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On the Convexity of the Production Possibility Set under General Production Conditions

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ON THE CONVEXITY OF THE PRODUCTION POSSIBILITY SET UNDER GENERAL PRODUCTION CONDITIONS*

by

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The main purpose of this note is to derive for general production possibility sets (p-p sets) sufficient conditions for convexity in the usual and also in a certain stricter sense.

In the economy to be analysed s+t goods are produced by q firms $h = 1,\dots, q$ using as primary inputs r factors $i = 1,\dots, r$. The goods are either final goods ($j=1,\dots,s$) that will at least during the time period under consideration never enter the production process as inputs, or intermediate goods ($k=1,\dots,t$) that are (net) inputs of some firms and (net) outputs of others.[1] It is assumed that all factors and goods are homogeneous and perfectly divisible, and that the factors are in fixed positive supply.

We shall use the following notation:[2,3]

$x_h = (x_{h1},\dots,x_{hs}) \geqq 0$: vector of the h^{th} firm's output of final goods

$y_h = (y_{h1},\dots,y_{ht})$: vector of the h^{th} firm's net input ($y_{hk} < 0$) or net output ($y_{hk} > 0$) of intermediate goods

$v_h = (v_{h1},\dots,v_{hr}) \leqq 0$: vector of the h^{th} firm's factor inputs

$w_h = (x_h, y_h, v_h)$: input-output vector of the h^{th} firm

$X = \sum_h x_h \geqq 0, \quad Y = \sum_h y_h, \quad V = \sum_h v_h \leqq 0, \quad W = (X, Y, V)$

$\bar{V} = (\bar{V}_1,\dots,\bar{V}_r) < 0$: vector of the total fixed factor supply

S_h = the h^{th} firm's production set

$S = \{W : W = \sum_h w_h, \; w_h \in S_h, \; h = 1,\dots,q\}$: the economy's production set

$T = \{(X,Y) : (X,Y,V) \in S, \; V \geqq \bar{V}\}$: the economy's production possibility set

We shall restrict our analysis largely to closed, convex individual production sets that also satisfy at least some of the following assumptions:

Assumption PI (possibility of inaction): $0 \in S_h$.

Assumption FD (free disposal): $w_h \in S_h$ implies $w_h^* \in S_h$ for all $w_h^* \leqq w_h$.

Assumption II (indispensability of some input): $w_h \geq 0$ implies $w_h \notin S_h$.

Assumption IPF (indispensability of primary factors): $w_h \in S_h$ with $v_h = 0$ implies $w_h \leqq 0$.

Assumption QB (quasi-boundedness from above): the subset of S_h with $v_h \geqq \bar{v}_h$ is bounded from above for every fixed $\bar{v}_h \leqq 0$.

Assumption BLI (boundedness with limited inputs): the subsets of S_h for which all input components do not exceed fixed finite limits are bounded.

Assumption SC (s-convexity with respect to the j^{th} component): S_h is convex and, in addition, for every pair of vectors w_h^1, w_h^2 in S_h with $w_{hj}^1 \neq w_{hj}^2$ and every α with $0 < \alpha < 1$ there exists a vector δ^{α} the j^{th} component of which is positive while all others vanish such that $\alpha w_h^1 + (1-\alpha)w_h^2 + \delta^{\alpha} \in S_h$.

Definition 1: A set being s-convex with respect to all components of its vectors will simply be called s-convex.

Assumptions II and BLI are hardly debatable: they exclude the possibility of the Land of Cockaigne. Assumption FD will not seriously restrict the scope of our analysis as long as externalities do not enter the picture. Later, however, we shall analyse certain types of external effects that can be represented, as will be shown, by "dummy" intermediate inputs for which free disposal has to be ruled out.

Supposing that every production process needs some primary input seems to be reasonable if we include all kinds of labour into the group of primary factors.[4] Thus assumption IPF is stronger than the assumptions made in, e.g., [1,5] that neither a single firm nor the economy as a whole can reverse any production process (i.e., $S_h \cap -S_h \subset \{0\}$, $S \cap -S \subset \{0\}$) and, hence, is also connected to some extent with the closedness of the economy's production set S and p-p set T (cf. [1], p.41).

Assumption QB together with IPF to which it is closely related implies indispensability of primary factors in a rather strict sense: with limited amounts of primary inputs a firm can only produce limited amounts of its outputs whatever amounts of intermediate inputs it uses. In the case of one primary and one intermediate input and CES production functions the elasticity of substitution has to be positive but less than unity for QB to be satisfied. This indicates that QB poses a rather severe restriction on our analysis.

Of all assumptions stated above only the last one, SC, introduces a somewhat new concept. Clearly, s-convexity falls in between convexity and strict convexity[5] of sets. For example, assume that the sets in Figure 1 are closed.[6] Then all of them are convex but only M_1 is strictly convex and M_1- M_4 are s-convex. As will be seen later, the notion of s-convexity with respect to some or all output components will prove to be useful when discussing production and p-p sets. To clarify its meaning suppose that the production set S^f is defined by a continuous production

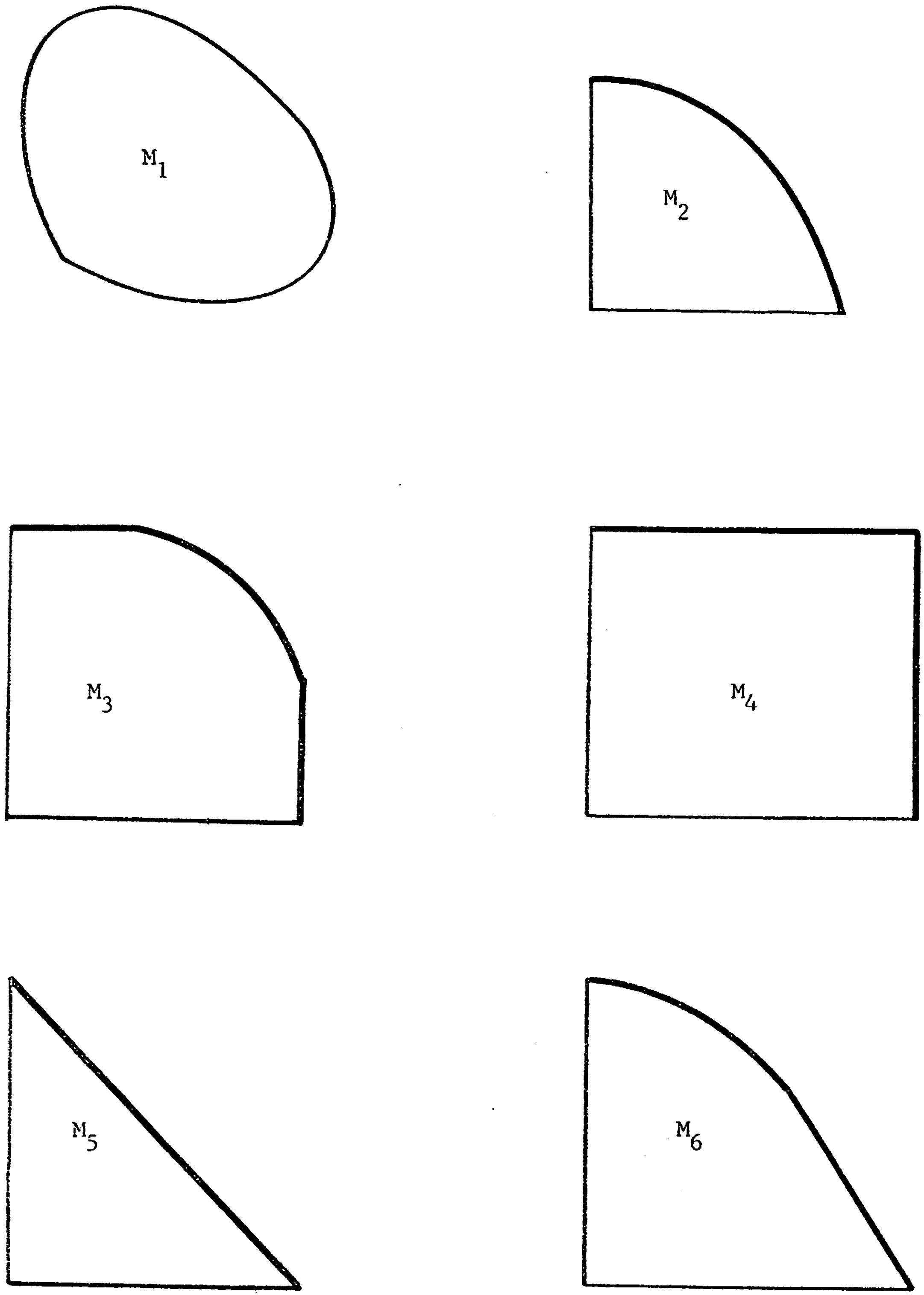

Figure 1

function f with a set K as domain of definition and that free disposal is possible:

$$S^f = \{(x,v) : (x,v) \leqq (x^*,v^*) \text{ with } x^* = f(-v^*), \text{ and } v^* \in K\}$$

It seems worth mentioning that only if the isoproduct surface $I_\alpha = \{v : f(-v) = \alpha\}$, $\alpha \geq 0$, is nowhere upward sloping in any direction it forms the boundary of the set $M_\alpha = \{v : (x,v) \in S^f, x \geq \alpha\}$. Otherwise, the possibility of free disposal implies that the upward sloping parts of I_α belong to the interior of M_α. In the case of two primary inputs both these possibilities are shown in Figure 2. (The heavy line represents an isoquant I_α and the hatched area the corresponding set M_α.)

It is fairly easy to prove that the following two propositions hold:

Proposition 1: Assume that f is a strictly quasi-concave[7] production function with a convex set K as domain of definition. Then all sets M_α, $\alpha \geq 0$, are s-convex. Moreover, the function f is concave or strictly concave if, and only if, the production set S^f is convex, or respectively, s-convex with respect to output. Finally, strict quasi-concavity of f and s-convexity of S^f with respect to output imply s-convexity of S^f with respect to all components.

Proposition 2: Assume that f is a continuously differentiable, homogeneous quasi-concave production function the domain of definition of which is a convex cone K with the origin as vertex. The degree of homogeneity of f is then not larger than unity (non-increasing returns to scale) or smaller than unity (decreasing returns to scale) if, and only if, the production set S^f is convex or, respectively, s-convex with respect to output.

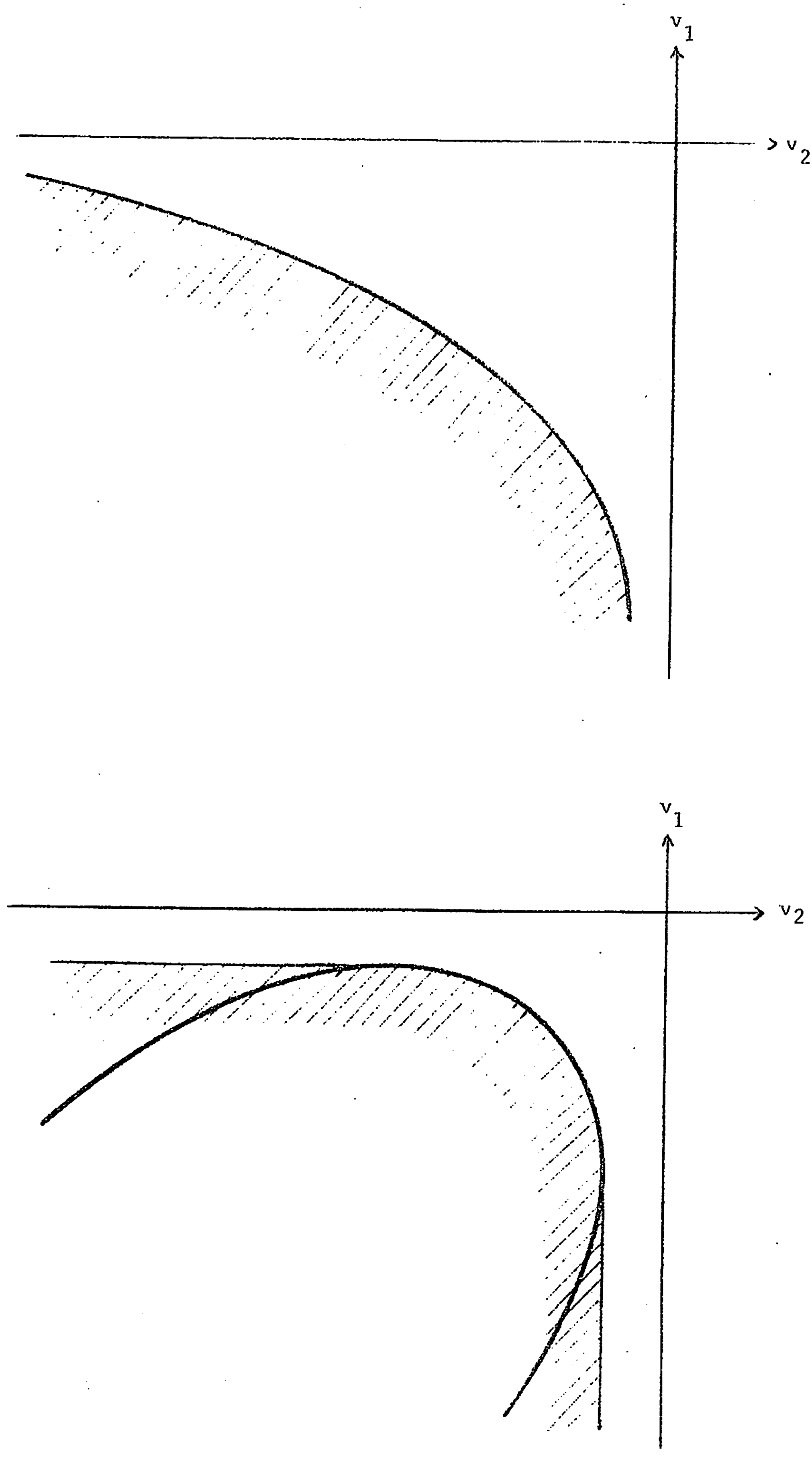

Figure 2

Professor Lancaster defines in [5], p. 136, that a production set S shows weakly decreasing returns to scale if it contains αw for every $w \in S$ and $0 \leq \alpha \leq 1$, but not for some $w \in S$ and $\alpha > 1$.

Proposition 3: A sufficient but not necessary condition for a production set S to display weakly decreasing returns to scale is that S satisfies assumptions PI and FD and is s-convex with respect to all components.

The proof is simple and, therefore, will be omitted. It seems natural to define non-increasing and decreasing returns to scale under general production conditions in such a way that these terms have the usual meaning in case of quasi-concave, homogeneous functions and that, moreover, decreasing returns are a special form of weakly decreasing returns.

Definition 2: A production set S satisfying assumption PI and being convex is said to show non-increasing returns to scale (for all goods).[8] If, in addition, S is s-convex with respect to certain goods it is said to display decreasing returns for these goods.[9]

To state a result concerning the shape of a s-convex p-p set we first have to introduce some new terms:

Definition 3: The p-p surface T^s of a closed p-p set consists of all weakly efficient points in T, i.e., all points $(X,Y) \in T$ such that there exists no $(X^*,Y^*) \in T$ with $(X,Y) < (X^*,Y^*)$.[10]

Definition 4: The p-p surface T^s of a closed p-p set T is called quasi-strictly convex from above if it is convex from above and if, in addition, every downward sloping hyperplane that is the boundary of a half-space $(p_X, p_Y)(X,Y)' \leq 0$, $(p_X, p_Y) \geq 0$, containing T has at most one point in common with T^s.[11,12]

<u>Definition 5</u>: A p-p set T is called complete if $(X,Y) \in T$ implies $(X^*,Y^*) \in T$ for all $(X^*,Y^*) \leqq (X,Y)$ and $X^* \geqq 0$.

<u>Proposition 4</u>: A closed, complete, convex p-p set T is s-convex if, and only if, its p-p surface T^s is quasi-strictly convex.

The central result to be proved in this note is:

<u>Lemma 1</u>: (i) <u>The production possibility set of an economy is complete and convex, or, respectively, s-convex if the production sets of all firms within the economy allow for inaction and free disposal and are convex or, respectively, s-convex with respect to the components representing goods, i.e., if assumptions PI and FD are satisfied and all goods are subject to non-increasing or, respectively, decreasing returns to scale</u>.

(ii) <u>The production possibility is closed if the production sets are closed and convex and if primary factors are indispensable</u>.

(iii) <u>The production possibility set is bounded from above if the production sets are quasi-bounded from above and primary factors are indispensable</u>.

(iv) <u>Every subset of the production possibility set with $Y \geqq \bar{Y}$ where $\bar{Y}$ is some fixed, finite vector of intermediate goods is bounded if some input is always indispensable and the individual production sets are bounded for limited inputs</u>.

Proof:

(i) a) Completeness of the p-p set T follows immediately from the fact that the production sets $S_1,\ldots,S_q$ satisfy assumptions PI and FD.

b) Convexity: Suppose (X^1,Y^1), $(X^2,Y^2) \in T$. Set $(X^\alpha,Y^\alpha) = \alpha(X^1,Y^1) + (1-\alpha)(X^2,Y^2)$. By definition of T there exist for p=1,2 input-output vectors $w_h^p \in S_h$, h=1,...,q with $\sum_h w_h^p = (X^p,Y^p,V^p)$ and $V^p \geq \overline{V}$. The convexity of S_h implies $w_h^\alpha = \alpha w_h^1 + (1-\alpha) w_h^2 \in S_h$, h=1,...,q, and $\sum_h v_h^\alpha \geq \overline{V}$ for $0 \leq \alpha \leq 1$. Since obviously $X^\alpha = \sum_h x_h^\alpha$, $Y^\alpha = \sum_h y_h^\alpha$ we find $(X^\alpha,Y^\alpha) \in T$ for $0 \leq \alpha \leq 1$. Thus T is convex.

c) s-Convexity: Suppose (X^1,Y^1) and (X^2,Y^2) are different vectors in T. Define (X^α,Y^α) and (x_h^p, y_h^p, v_h^p), p=1,2,α, as above. Clearly, $X_j^1 \neq X_j^2$ implies $x_{hj}^1 \neq x_{hj}^2$ for at least one h and, similarly, $y_{hk}^1 \neq y_{hk}^2$ for at least one h follows from $Y_k^1 \neq Y_k^2$. Suppose $S_1,\ldots,S_q$ are s-convex with respect to the components representing goods. Then for each α with $0 < \alpha < 1$ there exist non-negative vectors δ_h^α and η_h^α of dimension s and t, respectively, the components of which are equal to zero except that $\delta_{hj}^\alpha > 0$ if $x_{hj}^1 \neq x_{hj}^2$ and $\eta_{hk}^\alpha > 0$ if $y_{hk}^1 \neq y_{hk}^2$ such that $(x_h^\alpha + \delta_h^\alpha, y_h^\alpha + \eta_h^\alpha, v_h^\alpha) \in S_h$. Set $\Delta^\alpha = \sum_h \delta_h^\alpha$, $H^\alpha = \sum_h \eta_h^\alpha$. The components of Δ^α and H^α equal zero except that $\Delta_j^\alpha > 0$ if $X_j^1 \neq X_j^2$ and $H_k^\alpha > 0$ if $Y_k^1 \neq Y_k^2$. By construction, $(X^\alpha + \Delta^\alpha, Y^\alpha + H^\alpha) \in T$ for $0 < \alpha < 1$. Hence, T is s-convex.

(ii) Closedness. We recall that if the individual production sets satisfy assumptions IPF and PI production is irreversible and inaction possible on an economy-wide level (i.e., $S \cap -S = \{0\}$). This and the convexity and closedness of $S_1,\ldots,S_q$ implies that S is closed (cf. [1], pp. 41/2). Hence, the set $\hat{S} = \{W : W \in S, V \geq \overline{V}\}$ is closed. Since there exists a trivial continuous mapping of $\hat{S}$ onto T the p-p set is also closed.

(iii/iv) Boundedness: The assumptions made under (iii) and (iv) obviously imply the boundedness properties of T as stated in the lemma.

Q.E.D.

The proof indicates that we established only sufficient but not necessary conditions for the various properties of the p-p set. However, we shall not try to find weaker conditions. We also note that as far as our results are concerned there need not exist either final or intermediate goods while there must be at least one primary factor. Moreover, slight modifications of the proof will show that the following propositions are correct:

Corollary 1: s-convexity of all individual production sets with respect only to a certain group of goods implies s-convexity of the production possibility set with respect to these goods.[13]

Corollary 2: The economy's production set will be s-convex with respect to at least those goods with respect to which all individual production sets are s-convex.[14]

It seems worth mentioning that our assumptions do not guarantee that the p-p surface, if it is defined, has no parts being parallel to one or more coordinate axes (and thus do not rule out p-p sets like M_3 and M_4 in Figure 1). At first sight assuming full employment of all primary factors everywhere on the p-p surface might appear to fill this gap. However, little reflection shows that full employment is neither a necessary nor sufficient condition for the p-p surface having no parts parallel to coordinate axes.[15]

Comparing propositions (iii) and (iv) in lemma 1 the latter is of more interest since as long as nowhere in the world there exists a Land of Cockaigne, i.e., as long as everywhere assumptions II and BLI are satisfied, and, moreover, primary factors are in limited supply, (iv) implies that the world's p-p set is bounded. Hence the maximum total world output of any good and thus the maximum quantity of any intermediate good a single economy could produce or import are finite. This implies that the relevant part of the economy's p-p set is subject to a restriction of the kind $Y \geq \bar{Y} > \infty$ and, consequently, is bounded. Under those circumstances we need not worry that the whole p-p set might be not bounded from above.

Lemma 1 generalizes and unifies most of the results concerning the convexity, closedness and boundedness properties of p-p sets that have been published so far.

For example, nearly all of the propositions about the convexity of these sets proved, e.g., in [2, 3, 5, p. 133, 6, 7, 8, 10] for homothetic production functions with constant or decreasing returns to scale are covered by lemma 1, part (i).[16] Only the sufficient conditions for strict convexity from above of p-p curves or surfaces for linear homogeneous functions in [2, 4] are outside the scope of our lemma.[17] Furthermore, the non-homothetic production functions analysed by Professor Melvin [9] define production sets that are s-convex with respect to output, and hence the conclusion that the corresponding p-p curve is strictly convex from above follows from lemma 1 since evidently this curve is downward sloping throughout. Finally, Lancaster's (sufficient) Condition I ([4], p. 130) for a p-p surface in the case of general production functions to be convex from above is also a consequence of our results while we have not derived a counterpart of his (necessary) Condition II (ibid.).[18]

In the case of homothetic production functions and in the absence of intermediate goods the closedness and boundedness of p-p sets has found no special attention since both properties are trivial consequences. If, however, intermediate goods enter the scene the situation is not that simple. This has been lucidly discussed by Melvin [7] for an economy with two intermediate and no final goods, one indispensable primary factor, labour, and linear homogeneous production functions. His results concerning the closedness and boundedness of the p-p set are more general than ours since he need not make assumption QB or BLI but can rely on assumption IPF only as this has for linear homogeneous functions rather strong implications.

It will be noted that in some respects the sufficient conditions for convexity or s-convexity of the p-p set are quite weak since they are compatible with, for example,

(1) single firms producing one or more goods using any number of intermediate goods or primary factors as inputs;

(2) the output proportions of a firm being either variable or fixed;[19]

(3) different firms having identical, partly overlapping or totally different production programs;

(4) a primary factor or an intermediate good being used as an input by only some firms;

(5) an intermediate good being produced by a firm under some circumstances but used as an input under different ones;

(6) primary factors being underemployed or firms remaining idle even if the economy's resources are optimally allocated.

At first sight a major deficiency of our approach might seem to be that it does not cover the case of externalities in production. However, a simple trick enables us to introduce output-generated externalities[20,21] into the picture. The following example shows how that can be done:

Consider the case of q industries each producing one different final good using only primary inputs. Suppose the output x_{hh} of the h^{th} industry depends on the outputs of other industries and that this relationship is as follows:

$$x_{hh} = g_h(\xi_h)\, F_h(-v_h), \qquad F_h(0) = 0,\ h=1,\ldots,q \tag{1}$$

with

$$\xi_{hh} = 0, \quad \xi_{hk} = x_{kk}, \quad k \neq h \tag{2}$$

where the F_h's are continuous, quasi-concave production functions and the g_h's continuous, positive functions representing the existing externalities. At a given level of the economy's activities described by the total output vector $X = (x_{11},\ldots,x_{qq})$ the h^{th} industry is (locally) subject to external economies or, respectively, diseconomies originating in the k^{th} industry if g_h is a monotonically increasing or, respectively, decreasing function of x_{kk} in the neighbourhood of X.

Now assume that the h^{th} industry produces as a perfect complement of its final good, without using additional input amounts, a "dummy" non-marketable intermediate good the output y_{hh} of which is equal to $(q-1)x_{hh}$. Supposing that equal parts of y_{hh} are forced as inputs upon all other industries we can rewrite the production functions in (1) as follows:

$$x_{hh} = \frac{1}{q-1}\, y_{hh} = g_h(-\eta_h)\, F_h(-v_h) = G_h(-\eta_h, -v_h), \quad h = 1,\ldots,q \tag{3}$$

with

$$\eta_{hh} = 0, \quad \eta_{hk} = -\frac{1}{q-1}\, y_{kk} = -x_{kk}, \quad k \neq h \tag{4}$$

Denote by x_h the vector consisting of zeros except for the h^{th} component which equals x_{hh} and by y_h the vector having y_{hh} as its h^{th} and y_{kk} as its k^{th} component ($h \neq k$). Then the individual production sets and the p-p set can be defined as follows:

$$S_h^F = \{(x_h, v_h) : (x_h, v_h) \leqq (x_h^*, v_h^*) \text{ with } x_h^* = F_h(-v_h^*),\ v_h^* \in K \quad (5)$$

$$S_h^G = \{(x_h, y_h, v_h) : (x_h, y_h, v_h) \leqq (x_h^*, y_h, v_h^*)$$

$$\text{with } x_h^* = \frac{1}{q-1} y_{hh} = G_h(-\eta_h, -v_h^*)\} \quad (6)$$

$$T^G = \{X : X = \sum_h x_h,\ \sum_h y_h \geqq 0,\ \sum_h v_h \leqq \bar{V}$$

$$\text{with } (x_h, y_h, v_h) \in S_h^G,\quad h = 1, \ldots, q\} \quad (7)$$

Clearly, the production set S_h^G is always closed and it will be convex or, respectively, s-convex with respect to the final goods components if, and only if, the function g_h is concave and the production set S_h^F is convex or, respectively, s-convex with respect to the final goods components. Moreover, the p-p set T^G is obviously complete and bounded.

Lemma 2: Suppose there are q industries each producing one different final good using only primary inputs. Suppose that the production functions of these industries are described by Equations (1) and (2) where the function F_h shows the influence of primary inputs on the output of the h^{th} industry and the function g_h reflects the output-generated external effects originating in the other industries to which the h^{th} industry is subject. There the economy's production possibility set is complete, bounded, closed and also convex or, respectively, s-convex if the functions g_h are concave and the production sets defined by the function F_h are convex or, respectively, s-convex with respect to final goods.

Notes:

* I am grateful to Murray Kemp and Jim Melvin for helpful suggestions and criticism on an earlier draft. All errors are, of course, solely mine.

1. Making a distinction between final and intermediate goods follows a tradition, e.g., in international trade theory. In the present context it could be abolished but it is nevertheless retained since it has some expositional advantages.

   In passing it should be mentioned that an intermediate good may be put to final use, e.g., consumed, provided its total net output is positive. But that does not mean that it is considered partly as an intermediate and partly as a final good. On the other hand, part of the current output of a final good may be being added to the next period's supply of some primary factor. Hence, primary factors are either non-reproducible or their supply cannot be increased out of current production.

2. We follow the traditional convention by attributing positive values to (net) outputs and negative values to (net) inputs.

3. Suppose $y = (y_1,\ldots,y_n)$ is a real-valued vector. As usual, $y \geqq 0$ means $y_i \geq 0$, $i = 1,\ldots,n$, $y \geq 0$ means $y \geqq 0$ and $y \neq 0$, and $y > 0$ means $y_i > 0$, $i = 1,\ldots,n$.

4. For example, J. R. Melvin [7] assumes that labour which he considers as the only primary input is indispensable.

5. A closed set M is called strictly convex if for $x^1, x^2 \in M$, $x^1 \neq x^2$, all points $\alpha x^1 + (1-\alpha)x^2$, $0 < \alpha < 1$, are interior points of M, i.e., belong to M but not to its boundary.

$$H'' = \frac{H}{z}\left(\frac{d\eta}{dz} + \frac{\eta}{z}(\eta - 1)\right)$$

$H'' < 0$ for $z > 0$ neither does imply nor is implied by $0 < \eta < 1$ for $z > 0$, except if $d\eta/dz = 0$, i.e., G is homogeneous.

The new definition has been introduced here since it seems superior to the old one: it is more in accordance with Lancaster' and Debreu's definitions referred to above, it allows for a simpler formulation of the main result of this paper (cf. lemma 1, (i)), and it also could be used to re-state Theorems 2 and 2' in [3] in such a way that the relationship between the properties of homothetic production functions and the shape of the p-p surface becomes clearer.

10. A point $(X,Y) \in T$ should be called efficient if there exists no point $(X^*,Y^*) \in T$ with $(X,Y) \leq (X^*,Y^*)$. Suppose M_2- M_6 in Figure 1 represent p-p sets of an economy producing only two final goods. The p-p curves then are the heavily drawn-out parts of the boundaries. Clearly, the p-p curves of M_2, M_5 and M_6 consist of efficient points only while the p-p curves of M_3 and M_4 contain also weakly efficient points.

11. While the p-p curves of all the p-p sets M_2 - M_6 shown in Figure 1 are convex from above only those of M_2 - M_4 are quasi-strictly convex and that of M_2 is even strictly convex.

12. It can be shown that the set of efficient points on a quasi-strictly convex p-p surface is strictly convex from above.

13. This result can be readily applied, e.g., in case of an economy in which several final goods are produced by different single-product industries using only primary factors and having quasi-concave, homothetic production functions some of which display constant and others decreasing returns to scale.

14. The well known result (cf. e.g., [1], p. 42) that convexity of all individual production sets implies the convexity of the economy's production set follows once again from corollary 2.

15. Moreover, full employment of primary factors on the p-p surface should rather be a consequence of other assumptions than an assumption itself.

16. Using the same notation as in note 9 and setting

$$g = \frac{x}{H^{-1}}, \quad \omega = \frac{dg}{dx}\frac{x}{g}$$

where H^{-1} is the inverse of H we find

$$\eta = \frac{1}{1-\omega}, \quad \frac{d\eta}{dz} = \frac{\eta x}{z(1-\omega)^2}\frac{d\omega}{dx}$$

and thus

$$\text{sign } H'' = \text{sign } \left(\frac{d\eta}{dz} + \frac{\eta}{z}(\eta-1)\right) = \text{sign } \left(\frac{\omega}{x} + \frac{1}{1-\omega}\frac{d\omega}{dx}\right)$$

Obviously, the term within the last brackets is equivalent to the "critical" terms in Equation (26) of [3]. Taking note 9 into account it is immediately clear that theorems 2(b) and 2'(b) in [3] follow from lemma 1 (i).

17. However, taking the special properties of production sets defined by linear homogeneous production functions with constant or decreasing returns to scale into account one can give a topological proof similar to that for lemma 1 for the stronger propositions about the p-p set corresponding to such functions as stated in [2]. Cf. [3], p. 406.

18. Lancaster refers to convex p-p surfaces when stating these two conditions but in fact they are sufficient or, respectively, necessary for strict convexity to prevail.

19. Thus our approach covers possible internal economies or diseconomies arising from joint production within a firm.

20. Since factor-generated externalities are more or less a dynamic phenomenon they cannot be treated by a static analysis of the kind used here.

21. Effects that are external to single firms but internal to their respective industries have been analysed in [3].

22. Cf. also [2].

23. In the notation of note 9 the homothetic production function G displays increasing returns to scale wherever H'' [illegible] 0.

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