. Thus, regardless of i's action, the same all-pay auction is triggered. So both players are indifferent between war and peace, and the war equilibrium exists.

[20] A quote from Niccolò Machiavelli's Florentine Histories (1526).

This war equilibrium is reminiscent of a zero capital steady-state equilibrium in Solow growth models and is ignored for the remainder of this chapter.[21] That is, if there is only one equilibrium besides the war equilibrium, I call this equilibrium **essentially unique**.

3.4 Pure Common-Value Model

This section focuses on the common-value assumption by assuming away the power dimension. I find that for any $\upsilon \in (0, 1)$, a peace equilibrium exists, which is essentially unique.[22] This section is organized as follows. First, I show that neither player has an atom at zero in the pure common-value model in the all-pay auction. No atom implies that neither player's lowest type of the posterior support has a positive expected payoff in the all-pay auction. In turn, this implies that this lowest type plays peace with certainty whenever his opponent players peace with positive probability. Second, I show that a peace equilibrium exists in this model. Lastly, I present the main theorem of this section, which proves there can be no other equilibrium than war and peace.

Pure Common-Value Assumption Let $p_1 = p_2 = 1$ be known, such that $T_i = S_i$. Further, $s_i^k < s_i^{k+1}$.[23]

3.4.1 No Atom at Zero

The following subsection establishes that neither player has an atom at zero in the common value all-pay auction. I show that both players have the same unconditional bid distribution. Since their unconditional bid distributions are the same, neither player has a mass point (called an atom) at zero, or both have an atom at zero. The latter is impossible hence neither player has an atom at zero.

Siegel (2014) also made this claim by observing that the derivative of the ex-ante unconditional bid distribution is almost everywhere the same for both agents. This observation is not sufficient to claim that neither player has an atom at zero, but it is

[21] There is a steady-state with zero capital stock in the Solow growth model. This equilibrium is unstable in that if the capital stock increases above zero, it will never return to zero. The war equilibrium is unstable in a similar way. If player 1's strategy is perturbed by playing peace with ϵ probability, player 2 has an incentive to deviate to peace for at least one type. Hence, no player plays war with certainty if there is a chance that peace can be achieved.

[22] A peace equilibrium is an equilibrium in which war occurs with zero probability, and a peace equilibrium is the same as a peaceful equilibrium in Zheng (2019). When $\upsilon = 1$ and $\upsilon = 0$, a continuum of equilibrium exists, although counter-intuitive, this appears to be true. Proof of this claim can be found in the appendix.

[23] $p_i = 1$, is a normalisation which has no bearing on the results. As long as $p_1 = p_2 > 0$, everything in this section remains true.

sufficient if the bid distributions are absolutely continuous. This section provides proof that the unconditional bid distribution is absolutely continuous.

First, Lemma 3.4.1 establishes necessary condition for any Bayesian Nash Equilibrium of the all-pay auction. In equilibrium, all bids for both players must belong to the same closed interval (the bid space). On the bid space, the bid distributions of all types are continuous. Continuity implies no sudden jumps (atoms) except possibly at the zero bid. At the zero bid, at most, one player can have such an atom at zero. The atom at zero is economically relevant since the atom player is discouraged from participating in the auction.[24]

Lemma 3.4.1. *Both players have the same bid space:* $[0, x_h^\sigma]$. *Where* x_h^σ *is the highest possible bid in equilibrium. The equilibrium bid distributions for each type;* $\sigma_{i,h}(b|s_i^j)$ *is continuous on* $(0, x_h^\sigma]$. *In equilibrium at most one player has an atom at zero.*

Proof. The proof follows directly from Lemma 1 in Siegel (2014). □

Define the *unconditional bid distribution* of player i as follows:

$$H_{i,h,\sigma}(b) = \sum_{j=1}^{n_i} \mu_{h,\sigma}(s_i^j)\sigma_{i,h}(b|s_i^j). \tag{3.8}$$

Since $\sigma_{i,h}(b|s_i^j)$ is continuous on $(0, x_h^\sigma]$, $H_{i,h,\sigma}(b)$ is continuous on $(0, x_h^\sigma]$. Denote the CDF of s_i by $\Phi_i(s)$ and it's generalized inverse by $\Phi^{-1}(s)$.[25]

$$\begin{aligned} \Phi_i(s) &:= \sum_{s_i \leq s} \mu_{h,\sigma}(s_i) \\ \Phi^{-1}(s) &:= \inf\{z \in \operatorname{supp} \Phi_\mathrm{i} : \Phi_i(z) \geq s\} \end{aligned} \tag{3.9}$$

In a monotone equilibrium for any bid greater than zero, there is only one type that bids there. Let that type be denoted by $\tilde{s}_i(b)$.

$$\tilde{s}_i(b) := \Phi_i^{-1}(H_{i,h,\sigma}(b)) \tag{3.10}$$

[24] Bidding zero can be thought of as not participating in the auction (since ties have zero measure in equilibrium). For example, if player 2 has an atom of 0.2 at zero. Then player 2 bids zero 20% of time and therefore is not participating 20% of the time. This behaviour occurs if player 2 is somehow discouraged by player 1 to enter, for example, when player 1 is much stronger than player 2.

[25] Zheng (2019) introduces the generalized inverse to deal with type distributions that have atoms.

Let $D\sigma_{i,h}(\cdot|s_i)$ denote the derivative of $\sigma_{i,h}(\cdot|s_i)$ when it exists. Then from Lemma D.2.4 (in the appendix) one can find the derivative of the unconditional bid distribution almost everywhere.

$$H'_{i,h,\sigma}(b) = \mu_{h,\sigma}(\tilde{s}_i(b))D\sigma_{i,h}(b|\tilde{s}_i(b)) \text{ } a.e. \tag{3.11}$$

Lemma 3.4.2. *$H'_{1,h,\sigma}(b) = H'_{2,h,\sigma}(b)$ for almost every $b \in (0, x_h^\sigma]$*

Proof. First note that in equilibrium:

$$\begin{aligned} D\sigma_{1,h}(b|\tilde{s}_1(b))) &= \frac{1}{\mu_{h,\sigma}(\tilde{s}_1(b)|\tilde{s}_2(b))V(\tilde{s}_2(b),\tilde{s}_1(b))}, \\ D\sigma_{2,h}(b|\tilde{s}_2(b)) &= \frac{1}{\mu_{h,\sigma}(\tilde{s}_2(b)|\tilde{s}_1(b))V(\tilde{s}_1(b),\tilde{s}_2(b))}. \end{aligned} \tag{3.12}$$

Since we have common values: $V(\tilde{s}_1(b),\tilde{s}_2(b)) = V(\tilde{s}_2(b),\tilde{s}_1(b))$.Then,

$$\begin{aligned} H'_{1,h,\sigma}(b) = \mu_{h,\sigma}(\tilde{s}_1(b))D\sigma_{1,h}(b|\tilde{s}_1(b)) &= \frac{\mu_{h,\sigma}(\tilde{s}_1(b))}{\mu_{h,\sigma}(\tilde{s}_1(b)|\tilde{s}_2(b))V(\tilde{s}_2(b),\tilde{s}_1(b))} \\ =\frac{\mu_{h,\sigma}(\tilde{s}_1(b))}{\frac{\mu_{h,\sigma}(\tilde{s}_1(b),\tilde{s}_2(b))}{\mu_{h,\sigma}(\tilde{s}_2(b))}V(\tilde{s}_2(b),\tilde{s}_1(b))} &= \frac{\mu_{h,\sigma}(\tilde{s}_1(b))\mu_{h,\sigma}(\tilde{s}_2(b))}{\mu_{h,\sigma}(\tilde{s}_1(b),\tilde{s}_2(b))V(\tilde{s}_2(b),\tilde{s}_1(b))} \\ =\frac{\mu_{h,\sigma}(\tilde{s}_2(b))}{\frac{\mu_{h,\sigma}(\tilde{s}_1(b),\tilde{s}_2(b))}{\mu_{h,\sigma}(\tilde{s}_1(b))}V(\tilde{s}_1(b),\tilde{s}_2(b))} &= \mu_{h,\sigma}(\tilde{s}_2(b))D\sigma_{2,h}(b|\tilde{s}_1(b)) = H'_{2,h,\sigma}(b) \end{aligned} \tag{3.13}$$

□

Lemma 3.4.2 is not sufficient to conclude that there are no atoms at zero for both players, even though $H_{i,h,\sigma}$ is continuous, differentiable almost everywhere, and $H_{i,h,\sigma}(x_h^\sigma) = 1$.[26] This is because the agents may differ on the set of measure zero. Any measure associated with a distribution function can be uniquely decomposed into the sum of three measures: an absolutely continuous part fdx, the countable sum of atoms, and a continuous singular part. If the singular part cannot be ruled out, the bid density might differ for some set with measure zero. The singular part can imply that one player has an atom at zero. To conclude, what is needed is that H is absolutely continuous.

Lemma 3.4.3. *$H_{i,\sigma}(b) = \int_0^b \mu_{h,\sigma}(\tilde{s}_i(b))D\sigma_{i,h}(b|\tilde{s}_i(b)))dx$ everywhere on $(0, x_h^\sigma]$.*

Proof. In the appendix. □

[26] An example of function that is both continuous and differentiable almost everywhere but not absolutely continuous is the Cantor function.

Proposition 3.4.4. *In the pure common-value model, neither player has an atom at zero.*

Proof. By Lemmas 3.4.2 and 3.4.3, $H_{1,\sigma}(b) = H_{2,\sigma}(b)$ everywhere on $(0, x_h^\sigma]$. By Lemma 3.4.1 of the appendix at most one player can have an atom at zero. This implies that neither player has an atom at zero. □

3.4.2 Equilibria of the Pure Common Value Model

The no atom at zero is critical, and this drives the main result of this section that peace equilibrium is the essentially unique equilibrium of the Pure Common Value Model. No atom at zero implies that both players' lowest type in all-pay auction games has zero expected payoff. Therefore, this type prefers peace over war, and hence there is no equilibrium where this type plays war with positive probability when the opponent plays peace with positive probability.

Lemma 3.4.5 establishes that in a monotone equilibrium of the common-value all-pay auction $\min T_i(\mu)$ gets zero expected payoff. The lowest type in the posterior type support is $\min T_i(\mu)$ and it can be that $s_i^1 < \min T_i(\mu)$. To clarify, s_i^1 is the lowest type of the prior distribution. On-path a monotone equilibrium always exists, so on-path lemma 3.4.5 always applies.

When using an arbitrary μ, the existence of a monotone equilibrium needs to be established. It is helpful to note that if a player makes a unilateral deviation and the non-deviator believes him to be one type with certainty, there exists a monotone equilibrium in the game where those beliefs are the true beliefs.

Lemma 3.4.5. *In any monotone equilibrium, if $t_i = \min T_i(\mu)$ and $-i$ has no atom at zero, then $U_i(t_i, b) = 0$ for all $b \in BR_i(t_i)$.*

Proof. In a monotone equilibrium, $0 \in BR_i(t_i)$. Since the opponent has no atom at zero, $U_i(t_i, 0) = 0$. Thus, $U_i(t_i, b) = 0$ for all $b \in BR_i(t_i)$. □

Proposition 3.4.6. *There exists a peace equilibrium exists for every v.*

Proof. Suppose not and i finds it profitable to deviate for some s_i^k. If i deviates, she unilaterally triggers the all-pay auction. Since Bayes's rule does not apply, $-i$ forms arbitrary beliefs over i's type. Assume that $-i$ believes that $\mu(s_i^{n_i}|s_{-i}) = 1$ for all s_{-i}. This system of beliefs will induce an common value all-pay auction game in which $-i$ keeps $s_i^{n_i}$ indifferent. Player $-i$ plays the equilibrium strategy for the game in which $\mu(s_i^{n_i}|s_{-i}) = 1$ is the true belief.

This hypothetical game has a unique monotone equilibrium since PM is satisfied for $-i$ by assumption, and for i it is trivially satisfied with only one type. By lemma 3.4.5, the expected payoff for $s_i^{n_i}$ is zero in this game. By lemma 3.3.1, the expected payoff is strictly decreasing for all other types when $b > 0$. Therefore, given $-i$'s strategy, it must be that if i actually has a lower type than $s_i^{n_i}$, she bids zero, thus for all $s_i^k \in T_i$ expected payoff is zero.

Abiding by the peaceful σ yields positive expected payoff for all $s_i^k \in T_i$. Hence, this deviation is not profitable, a contradiction as required. This argument is true for every possible peace deal υ. □

Denote π_{σ,h,t_i} as the interim expected payoff for t_i given h and σ. If $h = (peace, peace) = (p, p)$, then the all-pay auction is not triggered and $\pi_{\sigma,(p,p),t_1} = \upsilon V(s_i, s_{-i})$ and $\pi_{\sigma,(p,p)t_2} = (1-\upsilon)V(s_i, s_{-i})$. For $h \neq (p, p)$, π_{σ,h,t_i} is determined in the all-pay auction.

Let $U_i(\sigma, t_i)$ be the expected payoff of abiding by strategy profile σ and $A := \{peace, war\} = \{p, w\}$.

$$U_1(\sigma, t_1) = \sum_{a_1 \in A} \sigma_{1,\emptyset}(a_1|t_1) \left[\sum_{a_2 \in A} \left(\sum_{t_2 \in T_2} \sigma_{2,\emptyset}(a_2|t_2) f(t_2|t_1) \pi_{\sigma,(a_1,a_2),t_1} \right) \right] \tag{3.14}$$

Note that in (3.14), conditional on player 1's type, $\sum_{t_2 \in T_2} \sigma_{2,\emptyset}(a_2|t_2) f(t_2|t_1)$ is the probability that the opponent makes announcement a_2.

On-path, $\min T_i(\mu_{(w,a_{-i}),\sigma})$ is the independent of a_{-i}. So through σ_i, i has complete control over his posterior on-path support when announcing war. The following lemma states that if my opponent plays peace with positive probability, and I have a strategy where I play war with positive probability. My lowest type that is supposed to play war with positive probability in accordance with this strategy has an incentive to play peace with certainty instead.

Lemma 3.4.7. *Let* $s_i = \min T_i(\mu_{(w,a_{-i},\sigma)})$, $\upsilon \in (0, 1)$ *and* $\exists\, s_{-i} \in S_{-i}$ *s.t.* $\sigma_{-i,\emptyset}(peace|s_{-i}) > 0$ *then* $\sigma_{i,\emptyset}(peace|s_i) < 1$ *is a not best response for* s_i.

Proof. By lemma 3.4.5, $U_{i,(w,a_2),\sigma}(s_i, b) = 0$. This implies that $\pi_{\sigma,(w,a_2),s_i} = 0$. Hence, we can rewrite (3.14) as follows.

$$U(\sigma, s_i) = \sigma_{i,\emptyset}(peace|s_i) \left[\sum_{s_{-i} \in S_{-i}} \sum_{a_2 \in A} \sigma_{-i,\emptyset}(a_2|s_{-i}) f(s_{-i}|s_i) \pi_{\sigma,(p,a_2),s_i} \right] \tag{3.15}$$

Because there is atleast one s_{-i} such that $\sigma_{-i,\emptyset}(peace|s_{-i}) > 0$ and $\upsilon \in (0, 1)$, the expected payoff of playing peace is positive. That is, $\sigma_{-i,\emptyset}(peace|s_{-i})f(s_{-i}|s_i)\pi_{\sigma,(p,p),s_i} > 0$. This implies that (3.15) is strictly increasing in $\sigma_{i,\emptyset}(peace|s_i)$. □

Lemma 3.4.7 implies there is no other equilibrium except for war and peace. This is because, in every other candidate equilibrium, there is at least one player who has a type that is the lowest type playing war with positive probability. This type has zero expected payoff playing war and positive expected payoff playing peace, and thus this candidate is not an equilibrium.

To relate this game to the signalling literature, notice that both players are "pooling" peace in the peace equilibrium. Likewise, both players are "pooling" war in the war equilibrium. That there are no separating equilibria makes sense. In the common value model, both players are trying to learn the true value of the prize, and signaling the type provides no advantage to the player—all the while, the opponent learns more about the value of V.

Theorem 3.4.8. *No equilibrium exists except for war and peace when $\upsilon \in (0, 1)$*

Proof. The proof is by means of contradiction. Suppose another equilibrium exists besides war and peace.

If in this equilibrium $-i$ plays anything but pool war, it must be that i is pooling peace. Suppose not, and i plays war with positive probability. There exists a type $s_i = \min T(\mu_{(w,a_{-i}),\sigma})$. Type s_i wants deviate to peace by lemma 3.4.7, a contradiction.

Therefore, it must be that if $-i$ plays anything but pool war, i pools peace. But if i pools peace, $-i$ will also want to pool peace by the same argument used for i. Thus, no other equilibrium exists than the peace equilibrium in which $-i$ plays anything but pool war.

Therefore, this other equilibrium must be such that $-i$ is pooling war. But if i plays anything but pool war himself, s^1_{-i} wants to deviate to peace by lemma 3.4.7, a contradiction. □

Theorem 3.4.8 hinges on the fact that neither player has an atom at zero, which induces that the lowest type of each player's posterior support gets zero expected pay-off in equilibrium. It is important to reiterate that this is due to the common value. If we add private values through power differences, it is easy to come up with a counterexample, as is shown in the next section.

3.5 Adding Power to the Common-Value Model

In this section, power is introduced to the model. In the all-pay auction game, adding power is equivalent to adding a private value element to the model. In this setting, the weaker player has an atom at zero in the all-pay auction, which implies that not every peace deal is implementable.

Once you depart from the "pure" common-value model, you need to know which beliefs players have when they unilaterally deviate away from the peace deal. Specifically, you need to know what the non-deviator beliefs are over the deviator's type since his beliefs are not subject to Bayes's rule. Since this chapter is concerned with the existence of peace equilibrium, I want to find the beliefs that are most permissive for peace. When no peace equilibrium exists for the most permissive beliefs, there exists no peace equilibrium.

The most permissive belief turns out to be where the deviator is believed to be the highest type with certainty. Mathematically this means that the deviator's highest type will be kept indifferent for all bids on the bid space. The deviator's payoff is entirely determined by the atom at zero of the non-deviating player.[27]

This feature is a blessing for analysis; if one knows the size of the atom at zero for the non-deviator given these beliefs, one knows the deviator's expected payoff choosing war. The approach for this section is as follows. Find the atom at zero for both players, which determines their incentive to deviate from the peace equilibrium by choosing war. Once the incentive of choosing war is known, it is clear which peace deals are acceptable for the players.

3.5.1 Player 1 is Known to be Stronger than Player 2.

I start my analysis by adding a commonly known difference in power to the common-value model. In this setting, I can find the size of the atom at zero for the weaker player in the all-pay auction. Further, I show that the stronger player will never have an atom at zero. No atom implies the weaker player will have zero expected payoff deviating to war. This makes existence trivial since the whole prize can be gifted to the stronger player. Unlike the pure common-value model, not every peace deal is implementable. There are certain peace deals that the stronger player will not accept. This section gives the minimal peace deal required for peace to be accepted by the stronger player.

[27] When a player's type is indifferent between all bids in the bid space, the payoff is the same for all bids. Therefore, it is sufficient to calculate the payoff at zero, which is only positive if the opponent has an atom at zero.

Assumption: Common Values with Known Differences in Power Let $p_1 > p_2 = 1$ is known, such that $T_i = S_i$. Further, $s_i^k < s_i^{k+1}$.

Equation(3.16) ex-post payoff of the all-pay auction if you win the object and bid b. The known differences in power implies that player 1 faces a lower marginal cost of bidding b.

$$V(s_i, s_{-i}) - \frac{b}{p_i} \tag{3.16}$$

Gunboat Diplomacy

Lemma 3.5.1 (Unconditional Surrender). *A peace equilibrium exists in which player 2 receives nothing.*

The proof of Lemma 3.5.1 is in the appendix. The weaker player accepting any peace deal is reminiscent of gunboat diplomacy. Gunboat diplomacy was a show of force to obtain favorable treaties for Western powers in the 19th century. With their powerful modern navies, they would essentially bully other countries into deals of their choice.

A famous example is the opening of Japan to trade by Commodore Perry. They forced Japan into trade with the United States by displays of power. At the time, the Japanese government was very much against trade with Western powers. However, the difference in power was clear to the Japanese government, and they eventually capitulated to Perry's demands.

No Unconditional Surrender

Throughout history, unconditional surrender is rare prior to any conflict, which suggests that υ cannot be set freely. Suppose that a peace deal is fixed between zero and one, i.e $\upsilon \in (0, 1)$. In particular, the assumption that $\upsilon < 1$ is important because now player 1 (the strong player) may reject this peace offer.

For Player 1, the profitability to deviate to war increases with power, while it has no effect on the expected payoff of picking peace. At the same time, this increase does not affect player 2's profitability to deviate to war. Thus, an increase in the power of player 1 hurts the prospect of peace. If we increase player 2's power, the opposite happens. It decreases the profitability to deviate for player 1, yet again it has no impact on player 2's profitability to deviate.[28] In other words, asymmetry in power allows for war to occur.

[28] It does not affect player 2's profitability as long as $p_1 \geq p_2$. When $p_1 = p_2$ then we are back in the pure common value model, and war can't occur on-path for any peace deal υ.

In general, it is possible to derive which peace deals are acceptable given primitives (the beliefs, the values of the prize, and the power of player 1). Theorem 3.5.2 provides this. However some notation is introduced, which first needs to be explained.

$$\mu_{k|m} V_{km} := \mu(s_2^k | s_1^m) V(s_1^m, s_2^k) \tag{3.17}$$

In the appendix,corollary D.2.8, it is shown that the size of the atom at zero for player 2 is $1 - \frac{p_2}{p_1}$. It matters to player 1 which types of player 2 have the atom, since player 1's payoff depends player 2's type. By the monotonicity of the equilibrium, all types below and equal to $\Phi_2^{-1}(1 - \frac{p_2}{p_1})$, have an atom. $\Phi_2(s_2^t) = \sum_{j=1}^{t} \mu_{t|n_1}$ is player 2's CDF when player 1 deviates.[29]

Denote the highest type of player 2 with an atom at zero as: $s_2^k := \Phi_2^{-1}(1 - \frac{p_2}{p_1})$. This type may also bid above zero, and spend less than measure 1 on the atom at zero. The size of atom of k is given by $\sigma_{2,h}(0|s_2^k)$. All types j below s_2^k have an atom of measure 1 at zero, i.e. $\sigma_{2,h}(0|s_2^j) = 1$.

Theorem 3.5.2. *For a fixed peace deal υ, a peace equilibrium exists iff for all types of player 1 $m \leq n_1$ the inequality in (3.18) holds.*

$$\upsilon \geq \frac{\sum_{s_2^j < \Phi_2^{-1}(1 - \frac{p_2}{p_1})} \mu_{j|m} V_{mj} + \mu_{k|m} V_{mk}\, \sigma_{2,h}(0|\Phi_2^{-1}(1 - \frac{p_2}{p_1}))}{\mathbb{E}[V|s_1^m]}. \tag{3.18}$$

Notice that if $p_1 = p_2 \implies \upsilon \geq 0$, we are back in the pure common value, in which player 1 accepts any deal. If $p_1 \to \infty \implies \upsilon \geq 1$. Hence, if player 1 is infinitly strong, she requires unconditional surrender.

The prior distribution f and the functional form of V matter. For example, given certain assumptions, a lower type (player 1 with a low signal of the value of the prize) may be more inclined to start a war than if she has a higher type. This non-monotone behavior is studied in the next subsection.

Non-Monotone Behaviour

Theorem 3.5.2 gives the condition on the primitives when a fixed peace deal is implementable. Since the expected payoff for war and the expected payoff for peace are increasing in type s_1 by condition PM, a lower type may choose war where a higher type

[29] This is what player 2 believes to be players 1's belief over player 2's type, that is, player 2's probability of being type s_2^t or higher given that he beliefs player 1 to be type $s_1^{n_1}$. By monotonicity the types are ordered on the bidspace from low to high, thus if player 2 bids zero with $1 - \frac{p_2}{p_1}$ probability then the highest type with an atom is $\Phi_2^{-1}(1 - \frac{p_2}{p_1}) := \inf \left\{ z \in \operatorname{supp} \Phi_2 : \Phi_2(z) \geq (1 - \frac{p_2}{p_1}) \right\}$.

would not. This possible if the value of peace increases faster than war when the type is increases. In this case non-monotone behaviour can occur in equilibrium.

If either the value of peace or war were constant across types, it would be sufficient to look at the highest type of player 1 and see if this type has an incentive to deviate.[30] However, since both payoffs are increasing in s_1, it could be that the difference is not increasing in type and that a lower type requires a higher υ to accept peace. To show this kind of non-monotone behavior, consider the following lemma.[31]

Lemma 3.5.3. *If $\forall i = 1, 2$, $S_i := \{L, H\}$. Furthermore, f is affiliated and $V(s_1, s_2)$ is log-supermodular in (s_1, s_2) then the minimal acceptable peace deal is decreasing in s_1.*

Proof. Let $\upsilon(L)$ and $\upsilon(H)$ denote the minimal acceptable peace deal for a low type and high type respectively. By Theorem 3.5.2, $\upsilon(L)$ and $\upsilon(H)$ are given by (3.19).

$$\begin{aligned}\upsilon &\geq \frac{\mu_{L|L}V_{LL} + \mu_{H|L}V_{HL}\sigma_{2,h}(0|\Phi_2^{-1}(c_2))}{\mu_{L|L}V_{LL} + \mu_{H|L}V_{HL}} = \upsilon(L) \\ \upsilon &\geq \frac{\mu_{L|H}V_{LH} + \mu_{H|H}V_{HH}\sigma_{2,h}(0|\Phi_2^{-1}(c_2))}{\mu_{L|H}V_{LH} + \mu_{H|H}V_{HH}} = \upsilon(H)\end{aligned} \tag{3.19}$$

$$\begin{aligned}&\upsilon(L) - \upsilon(H) = \\ &\frac{\mu_{L|L}V_{LL} + \mu_{H|L}V_{HL}\sigma_{2,h}(0|\Phi_2^{-1}(c_2))}{\mu_{L|L}V_{LL} + \mu_{H|L}V_{HL}} - \frac{\mu_{L|H}V_{LH} + \mu_{H|H}V_{HH}\sigma_{2,h}(0|\Phi_2^{-1}(c_2))}{\mu_{L|H}V_{LH} + \mu_{H|H}V_{HH}} \\ &= \frac{\left[\mu_{H|H}V_{HH}\mu_{L|L}V_{LL} - \mu_{H|L}V_{HL}\mu_{L|H}V_{LH}\right]\left(1 - \sigma_{2,h}(0|\Phi_2^{-1}(c_2))\right)}{\left(\mu_{L|L}V_{LL} + \mu_{H|L}V_{HL}\right)\left(\mu_{L|H}V_{LH} + \mu_{H|H}V_{HH}\right)} \geq 0\end{aligned} \tag{3.20}$$

Where the inequality follows from the fact that $\left[\mu_{H|H}V_{HH}\mu_{L|L}V_{LL} - \mu_{H|L}V_{HL}\mu_{L|H}V_{LH}\right] \geq 0$ by log-supermodularity of f (also called affiliation) and the log-supermodularity of $V(s_1, s_2)$. Further, $\sigma_{2,h}(0|\Phi_2^{-1}(c_2)) \leq 1$, and this inequality is strict if $p_1 < \infty$. □

Why does a low type need a better peace deal than a high type? Essentially, it is because of the information that affiliation provides. When player 1 receives a low type, not only does she know the common value prize is less valuable. She also knows that it is more likely that the opponent has a low signal because of affiliation. Because player

[30]For example, in Zheng (2019), the value of the prize peace a constant. In this case, it suffices to study the deviations of the highest type.

[31]It is important to clarify I mean non-monotone increasing. It also implicitly means that war is a "higher" action than peace, so perhaps a better definition is in order

2's low type has a strictly larger atom at zero than his high type, it is more likely that player one faces little resistance in war. Hence, war becomes relatively more profitable when player 1 has a low type.

As a corollary, if f is reverse affiliated and $V(s_1, s_2)$ is log-submodular, the conclusion of Lemma 3.5.3 is reversed. When $V(s_1, s_2) = s_1$, then V is trivially log-submodular. When f is independent, it is trivially reverse affiliated. Thus, checking the highest type for deviations is sufficient in an independent private value model.

Corollary 3.5.4. *If* $\forall i = 1, 2$, $S_i := \{L, H\}$*. Furthermore,* f *is reverse affiliated and* $V(s_1, s_2)$ *is log-submodular in* (s_1, s_2) *then the acceptable peace deal is increasing in* s_1*.*

3.5.2 Two-Dimensional Types

Recall that T_i denotes the set of private types, but now $T_i = P_i \times S_i \subset \mathbb{R}^2$ is two-dimensional. Like before, I characterize the size of the atom the non-deviator has when an unilateral deviation occurs, given the most permissive beliefs.[32] This yields each types' expected payoff when deviating to war, which determines if any fixed peace deal v is acceptable.

Potential Power

To the help analysis, I'm introducing the following notion of *Potential Power* of player i: (PP_i). Potential power is the supremum(sup) of i's set of power levels.

$$PP_i := \sup\ P_i \tag{3.21}$$

Potential power determines the profitability of war for both players, but it is not the only factor that determines if a peace equilibrium exists. Uncertainty over power also plays a role. What I find in this chapter is that certainty and symmetry in power lead to peace. In the model where there was certainty over power, peace was always implementable. This is because player 2's expected payoff from war was zero, so the prize could be gifted to player 1 and peace could be achieved. When there is uncertainty over power, player 2's payoff might be greater than zero, even when $PP_1 > PP_2$.

Atoms at Zero

With two dimensional types both players can have an atom at zero. The atom at zero for player 1 is going depend on the potential power of the player 2, and the likelihood of

[32] Just like in the previous subsection, the most permissive belief is $\mu(s_i^{n_i}) = 1$, when i deviates.

1's types conditional on player 2 achieving their potential power. It does not depend on V nor the signals over value of the prize S. This should not come as a surprise, because in the pure common-value model, which has no differences in power, there are no atoms. Therefore, only private value elements can lead to atoms at zero in the all-pay auction.

The likelihood of $-i$'s type j conditional on player i achieving their potential power is denoted by a^j_{-i}. Then with slight abuse of notation the CDF over the types for player $-i$ conditional on player i achieving their potential power is denoted by $\Phi_{-i}(t^k_{-i})$.

$$a^j_{-i} = f(t^j_{-i}|t_i^{n_1}) \text{ and } \Phi_{-i}(t^k_{-i}) = \sum_{j=1}^{k} a^j_{-i}. \tag{3.22}$$

Next, consider the projection map $\rho : T_{-i} \to P_{-i}$. The projection $\rho\left[t_{-i}\right]$ maps from a type t_{-i} to its corresponding power element. The atom player $-i$ has at zero when i deviates is given by (3.19). Proof of this is found in Lemma D.2.7.

$$c_{-i} = \inf \left\{ c \in [0,1] : \int_c^1 \frac{PP_i}{\rho\left[\Phi^{-1}_{-i}(u)\right]} du = 1 \right\} \tag{3.23}$$

Theorem 3.5.5

Given the primatives $(f, V, T, \succ)$, the atoms at zero are characterized. I define the expected payoff for deviating to war under the most permissive beliefs as $\omega_i(t_i^m, c_{-i})$. Notice that $\omega_i(t_i^m, c_{-i})$ is increasing in t_i by PM and increasing in c_{-i} (the opponents atom at zero).

$$\omega_i(t_i^m, c_{-i}) := \frac{1}{p_i\left[t_i^m\right]} \left[\sum_{t \prec \Phi^{-1}_{-i}(c_{-i})} f(t|t_i^m) V_i(t_i^m, t) + f(\Phi^{-1}_{-i}(c_{-i})|t_i^m) V_i(t_i^m, \Phi^{-1}_{-i}(c_{-i})) \sigma_{2,h}(0|\Phi_2^{-1}(c_2)) \right] \tag{3.24}$$

Theorem 3.5.5. *A peace equilibrium exists iff*

$$\forall m,\ \forall k,\ \omega_1(t_1^m, c_2) + \omega_2(t_2^k, c_1) \leq \upsilon \mathbb{E}[V|t_1^m] + (1-\upsilon)\mathbb{E}[V|t_2^k]. \tag{3.25}$$

Like in the previous section, there is uncertainty over which type has the most incentive to deviate, thus it must hold for every m and k. To illustrate theorem 3.5.5, consider the following examples.

Example 1. Let the players be ex-ante symmetric and $P = \{p^1, PP\}$ and $S = \{L, H\}$. Further, let the ordering $\succ$ be a lexicographic ordering. That is, $t \succ t' \iff t > t'$ or $t = t'$

and $s > s'$. This implies $t^1 = (p^1, L) \prec t^2 = (p^1, H) \prec t^3 = (PP, L) \prec t^4 = (PP, H)$.

Using (3.23), the atom both players face if they deviate is $c_{-i} = (1 - \frac{p^1}{PP}) \sum_{j=1}^{2} a_{-i}^j$. Notice that the atom is strictly smaller than $\sum_{j=1}^{2} a_{-i}^j$. Hence, the value of $\omega_i(t_i^m, c_{-i})$ is bounded by $\bar{c} = \sum_{j=1}^{2} a_{-i}$.

$$\omega_i(t_i^m, \bar{c}) = f(t^1|t^m)V(t^m, t^1) + f(t^2|t^m)V(t^m, t^2) \tag{3.26}$$

Equation (3.26) tells us that deviating to war becomes more profitable if the likelihood of facing a weak opponent $(f(t^1|t^m) + f(t^2|t^m))$ increases. Since, you always win the full prize if deviate to war and face weak opponent. Peace is implementable if this is not to likely. A sufficient conditions for peace is given by (3.27).

$$\begin{aligned} \omega_i(t_i^m, \bar{c}) &\leq \frac{1}{2}\mathbb{E}[V|t_i^m] \\ \iff \left[f(t^3|t^m) - f(t^1|t^m)\right] V(t^m, L) &+ \left[f(t^4|t^m) - f(t^2|t^m)\right] V(t^m, H) \geq 0 \end{aligned} \tag{3.27}$$

In the second line, the terms are collected by what the opponent signal s_{-i} is. Equation (3.27) reveals that if the types are independently uniformly distributed, that a peace equilibrium exists. Further, if the types are independently distributed, putting more weight on the strong types (t^3 and t^4) doesn't hurt the prospect of peace (war becomes more costly). Finally, positive correlation (for example affilation) reduces (3.27) for the weak types (t^1 and t^2). In fact if dependence is strong enough, (3.27) is negative for both weak types and a peaceful equilibrium can not be guaranteed.

Example 2. Consider the following private value model: $T_i = P_i = \{1, PP_i\}$. The players might differ in potential power, but are symmetric every other way, and being weak is normalized to 1. There is no private information on the common-value, so WLOG let $V = 1$. Now deviating to war simply yields c_2 for player 1 and c_1 for player 2. Thus, (3.25) becomes $c_1 + c_2 \leq 1$.

Using (3.23) to calculate c_{-i}.

$$c_{-i} = f(1|PP_i)\left(1 - \frac{1}{PP_{-i}}\right) + \frac{1}{PP_{-i}} - \frac{1}{PP_i} \tag{3.28}$$

Consider a baseline where the players are fully symmetric, i.e. $PP_1 = PP_2 = PP$. Then a peace equilibrium exists iff $(1 - \frac{1}{PP})f(p^1|PP) \leq \frac{1}{2}$. Suppose that peace was exactly implementable in the benchmark, i.e. $(1 - \frac{1}{PP})f(p^1|PP) = \frac{1}{2}$.

Now perturb this game in the following way. Increase player 1's potential power, $PP_1 = (1+\epsilon)PP$. Notice that by symmetry of the prior distribution: $f(p^1|PP_2) = f(p^1|PP_1) = f(p^1|PP)$.

$$\begin{aligned} c_1 + c_2 &= f(1|PP)\left(1 - \frac{1}{PP_1}\right) + \frac{1}{PP_1} - \frac{1}{PP_2} + f(1|PP)\left(1 - \frac{1}{PP_2}\right) + \frac{1}{PP_2} - \frac{1}{PP_1} \\ \iff c_1 + c_2 &= f(1|PP)\left(1 - \frac{1}{PP+\epsilon}\right) + f(1|PP)\left(1 - \frac{1}{PP}\right) > 1 \end{aligned} \tag{3.29}$$

If you stochastically increase the power of player 1 by increasing potential power, the old peace deal is no longer implementable. This shines a different light on the peace through deterrence claim in Zheng (2019) and Bester and Wrneryd (2006), that increase in stochastic power does not hurt the prospect for peace. They point out that if power is increased by reducing the probability $f(1|PP)$, the atoms reduce for the other player reduces, and hence a peace deal is still implementable. However, reducing $f(1|PP)$ reduces the chance of facing a weak type when a player realizes her type is strong. This reduces the incentive to go to war. Hence, what prevents war is the likelihood of facing a player that is more like me (also strong). Hence, the players are more symmetric (in the interim), and peace is implementable. Therefore, when the potential power for player 1 is increased in Example 2, the two players are more asymmetric in every "state of the world", and peace may not be implementable. To conclude, it is symmetry in power that prevents war.

3.5.3 The Atomic Age and the Fallacy of the Deterrence Through Strength

Ever since World War II, there has not been any direct war between two countries that had access to nuclear weapons. One might be tempted to conclude that more powerful countries imply a higher chance of peace. Furthermore, there is this idea that asymmetry in power leads to peace, for it will deter the weaker player from fighting and peace can be achieved. I reject these conjectures by giving an alternative explanation within the findings of my model.

First, what does uncertainty over power and potential power mean? I interpret potential power in countries as the known maximum level of access to manpower, equipment and military technology. The uncertainty can be explained as factors that hinder a country from monitoring another country's current state of manpower, equipment and military technology. Then having an arsenal of nuclear weapons in the context of traditional warfare implies a potential power limiting to infinity. The power of this weapon is so great that other random factors will hardly impact the realized power. Therefore, a country with access to nuclear weapons is country with a known power level. Naturally war will not occur between two countries with nuclear weapons, since this puts us in the pure common-value model. But it is not the destructiveness of the A-bomb but the symmetry of the players' power and the certainty over this power that led to peace.

There are plenty of examples of wars between atomic powers and those without them. Thus, increasing power unilaterally clearly does not lead to peace. This is particularly true when not all possible peace deals can be suggested to the players. In general, I think it is faulty to conclude that buildup of military power can prevent wars. Just prior to World War 1, Winston Churchill noted that "The world is arming as it has never armed before". Needless to say World War 1 did happen. I believe that the buildup of arms, which is an increase in potential power, but also increase in the elements of the support of power, contributes to increased uncertainty over another country's power. This allows countries to believe there is a chance they are stronger than their opponent, which leads to war.

3.6 Conclusion

Symmetry in power leads to peace, while asymmetry in power leads to war. Certainty leads to peace, while uncertainty leads to war. These two observations allow for an explanation of why the Atomic Age has been relatively peaceful. It also allows me to picture a world where peace does not rely on atomic weapons. If the world completely demilitarizes and there is strict global surveillance, I predict peace. I admit this is an unlikely equilibrium to be reached by countries of the world since those with weapons enjoy considerable advantages over those that do not. Furthermore, my model does not allow build-up or build down of armaments; it is a given in my model.

An avenue of further research is when potential power is endogenous. One might find it optimal to invest in power regardless of its cost in peacetime. It can perhaps explain the stockpiling of weapons in the Cold War and their dismantling afterward. Further, it would be interesting to design a mechanism to reduce the stockpile of weapons in such a setting.

Another direction to consider is a many-player negotiation to see how coalitions are formed. Many players is tricky in my model since many-player all-pay auctions are still in the frontier. However, one can imagine that it might be optimal to form an alliance to force the expected payoff of those outside the alliance to zero and then split the prize in the coalition. In this case, one could find what kind of alliances are feasible and stable.

Appendix A

All-Pay Auction Proofs

A.1 Necessary Conditions for Equilibrium in the All-Pay Auction

The following lemma's imply Lemma 1.3.1 in Chapter 1, Lemma 2.3.1 in Chapter 2, and Lemma 3.4.1 in Chapter 3. Omitted is the proof for the common supremum which can be found in Siegel (2014). The following results are true for any finite amount of types T_i.

Definition of a Gap $\operatorname{supp}\left[G_i^k\right] \subset \mathbb{R}$ has a gap if $\operatorname{supp}\left[G_i^k\right]$ is disconnected. A non-empty connected subset $\beta_\alpha \subset \mathbb{R}$ is a gap of $\operatorname{supp}\left[G_i^k\right]$ if 1. $\beta_\alpha \cap \operatorname{supp}\left[G_i^k\right] = \emptyset$, 2. $\beta_\alpha \subset \operatorname{conv}\left(\operatorname{supp}\left[G_i^k\right]\right)$ and 3. there exist no connected B s.t. $\beta_\alpha \subset B$ and B satisfies 1. and 2.

A gap is an interval and because the support is a closed set, any gap on the support is an open interval.

Lemma A.1.1. *Any gap β_α of $\operatorname{supp}\left[G_i^k\right]$ is a non-empty open interval and $\operatorname{supp}\left[G_i^k\right]$ can have at most countably many gaps.*

Proof. Since $\operatorname{supp}\left[G_i^k\right]$ by definition is the smallest closed set with full measure, it's complement is open. A non-empty subset of $\mathbb{R}$ is open iff it is the union of countable collection of disjoint open intervals. □

Thus, lemma A.1.1 implies that for each gap β_α of $\operatorname{supp}\left[G_i^k\right]$, $\inf \beta_\alpha \in \operatorname{supp}\left[G_i^k\right]$ and $\sup \beta_\alpha \in \operatorname{supp}\left[G_i^k\right]$.

Definition of an Atom $G_i^k(b)$ has atom at b if $Pr(b_i = b|k) > 0$. A player i has an atom at b if there exists a $k \in T_i$ s.t. $G_i^k(b)$ has atom at b.

Lemma A.1.2. *If $V_1^k(j) > 0 \implies V_2^j(k) > 0$ and if $V_1^k(j) = 0 \implies V_2^j(k) = 0$.*

Proof. Recall, $V_1^k(j) = f_1(j|k)v_1(k,j) = \frac{f(k,j)}{f_1(k)}v_1(k,j)$. $V_1^k(j) > 0 \implies f(k,j) > 0$, since by assumption $f_i(k) > 0$ and $v_i(k,j) > 0$. Then $V_2^j(k) = f_2(k|j)v_2(j,k) = \frac{f(k,j)}{f_2(j)}v_2(j,k) > 0$. □

Lemma A.1.3. *If $V_i^k(j) > 0$ then in equilibrium there exists no bid b^* s.t. both type k of i and type j of $-i$ have an atom at b^*.*

Proof. Suppose not, then k has an incentive to deviate to $b^*+\epsilon$ s.t. $0 < \epsilon < \frac{1}{2}V_i^k(j)Pr(b_{-i} = b|j)$. Because,

$$
\begin{aligned}
& U_i(b^* + \epsilon, k \,|\, G_{-i}) - U_i(b^*, k \,|\, G_{-i}) \\
\iff & \sum_{T_{-i}} V_i^k(j)\hat{G}_{-i}^j(b^* + \epsilon) - b^* - \epsilon - \sum_{T_{-i}} V_i^k(j)\hat{G}_{-i}^j(b^*) + b^* \\
& \geq \frac{1}{2}V_i^k(j)Pr(b_{-i} = b|j) - \epsilon > 0
\end{aligned}
\tag{A.1}
$$

So, b^* is not a best response. Because $Pr(b_i = b^*|k) > 0 \implies Pr\left(X \in \text{argmax}_b U_i(b, k \,|\, G_{-i})\right) < 1$, this is not an equilibrium, a contradiction as required.

□

Lemma A.1.4. *In equilibrium there exists no bid $b^* > 0$ s.t. both players have an atom.*

Proof. Suppose not and both players have an atom at $b^* > 0$. Then by the previous lemma, for all types k of i and j of $-i$ that have an atom at b^*, it must be that $V_i^k(j) = 0$ and $V_{-i}^j(k) = 0$. Now it can't be that there is an another type $d \neq j$ of player $-i$ s.t. d bids on the interval $(b^* - \delta, b^*)$ for some $\delta > 0$.[1] Thus, for all j that bid on $(b^* - \delta, b^*)$, it must be that $V_{-i}^j(k) = 0 \implies V_i^k(j) = 0$. If k reduces his bid b^* by ϵ he gains ϵ but doesn't lose expected payoff. Thus, b^* is not a best response for k. Since $Pr(b_i = b^*|k) > 0 \implies Pr\left(X \in \text{argmax}_b U_i(b, k \,|\, G_{-i})\right) < 1$, this contradicts equilibrium. □

Lemma A.1.5. *For both $i = 1, 2$, $\cup_{k=1}^{n_i} \text{supp}\left[G_i^k\right]$ has no gap.*

[1] This is because all bids on the interval $(b^* - \delta, b^*)$ for some $\delta > 0$ are not a best responses for type d by a similiar argument as in lemma A.1.3. Now either $(b^* - \delta, b^*)$ carries positive measure for supp $\left[G_{-i}^d\right]$ which leads to a contradiction of equilibrium, or it doesn't, which contradicts that its part of the support (the smallest closed set with full measure).

Proof. Suppose not and $\cup_{k=1}^{n_1}\text{supp}\left[G_1^k\right]$ has a gap. Let $(\underline{a},\bar{a})$ denote the gap and for this to be a gap $\bar{a} > 0$. Then either$(\underline{a},\bar{a})$ is part of a gap for $\cup_{k=1}^{n_2}\text{supp}\left[G_2^k\right]$ as well or player 1 has an atom at $\underline{a}$.[2] If there is an atom for type k of player i, and if $V_2^j(k) > 0$, player 2's type j has incentive to bid ϵ above $\underline{a}$, but for each chosen ϵ he can find a smaller δ which is more profitable. If $V_2^j(k) = 0$ for all k that have an atom at $\underline{a}$, then $\underline{a}$ is strictly better for j than any other bid on $(\underline{a},\bar{a})$. Thus, there can be no equilibrium where player 1's gap is $(\underline{a},\bar{a})$ and player 1 has an atom at $\underline{a}$.

Therefore, player 2 has a gap at $(\underline{a},\bar{a})$.[3] Since $\bar{a} \in \cup_{k=1}^{n_i}\text{supp}\left[G_1^k\right]$ there exists a k s.t. $\bar{a} \in \text{supp}\left[G_1^k\right]$ and k has an atom at $\bar{a}$, otherwise $U_2(\underline{a}, j \,|\, G_1) > U_2(\bar{a}, j \,|\, G_1)$ for all j. But since player 2 has the same gap, player 2 must also have an atom at $\bar{a}$. By lemma A.1.4 this contradicts equilibrium. This is true for all gaps, so $\cup_{k=1}^{n_i}\text{supp}\left[G_i^k\right]$ has no gap. □

Lemma A.1.6. *In equilibrium there exists no bid $b^* > 0$ s.t. either player has an atom.*

Proof. First note that if the full support assumption is satisfied i.e. $V_{-i}^j(k) > 0$ for all k and j then the claim is true.[4] Relaxing full support does not change it much, now if player 1 has an atom at $b^* > 0$ the only types of player 2 that will bid an open interval below $b^* > 0$ are such that $V_2^j(k) = 0$, thus player 1 has incentive to bid b^*, a contradiction. □

Lemma A.1.7. *For both $i = 1, 2$, $\inf\left(\cup_{k=1}^{n_i}\text{supp}\left[G_i^k\right]\right) = 0$.*

Proof. Let $a_1 = \inf\left(\cup_{k=1}^{n_1}\text{supp}\left[G_1^k\right]\right)$ and $\inf\left(\cup_{k=1}^{n_2}\text{supp}\left[G_2^k\right]\right) = a_2$. If $0 < a_1 \leq a_2$ then which ever type k of player 1 that has a_1 in its support would better off bidding zero instead of any bid in an epsilon radius around a_1. This is because k's payoff is continuous above zero since there are no atoms above zero. Now if $a_1 = 0$ and $a_2 > 0$, then on the interval $(0, a_2)$ player 2 does not bid. All of player 1' types strictly prefer bidding zero to any b on $(0, a_2)$ which contradicts no gap on $\cup_{k=1}^{n_i}\text{supp}\left[G_1^k\right]$. □

Lemma A.1.8. *If G is a weakly monotone equilibrium then* $\text{supp}\left[G_i^k\right]$ *is an interval.*

Proof. Suppose not, then any gap β_α of type k belongs to the support of j, i.e. $\beta_\alpha \subset \text{supp}\left[G_i^j\right]$. Take any bid $b^* \in \beta_\alpha$, since $\inf\, \beta_\alpha < b^*$ the pairwise maximum of $\{\inf\, \beta_\alpha, b^*\}$

[2] If player 1 has no atom at $\underline{a}$, then bidding $\underline{a}$ is strictly better than any other bid on $(\underline{a},\bar{a})$ for player 2.

[3] I showed that $(\underline{a},\bar{a})$ belongs to a gap of player 2, but if players 2's gap is a super set of player 1's gap, then player 1 has incentive to to deviate such that the gaps coincide. So in equilibrium if one player has a gap than the other player has the same gap.

[4] For proof see Siegel (2014).

does not belong to k's support so supp $\left[G_i^j\right] \nleq_s$ supp $\left[G_i^k\right]$. Since $b^* < \sup \beta_\alpha$ the pairwise minimum of $\{b^*, \sup \beta_\alpha\}$ doesn't belong to k's support so supp $\left[G_i^k\right] \nleq_s$ supp $\left[G_i^j\right]$. $\square$

A.2 Differentiability and the First Order Conditions

Proposition A.2.1 follows from lemma A.2.2, A.2.3, A.2.4 and A.2.5. The following results are true for any finite type space $T = (T_1, T_2)$.

Proposition A.2.1. *There is a open set $\mathbb{A} \subseteq \left[0, x^G\right]$ s.t. $\mathbb{A}$ has the full Lebesgue measure in $\left[0, x^G\right]$ and for both i and all k, $G_i^k(\cdot)$ and $U_i(\cdot, k|G_{-i})$ are differentiable for all $b \in \mathbb{A}$.*

Lemma A.2.2. *G_i^k is differentiable almost everywhere on $\left[0, x^G\right]$.*

Proof. This follows directly from Lebesgue theorem, since G_i^k is a monotone increasing function on the closed interval $[0, x^G]$. $\square$

This means that there exists a $A_i(k)$ for every type k s.t for every $b \in A(k)$, b is differentiable and $\mu(\left[0, x^G\right]) = \mu(A_i(k))$ where μ is the Lebesgue measure.

$$\mathbb{A} := (\cap_{T_1} A_1(k)) \cap (\cap_{T_2} A_2(k)) \tag{A.2}$$

Lemma A.2.3. *$\mathbb{A}$ has full measure in $[0, x^G]$.*

Proof. First I'll show that $(\cap_{T_i} A_i(k))$ has full measure. This means that for any j and k there exists no subset $S = A(k) \backslash A(j)$ s.t. $\mu(S) > 0$. So the sets $A(k)$ and $A(j)$ can only differ in $[0, x^G]$ on set with measure zero.

Suppose not, then there exist a j and k s.t. $S = A(k) \backslash A(j)$ and $\mu(S) > 0$. Thus $\mu(S \cup A(j)) = \mu(A(k)) = \mu(\left[0, x^G\right])$ and since S and $A(j)$ are disjoint $\mu(S) + \mu(A(j)) = \mu(\left[0, x^G\right])$ which implies that $\mu\left(A(j)\right) < \mu(\left[0, x^G\right])$ which contradicts lemma 2. Now let $X = (\cap_{T_1} A_1(k))$ and $Y = (\cap_{T_2} A_2(k))$ which both have full measure in $[0, x^G]$,then again it suffices to show that there exists no $S = X \backslash Y$ or $S = Y \backslash X$ s.t. $\mu(S) > 0$. If the former is true then $\mu\left(Y\right) < [0, x^G]$, a contradiction if the latter is true then $\mu\left(X\right) < [0, x^G]$ a contradiction. $\square$

Lemma A.2.4. *For all $b \in \mathbb{A}$, $U_i(\cdot, k|G_{-i})$ is differentiable.*

Proof. For all $b \in \mathbb{A}$,

$$\frac{d\, u_i(b, k|G_{-i})}{d\, b} = \sum_j^N V^k(j) \frac{d\, G_{-i}^j(b)}{d\, b} - 1 \tag{A.3}$$

Hence, (A.3) is differentiable since $G_{-i}^j(b)$ is differentiable on $\mathbb{A}$. $\square$

Lemma A.2.5. *If* $\mathbb{A} \subset \left[0, x^G\right]$ *then* $\mathbb{A}$ *is open and* $\bar{\mathbb{A}} = \left[0, x^G\right]$.

Proof. Since $\mathbb{A}$ has full measure in $\left[0, x^G\right]$ it must mean that $\left[0, x^G\right] \setminus \mathbb{A}$ has zero measure. This implies that any $b \in \left[0, x^G\right] \setminus \mathbb{A}$ is not an interior point of $\mathbb{A}$. In fact, $\left[0, x^G\right] \setminus \mathbb{A}$ is the boundary of $\mathbb{A}$, take any $b \in \left[0, x^G\right] \setminus \mathbb{A}$ and any epsilon neighborhood around b, then there is a point in this neighborhood, that belongs to $\mathbb{A}$. This implies that the set $\left[0, x^G\right] \setminus \mathbb{A}$ is a closed set the complement of this set $\mathbb{A}$ is therefore open. $\square$

Appendix B

Proofs for Chapter 1

B.1 First Order Conditions of the Binary Type Space

Recall that $\mathbb{A}$ is the set of differenentiable points on the bid space and that it is dense in the bid space by Proposition A.2.1. Further, the intersection of each bid-support with $\mathbb{A}$ is also dense in each support.

$$\text{Let } B_i(k) := \mathbb{A} \cap \text{supp}\left[G_i^k\right]. \tag{B.1}$$

Clearly, $B_i(k)\backslash\mathbb{A}$ has zero measure in $[0, x^G]$. Let $\bar{B_i}(k)$ denote the closure of a set $B_i(k)$.

Lemma B.1.1. $\bar{B}_i(k) = \text{supp}\left[G_i^k\right]$.

Proof. Recall, $\text{supp}\left[G_i^k\right] \subseteq \bar{\mathbb{A}} = \left[0, x^G\right]$ then $\bar{B}_i(k) = cl\left[\mathbb{A} \cap \text{supp}\left[G_i^k\right]\right] \subseteq \bar{\mathbb{A}} \cap \text{supp}\left[G_i^k\right] = \text{supp}\left[G_i^k\right]$. (The closure of the intersection is contained in the intersection of the closures.)

For $\bar{B}_i(k) \supseteq \text{supp}\left[G_i^k\right]$, suppose not and $\text{supp}\left[G_i^k\right] \not\supseteq \bar{B}_i(k)$. For $b \in \bar{B}_i(k)\backslash\text{supp}\left[G_i^k\right]$, b must be a limit point of $B_i(k) = \mathbb{A} \cap \text{supp}\left[G_i^k\right]$, but this implies that b a limit point of $\text{supp}\left[G_i^k\right]$, since $\text{supp}\left[G_i^k\right]$ is closed it contains all its limit points and therefore $\bar{B}_i(k)\backslash\text{supp}\left[G_i^k\right]$ is empty, a contradiction. □

In equilibrium the FOCs imply that for any $b \in \mathbb{A}$ and for each i:

$$\frac{d\,U_i(b, H|G_{-i})}{d\,b} = V^H(L)\frac{dG_{-i}^L(b)}{d\,b} + V^H(H)\frac{dG_{-i}^H(b)}{d\,b} - 1 = 0 \tag{B.2}$$

$$\frac{d\,U_i(b, L|G_{-i})}{d\,b} = V^L(L)\frac{dG_{-i}^L(b)}{d\,b} + V^L(H)\frac{dG_{-i}^H(b)}{d\,b} - 1 = 0 \tag{B.3}$$

$$\text{Let } B^* := \cap_i \left[B_i(L) \cap B_i(H)\right] \tag{B.4}$$

The set B^* is called the mutual overlap and if $V^L(L)V^H(H) - V^L(H)V^H(L) \neq 0$ then for all $b \in B^*$, Cramer's rule solves gives the unique solution to the linear system (B.2) and (B.3). That is for each i:

$$\frac{d\,G_i^L(b)}{d\,b} = \frac{V^H(H) - V^L(H)}{V^L(L)V^H(H) - V^L(H)V^H(L)} \tag{B.5}$$

$$\frac{d\,G_i^H(b)}{d\,b} = \frac{V^L(L) - V^H(L)}{V^L(L)V^H(H) - V^L(H)V^H(L)} \tag{B.6}$$

Notice that generic condition guarantees that the bid-densities are uniquely determined by Cramer's rule. This holds true except when $V^H(H) = V^H(L) > V^L(H) = V^L(L)$. However, in this case the unique equilibrium is monotone by KMS and there is no overlap. Hence, this one exception is not an interesting case.

Lemma B.1.2. *If $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$ then either (B.5) and (B.6) are well defined or the equilibrium is monotone.*

Proof. Recall that by assumption $\mathbb{E}\left[v|\ H\right] \geq \mathbb{E}\left[v|\ L\right] \iff V^H(H) + V^H(L) \geq V^L(H) + V^L(L)$. Then notice that $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$ implies $V^L(L)V^H(H) - V^L(H)V^H(L) \neq 0$ unless there exist some $\theta > 1$ such that $V^H(j) = \theta V^L(j)$ for all $j \in \{L, H\}$. However, when $V^H(j) > V^L(j)$ for all $j \in \{L, H\}$, Siegel (2014) results apply, which means that there is a unique equilibrium which is monotone. □

The probability mass each player i and type k expends for G_i^k on any set $I \subseteq \mathbb{A}$ with positive Lebesgue measure is known. That is let μ be the Lebesgue measure then $\mu(I)$ is measure of I, then the probability mass expended on I for type k of player i is $\frac{d\,G_i^k(b)}{d\,b} * \mu(I)$.

Lemma B.1.3. *In any equilibrium G if*

1. *If $V^H(L) = V^L(L)$ or if $V^H(H) = V^L(H)$ but not both, then $B^* = \emptyset$.*
2. *If $V^H(H) \neq V^L(H)$ then $B_i(H) \cap [B_{-i}(L) \cap B_{-i}(H)] = \emptyset$.*
3. *If $V^H(L) \neq V^L(L)$ then $B_i(L) \cap [B_{-i}(L) \cap B_{-i}(H)] = \emptyset$.*

Proof. 1. Since $V^L(L)V^H(H) - V^L(H)V^H(L) \neq 0$, Cramer's rule yields the unique solution to this system which implies that either $\frac{d\,G_1^H(b)}{d\,b} = 0$ or $\frac{d\,G_1^L(b)}{d\,b} = 0$, when the slope is zero at b it does not belong to the support thus $B^* = \emptyset$.

2. Suppose not and $B_1(H) \cap [B_2(L) \cap B_2(H)] \neq \emptyset$ then in order for this to be an equilibrium, the following has to be satisfied for all $b \in B_1(H) \cap [B_2(L) \cap B_2(H)]$: $\frac{du_2(b,L|G_1)}{db} = V^L(H)\frac{dG_1^H(b)}{db} - 1$ and $\frac{du_2(b,H|G_1)}{db} = V^H(H)\frac{dG_1^H(b)}{db} - 1$. But this impossible since $V^H(H) \neq V^L(H)$.

3. Suppose not and $B_1(L) \cap [B_2(L) \cap B_2(H)] \neq \emptyset$. Then for all $b \in B_1(L) \cap [B_2(L) \cap B_2(H)]$: $\frac{du_2(b,L|G_1)}{db} = V^L(L)\frac{dG_1^L(b)}{db} - 1$ and $\frac{du_2(b,L|G_1)}{db} = V^H(L)\frac{dG_1^L(b)}{db} - 1$. But this impossible since $V^H(L) \neq V^L(L)$.

□

Lemma B.1.4. *If $V^H(H) > (=)V^L(H)$ and $V^H(L) = (>)V^L(L)$, in equilibrium if for $x^G \geq b'' > b'$, $U_i(b'', H|G_{-i}) > U_i(b', H|G_{-i})$ then for all $0 \leq b < b'$, $U_i(b'', H|G_{-i}) > U_i(b, H|G_{-i})$.*

Proof. By (B.2) and (B.3), $\frac{du_i(b,H|G_{-i})}{db} \geq \frac{du_i(b,L|G_{-i})}{db}$, and in equilibrium either $\frac{dU_i(b,H|G_{-i})}{db} = 0$ or $\frac{dU_i(b,L|G_{-i})}{db} = 0$ (or both) for all $b \in \mathbb{A}$. Thus, in equilibrium for any $b < b'$, $U_i(b', H|G_{-i}) \geq U_i(b, H|G_{-i}) \geq 0$. □

Lemma B.1.5. *Let G be an equilibrium and $V^H(H) > V^L(H)$ and $V^H(L) = V^L(L)$ then $B_i(L) \cap [B_{-i}(L) \cap B_{-i}(H)] = \emptyset$.*

Proof. First note that if $V^L(L) = 0$ then $f(L, L) = 0$ and since $V^H(L) = V^L(L) \implies f(H, L) = f(L, H) = 0$, this would contradict $f(L) > 0$. Likewise, if $V^L(H) = 0 \implies f(L) = 0$. It is also impossible that $V^H(H) = 0$ since $V^H(H) > V^L(H)$. Thus, $V^k(j) > 0$ for all k and j and by lemma 1.3.1 at most one player has an atom at zero.

Let $I = B_1(L) \cap [B_2(L) \cap B_2(H)]$ and suppose the contrary and $I \neq \emptyset$. The proof has 5 steps.

1. By the first order conditions (B.2) and (B.3) for any $b \in I$,

$$\begin{aligned} \frac{dG_1^L(b)}{db} &= \frac{1}{V^L(L)} = \frac{1}{V^H(L)} \\ V^L(L)\frac{dG_2^L(b)}{db} + V^L(H)\frac{dG_2^H(b)}{db} &= 1. \end{aligned} \tag{B.7}$$

It must be that on I, $\frac{dG_2^L(b)}{db} > 0$ and $\frac{dG_2^H(b)}{db} > 0$, otherwise $I = \emptyset$. This implies that $Pr\,(X \in I) > 0$ with $X \sim G_2^H$, in other words player 2's high type must expend some bidding probability on I.

2. Notice that by equation (B.7) and because $V^H(H) > V^L(H)$ and $V^H(L) = V^L(L)$ that:

$$V^L(L)\frac{dG_2^L(b)}{d\,b} + V^L(H)\frac{dG_2^H(b)}{d\,b} < V^H(L)\frac{dG_2^L(b)}{d\,b} + V^H(H)\frac{dG_2^H(b)}{d\,b}. \tag{B.8}$$

   Therefore, $t_1 = H$ strictly prefers the supremum of I over any other $b \in I$. By lemma B.1.4 this means that any $b <$ sup I is not a best response and $t_1 = H$ can't have an atom at zero.

3. I establish there can be no other overlap besides I. Consider the set $I' = B_2(L) \cap [B_1(L) \cap B_1(H)] \neq \emptyset$, This is the same set as I, but the players are flipped. Any bid $b <$ sup I' is not a best-response for $t_2 = H$ by the same argument as in 2. above. This means that the bids in I can't be below I', otherwise $t_2 = H$ is not best responding. Likewise, it can't be that $b \in I'$ is below $\sup I$ otherwise any bid in I' is not a best response for $t_1 = H$. This combined with lemma B.1.3 means there can be no overlap besides I.

4. Since $t_1 = H$ must assign full measure to his best-response set, it must be that for some $b \geq \sup I$, $b \in (B_1(H) \cap B_2(L)) \setminus B^*$ or $b \in (B_1(H) \cap B_2(H)) \setminus B^*$. First, I show that $(B_1(H) \cap B_2(L)) \setminus B^* \cap \left(\sup I, x^G\right) \neq \emptyset$ is impossible. Since if it was possible then by the FOCs on $(B_1(H) \cap B_2(L)) \setminus B^* : \frac{dG_1^H(b)}{d\,b} = \frac{1}{V^L(H)}$ and $\frac{dG_2^L(b)}{d\,b} = \frac{1}{V^H(L)}$. This implies that for all $b \in (B_1(H) \cap B_2(L)) \setminus B^* \cap \left(\sup I, x^G\right)$, $t_2 = H$ makes an incremental gain on this set, then lemma B.1.4 implies that I can't be part of the equilibrium support for $t_2 = H$.

5. Thus, it must be that $(B_1(H) \cap B_2(H)) \setminus B^* \cap \left(\sup I, x^G\right) = \operatorname{int}\operatorname{supp}\left[\mathrm{G}_1^{\mathrm{H}}\right]$, on this set the slopes are $\frac{dG_1^H(b)}{d\,b} = \frac{1}{V^H(H)}$ and $\frac{dG_2^H(b)}{d\,b} = \frac{1}{V^H(H)}$, which implies that $t_i = L$ is making an incremental loss on this set, therefore $\sup B_i(L) = \sup I$. By symmetry both player's high type expend the same amount of probability on $(B_1(H) \cap B_2(H)) \setminus B^* \cap \left(\sup I, x^G\right)$. In fact they spend $\frac{\mu\left[(B_1(H) \cap B_2(H)) \setminus B^* \cap \left(\sup I, x^G\right)\right]}{V^H(H)}$ probability mass each, but recall that player 2's high type expended some bidding probability on I. Thus, if $Pr\left(X \in \operatorname{supp}\left[G_2^H\right]\right) = 1 \implies Pr\left(X \in \operatorname{supp}\left[G_1^H\right]\right) < 1$. Since G_1^H can't have an atom anywhere, this can't be an equilibrium, a contradiction.

□

Lemma B.1.6. *Let G be an equilibrium and $V^H(H) = V^L(H)$ and $V^H(L) > V^L(L)$ then $B_i(H) \cap [B_{-i}(L) \cap B_{-i}(H)] = \emptyset$.*

Proof. The proof follows a similar structure as the previous lemma, the only difference is step 4. First, it can't be that $(B_i(L) \cap B_{-i}(H)) \setminus B^*$ precedes I, since $t_{-i} = L$ makes an incremental loss on that set. Thus, only $(B_1(L) \cap B_2(L)) \setminus B^*$ could possibly precede I, but this means that both H's make an incremental gain on that set and won't bid below $\inf I$ in equilibrium. Thus, this leads to the same contradiction as in 5. □

Lemma B.1.3,B.1.5 and B.1.6 imply the following lemma B.1.7.

Lemma B.1.7. *$V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$, then in any equilibrium G, there exists a mutual overlap (B^*) or no overlap at all. Furthermore, if there is one $j \in \{L, H\}$ s.t. $V^H(j) = V^L(j)$, there is no overlap.*

By lemma B.1.7, only the following subsets of $\mathbb{A}$ are consequential in equilibrium.

$$\begin{aligned} S_1 &:= (B_1(H) \cap B_2(H)) \setminus B^* \\ S_2 &:= (B_1(H) \cap B_2(L)) \setminus B^* \\ S_3 &:= (B_1(L) \cap B_2(H)) \setminus B^* \\ S_4 &:= (B_1(L) \cap B_2(L)) \setminus B^* \\ S_5 &:= B^* \end{aligned} \tag{B.9}$$

Clearly these sets are disjoint, I'll demonstrate that the union of the S's is $\mathbb{A}$.

Lemma B.1.8. *If $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$ then $\cup_{n=1}^{5} S_n = \mathbb{A}$.*

Proof. By lemma B.1.7 all other possible subsets of $\mathbb{A}$ are empty. It's is easy to show that $\mathbb{A} = B_i(L) \cup B_i(H)$. In equilibrium $x^G > 0$ and $\mathbb{A}$ is dense in $\left[0, x^G\right]$ which implies that $B_i(k)$ is nonempty for at least one k, thus by the definition of $B_i(k)$ if $b \in B_i(L) \cup B_i(H) \implies b \in \mathbb{A}$. If $b \in \mathbb{A}$ then $b \in \left[0, x^G\right]$ and by lemma 1.3.1 this means that $b \in \operatorname{supp}\left[G_i^k\right]$ for at least k thus $b \in B_i(L) \cup B_i(H)$.

$$\begin{aligned} \cup_{n=1}^{5} S_n &= (B_1(H) \cap B_2(H)) \cup (B_1(H) \cap B_2(L)) \cup (B_1(L) \cap B_2(H)) \cup (B_1(L) \cap B_2(L)) \\ &= (B_1(H) \cap B_2(H)) \cup (B_1(H) \cap (A \setminus B_2(H))) \cup (B_1(L) \cap B_2(H)) \cup (B_1(L) \cap (A \setminus B_2(H))) \\ &= B_1(H) \cup B_1(L) = \mathbb{A} \end{aligned} \tag{B.10}$$

□

B.2 Proof of Proposition 1.3.2

Recall that $\mathbb{A}$ is the set of differentiable points in $\left[0, x^G\right]$ and that $B_i(k) := \mathbb{A} \cap \text{supp}\left[G_i^k\right]$ are all the differentiable points in the support of type k of player i. Further, recall that $B^* := \cap_i \left[B_i(L) \cap B_i(H)\right]$ is the set of differentiable points on the mutual overlap.

Definition $I \subset \mathbb{A}$ is a non-overlapping set if, 1) if for both i, $I \cap (B_i(L) \cap B_i(H)) = \emptyset$ and 2) for both i, either $I \subset \text{supp}\left[G_i^L\right]$ or $I \subset \text{supp}\left[G_i^H\right]$, and there exist no $I^* \supset I$ s.t. I^* satisfied 1 and 2.

By definition, I is the union of all sets s.t. 1 and 2 is true, since it's the largest possible non-overlapping set. Then from (B.9), S_1, S_2, S_3 and S_4 are the non-overlapping sets of this game in equilibrium.

Definition $I \subset \mathbb{A}$ is non-overlapping asymmetric set if I is non-overlapping and if $I \subset \text{supp}\left[G_i^k\right] \implies I \subset \text{supp}\left[G_{-i}^j\right]$ for $j \neq k$. $I \subset \mathbb{A}$ is non-overlapping symmetric set if I is non-overlapping and if for $k \in T$, $I \subset \text{supp}\left[G_i^k\right] \implies I \subset \text{supp}\left[G_{-i}^k\right]$.

Thus, S_2 and S_3 are non-overlapping asymmetric sets, and S_1 and S_4 are non-overlapping symmetric sets. The players spend the same amount of bidding probability on B^* and on all symmetric non-overlapping sets since the slope is uniquely determined by Cramer's rule. Since both players spend the same bid probability on all "symmetric sets" it can't be that there are odd amounts of non-overlapping asymmetric sets in equilibrium. Since both players would have a type left out on a non-overlapping asymmetric set, both types would need an atom at zero, which can't happen in equilibrium, this is what lemma B.2.1 shows.

Definition Define I^{-1} as the mirror of I. If $I \subset \mathbb{A}$ is a non-overlapping set s.t., $I \subset \text{supp}\left[G_1^k\right]$ and I $\subset \text{supp}\left[G_2^j\right]$ then I^{-1} is a non-overlapping set s.t. $I^{-1} \subset \text{supp}\left[G_1^j\right]$ and $\mathrm{I}^{-1} \subset \text{supp}\left[G_2^k\right]$ for $k \neq j$.

Therefore, the mirror of S_2 is S_3 and vice versa.

Lemma B.2.1. *Let G be an equilibrium and $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$, If I is a non-overlapping asymmetric set with length $\mu(I) > 0$, then the mirror has equal length, $\mu(I^{-1}) = \mu(I)$.*

Proof. Note that on all symmetric sets, the high types from each players spend the same amount of probability mass for G_i^H, likewise on all symmetric sets both low types spend the same amount of mass for G_i^L. Second, for an asymetric set I if $\mu(I) > 0$ then $V^H(L) > 0$ and $V^L(H) > 0$.[1]

Let $I = (B_1(H) \cap B_2(L)) \setminus B^*$. Suppose not, and $\mu(I^{-1}) < \mu(I)$. Since $\mu(I) > 0$ it must that $t_1 = H$ spends more probability mass than $t_2 = H$ because $\frac{\mu(I)}{V^L(H)} > \frac{\mu(I^{-1})}{V^L(H)} \geq 0$. Likewise, $t_2 = L$ spends more probability mass than $t_1 = L$ since $\frac{\mu(I)}{V^H(L)} > \frac{\mu(I^{-1})}{V^H(L)} \geq 0$. Since on all symmetric sets they spend the same amount of probability, it must be that $G_1^L(0) > 0$ and $G_2^H(0) > 0$, which contradicts equilibrium. By lemma 1.3.1, if $V^L(H) > 0$ and $G_1^L(0) > 0 \implies G_2^H(0) = 0$. Similar logic applies when $\mu(I^{-1}) > \mu(I)$, then $G_2^L(0) > 0$ and $G_1^H(0) > 0$, a contradiction. □

Lemma B.2.2. *If G is an equilibrium and $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$, then G is symmetric.*

Proof. Suppose not, and G is an asymmetric equilibrium, which means there is at least one non-empty non-overlapping asymmetric set I, and that $\mu(I) > 0$. Asymmetric sets come in pairs by lemma B.2.1 and $\mu(I^{-1}) = \mu(I)$.

Since $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$, let $V^H(H) > V^L(H)$. Let $I = (B_1(H) \cap B_2(L)) \setminus B^*$, then on this set $\frac{dG_1^H(b)}{db} = \frac{1}{V^L(H)}$ and $\frac{dG_2^L(b)}{db} = \frac{1}{V^H(L)}$. This means that $t_2 = H$ is making an incremental gain on I since $\frac{dU_2(b,H|G_1)}{db} = \frac{V^H(H)}{V^L(H)} - 1 > 0$. This means $t_2 = H$ strictly prefers the supremum of I over any other $b \in I$ and so bids in I are not best responses for this type. Since $\inf I$ is not a best response for $t_2 = H$ any bid arbitrary close to $\inf I$ is not a best response either, so if $\inf I > 0$, $t_2 = L$ must bid just below $\inf I$. Further, by lemma B.1.7 there is no overlap without $t_2 = H$, therefore, these bids just below $\inf I$ must belong to $(B_1(L) \cap B_2(L)) \setminus B^*$ otherwise it would contradict $\inf I$.[2] If $V^H(L) < V^L(L)$, then for all bids $b \in (B_1(L) \cap B_2(L)) \setminus B^*$, $t_2 = H$ and $t_1 = H$ are making an incremental loss since $\frac{dU_i(b,H|G_{-i})}{db} = \frac{V^H(L)}{V^L(L)} - 1 < 0$, but implies that any $b \in I$ is not a best response for $t_1 = H$, since he is indifferent on I and makes an incremental loss leading up to it. So it must be that $V^H(L) \geq V^L(L)$ or $\inf I = 0$. If $V^H(L) \geq V^L(L)$ then for $b \in (B_1(L) \cap B_2(L)) \setminus B^*$, $t_2 = H$ is making a incremental gain or is indifferent. Thus, $[0, \sup I]$ are not best responses for $t_2 = H$, then $[0, \sup I] <_s \operatorname{supp}\left[G_2^H\right]$.

On I^{-1} the same logic applies as on I, and therefore, $[0, \sup I^{-1}] <_s \operatorname{supp}\left[G_1^H\right]$. So $[0, \sup I] <_s \operatorname{supp}\left[G_2^H\right]$ and $[0, \sup I^{-1}] <_s \operatorname{supp}\left[G_1^H\right]$, but this impossible since $I \subset$

[1] First both have to be positive by the symmetry of f, that is $V^H(L) = 0 \iff V^L(H) = 0$. So if they are zero, by (B.2) and (B.3) then I is empty. Thus, if in any equilibrium $\mu(I) > 0$, then I is non empty hence both primal objects must be strictly positive.

[2] Since there is no overlap for just one player by lemma B.1.7, it must be that $b \in (B_1(L) \cap B_2(L)) \setminus B^*$, otherwise $b \in I$.

$\text{supp}\left[G_1^H\right]$ and $I^{-1} \subset \text{supp}\left[G_2^H\right]$. Similar proof when $V^H(H) < V^L(H)$ or $V^H(L) \neq V^L(L)$.

□

B.3 Proof of Proposition 2.4.17

Since any equilibrium G is symmetric by lemma B.2.2, I confine attention to symmetric strategies and drop the subscript i on the strategies.

Lemma B.3.1. *If $V^H(j) \geq V^L(j)$ for all $j \in \{L, H\}$ where at least one $\geq$ is strictly greater, then any equilibrium G is monotone.*

Proof. By lemma B.1.7, there either is a mutual overlap or no overlap, by Cramer's rule on (B.2) and (B.3) there can't be a mutual overlap so $B^* = \emptyset$. Thus, for all $b \in \mathbb{A}$, either $b \in B_1(H) \cap B_2(H)$ or $b \in B_1(L) \cap B_2(L)$. This means the interior of the supports are disjoint. Now what remains to be shown is that $\text{supp}\left[G^L\right] \leq_s \text{supp}\left[G^H\right]$. Since at least one inequality is strict lets say $V^H(H) > V^L(H)$ then this implies that if $b \in B_1(H) \cap B_2(H)$ then for all $b' > b$, $b' \notin \text{supp}\left[G^L\right]$.[3] Thus, the only strategy consistent with equilibrium is that both $B_i(L)$ and $B_i(H)$ are intervals s.t. for both i, $\text{supp}\left[G^L\right] = [0, \sup B_i(L)]$ and $\text{supp}\left[G^H\right] = [\sup B_i(L), \sup B_i(H)]$, thus $\text{supp}\left[G^L\right] \leq_s \text{supp}\left[G^H\right]$ as required. Likewise when $V^H(L) > V^L(L)$, if $b \in B_1(L) \cap B_2(L) \backslash B^*$ then for any $b' < b$, $b' \notin \text{supp}\left[G^H\right]$, and $\text{supp}\left[G^L\right] \leq_s \text{supp}\left[G^H\right]$. □

Since the equilibrium is symmetric the first order conditions (B.2) and (B.3) imply lemma B.3.2.

Lemma B.3.2. *If G is a symmetric equilibrium and $b \in \text{int}\left(\text{supp}\left[G^k\right]\right)$ and $b \notin \text{supp}\left[G^j\right]$ then $\frac{\partial G^k(b^*)}{\partial b} = \frac{1}{V^k(k)}$.*

Lemma B.3.3. *If $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$ then in any symmetric equilibrium G, $\text{supp}\left[G^k\right]$ is in an interval for all $k \in \{L, H\}$.*

Proof. Suppose not and $\text{supp}\left[G^k\right]$ has a gap β_α. Lemma 1.3.1 implies that a gap of k belongs to the support of type j, i.e. $\beta_\alpha \subseteq \text{supp}\left[G^j\right]$. By lemma B.3.2 for all $b \in \beta_\alpha$, $\frac{\partial G^j(b)}{\partial b} = \frac{1}{V^j(j)}$. Since $\inf \beta_\alpha$ and $\sup \beta_\alpha \in \text{supp}\left[G^k\right]$, it must be that $V^j(j) = V^k(j)$,

[3] Since type L is indifferent between all bids $B_1(L) \cap B_2(L) \backslash B^*$ and makes an incremental loss on all $b \in B_1(H) \cap B_2(H) \backslash B^*$. The incremental loss is because on $B_1(H) \cap B_2(H) \backslash B^*$ the slope of G^H uniquely determined by the first order conditions and is $\frac{1}{V^H(H)}$, since type L makes $V^L(H)G(b)$ here and pays b, he makes a net loss since $\frac{V^L(H)}{V^H(H)} - 1 < 0$.

if not and $V^j(j) \neq V^k(j)$ then $U(\inf \beta_\alpha, k \,|\, G) \neq U(\sup \beta_\alpha, k \,|\, G)$, which contradicts equilibrium. By assumption, either $V^j(k) > V^k(k)$ or $V^j(k) < V^k(k)$, regardless of the sign, lemma B.3.1 implies that any equilibrium that exists is monotone. By lemma A.1.8, if G is monotone then supp $\left[G^k\right]$ is connected, a contradiction as required. □

Lemma B.3.4. *If $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$, then any symmetric equilibrium G is weakly monotone.*

Proof. Since in equilibrium supp $\left[G^k\right]$ is an interval for all $k \in \{L, H\}$, the only candidate consistent with a non-monotone equilibrium is characterized by the following construction: supp $\left[G^k\right] \subset$ supp $[G^j] = \left[0, x^G\right]$, s.t. that $\underline{B}^k > 0$ and $\bar{B}^k < x^G$. Now suppose such a non-monotone equilibrium exists, let $k = H$ and $j = L$. In equilibrium for all $b \in \left[0, \underline{B}^H\right]$ and $b \in \left[\bar{B}^H, x^G\right]$, $\frac{\partial G^L(b)}{\partial b} = \frac{1}{V^L(L)}$ by lemma B.3.2. Thus, for this to be an equilibrium, $V^H(L) = V^L(L)$. Since if $V^H(L) < V^L(L)$, then H has a unique best response: zero, a contradiction. If $V^H(L) > V^L(L)$ then x^G is the unique best response, a contradiction. But if $V^H(L) = V^L(L)$ then $V^L(H) < V^H(H)$ by the generic condition and since $\mathbb{E}[v|\ H] \geq \mathbb{E}[v|\ L]$, which means that any equilibrium is monotone by lemma B.3.1, a contradiction. Likewise, if $k = L$ and $j = H$ then it must be that $V^L(H) = V^H(H)$, which means that $V^H(L) > V^L(L)$, which again implies that any equilibrium is monotone, a contradiction. □

B.4 Proof of Theorem 1.3.8

The following collection of lemmas proof Theorem 1.3.8. Consider the following 4 forms:

- **a)** $\underline{B}^L = \underline{B}^H = 0$ & $\bar{B}^L = \bar{B}^H$
- **b)** $\underline{B}^L = 0$ and $\bar{B}^L = \underline{B}^H < \bar{B}^H = x^G$.
- **c)** $\underline{B}^L = \underline{B}^H = 0$ and $\bar{B}^L < \bar{B}^H = x^G$.
- **d)** $\underline{B}^H > \underline{B}^L = 0$ and $\bar{B}^L = \bar{B}^H = x^G$.

Lemma B.4.1. *If if $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$ then the 4 forms above are the only equilibrium bid-supports that can exist.*

Proof. Symmetry implies we only have to look at on player. Weak monotonicity and lemma 1.3.1 implies that $\underline{B}^L \leq \underline{B}^H \leq \bar{B}^L \leq \bar{B}^H$. Observe every form is characterized by different inequalities being strict. Then the only combination of strict inequalities not present in the 4 forms is a form such that:

$$\underline{B}^L < \underline{B}^H < \bar{B}^L < \bar{B}^H. \tag{B.11}$$

In equilibrium by lemma 1.3.1, notice that $\underline{B}^L = 0$. Further, by lemma B.3.2 on the interval $\left(0, \underline{B}^H\right)$ that $\frac{G^L(\cdot)}{\partial b} = \frac{1}{V^L(L)}$ and that on $\left(\bar{B}^L, \bar{B}^H\right)$, $\frac{G^H(\cdot)}{\partial b} = \frac{1}{V^H(H)}$. Then, notice that in any equilibrium where (B.11) is holds, that $V^H(L) \geq V^L(L)$ has to be true otherwise $\underline{B}^H$ is not a best response for H. Likewise, it implies that $V^L(H) \leq V^H(H)$, otherwise L deviates to $\bar{B}^H$. Further, $\underline{B}^H < \bar{B}^L$ implies there is an overlap. For an overlap to exists, Cramer's rule requires that either one of the following two equations are satisfied.

$$V^H(H) > V^L(H) \text{ and } V^L(L) > V^H(L), \tag{B.12}$$

$$V^H(H) < V^L(H) \text{ and } V^L(L) < V^H(L). \tag{B.13}$$

It is impossible that $V^H(L) \geq V^L(L)$ and $V^L(H) \leq V^H(H)$ and (B.12) or (B.13) are simultaneously satisfied.

□

Lemma B.4.2. $\mathbb{E}[v|\ H] = \mathbb{E}[v|\ L] \iff$ *existence of an equilibrium of form (a).*

Proof. ($\implies$)Let $x = \frac{\partial G^H(b)}{\partial b} = \frac{\partial G^L(b)}{\partial b} = \frac{1}{\mathbb{E}[v|\ H]}$, $x^G = \mathbb{E}[v|\ H]$ then for both $k \in \{H, L\}$,

$$G^k(b) = \begin{cases} 0 & b < 0 \\ bx & b \in [0, x^G] \\ 1 & b > x^G \end{cases},$$

This is an equilibrium. Since, both types are indifferent on the entire bid space $\left[0, x^G\right]$ and it is never a best response bid above x^G.

($\impliedby$) Let strategy G below be an equilibrium. For some $x > 0$ and $x^G = \frac{1}{x}$ and every $k \in \{K, L\}$.

$$G^k(b) = \begin{cases} 0 & b < 0 \\ bx & b \in [0, \frac{1}{x}] \\ 1 & b > \frac{1}{x} \end{cases}$$

Suppose that $V^H(H) - V^L(H) > V^L(L) - V^H(L)$. Since zero belongs to both types support it must be that $U(x^G, H|\ G) = 0$, but this implies that $x^G = \mathbb{E}[v|\ H]$ and $U(x^G, L|\ G) = \mathbb{E}[v|\ L] - \mathbb{E}[v|\ H] < 0$, which contradicts equilibrium. Likewise if $U(x^G, L|\ G) = 0, U(x^G, H|\ G) = \mathbb{E}[v|\ H] - \mathbb{E}[v|\ L] > 0$, a contradiction.

□

Lemma B.4.3. *If $V^H(j) \neq V^L(j)$ for all $j \in \{L, H\}$ then construction (c) produces the unique candidate for an equilibrium of form (c).*

Proof. Suppose not, and there exists an other construction s.t. $\underline{B}^L = \underline{B}^H = 0$ and $\bar{B}^L < \bar{B}^H = x^G$. Since $= \underline{B}^H = 0$, and $\bar{B}^H = x^G$, in equilibrium it must be that $x^G = \mathbb{E}[v|\ H] = V^H(L) + V^H(H)$, otherwise H is not indifferent between zero and x^G. By lemma B.3.2, $\frac{\partial G^H(b)}{\partial b} = \frac{1}{V^H(H)}$ for all $b \in [\bar{B}^L, x^G]$. Therefore, the only thing that could possibly be different from construction (c) is $\bar{B}^L$. Since L has zero in his support, it must be that $\frac{\partial G^L(b)}{\partial b} = \frac{1}{\bar{B}^L}$ for all $b \in \left[0, \bar{B}^L\right]$. But since $\left[0, \bar{B}^L\right]$ is the overlap between H and L, $\bar{B}^L = \frac{1}{x_L}$, a contradiction, the construction is the same. □

Lemma B.4.4. *If $V^H(j) \neq V^L(j)$ for all $j \in \{L, H\}$ then construction (d) produces the unique candidate for an equilibrium of form (d).*

Proof. Suppose not, and there exists an other construction s.t. $\underline{B}^H > \underline{B}^L = 0$ and $\bar{B}^L = \bar{B}^H = x^G$. Since $\underline{B}^L = 0$, and $\bar{B}^L = x^G$, in equilibrium it must be that $x^G = \mathbb{E}[v|\ L]$, otherwise L is not indifferent between zero and x^G. For $G^H(b)$ on the interval $[\underline{B}^H, x^G]$, in equilibrium $\underline{B}^H = x^G - \frac{1}{\frac{\partial G^H(b)}{\partial b}}$, if not then $G^k(x^G) \neq 1$. By FOC, $\frac{\partial G^k(b)}{\partial b} = x_H$, which means that $\underline{B}^H = x^G - \frac{1}{x_H}$. Thus the only possible difference has to be that for $b \in [0, \underline{B}^H]$, $\frac{\partial G^L(b)}{\partial b} \neq \frac{1}{V^L(L)}$. But this contradicts lemma B.3.2. Thus, there exists no other construction. □

Lemma B.4.5. *If $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$ and $\mathbb{E}[v|\ H] = \mathbb{E}[v|\ L]$ neither form (b), (c) and (d) is an equilibrium.*

Proof. Form (b) follows from the necessity claim of lemma 1.3.4. The non existence of (c) and (d) follow from the fact on the overlap the bid densities for both types are the same when $\mathbb{E}[v|\ H] = \mathbb{E}[v|\ L]$.[4] □

The non existence of (a) when $\mathbb{E}[v|\ H] > \mathbb{E}[v|\ L]$ follows from lemma 2.4.2.

Lemma B.4.6. *If (1.9) or (1.10) is true the there exists no equilibrium of form (b).*

Proof. Suppose (1.9) is true and supp $\left[G^L\right] <_s$ supp $\left[G^H\right]$. Type $t_i = H$ is making an incremental loss on supp $\left[G^L\right]$, contradicting that $b \in$ supp $\left[G^H\right]$ are best responses. Now suppose (1.10) is true and supp $\left[G^L\right] <_s$ supp $\left[G^H\right]$. Then $t_i = L$ is making an incremental gain on supp $\left[G^H\right]$, contradicting that supp $\left[G^L\right]$ are best responses. □

The proof of lemma B.4.7 is verbatim to the first part of the proof of lemma B.4.1.

[4]For this binary signal structure observe that (B.5) and (B.6) coincide. For a general proof of this claim for N types see Schouten (2021) in the open question appendix.

Lemma B.4.7. *In any symmetric equilibrium G, for $k \neq j$*

(a) $V^j(j) > V^k(j) \implies 0 \in \text{supp}\left[G^k\right]$

(b) $V^j(j) < V^k(j) \implies x^G \in \text{supp}\left[G^k\right]$

Lemma B.4.8. *If (1.9) is true then form(d) is not an equilibrium. If (1.10) is true then form (c) is not an equilibrium.*

Proof. If (1.9) is true, by lemma B.4.7, $0 \in \text{supp}\left[G^H\right]$. Hence (d) can't be an equilibrium. If (1.10) is true, by lemma B.4.7, $x^G \in \text{supp}\left[G^L\right]$. Hence (c) can't be an equilibrium. □

Uniqueness of the algorithms has been established for (c) and (d), see lemma 1.3.6 and lemma 1.3.7 respectively. Uniqueness of construction of (a) is obvious, and Siegel (2014) claims uniqueness of construction of form (b). The what remains to be shown for theorem one is sufficiency.

Lemma B.4.9. *If $V^H(j) \neq V^L(j)$ for at least one $j \in \{L, H\}$.Then,*

a) *if $\mathbb{E}[v|\ H] = \mathbb{E}[v|\ L]$ an equilibrium of form (a) exists.*

b) *if $V^H(j) \geq V^L(j)$ for all $j \in \{L, H\}$ an equilibrium of form (b) exists.*

c) *If $V^H(H) > V^L(H)$ and $V^L(L) > V^H(L)$ an equilibrium of form (c) exists.*

d) *If $V^H(H) < V^L(H)$ and $V^L(L) < V^H(L)$ an equilibrium of form (d) exists.*

Proof. (a) and (b) follow from lemma 2.4.2 and lemma 1.3.4 respectively. In the construction of forms (c) and (d), each type is indifferent on its support, which follows from the FOCs. What remains to be shown is that there exists no profitable deviation outside the prescribed support.

For form (c), observe that the low type's bid support is an interval starting from zero and that $\bar{B}^L < \bar{B}^H = x^G$. Since $V^H(H) > V^L(H)$, the low type makes an incremental loss on the support of the high type above her own support. That is, for any bid $b \in \left(\bar{B}^L, x^G\right]$,

$$
,\quad \begin{aligned}
U(b, L\,|\,G_{-i}) - U(\bar{B}^L, L\,|\,G_{-i}) =& V^L(H)\left[G^H(b) - G^H(\bar{B}^L)\right] - \left(b - \bar{B}^L\right) \\
\iff& V^L(H)\left[\frac{b - V^H(L)}{V^H(H)} - \frac{\bar{B}^L - V^H(L)}{V^H(H)}\right] - \left(b - \bar{B}^L\right) \\
\iff& \frac{V^L(H)}{V^H(H)}\left[b - \bar{B}^L\right] - \left(b - \bar{B}^L\right) < 0.
\end{aligned} \tag{B.14}
$$

Thus, the low type has no incentive to deviate. On the other hand, the high type has no such deviation since her bid support coincides with the entire bid space, $\left[0, x^G\right]$. This implies that she can only deviate above the bid-space, which is never a best response. Thus, the construction produces an equilibrium.

Similar logic applies to form (d). For this form, H's bid support is an interval including x^G but now $\underline{B}^L < \underline{B}^H$. $V^L(L) < V^H(L)$ implies that the high type makes an incremental gain on the low type support below her own, hence she has no incentive to deviate below her support. The low type has no such deviation since her bid support coincides with the entire bid space. Thus, the construction produces an equilibrium.

□

B.5 Proofs of Propositions 1.4.1 and 1.4.2

Let $g^k(b) := \frac{\partial G^k(b)}{\partial b}$. Notice that for $b \in \left[\underline{B}^H, x^G\right]$,

$$\begin{aligned} g^L(b) &= \frac{V^L(H) - V^H(H)}{V^L(H)V^H(L) - V^L(L)V^H(H)} = x_L \\ g^H(b) &= \frac{V^H(L) - V^L(L)}{V^L(H)V^H(L) - V^L(L)V^H(H)} = x_{H.} \end{aligned} \tag{B.15}$$

Further for $b \in \left[0, \underline{B}^H\right]$, $g^H(b) = 0$. Lastly, recall that $\frac{1}{x_H} = x^G - \underline{B}^H$.

Lemma B.5.1. *If* $V^H(H) < V^L(H)$ and $V^L(L) < V^H(L)$, *then in equilibrium* $U(b, H|\ G) = \mathbb{E}\left[v|\ H\right] - \mathbb{E}\left[v|\ L\right]$.

Proof. Expected payoff is given by:

$$U(b, H|\ G) = \left(\frac{V^H(L)}{V^L(L)} - 1\right) \underline{B}^H \tag{B.16}$$

$$\begin{aligned} \underline{B}^H &= x^G - \frac{1}{x_H} \\ = V^L(L) + V^L(H) - &\left(\frac{V^L(H)V^H(L) - V^L(L)V^H(H)}{V^H(L) - V^L(L)}\right) \\ &= V^L(L)\left[\frac{\mathbb{E}\left[v|\ H\right] - \mathbb{E}\left[v|\ L\right]}{V^H(L) - V^L(L)}\right] \end{aligned} \tag{B.17}$$

$$U(b, H|\ G) = \left(V^H(L) - V^L(L)\right)\left[\frac{\mathbb{E}\left[v|\ H\right] - \mathbb{E}\left[v|\ L\right]}{V^H(L) - V^L(L)}\right] = \mathbb{E}\left[v|\ H\right] - \mathbb{E}\left[v|\ L\right] \tag{B.18}$$

□

Lemma B.5.2. *If* $V^H(H) < V^L(H)$ and $V^L(L) < V^H(L)$, *then in equilibrium* $Prob\{win = L|lose = H\} = \frac{1}{2}\left[\frac{V^L(H)-V^H(H)}{V^H(L)-V^L(L)}\right]$.

Proof.

$$
\begin{aligned}
Prob\{win = L|lose = H\} =& \int_{\underline{B}^H}^{x^G} g^L(y)G^H(y)dy \\
=& x_L \int_{\underline{B}^H}^{x^G} \int_0^y g^H(z)dzdy \\
=& x_L \int_{\underline{B}^H}^{G} \left[\int_0^{\underline{B}^H} 0dz + \int_{\underline{B}^H}^{y} x_H dz\right] dy \\
=& x_L x_H \int_{\underline{B}^H}^{x^G} \int_{\underline{B}^H}^{y} dzdy \\
=& \frac{x_L x_H}{2}\left(x^G - \underline{B}^H\right)^2 \\
=& \frac{1}{2}\left[\frac{V^L(H) - V^H(H)}{V^H(L) - V^L(L)}\right]
\end{aligned}
$$

□

Lemma B.5.3. *If* $V^H(H) > V^L(H)$ and $V^L(L) > V^H(L)$, *then in equilibrium* $Prob\{win = L|lose = H\} = \frac{1}{2}\left[\frac{V^L(L)-V^H(L)}{V^H(H)-V^L(H)}\right]$.

Proof. In the equilibrium of form (c), $\bar{B}^L = \frac{1}{x_L}$.

$$
\begin{aligned}
Prob\{win = L|lose = H\} =& \int_0^{\bar{B}^L} g^L(y)G^H(y)dy \\
=& x_L \int_0^{\bar{B}^L} \int_0^y g^H(z)dzdy \\
=& x_L x_H \int_0^{\bar{B}^L} ydy \\
=& \frac{x_L x_H \left(\bar{B}^L\right)^2}{2} \\
=& \frac{x_H}{2x_L} = \frac{1}{2}\left[\frac{V^L(L) - V^H(L)}{V^H(H) - V^L(H)}\right]
\end{aligned}
$$

□

B.6 Miscellaneous

B.6.1 Calculations for Proposition 1.5.2

Proof. $f(H,H|\omega_i) = f_i(H|\omega_i)f_{-i}(H|\omega_i) = \alpha(1-\alpha)$. Hence, $f(H,H,\omega_i) = f(H,H|\omega_i)f(\omega_i) = \frac{\alpha(1-\alpha)}{2}$. Likewise, $f(L,L,w_i) = \frac{\alpha(1-\alpha)}{2}$. Then, $f(H,H) = f(H,H,\omega_i) + f(H,H,\omega_{-i}) = \alpha(1-\alpha)$. Then, $f(\omega_i|H,H) = f(\omega_{-i}|H,H) = f(\omega_i|L,L) = f(\omega_{-i}|L,L) = \frac{1}{2}$.

$$\begin{aligned} v_i(H,H) &= \left(f(\omega_i|H,H)y^H + f(\omega_{-i}|H,H)y^L\right) \\ &= \frac{1}{2}\left(y^H + y^L\right) \end{aligned} \tag{B.19}$$

$$\begin{aligned} v_i(L,L) &= \left(f(\omega_i|L,L)y^H + f(\omega_{-i}|L,L)y^L\right) \\ &= \frac{1}{2}\left(y^H + y^L\right) \end{aligned} \tag{B.20}$$

$f(H,L|\omega_i) = f_i(H|w_i)f_{-i}(L|w_i) = \alpha^2$, then $f(H,L,w_i) = \frac{\alpha^2}{2}$. Likewise, $f(L,H,w_i) = \frac{(1-\alpha)^2}{2}$. Then, $f(H,L) = f(H,L,\omega_i) + f(H,L,\omega_{-i}) = \frac{\alpha^2+(1-\alpha)^2}{2}$ and $f(L,H) = \frac{\alpha^2+(1-\alpha)^2}{2}$. Thus, $f(\omega_i|H,L) = f(\omega_{-i}|L,H) = \frac{\alpha^2}{\alpha^2+(1-\alpha)^2}$ and $f(\omega_{-i}|H,L) = f(\omega_i|L,H) = \frac{(1-\alpha)^2}{\alpha^2+(1-\alpha)^2}$.

$$\begin{aligned} v_i(H,L) &= \left(f(\omega_i|H,L)y^H + f(\omega_{-i}|H,L)y^L\right) \\ &= \left(\frac{\alpha^2 y^H + (1-\alpha)^2 y^L}{\alpha^2 + (1-\alpha)^2}\right) \end{aligned} \tag{B.21}$$

$$\begin{aligned} v_i(L,H) &= \left(f(\omega_i|L,H)y^H + f(\omega_{-i}|L,H)y^L\right) \\ &= \left(\frac{(1-\alpha)^2 y^H + \alpha^2 y^L}{\alpha^2 + (1-\alpha)^2}\right) \end{aligned} \tag{B.22}$$

$$\begin{aligned} f(H|H) = 2\alpha(1-\alpha),&\ f(L|H) = \alpha^2 + (1-\alpha)^2 \\ f(H|L) = \alpha^2 + (1-\alpha)^2,&\ f(L|L) = 2\alpha(1-\alpha) \end{aligned} \tag{B.23}$$

$$\begin{aligned} V^H(H) &= \alpha(1-\alpha)\left(y^H + y^L\right) \\ V^L(L) &= \alpha(1-\alpha)\left(y^H + y^L\right) \\ V^H(L) &= \alpha^2 y^H + (1-\alpha)^2 y^L \\ V^L(H) &= (1-\alpha)^2 y^H + \alpha^2 y^L \end{aligned} \tag{B.24}$$

□

B.6.2 Asymmetric Example

The strategies for player 1 and player 2 in the example are given by (D.21) and (B.26) respectively. Proof of equilibrium is provided in the figure below. Player 1's low type and player 2's high type both have a constant slope of 1 on (0,2/5). Player 1's low type has a a constant slope of a half and player 2's low type has a constant slope of 1 on (2/5,6/5). Player 1's high type has a slope of 1 and player 2's low type has a slope of a half on (6/5,8/5). Both high types have constant slope of a half on (8/5,14/5).

$$G_1^L(b) = \begin{cases} \frac{1}{5} + b & b \leq \frac{2}{5} \\ \frac{2}{5} + \frac{b}{2} & \frac{2}{5} \leq b \leq \frac{6}{5} \\ 1 & b \geq \frac{6}{5} \end{cases}, \quad G_1^H(b) = \begin{cases} 0 & b \leq \frac{6}{5} \\ -\frac{6}{5} + b & \frac{6}{5} \leq b \leq \frac{8}{5} \\ -\frac{2}{5} + \frac{b}{2} & \frac{8}{5} \leq b \leq \frac{14}{5} \\ 1 & b \geq \frac{14}{5} \end{cases}. \tag{B.25}$$

$$G_2^L(b) = \begin{cases} 0 & b \leq \frac{2}{5} \\ b - \frac{2}{5} & \frac{2}{5} \leq b \leq \frac{6}{5} \\ \frac{b}{2} + \frac{1}{5} & \frac{6}{5} \leq b \leq \frac{8}{5} \\ 1 & b \geq \frac{8}{5} \end{cases}, \quad G_2^H(b) = \begin{cases} b & b \leq \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} \leq b \leq \frac{8}{5} \\ \frac{b}{2} - \frac{4}{5} & \frac{8}{5} \leq b \leq \frac{14}{5} \\ 1 & b \geq \frac{14}{5} \end{cases}. \tag{B.26}$$

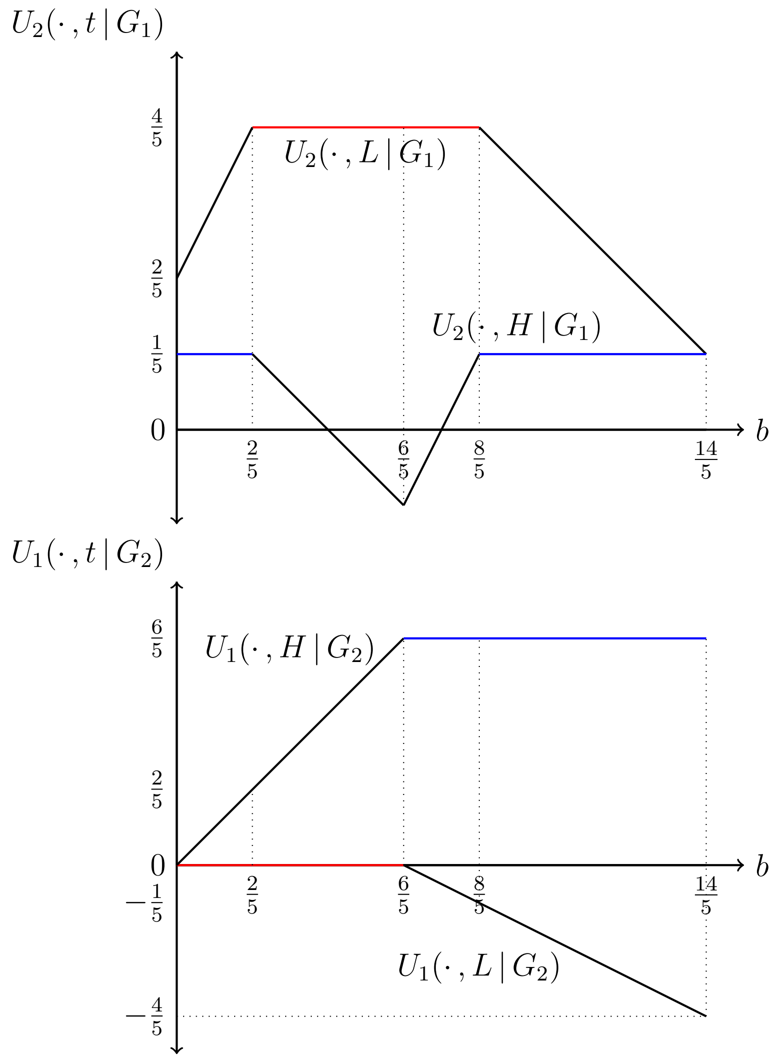

Figure B.1: The blue lines indicate the high type's equilibrium support and the red the lows type's equilibrium support.

Appendix C

Proofs for Chapter 2

C.1 Proof of Lemma 2.4.5

Proof. When $N = 2$ and $k = 1$, then $S = \{2\}$ is the only subset of $T\backslash\{k\}$. Hence, (2.30) is equivalent to $V^1(2) \neq V^2(2)$. When $k = 2$, then $S = \{1\}$ is the only subset of $T\backslash\{k\}$. Therefore, (2.30) is equivalent to $V^2(1) \neq V^1(1)$.

When $N = 3$, each type has three subsets to check. If $k = 3$, then $S = \{1\}$, $S = \{2\}$, or $S = \{1, 2\}$. If $S = \{1\}$, (2.30) is equivalent to $V^3(1) \neq V^1(1)$. While if $S = \{2\}$, (2.30) is equivalent to $V^3(2) \neq V^2(2)$. If $S = \{1, 2\}$, then (2.30) is equivalent to (C.1).

$$\begin{aligned} V^2(2)a_2 + V^1(2)a_1 &= V^3(2) \\ V^2(1)a_2 + V^1(1)a_1 &= V^3(1) \end{aligned} \tag{C.1}$$

Since the Generic Condition is satisfied, this system has a unique solution. Using Cramer's Rule the coefficients are provided by (C.2).

$$\begin{aligned} a_2 &= \frac{V^3(2)V^1(1) - V^1(2)V^3(1)}{V^2(2)V^1(1) - V^1(2)V^2(1)} \\ a_1 &= \frac{V^2(2)V^3(1) - V^3(2)V^2(1)}{V^2(2)V^1(1) - V^1(2)V^2(1)} \end{aligned} \tag{C.2}$$

If Condition 2 is satisfied the coefficients do not sum to one.

$$\begin{aligned} &\frac{V^3(2)V^1(1) - V^1(2)V^3(1) + V^2(2)V^3(1) - V^3(2)V^2(1)}{V^2(2)V^1(1) - V^1(2)V^2(1)} \neq 1 \\ \iff &\left(V^1(1) - V^2(1)\right)\left(V^3(2) - V^2(2)\right) \neq \left(V^3(1) - V^2(1)\right)\left(V^1(2) - V^2(2)\right) \end{aligned} \tag{C.3}$$

The exact same approach is used to find the conditions for the other 2 types.

□

C.2 Example 3: Correlated Private Values

Let $T = \{1, 2, 3\}$. Let the private value of winning be $v(3) = 4$, $v(2) = \theta 4$ and $v(1) = \theta^2 4$ with $\theta \in (0, 1]$. Thus, if your type increases by 1, the private value increases by a factor of $\frac{1}{\theta}$. Furthermore, conditional on receiving the type t, it is *two times* more likely that the opponent's type is t than any adjacent type and it is *four times* more likely than types that are twice removed.

$$M(T) = 4 * \begin{pmatrix} 4/7 & 2/7 & 1/7 \\ \theta/4 & \theta/2 & \theta/4 \\ \theta^2/7 & 2\theta^2/7 & 4\theta^2/7 \end{pmatrix}$$

The unique symmetric equilibrium is of form (a) if $\theta = 1$, of form (c) if $\theta > \frac{4}{7}$, of the (b-c) mixed-form if $\frac{7}{16} < \theta < \frac{4}{7}$ and of form (b) if $\theta \leq \frac{7}{16}$.

Proof. When $\theta = 1 \implies \mathbb{E}[v \,|\, N] = \mathbb{E}[v \,|\, k]$ for all $k \in T \iff$ the construction of form (a) is an equilibrium by lemma 2.4.1. Furthermore, lemma C.2.1 $\implies$ Condition c is satisfied for all types, then by theorem 2.4.11 this is the unique symmetric equilibrium.

When $\theta > \frac{4}{7}$ lemma C.2.1 $\implies$ Condition c is satisfied for all types. By lemma C.2.2, for all $n < N$ and all $b \in \beta(T^n)$ $\frac{dG^t(b)}{db}$ is strictly decreasing in t. Clearly, $\mathbb{E}[v \,|\, t]$ is strictly increasing in t. Then corollary 2.4.12 implies the unique symmetric equilibrium is of form (c).

If $\frac{7}{16} < \theta < \frac{4}{7}$ then the (b-c) mixed form of figure 2.7 is an equilibrium. First notice, that that type 3 dominates the other two types in a KMS sense. This means that for $t \in \{1, 2\}$, $b \in \operatorname{supp}[G_i^t]$ and $b' \in \operatorname{supp}\left[G_i^{3'}\right]$ implies that $b' \geq b$. Thus, the bid support of type 3 has to be right of the supports of the other two with no overlap.

To find the supports of type 1 and 2, suppose the type space is just $T^* = \{1, 2\}$. Then lemma C.2.1 Condition c is satisfied for all types. Further, for $b \in \beta(T^*)$:

lap

$$\frac{d\,G^2}{d\,b} = \frac{16\theta - 7}{6\theta^2} \qquad \frac{d\,G^1}{d\,b} = \frac{14 - 8\theta}{6\theta^2} \tag{C.4}$$

When $\theta > \frac{7}{16}$ the slopes are well defined, further it implies that $\frac{d\,G^2}{d\,b} < \frac{d\,G^1}{d\,b}$. Thus, corollary 2.4.12 implies that for T^* the unique symmetric equilibrium is of form (c). Then, combine it with the uniquely determined support of type 3 ”above” the bid supremum of the game with T^*. Then the unique symmetric equilibrium is the (b-c) mixed form.

If $\theta \leq \frac{7}{16}$ then KMS is satisfied and the unique equilibrium if of form (b).

□

Lemma C.2.1. *Condition c is satisfied for type 1 for all θ. Condition c is satisfied for for type 2 if $\theta > \frac{7}{16}$. Condition c is satisfied for type 3 if if $\theta > \frac{4}{7}$.*

Proof. For ***type* 1**, condition c needs to hold for $S \in \{\{2\}, \{3\}, \{2, 3\}\}$. Clearly it holds for the two singleton sets: $V^1(2) = 2\theta^2/7 < \theta/2 = V^2(2)$ and $V^1(3) = \frac{\theta^2}{7} < \frac{4}{7} = V^3(3)$. It is also easy to check that it holds for $S = \{2, 3\}$, since $V^3(2) > V^1(2)$ and $V^3(3) > V^1(3)$. To see this notice that if there exist a type (Like type 3 here) whose coefficients are strictly higher for some S, then on for this S condition c is satisfied.

For ***type* 2**, condition c needs to hold for $S \in \{\{1\}, \{3\}, \{1, 3\}\}$. First, condition c is always satisfied for $S = \{3\}$, since $V^2(3) = \frac{\theta}{4} < \frac{4}{7} = V^3(3)$. For $S = \{1\}$, condition (c) holds if:

$$\begin{aligned} V^2(1) - V^1(1) = \frac{\theta}{4} - \frac{4\theta^2}{7} < 0 \\ \iff \theta > \frac{7}{16} \end{aligned} \tag{C.5}$$

For $\{1, 3\}$, the unique linear combination is such that the coefficients sum to $\alpha_1 + \alpha_3 =$

$\frac{7(1+\theta^2)}{20\theta}$.

$$\begin{aligned} \frac{7(1+\theta^2)}{20\theta} &< 1 \\ \iff 7 + 7\theta^2 - 20\theta &< 0 \\ \theta &> \frac{10-\sqrt{51}}{7} \end{aligned} \tag{C.6}$$

Since $\frac{10-\sqrt{51}}{7} < \frac{7}{16}$, $\frac{7}{16}$ is the lower bound such that condition (c) is satisfied for type 1.

For ***type* 3**, condition c needs to hold for $S \in \{\{1\},\{2\},\{1,2\}\}$. Note that it is satisfied for $S = \{2\}$ iff $\theta > \frac{4}{7}$, this also sufficient for $S = \{1,2\}$, since this also implies $V^2(1) > V^3(1)$. lastly, $V^3(1) < V^1(1) \iff \theta > \frac{1}{2}$, so if $\theta > \frac{4}{7}$ is sufficient for condition c on $S = \{1\}$.

□

Lemma C.2.2. *If $\theta \in (\frac{1}{3}, 1)$ then for all $b \in \beta(T)$: $\frac{dG^3}{db} < \frac{dG^2}{db} < \frac{dG^1}{db}$, and for all $b \in \beta(\{2,3\})$: $\frac{dG^3}{db} < \frac{dG^2}{db}$. Furthermore, if $\theta > \frac{4}{7}$ then for all $b \in \beta(S)$ and $b \in \beta(T)$: $\frac{dG^3}{db} > 0$.*

Proof. By using Cramer's rule for the system $M(T)\boldsymbol{x} = \mathbf{1} \implies \forall b \in \beta(T):$

$$\begin{aligned} \frac{dG^3}{db} &= \frac{7\theta - 4}{12\theta} \\ \frac{dG^2}{db} &= \frac{60\theta - 21(1+\theta^2)}{72\theta^2} \\ \frac{dG^1}{db} &= \frac{7 - 4\theta}{12\theta} \end{aligned} \tag{C.7}$$

$$\begin{aligned} \frac{dG^3}{db} - \frac{dG^2}{db} &< 0 \\ \iff 3\theta^2 - 4\theta + 1 &< 0 \end{aligned} \tag{C.8}$$

This difference is a parabola with the roots at $\frac{1}{3}$ and at 1.

$$\begin{aligned} \frac{dG^2}{db} - \frac{dG^1}{db} &< 0 \\ \iff \theta^2 + 6\theta - 7 &< 0 \end{aligned} \tag{C.9}$$

This difference is a parabola with the roots at -7 and at 1.

By using Cramer's rule for the system $M(\{2,3\})\boldsymbol{x} = \mathbf{1} \implies \forall b \in \beta(\{2,3\})$:

$$\begin{aligned} \frac{d\,G^3}{d\,b} &= \frac{7\theta - 4}{12\theta} \\ \frac{d\,G^2}{d\,b} &= \frac{16 - 7\theta}{24\theta} \end{aligned} \tag{C.10}$$

$$\begin{aligned} \frac{d\,G^3}{d\,b} - \frac{d\,G^2}{d\,b} &< 0 \\ \iff \theta &< \frac{24}{21} \end{aligned} \tag{C.11}$$

It is obvious from the equations (C.7) and (C.10) $\frac{d\,G^3}{d\,b} > 0$ if $\theta > \frac{4}{7}$. □

C.3 Applying Condition b to Chen (2021)

The environment of Chen (2021) has four types, which he orders in a specific way, see (C.12). Each type has a independely drawn private value which can be either high (v_h) with probability p_h or low (v_l) with probability p_l. Each type also has a private binary signal which is correlated with opponents private value which can be high (h) or low (l), such that $pr(h|v_h) = pr(l|v_l) = q$.

His notation mapped to mine implies that:

$$\begin{aligned} t_4 = (v_h, h) = 4 \text{ and } t_3 = (v_h, l) = 3. \\ t_2 = (v_l, l) = 2 \text{ and } t_1 = (v_l, h) = 1. \end{aligned} \tag{C.12}$$

To calculate a primal objects, the following formula is used.[1]

$$\begin{aligned} V^k(j) = f(j|k)v_h \ for\ k \in \{3,4\} \\ V^k(j) = f(j|k)v_l \ for\ k \in \{1,2\} \end{aligned} \tag{C.13}$$

[1]Where $f(j|k)$ can be easily calculated using the formula's given in Chen (2021), for example: $f(4|4) = pr(v_h|h)pr(h|v_h) = \frac{p_h q^2}{p_h q + p_l(1-q)}$.

$$M(T) = \begin{pmatrix} \frac{p_h q^2 v_h}{p_h q + p_l(1-q)} & \frac{p_h q(1-q) v_h}{p_h q + p_l(1-q)} & \frac{p_l(1-q)^2 v_h}{p_h q + p_l(1-q)} & \frac{p_l(1-q) q v_h}{p_h q + p_l(1-q)} \\ \frac{p_h(1-q) q v_h}{p_h(1-q) + p_l q} & \frac{p_h(1-q)^2 v_h}{p_h(1-q) + p_l q} & \frac{p_l q(1-q) v_h}{p_h(1-q) + p_l q} & \frac{p_l q^2 v_h}{p_h(1-q) + p_l q} \\ \frac{p_h(1-q)^2 v_l}{p_h(1-q) + p_l q} & \frac{p_h(1-q) q v_l}{p_h(1-q) + p_l q} & \frac{p_l q^2 v_l}{p_h(1-q) + p_l q} & \frac{p_l(1-q) q v_l}{p_h(1-q) + p_l q} \\ \frac{p_h q(1-q) v_l}{p_h q + p_l(1-q)} & \frac{p_h q^2 v_l}{p_h q + p_l(1-q)} & \frac{p_l(1-q) q v_l}{p_h q + p_l(1-q)} & \frac{p_l(1-q)^2 v_l}{p_h q + p_l(1-q)} \end{pmatrix}$$

I reprove proposition 1 of Chen (2021) by showing that the condition b is satisfied when $q \geq \frac{2}{3}$.

Lemma C.3.1. *The environment in Chen (2021) satisfies condition b If* $\frac{1}{2} \leq q \leq \frac{v_h}{v_l + v_h}$..

Proof. I show that condition b holds for type 4, the proof for the other types is almost identical algebra and brings no new conditions on the primitives.

$$\begin{aligned} & J_L = 1 \ : \ V^4(3) \geq V^3(3) \\ \iff & \frac{p_h q(1-q) v_h}{p_h q + p_l(1-q)} \geq \frac{p_h(1-q)^2 v_h}{p_h(1-q) + p_l q} \\ \iff & \frac{q}{p_h q + p_l(1-q)} \geq \frac{(1-q)}{p_h(1-q) + p_l q} \\ \iff & q^2 \geq (1-q)^2 \iff q \geq \frac{1}{2} \end{aligned} \tag{C.14}$$

$$\begin{aligned} & J_L = 2 \ : \ V^4(3) + V^4(2) \geq V^3(3) + V^2(2) \\ \iff & \frac{p_h q(1-q) v_h + p_l(1-q)^2 v_h}{p_h q + p_l(1-q)} \geq \frac{p_h(1-q)^2 v_h + p_l q^2 v_l}{p_h(1-q) + p_l q} \\ \iff & \frac{v_h(1-q)\left(p_h q + p_l(1-q)\right)}{p_h q + p_l(1-q)} \geq \frac{p_h(1-q)^2 v_h + p_l q^2 v_l}{p_h(1-q) + p_l q} \\ \iff & v_h(1-q)\left(p_h(1-q) + p_l q\right) \geq p_h(1-q)^2 v_h + p_l q^2 v_l \\ \iff & v_h(1-q) \geq q v_l \\ \iff & q \leq \frac{v_h}{v_l + v_h} = q^* \end{aligned} \tag{C.15}$$

For $J_L = 3$ it is sufficient to note that $V^1(1) \leq V^4(1)$ for $q \geq \frac{1}{2}$. Such that: $V^4(3) + V^4(2) + V^4(1) \geq V^3(3) + V^2(2) + V^1(1)$ when $\frac{1}{2} \leq q \leq \frac{v_h}{v_l + v_h}$.

□

Hence, I corroborate Chen (2021) finding of a monotone equilibrium by using my approach. His environment is particularly interesting because it is first published paper (that I am aware off) that has a monotone equilibrium but KMS is not satisfied, except for $q = \frac{1}{2}$.[2] Since, a monotone equilibrium exists for measurable set of q, and KMS is satisfied only for a point in q, I assert that KMS is almost never satisfied.

Lemma C.3.2. *For the model in Chen (2021) KMS is almost never satisfied.*

Proof. KMS implies that $V^k(j)$ is increasing in k for all j.

$$\begin{aligned} & V^4(2) \geq V^3(2) \\ & \iff \frac{p_l(1-q)^2 v_h}{p_h q + p_l(1-q)} - \frac{p_l q(1-q) v_h}{p_h(1-q) + p_l q} \\ & \iff q \leq \frac{1}{2} \end{aligned} \tag{C.16}$$

But equation (C.14) implies that $q \geq \frac{1}{2} \iff V^4(3) \geq V^3(3)$. Hence, only for $q = \frac{1}{2}$ is KMS satisfied. □

C.4 Open Question in Linear Algebra

Theorem 2.4.11 and theorem 2.4.15 could be sharper if it was known in terms of the primitives of the model when the slopes of the bid distribution are ordered. This is an open question in linear algebra.

The Problem Suppose that the matrix A is an invertible $N \times N$ matrix with all positive entries i.e. $a_{i,j} \geq 0$. Then define the row-sum of row i as follows.

$$x_i := \sum_{j=1}^{N} a_{i,j} \tag{C.17}$$

Then let $B = A^{-1}$, then the row-sum of row i of B is $y_i := \sum_{j=1}^{N} b_{i,j}$.

Let x_i be increasing in i then under what conditions on A is y_i decreasing in i and when is y_i increasing in i?

If $N = 2$, the answer is simple. If A is a positive definite then $y_1 \leq y_2$, and if it is a negative definite $y_1 \geq y_2$.

[2]But $q = \frac{1}{2}$, reduces Chen's model to an independent private value model which removes all the interesting components of his model. Furthermore, the generic condition is not satisfied, hinting that the type space is better of being reduced to 2 types in this case.

Proof. x_i be increasing in i implies that $a_{1,1}+a_{1,2} \geq a_{2,1}+a_{2,2} \iff a_{1,1}-a_{2,1} \geq a_{2,2}-a_{1,2}$. If, $\det(A) > 0 \implies a_{1,1}a_{2,2} > a_{1,2}a_{2,1} \implies y_1 = \frac{a_{2,2}-a_{1,2}}{a_{1,1}a_{2,2}-a_{1,2}a_{2,1}} \leq \frac{a_{1,1}-a_{2,1}}{a_{1,1}a_{2,2}-a_{1,2}a_{2,1}} = y_2$. If, $\det(A) < 0 \implies a_{1,1}a_{2,2} < a_{1,2}a_{2,1}$ then $y_1 = \frac{a_{2,2}-a_{1,2}}{a_{1,1}a_{2,2}-a_{1,2}a_{2,1}} \geq \frac{a_{1,1}-a_{2,1}}{a_{1,1}a_{2,2}-a_{1,2}a_{2,1}} = y_2$. □

An other simple result can found if $x_i = x_j$ for all j.

Lemma C.4.1. *If $x_i = x_j$ for all $j \neq i \implies y_i = y_j$ for all $j \neq i$.*

Proof. Let $\mathbf{1}$ denote $N \times 1$ vector of ones. Let $\boldsymbol{x}$ be a vector of the row sums of A and let $\boldsymbol{y}$ be a vector of the row sums of B. Then clearly, $A \cdot \mathbf{1} = \boldsymbol{x}$ and $B \cdot \mathbf{1} = \boldsymbol{y}$. Since $x_i = x$ for all i, then $A \cdot \mathbf{1} = x\mathbf{1}$ where x is a scalar.

$$\begin{aligned} A \cdot \mathbf{1} &= x\mathbf{1} \\ \iff \frac{1}{x}\mathbf{1} &= A^{-1} \cdot \mathbf{1} \\ \iff \frac{1}{x}\mathbf{1} &= B \cdot \mathbf{1} = \boldsymbol{y} \end{aligned} \tag{C.18}$$

□

Appendix D

Proofs for Chapter 3

D.1 Proof of Proposition 3.3.1

The collection of lemmas in this appendix together form Proposition 3.3.1.

KMS condition A game $(2, (\mathbb{R}_+)_{i\in N}, (T_i(\mu))_{i\in N}, (\mu), (w_i)_{i\in N})$, satisfies KMS if $\forall i = 1, 2, \forall t_{-i} \in T_{-i}, if t_i \prec t'_i$ then, $\mu(t_{-i}|t_i)V(t_i, t_{-i}) - \mu(t_{-i}|t'{}_i)V_i(t'_i, t_{-i}) < 0$.

KMS guarantees that $\mu(t_{-i}|t_i)V(t_i, t_{-i})$ is increasing in t_i. This makes the expected payoff of the all-pay auction increasing in type. Furthermore, Siegel (2014) provides the following result.

Lemma D.1.1. *If an all-pay auction game satisfies KMS there exists a monotone equilibrium. Furthermore, that equilibrium is unique.*

In this chapter we want to guarantee that condition KMS is satisfied, but $\mu(t_{-i}|t_i)$ is an endogenous object, i.e. on path it is subject to Bayes's rule.[1] Even if we assume KMS to hold for the prior distribution that is in stage one of the game $f(t_{-i}|t_i)V(t_i, t_{-i}) - f(t_{-i}|t'{}_i)V_i(t'_i, t_{-i}) < 0$, this does not imply that KMS holds in the second stage of the game. What is sufficient is condition Preserve Montononicity.

Preserving Monotonicity Condition (PM) PM is satisfied If for both players i the following condition is satisfied. For any two consecutive signals $t'_i \prec t''_i$ of agent i and any two signals of agent $-i$, $t'_{-i} \prec t''_{-i}$. The following 2 inequalities hold:

$$\frac{V_i(t'_i, t''_{-i})}{V_i(t''_i, t''_{-i})} \leq \frac{f(t'_i, t'_{-i})f(t''_i, t''_{-i})}{f(t''_i, t'_{-i})f(t'_i, t''_{-i})} \leq \frac{V_i(t''_i, t'_{-i})}{V_i(t'_i, t'_{-i})} \tag{D.1}$$

[1] Off-path is no concern, since beliefs are arbitrary.

PM has 2 components:

1. The valuation ratio, $V_i(t'_i, t''_{-i})$ is strictly increasing in t_i thus $\frac{V_i(t'_i,t''_{-i})}{V_i(t''_i,t''_{-i})} < 1$ and $\frac{V_i(t''_i,t'_{-i})}{V_i(t'_i,t'_{-i})} > 1$.

2. The correlation object $\frac{f(t'_i,t'_{-i})f(t''_i,t''_{-i})}{f(t''_i,t'_{-i})f(t'_i,t''_{-i})}$.[2]

Lemma D.1.2. *Condition PM implies KMS to be satisfied on-path.*

Proof. Let $\hat{t}_{-i} \in T(\mu_{h,\sigma})$.

$$\begin{aligned}
&\mu_{h,\sigma}(\hat{t}_{-i}|t'_i)V(t'_i,\hat{t}_{-i}) - \mu_{h,\sigma}(\hat{t}_{-i}|t''{}_i)V_i(t''_i,\hat{t}_{-i}) \\
&\qquad \iff \frac{\sigma_{-i,\emptyset}(a_{-i}|\hat{t}_{-i})\, f(\hat{t}_{-i}|t'_i)V(t'_i,\hat{t}_{-i})}{\sum_{t_{-i}\in T_{-i}} \sigma_{-i,\emptyset}(a_{-i}|t_{-i})\, f(t_{-i}|t'_i)} - \frac{\sigma_{-i,\emptyset}(a_{-i}|\hat{t}_{-i})\, f(\hat{t}_{-i}|t''_i)V(t''_i,\hat{t}_{-i})}{\sum_{t_{-i}\in T_{-i}} \sigma_{-i,\emptyset}(a_{-i}|t_{-i})\, f(t_{-i}|t''_i)} \\
&\qquad \iff \frac{\sum_{t_{-i}\in T_{-i}} \sigma_{-i,\emptyset}(a_{-i}|t_{-i})\, f(t_{-i}|t''_i)\, f(\hat{t}_{-i}|t'_i)V(t'_i,\hat{t}_{-i})}{\sum_{t_{-i}\in T_{-i}} \sigma_{-i,\emptyset}(a_{-i}|t_{-i})\, f(t_{-i}|t'_i)\, f(\hat{t}_{-i}|t''_i)V(t''_i,\hat{t}_{-i})} - 1
\end{aligned} \tag{D.2}$$

PM implies that every element of the previous sum's numerator, is smaller than that of the same element in denominator.

1. If $t_{-i} > \hat{t}_{-i}$, PM implies that $\frac{f(t_{-i}|t''_i)\, f(\hat{t}_{-i}|t'_i)}{f(t_{-i}|t'_i)\, f(\hat{t}_{-i}|t''_i)} = \frac{f(t'_i,t'_{-i})f(t''_i,t''_{-i})}{f(t''_i,t'_{-i})f(t'_i,t''_{-i})} \leq \frac{V_i(t''_i,t'_{-i})}{V_i(t'_i,t'_{-i})} = \frac{V_i(t''_i,\hat{t}_{-i})}{V_i(t'_i,\hat{t}_{-i})}$.[3]

2. If $t_{-i} < \hat{t}_{-i}$, PM implies that $\frac{f(t_{-i}|t''_i)\, f(\hat{t}_{-i}|t'_i)}{f(t_{-i}|t'_i)\, f(\hat{t}_{-i}|t''_i)} = \frac{f(t''_i,t'_{-i})f(t'_i,t''_{-i})}{f(t'_i,t'_{-i})f(t''_i,t''_{-i})} \leq \frac{V_i(t''_i,t''_{-i})}{V_i(t'_i,t''_{-i})} = \frac{V_i(t''_i,\hat{t}_{-i})}{V_i(t'_i,\hat{t}_{-i})}$.[4]

3. If $t_{-i} = \hat{t}_{-i}$, then since V_i is strictly increasing $\frac{V_i(t'_i,\hat{t}_{-i})}{V_i(t''_i,\hat{t}_{-i})} < 1$.

Thus, PM implies that:

$$\begin{aligned}
&\frac{\sum_{t_{-i}\in T_{-i}} \sigma_{-i,\emptyset}(a_{-i}|t_{-i})\, f(t_{-i}|t''_i)\, f(\hat{t}_{-i}|t'_i)V(t'_i,\hat{t}_{-i})}{\sum_{t_{-i}\in T_{-i}} \sigma_{-i,\emptyset}(a_{-i}|t_{-i})\, f(t_{-i}|t'_i)\, f(\hat{t}_{-i}|t''_i)V(t''_i,\hat{t}_{-i})} < 1 \\
&\implies \mu_{h,\sigma}(\hat{t}_{-i}|t'_i)V(t'_i,\hat{t}_{-i}) < \mu_{h,\sigma}(\hat{t}_{-i}|t''{}_i)V_i(t''_i,\hat{t}_{-i})
\end{aligned} \tag{D.3}$$

□

Lemma D.1.3. *PM implies that $U_{i,h,\sigma}(t_i, b)$ is strictly increasing in t_i for all $b > 0$.*

[2]By the full suport assumption this object is well defined.
[3]$\hat{t}_{-i} = t'_{-i}$ and $t_{-i} = t''_{-i}$ in (D.1), using the upper bound.
[4]$\hat{t}_{-i} = t''_{-i}$ and $t_{-i} = t'_{-i}$ in (D.1), using the lower bound.

Proof. PM implies KMS, i.e. that for both i, $\mu(t_{-i}|t_i)V_i(t_i, t_{-i})$ is strictly increasing in t_i. Thus for $t_i \prec t_i'$, and any bid $b \in (0, x^\sigma]$:

$$\begin{aligned} U_{i,h,\sigma}(t_i, b) &= \sum_{t_{-i} \in T_{-i}} \mu(t_{-i}|t_i)V_i(t_i, t_{-i})\sigma_{-i,}(b|t_{-i}) - b \\ &< \sum_{t_{-i} \in T_{-i}} \mu(t_{-i}|t_i')V_i(t_i', t_{-i})\sigma_{-i,}(b|t_{-i}) - b = U_{i,h,\sigma}(t_i', b) \end{aligned} \tag{D.4}$$

□

D.2 Proof for Atoms at Zero

Atoms (see definition A.1) at zero are essential for the analysis of this chapter. In this appendix, I establish that if the opponent has no atom at zero, then the expected payoff of bidding zero is zero (Lemma D.2.1). Further, I show that unconditional bid distribution is absolutely continuous.

Lemma D.2.1. *In any monotone equilibrium, if $t_i = \min T_i(\mu)$ and $-i$ has no atom at zero, then $U_i(t_i, b) = 0$ for all $b \in BR_i(t_i)$.*

Proof. In a monotone equilibrium the lowest type t_i must have $0 \in BR_i(t_i)$, which implies that for all $b \in BR_i(t_i)$ the payoff is the same as for $b = 0$. When $-i$ has no atom at zero, $\sigma_{-i,}(0|t^j_{-i}) = 0$ for all j. Hence, $U_i(t_i, 0) = \sum_{j=1}^{n_{-i}} V(t_i, t^j_{-i})\mu(t^j_{-i}|t_i)\sigma_{-i,}(0|s^j_{-i}) = 0$. □

D.2.1 Proof that $H_{i,\sigma}(b)$ is Absolutely Continuous

Consider the following standard definition. F is called a distribution function of a measure μ if it has the following 3 properties. (i) F is monotone increasing, (ii) F is continuous from the right and (iii) $F(0) = 0$.

Lemma D.2.2. *Let F be a continuous distribution function that has a continuous derivative except at finitely many point then $F(x) = \int_0^x f(x)dx$ everywhere on $\mathbb{R}^+$.*[5]

Proof. (by induction) If x is a point of continuity of f then by fundamental theorem of calculus $\frac{d}{dx}\int_0^x f(t)dt = f(x)$, thus $F(x) = \int_0^x f(t)dt$ a.e on $\mathbb{R}^+$. Then define a continuous distribution function $G(x) = \int_0^x f(t)dt$. $F' = G'$ except at finitely many points.

[5] This proof is based on a sketch of a proof given in example (1) at page 379 chapter 8 Knapp (2016).

Suppose we have k of these points and we order them in set $a := \{a_1, a_2 \cdots, a_k\}$. By the mean value theorem, on the interval $[0, a_1]$ there exists a point $c_1 \in (0, a_1)$ s.t

$$F'(c_1) - G'(c_1) = \frac{F(a_1) - F(0)}{a_1} - \frac{G(a_1) - G(0)}{a_1} = \frac{F(a_1) - G(a_1)}{a_1} = 0 \tag{D.5}$$

This implies that $F(a_1) - G(a_1) = 0$, since $F'(c) - G'(c) = 0$ everywhere on the interior of the interval. Now suppose that $F(a_j) - G(a_j) = 0$ for $j > 1$ by the mean value theorem on $[a_j, a_{j+1}]$ there exists a point $c \in (a_j, a_{j+1})$ s.t

$$F'(c) - G'(c) = \frac{F(a_{j+1}) - F(a_j)}{a_{j+1} - a_j} - \frac{G(a_{j+1}) - G(a_j)}{a_{j+1} - a_j} = \frac{F(a_{j+1}) - G(a_{j+1})}{a_{j+1} - a_j} = 0 \tag{D.6}$$

This implies that $F(a_{j+1}) - G(a_{j+1}) = 0$, and this true for all $j > 1$. Thus $F(x) = G(x) = \int_0^x f(t)dt$ everywhere on $\mathbb{R}^+$. $\square$

Let $D\sigma_i(b|t_i)$ denote the derivative of $\sigma_i(b|t_i)$ at b if it exists. In any equilibrium,supp $D\sigma_i(b|t_i) = BR_i(t_i)$. Furthermore if we are in monotone equilibrium $D\sigma_i(b|t_i)$ is a step function with finitely many discrete jumps. By the previous lemma this implies that $\sigma_i(b|t_i)$ is absolutely continuous.

Lemma D.2.3. *The bid distribution $\sigma_i(b|t_i)$ is absolutely continuous for all all $b \in (0, x^\sigma]$.*

Proof. By Lemma A.1.6, atoms are only possible at the zero bid. Hence, redefine $\sigma_i(b|t_i) := c_0 + \sigma_i^*(b|t_i)$, where c_0 is the possible atom at zero and $\sigma_i^*(x|t_i)$ is the continuous atom-less part, s.t. $\sigma_i^*(0|t_i) = 0$. Thus $\sigma_i^*(x|t_i')$ is a distribution function, and D.2.2 applies thus $\sigma_i^*(x|a^1, s_i) = \int_0^x D\sigma_i(x|a^1, s_i)dx$ for all $x \in (0, x^\sigma]$. $\square$

Recall the unconditional bid distribution of player i.

$$H_{i,h,\sigma}(b) = \sum_{j=1}^{n_i} \mu_{h,\sigma}(s_i^j)\sigma_{i,h}(b|s_i^j). \tag{D.7}$$

Remember the CDF of t_i is $\Phi_i(t)$ and it's generalized inverse by $\Phi^{-1}(t)$.

$$\begin{aligned} \Phi_i(t) &:= \sum_{t_i \leq t} \mu_{h,\sigma}(t_i) \\ \Phi^{-1}(t) &:= \inf \{z \in \text{supp}\, \Phi_\text{i} : \Phi_i(z) \geq t\} \end{aligned} \tag{D.8}$$

In a monotone equilibrium for any bid greater than zero, there is only one type that bids there. Let that type be denoted by $\tilde{t}_i(b) = \tilde{t}_i(b) := \Phi^{-1}(H_i(b))$.

Lemma D.2.4. *$H'_{i,\sigma}(b) = \mu(\tilde{t}_i(b))D\sigma_i(b|\tilde{t}_i(b)))$ almost everywhere.*

Proof.

$$\text{Let } H_i^j(b) := \mu(t_i^j)D\sigma_i(b|t_i^j). \tag{D.9}$$

$H_i^j(b)$ is a monotone increasing function since $\sigma_i(b|t_i^j)$ is a monotone increasing function and $\mu(t_i^j) > 0$. Clearly, $H_{i,\sigma}(b) = \sum_{j=1}^{n_i} H_i^j(b)$ is an absolutely converging sequence of monotone increasing functions. Then Fubini's theorem on the differentiation of series of monotone functions states that if $H_{i,\sigma}(b) = \sum_{j=1}^{n_i} H_j(b)$ is absolutely convergent sequence of monotone increasing function on $[0, x^\sigma]$, then $H'_{i,\sigma}(b) = \sum_{j=1}^{n_i} H'_j(b)$ at almost every b.

$$H'_{i,\sigma}(b) = \sum_{j=1}^{n_i} H'_j(b) = \sum_{j=1}^{n_i} \mu(t_i^j)D\sigma_i(b|t_i^j) \tag{D.10}$$

$D\sigma_i(\cdot|t_i^j) = 0$ for all bids outside its support, and for almost every $b \in [0, x^\sigma]$ only one type , $\tilde{t}_i(b)$, $D\sigma_i(\cdot|\tilde{t}_i(b)) > 0$. Hence, the following statement is true almost everywhere.

$$H'_{i,\sigma}(b) = \mu(\tilde{t}_i(b))D\sigma_i(b|\tilde{t}_i(b)) \tag{D.11}$$

□

Lemma D.2.5. *$H_{i,\sigma}(b)$ is absolutely continuous for all $b \in (0, x^\sigma]$.*

Proof. Redefine $H_{i,\sigma}(b) := c_i + H^*_{-i,\sigma}(b)$, where c_i is a possible mass point at zero and $H^*_{-i,\sigma}(0) = 0$. $H^*_{-i,\sigma}(b)$ is continuous on $(0, x^\sigma]$ since $\mu(t_i)\sigma_i(b|t_i)$ is continuous and the finite sum of continuous function is continuous. Since $D\sigma_i(b|\tilde{t}_i(b))$ has only finitely many discontinuities, (D.11) has only finitely many discontinuities.[6]

To conclude, $H^*_{-i,\sigma}(b)$ is continuous on $(0, x^\sigma]$, $H_{-i,\sigma}(b)$ is increasing on $[0, x^\sigma]$ and $H^*_{-i,\sigma}(0) = 0$ thus $H^*_{-i,\sigma}(b)$ is a continuous distribution function further $H'_{-i,\sigma}(b)$ is continuous with finitely many discontinuities thus Lemma D.2.2 applies. □

D.2.2 Atoms Due to Power

Recall by Lemma 3.4.1, that at most one player has an atom at zero. Further, Lemma D.2.1 and Lemma D.1.3 imply that if the opponent has no atom at zero the payoff for deviating to war is zero.

Lemma D.2.6. *When $p_1 > p_2 = 1$ is known, player 2 has an atom at zero.*

[6] $H'_{i,\sigma}(b)$ has finitely many discontinuities since it has all the discontinuities from $D\sigma_i(b|\tilde{t}_i(b))$ and whenever $H'_{i,\sigma}(b)$ changes to a new type's $D\sigma_i(b|\tilde{t}_i(b))$. These changes in type can lead to at most n_i extra discontinuities.

Proof. (by contradiction) Suppose not, then neither player or player 1 has an atom at zero.

Recall in this model $T_i = S_i$, and $V_i(s_i, s_{-i}) = p_i V(s_i, s_{-i})$. For all bids $b \in (0, x^\sigma]$, the following is true:

$$
\begin{aligned}
H'_{1,\sigma}(b) = \mu_1(\tilde{s}_1(b)) D\sigma_1(b|\tilde{s}_1(b)) &= \frac{\mu(\tilde{s}_1(b))}{\mu(\tilde{s}_1(b)|\tilde{s}_2(b))V_2(\tilde{s}_2(b), \tilde{s}_1(b))} \\
= \frac{\mu_2(\tilde{s}_2(b))\mu_1(\tilde{s}_1(b))}{\mu(\tilde{s}_1(b), \tilde{s}_2(b))V(\tilde{s}_1(b), \tilde{s}_2(b))} &> \frac{1}{p_1} \frac{\mu_2(\tilde{s}_2(b))\mu_1(\tilde{s}_1(b))}{\mu(\tilde{s}_1(b), \tilde{s}_2(b))V(\tilde{s}_1(b), \tilde{s}_2(b))} \\
= \frac{\mu_2(\tilde{s}_1(b))}{\mu_1(\tilde{s}_2(b)|\tilde{s}_1(b))V_1(\tilde{s}_1(b), \tilde{s}_2(b))} &= D\sigma_2(b|\tilde{s}_2(x))\mu_1(\tilde{s}_2(b)) = H'_{2,\sigma}(b)
\end{aligned}
\tag{D.12}
$$

In equilibrium, $H_{1,\sigma}(x^\sigma) = H_{2,\sigma}(x^\sigma) = 1$. Because $H_{i,\sigma}(b) = \int_0^b H'_{i,\sigma}(x)dx$ everywhere and the slope of $H_{1,\sigma}$ is greater, it must be that for all bids $b < x^\sigma$: $H_{1,\sigma}(b) < H_{2,\sigma}(b)$. A contradiction as required. □

The following lemma is used to calculate the atom the non-deviating player has when the deviator is believed to be a single type with certainty. Since this is an off-path event, the non-deviating player beliefs are arbitrary.

Lemma D.2.7. *If i has one type t_i then then $c_{-i} = \inf\left\{c \in [0,1] : \int_c^1 \frac{p_i}{\rho\left[\Phi_{-i}^{-1}(u)\right]} du = 1\right\}$*

Proof.

$$
\begin{aligned}
H_{i,\sigma}(b) = \sum_{t_i} \mu(t_i)\sigma_i(b|t_i) &= c_i + \int_0^b \mu(\tilde{t}_i(x)) D\sigma_i(x|\tilde{t}_i(x)))dx \\
=& c_i + \int_0^b \frac{\mu(\tilde{t}_i(x))}{\mu(\tilde{t}_i(x)|\tilde{t}_{-i}(x))V_{-i}(\tilde{t}_{-i}(x), \tilde{t}_i(x))} dx
\end{aligned}
\tag{D.13}
$$

For both players $\mu(\tilde{t}_i(x)|\tilde{t}_{-i}(x)) = \mu(\tilde{t}_i(x))$, because i has only 1 type. Further, $\tilde{t}_{-i}(x) = \Phi_{-i}^{-1}(H_{-i}(x))$. Hence, we can rewrite ().

$$
H_{i,\sigma}(b) = c_i + \int_0^b \frac{1}{V_{-i}(\Phi_{-i}^{-1}(H_{-i}(x)), t_i)} dx \tag{D.14}
$$

$$
\begin{aligned}
V(b) &:= \int_0^b \frac{1}{V(t_i, \Phi_{-i}^{-1}(H_{-i}(x)))} dx \\
\implies V'(b) &= \frac{1}{V(t_i, \Phi_{-i}^{-1}(H_{-i}(x)))}
\end{aligned}
\tag{D.15}
$$

Recall, $t_1 = (s_1, p_1)$, since $V_1(t_1, \Phi_2^{-1}(H_2(x))) = p_1 V(t_1, \Phi_2^{-1}(H_2(x)))$. Where $V(t_1, \Phi_2^{-1}(H_2(x)))$ is the common-value of the prize separated from the power element.[7] For player 2, (D.14) becomes:

$$H_{2,\sigma}(b) = c_2 + \frac{1}{p_1} V(b). \tag{D.16}$$

Consider the projection map $\rho\,[t_{-i}]$, that maps from a type t_{-i} to its corresponding power component: p_{-i}.

$$\begin{aligned} V_{-i}(\Phi_{-i}^{-1}(H_{-i}(b)), t_i) &= \frac{\rho\left[\Phi_{-i}^{-1}(H_{-i}(b))\right]}{V'(b)} \\ \implies H_{i,\sigma}(b) = c_i &+ \int_0^b \frac{V'(x)}{\rho\left[\Phi_{-i}^{-1}(H_{-i}(x))\right]} dx \end{aligned} \tag{D.17}$$

Only one player can have an atom at zero in equilibrium. Now suppose $c_1 = 0$ then one can calculate the atom c_2.

$$\begin{aligned} 1 = H_{1,\sigma}(x^\sigma) &= \int_0^{x^\sigma} \frac{V'(x)}{\rho\left[\Phi_2^{-1}(H_2(x))\right]} dx \\ &= \int_0^{x^\sigma} \frac{V'(x)}{\rho\left[\Phi_2^{-1}(c + \frac{1}{p_1} V(x))\right]} dx \\ &= \int_{c_2}^{1} \frac{p_1}{\rho\left[\Phi_2^{-1}(u)\right]} du \end{aligned} \tag{D.18}$$

Where the last equality follows from the change of variables $\varphi(x^\sigma) = c_2 + \frac{1}{p_1} V(x^\sigma)) = 1$ such that $\varphi'(x)dx = \frac{1}{p_1} V'(x))dx$ and $f(u) = \frac{1}{\rho\left[\Phi_2^{-1}(u)\right]}$.

$\square$

Corollary D.2.8. *If $p_1 > p_2$ is known then $c_2 = 1 - \frac{p_2}{p_1}$.*

[7] To be precise, the common-value of the prize depends only on the signals s_1 and s_2 and not the entire type, t_1 and t_2. However, In order to use the signal notation in this proof, I would have to introduce another projection map, which is costly notation. Furthermore, the signals are not going to play any role in determining the size of atom.

Proof.

$$\begin{aligned}
\int_{c_2}^{1} \frac{p_1}{\rho\left[\Phi_2^{-1}(u)\right]} du = \int_{c_2}^{1} \frac{p_1}{p_2} du = 1 \\
\iff (1 - c_2)\frac{p_1}{p_2} = 1 \\
\iff c_2 = 1 - \frac{p_2}{p_1}
\end{aligned} \tag{D.19}$$

□

D.3 Proof of Continuum in Pure Common-Value Model

Proposition D.3.1. *When $\upsilon = 1$ or $\upsilon = 0$ there exists a continuum of equilibria in the pure common-value Model.*

Proof. If i plays peace with certainty, then any cut off strategy from $-i$ is an equilibrium. That is, for any cutoff type k, all types $j < k$ play peace with certainy: $\sigma_{-i}(peace|s_{-i}^{j}) = 1$. All types $j > k$ play warr with certainty: $\sigma_{-i}(war|s_{-i}^{j}) = 1$. Further let type k play an arbitrary probability distribution over war and peace: $\sigma_{-i}(war|s_{-i}^{k}) \in [0,1] \implies 1 - \sigma_{-i}(peace|s_{-i}^{k})$.

Since s_{-i}^{k} is the lowest type in the posterior choosing war, she gets zero payoff choosing war by Lemma 3.4.5. Therefore, s_{-i}^{k} is indifferent between war and peace. In a monotone equilibrium of the all-pay auction all types $j > k$ don't have zero bid in their bid-support. Hence, for all bids in their support, they have a strictly positive expected payoff by Lemma D.1.3. Therefore, these types have no incentive to deviate from war.

The types $j < k$ can also not benefit from war. If they choose war, the opponent believes this type to be k or higher. Bayes rule applies and by Lemma D.1.3, j has a negative payoff for all $b > 0$. Thus, the best j can do is bid zero, which implies zero payoff due to lack of atoms at zero from the opponent.

There this cutoff strategy is an equilibrium. For each k, there is continuum of values of $\sigma_{-i}(war|s_{-i}^{k}) \in [0,1]$ that yield an equilibrium. Notice that this is true for any type k. This means that there are n_{-i} times infinity equilibria.[8]

□

[8]Notice that there are no equilibria where types $j < k$ also mix. Even though the types below the cutoff are indifferent, if they mix, type k has an incentive to deviate to war with certainty. Hence. only the cutoff type can mix.

D.4 Proof of Theorem 3.5.2

Let $\omega_i(s_i^m)$ denote expected payoff deviating to war for type s_i^m of player i.

Lemma D.4.1. *When $p_1 > p_2 = 1$ is known, player 1 has a strictly positive expected payoff picking war. Futhermore, $\omega_1(s_1^m) = \sum_{S_2} V(s_1^m, s_2)\mu(s_2|s_1^m)\sigma_{2,(w,p)}(b = 0|s_2)$.*

Proof. By Lemma D.2.6, player 2 has an atom at zero. Hence, s_1^m can bid arbitrary close to zero and her expected payoff is at least (D.20).

$$\omega_1(s_1^m) \geq \sum_{S_2} V(s_1^m, s_2)\mu(s_2|s_1^m)\sigma_{2,(w,p)}(b = 0|s_2) > 0 \tag{D.20}$$

If player 1 deviates, then $\mu(s_1^{n_1}) = 1$. Which means that in the all-pay auction player 2 is going to keep $s_1^{n_1}$ indifferent over all bids in $[0, x^\sigma]$. Because, $s_1^{n_1}$ can get atleast (D.20, where $m = n_1$), and she is indifferent between all other bids, you get (D.21).

$$\begin{aligned} U(s_1^{n_1}, b) - U(s_1^{n_1}, 0) &= \sum_{S_2} V(s_1^{n_1}, s_2)\mu(s_2|s_1^{n_1}) \left(\sigma_{2,(w,p)}(b|s_2) - \sigma_{2,(w,p)}(0|s_2)\right) - b = 0 \\ \iff b &= \sum_{S_2} V(s_1^{n_1}, s_2)\mu(s_2|s_1^{n_1}) \left(\sigma_{2,(w,p)}(b|s_2) - \sigma_{2,(w,p)}(0|s_2)\right) \end{aligned} \tag{D.21}$$

Because the payoffs display increasing differences when KMS is satisfied (KMS implies $V(s_1^m, s_2)\mu(s_2|s_1^m) - V(s_1^{n_1}, s_2)\mu(s_2|s_1^{n_1}) < 0$). Using KMS and (D.21) you get (D.22).

$$\begin{aligned} U(s_1^m, b) - U(s_1^m, 0) &= \sum_{S_2} V(s_1^m, s_2)\mu(s_2|s_1^m) \left(\sigma_{2,(w,p)}(b|s_2) - \sigma_{2,(w,p)}(0|s_2)\right) - b \\ \iff \sum_{S_2} &\left[V(s_1^m, s_2)\mu(s_2|s_1^m) - V(s_1^{n_1}, s_2)\mu(s_2|s_1^{n_1})\right] \left(\sigma_{2,(w,p)}(b|s_2) - \sigma_{2,(w,p)}(0|s_2)\right) < 0 \end{aligned} \tag{D.22}$$

Thus, for any $b > 0$:

$$\omega_1(s_1^m) \leq \sum_{S_2} V(s_1^m, s_2)\mu(s_2|s_1^m)\sigma_{2,(w,p)}(b = 0|s_2) \tag{D.23}$$

□

Lemma D.4.2. *A peace equilibrium exists in which $\upsilon = 1$ (player 2 gets nothing).*

Proof. You can use the same argument in D.4.1 to see that $\omega_2(s_2^m) = 0$ for all m because player 1 has no atom at zero. Because of this, player 2 is indifferent between peace and war.

If player 1 abides by peace she gets:

$$\mathbb{E}[V|s_1^m] \geq \sum_{S_2} V(s_1^m, s_2)\mu(s_2|s_1^m)\sigma_{2,(w,p)}(b=0|s_2) = \omega_1(s_1^m) \tag{D.24}$$

□

Theorem 3.5.2 For a fixed peace deal υ, a peace equilibrium exists iff for all types of player 1 $m \leq n_1$ the inequality in (D.25) holds.

$$\upsilon \geq \frac{\sum_{s_2^j < \Phi_2^{-1}(1-\frac{p_2}{p_1})} \mu_{j|m} V_{mj} + \mu_{k|m} V_{mk}\, \sigma_{2,h}(0|\Phi_2^{-1}(1-\frac{p_2}{p_1}))}{\mathbb{E}[V|s_1^m]}. \tag{D.25}$$

Proof. By Corollary D.2.8, we can get the atom which is denoted as c_2.

$$\omega_1(s_1^m) = \sum_{s_2^j < \Phi_2^{-1}(c_2)} \mu_{j|m} V_{mj} + \mu_{k|m} V_{mk} \sigma_{2,h}(0|\Phi_2^{-1}(c_2)) \tag{D.26}$$

The expected value of accepting peace is given by the following equation.

$$\upsilon * \sum_{s_2^j} \mu_{j|m} V_{mj} \tag{D.27}$$

Take the difference between deviating to war and abiding by peace, and solve for which υ this is negative yields (D.25). □

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