5-16-2022 1:00 PM

Essays On Market Design And Auctions

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A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Economics

Abstract

My thesis consists of three chapters contributing to redistribution-driven market design and sponsored link auctions.

Chapter 2 and 3 (co-authored with Charles Zheng) study redistribution-driven market design with endogenous buyers and sellers. Chapter 2 considers a large market environment with each individual endowed with equal shares of limited resources and allowed to buy or sell the shares. We characterize the interim (incentive-constrained) Pareto frontier subject to market clearance and budget balance, and find that at most two prices are needed to attain any (interim) Pareto optimum. Under robust conditions of the primitives, the Pareto optimal allocation is unique, and a single price — without the help of rationing or lump-sum transfer — implements the optimal allocation. We find which types gain, and which types lose, when the social planner chooses a rationing mechanism over the single-price solution, as well as which type's welfare weight is crucial to the choice. The finding suggests a market-like mechanism to distribute Covid-vaccines optimally among the population that belongs to the same priority group.

In Chapter 3, we study a quasilinear independent private values set-up to allocate a commonly desirable item (*the good*) and a commonly undesirable item (*the bad*). We prove a necessary and sufficient condition for all interim Pareto optimal mechanisms to allocate the bad with strictly positive probability, despite that not allocating it at all is part of an ex-ante incentive efficient mechanism. The condition holds when types near the low end carry sufficiently high welfare densities. Replacing the welfare weight distribution by a second-order stochastically dominated one improves the prospect of the condition. The Kuhn-Tucker method in the literature is inapplicable because when our condition holds, the monotonicity constraint the method sets aside is binding unless the method suffers indeterminacy in admitting a continuum of solutions to the relaxed problem.

Chapter 4 investigates a sponsored link auction game in which consumers search one set of products (*block*) before the other, and sellers compete in bids to place their product links to the first block. Consumers are assumed to be unaware of the products in the second block when searching in the first block, search in each block optimally, according to Weitzman (1979), and update the current best option during the search. I characterize consumers' shopping outcomes with block-by-block search behavior. Letting sellers choose product prices and auction bids together, I find the equilibrium of the complete information second price auction with two payment schemes: fixed payment and per-transaction payment. I find auction revenue and consumer surplus are larger under the fixed payment, and seller profits are larger under the per-transaction payment because the latter distorts the winner's pricing strategy. If a social planner runs the platform, I find a consumer optimal positioning of products if sellers commit to prices before the position allocation.

Keywords: Market Design, Redistribution, Interim Incentive Efficiency, Vaccine Distribution, Online Shopping, Advertising

Summary for Lay Audience

A market allocates items to the individuals with the highest valuation. Since the valuations are unknown (private information) to a planner, the planner needs to design the allocation and payment rules to reveal the private information to achieve allocation efficiency. When people value money differently (i.e., the poor value a dollar more than the rich on average), the payment rule in a market becomes a tool to redistribute among individuals. Price control is often observed in markets with poor participants, e.g., price floor for agricultural products and rent ceiling for student accommodation.

Chapter 2 and chapter 3 (co-authored with Charles Zheng) study the redistribution-driven design problem. Chapter 2 considers a large market setting where individuals are endowed with equal shares of limited resources and are allowed to buy or sell the shares. We find that the optimal market structure has at most two prices. It is either a competitive market or involves rationing on one side of the market. The finding suggests using a simple market structure to allocate covid vaccine efficiently.

Chapter 3 studies the redistribution problem in an auction design setup. The planner has a commonly desirable item (*the good*) and a commonly undesirable item (*the bad*) to allocate. Although the bad itself imposes a cost to society and allocating it is not mandatory, we find a planner can use the bad to achieve redistribution. A sufficient condition to allocate the bad with a positive probability is when the planner sufficiently cares for the poor, because it increases the transfer from the rich to the poor.

In Chapter 4, I study sponsored link auctions used by online shopping platforms like Amazon and eBay. The platforms use the auction to sell the top positions on their webpages as advertisement spots, so how consumers search online is crucial to determine the value of the top positions. I incorporate a flexible search behavior and find that the surplus split between the participants (i.e., the platform, buyers, and sellers) depends on the payment rules of the auction. The platform and the buyers prefer a fixed bid payment rule, while the sellers prefer a transaction-based bid payment rule.

Co-Authorship Statement

This thesis contains co-authored material. Chapter 2 and 3 are co-authored with Charles Zheng. All authors are equally responsible for the work.

A version of Chapter 2 is resubmitted to *Economic Theory*. Chapter 3 and 4 are working papers.

Acknowledgements

This thesis is a collection of the joint works with Charles Zheng on redistribution-driven mechanism design, coupled with a topic of my interest in sponsored link auctions. Writing down the collections rewinds me to the summer of 2018 when Charles and I discussed the Babylonia marriage market and started thinking about using mechanism design to redistribute between agents with different income levels. The development of my thesis benefits a lot from the discussions with Charles day and night. Charles is my advisor, coauthor, and my friend. There is no word to express how grateful I am to Charles. The piece of wisdom I learned from him shall always guide me in my life.

I would also like to express my sincere gratitude to my committee members, Bruno Salcedo and Victor Aguiar. Bruno gave me helpful advice on writing and presentations, enabling me to introduce my research to a lay audience. He shows me the necessity to be a good presenter as a researcher. Victor is the person who gave me the first micro class at Western and opened the gate of my path to be a micro theorist. His passion for research and knowledge on various topics spurs me to think more and be creative. He encourages me to pursue challenging fields, and he is always the exemplary researcher in my mind. Bruno and Victor's support is indispensable for me to finish my doctoral study and open a new page in my career as an economist.

My thesis benefits significantly from the microeconomics faculty at Western, including Gregory Pavlov, Maria Goltsman, and Al Slivinski. I would give special thanks to Audra Bowlus, Juan Carlos Hatchondo, and Peter Streufert for their helpful advice during my mock interviews. I am grateful for all the work done by the staff in the economics department. I cannot count the times Sandra helped me with the administrative document and job applications. Debra has helped us with various contracts and arranged facilities in our office.

Meeting my fellow students at Western is one of the best parts of my graduate study. Ali Kamranzadeh and Henk Schouten used to discuss papers with me in our reading group. Cecilia Diaz Campo, Martin Luccion, and Ali shared valuable information with me during the job search, and we enjoyed lunch together. I would like to thank my cohort, Cecilia Diaz Campo, Rowena Jane Cornelius, Charles Gauthier, Marco Pariguana, and Robert Millard. I miss the class time we spent together, and I learned a lot from them. I also owe many thanks to the supportive Chinese community, Yina Qiao, Yifan Gong, Yuxi Yao, Qian Liu, Zhuang Liu, Tian Liu, Zijian Wang, Ziyu Zhang, Xun Chen, Kun zhu, and my roommate Chong Wang.

I am deeply grateful to my parents and my partner. It has been a long time since I visited them last time. I know how badly they miss me as I miss them. Their encouragement and unconditional love support every step of my life. My parents let me know that they are always my safe harbor no matter how far I go. I am fortunate to have had my grandparents in my childhood. They taught me how to be a person, respect others, and be respected. Their influence on my personality will shine throughout my life. I am also grateful to my aunt and uncle, who gave me constant care during my Ph.D. study. The winter break we spent together is part of my best memory in Canada.

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Chapter 1

Introduction

The fundamental role of a market is to allocate resources efficiently. Yet we often observe price and transaction quantity regulations that potentially distort allocative efficiency. The government imposes regulations in markets where some participants are extremely poor. Standard regulations include the price floor to sell agricultural products and the rent ceiling of apartments for students and the disabled. Conceptually, these regulations protect the poor by either increasing their income or decreasing their spending. But less has been done theoretically on how regulations should be designed based on the planner's preference.

My thesis contains two chapters on redistribution-driven mechanism design. Chapter 2 and chapter 3 of my thesis, coauthored with Charles Zheng, study the problem with endogenous buyers and sellers. Recent literature on redistribution-driven market design started using the techniques in mechanism design to explain why the government should impose price and quantity control in specific markets to protect the poor. Relevant works include Akbarpour, Dworczak and Kominers [1], Dworczak, Kominers and Akbarpour (henceforth DKA) [3], and Kang [6]. They made sectorial restrictions such that an individual is predetermined as a buyer or seller. In contrast, we consider a different setup where the buyer-seller role of an individual is endogenous. Players' endogenous role rises in markets when they share public resources, like health care, road use, and stock shares (partnership dissolution). Chapter 2 considers a large market environment where each individual is endowed with equal shares of a limited resource, and each is allowed to buy or sell the shares. Our goal is to characterize the interim Pareto frontier subject to incentive compatibility, individual rationality, market clearing, and budget balance.

An optimal mechanism in such a setting needs to stratify the space of type, which is the player's private information, into more than two tiers. And each tie is coupled with a price, *rationing* that restrict the quality to supply or demand, and *lump-sum transfer* to achieve redistribution to a group of individuals. In DKA, the optimal allocation stratifies the type space into at most five tiers, with rationing on at most two of the tiers and a potential lump-sum transfer. In Kang [6], there exists an optimal allocation that stratifies the type space into at most four tiers, and rationing is necessary to attain optimality. In contrast to the literature above with sectorial restrictions, we find that the optimal mechanism with endogenous buyers and sellers is simpler with at most three ties and rationing at most one side of the market. Thus, the optimal mechanism is either a *posted price* equilibrium with a single price clearing the market or rationing on one side of the market.

The simplicity of our solution is driven by a new observation due to the players' endogenous role as buyers and sellers. The observation is that any incentive-compatible and market clearing allocation can always be implemented in a budget-balanced and individual rational manner. Thus, the last two constraints of our problem can be ignored without loss. With only incentive-compatible and market clearing valid, our problem is similar to a monopolist's second-price discrimination problem with a capacity constraint or the fundamental Bayesian persuasion problem.

We summarize the conditions determining how the optimal mechanism moves on the Pareto frontier when the primitives change. That is, which type gains, which type loses when the planner changes the market structure from rationing to posted-price. When the mechanism switches from the rationing to the posted-price one, all types sufficiently close to the market-clearing price are worse off. Moreover, if the type distribution is convex (concave) on the rationed interval, switching from rationing to posted-price makes the low (high) types better off. Thus, the planner prefers rationing to posted-price if she puts sufficiently high welfare weight on the types close to the market-clearing price, and prefers posted-price to rationing if she cares about the high types and the low types.

Our model is applicable to large-market exchange economies where individuals have equal entitlements to a limited resource and are heterogeneous in their willingness to pay for access to the resource. For example, the planner may issue vaccine coupons to individuals in the same priority group. Because of the shortage of vaccines, each coupon generates a positive but less than one probability of getting vaccinated. Our findings suggest a market-like mechanism to distribute the covid vaccine optimally.

Chapter 3 considers the problem of allocating a commonly desirable item, which we call *the good*, and a commonly undesirable item, which we call *the bad*, to a finite number of individuals with quasi-linear independent private value (cost) of the good (the bad). Allocating a bad happens when the government decides where to construct a Nimby ("Not in my backyard") type of facility, such as trash disposal plants and oil pipeline terminals, which is commonly perceived as undesirable to its host. A Nimby facility absorbs the environmental cost in a region and lets the rest of the community enjoy its benefits. So it has a nature of private bad and public good. The allocation of Nimby has been studied in a procurement auction setup by Kunreuther and Kleindorfer [7]. But the question that remains is whether the planner can use the private bad nature of the Nimby facility to facilitate the planner's redistribution goal. And if the planner can choose to decrease the cost of the bad, should the planner do so?

Charles and I answer the questions above in chapter 3. Similar to chapter 2, individuals have the endogenous role of buyers and sellers. However, the mechanism design problem with finitely many asymmetric players is more complicated than that in Chapter 2 because of the existence of aggregate uncertainty of players' type. We characterize the interim incentive efficiency (IIE) mechanism in the spirit of Holmström and Myerson's [4]. Characterizing IIE with both a good and a bad requires us to handle the discontinuity of the virtual surplus function at any type whose allocation switches from getting the bad in expectation to getting the good in expectation. This makes the virtual surplus function deviate from the standard linear functional of allocations.

Without the linear structure, the characterization of the optimal mechanism usually uses the local Kuhn-Tucker method. However, the local Kuhn-Tucker condition can not solve our problem as the method needs to be applied to a relaxed problem, setting aside the monotonicity constraint (e.g., Ledyard and Palfrey [5]). When a bad needs to be allocated, we find the monotonicity constraint is generically binding in our problem, and the Kuhn-Tucker condition is not valid in our setup. Our paper thus adopts a global approach and characterizes the optimal mechanisms by a saddle point condition, which is necessary and sufficient for any mechanism to be Pareto optimal. We obtain the saddle point condition by formulating a *two-part operator*, with one part integrating the positive part of an allocation and the other integrating the negative part. We find a necessary and sufficient condition to allocate the bad, despite that not allocating it at all is part of the ex ante incentive efficient mechanism. The condition holds when the types close to the low end of the type space carry sufficiently high welfare weight.

Chapter 4 of my thesis studies the sponsored link auction. Online shopping platforms like Amazon and eBay use the auction to sell the top positions on their web pages. Unlike a standard auction selling concrete items, sponsored link auctions sell advertisement positions, whose value depends on consumers' search behavior during web browsing. My thesis contributes to the literature by introducing two novel elements in the sponsored link auctions. First, I assume consumers have partial information on product values, observe product prices during the search, and use the information to search optimally. Second, sellers in my model choose both product prices and auction bids optimally to maximize their profits. The questions are: How do sellers' pricing strategies and bidding strategies interact in such an environment. And how the interaction affects the surplus split between consumers, sellers, and the shopping platform.

I find that the split of surplus depends on the auction payment rule. If the auction winner's payment is fixed in a lump-sum manner, sellers' pricing and bidding strategies are independent. However, if the auction winner pays to the platform every time the winner's product is sold, product prices increase under the equilibrium. Thus, a shift from the fixed payment to the per-transaction payment decreases consumer surplus and increases the sellers' profits. Moreover, I find that sponsored link auction revenue falls from changing the payment rules.

Consumer search in my model is based on Weitzman's [8] framework where an individual's value of an item is separated into two parts; one is known before the search, and the other needs to be discovered from the search. However, the search outcome in Weitzman is independent of product position since consumers are assumed to be aware of all options at the beginning of the search. Such an assumption is unrealistic when consumers' menu size is large, especially in the context of online shopping.

I introduce the role of product position by introducing the concept of product *blocks*. A block of products is a set of adjacent products on the search webpage. Blocks are ordered and mutually exclusive, such that a higher block represents a higher position. For example, the first block is the first page of the shopping web, and the second block is the second page. I assume consumers search the first block before the second and update the current best option during the search. The block-by-block search deviates from Weitzman's well-known solution in two ways: First, the search order gives higher priorities to products in the first block than Weitzman's solution would, because a consumer is unaware of any product in the second block at the start of the search. Second, the search in the second block stops earlier than Weitzman's solution because the consumer's fallback value is larger due to the search in the first block. These make the demand for any product larger if its link is in the first block, and hence a sponsored link position is valuable. I characterize consumers' shopping outcomes by adapting Choi et al.'s (2018) [2] eventual purchase condition into the block-by-block search setup.

Letting sellers choose product prices and auction bids together, I characterize the equilibrium of the complete information second-price auction with two payment schemes: fixed payment and per-transaction payment. If a social planner runs the platform, I find the consumer-optimal positioning of products if sellers commit to their prices before the position allocation. Optimal positioning requires placing products into the first block with high expected values and low search costs.

Bibliography

[1] Mohammad Akbarpour, Piotr Dworczak, and Scott Duke Kominers. Redistributive allocation mechanisms. Working Paper, December 16, 2020.

[2] Michael Choi, Anovia Yifan Dai, and Kyungmin Kim. Consumer search and price competition. *Econometrica*, Vol. 86, No. 4, pages 1257—1281, July, 2018.

[3] Piotr Dworczak, Scott Duke Kominers, and Mohammad Akbarpour. Redistribution through markets. *Econometrica*, Vol. 89, No. 4, pages 1665–1698, July, 2021.

[4] Bengt Holmström and Roger Myerson. Efficient and durable decision rules with incomplete information. *Econometrica*, 51(6):1799–1820, 1983.

[5] John Ledyard and Thomas Palfrey. A general characterization of interim efficient mechanisms for independent linear environments. *Journal of Economic Theory*, 133:441–466, 2007.

[6] Zi Yang Kang. Optimal public provision of private goods. Mimeo, Stanford USB, December 2020.

[7] Houward Kunreuther and Paul Kleindorfer. A sealed-bid auction mechanism for siting noxious facilities. *American Economic Review*, 76:295–299, 1986.

[8] Martin L. Weitzman. Optimal Search for the Best Alternative. *Econometrica*, Vol. 47, No. 3, pages 641–654, 1979.

Chapter 2

Optimal Design for Redistributions among Endogenous Buyers and Sellers

2.1 Introduction

Ever since Myerson and Satterthwaite [20] discovered the impossibility of fully efficient bilateral trades given asymmetric information, much has been done theoretically to characterize the mechanisms that achieve the incentive-constrained Pareto frontier. New characterization has been obtained by an emerging literature of redistribution-driven market design such as Akbarpour, Dworczak and Kominers [2] (henceforth ADK), Dworczak, Kominers and Akbarpour [6] (henceforth DKA), and Kang [11].[1] Despite their large-market assumption (continuum of atomless individuals) that eliminates the market power of individuals, the general message from this literature is that a single market price is insufficient to implement allocations on the incentive-constrained Pareto frontier. An optimal mechanism needs to stratify the space of types (private valuations) into more than two tiers through tier-specific prices, augmented with *rationing* that restricts the quantity of demand or supply for individuals, as well as *lump sum transfers* among individuals to achieve redistribution objectives. Meanwhile, they observe upper bounds for the number of such instruments. In DKA [6], there exists an optimal allocation that stratifies the type space into at most five tiers, implemented through rationing on at most two of the tiers together with a lump sum transfer and tier-specific prices. In Kang [11], there exists an optimal allocation that stratifies the type space into at most four tiers, and rationing is necessary to attain optimality.[2]

However, all the studies cited above impose on the market some kind of sectorial restrictions, predetermined exogenously before the realization of types. An individual is exogenously

[1] Kang and Zheng [10], and Reuter and Groh [23], consider redistribution-driven mechanism design without the large-market assumption.

[2] DKA [6] observe that there exists an optimal mechanism whose allocation for the buyers, and allocation for the sellers, are each a monotone step function such that their total number of jumps or drops is at most four. In other words, if the lowest tier among the buyers and that among the sellers are combined into one tier (both excluded from trading), the total number of tiers, from the highest tier among the buyers to the highest tier among the sellers, is at most five.

Kang [11] observes that there exists an optimal mechanism that partitions the public-sector buyers into at most three tiers. This combined with the private-market buyers means four tiers.

assigned the role of a buyer or that of a seller in DKA [6] and Myerson and Satterthwaite [20], or exogenously assigned to a group in ADK [2]. In Kang [11], where individuals choose between the public and the private sectors to trade, the private sector is restricted to be operated under a competitive market price. If such exogenous restrictions are removed, with individuals free to choose a sector conditional on their types, would the policy instruments necessary to attain the Pareto frontier be simplified so that merely a single market price could suffice, or would they get complicated because the endogenous grouping of individuals now becomes an additional dimension in the policymaker's choice variable?

We therefore consider a large market with endogenous buyers and sellers. It is a continuum of individuals each allowed to buy or sell a good at a marginal utility or marginal cost equal to one's type. The planner has the same set of instruments for the buyers and for the sellers. As in the aforementioned studies on redistribution-driven market or mechanism design, we consider the entire incentive-constrained Pareto frontier through examining the optimization problem of a social planner who can take as primitive any type-dependent welfare weight. The welfare weight may vary with the type in whatever fashion because a social planner with redistributive motives may favor one type against another. In allowing for such arbitrary welfare weights the observations would then be applicable to the various social welfare criteria according to which the planner designs her mechanism.

Across such arbitrary welfare weights our characterization of optimal mechanisms turns out to be simpler than those in the above cited. We find a tighter upper bound, three, of the number of tiers that optimal mechanisms have to stratify the type space into (Theorem 2). Furthermore, given a robust condition of the primitives, we obtain the exact number of tiers, rather than only an upper bound thereof, in any optimal mechanism. Given such primitives, the optimal allocation is unique, and the associated optimal number of tiers is equal to either two or three (Theorem 3). The two results combined, we obtain a robust condition of the primitives under which a competitive price alone—offering a single price to all types, be they sellers or buyers, without any other instrument such as rationing, lump sum transfers or tier-specific prices—implements the optimal allocation. In particular, this is true even when the virtual surplus function is non-monotone. In the related literature, by contrast, the only case where a single price is known to implement interim Pareto optimality is the exogenous buyer-seller bilateral trade model of DKA [6], where the conditions they require together imply that the endogenous virtual surplus functions in their model be monotone.[3] Our optimality observation of a single market price without rationing is also opposite to the finding in Kang's [11] model that rationing is in general necessary.

The simplicity of our characterization is due to a new observation, mainly driven by the endogeneity of one's buyer- or seller-role. The observation is that any incentive compatible and market-clearing allocation can always be implemented in a budget-balanced (BB) and individually rational (IR) manner (Theorem 1). Thus, in contrast to the models of DKA [6], Kang [11], and Myerson and Satterthwaite [20], the BB and IR constraints can be removed without loss in our model. That eliminates the two-sidedness (buyer- and seller-sides) of the information asymmetry, a main driving force of the Myerson-Satterthwaite impossibility theorem. Then,

[3] DKA's Theorem 2 requires quasi-convexity of virtual surplus functions and positive derivative of the virtual surplus function at the minimum type ("low same side inequality" in their language). The two together imply that the virtual surplus function is monotone for both buyers and sellers.

as in a monopolist's capacity-constrained second-degree price discrimination problem (Bulow and Roberts [3], or the one-group special case in ADK [2]), or in the basic Bayesian persuasion problem (Kamenica and Gentzkow [9]), rationing is needed to attain optimality only when the market-clearing cutoff type is interior to an interval on which the virtual surplus function has to be ironed.[4] If that happens, rationing is needed only on that interval, and the type space is stratified to at most three tiers. If that does not happen, a single market price suffices optimality, and the type space is stratified into only two tiers, one being all the buyers (the "haves"), the other all the sellers (the "have-nots").

An implication of the characterization result is that the welfare weight of the types near the buyer-seller cutoff in the posted-price system is crucial to the social planner's choice between rationing versus posting a single price. We find that such types are better-off in a mechanism with rationing than given a posted price, and the planner prefers the former to the latter if the welfare weight on such types is sufficiently heavy. We can also tell which among the other types are definitely worse-off under rationing than under the posted price based on the curvature of the type-distribution on the types near the said buyer-seller cutoff (Theorems 4 and 5). Our model can also be modified to capture a kind of externalities without altering any of the results (Corollary 2).

With endogenous buyers and sellers, our model is applicable to large-market exchange economies where individuals have equal entitlements to a limited resource and are heterogeneous in their willingness to pay for the access to the resource. Applied to such situations, our finding implies that, given any redistributive preferences, the social planner can attain optimality through a market-like mechanism for individuals to trade their shares. It would issue to each eligible individual a coupon that represents the person's initial equal access to the limited resource, and the coupon trading would eventually stratify the individuals into at most three tiers in terms of their final shares of the limited resource, those who give up their shares completely, those who max out their acquisition of shares, and those in between. Given robust conditions of the primitives, the coupon-trading mechanism reduces to a single competitive price for the coupon. We illustrate this application in the context of Covid-vaccine allocation within a group of individuals of the same priority (Section 2.6.2).

Our model is similar to partnership dissolution models in that the roles of buyers and sellers are endogenous (e.g., [4], [5], [12] , [17], [18], [21], and [25]). Theorem 1 can be extended to those models provided that the values are private and partners are not overly asymmetric ex ante (though we find no precedent thereof in that literature).[5] That is consistent with Cramton, Gibbons and Klemperer's [2] observation that full efficiency can be attained in some partnership dissolution cases where the initial ownership is nearly equal across partners. However, the full efficiency result in partnership dissolution is based on a particular welfare weight that is neutral across types, while the counterpart in our model is valid for a nondegenerate set of welfare weights that may favor one type or another in various manners. In our model, it is trivial that a single market price implements optimality if our design objective is fixed to be the

[4]The Bayesian consistency condition in Bayesian persuasion models corresponds to the market clearance in Bulow and Roberts's [3] second-degree price discrimination problem, with the prior probability that a sender is supposed to split in the former corresponding to the market equilibrium quantity in the latter. If BB is binds under optimal mechanism, the solution contains at most four tiers as that in Le Treust and Tomala [16].

[5]The extension requires that the sets of no-trade types according to an allocation should have nonempty intersection across all ex ante asymmetric partners.

neutral welfare weight. Recently, full efficiency is shown to be implementable by Yang, Debo and Gupta [28] in their endogenous buyer-seller queuing model, where customers can trade their queuing positions.[6] Our model differs from their work in a similar way that ours differs from the partnership dissolution models.

Our design objective, maximizing the integral of agents' interim expected payoffs across all types measured by any welfare weight distribution, is in the spirit of Holmström and Myerson's [8] notion of interim incentive efficiency. This notion has been considered by a long strand of literature including Gresik [7], Laussel and Palfrey [13], Ledyard and Palfrey [14, 15], Pérez-Nievas [22], Wilson [23] and, recently, ADK [2], DKA [6], Kang and Zheng [10], Kang [11], and Reuter and Groh [23]. Our focus is the endogenoeity of an agent's buyer- or seller-role, which has not been the focus of the literature except Kang and Zheng [10].[7] In [10], we consider a design problem with finitely many players and without the market-clearing condition.

A main perspective of the above literature is that the welfare weight according to which the social planner maximizes the social welfare should be allowed to vary with individuals' types. The importance of this perspective is renewed by recent works on redistributional mechanisms such as ADK [2], DKA [6] and Kang [11], where the social planner's redistributive preferences need not be aligned with the distribution of types across individuals. Moreover, as DKA [6] have shown recently, even if the social planner is neutral across the fundamental characteristics of individuals, the planner would still be biased for some types against others when the type is not a sufficient statistic of the fundamental characteristics (cf. Section 2.2.1).

Allowing for all continuous welfare weight distributions, our characterization of the optimal mechanisms has the merit of being relatively value-free. Without making the absolute continuity assumption of the welfare weight distribution in the literature ([2], [6], [14] and [15]), our model allows for a larger variety of welfare weight distributions.

The next section defines the model. Section 2.3 observes that the budget balance constraint is never binding in our model. Section 2.4 characterizes the optimal mechanisms. Section 2.5 shows which types gain and which types lose when the planner chooses rationing over the posted-price solution, and whose welfare weight is crucial to the planner's choice between the two. Section 2.6 presents two examples, one to illustrate theoretical generality with the Cantor-Lebesgue function being the welfare weight distribution, the other to demonstrate applicability with the aforementioned Covid-vaccination distribution mechanism. Section 2.7 concludes.

2.2 The Model

There is a continuum of individuals, each characterized by a *type*. The type is distributed among the population according to a cdf F with support $[0, 1]$ and density f positive and continuous on the support. An individual of type t can produce up to one unit of a good at a marginal cost equal to t, and can acquire up to B units of the good at a marginal utility equal to t, with parameter $B \in \mathbb{R}_{++}$. (The case $B = \infty$ is considered in Appendix A.10.)

[6] A model of exogenous buyers and sellers of queuing positions has been considered by Yang, Wang and Cui [29].

[7] The model of [15] allows for such endogeneity but focuses on other topics.

By the revelation principle, a mechanism is modeled as a measurable function (q_1, q_2, p) : $[0, 1] \to [0, B]\times[0, 1]\times\mathbb{R}$ such that $q_1(t)$ is the expected quantity of the good that an individual of type t acquires, $q_2(t)$ the expected quantity that the individual supplies, and $p(t)$ the expected value of the net money transfer from the individual to others. We assume that each individual is risk neutral with preferences quasilinear in the net possession of the good and money. That is, a type-t individual acting as type t' gets an expected payoff equal to $(q_1(t') - q_2(t'))\, t - p(t')$ or, more succinctly, $tQ(t') - p(t')$ with the notation

$$Q := q_1 - q_2.$$

Of particular interest is a kind of payment rules such that $p(t)$ is a piecewise affine function of $Q(t)$ with the same constant term. That is, there exist $c \in \mathbb{R}$, integer n and mutually distinct $k_1, \ldots, k_n \in \mathbb{R}$ such that for any $t \in [0, 1]$, $p(t) = c + k_i Q(t)$ for some $i \in \{1, \ldots, n\}$. Given such a payment rule, c is the *lump sum transfer* to all types, and n is the number of *prices*. The payment rule is called *posted price* iff $n = 1$, namely, it offers a constant per-unit price to all types, be they buyers or sellers. If Q is equal to a constant on some nondegenerate interval S of $[0, 1]$ and the constant is neither -1 nor B, the mechanism is said to entail *rationing* on S.[8]

By *welfare weight distribution* we mean a cdf W with support $[0, 1]$ that is continuous on $\mathbb{R}$. Given any welfare weight distribution W, the design problem is to maximize

$$\int_0^1 (tQ(t) - p(t))\, dW(t) \tag{2.1}$$

among all mechanisms (q_1, q_2, p), with $Q = q_1 - q_2$, subject to incentive compatibility (IC) that $tQ(t) - p(t) \geq tQ(t') - p(t')$ for any $t, t' \in [0, 1]$, individual rationality (IR) that $tQ(t) - p(t) \geq 0$ for all t, budget balance (BB) that

$$\int_0^1 p(t) dF(t) \geq 0,$$

and market clearance that

$$\int_0^1 Q(t) dF(t) = 0.$$

Any solution (Q^*, p^*) to this design problem is called optimal mechanism, and Q^* *optimal allocation*.

2.2.1 Comments on the Welfare Weight Distribution

The welfare weight distribution W reflects the social planner's redistributive preferences across types. It corresponds to the supporting hyperplane at a point on the interim incentive-constrained Pareto frontier, as in the interim incentive efficiency literature initiated by Holmström and Myerson's [8]. DKA [6] interpret the Radon-Nikodym derivative of W (with respect to the type

[8]The term rationing makes sense because, by the envelope formula, the constancy of Q on S implies that $p(t)$ is an affine function of t, and so the per-unit price is constant, on S. Thus, for almost all types in S, marginal utilities are not equal to the (per-unit) price, while the individual is restricted from buying or selling up to the full capacity.

distribution F) at any type t as the expected value of an individual's marginal utility of money (MU_m) conditional on that the marginal rate of substitution (MRS) of the good relative to money is equal to the type t. They show that a social planner whose objective is (2.1) subject to such welfare weights is equivalent to the planner who is neutral across the underlying individual characteristics that determine the MU_m and MRS. Thus, the planner prefers redistributions from types with low MU_m in expectation ("the rich") to types with high MU_m in expectation (the "poor").

The continuity assumption of the welfare weight distribution W is consistent with the continuum-type (or large-market) model, as each type is supposed to be atomless. The assumption is weaker than its counterpart in the literature. For example, Ledyard and Palfrey [14, 15] and DKA [6] assume absolute continuity of W. Allowing for singular W, we can consider situations where the planner cares only about a measure-zero set of types.

2.2.2 Incorporation of Externalities

We can modify the model to incorporate a kind of externalities without altering any of our results. Let $\psi : [0,1]^2 \to \mathbb{R}$ be a measurable function. Suppose that, when an individual's net acquisition of the good is equal to a quantity $Q(t') \in [-1, B]$ (through acting as type t' given allocation Q), the externality of this acquisition spilled over to any other individual of type t, for any $t \in [0,1]$, is equal to $\psi(t,t')Q(t')$. This corresponds to a situation where the externality spilled over to an individual from the others' actions is evaluated according to the type of both players. For instance, in the context where the good is the access to Covid-vaccines, it is plausible that the more one is willing to get himself inoculated, the more strongly he believes that his health is affected by a person who is less willing to get inoculated since the latter cares less about the covid consequences and is more likely to spread the virus. In that case, $\phi(\cdot,\cdot)$ is increasing in the first term and decreasing in the second term. However, the model does not apply to situations where an individual's evaluation of the externality spilled over to him depends on some other personal characteristics in addition to his willingness to pay.[9]

In this model with externalities, the aggregate externality spilled over to an individual of type t, given allocation Q, is equal to $\int_0^1 \psi(t,t')Q(t')dF(t')$, which is a function of the player's own type t. The design constraints remain the same as in the original model since externalities are not controlled by any individual, and the social planner's objective becomes

$$\int_0^1 \left(tQ(t) - p(t) + \int_0^1 \psi(t,t')Q(t')dF(t') \right) dW(t). \tag{2.2}$$

2.3 The Budget Balance Condition

As is well-known in the market or mechanism design literature of bilateral trades (Myerson and Satterthwaite [20], Ledyard and Palfrey [15], DKA [6], etc.), the main source of complication in characterizing the optimal mechanisms is the budget balance (BB) constraint. In the literature, the possibility that the constraint is binding cannot be ruled out a priori. When it is binding, characterization of the optimal mechanisms depends on endogenous variables and

[9] See Akbarpour et al. [1] for a model that applies to such a situation.

hence cannot be generally described purely in terms of the primitives. In our model, by contrast, the constraint is never binding, as long as the allocation is incentive compatible (namely, weakly increasing) and market clearing:

Theorem 1 *For any weakly increasing allocation $Q : [0,1] \to \mathbb{R}$ that satisfies the market clearing condition, there exists a payment rule $p : [0,1] \to \mathbb{R}$ with which (Q, p) satisfies IR, IC and BB.*

Proved in Appendix A.1, Theorem 1 is driven by our assumption that each individual is free to choose between buying and selling. To understand the theorem, let us rewind the complication in the previous literature, where an individual is assumed to have no such freedom. Consider any IC allocation in the previous literature. Let c_1 be the highest type among those who get to sell a positive quantity of the good, and c_2 the lowest type among those who get to buy a positive quantity of the good, according to this allocation. If $c_1 \le c_2$ then the social planner can easily implement this allocation in a manner that satisfies BB and IR. For example, she can use the payment rule that maximizes her profit among those that implement the allocation in a manner that satisfies IR. Given this payment rule, one can show that the per unit price offered to any buyer is no less than c_2. The intuitive reason is that, by the definition of c_2, almost every buyer's type—marginal utility of the good—is above c_2. Likewise, the payment rule does not pay a seller a unit price more than c_1, as a seller's type—marginal cost—is below c_1. Thus the planner's profit is no less than c_2 times the aggregate demand subtracted by c_1 times the aggregate supply. With market clearance and $c_1 \le c_2$, this profit is nonnegative and hence the planner's budget is balanced.

The problem, however, is that $c_1 \le c_2$ cannot be guaranteed when individuals are not free to choose between buying and selling. Had they been free to do so, $c_1 > c_2$ would violate IC, as the buyer-types in (c_2, c_1) value the good less than some seller-types in (c_2, c_1) and so such buyers and sellers would rather switch roles. When individuals are not free to choose between buying and selling, by contrast, they cannot undo $c_1 > c_2$ by switching between buying and selling. When $c_1 > c_2$, the planner may need to pay more per unit of procurement than she charges per unit of sales. But that would break her budget unless the planner compromises on some other aspects of the allocation. Thus the BB constraint may be binding at an optimal mechanism. In our model, by contrast, individuals are free to switch between buying and selling and hence only allocations with $c_1 \le c_2$ can be IC. Thus, as in the previous paragraph, the BB constraint is automatically satisfied.

In a nutshell, Theorem 1 comes from the simple fact that, in a market where everyone is free to switch between buying and selling, any buyer's marginal value of the good is higher than any seller's marginal cost of supplying it. The social planner can therefore profit from buying the good from the sellers and selling it to the buyers.

Due to Theorem 1, our design problem is reduced to an optimization among allocations without the IR and BB constraints:

Corollary 1 *A mechanism (Q^*, p^*) is an optimal mechanism if and only if Q^* solves*

$$\begin{array}{ll} \max_Q & \int_0^1 QVdF \\ s.t. & Q : [0,1] \to [-1, B] \text{ is weakly increasing} \\ & \int_0^1 QdF = 0, \end{array} \tag{2.3}$$

where $V : [0,1] \to \mathbb{R}$ *is the* virtual surplus *function defined by, for any* $t \in [0,1]$,

$$V(t) := t - \frac{W(t) - F(t)}{f(t)}. \tag{2.4}$$

To prove the corollary, use the usual routine of envelope theorem and integration by parts to show (Lemma 11, Appendix A.2) that a mechanism (Q^*, p^*) is an optimal mechanism (a solution to the design problem defined in Section 2.2) if and only if Q^* maximizes $\int_0^1 Q(t)V(t)dF(t)$ among all weakly increasing allocations $Q : [0,1] \to [-1, B]$ subject to two conditions: (i) market clears ($\int_0^1 QdF = 0$), and (ii) there exists a payment rule that implements Q with respect to the IR and BB constraints. Condition (ii), by Theorem 1, is guaranteed by Condition (i) and the monotonicity of the allocation. Thus, the maximization problem is equivalent to Problem (2.3), and hence the corollary follows.

Following the same routine, one can prove (Appendix A.3) that the planner's problem is exactly the same as (2.3) in the modified model with externalities defined in Section 2.2.2.

Corollary 2 *In the modified model with externalities such that the social planner's objective is (2.2) instead of (2.1), an allocation is optimal if and only if it solves Problem (2.3) with the virtual surplus function be* $\tilde{V}(t) := V(t) + \int_0^1 \psi(t,t')Q(t')dF(t')$.

2.4 Optimal Mechanisms

Problem (2.3), with a harmless change of variables, is the same as the single-market monopoly problem considered by Bulow and Roberts [3] subject to the market clearing constraint. As has been understood in the literature, the virtual surplus $V(t)$ corresponds to the monopolist's marginal revenue extracted from type-t individuals. The problem can be solved by the standard ironing method.

If a single price τ per unit is offered to all individuals without quantity restrictions, so that every type above τ would buy B units of the good, and every type below τ would sell one unit thereof, then the market clearing condition $\int_0^1 QdF = 0$ is satisfied iff $\tau = F^{-1}\left(\frac{B}{B+1}\right)$. Thus we call $F^{-1}\left(\frac{B}{B+1}\right)$ *market-clearing price*. With the marginal revenue interpretation of $V(t)$, it is clear that a posted price equal to the market clearing price attains the optimality of (2.3) if the marginal revenue of any type below the market-clearing price is no higher than the marginal revenue of any type above the market-clearing price. In other words, the posted price is optimal if $V(\cdot) - V\left(F^{-1}\left(\frac{B}{B+1}\right)\right)$ is single-crossing on $[0,1]$.

Without the single-crossing condition, there may be a type below the market-clearing price that contributes a larger marginal revenue than some type above the price does. To exploit the larger marginal revenues of such lower types without violating the monotonicity (IC) condition of the allocation, the planner needs to find an appropriate interval $[a,b]$ that contains the market-clearing price and treat the types in $[a,b]$ equally. As long as the average marginal revenue in $[a,b]$ is not less than the average marginal revenue in $[0,a)$, and not greater than that in $(b,1]$, the planner can attain optimality through stratifying the types into at most three tiers:[10] Types in $[0,a)$ sell and types in $(b,1]$ buy, each in full capacity, while types in (a,b) are

[10] By a *tier* in an incentive compatible (and hence monotone) allocation $Q : [0,1] \to \mathbb{R}$, we mean the inverse image $Q^{-1}(s)$ of some s in the range of Q such that $Q^{-1}(s)$ is a nondegenerate interval.

rationed a constant quantity that clears the market. This characterization is formalized by the next theorem, proved in Appendix A.5.

Theorem 2 *(i) There exists an optimal mechanism consisting of an allocation*

$$Q^*(t) := \begin{cases} -1 & \textit{if } 0 \le t < a \\ \frac{F(a)-B(1-F(b))}{F(b)-F(a)} & \textit{if } a < t < b \\ B & \textit{if } b < t \le 1, \end{cases} \tag{2.5}$$

where $0 \le a \le F^{-1}\left(\frac{B}{B+1}\right) \le b \le 1$*, and a payment rule that has at most two prices and entails rationing only on* (a, b)*. (ii) If the function* $V(\cdot) - V\left(F^{-1}\left(\frac{B}{B+1}\right)\right)$ *is single-crossing on* $[0, 1]$*, then*

$$Q^*(t) = \begin{cases} -1 & \textit{if } 0 \le t < F^{-1}\left(\frac{B}{B+1}\right) \\ B & \textit{if } F^{-1}\left(\frac{B}{B+1}\right) < t \le 1, \end{cases} \tag{2.6}$$

and the payment rule becomes a posted price equal to $F^{-1}\left(\frac{B}{B+1}\right)$ *without rationing or lump sum rebate.*

Theorem 2 implies that an optimal allocation exists and it is a tiered allocation consisting of at most three tiers. Moreover, when the single crossing condition of $V - V\left(F^{-1}\left(\frac{B}{B+1}\right)\right)$ is satisfied, the optimal allocation has only two tiers and is implemented by offering the market-clearing price $F^{-1}\left(\frac{B}{B+1}\right)$ to everyone. Allocating the full capacity (-1 or B) to each buyer- or seller-type, the optimal allocation does not entail rationing. It is easy to check that the posted price yields zero profit for the planner (since the allocation satisfies the market clearing condition), and hence the optimal mechanism has no lump sum rebate. Note that the single-crossing condition can be satisfied by even non-monotone virtual surplus functions.

For the three-tier allocation (2.5), the interval (a, b) can be constructed from the primitives with the definition of ironing. As shown in the proof (Appendix A.5), when (2.6) is not optimal, the interval (a, b) for (2.5) to be optimal contains the market-clearing price $F^{-1}\left(\frac{B}{B+1}\right)$ as an interior point. From the envelope formula one can derive the optimal payment rule that implements (2.5), described by the next corollary (proved in Appendix A.6).

Corollary 3 *For any* $a, b \in [0, 1]$ *such that* $a < F^{-1}\left(\frac{B}{B+1}\right) < b$*, the mechanism that consists of allocation (2.5) and any optimal payment rule that implements (2.5) transfers a positive lump sum constant to all types and entails rationing only on* (a, b) *with the rationed quantity*

$$x := \frac{F(a) - B(1 - F(b))}{F(b) - F(a)}, \tag{2.7}$$

and the payment rule can be replaced without loss by the combination of the said lump sum transfer and two distinct per-unit prices specified below:

i. if $x \ge 0$*, each type in* $[0, a)$ *sells one unit at the price equal to a, each in* (a, b) *buys the quantity x at the price equal to a per unit, and each in* $(b, 1]$ *buys the quantity B at the price* $b - (b - a)x/B$ *per unit;*

ii. *if* $x \leq 0$*, each type in* $[0, a)$ *sells one unit at the price equal to* $a - (b - a)x$ *per unit, each in* (a, b) *sells the quantity* $|x|$ *at the price equal to* b *per unit, and each in* $(b, 1]$ *buys the quantity* B *at the price* b *per unit.*

The next theorem (proved in Appendix A.7) says that the optimal allocation characterized above is unique given a nondegenerate set of parameter values. Thus, not only is there no need to stratify the types into more than three tiers, often it is also suboptimal to do so. Combined with the single-crossing condition in the previous theorem, Theorem 3 implies that under robust conditions of the primitives—allowing for some cases of non-monotone virtual surplus functions—there exists a unique optimal allocation and it is implemented by a single posted price without the help of rationing or lump sum transfers.

Theorem 3 *If there exists no positive-measure subset* S *of* $[0, 1]$ *such that* V *is constant on* S*, the optimal allocation is unique (modulo measure zero).*

To see the role played by the non-constancy assumption of V, consider a case where the rationed interval (a, b) in (2.5) is a proper subset of another interval (a', b') in $[0, 1]$ such that V restricted on $(a', b') \setminus (a, b)$ happens to be constantly equal to the average marginal revenue on (a, b). Then the rationed interval can be extended from (a, b) to any (a'', b'') for which $a' \leq a'' \leq a < b \leq b'' \leq b'$ without undermining the optimality of the allocation, as it is the average marginal revenue within a set of types that determines how much the planner would prioritize the set. Such cases are ruled out by the non-constancy assumption.

Whether there can be multiple optimal allocations or not, any optimal allocation requires stratifying the type space into at least two tiers:

Corollary 4 *Egalitarian allocations (*$Q = 0$ *a.e.* $[0, 1]$*, or autarky) are never optimal.*

The proof of the corollary (Appendix A.6) uses an observation in the proof of Theorem 2. The intuition is simply that there is always a gain of trade between the sufficiently low types and the sufficiently high ones. When a type t near zero supplies a unit of the good, the cost to the society is $V(t) \approx 0$. When a type t' near the supremum type acquires a unit of the good, the social benefit is $V(t') \approx 1$.

2.5 Posted Price versus Rationing

The previous section shows that any optimal mechanisms—supported as a Pareto frontier point by some continuous welfare weight distribution—can be simplified to one of only two alternatives: It is either the posted-price system, implementing the two-tier allocation (2.6), or a rationing system that implements a three-tier allocation (2.5) and entails rationing on the middle tier. This section shows who gains, and who loses, when the mechanism switches from one kind to the other, each being Pareto optimal. We shall also see whose welfare weight plays a crucial role in the social planner's choice between the two alternatives.

2.5.1 Who Gains and Who Loses from Rationing

It is intuitive that the types near the market-clearing price $F^{-1}\left(\frac{B}{B+1}\right)$ gain when the mechanism switches from the posted price to rationing. Given the posted price, which is equal to $F^{-1}\left(\frac{B}{B+1}\right)$, the type $F^{-1}\left(\frac{B}{B+1}\right)$ gets zero net payoff whether it buys or sells the good, as its valuation of the good is equal to the type. The type has no other source of surplus because the posted-price system, essentially a competitive equilibrium, yields no profit for the planner to rebate to the individuals. Given rationing, by contrast, the type has at least a positive lump sum rebate as part of its surplus. The lump sum is positive because the planner gets a positive profit from rationing on the middle tier (a, b) that contains the market-clearing price $F^{-1}\left(\frac{B}{B+1}\right)$ as an interior point: The planner exploits her monopsony power in squeezing the range of her full-capacity procurement from $\left[0, F^{-1}\left(\frac{B}{B+1}\right)\right)$ to $[0, a)$, and monopoly power in squeezing the range of her full-capacity sales from $\left[F^{-1}\left(\frac{B}{B+1}\right), 1\right]$ to $(b, 1]$. Thus, the type $F^{-1}\left(\frac{B}{B+1}\right)$ gains strictly when the mechanism changes from the posted price to rationing. By continuity, so do the nearby types.

The more complicated question is which types get hurt in order for such middle types to gain. While the general answer may depend on the parameter values, we can tell whether the high or the low types are definitely worse-off based on the curvature of the type distribution F around the market-clearing price. According to the next theorem, if the distribution F of types is convex on the middle tier in a rationing mechanism, the low types—those who get to sell at full capacity in both mechanisms—are definitely worse-off when the posted price is replaced by the rationing mechanism: In Figure 2.1, on the set $[0, a)$ of low types, the red dotted line—the surplus given rationing—lies below the blue solid line—the surplus given the posted price. If F is concave on the middle tier, by contrast, the high types—those who get to buy at full capacity in both mechanisms—are definitely worse-off: In Figure 2.2, the red dotted line lies below the blue solid line on $(b, 1]$, the set of high types.

Theorem 4 *If the allocation in an optimal mechanism switches from the two-tier (2.6) to a three-tier (2.5) that entails rationing on some* (a, b) *for which* $0 < a < F^{-1}\left(\frac{B}{B+1}\right) < b < 1$*, then:*

a. all the types sufficiently near to the market-clearing price $F^{-1}\left(\frac{B}{B+1}\right)$ *are better-off;*

b. if F is convex on (a, b)*, any type in* $[0, a)$ *is worse-off;*

c. if F is concave on (a, b)*, any type in* $(b, 1]$ *is worse-off.*

Theorem 4 is proved in Appendix A.8. To understand the less intuitive parts, Claims (b) and (c), let us consider a stochastic counterpart to the rationing allocation (2.5): For each individual, the allocation is randomly selected independently so that it is

$$Q_a(t) := \begin{cases} -1 & \text{if } t \in [0, a] \\ B & \text{if } t \in (a, 1] \end{cases}$$

with probability $(1 + x)/(B + 1)$, and

$$Q_b(t) := \begin{cases} -1 & \text{if } t \in [0, b] \\ B & \text{if } t \in (b, 1] \end{cases}$$

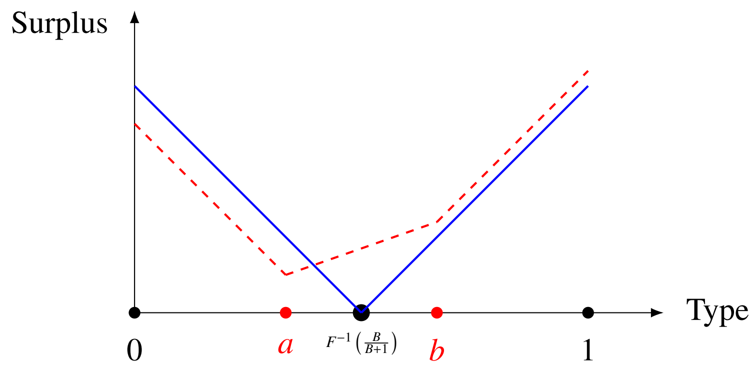

Figure 2.1: Posted-price (blue solid) vs. rationing (red dotted) given convex F

with probability $1-(1+x)/(B+1)$, where x is determined by (2.7). By the choice of x, $F(a)(1+x)/(B+1)+F(b)(1-(1+x)/(B+1)) = B/(B+1)$. That is, from the ex ante or the social planner's viewpoint, the expected quantity to procure from the individual is equal to $B/(B+1)$. By the same token, the expected quantity to sell to the individual is equal to $B/(B+1)$. Thus, when the same lottery is run independently for all individuals, supply is equal to demand in the aggregate level. Note that from each (privately informed) individual's viewpoint, the stochastic allocation is equivalent to the rationing allocation (2.5).

The stochastic allocation can be implemented by the corresponding stochastic payment rule: To each individual, if the lottery picks Q_a then offer to him a price equal to a per unit for the individual to buy or sell the good in full capacity; if the lottery picks Q_b then analogously offer him the price b per unit for buying and selling. One readily sees that this stochastic payment rule cannot generate any positive expected profit for the social planner, as a posted price leaves no rent to the planner.[11]

Thus, under any optimal mechanism that implements the stochastic allocation, which is required to be budget balanced, the payment rule differs from the stochastic payment rule, in expectation, only by a lump sum transfer to the individuals that is a nonpositive constant across types. Consequently, each individual's expected revenue under any optimal mechanism of the stochastic allocation, or equivalently the rationing allocation, is bounded from above by the expected revenue he receives from the stochastic payment rule, as the latter has yet to count the nonpositive lump sum transfer. In other words, the expected revenue for a type in $[0,a)$ is bounded from above by the convex combination between a and b according to the probability mix in the lottery. This convex combination is labeled by the red dot in Figure 2.3. As shown in Figure 2.3, when F is convex on (a,b) and hence F^{-1} concave on $(F(a),F(b))$, the convex combination is less than $F^{-1}\left(\frac{B}{B+1}\right)$. This is simply due to Jensen's inequality. Intuitively speaking, if F is the uniform distribution, the market-clearing cutoff value $F^{-1}\left(\frac{B}{B+1}\right)$ would be equal to the cutoff quantile $B/(B+1)$. Now make F convex around the cutoff. That means moving a mass of types from below the cutoff to above the cutoff. Consequently, the cutoff

[11] Furthermore, the expected profit generated by the stochastic payment rule is negative. To see that, notice there exists excess supply under the market with the high price b and excess demand under the market with the low price a. Thus, the planner buys the excess supply at a high price and sells the excess demand at a low price, so she must earns a negative profit with the market clearing condition.

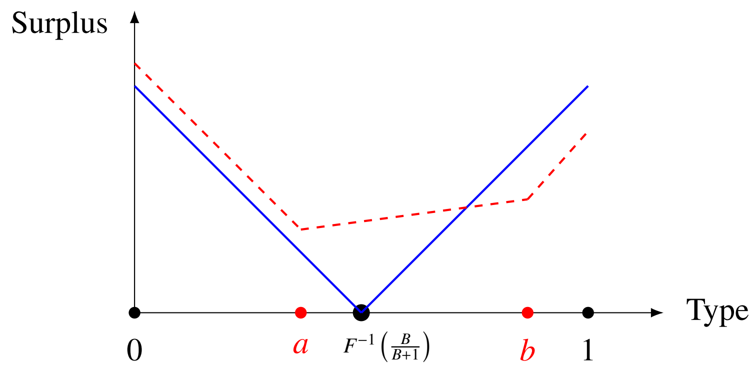

Figure 2.2: Posted-price (blue solid) vs. rationing (red dotted) given concave F

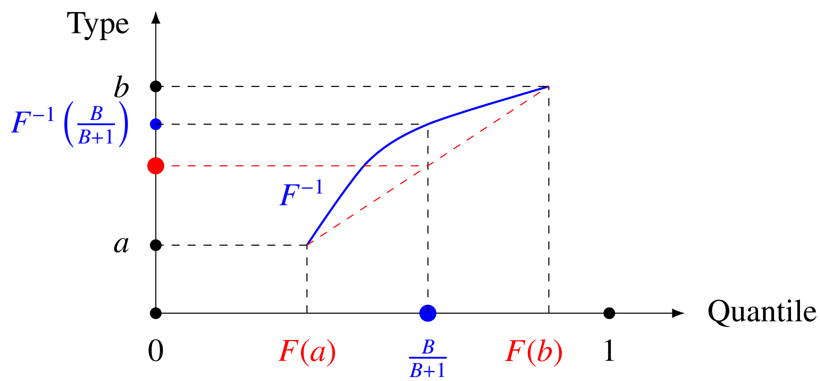

Figure 2.3: The red dot: Expected revenue upper bound for low types

value $F^{-1}\left(\frac{B}{B+1}\right)$, whose quantile is supposed to $B/(B+1)$, needs to be adjusted upward. Since $F^{-1}\left(\frac{B}{B+1}\right)$ is the revenue for any type in $[0, a)$ under the deterministic posted-price system without rationing, the type receives less expected surplus in the rationing mechanism than in the posted-price system, as claimed in Part (b) of the theorem. Part (c) of the theorem is analogous from the perspective of the high types in $(b, 1]$.

2.5.2 The Welfare Weight of the Market-Clearing Price

Since the middle types around the market-clearing price gain from rationing (Part (a) of Theorem 4), it is natural that the heavier is the welfare weight on such middle types, the more is the social planner leaning towards a rationing mechanism. To formalize that, for any welfare weight distributions W and W_*, let us say—

- W_* is a *spread* of W away from $F^{-1}\left(\frac{B}{B+1}\right)$ iff $W_* \geq W$ on $\left[0, F^{-1}\left(\frac{B}{B+1}\right)\right)$ and $W_* \leq W$ on $\left(F^{-1}\left(\frac{B}{B+1}\right), 1\right]$;
- W_* is a *contraction* of W towards $F^{-1}\left(\frac{B}{B+1}\right)$ iff $W_* \leq W$ on $\left[0, F^{-1}\left(\frac{B}{B+1}\right)\right)$ and $W_* \geq W$ on $\left(F^{-1}\left(\frac{B}{B+1}\right), 1\right]$.

Intuitively speaking, a spread away from $F^{-1}\left(\frac{B}{B+1}\right)$ moves some welfare weights around the market-clearing cutoff type to the higher and lower types, and a contraction towards $F^{-1}\left(\frac{B}{B+1}\right)$ does the opposite. The next theorem shows that the two operations have opposite effect on the optimality of the posted-price system.

Theorem 5 *Suppose that the posted-price system is optimal given a welfare weight distribution W. Then:*

a. *if W_* is a spread of W away from $F^{-1}\left(\frac{B}{B+1}\right)$, then the posted-price system is optimal given W_* being the welfare weight distribution;*

b. *for any $\epsilon > 0$ there exists a contraction W_* of W towards $F^{-1}\left(\frac{B}{B+1}\right)$ such that $\|W_* - W\|_{\max} \leq \epsilon$ and the posted-price system is not optimal given W_* being the welfare weight distribution.*

Theorem 5 is proved in Appendix A.9. Its intuition, as mentioned above, has been suggested by Part (a) of Theorem 4. We can get a more explicit intuition by adopting DKA's [6] rich-vs-poor interpretation of the welfare weight distribution. According to DKA, the density of the welfare weight distribution at a type t corresponds to the average marginal utility of money among the individuals whose marginal rate of substitution of the good relative to money is equal to t (cf. Section 2.2.1). When W spreads the weight away from $F^{-1}\left(\frac{B}{B+1}\right)$, the types near $F^{-1}\left(\frac{B}{B+1}\right)$ are having lower marginal utilities of money in average and hence there is less a need for redistributing money transfers to such types through deviating from the zero-rebate posted-price system. When W contracts the weight towards $F^{-1}\left(\frac{B}{B+1}\right)$, by contrast, the nearby types value money more in average, which strengthens the need to make money transfers to them through moving away from the posted-price system.

2.6 Examples

2.6.1 The Cantor Welfare Weight Distribution

Suppose that the type distribution F is the uniform distribution U[0, 1] on [0, 1], and that the welfare weight distribution W is the Cantor-Lebesgue function φ, so the support of the distribution is the (ternary) Cantor set.[12] By the well-known properties of the Cantor-Lebesgue function, φ is a continuous cdf that assigns positive welfare weights only to the (ternary) Cantor set, which is of zero (Lebesgue) measure, and φ increases at unbounded rates on the Cantor set. Thus the social planner cares only about a set of types of zero measure, and her redistributive preferences cannot be described by welfare *densities* (or the "Pareto weights" in DKA [6]). Plug $W = \varphi$ and $F = \text{U}[0, 1]$ into (2.4) to obtain the virtual surplus function V:

$$V(t) = 2t - \varphi(t)$$

for all $t \in [0, 1]$. It is obvious that V is not monotone, whose graph is depicted in Figure 2.4.

Nonetheless, our result applies. There are countably many intervals in [0, 1] on which

[12]See Royden and Fitzpatrick [24, Section 2.7] for the definition and properties of the Cantor set and the Cantor-Lebesgue function.

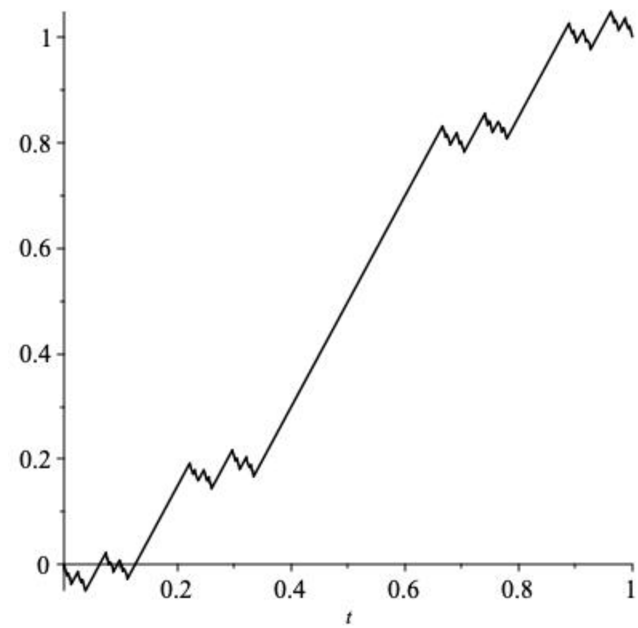

Figure 2.4: The virtual surplus given the Cantor welfare weight distribution

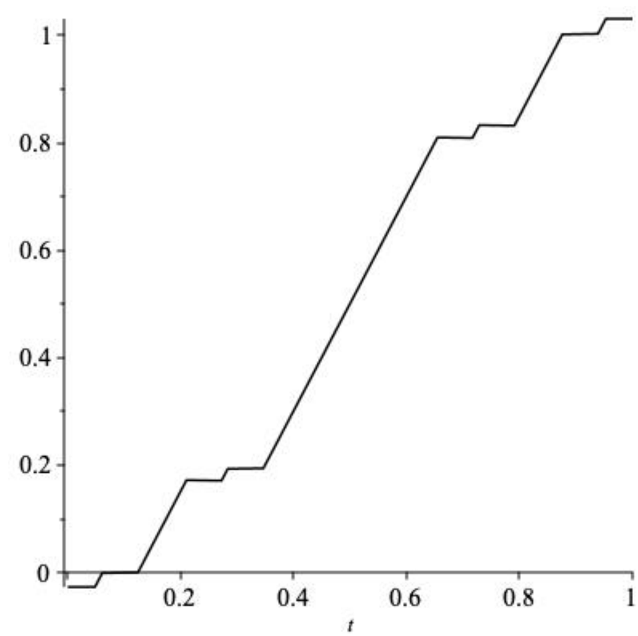

Figure 2.5: The ironed virtual surplus given the Cantor welfare weights

$V = \overline{V}$ because V is single-crossing at any point in those intervals. Examples of such monotone intervals are the one in the middle which is approximately $[0.4, 0.6]$, the one to the left, approximately $[0.15, 0.2]$, the one to the right, approximately $[0.825, 0.875]$, and so on (Figure 2.5). If the market-clearing price $\frac{B}{B+1}$ belongs to any of such monotone intervals, the optimal allocation is uniquely the two-tier stratification with $\frac{B}{B+1}$ being the buyer-seller cutoff. Else, $\frac{B}{B+1}$ is interior to an interval where V needs to be ironed, and a three-tier allocation is optimal. Furthermore, this is the unique optimal allocation by Theorem 3, as the inverse image $V^{-1}(x)$ is of zero measure for any $x \in [0, 1]$.

Although almost all types carry zero welfare weight according to the Cantor-Lebesgue function, one readily sees that the optimal mechanism gives positive surpluses to all types (except type $F^{-1}\left(\frac{B}{B+1}\right)$ when the posted-price system is optimal). That is because the IC condition requires that the surplus for any type be at least as large as the surplus for type $F^{-1}\left(\frac{B}{B+1}\right)$, the buyer-seller cutoff.

2.6.2 Queuing for Covid-Vaccination

Our model is equivalent to the following exchange economy up to normalization. Every (atomless) individual is endowed with one unit of the good, individuals can sell any fraction of their endowments for money, and each can consume up to $B + 1$ units of the good. The good can be interpreted as the access to a limited public resource an equal share of which everyone is initially entitled to. Individuals' types are their willingness to pay for the access to the resource. The social planner's welfare weight distribution W need not be aligned with the distribution F of the willingness to pay. Our model then applies and the planner's optimality can be achieved by a market-like mechanism where individuals trade their shares of the public resource given a menu containing at most three price-quantity contracts.

To be explicit, let us apply the idea to the allocation of Covid-vaccines. While a social planner often has explicit preferences over who should receive the vaccines before others and hence might want to prioritize vaccine allocation according to the groups (Akbarpour et al. [1]; Sömet et al. [26]), it has often been reported that individuals of the same priority level (e.g., healthcare workers) are heterogeneous in their vaccine willingness or hesitancy. Thus, let us focus on the issue about the limited supply of Covid-vaccines on one hand and the heterogeneous willingness to vaccination within the same priority group on the other.[13]

For a stylized model to capture this issue, normalize to one the measure of the population in a priority group, and suppose that the quantity of Covid-vaccines available to this population is equal to $\alpha \in (0, 1)$. (In other words, with the available quantity, only up to a fraction α of the population gets to be inoculated.) Suppose that an individual's willingness to get vaccinated is represented by a type $t \in [0, 1]$, drawn from a cdf F. Assume that, if q is the probability for an individual to get a full inoculation, m his net monetary receipt, and each individual of type t' is allocated a probability $\tilde{q}(t')$ of getting fully inoculated, then the type-t individual's gross payoff

[13] Akbarpour et al. [1] consider both the issue of within-group heterogeneity and the planner's cross-group preferences in vaccine distributions. We consider only the within-group heterogeneity issue to focus on how it can be solved by simple market mechanisms once the quantity of vaccines available to a group has been determined.

is equal to

$$qt + m + \psi(t) \int_0^1 \tilde{q}(t')dF(t').$$

That is, we adopt the modified model with externalities in Section 2.2.2, with the caveat noted in that section. For instance, a strictly increasing ψ corresponds to situations where the more individuals are willing to get vaccinated themselves the more they wish to push others to get vaccinated.

Since the population is of the same priority group, at the outset everyone is entitled to an equal access to vaccination. That is, any individual is entitled to a probability α of getting inoculated. Implicitly, each member of the population is initially issued a coupon, so that one coupon gives a person the probability α of getting vaccinated.

The Covid-vaccine distribution system in the real world often rely on individuals to sign up and wait for the call, or to search for available vaccination stations themselves. That, roughly speaking, corresponds in our stylized model to a mechanism that bans individuals from trading their coupons and rations the vaccines to those who show up after each having borne a sunk cost c. Consequently, only the types above c show up. Either the mass $1 - F(c)$ of those who show up exceeds the available quantity α of vaccines, or $1 - F(c) < \alpha$. In the former case, some of those who show up do not get inoculated and their sunk cost is wasted. In the latter case, the excess quantity of vaccines is wasted because Covid-vaccines, at least up to this time, cannot be stored for long.

Alternatively, the government could minimize the signup and search cost c to a negligible level—as the US government did in issuing the stimulus checks to taxpayers—and consider allowing eligible individuals to trade their entitled access to the vaccine. If trade is not allowed, with c reduced to zero, the entire population shows up and the vaccines are rationed to them randomly, which is not optimal (Corollary 4). To allow for trade, the government could issue digital coupons to individuals, one unit to each, to represent their initial entitlements to inoculation. Then individuals can trade any fraction of their coupons so that a person who shows up with a quantity q of coupons gets to be inoculated with probability equal to $q\alpha$. To ensure vaccination for sure, a person needs only to hold a quantity $1/\alpha$ of coupons. Thus, the quantity of coupons that a person needs to acquire, in addition to the one unit the person is endowed with, does not need to exceed $1/\alpha - 1$. That is,

$$B = \frac{1}{\alpha} - 1.$$

A mechanism can be represented by (Q, P) such that $Q(t)$ is the quantity of coupons that a type-t person acquires from others, and $P(t)$ his net payment.[14] Then a type-t individual's expected gross payoff from claiming to be type t' is equal to

$$\begin{aligned} &(Q(t') + 1)\,\alpha t - P(t') + \psi(t) \int_0^1 \alpha\,(Q(s) + 1)\,dF(s) \\ =\ & \alpha \underbrace{\left(Q(t')t - \frac{1}{\alpha}P(t')\right)}_{u(t',t|Q,P)} + \alpha t + \alpha\psi(t) + \alpha \int_0^1 \psi(t)Q(s)dF(s). \end{aligned}$$

[14] Negative $Q(t)$ means selling the corresponding quantity, and negative $P(t)$ means being paid.

Thus, any type t's decision is equivalent to maximizing the expected net payoff $u(t', t|Q, P)$ among all t'. This, coupled with Corollary 2, implies that the social planner's design objective is equivalent to an integral of the truthtelling expected net payoff $u(t, t|Q, P)$ across all types t measured by some welfare weight distribution W. Thus our result applies and, if it entails (Q^*, p^*) as the mechanism, the planner would construct the payment rule P^* by $P^* := \alpha p^*$.

If $W = F$, namely, the social planner is neutral across types, then $V(t) = t$ for all t and the optimal mechanism is a single posted price equal to the market-clearing price $F^{-1}\left(\frac{B}{B+1}\right)$ $(= F^{-1}(1-\alpha))$. That is, if the planner has no redistribution bias across types, she would opt for the free-market solution to distribute vaccines within the same priority group. In reality, however, the planner often puts heavier welfare weights on some types than on others. For instance, suppose that individuals' types represent their comorbidity-rates of Covid and that the type-distribution F is concave (say due to the Omicron variant, higher comorbidity occurs less frequently). Meanwhile, suppose that the social planner puts heavier weights on types of higher comorbidity rates, because higher comorbidity entails heavier social costs to healthcare, so much so that the welfare weight distribution W is convex.[15] Thus, F is concave and W convex. This, coupled with a technical condition that the Radon-Nikodym derivative w of W is bounded from above by 2, implies that the virtual surplus function is increasing and so the social planner would stay with the free-market solution (Theorem 2.ii). Even without the technical condition of w and the global concavity of F, as long as F is concave around the market-clearing cutoff type, Part (c) of Theorem 4 would still imply that the planner is unlikely to forgo the free-market solution, because any other mechanism, entailing rationing, would hurt the high-comorbidity types that she cares about.

Even when the free-market solution (posted-price system) is not optimal, the planner can still achieve optimality through a market-like mechanism that uses at most two distinct prices and entails rationing on only one tier among the types (Theorem 2 and Corollary 3). For example, when the rationed quantity x in the optimal allocation (2.5) is positive, the planner can set the price for coupons to be a dollars each for those who want to sell their coupons, and offer to those who want to acquire coupons a menu of two options, one to buy B coupons at the unit price equal to $b - x(b-a)/B$ (thereby guaranteeing vaccination for sure), the other to buy x coupons at the unit price a (thereby getting vaccinated with probability $(x+1)\alpha$).

Contrary to the vaccine wastefulness problem of the mechanisms in current practice, none of the optimal allocations prescribed above leaves any vaccine unused. That is due to the market clearing condition satisfied by the optimal allocations. According to (2.5), the total quantity of demand for vaccination within the priority group is equal to

$$\begin{aligned}
& ((B+1)(1-F(b)) + (x+1)(F(b)-F(a)))\,\alpha \\
= \;& (B(1-F(b)) + x(F(b)-F(a)) + 1 - F(b) + F(b) - F(a))\,\alpha \\
= \;& (1 \cdot F(a) + 1 - F(b) + F(b) - F(a))\,\alpha \quad \text{(market clearing)} \\
= \;& \alpha,
\end{aligned}$$

which is equal to the total quantity of vaccine supply available to the group.

The bottom line is: A market-like Covid-vaccine distribution mechanism, which sets at most two prices for vaccination entitlements to stratify the population of a same priority group

[15] We thank the associate editor for suggesting comorbidity as a direction to interpret the welfare weights.

into at most three tiers and uses rationing to at most one of the three, would outperform the current within-group egalitarian rationing mechanism.

2.7 Conclusion

It is common that individuals start on an equal footing and end with different outcomes, just because of the idiosyncrasy in one's ability, taste or pure luck. It is also common that such inequality in the outcomes, like it or not, is often class-oriented, grouping individuals into several tiers and treating the members of each roughly indiscriminately. This paper provides a mathematical fable for such stratification among humanity. It says that, even in the idealized situation where the society is framed in an interim incentive-constrained Pareto optimal manner, stratification is still unavoidable and, in fact, necessary for the social wellbeing. Meanwhile, our finding implies that stratification of more than three tiers is unnecessary, and often suboptimal. Consequently, while the people should be stratified into at least two tiers and, due to the market clearing condition, there should be at least one tier for the haves and another for the have-nots, oftentimes there should not be more than two subdivisions in either category. Thus, while the bisection of the rich into East Egg and West Egg, under the penetrating pen of F. Scott Fitzgerald, may be understood as part of a three-tier optimal allocation, any further subdivision of either Egg is likely suboptimal.

An insight from this study is that, in a large market where individuals are free to choose between buying and selling, a single competitive market price—without the help from any other instruments such as rationing, redistribution or tier-specific prices—is often capable of implementing interim Pareto optimality despite the presence of asymmetric information. It should be emphasized that such robustness of the competitive market is not an artifice of any specific social welfare criterion say a pro-market value system; but rather it holds true for a wide variety of welfare weight distributions that may favor one type or another, as long as the welfare weight is not overly contracted towards the market-clearing cutoff type. From such a relatively value-free perspective, one could understand the institutional evolution in the United States regarding the allocation of the radio frequency spectrum, from hearings and lotteries to market-like auctions, as a movement towards the Pareto frontier that should still have happened even if the policymaker's objective is something other than to raise revenues or to develop the wireless industries. From the same perspective one could see a robust normative force towards market-oriented solutions to problems of prioritizing citizens for the access to limited resources, be they Covid vaccines or magnet schools.

Bibliography

[1] Mohammad Akbarpour, Eric Budish, Piotr Dworczak, and Scott Duke Kominers. An economic framework for vaccine prioritization. Working Paper, December 18, 2021.

[2] Mohammad Akbarpour, Piotr Dworczak, and Scott Duke Kominers. Redistributive allocation mechanisms. Working Paper, December 16, 2020.

[3] Jeremy Bulow and John Roberts. The simple economics of optimal auctions. *Journal of Political Economy*, 97(5):1060–1090, 1989.

[4] Hung-Ken Chien. Incentive efficient mechanism for partnership. Mimeo, June 1, 2007.

[5] Peter Cramton, Robert Gibbons, and Paul Klemperer. Dissolving a partnership efficiently. *Econometrica*, 55(3):615–632, May 1987.

[6] Piotr Dworczak, Scott Duke Kominers, and Mohammad Akbarpour. Redistribution through markets. *Econometrica*, 89(4):1665–1698, 2021.

[7] Thomas Gresik. Incentive-efficient equilibria of two-party sealed-bid bargaining games. *Journal of Economic Theory*, 68:26–48, 1996.

[8] Bengt Holmström and Roger Myerson. Efficient and durable decision rules with incomplete information. *Econometrica*, 51(6):1799–1820, 1983.

[9] Emir Kamenica and Matthew Gentzkow. Bayesian persuasion. *American Economic Review*, 101:2590–2615, 2011.

[10] Mingshi Kang and Charles Z. Zheng. Pareto Optimality of Allocating the Bad. Mimeo, December 2020.

[11] Zi Yang Kang. Optimal public provision of private goods. Mimeo, Stanford USB, December 2020.

[12] Thomas Kittsteiner. Partnerships and double auctions with interdependent valuations. *Games and Economic Behavior*, 44:54–76, 2003.

[13] Didier Laussel and Thomas Palfrey. Efficient equilibria in the voluntary contributions mechanism with private information. *Journal of Public Economic Theory*, 5:449–478, 2003.

[14] John Ledyard and Thomas Palfrey. A characterization of interim efficiency with public goods. *Econometrica*, 67:435–448, 1999.

[15] John Ledyard and Thomas Palfrey. A general characterization of interim efficient mechanisms for independent linear environments. *Journal of Economic Theory*, 133:441–466, 2007.

[16] Maël Le Treust and Tristan Tomala. Persuasion with limited communication capacity. *Journal of Economic Theory*, 184:10940, 2019.

[17] Simon Loertscher and Cédric Wasser. Optimal structure and dissolution of partnerships. *Theoretical Economics*, 14:1063–1114, 2019.

[18] Hu Lu and Jacques Robert. Optimal trading mechanisms with ex ante unidentifiable traders. *Journal of Economic Theory*, 97:50–80, 2001.

[19] Roger Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, February 1981.

[20] Roger Myerson and Mark A. Satterthwaite. Efficient mechanisms for bilateral trading. *Journal of Economic Theory*, 29:265–281, 1983.

[21] Tymofiy Mylovanov and Thomas Tröger. Mechanism design by an informed principal: Private values with transferable utility. *Review of Economic Studies*, 81:1668–1707, 2014.

[22] Mikel Pérez-Nievas. Interim efficient allocation mechanisms. Working Paper 00-20, Departmento de Economia, Universidad Carlos III de Madrid, February 2000.

[23] Marco Reuter and Carl-Christian Groh. Mechanism design for unequal societies. Discussion Paper No. 228, November 2020.

[24] H.L. Royden and P.M. Fitzpatrick. *Real Analysis*. Pearson Education, Boston, 4th edition, 2010.

[25] Ilya Segal and Michael Whinston. Property rights and the efficiency of bargaining. *Journal of the European Economic Association*, 14:1287–1328, 2016.

[26] Tayfun Sömet, Parag A. Pathak, M. Utku Ünever, Govind Persad, Robert D. Truog, and Douglas B. White. Categorized priority systems. *Humanities: Vantage*, 159(3):1294–1299, 2021.

[27] Robert Wilson. Incentive efficiency of double auctions. *Econometrica*, 53:1101–1105, 1985.

[28] Luyi Yang, Laurens Debo, and Varun Gupta. Trading time in a congested environment. *Management Science*, 63:2377–2395, 2017.

[29] Luyi Yang, Zhongbin Wang, and Shiliang Cui. A model of queue scalping. *Management Science*, 2021. Forthcoming.

Chapter 3

Pareto Optimality of Allocating the Bad

3.1 Introduction

The decision on where to locate a Nimby—a "not in my backyard" type of noxious facility—has been considered as a procurement auction by Kunreuther and Kleindorfer [8], with alternative locations treated as bidders for the contract of hosting the Nimby. What has not been considered theoretically is why the Nimby should exist at all and be located somewhere. An answer to this question in general would require comparison between the Nimby's public good to the society and its private bad to its host. While the private bad is usually apparent, its public good is far from obvious. Debates about the net public benefit of a Nimby are often contentious (e.g., the location of an oil pipeline terminal in Canada), and the conclusion thereof results more from endogenous politics than exogenous nature. In this paper we treat a Nimby as purely a private bad to its host, assuming away its public good and allowing the planner to do away with a Nimby at no cost. Since our conclusion is that it is interim Pareto optimal to allocate such a pure bad to someone sometimes, the argument for the necessity of a Nimby can only strengthen if its public benefit is taken into account.

We thus abstract a Nimby into a *bad*, an item that gives its recipient a utility below the outside option and has no effect on the utility of anyone else. Other than Nimbies, a bad can also be the status of being excluded from an otherwise publicly available service that has been taken for granted. This paper develops a mechanism design method to characterize the set of primitives given which allocating a bad to some society members is necessary to achieve social optimality.

To reduce the arbitrariness in parameters, we assume quasilinear preferences and independent private values (IPV) as usual in mechanism design. Suppose that receiving the bad means a negative payoff $-t_i$ to player i whose realized type is $t_i \in \mathbb{R}_+$ and, if in addition i receives a money transfer m, i's payoff is equal to $-t_i + m$. For anyone to be willing to receive the bad, the society needs to compensate the recipient with money. To raise funds for such compensation, there needs to be a good available for allocation as well that gives its recipient a utility above the outside option. An example can be the privilege of hosting a popular game (whose externalities to other regions we assume away as we do to the Nimby). To minimize the departure from the standard, unidimensional-type framework, we assume that if player i gets the good with probability q_A and the bad with probability q_B, combined with net money receipt $m \in \mathbb{R}$,

i's expected payoff is equal to $(q_A - q_B)t_i + m$.[1]

Nothing in this setup forces the allocation of the bad. In fact, one readily sees that never allocating the bad, coupled with allocating the good to the highest realized type, is ex ante incentive efficient.[2] However, even if they have agreed on this allocation ex ante, some players may have second thoughts during the interim, when each is privately informed of one's own type. In the context of universal healthcare coverage, for instance, a player who turns out to be extremely healthy would rather opt out of the coverage in return for money and may push for such an alternative. Thus the allocation that the society ends with depends on the bargaining power across player-types during the interim. In the spirit of Holmström and Myerson's [5] interim incentive efficiency (IIE), let us think of any idealized outcome of this interim bargaining process as if it were an optimum chosen by a social planner whose objective, or social welfare, is a weighted sum of the interim expected payoffs across player-types, with the various welfare weights of player-types capturing their bargaining power relative to one another.[3] Thus the question becomes, in an environment where never allocating the bad is part of an ex ante incentive efficient allocation, under what condition of the welfare weight distribution is the bad allocated with strictly positive probabilities in *all* mechanisms that maximize the social welfare subject to incentive compatibility (IC), individual rationality (IR) and budget balance (BB)?

IIE has been investigated in various models by Dworczak, Kominers and Akbarpour [3], Gresik [4], Laussel and Palfrey [9], Ledyard and Palfrey [10, 11], Pérez-Nievas [20] and Wilson [23]. Although the bad has no counterpart in these models except [11], and allocation of the bad is not the focus in [11],[4] from their characterizations one could get an intuition that introducing a bad might enlarge the social welfare. The typical pattern is that types are ranked according to their virtual surpluses, with types of higher ranks getting the good with higher probabilities, provided that their virtual surpluses are nonnegative and the incentive constraints non-binding. Thus, it appears natural that types with negative virtual surpluses should be allocated a bad, if available, since the bad is opposite to the good.

For this intuition to lead to a primitive condition under which the bad is allocated, however, one needs to handle the discontinuity of the virtual surplus function at any type whose allocation switches from getting the good in expectation to getting the bad in expectation. Thus, if the designer allocates the bad to the types whose virtual surpluses from getting the good are negative, she may be making a mistake because the virtual surpluses of these types from getting the bad should have been calculated differently. The complication comes from the endogenous buyer-seller role for each player. If a player's type is likely to be allocated the good in the mechanism under consideration, the player acting as a buyer of the good would understate his type. By contrast, if his type is likely a recipient of the bad, the player acting as a

[1] The paper can be easily generalized to the case where $(q_A - q_B)t_i + m$ is replaced by $(q_A - \alpha_i q_B)t_i + m$ for some commonly known, player-specific parameter $\alpha_i \in (0, 1)$.

[2] See Footnote 14 or Corollary 7 for a proof on the ex ante incentive efficiency of this allocation. That implies the allocation is also interim and ex post incentive efficient given appropriate welfare weight distributions (cf. Holmström and Myerson's [5]).

[3] Ledyard and Palfrey [10] provide a forceful motivation for such positive interpretations of IIE. Even from a purely normative viewpoint, one can readily relate to real-world situations where the social planner favors some types against others—such as transferring money from the rich to the poor—and wants to choose an optimal allocation according to her biased value judgement, a particular welfare weight distribution.

[4] Ledyard and Palfrey [11] use the Kuhn-Tucker method. Due to its limitation according to our Theorem 8 (explained later), this method could not have led to our result that the bad is needed for social optimality.

seller of such a costly service would exaggerate his type. Consequently, the operation of integrating a player's surplus from an allocation bifurcates between different measures, depending on whether the player is in expectation allocated the good—so his reduced form allocation is positive—or in expectation allocated the bad—with reduced-form allocation negative. In other words, the expected surplus is not a linear functional of allocations.

Without a linear structure, the method to characterize optimal mechanisms is usually a local one à la the Kuhn-Tucker theorem, but the method is usually limited by its local nature and, as part of our results shows, cannot address our question. Since the virtual surplus intuition corresponds to the first-order condition derived from such methods, they cannot warrant its validity here.

This paper thus adopts a global approach. We characterize the optimal mechanisms by a saddle point condition, which is not only sufficient but also necessary for any mechanism to be optimal (Theorem 6). We obtain this condition through formulating the aforementioned integration into a *two-part operator* on the allocations. The operator integrates the positive part of an allocation with one measure and integrates the negative part thereof with another measure. The positive part of an allocation is defined on the types that are more likely to get the good than the bad, and the negative part, more likely to get the bad than the good. Albeit nonlinear, this operator is always concave on the space of allocations. This drives the necessity of the saddle point condition.

From the saddle point condition we obtain a necessary and sufficient condition, in terms of the distribution of types and that of welfare weights, for every optimal mechanism to allocate the bad with a strictly positive probability (Theorem 7). One corollary is that, if the welfare weight distribution is replaced by a second-order stochastically dominated one, the bad is allocated with strictly positive probabilities in all optimal mechanisms if it is so before the substitution (Corollary 5). Another corollary is an unrestrictive condition sufficient for all optimal mechanisms to allocate the bad with strictly positive probabilities: all that we need is that the welfare density (Radon-Nikodym derivative of the welfare weight distribution with respect to the type distribution) around the infimum of the type support be more than twice the average welfare density (Corollary 6). Thus, the social optimality of allocating the bad is not at all sensitive to the particular forms of the welfare density or the type distribution. In particular, a social planner would still allocate the bad even when she assigns most of the welfare weight to high types.[5]

As our model does not force the necessity of the bad, Corollary 7 says that no optimal mechanism allocates the bad at all if the welfare weight distribution second-order stochastically dominates the exogenous distribution of types. Hence a social planner allocates the bad only if she favors the low and high types against the middle ones.

To demonstrate the generality of our method, and to gain understanding of all optimal mechanisms that need the bad as an instrument, we prove that the Kuhn-Tucker method used in the literature could not have led to our finding: The literature applies the method to a relaxed problem that is valid only if its solution happens to satisfy a monotonicity condition—the second-order part of IC—set aside by the relaxed problem. We find that if the bad is allocated

[5]For example, suppose that almost the entire welfare weight is assigned to a small interval at the supremum type 1, with a tiny weight say 3ϵ uniformly distributed to the elements of a tiny interval $[0, \epsilon]$ at the infimum. Then Corollary 6 says that the social planner would still allocate the bad with a strictly positive probability.

with strictly positive probabilities in all optimal mechanisms then the relaxed problem has only two alternatives: either it admits no optimal mechanism as a solution, or it suffers indeterminacy in admitting a continuum of solutions to the relaxed problem (Theorem 8). Furthermore, the first alternative, which means that any solution to the relaxed problem violates the monotonicity constraint, is generic (Corollary 8). Thus, generically speaking, when the bad is needed for social optimality, the condition derived from the Kuhn-Tucker method is vacuous, and any optimal mechanism entails rationing across some types because the monotonicity constraint is binding.

Theorems 7 and 8 are established on the saddle point condition (Theorem 6) of any optimal mechanism. The complication is that the associated Lagrangian is a two-part operator acting on reduced-form allocations with the aforementioned, sign-specific measures. To exploit the saddle point condition despite the complication, we develop a perturbation method. The idea is to perturb an allocation without altering the sign of its reduced form at any type (nonnegative for buyer types, and nonpositive for seller types) so that the measure acting on the reduced form remains unchanged. Then the Lagrangian becomes linear in the perturbations. That allows us to characterize any maximizer of the Lagrangian through perturbations along the direction of the measure, as long as the desired direction can be achieved in an ex post feasible manner. To that end, we formulate a family of ex post feasible, sign-preserving perturbations, thereby obtaining conditions necessary for any maximizer of the Lagrangian associated with the saddle point condition (Section 3.4.2 and Appendix B.6.2).

By the monotonicity condition of the IC constraint, if the bad is allocated at all, it is allocated to low types. Thus, Corollaries 5 and 6 imply that a social planner should allocate the bad to low types if she either spreads more welfare weights to low types in general, or cares enough about the extreme low types in particular. For instance, since a player's type is equal to his marginal rate of substitution between consumption and money, one may think of a low type as a financially constrained consumer. In such a context, an implication is that a social planner who cares enough about the extreme poor should buy the poor out of the coverage of the benefit under consideration.

We are aware of two other global methods to handle nonlinear problems in mechanism design. One is Toikka's [22] generalized ironing technique to maximize a concave functional of monotone real functions. This could be relevant to the Lagrange problem associated with our saddle point condition, which has absorbed the budget balance constraint of our original problem. However, the Lagrangian does not satisfy the differentiability assumption in Toikka (nor the discreteness assumption in the online supplement). Moreover, it remains to be seen whether the single-agent assumption of the method can be removed to accommodate the ex post feasibility constraint in an auction model such as our Lagrange problem.

The other method is to turn the set of ex post feasible IC reduced form allocations into a family of monotone real functions majorized by a known function and characterize the maxima of a convex functional on this family as its extreme points (Kleiner, Moldovanu and Strack [7]). In our model, since the Lagrangian is not convex, the only case to which this method could be applicable is our original optimization problem subject to an additional restriction that mechanisms be symmetric across players (so that each allocation corresponds to a real function). However, we also require the budget balance condition, and a bad is to be allocated alongside with a good (rather than two goods to be allocated). It remains to be seen whether ex post feasibility can be captured by a single majorization relation and whether the choice set can be equal to a majorization family with respect to such a relation.

Our model shares a similar feature with the partnership dissolution literature in that the buyer or seller role of a player is endogenous (Cramton, Gibbons and Klemperer [2], Chien [1], Loertscher and Wasser [13], Lu and Robert [14], Mylovanov and Tröger [19], and Segal and Whinston [21]). Selling one's partnership share in that framework corresponds to being allocated the bad in our model. However, since partnership dissolution requires market clearance in the trading of shares, it is out of the question in that framework whether a bad should be allocated at all. Nevertheless, our saddle point characterization, giving a necessary and sufficient condition for any interim Pareto optimal mechanism subject to IC, IR and BB, is applicable to partnership dissolution given independent private values (Remark B.7, Appendix B.7). Another difference is in the design objectives. We consider interim Pareto optimality (namely, IIE), which allows for any welfare weights varying across player-types. By contrast, the design objective regarding partnership dissolution is a sum of surpluses with welfare weights uniform across types. It has been the simple sum of surpluses across players with uniform welfare weights in much of the literature.[6] Recently, the objective is a weighted average—with type-independent weights—between the expected revenue and the winner's surplus in Lu and Robert and in Loertscher and Wasser, and the ex ante surplus of one of the players (informed principal) in Mylovanov and Tröger.

With both the buyer and seller roles possible to each player, a player's type at which the participation constraint binds is not determined a priori. Thus our model is somewhat related to the countervailing incentives literature such as Lewis and Sappington [12], Maggi and Rodríguez-Clare [16], and Jullien [6]. Their focus is to address the issue that full participation may cause loss of generality and to exploit the curvature of the agent's reservation utility function for explanations of various pooling properties in the principal's optimal mechanism. By contrast, there is no loss to assume full participation in our model, and our focus is to resolve the nonlinearity problem caused by the endogenous buyer-seller role in an otherwise linear structure. In addition, given the endogenous discontinuity of the virtual surplus functions, the Hamiltonian technique in [6] is inapplicable to our model.

3.2 The Model

3.2.1 The Good, the Bad, and n Players

There are two items, named A and B, and n players ($n \geq 2$), each of whom can be allocated one or both or none of the items. Each player i's private information at the outset, or type t_i, is independently drawn according to the same cumulative distribution function F with density f strictly positive on the support $[0, 1]$.[7] Given type t_i, if player i gets item A with probability x_{iA}, item B with probability x_{iB}, and delivers money transfer in the amount $y_i \in \mathbb{R}$ (negative y_i meaning i being the recipient of money), player i's payoff is equal to

$$(x_{iA} - x_{iB})\, t_i - y_i. \tag{3.1}$$

[6]Much of the partnership dissolution literature focuses on the implementability of one particular winner-selection rule, the efficient allocation, which is optimal only if it is implementable and only if the design objective is the simple sum of the surpluses across players.

[7]See Appendix B.7 for a generalization that allows for player-specific distributions.

Hence item A is interpreted as a good, and item B a bad, to all players; t_i corresponds to the intensity of player i's preference for the good over the bad.

3.2.2 Allocations and Mechanisms

An *ex post allocation* means a list $(q_{iA}, q_{iB})_{i=1}^n$ of functions such that $q_{iA}, q_{iB} : [0,1]^n \to [0,1]$ for each i and, for each $t \in [0,1]^n$,

$$\sum_i q_{iA}(t) \leq 1 \quad \text{and} \quad \sum_i q_{iB}(t) \leq 1. \tag{3.2}$$

An *ex post payment rule* means a list $(p_i)_{i=1}^n$ of functions such that $p_i : [0,1]^n \to \mathbb{R}$ for each i. By the revelation principle, any equilibrium-feasible mechanism corresponds to a pair of ex post allocation $(q_{iA}, q_{iB})_{i=1}^n$ and ex post payment rule $(p_i)_{i=1}^n$, with $q_{ij}(t)$ interpreted as the probability with which item j ($j \in \{A, B\}$) is assigned to player i, and $p_i(t)$ the net money transfer from player i to others, when t is the profile of alleged types across players.

For each player i, denote F_{-i} for the product measure on $[0,1]^{n-1}$ generated by F on each subspace $[0,1]$. A *mechanism (in reduced form)* means a list $(Q_i, P_i)_{i=1}^n$, often abbreviated as (Q, P), of functions $Q_i : [0,1] \to \mathbb{R}$ and $P_i : [0,1] \to \mathbb{R}$ ($\forall i = 1, \ldots, n$) such that, for some ex post allocation-payment rule $(q_{iA}, q_{iB}, p_i)_{i=1}^n$, Q_i is the marginal of $q_{iA} - q_{iB}$, and P_i the marginal of p_i, onto the i^{th} dimension. That is, for any i and any $t_i \in [0,1]$,

$$Q_i(t_i) = \int_{[0,1]^{n-1}} (q_{iA}(t_i, t_{-i}) - q_{iB}(t_i, t_{-i}))\, dF_{-i}(t_{-i}) \tag{3.3}$$

and $P_i(t_i) = \int_{[0,1]^{n-1}} p_i(t_i, \cdot) dF_{-i}$ for any t_i and any i. The part $(Q_i)_{i=1}^n$ in $(Q_i, P_i)_{i=1}^n$ is called (reduced-form) *allocation*. Call $(Q_i)_{i=1}^n$ the *reduced form* of $(q_{iA}, q_{iB})_{i=1}^n$ if and only if (3.3) holds for all i and all t_i.

Remark *Eq. (3.3) is the construct that sets our model apart from the existing optimal auction framework. The feature of (3.3) is that the reduced form allocation to a player can be positive or negative, and the sign thereof is endogenous. As we will see in Section A.2, such endogenous signing of the allocation results in a nonlinear structure. This feature of endogenous signing stems from the assumption that the two items up for allocation point to opposite directions with respect to a player's nonparticipation payoff—which we normalize to zero without loss of generality—with the good generating a payoff larger than, and the bad less than, the nonparticipation payoff. That is why the role of the bad in our model cannot be replaced by a lesser good as long as the utility of the lesser good is still larger than the nonparticipation payoff.*

3.2.3 Constraints

Given any (reduced-form) mechanism $(Q_i, P_i)_{i=1}^n$, it follows from the quasilinear utility function postulated previously that the interim expected payoff for any type t_i of player i to act as a type $\hat{t}_i$, given truthtelling from others, is equal to $Q_i(\hat{t}_i)t_i - P_i(\hat{t}_i)$. Denote

$$U_i(t_i \mid Q, P) := \max_{\hat{t}_i \in [0,1]} Q_i(\hat{t}_i)t_i - P_i(\hat{t}_i). \tag{3.4}$$

As is routine in auction theory, *incentive compatibility* (IC) of $(Q_i, P_i)_{i=1}^n$ is equivalent to simultaneous satisfaction of two conditions for each player i: (i) Q_i is weakly increasing on $[0,1]$; (ii) for any $t_i, t_i^0 \in [0,1]$,

$$P_i(t_i) - P_i(t_i^0) = \int_{t_i^0}^{t_i} s dQ_i(s). \tag{3.5}$$

We assume that each player can opt out of a mechanism before it operates thereby getting zero as the outside payoff. Thus $(Q_i, P_i)_{i=1}^n$ is said *individually rational* (IR) if and only if $U_i(t_i|Q,P) \geq 0$ for all i and all $t_i \in T_i$.

For the society consisting of the n players to transfer wealth among themselves without relying on outside subsides, we require that a mechanism be budget-balanced: $(Q_i, P_i)_{i=1}^n$ satisfies *budget balance* (BB) if and only if $\sum_i \int_0^1 P_i(t_i)dF(t_i) \geq 0$.[8]

3.2.4 Interim Incentive-Constrained Pareto Optimality

While our method applies to cases where players may weigh differently in the social welfare (Appendix B.7), to focus on transfers across types, the main text presents only welfare weights that are neutral across players, who are assumed ex ante symmetric (again for notational simplicity). Thus, by *welfare density* we mean a function $w : [0,1] \to \mathbb{R}_{++}$ for which $\int_0^1 w dF = 1$. Given any welfare density w, the mechanism design problem is to maximize

$$\sum_{i=1}^n \int_0^1 U_i(t_i \mid Q,P) w(t_i) dF(t_i) \tag{3.6}$$

among all mechanisms (Q,P) that are IC, IR and BB.[9] It is obvious that any solution say (Q^*, P^*) to this problem is interim incentive-constrained Pareto optimal.[10] That is, there does not exist another IC, IR and BB mechanism (Q,P) for which $U_i(\cdot|Q,P) \geq U_i(\cdot|Q^*,P^*)$ a.e. on $[0,1]$ for all players i and, for some player i, $U_i(\cdot|Q,P) > U_i(\cdot|Q^*,P^*)$ on a positive-measure subset of $[0,1]$.[11]

3.3 Saddle Point Characterization

To state the saddle point characterization for all *optimal mechanisms* (maximizers of (3.6) subject to IC, IR and BB), we need to introduce a notation for a nonlinear, *two-part operator* on allocations. A crucial property of this operation is its concavity, which drives the necessity of

[8]There is no substantive difference between such ex ante condition for budget balance and its ex post counterpart. Mimicking the proof of Lemma 4 of Cramton, Gibbons and Klemperer [2], one can prove that if $\sum_i \int_0^1 P_i(t_i)dF(t_i) \geq 0$ then $(P_i)_{i=1}^n$ is the profile of the marginals of an ex post payment profile $(p_i)_{i=1}^n$ for which $\sum_i p_i(t) \geq 0$ for all $t \in [0,1]^n$.

[9]By definition, any mechanism is ex post feasible in the sense of being the reduced form of some ex post allocation-payment rule (Section 3.2.2).

[10]Interim incentive-constrained Pareto optimality is the same as interim incentive efficiency (IIE) if the IR and BB constraints are added to the IIE framework, which usually considers only IC.

[11]Since F is absolutely continuous in Lebesgue measure by assumption, the notion of measure zero with respect to F is equivalent to that with respect to Lebesgue measure restricted to $[0,1]$.

the saddle point condition. Section A.2 motivates the notation from the endogenous discontinuity of the virtual surplus function in our model. Section 3.3.2 defines the notation.

3.3.1 Nonlinearity from Having Both a Good and a Bad

Given any welfare density w and any IC mechanism (Q, P), the objective (3.6) is equal to

$$\sum_i \int_0^1 Q_i(t_i)t_i dW(t_i) - \sum_i \int_0^1 P_i(t_i)dW(t_i), \tag{3.7}$$

where we define

$$W(t_i) := \int_0^{t_i} w(s)dF(s) \tag{3.8}$$

for any $t_i \in [0, 1]$. Note that W is a cdf with support $[0, 1]$. We shall call W *welfare weight distribution*.

By the routine of envelope theorem and integration by parts, one obtains that (Appendix B.2), for any $t^0 \in [0, 1]$ and any IC mechanism (Q, P),

$$\begin{aligned}\int_0^1 P_i(t_i)dW(t_i) \;=\; & \int_0^1 Q_i(t_i)t_i dW(t_i) + \int_0^{t^0} Q_i(t_i)W(t_i)dt_i - \int_{t^0}^1 Q_i(t_i)\left(1 - W(t_i)\right) dt_i \\ & -U_i(t^0|Q, P).\end{aligned} \tag{3.9}$$

To keep track of the IR and BB constraints, this equation is useful only when $U_i(t^0|Q, P) = \min_{[0,1]} U_i(\cdot|Q, P)$. By IC and the envelope equation (3.5), $U_i(\cdot|Q, P)$ is convex and its derivative is equal to Q_i a.e. Thus, if $U_i(\cdot|Q, P)$ attains its minimum at t^0 then $Q_i \le 0$ on $[0, t^0)$, and $Q_i \ge 0$ on $(t^0, 1]$. Hence (3.9) implies

$$\begin{aligned}\int_0^1 P_i\, dW \;=\; & \int_0^1 Q_i(t_i)t_i\, dW(t_i) + \int_0^1 \left(-Q_i^-(t_i)\right) W(t_i)dt_i + \int_0^1 Q_i^+(t_i)\left(-1 + W(t_i)\right) dt_i \\ & - \min_{[0,1]} U_i(\cdot|Q, P),\end{aligned} \tag{3.10}$$

where

$$Q_i^-(s) := \max\{-Q_i(s), 0\} \quad \text{and} \quad Q_i^+(s) := \max\{Q_i(s), 0\}.$$

Eq. (3.10) reveals that the cdf W acts on an allocation Q_i in a bifurcated manner. When Q_i is positive (player i acting like a buyer in expectation), the marginal utility t_i is reduced by $1 - W(t_i)$; when Q_i is negative (i acting like a seller in expectation), the marginal cost t_i is increased by $W(t_i)$. Thus, $1 - W(t_i)$ and $W(t_i)$ correspond to a type-t_i player's information rent, bifurcated between the player's buyer and seller roles, that takes into account the weight $w(t_i)$ in the social welfare.[12] It is important to note that this action of W, or the right-hand side on the first line of (3.10), is not a linear functional on the space of Q_i unless the space is restricted by either nonnegativity (availability of only goods) or nonpositivity (availability of only bads). Such nonlinearity is precisely the effect of having both a good and a bad for allocation in our otherwise linear, standard model.

[12] In the special case where $w = 1$ on $[0, 1]$, $W = F$ and $1 - W(t_i)$ and $W(t_i)$ become the recognizable information rents in the optimal auction and optimal procurement models.

3.3.2 Two-Part Operators

To focus attention to the nonlinear action of W, and to apply the same kind of actions to other distributions, we abstract from (3.10) a two-part operator defined below. For any function $\varphi : \mathbb{R} \to \mathbb{R}^2$ denoted by $\varphi := (\varphi_+, \varphi_-)$ such that $\varphi_+, \varphi_- : \mathbb{R} \to \mathbb{R}$, denote

$$\langle Q_i : \varphi| := \int_0^1 Q_i^+(s)\varphi_+(s)ds + \int_0^1 \left(-Q_i^-(s)\right)\varphi_-(s)ds.$$

Note that the operator $Q_i \mapsto \langle Q_i : \varphi|$ acts on the function Q_i in two parts, one on the positive part Q_i^+, the other on the negative part $-Q_i^-$. Hence we call $\langle\cdot : \varphi|$ *two-part operator*. The asymmetric bracket of Q_i and φ is to highlight the asymmetry between the two arguments: $\langle Q_i : \varphi|$ is a nonlinear functional of Q_i and yet a linear functional of φ.

The φ in $\langle Q_i : \varphi|$ corresponds to the information rent density derived from an underlying distribution. In general, by *distribution* on $[0,1]$ we mean a function $G : \mathbb{R} \to \mathbb{R}_+$ that is weakly increasing, right-continuous, vanishing on $(-\infty, 0)$, and equal to $\max_{\mathbb{R}} G$ on $[1, \infty)$. For any distribution G on $[0,1]$, define a function $\rho(G) : \mathbb{R} \to \mathbb{R}^2$ by

$$\rho(G) := (\rho_+(G), \rho_-(G))$$

such that, for all $s \in \mathbb{R}$,

$$\rho_+(G)(s) := -G(1) + G(s) \quad \text{and} \quad \rho_-(G)(s) := G(s). \tag{3.11}$$

Thus, for any distribution G on $[0,1]$ and any $Q_i : [0,1] \to \mathbb{R}$, the above notation implies

$$\langle Q_i : \rho(G)| = \int_0^1 Q_i^+(s)\rho_+(G)(s)ds + \int_0^1 \left(-Q_i^-(s)\right)\rho_-(G)(s)ds. \tag{3.12}$$

By (3.12), we generalize (3.10) to all distributions G on $[0,1]$ and all IC mechanisms (Q, P):

$$\int_0^1 P_i\, dG = \int_0^1 Q_i(t_i)t_i\, dG(t_i) + \langle Q_i : \rho(G)| - G(1) \min_{[0,1]} U_i(\cdot|Q, P). \tag{3.13}$$

Comparing (3.13) with (3.10), we see that $\rho_+(G)$ reflects i's information rent density when i acts as a buyer in expectation, and $\rho_-(G)$, i's information rent density when i acts as a seller in expectation, had i's type been measured by G.[13]

3.3.3 The Lagrangian

Denote $\mathcal{Q}$ for the space of all reduced-form allocations $(Q_i)_{i=1}^n$ (each being the reduced form of an ex post allocation according to (3.3)). Denote $\mathcal{Q}_{\text{mon}}$ for the set of $(Q_i)_{i=1}^n \in \mathcal{Q}$ such that Q_i is weakly increasing on $[0,1]$ for any i. For any welfare density w, define W according to (3.8). For any $Q := (Q_i)_{i=1}^n \in \mathcal{Q}$ and any $\lambda \in \mathbb{R}_+$, define

$$\mathcal{L}(Q, \lambda) := \sum_i \int_0^1 Q_i(t_i)\left((1+\lambda)t_i - \frac{W(t_i) - F(t_i)}{f(t_i)}\right) dF(t_i) + \lambda \sum_i \langle Q_i : \rho(F)| \,. \tag{3.14}$$

[13] In the main text, G can be W as in (3.10), or F as in the next section. In general (Appendix B.7), G can be any multiple of F or W.

Theorem 6 *Given any welfare density w, there exists a payment rule P^* with which a mechanism (Q^*, P^*) maximizes (3.6) subject to IC, IR and BB if and only if there exists $\lambda \in \mathbb{R}_+$ such that, for all $Q \in \mathcal{Q}_{\text{mon}}$ and all $\lambda' \in \mathbb{R}_+$,*

$$\mathcal{L}(Q^*, \lambda') \geq \mathcal{L}(Q^*, \lambda) \geq \mathcal{L}(Q, \lambda). \tag{3.15}$$

Proof First, the problem of maximizing (3.6) subject to IC, IR and BB is equivalent to

$$\max_{Q \in \mathcal{Q}_{\text{mon}}} \quad \sum_i \int_0^1 Q_i(t_i)\,(t_i f(t_i) - W(t_i) + F(t_i))\,dt_i \tag{3.16}$$

$$\text{s.t.} \quad \sum_i \int_0^1 Q_i(t_i) t_i\,dF(t_i) + \sum_i \langle Q_i : \rho(F)| \geq 0. \tag{3.17}$$

The domain $\mathcal{Q}_{\text{mon}}$ captures the ex post feasibility requirement (that $Q \in \mathcal{Q}$) and the second-order part of IC (monotonicity of Q). Ineq. (3.17) is the joint constraint of IR, BB and the first-order part of IC (Lemma 20, Appendix B.3.1). It is obtained through applying (3.13) to the case $G = F$ for all players i, summing such equations across i, and then use IR and BB. The objective (3.16) is obtained through calculating the social welfare (3.6) generated by a mechanism (Q, P) that takes into account of the optimal choice of the payment rule among those that implement any given Q. The proof amounts to calculating the optimal amount of lump sum transfers to be redistributed (Lemma 21, Appendix B.3.2).

Second, the set of all $(Q_i)_{i=1}^n \in \mathcal{Q}_{\text{mon}}$ that satisfy (3.17) is a convex set. That is because the domain $\mathcal{Q}_{\text{mon}}$ is convex (Appendix B.3.3), and the mapping $(Q_i)_{i=1}^n \mapsto \sum_i \langle Q_i : \rho(F)|$ a concave functional on $\mathcal{Q}$ (Lemma 22, Appendix B.3.4). Such concavity is driven by the fact that a player tends to shade the marginal value by a price discount $\rho_+(F)$ when acting like a buyer (when $Q_i = Q_i^+$), and exaggerate the marginal cost by a price markup $\rho_-(F)$ when acting like a seller (when $Q_i = -Q_i^-$). One can also show that there exists a $(Q_i)_{i=1}^n \in \mathcal{Q}_{\text{mon}}$ such that (3.17) is satisfied strictly (Appendix B.3.5). Consequently, the conditions corresponding to those in Luenberger [15, Corollary 1, p219] are satisfied. Hence the saddle point condition is necessary and sufficient for any solution to Problem (3.16)–(3.17).

Remark *To appreciate the succinct two-part operator notation, consider the counterparts of (3.17) in the literature. In the bilateral trade model of Myerson and Satterthwaite [18], where a player's buyer or seller role is exogenous, the counterpart to this constraint is their Ineq. (2), the right-hand side of which can be split into two integrals, one being an integral of the valuations v_2 and v_1, the other an integral of the information rents $(1 - F_i(v_i))/f_i(v_i)$ and $F_i(v_i)/f_i(v_i)$. The first integral corresponds to our first integral in (3.17), and the second integral, our two-part operation (3.17). Note, however, that their counterpart to our two-part operation is a linear functional of their allocation p. That is because a player in their model is a priori either a buyer or a seller, hence the two-part operation reduces to*

$$\int Q_{\text{buyer}}(t_{\text{buyer}})\left(-1 + F_{\text{buyer}}(t_{\text{buyer}})\right) dt_{\text{buyer}} - \int Q_{\text{seller}}(t_{\text{seller}}) F_{\text{seller}}(t_{\text{seller}}) dt_{\text{seller}},$$

which is linear in $(Q_{\text{buyer}}, Q_{\text{seller}})$. By contrast, in the partnership dissolution model of Cramton et al. [2], where a player's buyer or seller role is endogenous, the counterpart to our (3.17) *is their Ineq. (I), which is nonlinear in their allocation S_i. It is nonlinear in S_i because their integration depends on the upper or lower limit v_i^*, which depends on S_i. Their counterpart to the first integral in (3.17) is zero because of their market clearance condition.*

Remark *For any $\lambda \in \mathbb{R}_+$, define $\left(V_+^\lambda, V_-^\lambda\right) : [0,1] \to \mathbb{R}^2$ by, for any $t_i \in [0,1]$,*

$$V_+^\lambda(t_i) \;:=\; (1+\lambda)t_i - \frac{W(t_i)-F(t_i)}{f(t_i)} - \lambda \cdot \frac{1-F(t_i)}{f(t_i)}, \tag{3.18}$$

$$V_-^\lambda(t_i) \;:=\; (1+\lambda)t_i - \frac{W(t_i)-F(t_i)}{f(t_i)} + \lambda \cdot \frac{F(t_i)}{f(t_i)}. \tag{3.19}$$

The counterpart when $\lambda = 0$ is, for any $t_i \in [0,1]$,

$$V(t_i) := t_i - \frac{W(t_i)-F(t_i)}{f(t_i)}. \tag{3.20}$$

Denote $(V_+^\lambda, V_-^\lambda)f := (V_+^\lambda f, V_-^\lambda f)$. Then Eq. (3.14) is equivalent to

$$\mathscr{L}(Q,\lambda) = \sum_i \left\langle Q_i : (V_+^\lambda, V_-^\lambda) f \right| . \tag{3.21}$$

Hence the vector-valued function $(V_+^\lambda, V_-^\lambda)$ is the virtual surplus in our model.

Remark *When the constraint (3.17)—the joint constraint of IR, BB and the first-order part of IC—is non-binding, $\lambda = 0$ and the virtual surplus is reduced to the real-value function V in (3.20), which is similar to those in standard models except for the influence $W(t_i)$ from the welfare weights.*[14] *When the constraint (3.17) is binding, however, $\lambda > 0$ and the meaning of virtual surplus is enriched by (3.18) and (3.19): The marginal contribution of type t_i is equal to $1+\lambda$ times its marginal gain t_i of trade subtracted by its net information rent $\frac{W(t_i)-F(t_i)}{f(t_i)}$ and plus its marginal contribution $\lambda\rho(F)(t_i)$ to budget balancing, with the information rent skewed by the welfare weight W, and the budget-balancing contribution determined by the vector-value function $\rho(F)$.*

3.4 The Condition for the Bad to Be Needed

Given any welfare density w, we say that *the bad is needed* if and only if, for any (Q^*, P^*) that maximizes (3.6) subject to IC, IR and BB, $Q_i^* < 0$ on a positive-measure subset of $[0,1]$

[14] To illustrate the impact of the welfare weights, suppose that the welfare density is uniform across types, namely, $w = 1$ on $[0,1]$. In that case, $W = F$ by (3.8), and the Lagrangian (3.14) becomes $(1+\lambda)\sum_i \int_0^1 Q_i(t_i)t_i \, dF(t_i) + \lambda \sum_i \langle Q_i : \rho(F)|$. Consequently, if the constraint (3.17) were set aside, the optimal allocation given the fact $t_i > 0$ for all $t_i \in (0,1]$ is to allocate the good to a player with the highest realized type and never allocate the bad at all. Auctioning off only the good and not at all the bad, the allocation satisfies (3.17) and hence is indeed optimal (Lemma 2). This confirms the intuition that the bad is not needed at all if the welfare density is constant across types. That is, not allocating the bad at all is ex ante efficient.

for some player i. In other words, the bad is needed if and only if every socially optimal mechanism given welfare density w allocates the bad with a strictly positive probability.[15]

Theorem 7 *The bad is needed if and only if*

$$\exists t_i \in (0,1) : \int_0^{t_i} V(s)dF(s) < 0. \tag{3.22}$$

The "if" part of Theorem 7 is proved in Section 3.4.1, and the "only if" part in Section 3.4.2. Before proving it, we present three corollaries of the theorem, each proved in Appendix B.4.

Condition (3.22) both necessary and sufficient, we obtain a sharp comparative statics result regarding the prospect that the bad is needed for social optimality. From any welfare density function $w : [0,1] \to [0,1]$ (with $\int_0^1 wdF = 1$), the corresponding welfare weight distribution W is derived according to (3.8). For any such welfare weight distributions W and $\widetilde{W}$, W is said to *second-order stochastically dominate* $\widetilde{W}$ if and only if

$$\forall r \in [0,1] : \int_0^r \widetilde{W}(s)ds \geq \int_0^r W(s)ds. \tag{3.23}$$

Corollary 5 *If the bad is needed given welfare weight distribution W and if W second-order stochastically dominates $\widetilde{W}$, then the bad is also needed given $\widetilde{W}$.*

The corollary implies that, if the social planner moves some welfare weight from the middle types to the low and high types, she would allocate the bad to someone if she did so previously.

Corollary 6 shows how little the condition (3.22) requires for the bad to be needed. All that it takes is for the types near 0 to carry welfare densities above twice the average density:

Corollary 6 *If f is differentiable at 0, the welfare density w is continuous at zero, and $w(0) > 2$, then the bad is needed.*

We see from both Corollaries 5 and 6 that the bad is allocated when the welfare densities on low types are sufficiently high. The intuition is that we can think of a player's type as the player's marginal rate of substitution between the net utility from the items and the money transfer, and hence low types value money transfers more than the cost of receiving a bad. Given the IC and IR constraint, whenever a bad is allocated, it is allocated to some low types with positive money transfers. Thus, when the planner puts a relatively high weight on low types, a bad should be allocated.

It should be noted that our model does not force the result that the bad is needed. As the next corollary shows, the model allows for a nondegenerate set of welfare densities given which the bad is not needed at all. It says that any welfare distribution that second-order

[15] The definition rules out the uninteresting case where an optimal mechanism allocates the bad with a strictly positive probability just to cancel out any such allocation outcome by simultaneously allocating the good so that $Q_i^* = 0$ whenever the bad is allocated. To see why this case does not count as the bad being needed, modify the mechanism so that it allocates neither item to i whenever $Q_i^* = 0$. The modification is ex post feasible, and it preserves the reduced form of the original mechanism. Thus it is an optimal mechanism that does not allocate the bad at all. That violates the condition that *every* optimal mechanism allocates the bad with a strictly positive probability.

stochastically dominates the exogenous distribution of types renders the bad unnecessary for social optimality. The corollary follows from Corollary 5 and the fact that a social planner who weighs all types uniformly does not need the bad for optimality.[16]

Corollary 7 *If the welfare weight distribution W second-order stochastically dominates the exogenous distribution F of types, the bad is not needed.*

3.4.1 Why the Bad Is Needed if (3.22) holds

The argument that (3.22) implies the necessity of the bad is a proof by contrapositive. Suppose that an optimal mechanism does not allocate the bad at all. Then there is no need to raise funds to pay someone to receive the bad. Thus budget balancing becomes a nonissue. That is, the constraint (3.17) is non-binding (Lemma 2) and so the Lagrangian (3.14) reduces to a linear form. It then follows from the saddle point characterization that the mechanism is a solution to a linear programming problem. Thus, one can apply the optimal auction technique to show that the mechanism would allocate the bad with a strictly positive probability unless the ironed copy of the virtual surplus is not negative enough for (3.22) to hold.

To formalize this argument, let us recall the notations of hierarchical allocations and ironing in the optimal auction theory. For each item j (which can be the good or the bad) and any function $\phi : [0,1] \to \mathbb{R}$, an allocation of item j is said *hierarchical according to ϕ* if and only if, for almost every $(t_k)_{k=1}^n \in [0,1]^n$, item j is allocated to player i if $\phi(t_i) > \max\{0, \max_{k\neq i} \phi(t_k)\}$, and the item is not allocated if $\phi(t_i) < 0$ for all players i.

For any integrable function $g : [0,1] \to \mathbb{R}$, define $H_g : [0,1] \to \mathbb{R}$ by

$$H_g(r) := \int_0^r g\left(F^{-1}(s)\right) ds \tag{3.24}$$

for all $r \in [0,1]$ and denote $\widehat{H}_g$ for the convex hull of H_g on $[0,1]$. The function $\overline{g} : [0,1] \to \mathbb{R}$ such that

$$\overline{g}(t_i) = \frac{d}{dr}\widehat{H}_g(r)\bigg|_{r=F(t_i)} \tag{3.25}$$

a.e. $t_i \in [0,1]$ is called *ironed copy* of g.

Note that the V defined by (3.20) is integrable, with both W and F continuous. Hence its ironed copy $\overline{V}$ is well-defined. The next lemma is a straightforward extension of Myerson's [17, §6] ironing technique. Hence we omit its proof.

Lemma 1 *If Q^* maximizes $\mathscr{L}(Q,0)$ among all $Q \in \mathscr{Q}_{\text{mon}}$, then Q^* is the reduced form of an ex post allocation $(q^*_{iA}, q^*_{iB})_{i=1}^n$ such that $(q^*_{iA})_{i=1}^n$ is a hierarchical allocation of the good according to $\overline{V}$, and $(q^*_{iB})_{i=1}^n$ a hierarchical allocation of the bad according to $-\overline{V}$.*

The next lemma, proved in Appendix B.5, formalizes the aforementioned intuition that if the bad is not allocated at all then budget balancing becomes a nonissue.

Lemma 2 *If $Q \in \mathscr{Q}_{\text{mon}}$ and if $Q_i \geq 0$ on $[0,1]$ for any i, then Q satisfies (3.17). If, in addition, Q solves Problem (3.16)–(3.17), then Q satisfies (3.17) strictly.*

[16]This fact is verified in the proof of Corollary 7 (Appendix B.4) succinctly, and in Footnote 14 intuitively.

Proof of the "If" Part of Theorem 7 First, suppose (3.22) and, to the contrary of the claim, that for some optimal mechanism (Q^*, P^*), $Q_i^* \geq 0$ a.e. on $[0, 1]$ for all players i. Then Lemma 2 implies that Q^* satisfies the constraint (3.17) strictly. Thus, by the saddle point condition in Theorem 6, $\lambda = 0$ and Q^* maximizes $\mathscr{L}(\cdot, 0)$ on $\mathscr{Q}_{\text{mon}}$. Then Lemma 1 implies that Q^* entails a hierarchical allocation of the bad according to $-\overline{V}$. That is, for almost every $(t_1, \ldots, t_n) \in [0, 1]^n$, Q^* awards the bad to a player i if $\overline{V}(t_i) < \min\{0, \min_{k\neq i} V(t_k)\}$. Thus, Q^* allocates the bad with a strictly positive probability if $\overline{V} < 0$ on a nondegenerate interval of $[0, 1]$ (as F is assumed strictly increasing on $[0, 1]$). By (3.22) and the definition of H_V ((3.20) and (3.24)), $H_V(F(t_i)) < 0$ for some $t_i \in (0, 1)$. This, coupled with the fact $H_V(0) = 0$ (due to (3.24)) implies that the convex hull $\widehat{H}_V$ of H_V is negatively sloped on $[0, F(t_i)]$. Then, by (3.25), $\overline{V} < 0$ on the nondegenerate interval $[0, t_i]$. Thus, in the positive-probability event where some player's type belongs to $[0, t_i]$, the bad is allocated to someone. This contradicts the supposition that $Q_i \geq 0$ a.e. on $[0, 1]$ for all i. ∎

3.4.2 Why (3.22) Is True if the Bad Is Needed

Suppose that (3.22) does not hold and yet the bad is needed. We shall derive a contradiction from this hypothesis through perturbing any optimal mechanism that allocates the bad with a strictly positive probability. The perturbation either enlarges the Lagrangian or renders another optimal mechanism that does not allocate the bad at all, hence a contradiction obtains in either case. The complication is that the Lagrangian is a two-part operation that switches between two integrations (V_-^λ versus V_+^λ, cf. (3.21)) depending on the signs of the reduced forms. Thus we want the perturbation to preserve the sign of each player's reduced-form allocation. Hence we start with Section 3.4.2 to formalize such perturbations.

Sign-Preserving Perturbations of Allocations

For any $Q := (Q_i)_{i=1}^n \in \mathscr{Q}$, a vector $(c_i)_{i=1}^n \in [0, 1]^n$ is called *crossing point* of Q if and only if, for each i, $Q_i \leq 0$ a.e. on $[0, c_i]$ and $Q_i \geq 0$ a.e. on $[c_i, 1]$. Obviously, if $Q \in \mathscr{Q}_{\text{mon}}$, then a crossing point of Q exists, each Q_i being weakly increasing. If $Q \in \mathscr{Q}$ has a crossing point $(c_i)_{i=1}^n \in [0, 1]^n$ then, for any $\lambda \geq 0$, Eqs. (3.18), (3.19) and (3.21) together imply that

$$\mathscr{L}(Q, \lambda) = \sum_i \int_0^{c_i} Q_i(t_i) V_-^\lambda(t_i) dF(t_i) + \sum_i \int_{c_i}^1 Q_i(t_i) V_+^\lambda(t_i) dF(t_i). \tag{3.26}$$

For any $Q \in \mathscr{Q}$ with any crossing point $c \in [0, 1]^n$, we are interested in perturbing the negative part of Q without upsetting its crossing point or the second sum in (3.26). Such perturbations transform Q into an element of—

$$\mathscr{Q}(Q, c) := \{(Q_i')_{i=1}^n \in \mathscr{Q} \mid \forall i\, [Q_i' \leq 0 \text{ on } [0, c_i), Q_i' = Q_i \text{ on } (c_i, 1]]\}. \tag{3.27}$$

For now, we need only to define one kind of such perturbations (more in Appendix B.6.2):

Reservation $R_{i,T}$: For any player i, any $T \subseteq [0, 1]^n$, and any $Q \in \mathscr{Q}$ that is the reduced form of an ex post allocation $(q_{kA}, q_{kB})_{k=1}^n$, define $R_{i,T}(Q)$ to be the reduced form of the ex post allocation $(\tilde{q}_{kA}, \tilde{q}_{kB})_{k=1}^n$ that is the same as $(q_{kA}, q_{kB})_{k=1}^n$ except $\tilde{q}_{ij}(t) := 0$ for any $t \in T$

and any item $j \in \{A, B\}$. That is, when T occurs, the planner keeps any item to herself whenever the original allocation would award it to player i.

Denote π_i for the projection of any n-vector $(t_1, \ldots, t_n)$ onto its i^{th} component t_i. The next lemma follows directly from the definition of $R_{i,T}$ and hence we omit its proof:

Lemma 3 *For any $Q \in \mathscr{Q}$ with crossing point $c := (c_i)_{i=1}^n \in [0,1]^n$, any player i and any $T \subseteq [0,1]^n$ for which $\pi_i(T) \subseteq [0, c_i]$, if $Q' := R_{i,T}(Q)$, then:*

a. $Q'_i = 0$ on $\pi_i(T)$, $Q'_i = Q_i$ on $[0,1] \setminus \pi_i(T)$, and $Q'_k = Q_k$ on $[0,1]$ for all $k \neq i$;

b. $Q' \in \mathscr{Q}(Q, c)$ and, if in addition $Q \in \mathscr{Q}_{\text{mon}}$ and $\pi_i(T) \supseteq [0, c_i]$, $Q' \in \mathscr{Q}_{\text{mon}}$;

c. Eq. (3.26) holds when Q is replaced by Q'.

Proof of the "Only If" Part of Theorem 7

Suppose, to the contrary, that the bad is needed and yet (3.22) does not hold. Then

$$\forall t_i \in (0,1) : \int_0^{t_i} V(s)dF(s) \geq 0. \tag{3.28}$$

Pick any optimal mechanism (Q^*, P^*). The saddle point condition in Theorem 6 implies that Q^* maximizes $\mathscr{L}(\cdot, \lambda)$ on $\mathscr{Q}_{\text{mon}}$ for some $\lambda \geq 0$. By (3.19) and (3.20),

$$V_-^\lambda \begin{cases} = V \text{ on } [0,1] & \text{if } \lambda = 0 \\ > V \text{ on } (0,1] & \text{if } \lambda > 0. \end{cases} \tag{3.29}$$

Thus (3.28) implies

$$\forall \lambda > 0 : \forall t_i \in (0,1) : \int_0^{t_i} V_-^\lambda(s)dF(s) > 0. \tag{3.30}$$

By hypothesis, Q^* allocates the bad with a strictly positive probability. Thus $Q_i^* < 0$ on a positive-measure subset of $[0,1]$ for some player i. Since $Q^* \in \mathscr{Q}_{\text{mon}}$, Q_i^* is weakly increasing, hence this subset is an interval $[0, c_i)$ or $[0, c_i]$ for some $c_i \in (0,1]$. Without loss of generality, let c_i be the maximum among all such upper bounds so that $Q_i^* < 0$ on $[0, c_i)$, and $Q_i^* \geq 0$ on $(c_i, 1]$. For any $k \neq i$, with Q_k^* weakly increasing, there exists $c_k \in [0,1]$ for which $c := (c_i, (c_k)_{k \neq i})$ is a crossing point of Q^*.

Consider an alternative allocation $R_{i,T}(Q^*)$ for which $T = \{(t_k)_{k=1}^n \in [0,1]^n \mid t_i \in [0, c_i]\}$. That is, modify Q^* by reserving both items from player i when i's type belongs to $[0, c_i]$. By Lemma 3.b, $R_{i,T}(Q^*) \in \mathscr{Q}_{\text{mon}}$ and also has c as a crossing point. Thus, given the same c, (3.26) holds whether $Q = R_{i,T}(Q^*)$ or $Q = Q^*$. Then by Lemma 3.a,

$$\mathscr{L}(Q^*, \lambda) - \mathscr{L}(R_{i,T}(Q^*), \lambda) = \int_0^{c_i} Q_i^*(t_i)V_-^\lambda(t_i)dF(t_i). \tag{3.31}$$

Since Q_i^* is weakly increasing, by Fubini's theorem we have

$$\int_0^{c_i} Q_i^*(t_i)V_-^\lambda(t_i)dF(t_i) = \left(\lim_{s \uparrow c_i} Q_i^*(s)\right) \int_0^{c_i} V_-^\lambda(r)dF(r) - \int_0^{c_i} \int_0^s V_-^\lambda(r)dF(r)dQ_i^*(s). \tag{3.32}$$

By (3.28), (3.29) and (3.30), together with Q_i^* being weakly increasing, the right-hand side of (3.32) is nonpositive. This coupled with (3.31) implies that

$$\mathscr{L}(R_{i,T}(Q^*), \lambda) \geq \mathscr{L}(Q^*, \lambda). \tag{3.33}$$

Furthermore, if $\lambda > 0$, the right-hand side of (3.32) is negative. That is because, by the choice of c_i, either (i) $\lim_{s\uparrow c_i} Q_i^*(s) < 0$ or (ii) $\lim_{s\uparrow c_i} Q_i^*(s) = 0$. In Case (i), the first term on the right-hand side of (3.32) is negative due to (3.30). In Case (ii), $\lim_{s\uparrow c_i} Q_i^*(s) - Q_i^*(0) = 0 - Q_i^*(0) > 0$ (since $Q_i^* < 0$ on $[0, c_i)$) and so Q_i^* as a distribution assigns a positive measure on $[0, c_i)$; hence the double integral on the right-hand side of (3.32) is positive. Thus, by (3.31),

$$\lambda > 0 \Rightarrow \mathscr{L}(R_{i,T}(Q^*), \lambda) > \mathscr{L}(Q^*, \lambda).$$

Consequently, by the saddle point condition that Q^* maximizes $\mathscr{L}(\cdot, \lambda)$ on $\mathscr{Q}_{\text{mon}}$, $\lambda = 0$. This coupled with (3.33) means that $R_{i,T}(Q^*)$ is a maximizer of $\mathscr{L}(\cdot, 0)$ on $\mathscr{Q}_{\text{mon}}$.

If there is another player $k \neq i$ to whom $R_{i,T}(Q^*)$ allocates the bad with a strictly positive probability, perturb $R_{i,T}(Q^*)$ by the reservation operator R_{k,T_k} such that $T_k = \{(t_l)_{l=1}^n \in [0,1]^n \mid t_k \in [0, c_k]\}$. By the previous reasoning, $R_{k,T_k}(R_{i,T}(Q^*))$ is a maximizer of $\mathscr{L}(\cdot, 0)$ on $\mathscr{Q}_{\text{mon}}$. Repeating this reservation procedure, we eventually obtain an allocation $\tilde{Q}$ that allocates the bad with zero probability and maximizes $\mathscr{L}(\cdot, 0)$ on $\mathscr{Q}_{\text{mon}}$. Since $\tilde{Q}$ is entirely nonnegative (c a crossing point of Q^*), the left-hand side of (3.17) is nonnegative (Lemma 2) and so $\lambda = 0$ is a minimum of the Lagrangian $\mathscr{L}(\tilde{Q}, \cdot)$ on $\mathbb{R}_+$. Thus $(\tilde{Q}, 0)$ is a saddle point and hence, by the sufficiency part of Theorem 6, $\tilde{Q}$ constitutes an optimal mechanism subject to IC, IR and BB. Since $\tilde{Q}$ does not allocate the bad at all, we obtain a contradiction to the premise that the bad is allocated with a strictly positive probability in every optimal mechanism. ∎

3.5 Why the Kuhn-Tucker Method Does Not Deliver

To solve a constrained optimization problem such as (3.16) with the Kuhn-Tucker theorem, a typical approach is to apply the theorm to a relaxed problem, which sets aside the monotonicity constraint (the second-order part of IC). For the solution thereby obtained to be valid to the original problem, the method would have to assume that any solution obtained through this method happens to satisfy the set aside monotonicity constraint (e.g., Ledyard and Palfrey [11], who refer to this assumption "regular case"). Could this method have delivered some counterpart to Theorem 7 such as the bad being needed given a nondegenerate set of parameter values? The answer is No. The next Theorem 8 says that if the bad is needed then either every solution to the relaxed problem violates the monotonicity constraint, or the relaxed problem suffers indeterminacy in the sense that it has a continuum of solutions.

To state the theorem, recall that an *optimal mechanism* means any maximizer of (3.6) subject to IC, IR and BB. As shown in Section 3.3, maximizing (3.6) subject to IC, IR and BB is equivalent to maximizing (3.16) among all $Q \in \mathscr{Q}$ subject to (3.17) and the monotonicity constraint $Q \in \mathscr{Q}_{\text{mon}}$. Call an allocation *optimal* if and only if it is a solution to this maximization problem. The *relaxed problem*, by contrast, is to maximize (3.16) among all $Q \in \mathscr{Q}$ subject to only (3.17). Explicitly put, the relaxed problem is

$$\begin{array}{ll} \max_{Q\in\mathscr{Q}} & \sum_i \int_0^1 Q_i(t_i)\,(t_i f(t_i) - W(t_i) + F(t_i))\,dt_i \\ \text{s.t.} & \sum_i \int_0^1 Q_i(t_i) t_i\,dF(t_i) + \sum_i \langle Q_i : \rho(F)| \geq 0. \end{array} \tag{3.34}$$

The next lemma, proved in Appendix B.6.1, provides the basis for the theorem.

Lemma 4 *Given any welfare density w:*

a. Q^* is a solution to (3.34) if and only if there exists $\lambda \in \mathbb{R}_+$ such that (Q^*, λ) is a saddle point with respect to $(\mathscr{L}, \mathscr{Q})$ *in that (3.15) holds for all $Q \in \mathscr{Q}$ and all $\lambda' \in \mathbb{R}_+$;*

b. if Q^* maximizes $\mathscr{L}(Q, \lambda)$ among all $Q \in \mathscr{Q}$, then:

i. if $\lambda = 0$ then, for almost all $(s, s') \in [0, 1]^2$, $V(s) > V(s') \Rightarrow Q_i^*(s) > Q_i^*(s')$;

ii. if $\lambda > 0$, then $Q_i^*(s)Q_j^*(s) \geq 0$ for all players i and j and almost every $s \in [0, 1]$;

iii. if $Q_i^* < 0$ on $[0, c_i)$ for some $c_i \in (0, 1]$ and some player i, then $V_-^\lambda \leq 0$ on $(0, c_i)$.

Theorem 8 *Assume that f is differentiable on* $[0, 1]$. *For any welfare density w given which the bad is needed, the relaxed problem (3.34) either (i) does not have any optimal allocation as a solution or (ii) has a continuum of solutions.*

Alternative (ii) in Theorem 8, where the relaxed problem admits a continuum of solutions, corresponds to a condition that the virtual surplus is constantly zero on a nondegenerate interval $(0, c_*)$. This condition can be violated with slight perturbations of the type-density f or the welfare density w near 0. Thus the next corollary obtains (proved in Appendix B.6.4).

Corollary 8 *In the parameter space consisting of all pairs (f, w) of type-density function f and welfare density w such that f is differentiable, it is generically true that if the bad is needed then the constraint $Q \in \mathscr{Q}_{\text{mon}}$ is binding for any optimal mechanism.*

Proof of Theorem 8 It suffices to prove that if statement (i) is not true then statement (ii) is true. Thus, let Q^* be an optimal allocation that is also a solution to (3.34), and we shall prove that (3.34) has a continuum of solutions. As part of the definition of optimality, $Q^* \in \mathscr{Q}_{\text{mon}}$. With Q^* a solution to (3.34), there exists a $\lambda \in \mathbb{R}_+$ for which (Q^*, λ) is a saddle point with respect to $(\mathscr{L}, \mathscr{Q})$ (Lemma 4.a). Thus, Q^* maximizes $\mathscr{L}(\cdot, \lambda)$ on $\mathscr{Q}$.

We claim that $\lambda > 0$. Suppose not, then Lemma 4.b.i implies that $Q_i^*(t_i)$ is a strictly increasing function of $V(t_i)$ a.e. $t_i \in [0, 1]$. Since the bad is needed by hypothesis, Theorem 7 implies that (3.22) holds, which in turn implies $V < 0$ somewhere in $[0, 1]$. This, coupled with the fact that $V(0) = 0$ and V is differentiable (as f is differentiable), implies that V is negative and strictly decreasing on (a, b) for some $0 \leq a < b \leq 1$. Then Q_i^* is strictly decreasing a.e. on (a, b), contradicting the monotonicity condition $Q^* \in \mathscr{Q}_{\text{mon}}$.

By the hypothesis that the bad is needed and the fact $Q^* \in \mathscr{Q}_{\text{mon}}$, $Q_i^* < 0$ on $[0, x)$ for some $x \in (0, 1]$ and some player i. Let

$$c_* := \max_{i=1,\ldots,n} \sup \{x \in [0, 1] : Q_i^* < 0 \text{ on } [0, x)\}.$$

Note $c_* > 0$. Let i_0 be a player that attains this maximum, so $Q_{i_0}^* < 0$ on $[0, c_*)$. Since $\lambda > 0$, Lemma 4.b.ii applies. Thus, for any player $k \neq i_0$, $Q_k^* Q_{i_0}^* \geq 0$ a.e., and hence $Q_k^* \leq 0$ a.e. on $[0, c_*)$. The definition of c_*, coupled with $Q^* \in \mathscr{Q}_{\text{mon}}$, also implies $Q_k^* \geq 0$ on $(c_*, 1]$ for all players k. Thus $c := (c_k)_{k=1}^n$ defined by $c_k := c_*$ for all k is a crossing point of Q^*.

There are only two possible cases: (i) $V_-^\lambda(x) < 0$ for some $x \in (0, c_*)$, or (ii) $V_-^\lambda \geq 0$ on $(0, c_*)$. The rest of the proof is to establish two observations:

a. Case (i) implies that Q^* violates the monotonicity constraint and hence (3.34) admits no optimal allocation as a solution.

b. Case (ii) implies that (3.34) has a continuum of solutions.

Both observations are based on the fact that Q^* maximizes $\mathscr{L}(\cdot, \lambda)$ on $\mathscr{Q}(Q^*, c)$, where $\mathscr{Q}(Q^*, c)$, according to (3.27) and the definition of c here, is the set of $Q \in \mathscr{Q}$ that result from some sign-preserving perturbations of Q^* that leave the positive part of Q_i^* ($\forall i$) unchanged. Note, for any $Q \in \mathscr{Q}(Q^*, c)$, (3.26) holds and

$$\mathscr{L}(Q, \lambda) - \mathscr{L}(Q^*, \lambda) = \sum_i \int_0^{c_*} (Q_i(s) - Q_i^*(s))\, V_-^\lambda(s) dF(s). \tag{3.35}$$

Thus, plug "$Q_i^- = -Q_i$ and $\left(Q_i^*\right)^- = -Q_i^*$ on $[0, c_*)$" into (3.35) to get that Q^* solves

$$\max_{Q \in \mathscr{Q}(Q^*, c)} \sum_i \int_0^{c_*} Q_i^-(s) \left(-V_-^\lambda(s)\right) dF(s). \tag{3.36}$$

In Case (i), since V_-^λ is differentiable and $V_-^\lambda(0) = 0$, there is a nondegenerate interval $I \subsetneq (0, c_*)$ on which V_-^λ is negative and strictly decreasing. Since Q^* is a solution to (3.36), it does not allocate the good to any player-type in I, nor the bad to any player-type in $(c_*, 1]$; furthermore, if the bad is to be allocated to some player-types in $[0, c_*]$, the bad goes to the one whose V_-^λ-value is the lowest among all negative ones. If these three properties are not all satisfied, one can construct a $Q \in \mathscr{Q}(Q^*, c)$ that outperforms Q^* in terms of the objective of (3.36) (Lemma 24 in Appendix B.6.2, where $g = -V_-^\lambda$). Thus, by (3.3), for any i and any $t_i \in I$, $Q_i^*(t_i)$ is equal to the negative of the marginal of the ex post allocation $q_{iB}^*(t_i, \cdot)$ of the bad, and $Q_i^*(t_i)$ is strictly increasing in $V_-^\lambda(t_i)$ for a.e. $t_i \in I$ (Lemma 25 in Appendix B.6.2, where $g = -V_-^\lambda$). But then Q_i^* is strictly decreasing a.e. on I, violating the monotonicity constraint. Hence Q^* cannot be an optimal allocation.

In Case (ii), $V_-^\lambda = 0$ on $[0, c_*)$ by Lemma 4.b.iii and the definition of c_*. Then (3.35) implies $\mathscr{L}(Q, \lambda) = \mathscr{L}(Q^*, \lambda)$ for any $Q \in \mathscr{Q}(Q^*, c)$. Thus, any $Q \in \mathscr{Q}(Q^*, c)$ is also a maximizer of $\mathscr{L}(\cdot, \lambda)$ on $\mathscr{Q}$. By Lemma 4.a, any such Q is a solution to the relaxed problem if λ minimizes $\mathscr{L}(Q, \cdot)$ on $\mathbb{R}_+$, which is true if the constraint in the relaxed problem (3.34) is binding for Q. By the definition of $\langle Q_i : \rho(F)|$, the constraint being binding for Q is equivalent to

$$\sum_i \int_0^1 Q_i^-(s)\,(sf(s) + F(s))\, dF(s) = \sum_i \int_0^1 Q_i^+(s)\,(sf(s) + F(s) - 1)\, dF(s).$$

Since c is a crossing point for all $Q \in \mathscr{Q}(Q^*, c)$, this equation is the same as

$$\begin{aligned}\sum_i \int_0^{c_*} Q_i^-(s)\,(sf(s) + F(s))\, dF(s) &= \sum_i \int_{c_*}^1 Q_i^+(s)\,(sf(s) + F(s) - 1)\, dF(s) \qquad (3.37)\\ &= \underbrace{\sum_i \int_{c_*}^1 (Q_i^*(s))^+\,(sf(s) + F(s) - 1)\, dF(s)}_{=:z},\end{aligned}$$

with the second line due to the fact that $Q_i(s) = Q_i^*(s)$ whenever $Q_i^*(s) > 0$ (by the definition of $\mathscr{Q}(Q^*, c)$). Thus, any $Q \in \mathscr{Q}(Q^*, c)$ for which

$$\sum_i \int_0^{c_*} Q_i^-(s)\,(sf(s) + F(s))\,dF(s) = z \tag{3.38}$$

is a solution to the relaxed problem. Since $c_* > 0$, $\mathscr{Q}(Q^*, c)$ is a convex set with nonempty interior. The left-hand side of (3.38) is a linear functional on $\mathscr{Q}(Q^*, c)$. Thus one can show that the set of $Q \in \mathscr{Q}(Q^*, c)$ that satisfies (3.38) is a hyperplane intersection of the interior of $\mathscr{Q}(Q^*, c)$ (Lemma 26, Appendix B.6.3). Hence there is a continuum of solutions to the relaxed problem, as asserted. ■

3.6 Conclusion

This paper asks a novel question: Under what primitive condition in a quasilinear independent private values model is a commonly undesirable item needed as an instrument to achieve interim Pareto optimality? The answer is a necessary and sufficient condition, which holds if the extreme low types weigh in the social welfare more than twice the average weight, or if the welfare weight distribution spreads out sufficiently. This result holds regardless of the particular functional forms of the social welfare distribution and the type distribution. The finding sheds a new light on policy issues regarding the location decision of Nimbies. Even if the public good effect of a Nimby were assumed away and there were no cost to do away with the Nimby completely, the Nimby is still needed to optimize the social welfare when the welfare weights of the high and low types are sufficiently large. Put differently, if we think of welfare weights as the bargaining power among various players in the interim, in any idealized outcome of the interim bargaining process, some low types have to end with the bad if the low and high types are sufficiently powerful relative to the middle types.

It is important to note that the purpose of this paper is not to find a tractable model where a bad is needed for social optimality. Our purpose, rather, is to identify the condition under which the bad is needed in an environment that does not at all force the usage of the bad, with never allocating the bad part of an ex ante incentive efficient allocation. Nevertheless, some models where the usage of the bad arises more easily are also interesting to study and could also use our method. For example, consider the provision of healthcare with congestion such that everyone is endowed with a basic amount of healthcare service. If someone wants a premium service, someone else has to give up the basic service thereby freeing up the facility for the former. The latter's action can be interpreted as receiving a unit of the bad, and this setup is subject to the market clearing condition that the quantity of the good (premium service) awarded be equal to the quantity of the bad received. Ex ante incentive efficiency would then entail assignment of the bad to some types. Given such a setup, our saddle point characterization remains valid.

This paper contributes a new method to the mechanism design problems where a player's role in the market is not exogenous but rather determined by the mechanism and the player's action. Such endogeneity upsets the linearity of a player's ex ante surplus as a function of the allocation in the mechanism. We restore the structure to a tractable, concave two-part operator thereby characterizing all the optimal mechanisms with a saddle point condition. Furthermore, to derive properties of all optimal mechanisms from the saddle point condition, we develop a perturbation method that uses a family of ex post feasible, sign-preserving perturbations of any optimal allocation. Preserving every player's endogenous role in the market in a type-by-type manner, such perturbations affect the associated Lagrangian linearly, because they do not alter the measure with which the two-part operator acts on the allocation. Considering such perturbations in the direction of the fixed measure, we obtain necessary conditions for all—rather than only for some—optimal mechanisms. Our method proves more applicable than the Kuhn-Tucker method given our environment, as it is generically impossible for the Kuhn-Tucker method to obtain a counterpart to our result.

Bibliography

[1] Hung-Ken Chien. Incentive efficient mechanism for partnership. Mimeo, June 1, 2007.

[2] Peter Cramton, Robert Gibbons, and Paul Klemperer. Dissolving a partnership efficiently. *Econometrica*, 55(3):615–632, May 1987.

[3] Piotr Dworczak, Scott Duke Kominers, and Mohammad Akbarpour. Redistribution through markets. *Econometrica*, Vol. 89, No. 4, pages 1665–1698, July, 2021.

[4] Thomas Gresik. Incentive-efficient equilibria of two-party sealed-bid bargaining games. *Journal of Economic Theory*, 68:26–48, 1996.

[5] Bengt Holmström and Roger Myerson. Efficient and durable decision rules with incomplete information. *Econometrica*, 51(6):1799–1820, 1983.

[6] Bruno Jullien. Participation constraints in adverse selection models. *Journal of Economic Theory*, 93:1–47, 2000.

[7] Andreas Kleiner, Benny Moldovanu, and Philipp Strack. Extreme points and majorization: Economic applications. Mimeo, May 25, 2020.

[8] Houward Kunreuther and Paul Kleindorfer. A sealed-bid auction mechanism for siting noxious facilities. *American Economic Review*, 76:295–299, 1986.

[9] Didier Laussel and Thomas Palfrey. Efficient equilibria in the voluntary contributions mechanism with private information. *Journal of Public Economic Theory*, 5:449–478, 2003.

[10] John Ledyard and Thomas Palfrey. A characterization of interim efficiency with public goods. *Econometrica*, 67:435–448, 1999.

[11] John Ledyard and Thomas Palfrey. A general characterization of interim efficient mechanisms for independent linear environments. *Journal of Economic Theory*, 133:441–466, 2007.

[12] Tracy R. Lewis and David E. M. Sappington. Countervailing incentives in agency problems. *Journal of Economic Theory*, 49:294–313, 1989.

[13] Simon Loertscher and Cédric Wasser. Optimal structure and dissolution of partnerships. *Theoretical Economics*, 14:1063–1114, 2019.

[14] Hu Lu and Jacques Robert. Optimal trading mechanisms with ex ante unidentifiable traders. *Journal of Economic Theory*, 97:50–80, 2001.

[15] David G. Luenberger. *Optimization by Vector Space Methods*. John Wiley & Sons, 1969.

[16] Giovanni Maggi and Andrés Rodríguez-Clare. On countervailing incentives. *Journal of Economic Theorey*, 66:238–263, 1995.

[17] Roger Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, February 1981.

[18] Roger Myerson and Mark A. Satterthwaite. Efficient mechanisms for bilateral trading. *Journal of Economic Theory*, 29:265–281, 1983.

[19] Tymofiy Mylovanov and Thomas Tröger. Mechanism design by an informed principal: Private values with transferable utility. *Review of Economic Studies*, 81:1668–1707, 2014.

[20] Mikel Pérez-Nievas. Interim efficient allocation mechanisms. Working Paper 00-20, Departmento de Economia, Universidad Carlos III de Madrid, February 2000.

[21] Ilya Segal and Michael Whinston. Property rights and the efficiency of bargaining. *Journal of the European Economic Association*, 14:1287–1328, 2016.

[22] Juuso Toikka. Ironing without control. *Journal of Economic Theory*, 146:2510–2526, 2011.

[23] Robert Wilson. Incentive efficiency of double auctions. *Econometrica*, 53:1101–1105, 1985.

[24] Charles Z. Zheng. Utilitarian representation of interim efficient mechanisms given continuum types. Mimeo, April 2020.

Chapter 4

Sponsored Link Auctions with Conusmer Search

4.1 Introduction

Amazon and eBay place "sponsored links" at the top of the webpages where customers can spot them immediately. Top positions on the product search results provide an instant visibility boost to customers. Sellers who own the sponsored link positions are determined by auctions.[1] Sponsored link auctions are important and fast-growing channels for online shopping platforms to collect revenues. Amazon reports auction revenues of 14.08 billion dollars in 2019 and 21.48 billion dollars in 2020. Unlike auctions selling concrete items, sponsored link auctions sell advertisement positions. The value of a sponsored link position thus depends on how it increases the auction winner's profit by affecting consumers' search outcomes.

This chapter contributes to the literature by introducing two novel elements in the sponsored link auctions. First, I assume consumers have partial information on product values, observe product prices during the search, and use the information to search optimally. Second, sellers in my model choose both product prices and auction bids optimally to maximize their profits. The questions are: How do sellers' pricing strategies and bidding strategies interact in such an environment. And how the interaction affects the surplus split between consumers, sellers, and the shopping platform. My main finding is that the answers depend on the payment rule of the auction. If the auction winner's payment is fixed in a lump-sum manner, sellers' pricing and bidding strategies are independent. However, if the auction winner pays to the platform every time a product is sold, product prices increase under the equilibrium. Thus, a shift from the fixed payment to the per-transaction payment decreases consumer surplus and increases the sellers' profits. Moreover, I find that sponsored link auction revenue decreases from such a change of the payment rules.

I consider an oligopoly market where the value of a product is separated into two parts: one is known prior search, the other needs to be discovered from search.[2] To be specific, a

[1] The auction rules used by Amazon and eBay are different. Amazon uses second price auctions, and the payments are made based on the number of clicks of product links. eBay uses first price auctions, and the payments are made based on the number of transactions. Criteria about the quality of the product exist to be placed on a sponsored link.

[2] The definition of search good, and how it distinguishes from experienced good is well-documented by Nelson

consumer's payoff from purchasing a product i is $v_i + z_i - p_i$, where v_i (prior value) and p_i (product price) are known before the search but a search cost s_i has to be paid in order to learn z_i (match value).[3] The payoff setting fits into the online shopping context reasonably well, as the shopping procedure can be described as follows. First, a consumer chooses a shopping platform and types the keywords the product. Second, based on the input keywords, the platform produces a webpage of multiple product links, with the corresponding product prices (p_i) and brief product descriptions (v_i). Third, the consumer sequentially clicks a set of links based on the prior information (v_i, p_i and belief on z_i), spends time (pays search cost s_i) to collect detailed information (learns z_i) on the product description page. If the product does not fit the consumer's preference, the consumer continues searching until a decision (purchase or quit) is made.[4]

Weitzman's (1979) [19] solution fully characterizes the optimal search rule in such an environment, with the optimal search order and the stopping rule. However, because a consumer is assumed to be aware of the existence of **all** products when the search starts, the consumer search is not affected by the product position. Awareness of all products initially thus makes product demands irrelevant to product positions on a webpage. Such an assumption is realistic in small markets, in which consumers view all the products immediately after they input the keywords. However, the search result of a particular keyword on Amazon usually consists of hundreds of links. Consumers need to read the search results in a top-to-down manner, scroll down, and flip pages to know the products' existence at lower positions.[5] With limited memory and the belief that links are sorted from high to low quality, most consumers only glance at the first few pages before stopping.

To capture the online search behavior and justify why sellers pay for sponsored link positions, I introduce the concept of *block*, which is a set of adjacent products in the keyword search result. For example, the first block is the first page, or the set of products that a consumer can see immediately (top of the first page), which consists of the sponsored links and the "best sellers." The second block consists of the products that a consumer needs to scroll down or flip the webpage to see. Blocks are mutually exclusive, such that one product belongs to one block. I assume all consumers search block-by-block, apply Weitzman's optimal search rule within each block and update the current best option when switching blocks. For instance, when searching the second block, consumers use the value of the best product searched in the first block as the new outside option, conditional on it being better than the original outside option.

Intuitively, block-by-block search distorts both search order and stopping rule compared to the case without blocks (Weitzman). The search order is distorted since consumers are unaware of any product in the second block at the beginning of the search, even if some products in the second block have a high priority to search. The stopping rule is distorted since consumers

(1970) [16].

[3]This payoff is also used in Armstrong and Zhou (2011) [1], Haan et al. (2017) [12] and Choi, Dai and Kim (2018) [7].

[4]Quit the search is equivalent to taking an outside option, which can be purchasing from another platform or purchasing in-store.

[5]Joachims et al. [13] provide experimental evidence supporting the block-by-block search by monitoring browser users' eyeballs movement when they glance over google search results online. They find users "first scan the viewable results quite thoroughly before resorting to scrolling".

update the best option according to the search outcome in the first block and thus search fewer products in the second block. Thus, the block-by-block search behavior increases the demand for products in the first block, decreases the demand for products in the second block, and creates incentives for sellers to win the sponsored link position in the first block.

For tractability, I focus on the environment with two blocks and one sponsored link position in the first block. Despite the limited number of blocks, the setup captures the critical features of sellers' pricing and bidding competition with consumer search. Having only two blocks is not an unreasonable abstraction from reality. First, to guarantee good users' experiences, Amazon and eBay impose a set of prerequisites, regarding product quality, on the participants of the sponsored link auction. Since the online platform ranks products from high quality to low quality in general, the sellers competing for the sponsored link position are likely from higher positions. Second, most consumers only search among the first few pages of a selling website.[6] Thus, focusing on the first two blocks does not affect the underlying implications the paper intends to study.

The timing of my model is the following. First, sellers bid the top position first. Second, given the position determined by the auction, sellers set prices optimally via oligopoly price competition. Last, given the position and prices, consumers search optimally. I solve the model by backward induction by characterizing consumer demand first, then sellers pricing equilibrium, and lastly the bidding equilibrium. The solution concept is subgame perfect Nash equilibrium.

Characterization of demand functions relies on the eventual purchase theorem by Choi, Dai and Kim [7] (henceforth CDK), who summarize the conditions of purchase outcomes in Weitzman under a discrete choice formulation. I adapt CDK's eventual purchase theorem to the context of block-by-block search (Theorem 9). With the revised eventual purchase theorem, I characterize the product demands under any product position. By assuming log-concave densities of product values, the demand functions are log-supermodular, and pricing equilibrium is uniquely pinned down by first-order conditions (Theorem 10). Hence, the equilibrium profit of any seller is uniquely determined under any auction outcome. Sellers then bid for the sponsored link position to increase their equilibrium profits.

I study the complete-information second price auctions with two payment schemes: fixed payment and per-transaction payment. Under fixed payment, the winner pays the second-highest bid once in a lump-sum manner. Under per-transaction payment, the winner pays the second-highest bid whenever a product is sold. Since sellers can update their bids frequently and experiment with them, the sponsored link auction is treated as a complete information game. Under both payment schemes, the unique pure strategy pricing equilibrium exists given the winner's identity and the corresponding bid payment (Theorem 10 applies). This is because the bid payment is independent of the winner's pricing strategy under the fixed payment, and can be regarded as the additional marginal cost paid by the winner to produce a product under per-transaction payment. I characterize the pure-strategy subgame perfect Nash equilibrium of the bidding game under both payment schemes (Theorem 11 and Theorem 12).

In the comparative statics analysis with symmetric sellers, I find the fixed payment leads to a higher auction revenue and consumer surplus, while the per-transaction payment provides

[6] Using detailed online browsing and transaction data of the book market, Delossantos et al. (2012) [9] find it is likely that consumers choose a fixed and small sample size to search when entering the market.

more seller profits (Theorem 13). The reason is that the per-transaction payment distorts the winner's optimal pricing rule and increases the prices of all products. Since a seller's profit increases if a competitor sets a higher price, and a product's equilibrium price increases in its marginal cost, non-winners in the per-transaction payment auction have incentives to increase their bids to increase the winner's marginal cost.

The paper also provides the consumer-optimal positioning of products if sellers commit to price before the position is allocated (Theorem 14). Intuitively speaking, the platform needs to help consumers find a high payoff product with a low search cost paid. Thus, allocating all products with high expected values into a higher position may fail to be optimal, because the search friction prevents consumers from purchasing the best option in general. Surprisingly, low search cost can be more important: Under an extreme situation where it is costlessly to learn the value of all products in the first block, consumer surplus equals that without blocks, which is the highest attainable surplus in the search environment.

Compiani et al. (2021) [8] find a consumer optimal positioning, called "diamonds in rough," in a different setup. Instead of assuming consumers search block-by-block, they assume that the reservation value, which determines the optimal search order, has an additive separable term decreasing in product position (larger for higher position). They find to maximize consumer surplus, platforms need to put products whose utility indexes exceed search indexes into higher positions. Their search environment is more general as the utility index and the search index are independent. In my model, subject to Weitzman's search framework, the counterparts of the utility index and search index are correlated through the prior and match values. Moreover, their model does not consider search cost explicitly. So the results are not directly comparable.

A mass body of literature studies sponsored link auctions under different setups, but most papers study auctions run by online search engines (e.g., Google and Yahoo!). Some of the differences between my paper and past papers are caused by the different auction contexts, especially that product prices are not observable on Google's search results. So I emphasize that the paper is not arguing that the sponsored link auction model with flexible search behavior is a superior one. Rather, I think it provides a different perspective to analyze the auction when consumers have more or less prior information. Varian (2007) [20], and Edelman, Ostrovsky, and Schwarz (2007) [11] study the generalized second price auction, in which the winner of the k's position pays the $k-1$th highest bid. In contrast to modeling the positions as blocks, positions in their paper are strictly ranked, and a higher position always have a higher number of clicks. Both papers assume a seller's payoff from a click depends only on the seller's identity but not on the seller's position, and the number of clicks received by a seller depends only on the seller's position but not on the seller's identity. They characterize a local condition for the equilibrium and find truth-telling is not an equilibrium in the generalized second price auction. Börgers et al. (2013) [4] extend the generalized second price auction by allowing both the value of a click and the expected number of clicks to be seller-position specific. They find a multiplicity of equilibria exist in general. By explicitly modeling consumer search behavior and letting sellers compete in prices, I capture the "allocative externality" that is not considered by the previous literature. Allocative externality is within block competition and exists when a seller's payoff depends not only on the position but also on other sellers' identities. For example, a seller gets more clicks if the competitors on the same webpage are weakly than if surrounded by well-known brands.

Chen and He (2006) [5] and Athey and Ellison (2011) [2] incorporate search behavior into sponsored link auction games in a different context. They assume the value of any link is draw from the same distribution, and consumers have no prior information on the value before clicking a link. Search is driven by consumers' belief in the links' qualities, which are the probabilities of satisfying a consumer's need, independent of the links' realized values. With no prior information on product values and prices, consumers always search for the first product that satisfies their needs. Identical value distribution implies that consumers have the same expected payoff from any link and that all sellers set the same equilibrium price and receive the same profit from a transaction. So the interaction between price and bid studied in this paper is not valid in their setup. By allowing heterogeneous consumer preferences and sellers' strategic pricing, the paper analyzes consumer surplus from a different perspective and studies how price and bid interact in the sponsored link auction.

The rest of the paper is organized as follows. Section 4.2 introduces the model and consumer's optimal search rule. Section 4.3 characterizes product demands in two blocks respectively. Section 4.4 establishes the existence and uniqueness of pricing strategy under any position outcome. The existence of pure strategy subgame perfect Nash equilibrium of the bidding game is established in Section 4.5. Section 4.6 analyzes comparative statics that compares different bid payment schemes. Section 4.7 provides the consumer optimal positioning of products. The last section concludes.

4.2 The Model

A market consists of n sellers indexed by $i \in N := \{1, \ldots, n\}$. Each seller i can produce a product at a marginal cost c_i and zero fixed cost. All products are in the same category and are horizontally differentiated. Each seller produces one type of product, so the index i also refers to seller i's product.

To sell their products, sellers need to put product links on an online shopping platform. A platform allocates all sellers into two *blocks*, indexed by $k \in \{1, 2\}$. Let $N_k \subset N$ be the set of sellers in block k, with $N_1 \cup N_2 = N$ and $N_1 \cap N_2 = \emptyset$. I call (N_1, N_2) the *position* of sellers. Sellers in the same block share the same position and the size of the two blocks are fixed, such that $|N_1| = n_1$ and $|N_2| = n_2$.

The platform sells one position in N_1 through a second price auction. Before the auction, the platform determines the *status quo* position of sellers, (N_1^0, N_2^0), and a positioning function $l : N_2^0 \to N_1^0$. The positioning function l decides which seller in N_1^0 to be moved to block 2 if $i \in N_2^0$ wins to maintain the sizes of the two blocks. If seller $i \in N_2^0$ wins the auction, the position is $(N_1, N_2) = (N_1^0 + i - l(i), N_2^0 - i + l(i))$.[7] If seller $i \in N_1^0$ wins, the position stays the same as the *status quo*, i.e., $(N_1, N_2) = (N_1^0, N_2^0)$. Without loss of generality, let $N_1^0 = \{1, \ldots, n_1\}$ and $N_2^0 = \{n_1 + 1, \ldots, n\}$, so the index i is generated such that the first n_1 sellers are in N_1^0 and the rests are in N_2^0.

[7] I use + and − as set operators: $(N_1^0 + i - l(i), N_2^0 - i + l(i)) = (N_1^0 \cup \{i\} \setminus \{l(i)\}, N_2^0 \setminus \{i\} \cup l(i))$.

position	seller
N_1^0	1
	2
N_2^0	3
	4

position	seller
N_1	1
	3
N_2	2
	4

Table 4.1: status quo position (left) and the position when seller 3 wins the auction with $l(3) = 2$ (right).

After positions are determined by the auction, all sellers simultaneously announce their prices. Define the price of product i as p_i. If $i \in N_1^0$ wins the auction, the price vector of products in block 1 is $\boldsymbol{p}_1 := (p_1, \ldots, p_{n_1})$ and that in block 2 is $\boldsymbol{p}_2 := (p_{n_1+1}, \ldots, p_n)$. Else, if $i \in N_2^0$ wins the auction, the price vectors are $\boldsymbol{p}_1 := (p_1, \ldots, p_{l(i)-1}, p_i, p_{l(i)+1}, \ldots, p_{n_1})$ and $\boldsymbol{p}_2 := (p_{n_1+1}, \ldots, p_{i-1}, p_{l(i)}, p_{i+1}, \ldots, p_n)$.

There is a unit mass of consumers with unit demands. A consumer's random utility from consuming product i is $V_i + Z_i$, where V_i is the prior value for product i, and Z_i is the additional value the consumer discovers from visiting product i's selling page. For each consumer, the realization of V_i and Z_i, denoted as v_i and z_i, are drawn from distributions $F_i(\cdot)$ and $G_i(\cdot)$ independently, with continuously differentiable densities $f_i(\cdot)$ and $g_i(\cdot)$, and supports $[\underline{v}_i, \overline{v}_i]$ and $[\underline{z}_i, \overline{z}_i]$. For all i, $F_i(\cdot)$ and $G_i(\cdot)$ are common knowledge among consumers and sellers.[8] I define $\boldsymbol{v}_k$ and $\boldsymbol{z}_k$ as the vectors of prior value and match value in block $k \in \{1, 2\}$ in the same way as how $\boldsymbol{p}_k$ is defined.

With product position and prices, a consumer searches in the following way:

1. A consumer starts the search and sees all products in block 1. The consumer knows $\boldsymbol{v}_1$ and $\boldsymbol{p}_1$ immediately.

2. The consumer can visit any product $i \in N_1$'s selling page and learn z_i at a search cost s_i on the first visit. Based on the realized z_i, the consumer either ends the search in block 1, or continues searching in block 1.

3. If the consumer finds no more product in block 1 worth visiting, he/she sees all the products in block 2 and knows $\boldsymbol{v}_2$ and $\boldsymbol{p}_2$ immediately. The consumer repeats the search process as that in block 1.

4. Recall a product is costless after the first visit. The consumer compares all visited products and makes the purchase decision.

The underlying assumption of the search behavior is that consumers are unaware of products in block 2 when search in block 1. Searching block 1 before block 2 is non-strategic, as block 1 is always shown to consumers before block 2.

Let $\widetilde{N}$ be the set of products whose links are clicked by a consumer before the purchase decision is made. The consumer who eventually purchases product i gets a payoff:

$$U(v_i, z_i, p_i, \widetilde{N}) = v_i + z_i - p_i - \sum_{j \in \widetilde{N}} s_j.$$

[8]A single consumer does not make decisions based on F_i, since v_i is realized from the consumer's point of view. Thus, assuming F_i is unknown to consumers does not change any result in this paper.

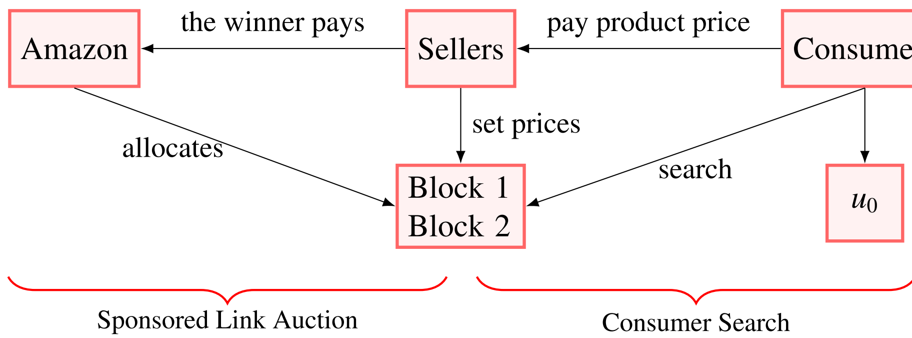

Figure 4.1: Interactions between sponosred link auction and search

If a consumer chooses to purchase nothing from the platform after search, the payoff is $u_0 - \sum_{j \in \widetilde{N}} s_j$, where u_0 is the outside option value and is assumed to be identical to all consumers. Consumers search the two blocks to maximize their expected payoffs, under the restriction that they can search block 2 only if they finish searching in block 1.

Figure 4.1 shows how the platform, sellers and consumers interact in my model.

Weitzman (1979) [19] characterizes the optimal search rule under the environment where consumers are aware of all products initially (without blocks). When two blocks exist, the optimal search rule in each block remains the same as that in Weitzman. The key difference is that when searching in the second block, consumers update the outside option according to the search outcome from the first block. The following lemma states the optimal search strategy.

Lemma 5 *For any consumer, given $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$, the optimal search strategy is as follows: for each i, let z_i^* be determined by:*

$$s_i = \int_{z_i^*}^{\bar{z}_i} (1 - G_i(z_i)) dz_i. \tag{4.1}$$

- *Order: In each block $k \in \{1, 2\}$, the consumer clicks the products in the descending order of $v_i + z_i^* - p_i$.*

- *Stopping: At any point during the search, let $\widetilde{N}_k \subseteq N_k$ be the set of visited products in block k.If $k = 1$, the consumer stops searching block 1 and switches to block 2 if and only if*

$$\max \left\{ u_0, \max_{i \in \widetilde{N}_1} \{v_i + z_i - p_i\} \right\} > \max_{j \in N_1 \setminus \widetilde{N}_1} \{v_j + z_j^* - p_j\}.$$

  If $k = 2$, the consumer stops searching and takes the current best option if and only if

$$\max \left\{ u_0, \max_{i \in \widetilde{N}_1 \cup \widetilde{N}_2} \{v_i + z_i - p_i\} \right\} > \max_{j \in N_2 \setminus \widetilde{N}_2} \{v_j + z_j^* - p_j\}.$$

Lemma 5 shows consumer's optimal search rule depends on product price and two types of value: *true value*, $v_i + z_i$, and *reservation value*, $v_i + z_i^*$. I define the corresponding *net value* as the value minus price, e.g., the *net true value* of product i is $v_i + z_i - p_i$. The ordering rule in

Lemma 5 says, for any consumer with realized value $(\boldsymbol{v}_k, \boldsymbol{z}_k)_{k=1}^2$ and price $(\boldsymbol{p}_k)_{k=1}^2$, the product with a higher net reservation value in each block has a higher priority to search, as it is more likely to have a higher net true value. Given block k is under search, the stopping rule says that a consumer stops searching block k if the value of the current best option is larger than the *net reservation values* of all products not visited in block k. Since $s_i = \mathbb{E}\left[\max\{Z_i - z_i^*, 0\}\right]$ (by Eq. (4.1) and integration by parts), the stopping rule can be intuitively explained as that the expected additional value discovered from one more visit is less than the search cost. For instance, if the current best option has a value u, and i is the next product to visit according to the order rule, the expected gain from an additional visit is

$$\mathbb{E}\left[\max\{v_i + Z_i - p_i - u, 0\}\right] - \underbrace{\mathbb{E}\left[\max\{Z_i - z_i^*, 0\}\right]}_{=:s_i},$$

which is strictly negative if $u > v_i + z_i^* - p_i$ and positive otherwise.

Given that all consumers search optimally, the mass of consumers who purchase product i is $D_i^k(\boldsymbol{p}_1, \boldsymbol{p}_2)$, which depends on the product position and product prices. The platform runs a second price auction to sell the sponsored link position (details in Section 4.5). Sellers simultaneously decide product prices and auction bids to maximize their expected profits net from auction payments.

Limitations / Alternative modelling choice Though consumer search behavior is more flexible comparing to past sponsored link auction literature, it is worth emphasizing the imposed assumptions and limitations.

First, all consumers search the first block before the second. This assumption is relatively intuitive and is supported by eye-tracking experiments in Joachims et al. [13].

Second, search costs are constant across all consumers, and consumers have to pay the search cost before purchasing. When shopping online, some consumers may have target items before the search. Namely, they may learn product characteristics offline (from friends' recommendations or past experiences) and not need to read the product description and pay the search cost before purchasing. A recent paper by Chen et al. [6] studies this scenario which they call "blind buying". This paper aims to build a tractable model to study the interaction between consumer search, sellers' pricing, and bidding in the sponsored link auction.

Third, I assume that one seller produces only one type of good. In reality, if a seller produces different products and owns more than one link on the platform, they can set prices and bid to maximize the joint profits from all her products. Studying the joint maximization problem is out of the scope of the paper. I leave the extensions for future research.

Lastly, consumers may find it costly to read through the search list and thus stop at some point without scrolling down. Stoping without reading all links is less likely to happen when the search list is short. And intuitively, adding this additional layer to stop the search would make higher positions more valuable but should not change the key results in this chapter.

4.3 Demand Characterization

Given $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$, CDK define the *effective value* of product i as $w_i := v_i + \min\{z_i, z_i^*\}$, which is the minimum between i's true value and reservation value. CDK find the eventual purchase condition without blocks: consumers eventually purchase the product with the highest net effective value $w_i - p_i$ among all $i \in N$, conditional on the net effective value is larger than the outside option value u_0. I adapt CDK's Theorem 1 into the context where consumers search block 1 first and update the outside option value according to the search outcome from block 1 when searching block 2.

Theorem 9 *For a consumer with* $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$*, let* $i^* = \arg\max_{i \in N_1}\{w_i - p_i\}$ *and* $j^* = \arg\max_{i \in N_2}\{w_i - p_i\}$*.*[9]

- *The consumer purchases product* $i \in N_1$ *if and only if* $i = i^*$, $w_i - p_i > u_0$, *and* $v_i + z_i - p_i > w_{j^*} - p_{j^*}$.
- *The consumer purchases product* $i \in N_2$ *if and only if* $i = j^*$ *and* $w_i - p_i > u_1$*, where* $u_1 := u_0$ *if* $w_{i^*} - p_{i^*} < u_0$ *and* $u_1 := v_{i^*} + z_{i^*} - p_{i^*}$ *if* $w_{i^*} - p_{i^*} \geq u_0$.

Proof Since the search rule in each block follows Weitzman, CDK's eventual purchase theorem applies in each block. Consumers always search block 1 before block 2 and update the current best option when switching blocks, so they compare the net realized value of i^* and the effective value of j^* to make the final purchase decision.

Demand for product i is the probability that the purchase conditions in Theorem 9 are satisfied. Theorem 9 says the product with the highest effective value in each block is the *candidate product* to purchase in that block. The consumer compares the outside option value, the **net true value** of the candidate product in block 1, and the **net effective value** of the candidate product in block 2 to make the purchase decision. The asymmetry between how consumers make the purchase decision in block 1 (using true value) and block 2 (using effective value) is the underlying force making a position in block 1 valuable.

Since consumers select the candidate product in each block based on the *net effective value*, it is useful to set up the distribution of the effective value of product i, $W_i = V_i + \min\{Z_i, z_i^*\}$, to characterize the demands. Denote the distribution of W_i as $H_i(\cdot)$:

$$H_i(w_i) := \int_{\underline{z}_i}^{\bar{z}_i} F_i(w_i - \min\{z_i, z_i^*\}) dG_i(z_i). \tag{4.2}$$

Notations Before proceeding, I introduce some notations to formulate demand functions. Define $\Gamma_i^k := \max_{j \in N_k \setminus \{i\}}\{W_j - p_j\}$ be the consumer's (random) net effective value of the best alternative to product i in block k, and define $\Gamma_0^k := \max_{j \in N_k}\{W_j - p_j\}$ be the (random) net effective value of the candidate product in block k. The table below summarizes notations used to characterize demand:

random variable	distribution	realization	support
$X_i := \max\{u_0, \Gamma_i^1\}$	$\widetilde{H}_i^1(\cdot)$	x_i	$[\underline{x}_i, \overline{x}_i]$
$Y_i := \max\{u_0, \Gamma_i^2\}$	$\widetilde{H}_i^2(\cdot)$	y_i	$[\underline{y}_i, \overline{y}_i]$
$X_0 := \Gamma_0^1$	$H_*^1(\cdot)$	x_0	$[\underline{x}_0, \overline{x}_0]$
$Y_0 := \Gamma_0^2$	$H_*^2(\cdot)$	y_0	$[\underline{y}_0, \overline{y}_0]$

[9] Since both f_i and g_i are continuous, the event $w_i - p_i = w_j - p_j$ for $i \neq j$ happens with zero probability.

4.3.1 Demand in Block 1

When product i belongs to block 1 ($i \in N_1$), for any $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$, x_i is the largest effective value in block 1 besides i. By Theorem 9, if $x_i \geq y_0$, a consumer never purchases from block 2, since according to the optimal search rule, the consumer cannot find any product in block 2 better than the candidate product in block 1. Thus, the sufficient condition for the consumer to purchase product i is $w_i - p_i > x_i$, so i is the candidate product in block 1. However, if $y_0 > x_i$, $w_i - p_i > x_i$ alone is no longer sufficient to guarantee the purchase of i, since the consumer may find a product better than i in block 2 if the net realized value of i is not large enough (i.e., $v_i + z_i - p_i < y_0$). The next lemma states a simple condition under which product $i \in N_1$ is purchased.

Lemma 6 *Given $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$, the consumer purchases product i in block 1 if and and only if*

$$v_i + \min\{z_i, z_i^* + (y_0 - x_i)^+\} - p_i > \max\{x_i, y_0\}, \tag{4.3}$$

where $(y_0 - x_i)^+ := \max\{y_0 - x_i, 0\}$.

Lemma 6 is proven in Appendix C.1. Given Lemma 6, it is useful to define the the distribution of $W_i(q) := V_i + \min\{Z_i, z_i^* + q\}$ for $q \geq 0$, which is the *distorted effective value* of product i that stems from the block-by-block search behavior. Lemma 6 thus says a product in block 1 is purchased if its distorted effective value is larger than the effective value of all other products.

Remark One can regard the change of effective value to the distorted effective value of product i as if the reservation value of i increases from $v_i + z_i^*$ to $v_i + z_i^* + q$. When there is no block, a consumer searches in the descending order of the net reservation values among all products. So an increase of i's reservation value means the consumer searches i earlier. Hence, the distorted effective value reflects that search order is distorted such that products in block 1 are searched earlier.

The distribution of the distorted effective value, $W_i(q)$, is

$$\widehat{H}_i(w_i, q) := \int_{\underline{z}_i}^{\bar{z}_i} F_i(w_i - \min\{z_i, z_i^* + q\}) dG_i(z_i). \tag{4.4}$$

Lemma 7 below states that $\widehat{H}_i(\cdot, q)$ becomes more first order stochastically dominant when q increases.

Lemma 7 *For any $q' \geq q \geq 0$, $\widehat{H}_i(\cdot, q')$ first order stochastically dominates (FOSD) $\widehat{H}_i(\cdot, q)$.*

Proof For any realized (v_i, z_i) and any $q' \geq q \geq 0$, $w_i(q') \geq w_i(q)$ by definition. The the difference between $\widehat{H}(\cdot, q')$ and $\widehat{H}(\cdot, q)$ is

$$\widehat{H}_i(w_i, q') - \widehat{H}_i(w_i, q) = \int_{\underline{z}_i}^{\bar{z}_i} \left[F_i(w_i - \min\{z_i, z_i^* + q'\}) - F_i(w_i - \min\{z_i, z_i^* + q\})\right] dG_i(z_i),$$

which is non-positive as F_i is a distribution and is increasing.

Let $\boldsymbol{p} := (\boldsymbol{p}_1, \boldsymbol{p}_2)$ be the price vector of all products. Given x_i and y_i, Ineq. (4.3) holds with probability $1 - \widehat{H}_i\left(\max\{x_i, y_0\} + p_i, (y_0 - x_i)^+\right)$. Thus, the demand for product i in block 1 is:

$$D_i^1(\boldsymbol{p}) \;:=\; \int_{\underline{y}_0}^{\bar{y}_0} \int_{\underline{x}_i}^{\bar{x}_i} \left(1 - \widehat{H}_i\left(\max\{x_i, y_0\} + p_i, (y_0 - x_i)^+\right)\right) d\widetilde{H}_i^1(x_i) dH_*^2(y_0). \qquad (4.5)$$

To compare the effect of block-by-block search on product demand to that without blocks, denote $\mathscr{D}_i(\boldsymbol{p})$ as the demand function without blocks, such that consumers are aware of all products when search starts. CDK's Eventual Purchase Theorem says consumers always purchase the item with highest net effective value. Thus,

$$\mathscr{D}_i(\boldsymbol{p}) \;:=\; \int_{\underline{y}_0}^{\bar{y}_0} \int_{\underline{x}_i}^{\bar{x}_i} \left(1 - H_i\left(\max\{x_i, y_0\} + p_i\right)\right) d\widetilde{H}_i^1(x_i) dH_*^2(y_0), \qquad (4.6)$$

which is the probability of $w_i - p_i > \max\{x_i, y_0\}$.

Taking the difference between (4.5) and (4.6), a position in block 1 increases product i's demand by:

$$\int_{\underline{y}_0}^{\bar{y}_0} \int_{\underline{x}_i}^{\bar{x}_i} \left(H_i\left(\max\{x_i, y_0\} + p_i\right) - \widehat{H}_i\left(\max\{x_i, y_0\} + p_i, (y_0 - x_i)^+\right)\right) d\widetilde{H}_i^1(x_i) dH_*^2(y_0),$$

which, by Lemma 7 and that $H_i(\cdot) = \widehat{H}_i(\cdot, 0)$, is non-negative and is strictly positive if the event $Y_0 - X_i > 0$ occurs with a strictly positive probability.

4.3.2 Demand in Block 2

A block-1 product's demand depends on the product's *distorted effective value*. In contrast, a block-2 product's demand involves net *true value* of the candidate product in block 1, which is treated as the updated outside option value when consumers search in block 2.

For any $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$, I use the notations in Theorem 9 and let $i^* := \arg\max_{i \in N_1}\{w_i - p_i\}$. I call "the consumer finds a *candidate product* in block 1" if $w_{i^*} - p_{i^*} \geq u_0$. That is, the consumer finds a better option than the original outside option after searching block 1. For any consumer searching in block 2, let u_1 denotes the updated outside option value from searching block 1. Thus, u_1 equals u_0 if the consumer does not find a candidate product in block 1 and equals $v_{i^*} + z_{i^*} - p_{i^*}$ otherwise. Theorem 9 says that product i in block 2 is purchased if and only if $w_i - p_i > \max\{u_1, y_i\}$. I denote U_1 as the random variable of u_1, since the updated outside option value is random from block-2 sellers' perspective.

To derive demand for products in block 2, it is essential to characterize the cumulative density function of U_1, denoted as $J(\cdot)$. For a fixed block-1 candidate product i, $J(u)$ is the probability of that $i \in N_1$ is the candidate product and that i's net true value is less than u, which is $\Pr(\{W_i - p_i > X_i\} \cap \{V_i + Z_i - p_i < u\})$, or equivalently,

$$\Pr\left(W_i - p_i > X_i\right) - \Pr\left(\{W_i - p_i > X_i\} \cap \{V_i + Z_i - p_1 > u\}\right).$$

Given x_i (the effective value of the best alternative of i in block 1), the former probability in the above expression is $1 - H_i(x_i + p_i)$ and the latter is $1 - \widehat{H}_i(u + p_1, u - x_i)$, conditional on

$u > x_i$.[10] As $u_1 = u_0$ if and only if a consumer does not find a *candidate product* in N_1, we have $J(u_0) = H^1_*(u_0)$. Thus, the probability of $u_0 < U_1 < u$ can be formulated as:

$$J(u) - H^1_*(u_0) \quad := \quad \sum_{i \in N_1} \int_{u_0}^{u} \left[\widehat{H}_i(u + p_i, u - x_i) - H_i(x_i + p_i)\right] d\widetilde{H}^1_i(x_i). \tag{4.7}$$

The summation is taken as every block-1 product is possible to be the candidate product. Notice that while $\widehat{H}(x, q) \leq H(x)$ for any x and $q \geq 0$, $\widehat{H}(u + p, u - x) \geq H(u + p)$ for any $u \geq x$, so the integrand in (4.7) is positive for all i.

By Theorem 9, the demand for a product $i \in N_2$ is:

$$D^2_i(\boldsymbol{p}) \quad := \quad \int_{\underline{y}_i}^{\bar{y}_i} \int_{u_0}^{\bar{u}_1} (1 - H_i(\max\{u_1, y_i\} + p_i))\, dJ(u_1) d\widetilde{H}^2_i(y_i), \tag{4.8}$$

which is the probability that $W_i - p_i$ is larger than $\max\{U_1, Y_i\}$. By definition, given $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p})^2_{k=1}$, if $w_{i^*} - p_{i^*} \geq u_0$, $u_1 := v_{i^*} + z_{i^*} - p_{i^*} \geq v_{i^*} + \min\{z_{i^*}, z^*_{i^*}\} - p_{i^*} =: x_0$; else, $u_1 = u_0 = x_0$. So $J(\cdot)$ FOSDs $H^1_*(\cdot)$. Notice that for any $u > u_0$, $H^1_*(u) - H^1_*(u_0)$ can be written as (shown in Appendix C.2):

$$H^1_*(u) - H^1_*(u_0) \quad = \quad \sum_{i \in N_1} \left[\int_{u_0}^{u} H_i(u + p_i) - H_i(x_i + p_i)\right] d\widetilde{H}^1_i(x_i). \tag{4.9}$$

Comparing (4.7) and (4.9), for any $u > u_0$, the difference between H^1_* and J is

$$\begin{aligned} \mathcal{K}(u) \quad &:= \quad H^1_*(u) - J(u), \\ &= \quad \sum_{i \in N_1} \int_{u_0}^{u} \left[H_i(u + p_i) - \widehat{H}_i(u + p_i, u - x_i)\right] d\widetilde{H}^1_i(x_i), \end{aligned} \tag{4.10}$$

for any $u \geq u_0$. $\mathcal{K}$ has an intuitive meaning: conditional on i is the candidate product in N_1, $\mathcal{K}(u)$ is $\Pr(\{W_i - p_i < u < V_i + Z_i - p_i\})$. That is, the probability of that u lies between i's effective value and true value. This is because the integrand in (4.10) is

$$\begin{aligned} & (1 - \widehat{H}_i(u + p_i, u - X_i)) - (1 - H_i(u + p_i)) \\ = \; & \Pr(\{W_i - p_i > X_i\} \cap \{V_i + Z_i - p_i > u\}) - \Pr(W_i - p_i > u) \\ = \; & \Pr(\{X_i < W_i - p_i < u < V_i + Z_i - p_i\}). \end{aligned}$$

By Lemma 7, $H_i(u + p_i) > \widehat{H}_i(u + p_i, u - x_i)$, so (4.10) is always positive and implies $J(u)$ first order stochastically dominates $H^1_*(u)$.

When there is no block, following CDK, the demand for product $i \in N_2$ is:

$$\mathscr{D}_i(\boldsymbol{p}) \quad = \quad \int_{\underline{y}_i}^{\bar{y}_i} \int_{\underline{x}_0}^{\bar{x}_0} (1 - H_i(\max\{x_0, y_i\} + p_i))\, dH^1_*(x_0) d\widetilde{H}^2_i(y_i). \tag{4.11}$$

Comparing (4.11) and (4.8), the demand for product $i \in N_2$ decreases if consumer search block-by-block, as J FOSDs H^1_* and $1 - H_i(\max\{u_1, y_i\} + p_i)$ is decreasing in u_1.

The lemma 8 below concludes the comparison of product i's demand when consumers search block-by-block ($D^k_i(\boldsymbol{p})$) and i's demand without blocks ($\mathscr{D}_i(\boldsymbol{p})$).

[10]By the definition of $\widehat{H}$ and Lemma 6, $\Pr(\{W_i - p_i > x_i\} \cap \{V_i + Z_i - p_1 > u\}) = 1 - \widehat{H}_i(u + p_1, u - x_i)$. The two conditions are exactly the conditions to purchase $i \in N_1$ if $y_0 = u$.

Lemma 8 *Given a set of products N and any $\boldsymbol{p} \in \mathbb{R}^n_{++}$, for any position (N_1, N_2), if $i \in N_1$, then $D^1_i(\boldsymbol{p}) \geq \mathscr{D}_i(\boldsymbol{p})$, while if $i \in N_2$, then $D^2_i(\boldsymbol{p}) \leq \mathscr{D}_i(\boldsymbol{p})$.*

Lemma 8 states that the demand for any product increases if its position is in the first block, and thus sellers have incentives to win the sponsored link position. The intuition behind Lemma 8 is that: when the search order is distorted, it is possible that product i in block 1 is searched earlier comparing to the case without blocks. Thus, even if a product j in block 2 has a net effective value higher than that of i, and therefore has a higher priority to purchase than i without block, a consumer may end up purchasing i instead of j if the realized match value of product i is large enough. This is because j is no longer considered when the consumer searches in the second block if $i \in N_1$ is found to be good enough.[11] Thus, the demand for product $i \in N_1$ increases, and the demand for product $j \in N_2$ decreases, compared to the case without blocks.

4.4 Pricing Equilibrium

Since sellers' bidding strategy depends on their profits increase from winning the sponsored link auction, it is crucial to construct the equilibrium profit from price competition. This section establishes the existence and uniqueness of the pure strategy pricing equilibrium under any product position. Given equilibrium prices, sellers decide their auction bids by comparing equilibrium profits under different positions resulting from the auction. Optimal bidding strategy and its effect on pricing strategy are studied in Section 4.5.

Given any position (N_1, N_2), each seller i chooses the product price p_i to maximize the profit, $(p_i - c_i)\, D^k_i(\boldsymbol{p})$. The following three assumptions are needed to establish the existence and uniqueness of the pricing equilibrium.

Assumption 1 *For all i, $f_i(\cdot)$ and $g_i(\cdot)$ are log-concave.*

Assumption 2 *For all i, the support of $f_i(\cdot)$ has no upper bound (i.e., $\overline{v}_i = \infty$). Moreover, the variance of V_i is sufficiently large, and either F_i has no lower bound (i.e., $\underline{v}_i = -\infty$) or $f_i(\underline{v}_i) = 0$ for all i.*

Assumption 3 *For all i, either $g'_i(\overline{z}_i) \leq 0$, or G_i has no upper bound (i.e., $\overline{z}_i = \infty$).*

Theorem 10 *Under Assumption 1, 2 and 3, for any $k \in \{1, 2\}$ and any $i \in N_k$, $D^k_i(\boldsymbol{p})$ is log-concave in p_i and $\log D^k_i(\boldsymbol{p})$ has strictly increasing differences in p_i and p_j for any $j \neq i$. And there is a unique pure strategy pricing equilibrium for all sellers.*

Theorem 10 is proved in Appendix C.3. Log-concavity is a standard assumption in literature of pricing equilibrium.[12] The necessary first-order condition (FOC) for pricing equilibrium is:

$$\frac{1}{p_i - c_i} = -\frac{dD^k_i(\boldsymbol{p})/dp_i}{D^k_i(\boldsymbol{p})}, \tag{4.12}$$

[11] This is the case where $v_i + \min\{z_i, z^*_i\} - p_i < v_j + \min\{z_j, z^*_j\} - p_j$ but $v_i + z_i - p_i > v_j + z^*_j - p_j$.

[12] A comprehensive list of log concave density is given by Quint (2014) [18] and Bagnoli and Bergstrom (2005) [3]. The latter shows a distribution function is log-concave if it has a differentiable and log-concave density.

which is also sufficient given Theorem 10. Assumption 2 guarantees any product has a strictly positive possibility to be purchased at any price, and eliminates the equilibrium with a zero demand. The intuition of Theorem 10 is similar to Theorem 2 in CDK and Theorem 1 in Quint (2014) [18]. Assumption 1 implies $1 - \widehat{H}_i(\cdot, q)$ is log-concave for all i and $q > 0$. Assumption 1 and 2 together implies H_i is also log-concave (See the right panel of Figure 4.2 and CDK's Proposition 2). All three assumptions together implies J is log-concave. Since log-concavity in preserved under marginalization (Prékopa-Leindler's inequality), by the demand functions in Section 4.3, $D_i^k(\boldsymbol{p})$ is log-concave for any $k \in \{1, 2\}$ and $i \in N_k$. Log-concavity of $D_i^k(\boldsymbol{p})$ guarantees the right-hand side of (4.12) being monotonically increasing in p_i, while the left-hand side is strictly decreasing in p_i. So an equilibrium exists. Strictly increasing difference of $\log D_i^k(\boldsymbol{p})$ in p_i and p_j guarantees the best respond is unique.

Figure 4.2 shows large variance of V_i ensures log-concavity of effective values. In both panels, $F_i(v_i) = 1/(1 + e^{-v_i/\alpha})$, $G_i(z_i)$ is standard normal and $s_i = 1$ (z_i^* is approximately -0.9) for all $i \in N$. In the left panel, $p_1 = 1$ and $n_1 = 3$.

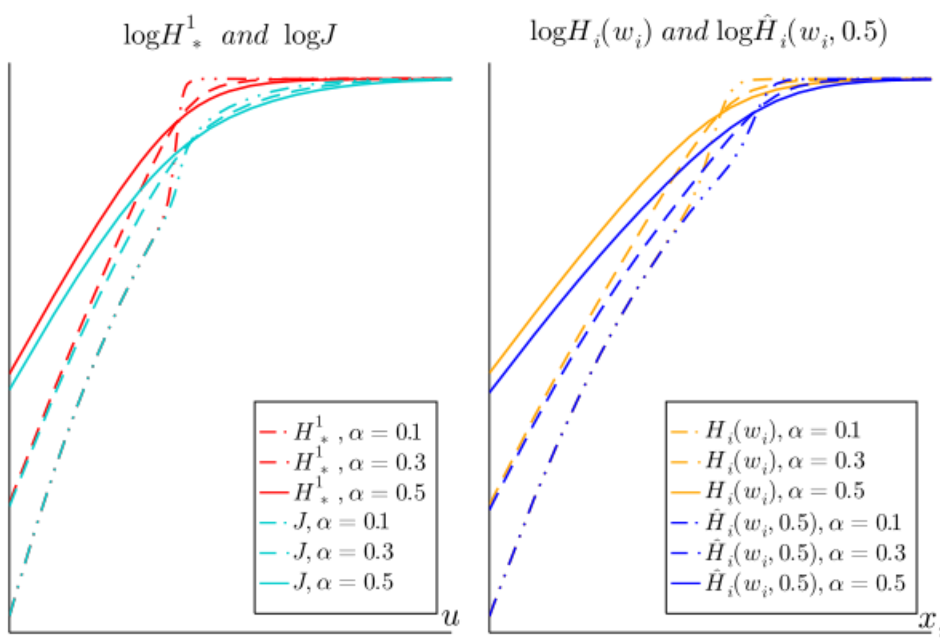

Figure 4.2: $\log H_*^1(u)$ and $\log J(u)$ (left), and $\log H_i(w_i)$ and $\log \widehat{H}_i(w_i, 0.5)$ (right) with different variances of V_i.

To prove Theorem 10, I take advantage of existing results from Quint (2014). With the discrete choice formulation of the demand (Theorem 9), Quint (2014)'s Theorem 1 implies it suffices to show: 1) for any $i \in N_1$ and $j \neq i$, $\Pr(\max\{X_i, Y_0\} - \min\{Z_i, z_i^* + (Y_0 - X_i)^+\} < t)$ is log-supermodular in t and p_j; and 2) $J(u)$ is log-supermodular in u and p_i for any $i \in N_1$.[13] Both are proved in Appendix C.3 given Assumptions 1, 2 and 3. In particular, by (4.10), $J(u) = H_*^1(u) - \mathcal{K}(u)$, and $H_*^1(u) = \prod_{i \in N_1} H_i(u + p_i)$ is log-concave provided H_i is log-concave. I show when the variance of V_i is sufficiently large and the density of Z_i is weakly decreasing

[13] Quint defines a function is log-supermodular if its log is supermodular. For simplicity, I abuse the notation by defining J as a single argument function, but p_i affects J for any $i \in N_1$.

at the upper bound (Asumption 3), $\mathcal{K}(u)$ is spread out and its effect on the shape of $\log J(u)$ vanishes (See the left panel of Figure 4.2). Assumption 3 guarantees there is no spike in J at the upper bound when the variance of V_i is large, and is relatively weak as it holds under many log-concave density functions.[14]

4.5 Sponsored Link Auctions

Since *status quo* position (N_1^0, N_2^0) is exogenously fixed, with one spot in block 1 been auctioned, only two forms of position exist: 1) If seller $i \in N_1^0$ wins the sponsored link auction, position stays the same as the *status quo*, i.e., $(N_1, N_2) = (N_1^0, N_2^0)$; 2) If seller $i \in N_2^0$ wins the sponsored link auction, position becomes $(N_1, N_2) = (N_1^0 + i - l(i), N_2^0 - i + l(i))$.

By Lemma 8, if the product position switches from (N_1^0, N_2^0) (*status quo* position) to $(N_1^0 + i - l(i), N_2^0 - i + l(i))$ for any $i \in N_2^0$, the demand for product i increases and that for $l(i)$ decreases. Thus, the sponsored link position has a strictly positive value for sellers in N_2^0. Seller $l(i) \in N_1^0$ thus bids in order to keep the position in block 1. Other sellers in N_1^0 also have incentives to submit a positive bid if i is a stronger competitor than $l(i)$, and i's entry decreases the profits of some sellers in $N_1^0 \setminus \{l(i)\}$.

I treat the sponsored link auction game as a complete information game, since given position (N_1, N_2) and price $(p_i)_{i=1}^n$, the demand for any product is completely determined by the priors $(F_i)_{i=1}^n$ and $(G_i)_{i=1}^n$, known to all sellers. The complete information assumption is also used by Varian (2007) [20], and Edelman, Ostrovsky, and Schwarz (2007) [11]. Since sponsored link auctions occur repeatedly and all sellers can update their bids frequently, they can easily learn the demands and competitors' bids by experimenting with their bids. Edelman and Schwarz (2010) [10] provide a detailed rationale to assume complete information.

The sponsored link auction is modelled as the second price auction.[15] The platform can choose between two different payment rules, fixed payment, and per-transaction payment. The winner pays the second-highest bid only once under the fixed payment and pays the second-highest bid whenever his/her product is purchased under the per-transaction payment. Since sellers' profits are uniquely determined under any position outcome by Theorem 10, a seller's gain from winning the sponsored auction is the difference between the profits under the losing position and that under the winning position. Same as Börgers et al. [4], I study the pure strategy subgame perfect Nash equilibrium (SPNE) of the bidding game. The bidding equilibrium exists if no seller has an incentive to revise the bid and switch the position unilaterally under the equilibrium. The concept is similar to the locally envy-free equilibrium defined in Edelman, Ostrovsky, and Schwarz (2007) [11], given the platform only sells one top position in my model.

For any i, let b_i be the equilibrium bid submitted by seller i, and the vector of bids is $\boldsymbol{b} := \{b_1, b_2, \ldots, b_n\}$. Denote $\boldsymbol{b}_{-i} = \{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n\}$ as the vector of bids besides i, and $\boldsymbol{b}_{-i}^{(1)} = \max \boldsymbol{b}_{-i}$ be the maximum bid in $\boldsymbol{b}_{-i}$.

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[14] For example, uniform, normal, logistic, and exponential distributions.

[15] Under complete information, the second price auction is equivalent to the ascending bid auction.

4.5.1 Fixed payment

Under the fixed payment, if $b_i = \max \boldsymbol{b}$, the winner i's payoff is $(p_i - c_i)D_i^1(\boldsymbol{p}) - \boldsymbol{b}_{-i}^{(1)}$.[16] Since $\boldsymbol{b}_{-i}$ and p_i are in additive separable terms in the payoff function, bidding and pricing strategies do not interact since the FOCs in (4.12) are not affected. Denote π_i as the equilibrium profit of seller $i \in N$ under *status quo* position (N_1^0, N_2^0), and π_i^j as the equilibrium profit of seller i if j wins the auction. So π_i^j is the profit of i under the position $(N_1^0 + j - l(j), N_2^0 - j + l(j))$ if $j \in N_2^0$, and $\pi_i^j = \pi_i$ if $j \in N_1^0$.

For any seller $i \in N$, let b_i be the change of i's profit from the *status quo* position (N_1^0, N_2^0) to the position in which i wins the sponsored link auction, under the pricing equilibrium. That is,

$$b_i := \pi_i^i - \pi_i,$$

which is zero if $i \in N_1^0$. Let b_i^j be the change of i's profit from $j \in N_2^0$'s winning position, $(N_1^0 + j - l(j), N_2^0 - j + l(j))$, to i's winning position, which is (N_1^0, N_2^0) if $i \in N_1^0$ and is $(N_1^0 + i - l(i), N_2^0 - i + l(i))$ if $i \in N_2^0$. That is,

$$b_i^j := \pi_i^i - \pi_i^j.$$

If $i \in N_1^0$ and $j \in N_2^0$, b_i^j is i's willingness to pay to deter j entering block 1.

Theorem 11 *Under the fixed payment, conditional on all sellers setting the equilibrium prices, a subgame perfect Nash equilibrium exists.*

Theorem 11 is proved in Appendix C.4. The basic idea is to construct a tâtonnement process that ends at the equilibrium.

4.5.2 Per-transaction payment

Under the per-transaction payment, the winner pays to the platform when a consumer purchases the winner's product, and the winner i's payoff is $(p_i - c_i - \boldsymbol{b}_{-i}^{(1)})D_i^1(p_i, \boldsymbol{p}_{-i})$, where $\boldsymbol{b}_{-i}^{(1)}$ is the runner-up's bid. Thus, if seller i wins the sponsored link auction, it is equivalent to regard product i's marginal cost as $c_i + \boldsymbol{b}_{-i}^{(1)}$, with position changed according to the auction rule. Hence, Theorem 10 applies, and the prices of all products are determined uniquely given the bid payment $\boldsymbol{b}_{-i}^{(1)}$.

It turns out that with per-transaction payment, there is no difference between using first or second price auction from the winner's point of view since under the bidding equilibrium, the highest bid and the second-highest bid coincides.[17] This is because the winner's optimal price increases in the marginal cost, which "consists of" the runner-up's bid. In the complete information game, non-winners benefit from increasing their bids to increase the winner's marginal cost.

Lemma 9 *The equilibrium demand and profit of seller i increase if the marginal cost of i decreases, or if the marginal cost of j increases for any j ≠ i.*

[16]Fixed payment here is equivalent to pay-per-impression in marketing literature.

[17]More precisely, the runner-up's bid is arbitrarily close to the winner's bid.

Lemma 9 follows from Quint's Theorem 4, which applies because of the log-supermodular demand structure.[18] Thus, all non-winner sellers benefit if the runner-up's bid becomes closer to the winner's bid, regardless of who is the runner-up. This process continues until the runner-up's bid matches the winner's bid. Thus, in the rest of the section, I take the auction payment as a first price auction. The only difference is that the runner-up's bid must match the winner's bid, and the tie-breaking rule is that the runner-up never wins.[19] This avoids the empty best response of the runner-up.

Similar to Section 4.5.1, I define $\pi_i^j(b)$ as the equilibrium profit of seller i conditional on j wins the auction with bid b. In particular, I define the winner's profit as that net from auction payments to simplify the notation. That is, if seller i submits the highest bid b, the equilibrium profit is $\pi_i^i(b) := (p_i - c_i - b)D_i^1(p_i, \boldsymbol{p}_{-i})$. The winner always pays his/her bid since the auction is treated as the first price for the reason discussed above.

To characterize the auction equilibrium, I consider the hypothetical scenario where the auction winner r, the winning bid b, and the runner-up j are given ex-ante. Under the bidding equilibrium that r wins with bid b, two conditions must be satisfied. First, no seller finds it profitable to beat r:

$$\pi_i^i(b) - \pi_i^r(b) \leq 0, \tag{4.13}$$

for all $i \in N \setminus \{r\}$. Lemma 9 states that any seller's equilibrium profit is decreasing in the marginal cost and is increasing in a competitor's marginal cost. Hence the left-hand side of (4.13) is decreasing in b, which pins down the lower bound of b. Second, it is not profitable for winner r to withdraw the bid and let the runner-up j wins:

$$\pi_r^r(b) - \pi_r^j(b) \geq 0, \tag{4.14}$$

for a given $j \in N \setminus \{r\}$, which characterizes the upper bound of b as (4.14) is decreasing in b. The bidding equilibrium exists if there exist r and j such that the upper bound of b characterized by (4.14) is larger than the lower bound of b characterized by (4.13).

Theorem 12 *Under the per-transaction payment, conditional on all sellers setting the equilibrium prices, a subgame perfect Nash equilibrium exists.*

One special case is where sellers are symmetric (Section 4.6), such that the cycle holds whenever the bid is lower than the equilibrium bid. But once the bid reaches the equilibrium bid, all strict inequalities in the cycle become equality, so (4.13) is satisfied.

4.6 Comparative Statics

This section shows a set of comparative static results when sellers are symmetric. To be specific, for any $i \in N$, let $F_i = F$, $G_i = G$, $c_i = c$, $s_i = s$. I compare the auction revenue, consumer surplus, and seller profits between the fixed payment and the per-transaction payment rule. Assumption 1, 2 and 3 are kept in this section.

[18] I do not need Quint's Assumption 3 since each product only has one component in my model.

[19] Under complete information, the runner-up can submit and retrieve the bid until the tie is broken such that he/she does not win.

With Theorem 10, equilibrium prices are pinned down by FOCs. Under the symmetric environment, it is without loss of generality to fix the winner as $r \in N$. If the bid payment is fixed, by Theorem 10, all sellers in block 1 (include the winner r) set the price at p_1^f and all sellers in block 2 set the price at p_2^f, such that p_1^f and p_2^f satisfy:

$$\frac{1}{p_k^f - c} = -\frac{dD_i^k(p, \boldsymbol{p}_{-i}^f)/dp}{D_i^k(p, \boldsymbol{p}_{-i}^f)}\bigg|_{p=p_k^f},$$

for $k \in \{1,2\}$, where $D_i^k(p, \boldsymbol{p}_{-i}^f)$ is the demand for $i \in N_k$ conditional on i sets a price p and all other sellers set the equilibrium price $\boldsymbol{p}_{-i}^f$ under the fixed payment. If the bid payment is based on the number of transactions and the winner's bid is b, the equilibrium is that the winner sets a price p_r, all sellers in $N_1 \setminus \{r\}$ set price at p_1^t, and all sellers in N_2 set price at p_2^t. And p_r, p_1^t and p_2^t satisfy

$$\frac{1}{p_r - c - b} = -\frac{dD_r^k(p, \boldsymbol{p}_{-i}^t)/dp}{D_r^k(p, \boldsymbol{p}_{-i}^t)}\bigg|_{p=p_r}, \quad \frac{1}{p_k^t - c} = -\frac{dD_i^k(p, \boldsymbol{p}_{-i}^t)/dp}{D_i^k(p, \boldsymbol{p}_{-i}^t)}\bigg|_{p=p_k^t},$$

for $k \in \{1,2\}$, where $D_i^k(p, \boldsymbol{p}_{-i}^t)$ is the demand for $i \in N_k$ conditional on i set price at p and all other sellers set the equilibrium price $\boldsymbol{p}_{-i}^t$ under the per-transaction payment. The proof of the uniqueness of the equilibrium is provided in Appendix C.6.

For notation simplicity, I denote D_f^1 and D_f^2 as the equilibrium demands for a product in block 1 and block 2 under the fixed payment. And define $D_r(b)$, $D_t^1(b)$ and $D_t^2(b)$ as the equilibrium demand for the winner's product, a product in block 1 and block 2 under the per-transaction payment when the winner's bid is b.

With the notations from Section 4.5.1, we cans show

$$b_i = \begin{cases} 0 & \text{if } i \in N_1^0 \\ (p_1^f - c)D_f^1 - (p_2^f - c)D_f^2 & \text{if } i \in N_2^0 \end{cases},$$

$$b_i^j = \begin{cases} (p_1^f - c)D_f^1 - (p_2^f - c)D_f^2 & \text{if } j \in N_0^2 \text{ and } i = l(j) \\ b_i & \text{else} \end{cases}.$$

Thus, the auction revenue under fixed payment is $(p_1^f - c)D_f^1 - (p_2^f - c)D_f^2$.

I use the notations in Section 4.5.2 to find the bidding equilibrium with per-transaction payment. With the symmetric assumption, the winner r's net profit is $\pi_r^r(b) = (p_r^t - c - b)D_r(b)$, and

$$\pi_i^j(b) = \begin{cases} (p_1^t - c)D_t^1(b) & \text{if } i \in N_1^0 \text{ and } i \neq l(j) \\ (p_2^t - c)D_t^2(b) & \text{else} \end{cases}.$$

Because of the "for all" quantifier, (4.13) becomes:

$$(p_r^t - c - b)D_r(b) \leq (p_2^t - c)D_t^2(b) \quad \textit{and} \quad (p_r^t - c - b)D_r(b) \leq (p_1^t - c)D_t^1(b),$$

And because of the "there exists" quantifier, (4.14) becomes

$$(p_r^t - c - b)D_r(b) \geq (p_2^t - c)D_t^2(b) \quad \textit{or} \quad (p_r^t - c - b)D_r(b) \geq (p_1^t - c)D_t^1(b).$$

The two conditions in (4.13) and (4.14) together imply, under the equilibrium,

$$(p_r^t - c - b)D_r(b) = (p_2^t - c)D_t^2(b), \tag{4.15}$$

and

$$(p_r^t - c - b)D_r(b) \leq (p_1^t - c)D_t^1(b). \tag{4.16}$$

Condition (4.15) thus determines the equilibrium bid and the auction revenue is

$$bD_r(b) = (p_r^t - c)D_r(b) - (p_2^t - c)D_t^2(b). \tag{4.17}$$

Theorem 13 *In the symmetric environment, both consumer surplus and auction revenue are higher under the fixed payment than under the per-transaction payment; profits of non-winners are higher under the per-transaction payment than that under the fixed payment.*

Proof By the Monotone Selection Theorem of Milgrom and Shannon [15], the product price of the winner r increases when the payment rule switches from the fixed to the per-transaction ($p_r^t > p_1^f$). Since prices are complements in the supermodular game, prices of all products increase. Lemma 9 implies an increase of winner r's marginal cost increases the equilibrium profits of non-winners.

The winner r's profit, $(p_r^t - c)D_r(b)$, must be weakly smaller. This is because non-winners' FOCs are the same under the fixed payment and per-transaction payment. If $(p_r^t - c)D_r(b) > (p_1^f - c)D_1(b)$, winner's profit under the fixed payment can be increased if the winner switches the price from p_1^f to p_r^t. Contradict p_1^f is the best response of the winner under the fixed payment.

These together imply $(p_r^t - c)D_r(b) < (p_1^f - c)D_f^1$ and $(p_2^t - c)D_t^2(b) > (p_2^f - c)D_f^2$ for any $b > 0$. By (4.17), the total bid payment of winner r decreases. The consumer surplus decreases as the prices of all products increase, i.e., $p_r^t > p_1^f$, $p_1^t > p_1^f$ and $p_2^t > p_2^f$.

The intuition of Theorem 13 is comparable to the distortion effect of proportional income/consumption tax. The trade-offs between using the proportional tax and the lump-sum tax are efficiency (higher tax income) and redistribution (tax more on high income).[20] While in a sponsored link auction, the trade-off between using the fixed payment and using the per-transaction payment is how to split the total surplus among the platform, buyers, and sellers: fixed payment leads to higher auction revenue and consumer surplus, and per-transaction payment leads to higher seller surplus. In reality, an online shopping platform may care both consumer surplus and seller surplus for a long-run objective. Amazon also collects revenue from Amazon Prime membership subscriptions, which relates to customers' shopping experience or consumer surplus. The per-click payment used by Amazon is close to a mixture between the fixed payment and the per-transaction payment since every click has a probability of converting to a transaction. Theorem 13 thus implies the per-click payment used by Amazon splits the total surplus from the online transaction between consumers and sellers in a relatively equal way. While the per-transaction payment used by eBay is a seller-friendly rule.

A recent paper by Ostrovsky (2021) [17] studies choice screen auctions with different payment methods. Though the underlying mechanisms are different, he finds a similar result in

[20] See Chapter 6 in Kaplow (2008) [14].

consumer surplus. The choice screen auction is operated on Android platform smartphones, and search engine providers compete to place their apps on the screen the first time consumers use the smartphone to pick a default search engine. Search engine providers can choose the popularity (demand) of the search engine and the revenue-per-use (product price), which are negatively correlated. Ostrovsky (2021) shows that "per-install" (per-transaction) payment distorts the search engine toward extracting as much revenue from its users (higher price and lower consumer surplus) compared to "per-appearance" (fixed) payment, as the winner has the incentive to decrease its number of installs (transactions).

4.7 Consumer Optimal Positioning

If the platform aims to improve the users' experiences for a long-term goal (e.g., attract more users and more membership subscriptions like Amazon Prime), or if a social planner regulates the platform, consumer surplus maximization may become a part of the goals the platform target. This section assumes all sellers commit to product prices before the position allocation.

To maximize consumer surplus, a naive platform may intend to allocate products with higher expected net true value into block 1. But such a practice may fail to be optimum. The reason is that under the search environment, the match value z_i is unknown a priori, and consumers, in general, do not purchase the product with the highest net true value (i.e., the existence of the search friction). As search is costly, consumers' payoffs increase from a lower search cost paid during the search if the purchase decision remains the same. The question is how the platform should allocate products to minimize the search frictions, which requires balancing consumers' payoff from the final purchasing and the cost paid during the search. This section provides a consumer optimal position rule that answers the question.

As consumers pay search costs whenever they visit (click) a product selling page. To calculate the total search cost, I need to formalize the conditions under which a consumer clicks i's link.

Lemma 10 *Given $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$, product $i \in N_k$ for $k \in \{1, 2\}$ is clicked if and only if both conditions below are satisfied.*

1. $v_i + z_i^* - p_i > u_{k-1}$,
2. $\forall j \in N_k \setminus \{i\}$, $v_j + \min\{z_j, z_j^*\} - p_j < v_i + z_i^* - p_i$.

Proof The proof directly follows from the optimal search rule in Lemma 5. The first condition states the consumer never click a link with a reservation value less than the outside option value, u_0 if searching in block 1 and u_1 if searching in block 2. The second condition is: $\forall j \in N_1 \setminus \{i\}$ such that $v_j + z_j^* - p_j > v_i + z_i^* - p_i$, it has to be $v_j + z_j - p_j < v_i + z_i^* - p_i$. That is, for any product j that is searched before i according to the optimal search rule, the net realized value of j must be less than the net reservation value of i. So the consumer continues searching to product i.

For any position (N_1, N_2), I use the definition of X_i, X_0, Y_i, Y_0 and U_1 as that in Section 4.3. Since X_i, X_0, Y_i, Y_0 and U_1 purely depend on V_i, Z_i and product position, all expectations in this section are taken with respect to V_i and Z_i for all $i \in N$. Under the optimal search rule, consumer surplus is a function $W : (N_1, N_2) \to \mathbb{R}$ such that

$$\begin{aligned} W(N_1, N_2) \quad := \quad & \mathbb{E}\left[\mathbb{1}_{\{\max\{X_0,Y_0\}<u_0\}}\right] u_0 \\ + \quad & \sum_{i\in N_1} \mathbb{E}\left[\mathbb{1}_{\{X_i<W_i-p_i\}\cap\{Y_0<V_i+Z_i-p_i\}\}}(V_i + Z_i - p_i) - \mathbb{1}_{\{X_i<V_i+z_i^*-p_i\}} s_i\right] \\ + \quad & \sum_{i\in N_2} \mathbb{E}\left[\mathbb{1}_{\{\max\{U_1,Y_i\}<W_i-p_i\}}(V_i + Z_i - p_i) - \mathbb{1}_{\{\max\{U_1,Y_i\}<V_i+z_i^*-p_i\}} s_i\right]. \end{aligned} \tag{4.18}$$

The first term in (4.18) is the payoff of consumers who pick the outside option eventually. In the two summations (line 2 and line 3), the first terms are payoffs of consumers purchasing product i in block 1 and block 2, respectively, where the indicators incorporate the purchase conditions from Theorem 9. Lemma 10 says that product $i \in N_k$ is visited (clicked) if and only if i's net reservation value is larger than the highest effective value in N_k. This forms the second term in the two summations.

Given sizes of the two blocks n_1 and n_2, and sellers' committed prices $(p_i)_{i=1}^n$, the planner solves the following consumer surplus maximization problem:

$$\begin{array}{rl} \max_{N_1,N_2} & W(N_1, N_2) \\ \text{s.t.} & N_1 \cup N_2 = N, N_1 \cap N_2 = \emptyset \\ & |N_1| = n_1, |N_2| = n_2. \end{array} \tag{4.19}$$

Theorem 14 *Given F_i, G_i for all i, and the size of the two blocks n_1, n_2, and if each seller i commits to a price p_i, the problem (4.19) is equivalent to minimizing*

$$W^L(N_1, N_2) := \mathbb{E}\left[(Y_0 - X_0)^+\right] - \mathbb{E}\left[(Y_0 - U_1)^+\right], \tag{4.20}$$

subject to the constraints in (4.19).

Theorem 14 is proved in Appendix C.7, which shows

$$W(N_1, N_2) = \overline{W} - W^L(N_1, N_2),$$

where $\overline{W} = \mathbb{E}[u_0, X_0, Y_0]$ is the expectation of the highest net effective value among all products (including the outside option) and is uncorrelated with product position. $\overline{W}$ is exactly the consumer surplus without blocks.[21] Notice that W^L is non-negative given U_1 FOSDs X_0, so the consumer surplus is indeed lower comparing to the case without blocks (i.e., $W \leq \overline{W}$). W^L can thus be regarded as the welfare loss from the block-by-block search behavior.

To get implications of Theorem 14, rewrite W^L as:

$$W^L(N_1, N_2) = \mathbb{E}\left[\mathbb{1}_{\{U_1 \geq Y_0\}} (Y_0 - X_0)^+\right] + \mathbb{E}\left[\mathbb{1}_{\{U_1 < Y_0\}} (U_1 - X_0)\right]. \tag{4.21}$$

Theorem 14 thus tells us that the platform maximizes the consumer surplus under two criteria. First,
$\mathbb{E}\left[\mathbb{1}_{\{U_1 \geq Y_0\}} (Y_0 - X_0)^+\right] = \mathbb{E}\left[\mathbb{1}_{\{U_1 \geq Y_0 \geq X_0\}} (Y_0 - X_0)\right]$ should be small. That is, conditional on consumers purchase from block 1, or $U_1 \geq Y_0$, products with high (low) net effective values should

[21] See Corollary 1 of Choi, Dai and Kim (2018) [7]

be placed at block 1 (block 2). Second, $\mathbb{E}\left[\mathbb{1}_{\{U_1<Y_0\}}(U_1 - X_0)\right]$ should also be small. That is, conditional on consumers purchasing from block 2 ($U_1 < Y_0$), candidate product in block 1 should have a low search cost. This is because $U_1 - X_0 := (Z_{i^*} - z^*_{i^*})^+$, conditional on i^* is the candidate product in block 1. By (4.1), $\mathbb{E}\left[(Z_{i^*} - z^*_{i^*})^+\right] = s_{i^*}$.

There are two extreme cases where the loss from block-by-block search, $W^L(N_1, N_2)$, achieves zero under the optimal positioning. The first case is $w_{i^*} - p_{i^*} \geq w_{j^*} - p_{j^*}$ for any candidate product i^* in block 1 and any candidate product j^* in block 2 under any realized $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$. That is, $\Pr(X_0 \geq Y_0) = 1$: any product in block 1 has a net effective value larger than that in block 2. Under this case, the block-by-block search aligns perfectly the optimal search without blocks, since products in the second block are never visited. $W^L = 0$ since both terms in (4.20) are zeros.

The second case is $\Pr(U_1 = X_0) = 1$, which is equivalent to $\Pr\left(Z_{i^*} = z^*_{i^*}\right) = 1$. Under this case, the candidate product in block 1 has zero uncertainty in match value and incurs a zero search cost to click. Thus, although the block-by-block search distorts the optimal search order, it has no impact on consumer surplus since consumers pay no search cost in the first block. $W^L = 0$ since the two terms in (4.20) are identical. The two cases mentioned above have strong requirements on products' priors (non-overlapping effective value supports and zero search cost) and are not likely to happen. A consumer surplus maximizing platform needs to consider both criteria when positioning products.

Notice that the algorithm provided by Theorem 14 requires all sellers to commit to their prices before the allocation of product positions. If sellers are unable to commit to prices, a consumer surplus maximizing platform needs complete a much more complicated task by enumerating and comparing $W = \overline{W} - W^L$ under all possible positions with the corresponding equilibrium prices.

4.8 Conclusion

Internet changes every aspect of economics as a low-cost communication method. Since the pandemic, more and more people have shifted from in-store shopping to online shopping to keep social distance. Compared to in-store advertising, advertising a product online is much more straightforward: a seller can buy a position at the top of the product webpage. Online shopping platforms use sponsored link auctions to determine who gets the limited number of positions and how much to pay. A typical online shopping activity is an interaction between three parties: consumers, sellers, and the shopping platform, with each party a different objective: consumers maximize the consumption payoffs from search, sellers maximize the selling profits, and the platform either maximizes the sponsored link auction revenue or coordinates on the surplus split between the three parties as a social planner.

By assuming consumers follow a block-by-block searching behavior, this paper formulates consumers' discrete choice shopping problem and the demand functions. The paper also establishes pricing and bidding equilibrium and finds how pricing strategy, bidding strategy, and product position interact in a complete information second price auction. A comparison between the fixed payment and per-transaction payment in the symmetric environment shows that consumer surplus and auction revenue are higher under fixed payment. By contrast, sellers' profits are higher under per-transaction payment. The paper also finds a simple position rule to maximize consumer surplus if sellers commit to price before the position allocation. The demand functions in my model have standard properties, and the discrete choice formulation may apply to empirical research.

Many interesting questions remain to be answered. A natural question is how results might change if the platform auctions more than one position in the first block. With more than one position, the pricing equilibrium exists. Still, the allocative externality complicates bidding equilibrium as the winner of one position needs to make the bidding decision based on the identities of other winners. The other is to allow “blind buying” in Chen et al. [6] such that the consumer need not learn the match value before purchase.

Bibliography

[1] Mark Armstrong and Jidong Zhou. Paying for prominence. *The Economic Journal*, Vol. 121, No. 556, pp. F368- F395, 2011.

[2] Susan Athey and Glenn Ellison. Position auctions with consumer search. *Quarterly Journal of Economics*, pages 1213-1270, 2011.

[3] Mark Bagnoli and Ted Bergstrom. Log-concave probability and its applications. *Economic Theory*, Vol. 26, No. 2, pages 445–469, 2005.

[4] Tilman Börgers, Ingemar Cox, Martin Pesendorfer, and Vaclav Petricek. Equilibrium Bids in Sponsored Search Auctions: Theory and Evidence. *American Economic Journal: Microeconomics*, Vol. 5, No. 4, pages 163-187, 2013.

[5] Yongmin Chen and Chuan He. Paid Placement: Advertising and Search on the internet. *The Economic Journal*, 121 , F309-–F328, November 2011.

[6] Yanbin Chen, Sanxi Li, Kai Lin and Jun Yu. Consumer Search with Blind Buying. *Games and Economic Behavior*, 126, 402–427, 2021.

[7] Michael Choi, Anovia Yifan Dai, and Kyungmin Kim. Consumer search and price competition. *Econometrica*, Vol. 86, No. 4, pages 1257—1281, July, 2018.

[8] Giovanni Compiani, Gregory Lewis, Sida Peng and Peichun Wang. Online Search and Product Rankings: A Double Index Approach *Working Paper*, August 2021.

[9] Babur De Los Santos, Ali Hortaçsu, and Matthijs R. Wildenbeest. Testing Models of Consumer Search Using Data on Web Browsing and Purchasing Behavior. *American Economic Review*, Vol. 102, No. 6: pages 2955-–2980, 2012.

[10] Benjamin Edelman and Michael Schwarz. Optimal Auction Design and Equilibrium Selection in Sponsored Search Auctions *American Economic Review*, Vol. 100, No. 2, pages 597–602, 2010

[11] Benjamin Edelman, Michael Ostrovsky, and Michael Schwarz. Internet Advertising and the Generalized Second-Price Auction: Selling Billions of Dollars Worth of Keywords. *American Economics Review*, Vol. 97, No. 1, pages 242–259, March, 2007.

[12] Marco A. Haana, José L. Moraga-González and Vaiva Petrikaitė. A Model of Directed Consumer Search. *International Journal of Industrial Organization*, Vol. 61, Pages 223–255, November 2018.

[13] Thorsten Joachims, Laura Granka, Bing Pan, Helene Hembrooke and Geri Gay. Accurately Interpreting Clickthrough Data as Implicit Feedback. *Acm Sigir Forum*, Vol. 55, pages 4-–11, 2017.

[14] Louis Kaplow. The Theory of Taxation and Public Economics. *Princeton University Press*, 2008.

[15] Paul Milgrom and Chris Shannon. Monotone Comparative Statics. *Econometrica*, Vol. 62, No. 1, 157–180, January, 1994.

[16] Phillip Nelson. Information and Consumer Behavior. Journal of Political Economy, 78(2), pages 311—329, 1970.

[17] Michael Ostrovsky. Choice Screen Auctions Working paper, May 2021.

[18] Daniel Quint. Imperfect competition with complements and substitutes. *Journal of Economic Theory.* 152, pages 266–290, 2014.

[19] Martin L. Weitzman. Optimal Search for the Best Alternative. *Econometrica*, Vol. 47, No. 3, pages 641—654, 1979.

[20] Hal R. Varian. Position auctions. *International Journal of Industrial Organization*, 25, pages 1163-–1178, 2007.

Appendix A

Appendices to Chapter 2

A.1 Proof of Theorem 1

Let $Q : [0,1] \to \mathbb{R}$ be weakly increasing and market clearing. Denote

$$\begin{aligned} c_1 &:= \sup\{t \in [0,1] \mid Q(t) < 0\}, && \text{(A.1)} \\ c_2 &:= \inf\{t \in [0,1] \mid Q(t) > 0\}. && \text{(A.2)} \end{aligned}$$

Since Q weakly increasing, we have $c_1 \leq c_2$. Furthermore, by the envelope theorem, one can construct a payment rule p that implements Q such that IC is satisfied and the surplus $cQ(c) - p(c)$ is equal to zero for some $c \in [c_1, c_2]$. With $c_1 \leq c_2$ and Q weakly increasing, $tQ(t) - p(t)$ is weakly decreasing on $[0, c]$ and weakly increasing on $[c, 1]$. Thus (Q, p) also satisfies IR. The rest of the proof shows that (Q, p) satisfies BB.

First, we claim that (Q, p) satisfies BB if the following condition holds:

$$t < c_1 \leq c_2 < t' \Longrightarrow \frac{p(t)}{Q(t)} \leq c_1 \leq c_2 \leq \frac{p(t')}{Q(t')}. \quad \text{(A.3)}$$

By the definitions of c_1 and c_2, $Q = 0$ on (c_1, c_2). Thus, $tQ(t) - p(t) = 0$ and $p(t) = 0$ for all $t \in (c_1, c_2)$ by the envelope theorem and $cQ(c) - p(c) = 0$. It follows that

$$\begin{aligned} \int_0^1 p(t)dF(t) &= \int_0^{c_1} p(t)dF(t) + \int_{c_2}^1 p(t)dF(t) \\ &= \int_0^{c_1} \frac{p(t)}{Q(t)} Q(t)dF(t) + \int_{c_2}^1 \frac{p(t)}{Q(t)} Q(t)dF(t) \\ &\geq \int_0^{c_1} c_1 Q(t)dF(t) + \int_{c_2}^1 c_2 Q(t)dF(t) \\ &\geq c_1 \left(\int_0^{c_1} Q(t)dF(t) + \int_{c_2}^1 Q(t)dF(t) \right) \\ &= 0, \end{aligned}$$

where the third line is due to (A.3) and the fact that $Q < 0$ on $[0, c_1)$ and $Q > 0$ on $(c_2, 1]$, the fourth line due to $c_2 \geq c_1$ and $Q > 0$ on $(c_2, 1]$, and the last line due to market clearing. Thus, (Q, p) satisfies BB if (A.3) holds.

To prove (A.3), it suffices to prove that $p(t)/Q(t)$ is a weakly increasing function of t on $[0, c_1)$ and $(c_2, 1]$ and

$$\max_{t\in[0,c_1)} \frac{p(t)}{Q(t)} \leq c_1 \leq c_2 \leq \min_{t\in(c_2,1]} \frac{p(t)}{Q(t)}. \tag{A.4}$$

To prove the monotonicity of p/Q on $(c_1, 1]$ pick any $t, t' \in (c_1, 1]$ and $t > t'$. IC requires

$$\begin{aligned} & t'Q(t') - p(t') \geq t'Q(t) - p(t) \\ \iff \quad & t' - \frac{p(t')}{Q(t')} \geq \frac{Q(t)}{Q(t')}\left(t' - \frac{p(t)}{Q(t)}\right) \\ \iff \quad & t' - \frac{p(t')}{Q(t')} \geq t' - \frac{p(t)}{Q(t)}, \end{aligned}$$

where the second line follows by dividing $Q(t')$ on the both side of inequality. If $t' - p(t)/Q(t) \leq 0$ the last line follows directly since $t' - p(t')/Q(t') \geq 0$ by IR and $Q(t') > 0$. Else, $t' - p(t)/Q(t) > 0$ and the last line is due to $Q(t)/Q(t') \geq 1$. We can get the same result for $t, t' \in [0, c_2)$ and $t > t'$ by

$$\begin{aligned} & tQ(t) - p(t) \geq tQ(t') - p(t') \\ \iff \quad & t - \frac{p(t)}{Q(t)} \leq \frac{Q(t')}{Q(t)}\left(t - \frac{p(t')}{Q(t')}\right) \\ \iff \quad & t - \frac{p(t)}{Q(t)} \leq t - \frac{p(t')}{Q(t')}, \end{aligned}$$

where the last line is due to $Q(t')/Q(t) \geq 1$ if $t - p(t')/Q(t') \leq 0$ and follows directly if $t - p(t')/Q(t') > 0$ since $t - p(t)/Q(t) \leq 0$ by IR and $Q(t) < 0$. Thus, p/Q is weakly increasing on $[0, c_1)$ and $(c_2, 1]$.

To complete the proof, we show (A.4). For any $t < c_1$, IC implies $0 = c_1 Q(c_1) - p(c_1) \geq c_1 Q(t) - p(t)$, which implies $p(t)/Q(t) \leq c_1$ given $Q(t) < 0$ on $[0, c_1)$ Analogously, for any $t > c_2$, IC implies $0 = c_2 Q(c_2) - p(c_2) \geq c_2 Q(t) - p(t)$, which implies $p(t)/Q(t) \geq c_2$ given $Q(t) > 0$ on $(c_2, 1]$. Since $c_1 \leq c_2$, we have (A.4). This completes the proof of the theorem.

A.2 Derivation of the Virtual Surplus Function

While various forms of the routine have appeared numerously in the literature, it is helpful to formalize it so as to clarify the role of the constraints.

Lemma 11 *For any nonempty subset S of $\mathbb{R}$, the problem*

$$\begin{array}{rl} \max_{(Q,p)} & \int_0^1 (tQ(t) - p(t))\, dW(t) \\ s.t. & Q : [0, 1] \to S \textit{ is weakly increasing} \\ & p(t') - p(t) = \int_t^{t'} s dQ(s) \quad (\forall t, t' \in [0, 1]) \\ & \int_0^1 p dF \geq 0 \end{array} \tag{A.5}$$

is equivalent to

$$\begin{array}{ll} \max_{(Q,p)} & \int_0^1 Q(t)V(t)dF(t) \\ s.t. & Q:[0,1]\to S \text{ is weakly increasing} \\ & p(t')-p(t)=\int_t^{t'} sdQ(s) \quad (\forall t,t'\in[0,1]) \\ & \int_0^1 pdF=0. \end{array} \tag{A.6}$$

Proof First, there is no loss of generality to replace the constraint $\int_0^1 pdF \geq 0$ in (A.5) by $\int_0^1 pdF = 0$: If $\int_0^1 pdF > 0$, we can modify the payment rule by rebating the positive money surplus $\int_0^1 pdF$ back to the types uniformly. That enlarges $tQ(t) - p(t)$ for all $t \in [0,1]$ and hence enlarges the objective $\int_0^1 (tQ(t) - p(t))\, dW(t)$ because the distribution W assigns a positive measure on $[0,1]$. Thus, in any optimum, $\int_0^1 pdF > 0$ does not hold.

Second, with $\int_0^1 pdF = 0$, we show that the objective in (A.5) is equal to that in (A.6).[1] Denote $U(t) := tQ(t) - p(t)$ for all t. By the envelope theorem, $dU(t) = Q(t)dt$. This coupled with integration-by-parts gives

$$\int_0^1 UdF = U(1) - \int_0^1 FdU = U(1) - \int_0^1 F(t)Q(t)dt.$$

Likewise,

$$\int_0^1 UdW = U(1) - \int_0^1 W(t)Q(t)dt.$$

Plug the expression of $U(1)$ from the former equation into the latter equation to obtain

$$\begin{aligned} \int_0^1 UdW &= \int_0^1 UdF + \int_0^1 F(t)Q(t)dt - \int_0^1 W(t)Q(t)dt \\ &= \int_0^1 tQ(t)dF(t) - \int_0^1 p(t)dF(t) - \int_0^1 \frac{W(t)-F(t)}{f(t)}Q(t)dF(t) \\ &= \int_0^1 Q(t)V(t)dF(t), \end{aligned}$$

with the last equality due to $\int_0^1 pdF = 0$ and the definition of V.

A.3 Proof of Corollary 2

Given any mechanism (Q, p) in the modified model, a type-t individual's expected payoff from acting as type t' is equal to

$$tQ(t') - p(t') + \int_0^1 \psi(t,s)Q(s)dF(s).$$

[1] We thank the associate editor for suggesting the following short proof. Our previous proof is longer and suitable to asymmetric models where individuals' types are drawn from different distributions.

The integral, constant to t', is immaterial to the incentive constraints. Clearly it does not affect the market clearing and BB constraints either. Thus we need only to show that the objectives in the two problems are equivalent. In the modified model, the objective (2.2) is equal to (2.1) plus

$$\int_0^1 \int_0^1 \psi(t,t')Q(t')dF(t')dW(t).$$

Switch the order of integration to rewrite this double integral as

$$\int_0^1 Q(t') \underbrace{\int_0^1 \psi(t,t')dW(t)}_{=:\Psi(t')} dF(t').$$

Thus, following the same routine of envelope theorem and integration-by-parts that results in Corollary 1, one can prove that an allocation is optimal if and only if it solves Problem (2.3) such that the objective $\int_0^1 QVdF$ therein is replaced by $\int_0^1 Q\tilde{V}dF$ where

$$\tilde{V}(t) := V(t) + \Psi(t)$$

for all $t \in [0,1]$. To incorporate the externality, one needs to replace the virtual surplus $V(t)$ by $\tilde{V}(t)$.

A.4 Lemmas of Ironing

Define for each $s \in [0,1]$

$$H(V)(s) := \int_0^s V\left(F^{-1}(r)\right)dr. \tag{A.7}$$

Denote $\widetilde{H}(V)$ for the convex hull of $H(V)$ on $[0,1]$ (cf. Myerson [17]). Then the *ironed virtual surplus* $\overline{V} : [0,1] \to \mathbb{R}$ is defined by

$$\overline{V}(t) = \frac{d}{ds}\left(\widetilde{H}(V)\right)(s)\Big|_{s=F(t)} \tag{A.8}$$

whenever $\widetilde{H}(V)$ is differentiable at $F(t)$, and extended to all of $[0,1]$ by one-sided continuity.

If $0 \le a < b \le 1$, (a,b) is called *ironed interval* iff $\widetilde{H}(V) < H(V)$ on $(F(a), F(b))$, $\widetilde{H}(V)(F(a)) = H(V)(F(a))$ and $\widetilde{H}(V)(F(b)) = H(V)(F(b))$. That is, an ironed interval is an inclusion-maximal open interval on which $H(V)(F(\cdot)) > \widetilde{H}(V)(F(\cdot))$. As is well-known, on an ironed interval the monotonicity condition of Q is binding, and $\overline{V}$ is constant.[2]

Lemma 12 *For any $a, b \in [0,1]$ such that $a \le F^{-1}\left(\frac{B}{B+1}\right) \le b$, if $\overline{V}$ is constant on (a,b) (unless $(a,b) = \emptyset$) and neither a nor b is an interior point of any ironed interval, then (2.5) is an optimal solution for (2.3) and, for any Q that is feasible to (2.3), $\int_0^1 QVdF < \int_0^1 Q^*VdF$ in any of the following three cases:*

[2] While $\overline{V}$ is constant on any ironed interval, an interval on which $\overline{V}$ is constant need not be an ironed interval, as it is possible that $\overline{V} = V$ on some interval where V happens to be constant.

i $a \le a' < b' \le b$, (a', b') *is an ironed interval, and* Q *is not constant on* (a', b')*;*

ii $Q \ne Q^*$ *on a positive-measure subset* S *of* $[0, a)$ *for which* $\overline{V} < \overline{V}\big|_{(a,b)}$ *on* S*;*

iii $Q \ne Q^*$ *on a positive-measure subset* S *of* $(b, 1]$ *for which* $\overline{V} > \overline{V}\big|_{(a,b)}$ *on* S*.*

Proof By (2.5), $\int_0^1 Q^* dF = 0$. Since $a \le F^{-1}\left(\frac{B}{B+1}\right) \le b$, it follows again from (2.5) that Q^* is weakly increasing. Thus Q^* is feasible to (2.3). To prove that it is optimal for (2.3), use Myerson's [17] equation (from (A.7), (A.8), and integration by parts)

$$\int_0^1 Q(t)V(t)dF(t) = \int_0^1 Q(t)\overline{V}(t)dF(t) - \int_0^1 \left(H(V)(F(t)) - \widetilde{H}(V)(F(t))\right) dQ(t) \tag{A.9}$$

for any weakly increasing function Q on $[0, 1]$. Observe that the second integral on the right-hand side of (A.9) is nonnegative as Q is weakly increasing, and that it is strictly positive if and only if Q is not constant on some ironed interval. It then follows from the definition of Q^* that the said integral is zero when $Q = Q^*$, because Q^* by construction has only a and b as jump points and, by the hypothesis of the lemma, neither a nor b is interior to any ironed interval. Thus, to prove optimality of Q^* it suffices to show $\int_0^1 Q^*\overline{V}dF \ge \int_0^1 Q\overline{V}dF$ for any Q feasible to (2.3). To show that, note

$$\int_0^1 Q^*\overline{V}dF - \int_0^1 Q\overline{V}dF = \underbrace{\int_0^a (Q^* - Q)\overline{V}dF}_{X} + \underbrace{\int_a^b (Q^* - Q)\overline{V}dF}_{Y} + \underbrace{\int_b^1 (Q^* - Q)\overline{V}dF}_{Z}.$$

If $(a, b) \ne \emptyset$, let v be the constant that $\overline{V}$ is equal to on (a, b) by the hypothesis of the lemma; else let $v := \overline{V}(b)$. Note: $X \ge v \int_0^a (Q^* - Q)dF$ because $Q^* - Q = -1 - Q \le 0$ and $\overline{V} \le v$ on $[0, a)$; $Y = v \int_a^b (Q^* - Q)dF$ because either $\overline{V} = v$ on (a, b) or $a = b$; and $Z \ge v \int_b^1 (Q^* - Q)dF$ because $Q^* - Q = B - Q \ge 0$ and $\overline{V} \ge v$ on $(b, 1]$. Thus,

$$\int_0^1 Q^*\overline{V}dF - \int_0^1 Q\overline{V}dF \ge v \int_0^a (Q^* - Q)dF + v \int_a^b (Q^* - Q)dF + v \int_b^1 (Q^* - Q)dF = 0, \tag{A.10}$$

with the equality due to $\int Q^* dF = 0 = \int Q dF$. Thus, Q^* is optimal for (2.3).

To prove the rest of the lemma, pick any Q feasible to (2.3). Then $Q : [0, 1] \to [-1, B]$ is weakly increasing and $\int Q dF = 0$. In Case (i), $a' < b'$ and Q is not constant on (a', b'). Then Q, weakly increasing, is strictly increasing on a positive-measure subset of (a', b'). Thus, the distribution induced by Q assigns a positive measure on (a', b'). This, coupled with the hypothesis that (a', b') is an ironed interval, implies that the second integral on the right-hand side of (A.9) is strictly positive given Q. By contrast, the integral given Q^* is zero. This, coupled with $\int_0^1 Q^*\overline{V}dF \ge \int_0^1 Q\overline{V}dF$ proved above, implies $\int_0^1 VQdF < \int_0^1 VQ^*dF$.

In Case (ii), since $Q^* = 0$ on $[0, a)$, the hypothesis $Q \ne Q^*$ on $S \subseteq [0, a)$ implies that $Q^* - Q < 0$ on the positive-measure subset S of $[0, a)$. This, combined with $Q^* - Q \le 0$ on $(0, a)$ and $\overline{V} < V(a) \le v$ on S, implies $\int_0^a (Q^* - Q)\,\overline{V}dF > v \int_0^a (Q^* - Q)\,dF$. Thus the inequality in (A.10) is strict. Case (iii) is analogous to Case (ii).

Since (A.8) defines $\overline{V}$ only at t for which $\widetilde{H}(V)$ is differentiable at $F(t)$, let us specify the extension of $\overline{V}$ to the two endpoints:

$$\overline{V}(1) := \sup_{t'\uparrow 1} \overline{V}(t') \quad \text{and} \quad \overline{V}(0) := \inf_{t'\downarrow 0} \overline{V}(t'). \tag{A.11}$$

Lemma 13 $\overline{V}(0) < \overline{V}(1)$.

Proof First, we observe that $\overline{V}(0) \leq 0$. To see that, note from the definition of ironing that $\overline{V}(0)$ is the slope of the supporting line at the point 0 of the epigraph of $H(V)$. Since $V(0) = 0$ by the definition of V and V is continuous by the assumption that both f and W are continuous, the right-derivative of $H(V)$ at point 0 is well defined and is equal to 0. Thus, the slope $\overline{V}(0)$ of the supporting line of $H(V)$ at point 0 is less than or equal to 0.

Now that $\overline{V}(0) \leq 0$, we need only to show $\overline{V}(1) > 0$. Suppose not, then $\overline{V} \leq 0$ on $[0, 1]$ by its monotonicity. Then

$$0 \geq \int_0^1 \overline{V}(t)dF(t) = \widetilde{H}(V)(1) = H(V)(1) = \int_0^1 V(t)dF(t),$$

where the first equality is due to (A.8), the absolute continuity of $\widetilde{H}(V)$ and $\widetilde{H}(V)(0) = 0$, the second equality due to $\widetilde{H}(V)$ being the convex hull of $H(V)$ on $[0, 1]$ and $H(V)(0) = 0$, and the last equality due to (A.7). Thus, $0 \leq \int_0^1 V(t)dF(t)$ and hence, by the definition of V,

$$\int_0^1 W(t)dt \geq \int_0^1 tdF(t) + \int_0^1 F(t)dt = 1,$$

with the equality due to integration by parts. Since W is a cdf that is supported by $[0, 1]$ and continuous on $\mathbb{R}$, $W \leq 1$ on $[0, 1]$ and strictly so on a positive-measure subset thereof. Thus $\int_0^1 W(t)dt < 1$ and the above-displayed inequality is impossible, which leads to the desired contradiction.

Lemma 14 *$\overline{V}$ is continuous at the points 0 and 1.*

Proof We shall prove that $\overline{V}$ is continuous at point 1. The case of point 0 is symmetric.

If $\widetilde{H}(V) = H(V)$ on $(F(1-\delta), 1]$ for some $\delta > 0$, then by (A.8) $\overline{V} = V$ on $(1-\delta, 1]$, and hence the continuity $\overline{V}$ at point 1 follows from the continuity of V. If $\widetilde{H}(V) < H(V)$ on $(F(1-\delta), 1)$ for some $\delta > 0$, then $(1-\delta, 1)$ is contained in an ironed interval, so $\overline{V}$ is constant on $(1-\delta, 1)$ and hence $\sup_{t'\uparrow 1} \overline{V}(t')$ is equal to this constant. Then (A.11) implies that $\overline{V}(1)$ is equal to the constant; thus again $\overline{V}$ is continuous at 1.

Thus, suppose that neither of the previous cases hold. That is, there exists an ironed interval (a_1, b_1) such that $0 \leq a_1 < b_1 < 1$, there exists another ironed interval (a_2, b_2) for which $b_1 \leq a_2 < b_2 < 1$, and furthermore for any ironed interval (a_k, b_k) for which $b_k < 1$, there exists another ironed interval (a_{k+1}, b_{k+1}) for which $b_k \leq a_{k+1} < b_{k+1} < 1$. Thus, by recursion, $[0, 1]$ is partitioned by

$$0 \leq a_1 < b_1 \leq a_2 < b_2 \leq a_3 < \cdots \leq a_k < b_k \leq a_{k+1} < \cdots < 1$$

such that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 1,$$
$$\forall k\, \exists v_k \in \mathbb{R} : \left[\overline{V} = v_k \text{ on } (a_k, b_k)\right],$$

with the last line due to (a_k, b_k) being an ironed interval. Since $\overline{V}$ is weakly increasing,

$$v_1 \leq v_2 \leq v_3 \leq \cdots v_k \leq v_{k+1} \leq \cdots .$$

Within this case, to prove the continuity of $\overline{V}$ at point 1, we start by observing that

$$\lim_{k\to\infty} v_k = V(1). \tag{A.12}$$

To show that, for each k pick any $t_k \in [b_k, a_{k+1}]$. Then $\overline{V}(t_k) = V(t_k)$. With $\overline{V}$ weakly increasing,

$$v_1 \leq V(t_1) \leq v_2 \leq V(t_2) \leq v_3 \leq V(t_3) \leq \cdots .$$

Thus $\lim_{k\to\infty} v_k = \lim_{k\to\infty} V(t_k) = V(1)$, with the second equality due to $t_k \to 1$ and V being continuous at 1.

Next, pick any sequence $(t'_j)_{j=1}^{\infty}$ converging to 1 such that $t'_1 \leq t'_2 \leq t'_3 \leq \cdots$. For each j, either $t'_j \in (a_{k_j}, b_{k_j})$ for some k_j, or $t'_j \in [b_{k_j}, a_{k_j+1}]$ for some k_j. In the former case, $\overline{V}(t'_j) = v_{k_j}$; in the latter, $v_{k_j} \leq \overline{V}(t'_j) \leq v_{k_j+1}$. Both cases considered,

$$\lim_{j\to\infty} \overline{V}(t'_j) = \lim_{j\to\infty} v_{k_j} = V(1),$$

with the second equality due to (A.12). Since $(t'_j)_{j=1}^{\infty}$ can be any sequence converging to 1 from below, the above equation also implies $\lim_{t'\uparrow 1} \overline{V}(t') = V(1)$. Consequently, $\sup_{t'\uparrow 1} \overline{V}(t') = \lim_{t'\uparrow 1} \overline{V}(t') = V(1)$. This coupled with (A.11) implies $\overline{V}(1) = \lim_{t'\uparrow 1} \overline{V}(t')$, namely, $\overline{V}$ is continuous at 1.

Lemma 15 *If $V(1) > V(t)$ for all $t \in [0, 1)$, the followings are true for $\overline{V}$:*

a. there is no $x \in [0, 1)$ for which $\overline{V} \neq V$ on $(x, 1)$;

b. $\overline{V}(1) = V(1)$;

c. $\overline{V}(1) > \overline{V}(t)$ for all $t \in [0, 1)$.

Proof Proof of (a): Suppose, to the contrary, that there exists an $x \in [0, 1)$ for which $\overline{V} \neq V$ on $(x, 1)$. Then $(x, 1)$ is contained in some ironed interval say $(x_*, 1)$ such that $\widetilde{H}(V)(F(t)) < H(V)(F(t))$ for all $t \in (x_*, 1)$, $\widetilde{H}(V)(F(x_*)) = H(V)(F(x_*))$, $\widetilde{H}(V)(F(1)) = H(V)(F(1))$, and $\widetilde{H}(V)$ has a constant slope β on $[F(x_*), 1]$. Since $\widetilde{H}(V)(F(t)) < H(V)(F(t))$ for all $t \in (x_*, 1)$, for any $t < 1$ sufficiently close to 1,

$$\frac{1}{1 - F(t)} \int_t^1 V(s)dF(s) = \frac{1}{1 - F(t)} \left(H(V)(F(1)) - H(V)(F(t))\right) \leq \beta.$$

Taking the limit of the inequality as $t \to 1$ and noting continuity of V at 1, we have $V(1) \le \beta$. Meanwhile, since $\widetilde{H}(V)(F(t)) < H(V)(F(t))$ for all $t \in (x_*, 1)$, there exists $t' \in (x_*, 1)$ for which the slope of $H(V)$ at $F(t')$ is greater than β. That is, $V(t') > \beta$, which coupled with $V(1) \le \beta$ implies $V(t') > V(1)$, contradicting the hypothesis that V is maximized at 1.

Proof of (b): Note, from the proof of Lemma 14, that $\overline{V}(1) = V(1)$ unless $\widetilde{H}(V) < H(V)$ on $(F(1-\delta), 1)$ for some $\delta > 0$, namely, $(1-\delta, 1)$ is contained in an ironed interval. Thus $\overline{V} \neq V$ on $(x, 1)$ for some $x \in (1-\delta, 1)$, contradicting (a). Thus (b) holds.

Proof of (c): Suppose, to the contrary, that $\overline{V}(1) \le \overline{V}(t_0)$ for some $t_0 \in [0, 1)$. Then, with $\overline{V}$ weakly increasing, $\overline{V}(t) = \overline{V}(1)$ for all $t \in [t_0, 1]$. By (a), there exists $t_1 \in (t_0, 1)$ for which $\overline{V}(t_1) = V(t_1)$. Since $t_1 \in (t_0, 1)$, $\overline{V}(t_1) = \overline{V}(1)$. Then (b) implies $V(1) = \overline{V}(1) = \overline{V}(t_1) = V(t_1)$, contradicting the hypothesis that $V(1) > V(t)$ for all $t \in [0, 1)$.

A.5 Proof of Theorem 2

Recall the definition of ironed interval from Appendix A.4.

Lemma 16 *For any welfare weight distribution W, the two-tier allocation (2.6) is optimal if and only if $F^{-1}\left(\frac{B}{B+1}\right)$ is not interior to any ironed interval.*

Proof If $F^{-1}\left(\frac{B}{B+1}\right)$ is not interior to any ironed interval, then Lemma 12 applies to the case where $a = b = F^{-1}\left(\frac{B}{B+1}\right)$ so that the Q^* defined in (2.5) specializes to (2.6), the two-tier allocation implemented by the posted-price system. Thus Lemma 12 implies that (2.6) is optimal, and the "if" part of the claim is true. To prove the "only if" part, suppose that $F^{-1}\left(\frac{B}{B+1}\right)$ is interior to some ironed interval. Then Part (i) in Lemma 12 implies that no optimal allocation has a jump point in the ironed interval and hence the allocation (2.6), whose jump point is $F^{-1}\left(\frac{B}{B+1}\right)$, is not optimal.

Lemma 17 *If the function $V(\cdot) - V\left(F^{-1}\left(\frac{B}{B+1}\right)\right)$ is single-crossing on $[0, 1]$, then $F^{-1}\left(\frac{B}{B+1}\right)$ is not interior to any ironed interval.*

Proof Denote $m := F^{-1}\left(\frac{B}{B+1}\right)$ and suppose that $V(\cdot) - V(m)$ is a single-crossing function on $[0, 1]$. By the definition of ironing, it suffices to prove that $F(m)$ is a *convex point* of $H(V)$ in the sense that at no point below $F(m)$ is $H(V)$ steeper than it is at $F(m)$, and at no point above $F(m)$ is $H(V)$ less steep than it is at $F(m)$. Since $V(\cdot) - V(m)$ single-crossing on $[0, 1]$,

$$s < F(m) < s' \Longrightarrow V\left(F^{-1}(s)\right) \le V\left(F^{-1}(F(m))\right) = V(m) \le V\left(F^{-1}(s')\right).$$

By (A.7) the definition of $H(V)$, the derivative of $H(V)$ at any $s \in (0, 1)$ is $D(H(V))(s) = V\left(F^{-1}(s)\right)$. Plug this into the above-displayed formula to obtain

$$s < F(m) < s' \Longrightarrow D(H(V))(s) \le D(H(V))(F(m)) \le D(H(V))(s').$$

Thus, $F(m)$ is a convex point of $H(V)$, as desired.

Proof of Theorem 2 Let a be the infimum, and b the supremum, of

$$\left\{t \in [0,1] \mid \overline{V}(t) = \overline{V}\left(F^{-1}(B/(B+1))\right)\right\}. \tag{A.13}$$

Then neither a nor b is interior to an ironed interval, and hence Lemma 12 implies that the allocation (2.5) is an optimal allocation. In the case where $V(\cdot) - V\left(F^{-1}\left(\frac{B}{B+1}\right)\right)$ is single-crossing on $[0,1]$, Lemmas 16 and 17 together imply that the two-tier allocation (2.6) is optimal. The payment rules that the theorem asserts implement (2.5) and (2.6) respectively can be derived from the allocations according to the envelope formula, as explained in the comments around the theorem. ∎

A.6 Proofs of Corollaries 3 and 4

Corollary 3 First, consider the case $x \geq 0$. By $a < F^{-1}\left(\frac{B}{B+1}\right)$ and (2.7), $x < B$ and hence the allocation (2.5) is weakly increasing. Thus, the allocation can be implemented by a payment rule. Consider the one that maximizes the planner's profit among all that implement the allocation. By the envelope formula, one can show that this payment rule is the same as the one described in Part (i) of the corollary. For instance, an individual of type $t \geq b$ gets the surplus $(b-a)x + (t-b)B$ by the envelope theorm and hence needs to deliver a total payment equal to $bB - (b-a)x$ for the quantity B of the good. That implies the per-unit price $b - (b-a)x/B$ stated in the corollary. Note $b - (b-a)x/B > a$ (because $a < F^{-1}\left(\frac{B}{B+1}\right)$). Thus, the profit generated by this payment rule is greater than

$$-aF(a) + a\left(F(b) - F(a)\right)x + a\left(1 - F(b)\right)B = 0,$$

with the equality due to the market clearing condition. By the envelope formula, the payment rules that implement the allocation differ from one another only by a constant. Thus, since the planer would rebate all her profit to the individuals to achieve the optimality of (2.1), the optimal payment rule that implements (2.5) is the profit-maximizing one among those that implement (2.5), augmented with a lump sum transfer to the individuals. Since the allocation restricted to (a,b) is equal to the constant x and $0 \leq x < B$, the mechanism entails rationing on (a,b). Thus the corollary is true in the case $x \geq 0$.

The case $x \leq 0$ is symmetric. By $F^{-1}\left(\frac{B}{B+1}\right) < b$ and (2.7), we have $-1 < x$ and hence (2.5) is weakly increasing and entails rationing on (a,b). The profit-maximizing payment rule follows similarly from the envelope formula. Since $-1 < x$, the per-unit price $a - (b-a)x$ for the seller-types in $[0,a)$ is strictly less than b. This coupled with the market clearing condition implies that the profit generated by the payment rule is positive and hence the optimal payment rule makes a positive lump sum transfer to the individuals. ∎

Corollary 4 By Lemmas 13 and 14 (Appendix A.4), the ironed virtual surplus function $\overline{V}$ is continuous at both points 0 and 1, and $\overline{V}(0) < \overline{V}(1)$. This fact implies that the conditions (ii) and (iii) in Lemma 12 are true when the egalitarian allocation Q is compared to the optimal allocation Q^*, and hence Q is strictly outperformed by Q^*. ∎

A.7 Proof of Theorem 3

Theorem 3 follows directly from the next lemma, as the non-constancy assumption of V in the theorem implies the condition (A.14) in the lemma. Recall the definition of ironed interval in Appendix A.4.

Lemma 18 *There exists at most one (modulo measure zero) optimal allocation if*

$$\overline{V} = \overline{V}\left(F^{-1}\left(\tfrac{B}{B+1}\right)\right) \text{ on } (a,b) \neq \varnothing \Longrightarrow (a,b) \text{ is a subset of an ironed interval.} \tag{A.14}$$

Proof Denote a' for the infimum, and b' the supremum, of the set (A.13) defined in Apendix A.5. By the condition (A.14) of the theorem, either $a' = b'$ or $a' < b'$ and (a', b') is an ironed interval. Thus, by Lemma 12, the allocation Q^* defined by (2.5) such that $a = a'$ and $b = b'$ is an optimal allocation. By the definition of the set (A.13), any type below a has a lower $\overline{V}$-value, and any type above b has a higher $\overline{V}$-value. This, coupled with the fact that (a', b') is an ironed interval unless $a' = b'$, implies that the conditions (i), (ii) and (iii) in Lemma 12 hold for any feasible allocation Q that differs from Q^* by a positive measure. It follows that any feasible allocation Q is strictly outperformed by Q^*. Thus the optimal allocation is unique.

A.8 Proof of Theorem 4

Lemma 19 *In any optimal mechanism where the allocation is the three-tier (2.5) that rations a quantity x—defined by (2.7)—on some (a,b) for which $0 < a < F^{-1}\left(\frac{B}{B+1}\right) < b < 1$:*

a. the surplus for type zero is equal to $a + (b-a)(B-x)(1-F(b))$;

b. the surplus for type one is equal to $(1-b)B + (b-a)(1+x)F(a)$.

Proof To prove Claim (a), consider first the case where $x \geq 0$. Recall from Corollary 3 for the payment rule in this case. The surplus for type zero is equal to a (the revenue the type receives from selling his one unit endowment) plus the (per-capita) lump sum rebate from the planner. The lump sum rebate is equal to the planner's profit from implementing (2.5) through the profit-maximizing payment rule. Note that the planner cannot profit from selling the good to the types in (a,b), as the per-unit revenue extracted from them is equal to a, the per-unit cost from procuring the good (from the seller-types in $[0,a)$). Thus, the planner can profit only from the sales to the buyer-types in $(b,1]$. The per-unit profit is the price difference $b-(b-a)x/B-a$ between the price $b-(b-a)x/B$ offered to $(b,1]$ and the price a to the seller-types. Since the amount of sales to $(b,1]$ is $(1-F(b))B$, the profit is equal to

$$(b-(b-a)x/B-a)(1-F(b))B = (b-a)(B-x)(1-F(b)). \tag{A.15}$$

Thus the surplus for type zero is equal to a plus the above expression, as in Claim (a).

Next consider the other case, $x < 0$. Again recall from Corollary 3 for the payment rule in this case. The planner can profit only from the quantity she procures from the seller-types in $[0,a)$. The per-unit profit from this quantity is the price difference $b-(a-(b-a)x)$ between the sales price b and the procurement price $a-(b-a)x$. The quantity is equal to the mass $F(a)$ of $[0,a)$. Thus the profit is equal to

$$(b-(a-(b-a)x))\,F(a). \tag{A.16}$$

Note that the revenue a type zero receives from selling his one unit of the good is equal to $a-(b-a)x$. Therefore, the surplus for type zero is equal to

$$\begin{aligned} a-(b-a)x+(b-(a-(b-a)x))\,F(a) &= a-(b-a)x+(b-a)(1+x)F(a) \\ &= a+(b-a)(B-x)\,(1-F(b)), \end{aligned}$$

where the second line is equivalent to (2.7), the definition of x. Thus the surplus for type zero in the case $x<0$ is also equal to the expression in Claim (a). Hence Claim (a) is true.

To prove Claim (b), consider first the case $x<0$. Given the payment rule characterized in Corollary 3 for the mechanism of (2.5), type one buys the quantity B of the good at the price b per unit and receives a lump sum rebate, which has been shown to be equal to (A.16) in the proof of Claim (a). Thus, the surplus for type one given (2.5) is equal to

$$(1-b)B+(b-(a-(b-a)x))\,F(a)$$

when $x<0$, as asserted by Claim (b).

Next consider the other case, $x\geq 0$. Given the payment rule characterized in Corollary 3 for the mechanism of (2.5), type one buys the quantity B of the good at the price $b-(b-a)x/B$ per unit and receives a lump sum rebate, which has been shown to be equal to (A.15) in the proof of Claim (a). Thus, the surplus for type one given (2.5) is equal to

$$\begin{aligned} & (1-(b-(b-a)x/B))\,B+(b-a)(B-x)\,(1-F(b)) \\ = \;& (1-b)B+(b-a)x+(b-a)(B-x)\,(1-F(b)) \\ = \;& (1-b)B+(b-a)(1+x)F(a), \end{aligned}$$

with the last line equivalent to (2.7), the definition of x. Thus, the surplus for type one in the case $x\geq 0$ is also equal to the expression asserted by Claim (b). Hence Claim (b) is true.

Proof of Theorem 4 Claim (a) is intuitive. By Corollary 3, the mechanism of allocation (2.5) transfers a (strictly) positive lump sum rebate to all types, and hence the surplus for the type $F^{-1}\left(\frac{B}{B+1}\right)$ given allocation (2.5) is positive. By contrast, the surplus for the type $F^{-1}\left(\frac{B}{B+1}\right)$ is equal to zero in the mechanism of the allocation (2.6): By the envelope theorem, one readily sees that the surplus function given allocation (2.6) attains its minimum at the type equal to $F^{-1}\left(\frac{B}{B+1}\right)$. Meanwhile, it is easy to show that any payment rule that implements a market-clearing two-tier allocation such as (2.6) generates zero profit for the planner and hence zero lump sum rebate to the individuals. Thus type $F^{-1}\left(\frac{B}{B+1}\right)$ gets zero surplus under the allocation (2.6). It follows that the surplus for type $F^{-1}\left(\frac{B}{B+1}\right)$ given allocation (2.5) is greater than that given (2.6). By continuity of surplus as a function of types, this strict inequality extends to types sufficiently near to $F^{-1}\left(\frac{B}{B+1}\right)$, and hence Claim (a) is true.

To prove Claims (b) and (c), note from (2.7), the definition of x, that

$$-F(a)+B(1-F(b))+x(F(b)-F(a))=0,$$

or equivalently,

$$\frac{B}{B+1} = \frac{1+x}{B+1}F(a) + \frac{B-x}{B+1}F(b). \tag{A.17}$$

Since $(1+x)/(B+1)$ and $(B-x)/(B+1)$ are between zero and one and sum up to one, $B/(B+1)$ is a convex combination between $F(a)$ and $F(b)$. When F is convex on (a,b), F^{-1} is concave on $(F(a), F(b))$ because F is strictly increasing by assumption. Thus, a simple application of Jensen's inequality implies:

$$F^{-1}\left(\frac{B}{B+1}\right) \geq \frac{1+x}{B+1}a + \frac{B-x}{B+1}b = a + \frac{B-x}{B+1}(b-a). \tag{A.18}$$

By Lemma 19.a, the surplus for type zero under the optimal mechanism of the rationing allocation (2.5) is equal to

$$\begin{aligned} a + (b-a)(B-x)(1-F(b)) &< a + (b-a)\frac{B-x}{B+1} \\ &\leq F^{-1}\left(\frac{B}{B+1}\right), \end{aligned}$$

where the first inequality follows from $F(b) > B/(B+1)$, and the second inequality from (A.18). Thus, since the surplus for type zero given the optimal mechanism of allocation (2.6) is equal to the market-clearing price $F^{-1}\left(\frac{B}{B+1}\right)$, type zero is worse-off in the rationing mechanism of allocation (2.5) than in the mechanism of the allocation (2.6). Since the allocations (2.5) and (2.6) are identically equal to -1 for all types in $[0, a)$, the envelope theorem implies that in both allocations, the surplus decreases at the same rate -1 when the type increases from zero to a. Consequently, Claim (b) of the theorem follows.

Similarly, when F is concave on (a,b), F^{-1} is convex on $(F(a), F(b))$. Thus

$$F^{-1}\left(\frac{B}{B+1}\right) \leq \frac{1+x}{B+1}a + \frac{B-x}{B+1}b = b - \frac{1+x}{B+1}(b-a). \tag{A.19}$$

By Lemma 19.b, the surplus for type one under the optimal mechanism of the rationing allocation (2.5) is equal to

$$\begin{aligned} (1-b)B + (b-a)(1+x)F(a) &= B - B\left(b - (1+x)(b-a)\frac{F(a)}{B}\right) \\ &< B - B\left(b - \frac{1+x}{B+1}(b-a)\right) \\ &\leq B\left(1 - F^{-1}\left(\frac{B}{B+1}\right)\right), \end{aligned}$$

where the first inequality follows from $F(a) < B/(B+1)$, and the second inequality from (A.19). Thus, since the surplus for type one given allocation (2.6) is equal to $B\left(1 - F^{-1}\left(\frac{B}{B+1}\right)\right)$, type one is worse-off in the allocation (2.5) than in the allocation (2.6). Since the allocations (2.5) and (2.6) are identically equal to B for all types in $[0, a)$, the envelope theorem implies that in both allocations, the surplus increases at the same rate B when the type increases from b to 1. Consequently, Claim (c) of the theorem follows. ■

A.9 Proof of Theorem 5

Denote V for the virtual surplus function given W, and V_* the virtual surplus function given W_*. Also $m := F^{-1}\left(\frac{B}{B+1}\right)$.

Part (a) By the definition of ironing (Appendix A.4), m is interior to an ironed interval given W if and only if $\widetilde{H}(V)(F(m)) < H(V)(F(m))$, which in turn holds if and only if there exist $a, b \in [0,1]$ for which $F(a) < B/(B+1) < F(b)$ and

$$\begin{aligned} H(V)(F(m)) &> H(V)(F(a)) + \frac{F(m)-F(a)}{F(b)-F(a)}\left(H(V)(F(b)) - H(V)(F(a))\right), \\ &= \frac{F(b)-F(m)}{F(b)-F(a)} H(V)(F(a)) + \frac{F(m)-F(a)}{F(b)-F(a)} H(V)(F(b)). \end{aligned}$$

The above condition is equivalent to that, for some $a < m < b$ ($m = F^{-1}\left(\frac{B}{B+1}\right)$),

$$\frac{F(b)-F(m)}{F(b)-F(a)} \int_a^m V(s)dF(s) - \frac{F(m)-F(a)}{F(b)-F(a)} \int_m^b V(s)dF(s) > 0,$$

which one can simplify, by dividing $(F(b)-F(m))(F(m)-F(a))/(F(b)-F(a))$, to

$$\frac{1}{F(m)-F(a)} \int_a^m V(s)dF(s) - \frac{1}{F(b)-F(m)} \int_m^b V(s)dF(s) > 0.$$

It follows that m is not interior to any ironed interval given V if and only if, for any $a \in [0,m)$ and any $b \in (m,1]$,

$$\frac{1}{F(m)-F(a)} \int_a^m V(s)dF(s) - \frac{1}{F(b)-F(m)} \int_m^b V(s)dF(s) \le 0. \tag{A.20}$$

By the hypothesis in the theorem that the posted-price system is optimal given W, Lemma 16 implies that m is not interior to any ironed interval given V, and hence (A.20) holds for any $a \in [0,m)$ and any $b \in (m,1]$. Now let W_* be any spread of W away from m, namely, $W_* \ge W$ on $[0,m)$ and $W_* \le W$ on $(m,1]$. Then, by (2.4) the definition of virtual surplus,

$$V_*(t) - V(t) = \frac{W(t)-W_*(t)}{f(t)} \begin{cases} \le 0 & \text{if } t \in [0,m) \\ \ge 0 & \text{if } t \in (m,1]. \end{cases}$$

Thus, when the V in (A.20) is replaced by V_*, the inequality (A.20) remains to be true for any $a \in [0,m)$ and any $b \in (m,1]$. In other words, m is not interior to any ironed interval given W_*. Then Lemma 16 applied to the case of W_* implies that the posted-price system is optimal given W_*.

Part (b) Let $\epsilon > 0$. Let

$$\begin{aligned} a &:= \inf\{t \in [0,m) \mid W(t) > W(m) - \epsilon/2\}, \\ b &:= \sup\{t \in (m,1] \mid W(t) < W(m) + \epsilon/2\}. \end{aligned}$$

Since welfare weight distributions are assumed continuous on $\mathbb{R}$, $W(a) < W(m)$ and there is a positive-measure subset of (a, m) on which $W > W(a)$. For any $t \in \mathbb{R}$ define

$$W_*(t) := \begin{cases} W(t) & \text{if } t < a \\ W(a) & \text{if } a \leq t < m \\ W(b) & \text{if } m \leq t < b \\ W(t) & \text{if } t \geq b. \end{cases}$$

By the construction of W_* and the monotonicity of W, W_* is a cdf, $W_* \leq W$ on $[0, m)$, $W_* \geq W$ on $(m, 1]$, and $\|W_* - W\|_{\max} \leq W(b) - W(a) \leq \epsilon$. Since W_* has a jump at $t = m$, the virtual surplus function V_* given W_*, by (2.4), has a drop at $t = m$. Thus one can modify W_* into a continuous function (just to satisfy our assumption that welfare weight distributions are continuous) by replacing the jump at $t = m$ with a sufficiently steep affine segment, so that V_* after modification remains to be decreasing strictly at $t = m$. It follows from the definition of ironing that m is interior to an ironed interval given W_* (with or without the continuity modification of the jump). Then Lemma 16 implies that no optimal allocation given W_* can be implemented by the posted-price system.

A.10 Unbounded Acquisition

The main model assumes that the upper bound B for acquisition quantity per type is finite. Here we consider an extension where $B = \infty$. This case reflects a world with severe inequalities and insatiable demands for the good. For example, it could be an exchange economy where the endowment is an individual's initially acquired tract of land when a group of colonists arrive at a new, unoccupied place, or one's own private information in digital format that can be traded off for convenience, or a citizen's initial voting power in a fledging republic say the early Roman Republic. The following extension sheds light on the tendency that such resources are concentrated to a tiny few of the society.

Now that there is no upper bound on the quantity that a type is allowed to acquire, the buyer-types in this case should only be those types that maximize the ironed virtual surplus $\overline{V}$—selling the good to any type with lower $\overline{V}$-value would be a waste—and all other types should be sellers. The outcome in this case is therefore intuitive. Either the ironed virtual surplus $\overline{V}$ attains its maximum at a unique point (the highest type, as $\overline{V}$ is weakly increasing by construction), or $\overline{V}$ is maximized by multiple points, which constitute an upper interval in the type space. In the former case, all members of the society supply the good to the single, highest type. In the latter case, the optimal allocation entails two tiers, the "haves" consisting of the $\overline{V}$-maximizers, and the "have-nots" consisting of all the other types.

The former case, the utmost form of inequalities, needs to be formalized because the corresponding optimal allocation is not a real function. We say that the optimal allocation is *singular* iff there exists a sequence $(Q^n)_{n=1}^{\infty}$ of functions $Q^n : [0, 1] \to [-1, \infty)$, each weakly increasing and market clearing (and hence budget balancing by Theorem 1, which remains intact when $B = \infty$), such that Q^n converges pointwise to the extended-real function Q^{∞} defined by

$$Q^{\infty}(t) := \begin{cases} -1 & \text{if } t \in [0, 1) \\ \infty & \text{if } t = 1 \end{cases} \tag{A.21}$$

and, for any function $Q : [0,1] \to [-1,\infty)$ that is weakly increasing and market clearing (and hence budget-balancing), there exists N for which Q^n outperforms Q in terms of the design objective in Section 2.2 for all $n \geq N$. Then one can prove (Appendix A.11) the following characterization of the optimal allocation.[3]

Theorem 15 *When $B = \infty$:*

a. if $V(1) > V(t)$ for all $t \in [0,1)$, then the optimal allocation is singular;

b. else then there exists an optimal allocation and it is a two-tier allocation defined by

$$Q^*(t) := \begin{cases} -1 & \text{if } 0 \leq t < c^* \\ \frac{F(c^*)}{1-F(c^*)} & \text{if } c^* < t \leq 1, \end{cases} \tag{A.22}$$

where

$$c^* := \inf\left(\arg\max_{[0,1]} \overline{V}\right).$$

Clearly, rationing is needed to implement the optimal allocation (A.22) in case (b). Such necessity of rationing among the "haves" makes sense realistically: unchecked concentration begets social upheavals. Nonetheless, case (a) in a sense corresponds to the optimality of the posted-price system: Since $V(1) > V(t)$ for all $t \in [0,1)$, one can construct a sequence $(B_n, Q^n)_{n=1}^{\infty}$ such that $Q^n \to Q^\infty$ pointwise, $B_n \to_n \infty$, and for each n, Q^n is the optimal allocation in the basic model given upper bound B_n, implemented by posting the market-clearing price $F^{-1}\left(\frac{B_n}{B_n+1}\right)$. Since Q^n in the sequence attains the optimality given B_n, Q^∞ can be viewed as the limit of the optimum-implementing posted-price system when the acquisition cap rises without bound.

A sufficient condition to rule out the singularity case in Theorem 15 is that V be strictly decreasing at 1, which means $2 < w(1)$ if w is the Radon-Nikodym derivative of the welfare weight distribution W with respect to F. Intuitively speaking, had the optimal allocation been singular, the surplus for a type-1 player, whose type is the highest, would be infinitesimal, since the price for the good converges to one. But if the designer rations the quantity to an interval $(c, 1]$, the trading price is $c < 1$, and so the type-1 player gets a strictly positive surplus. Thus if the welfare weight density on type one is sufficiently large, the optimal allocation is to ration the good to some interval $(c^*, 1]$.

A.11 Proof of Theorem 15

Lemma 12 remains intact except that the allocation Q^* defined in (2.5) is modified into

$$Q^*(t) := \begin{cases} -1 & \text{if } 0 \leq t < a \\ x & \text{if } a < t < b \\ y & \text{if } b < t \leq 1 \end{cases} \tag{A.23}$$

for any $-1 \leq x \leq y$ that satisfies market clearing, and the Q that is compared against Q^* modified to any $Q : [0,1] \to [-1, y]$. Theorem 1 remains the same.

[3] With a condition similar to (A.14), one can also establish a uniqueness claim of the optimal allocation.

Claim (a) Assume the premise of this claim, that $V(1) > V(a)$ for all $a \in [0, 1)$. To satisfy the condition for singularity, we start by constructing a sequence of allocations that converges to Q^∞. Since $\overline{V}$ is monotone, for any $x \in \mathbb{R}$ the inverse image $\overline{V}^{-1}(x)$ is a nondegenerate interval if it contains more than one point. There are at most countably many such nondegenerate intervals. Thus, either $\overline{V} = V$ and is strictly increasing on $[0, 1]$, or $[0, 1]$ is partitioned by a sequence $(\tau_k, \theta_k)_{k=1}^K$, for some $K \in \{1, 2, 3, \ldots\} \cup \{\infty\}$, such that

$$0 \leq \tau_1 < \theta_1 \leq \tau_2 < \theta_2 \leq \tau_3 < \theta_3 \leq \cdots \leq 1,$$

for each k there is $v_k \in \mathbb{R}$ for which $\overline{V} = v_k$ on (τ_k, θ_k), and $k < j \Rightarrow v_k < v_j$. Note that any ironed interval is contained in $[\tau_k, \theta_k]$ for some k. For any $n = 2, 3, 4, \ldots$, define

$$Q^n(t) := \begin{cases} -1 & \text{if } 0 \leq F(t) < 1 - 1/n \\ n-1 & \text{if } 1 - 1/n < F(t) \leq 1. \end{cases}$$

For each n, Q^n is weakly increasing and market clearing by construction and hence is also budget balancing by Theorem 1. Clearly $(Q^n)_{n=2}^\infty$ converges to Q^∞ pointwise. We shall extract an infinite subsequence of $(Q^n)_{n=2}^\infty$ whose jump points do not belong to the interior of any ironed interval, which is contained by $[\tau_k, \theta_k]$ for some k. Start with the smallest n for which $F^{-1}(1 - 1/n) \in (\tau_k, \theta_k)$ for some k. Replace the jump point $F^{-1}(1 - 1/n)$ for Q^n by θ_k, and raise the level of Q^n on $(\theta_k, 1]$ to $F(\theta_k)/(1 - F(\theta_k))$ to preserve market clearing. Remove all the Q^m in the original sequence such that

$$F^{-1}(1 - 1/(n-1)) < F^{-1}(1 - 1/m) < \theta_k.$$

Since $V(1) > V(a)$ for all $a \in [0, 1)$ by hypothesis, $\theta_k < 1$ (Lemma 15.a), thus there exists an integer M that is the largest among such m. Then, starting from Q^{M+1}, modify the sequence $(Q^n)_{n=M+1}^\infty$ as we do $(Q^n)_{n=2}^\infty$. By recursion, we obtain an infinite subsequence $(Q^{n_j})_{j=1}^\infty$ of $(Q^n)_{n=2}^\infty$ such that for any j and any k the jump point of Q^{n_j} does not belong to (τ_k, θ_k).

Pick any feasible allocation Q. We shall prove that Q is outperformed by the Q^{n_j} in $(Q^{n_j})_{j=1}^\infty$ for all sufficiently large j. Since the elements in the sequence are both market clearing and budget balancing, it suffices, as in the proof of Lemma 12, to prove that $\int Q^{n_j} V dF > \int Q V dF$. To that end, recall from Lemmas 14 and 15 that $\overline{V}$ is continuous at $t = 1$ and

$$\forall t < 1 : \overline{V}(t) < \overline{V}(1) = V(1). \tag{A.24}$$

This coupled with $V(1) = 1 > 0$, implies

$$\exists \delta > 0 : \forall t \in (1 - \delta, 1] : \overline{V}(t) > 0. \tag{A.25}$$

Since the range of Q is contained in $[-1, \infty)$, the market clearing condition implies that $Q > -1$ on a positive-measure subset of $[0, 1]$. Consequently, with Q weakly increasing,

$$\theta := \inf\{t \in [0, 1] \mid Q(t) > -1\} < 1.$$

Since the range of Q is $[-1, \infty)$ and Q is weakly increasing, $\max_{[0,1]} Q = Q(1) < \infty$. Thus there exists J such that for any $j \geq J$ we have

$$\max\{\theta, 1 - \delta\} < F^{-1}\left(1 - \frac{1}{n_j}\right) \quad \text{and} \quad n_j - 1 > Q(1).$$

For any $j \geq J$, denote the jump point of Q^{n_j} by x_j. Then either $x_j = F^{-1}(1 - 1/n_j)$, or $x_j = \theta_k$ such that θ_k is the right endpoint of the interval (τ_k, θ_k) to which $F^{-1}(1 - 1/n_j)$ belongs. Let $v := \overline{V}(x_j)$. Thus, $1 > x_k \geq F^{-1}(1 - 1/n_j)$ and $v \geq \overline{V}\left(F^{-1}(1 - 1/n_j)\right)$. Since $1 - \delta < F^{-1}\left(1 - 1/n_j\right)$, (A.25) implies $v > 0$. With $\overline{V}$ weakly increasing, $\overline{V}(t) \geq v$ for all $t \in \left[x_j, 1\right]$. Furthermore, (A.24) implies $\overline{V}(1) > v$; since $\overline{V}$ is continuous at $t = 1$, there exists a positive-measure subset E of $\left[x_j, 1\right]$ such that $\overline{V}(t) > v$ for all $t \in E$. This, coupled with the fact $Q^{n_j} \geq n_j - 1 > Q$ on $(x_j, 1]$ (by the construction of Q^{n_j} and the choice of J), implies that Lemma 12.b.iii applies to the case where $Q^* = Q^{n_j}$, with x_j here playing the role of a and b there and, by construction of Q^{n_j}, not interior to any ironed interval. Thus, Lemma 12.b.iii implies $\int_0^1 Q^{n_j} V dF > \int_0^1 Q V dF$, as desired.

Claim (b) Since $\overline{V}$ is weakly increasing, $\arg\max_{[0,1]} \overline{V}$ is an interval $[c^*, 1]$ or $(c^*, 1]$ for some $c^* \leq 1$. Since $V(1) \leq V(a)$ for some $a < 1$, $c^* < 1$. Thus, the allocation Q^* is well-defined by (A.22). It is a a two-tier allocation because $c_* > 0$ due to the fact $\overline{V}(0) < \overline{V}(1)$ (Lemma 13). By (A.22), Q^* is market clearing. It is also budget balancing by Theorem 1. Thus, it suffices to show that Q^* maximizes $\int_0^1 Q V dF$ among all weakly increasing $Q : [0, 1] \to [-1, \infty)$ subject to the market clearing condition. Thus pick any such Q. Note that Q^* corresponds to the special case of the Q^* defined in (A.23) where $a = c^*$ and $b = 1$. By the definition of c^*, c^* is not interior to any ironed interval. Thus Lemma 12 (modified according to the start of this proof) applies, and hence $\int_0^1 Q^* V dF \geq \int_0^1 Q V dF$, as desired.

Appendix B

Appendices to Chapter 3

B.1 A Binary Example

?? Suppose that there are two players, each having a type equal to either 1 or 6 with equal probabilities, and that the the welfare weights are $w(1) = 12/7$ and $w(6) = 2/7$.

For simplicity of illustration within this example, let us consider only symmetric mechanisms. First, let us find out the constrained optimum subject to the restriction that the bad be never assigned. Given any incentive feasible symmetric revelation mechanism, let Q_t denote a player's expected probability of getting the good given his type being t, and let P_t denote the expected value of the money transfer from him to others. Incentive compatibility (IC) for both $t \in \{1, 6\}$ means

$$\begin{aligned} 6Q_6 - P_6 &\geq 6Q_1 - P_1, \\ Q_1 - P_1 &\geq Q_6 - P_6. \end{aligned}$$

Ex post budget balancing (BB), combined with symmetry of the mechanism and equal probabilities of the two types, implies that $P_1 + P_6 = 0$. This, coupled with the IC conditions displayed above, implies

$$(Q_6 - Q_1)/2 \leq P_6 \leq 3(Q_6 - Q_1).$$

This implies $Q_6 - Q_1 \geq 0$ and hence $P_6 \geq 0$ and $P_1 = -P_6 \leq 0$. That means the individual rationality (IR) constraint for type 1, $Q_1 - P_1 \geq 0$, is non-binding. The IR for type 6, $6Q_6 - P_6 \geq 0$, is also non-binding due to the above-displayed inequality. Thus, it is necessary for any constrained optimum that $P_6 = 3(Q_6 - Q_1)$. Hence the social welfare is

$$2\left(\frac{1}{2}(6Q_6 - P_6)\cdot\frac{2}{7} + \frac{1}{2}(Q_1 - P_1)\cdot\frac{12}{7}\right) = \frac{12}{7}\left(\frac{7}{2}Q_6 - \frac{3}{2}Q_1\right),$$

which is maximized when Q_6 is maximized and Q_1 minimized. Thus following is an optimum among symmetric mechanisms that do not assign the bad at all:

a. When both players report the high type, allocate the good randomly to one of them with equal probability and make no money transfer.

b. If both players report the low type, make no allocation and no money transfer.

c. If one player reports the high type and the other reports the low type, assign the good to the high-type player and have him transfer to the other player an amount equal to

$$3(Q_6 - Q_1) \div \underbrace{(1/2)}_{\Pr(t_{-i}=1|t_i=6)} = 6\left(\underbrace{(1/2)(1/2) + 1/2}_{Q_6} - 0\right) = \frac{9}{2}.$$

Thus, given that the bad is never assigned, the optimal social welfare is equal to

$$\frac{12}{7}\left(\frac{7}{2}\underbrace{((1/2)(1/2) + 1/2)}_{Q_6} - \frac{3}{2} \cdot 0\right) = \frac{9}{2}.$$

By contrast, consider another symmetric mechanism that assigns the bad sometimes. It stipulates the same rule as the previous mechanism except—

c*. If one player reports the high type and the other reports the low type, allocate the good to the high-type player and the bad to the low-type player, and have the high-type player transfer an amount of money equal to 6 to the low-type player.

It is easy to verify incentive feasibility of this mechanism. The social welfare becomes larger:

$$\left(\frac{1}{2}\right)^2 6 \cdot \frac{2}{7} + \underbrace{\frac{1}{2}\left((6-6) \cdot \frac{2}{7} + (-1+6) \cdot \frac{12}{7}\right)}_{\text{different types}} = \frac{33}{7}.$$

B.2 Proof of (3.9): An Integration-by-Part Routine

Pick any $t^0 \in [0, 1]$. Since (Q, P) is IC, (3.5) implies

$$\begin{aligned} \int_0^1 P_i \, dW &= \int_0^1 \left(t_i Q_i(t_i) - \int_{t^0}^{t_i} Q_i(s) ds - U_i(t^0 \mid Q, P)\right) dW(t_i) \\ &= \int_0^1 t_i Q_i(t_i) dW(t_i) - U_i(t^0 \mid Q, P) - \int_0^1 \int_{t^0}^{t_i} Q_i(s) ds \, dW(t_i). \end{aligned}$$

Decompose the last double integral and use Fubini's theorem to obtain

$$\begin{aligned} \int_0^1 \int_{t^0}^{t_i} Q_i(s) ds \, dW(t_i) &= -\int_0^{t^0} \int_{t_i}^{t^0} Q_i(s) ds \, dW(t_i) + \int_{t^0}^1 \int_{t^0}^{t_i} Q_i(s) ds \, dW(t_i) \\ &= -\int_0^{t^0} \int_0^{s} Q_i(s) dW(t_i) ds + \int_{t^0}^1 \int_s^1 Q_i(s) dW(t_i) ds \\ &= -\int_0^{t^0} Q_i(s) W(s) ds + \int_{t^0}^1 Q_i(s) (1 - W(s)) \, ds. \end{aligned}$$

Plugging the second multiline formula into the first one for $\int_{T_i} P_i dW$, we get (3.9).

B.3 Details of Theorem 6

B.3.1 The Joint Constraint for IC, IR and BB

Lemma 20 *For any allocation* $(Q_i)_{i=1}^n$ *such that* Q_i *is weakly increasing on* $[0,1]$ *for any* i*, there exists a payment rule* $(P_i)_{i=1}^n$ *with which* $(Q_i)_{i=1}^n$ *constitutes an IC, IR and BB mechanism if and only if (3.17) is true.*

Proof Applying (3.13) to the case $G = F$ for all players i and summing the equations thereby obtained across i, we get the total expected money surplus from any IC mechanism (Q, P):

$$\sum_i \int_0^1 P_i \, dF = \sum_i \int_0^1 Q_i(t_i) t_i \, dF(t_i) + \sum_i \langle Q_i : \rho(F)| - \sum_i \min_{[0,1]} U_i(\cdot|Q, P). \tag{B.1}$$

Thus, BB ($\sum_i \int_0^1 P_i \, dF \geq 0$), IR ($\min_{[0,1]} U_i(\cdot|Q, P) \geq 0$ for all i) and IC together imply (3.17).

Conversely, suppose (3.17). With Q_i weakly increasing, pick $t_i^0 \in [0, 1]$ for which $Q_i(t_i) \geq 0$ for all $t_i \in (t_i^0, 1]$ and $Q_i(t_i) \leq 0$ for all $t_i \in [0, t_i^0)$. With such t_i^0, construct P_i via (3.5) so that $U_i(t_i^0|Q, P) = 0$ for all i. This, with Q_i weakly increasing, implies IC. Since P_i is constructed via (3.5), the derivative of $U_i(\cdot|Q, P)$ is equal to Q_i and hence, by the choice of t_i^0, $U_i(\cdot|Q, P)$ attains its minimum at t_i^0. Hence IR obtains by construction of P. Now that $(Q_i, P_i)_{i=1}^n$ is IC, (B.1) holds. Then (3.17) coupled with $\sum_i \min_{[0,1]} U_i(\cdot|Q, P) = 0$ implies BB.

B.3.2 The Social Welfare with Optimal Lump Sum Rebate

Lemma 21 *Given* (Q, P)*, the value of (3.6) subject to IC, IR and BB is equal to that of (3.16).*

Proof Let (Q, P) be any mechanism subject to IC, IR and BB. By IC, (3.10) holds for all i. Plug (3.10) into (3.7) and apply the two-part operator notation to see that the social welfare (3.6) generated by (Q, P) is equal to

$$\sum_i \min_{[0,1]} U_i(\cdot|Q, P) - \sum_i \langle Q_i : \rho(W)| \, . \tag{B.2}$$

By (B.1) and BB ($\sum_i \int_0^1 P_i \, dF \geq 0$),

$$\sum_i \min_{[0,1]} U_i(\cdot|Q, P) \leq \sum_i \int_0^1 Q_i(t_i) t_i \, dF(t_i) + \sum_i \langle Q_i : \rho(F)| \, . \tag{B.3}$$

The right-hand side of (B.3) can be attained by a payment rule that implements $(Q_i)_{i=1}^n$: Construct a payment rule P_i^Q via (3.5) such that $\min_{[0,1]} U_i(\cdot|Q, P^Q) = 0$ for all i. With (Q, P) IC, Q_i is weakly increasing for each i. This coupled with the construction of P_i^Q implies that $(Q_i, P_i^Q)_{i=1}^n$ is IC. Thus $\sum_i \int_0^1 P_i^Q dF$ is equal to the right-hand side of (B.1) with the role of P there played by P^Q here. Consequently, since $\min_{[0,1]} U_i(\cdot|Q, P^Q) = 0$, $\sum_i \int_0^1 P_i^Q dF$ is equal to the right-hand side of (B.3). Then define P^* to be the payment rule that combines $(P_i^Q)_{i=1}^n$ with the lump sum transfer back to the players in the amount equal to $\sum_i \int_0^1 P_i^Q dF$. With $P := P^*$, the left-hand side of (B.3) is equal to $\sum_i \int_0^1 P_i^Q dF$. It follows that P^* satisfies (B.3) as an equality.

Thus, given any implementable Q, the maximand of $\sum_i \min_{[0,1]} U_i(\cdot|Q,P)$ is equal to the right-hand side of (B.3). Substitute the right-hand side of (B.3) for the $\sum_i \min_{[0,1]} U_i(\cdot|Q,P)$ in (B.2) to see that the social welfare (3.6) generated by an optimal (Q,P) is equal to

$$\begin{aligned}
& \sum_i \int_0^1 Q_i(t_i)t_i\,dF(t_i) + \sum_i \langle Q_i : \rho(F)| - \sum_i \langle Q_i : \rho(W)| \\
= \; & \sum_i \int_0^1 Q_i(t_i)t_i\,dF(t_i) + \sum_i \langle Q_i : \rho(F) - \rho(W)|\,,
\end{aligned}$$

with the equality due to linearity of $\varphi \mapsto \langle Q_i : \varphi|$. The right-hand side of the above equation, by (3.11), is equal to (3.16).

B.3.3 Convexity of $\mathscr{Q}_{\text{mon}}$

Let $\gamma \in [0,1]$ and $Q, \hat{Q} \in \mathscr{Q}_{\text{mon}}$. Since $Q \in \mathscr{Q}_{\text{mon}}$, it is generated by a $(q_{iA}, q_{iB})_{i=1}^n$ with $\sum_i q_{iA}(\cdot) \le 1$ and $\sum_i q_{iB}(\cdot) \le 1$ via (3.3), and Q_i is weakly increasing for all i. Likewise, $\hat{Q} = (\hat{Q}_i)_{i=1}^n$ is generated by a $(\hat{q}_{iA}, \hat{q}_{iB})_{i=1}^n$ with each $\hat{Q}_i$ weakly increasing. Then $\sum_i (\gamma q_{iA} + (1-\gamma)\hat{q}_{iA}) \le 1$ and $\sum_i (\gamma q_{iB} + (1-\gamma)\hat{q}_{iB}) \le 1$; furthermore, for each i, $\gamma Q_i + (1-\gamma)\hat{Q}_i$ satisfies (3.3) with respect to $(\gamma q_{iA} + (1-\gamma)\hat{q}_{iA}, \gamma q_{iB} + (1-\gamma)\hat{q}_{iB})$, and is weakly increasing because both Q_i and $\hat{Q}_i$ are so. Thus $(\gamma Q_i + (1-\gamma)\hat{Q}_i)_{i=1}^n \in \mathscr{Q}_{\text{mon}}$.

B.3.4 Concavity of Two-Part Operators

Lemma 22 *$(Q_i)_{i=1}^n \mapsto \sum_i \langle Q_i : \rho(F)|$ is a concave functional on $\mathscr{Q}$. Furthermore, for any $Q, Q' \in \mathscr{Q}$, if $Q_i(t_i)Q'_i(t_i) < 0$ for all t_i in a positive-measure subset of $[0,1]$ for some i and if $\alpha \in (0,1)$, then*

$$\alpha \sum_i \langle Q_i : \rho(F)| + (1-\alpha)\sum_i \langle Q'_i : \rho(F)| < \sum_i \left\langle \alpha Q_i + (1-\alpha)Q'_i : \rho(F)\right|. \tag{B.4}$$

Proof It suffices to prove, for each i, that $\langle Q_i : \rho(F)|$ is a concave functional of Q_i, and strictly so if $Q_i^+ \neq 0$ on a positive-measure subset of $[0,1]$. By (3.11), the definition of $\rho(F)$, $\rho_+(F) < \rho_-(F)$ on $[0,1]$. By the definition of two-part operators and the fact $Q_i = Q_i^+ - Q_i^-$,

$$\begin{aligned}
\langle Q_i : \rho(F)| &= \int_0^1 Q_i^+(t_i)\rho_+(F)(t_i)dt_i - \int_0^1 Q_i^-(t_i)\rho_-(F)(t_i)dt_i \\
&= \int_0^1 Q_i(t_i)\rho_-(F)(t_i)dt_i + \int_0^1 Q_i^+(t_i)\left[\rho_+(F)(t_i) - \rho_-(F)(t_i)\right]dt_i.
\end{aligned}$$

On the second line, the first integral is linear in Q_i; the second integral is concave in Q_i because $Q_i(t_i) \mapsto Q_i^+(t_i)$ is convex, $\rho_+(F) - \rho_-(F) \le 0$ on $[0,1]$, and hence $Q_i(t_i) \mapsto Q_i^+(t_i)\,(\rho_+(F)(t_i) - \rho_-(F)(t_i))$ is concave for all $t_i \in [0,1]$. Thus $\langle Q_i : \rho(F)|$ is concave in Q_i. To prove the second statement of the lemma, note that the convex mapping $x \mapsto x^+$ is strictly convex for those $x, y \in \mathbb{R}$ such that $xy < 0$ in the sense that $xy < 0$ implies

$$\alpha x^+ + (1-\alpha)y^+ > (\alpha x + (1-\alpha)y)^+$$

for all $\alpha \in (0,1)$. This coupled with the fact $\rho_+(F) - \rho_-(F) < 0$ on $[0,1]$ implies that

$$\left(\alpha Q_i^+ + (1-\alpha)(Q')_i^+\right)(\rho_+(F) - \rho_-(F)) < \left(\alpha Q_i + (1-\alpha)Q_i'\right)^+ (\rho_+(F) - \rho_-(F))$$

on the subset of $[0,1]$ where $Q_i Q_i' < 0$. Thus, if this subset, denoted by E, is of positive measure, the above strict inequality is preserved by integration on E. When the integration domain extends from E to $[0,1]$, the strictly inequality is again preserved because $Q_i(t_i) \mapsto Q_i^+(t_i)(\rho_+(F)(t_i) - \rho_-(F)(t_i))$ is concave for every $t_i \in [0,1]$. Thus (B.4) follows.

B.3.5 Existence of Interior Solutions for (3.17)

Let $(Q_i)_{i=1}^n$ be the allocation of auctioning off the good through an expected-revenue-maximizing auction (cf. Myerson [17]) and never assigning the bad at all. Hence Q_i is never negative, $\langle Q_i : \rho(f)| = \int_0^1 Q_i(s)\rho_+(F)(s)ds$, and so the left-hand side of (3.17) is equal to

$$\sum_i \int_0^1 Q_i(t_i)\left(t_i - \frac{1-F(t_i)}{f(t_i)}\right) dF(t_i),$$

which, by Myerson [17], is equal to the expected value of the pointwise maximum among the nonnegative ironed virtual utilities of the good. Since F has no gap in $[0,1]$, this expected value is strictly positive. Thus (3.17) is satisfied strictly.

B.4 Proof of Corollaries 5, 6 and 7

Corollary 5 Use (3.20) and integration-by-parts to obtain

$$\begin{aligned} \int_0^{t_i} V(r)dF(r) &= t_iF(t_i) - t_iW(t_i) + \int_0^{t_i} r dW(r) \\ &= t_iF(t_i) - \int_0^{t_i} (t_i - r)w(r)f(r)dr \end{aligned}$$

for any $t_i \in [0,1]$, with the second line due to (3.8). For each $t_i \in [0,1]$, define

$$\begin{aligned} R(t_i) &:= \int_0^{t_i} (t_i - r)w(r)f(r)dr, \\ \widetilde{R}(t_i) &:= \int_0^{t_i} (t_i - r)\widetilde{w}(r)f(r)dr. \end{aligned}$$

By this definition and the above calculation, (3.22) is equivalent to "$t_iF(t_i) < R(t_i)$ for some $t_i \in (0,1)$." By hypothesis, the bad is allocated with a strictly positive probability given welfare density w, thus the "only if" part of Theorem 7 implies that $t_iF(t_i) < R(t_i)$ for some $t_i \in$

$(0, 1)$. The "if" part of the theorem implies that the bad is also allocated with a strictly positive probability given $\widetilde{w}$ if $t_i F(t_i) < \widetilde{R}(t_i)$ for some $t_i \in (0, 1)$. Thus, we complete the proof by showing that $\widetilde{R} \geq R$ on $(0, 1]$. To show that, pick any $t_i \in (0, 1]$. We have:

$$\begin{aligned}
\widetilde{R}(t_i) - R(t_i) &= \int_0^{t_i} (t_i - r)\,(\widetilde{w}(r) - w(r))\, f(r) dr \\
&= \int_0^{t_i} (t_i - r)\, d\left(\widetilde{W}(r) - W(r)\right) \\
&= -t_i \left(\widetilde{W}(0) - W(0)\right) - \int_0^{t_i} \left(\widetilde{W}(r) - W(r)\right) d(t_i - r) \\
&= \int_0^{t_i} \left(\widetilde{W}(r) - W(r)\right) dr,
\end{aligned}$$

which is nonnegative by (3.23), as $\widetilde{W}$ is second-order stochastically dominated by W. ∎

Corollary 6 By Theorem 7, it suffices to prove (3.22). By (3.20) and differentiability of f at 0, the derivative of V at 0 is equal to $2 - w(0)$, which is negative by the hypothesis of the corollary. Thus, with $V(0) = 0$ by (3.20), (3.22) holds for some t_i near 0. ∎

Corollary 7 By Corollary 5, it suffices to prove that the bad is not needed when the welfare weight distribution is F. Given such welfare weight distribution, $V(t_i) = t_i$ for all $t_i \in [0, 1]$ by (3.20). Hence (3.22) does not hold. By Theorem 7, the bad is not needed. ∎

B.5 Details of Theorem 7

Proof of Lemma 2 Since $Q_i \geq 0$ by hypothesis, $\langle Q_i : \rho(f) | = \int_0^1 Q_i(s)\rho_+(F)(s) ds$, and so the left-hand side of (3.17) is equal to

$$\sum_i \int_0^1 Q_i(s)\,(s f(s) - (1 - F(s)))\, ds = \sum_i \int_0^1 Q_i(s) d\,(-s(1 - F(s)))\,.$$

With integration by parts,

$$\int_0^1 Q_i(s) d\,(-s(1 - F(s))) = \int_0^1 s(1 - F(s)) dQ_i(s).$$

Since $s(1 - F(s)) > 0$ for all $s \in (0, 1)$, and Q_i weakly increasing, the above integral is nonnegative for all i. Hence (3.17) is satisfied. Furthermore, the above integral is strictly positive unless Q_i is constant a.e. on $[0, 1]$. Thus, the proof is complete if it is impossible to have a solution $(Q_i)_{i=1}^n$ to problem (3.16) such that, for each i, Q_i is equal to a nonnegative constant a.e. on $[0, 1]$. To that end, let Q be such an allocation: for each i, $Q_i = a_i$ a.e. on $[0, 1]$ for some

$a_i \in [0,1]$. Then, by (3.3),

$$\begin{aligned}\sum_i a_i &= \sum_i \int_0^1 \int_{[0,1]^{n-1}} (q_{iA}(t_i,t_{-i}) - q_{iB}(t_i,t_{-i}))\, dF_{-i}(t_{-i}) dF(t_i) \\ &\le \sum_i \int_0^1 \int_{[0,1]^{n-1}} q_{iA}(t_i,t_{-i}) dF_{-i}(t_{-i}) dF(t_i) \\ &= \int_{[0,1]^n} \sum_i q_{iA}(t_1,\dots,t_n)\, dF(t_1)\cdots dF(t_n) \\ &\le 1.\end{aligned}$$

Given this allocation, the objective in problem (3.16) is equal to

$$\sum_i a_i \int_0^1 V(t_i) dF(t_i) = \left(\sum_i a_i\right) \int_0^1 V(t_i) dF(t_i) \le \int_0^1 V(s) dF(s) = \int_0^1 \overline{V}(s) dF(s), \quad \text{(B.5)}$$

with the last "=" due to the definition of ironing, (3.24)—(3.25). By contrast, consider the allocation that never allocates the bad and allocates the good hierarchically according to the ironed copy $\overline{V}$ of V (cf. Section 3.4.1). Never allocating the bad, this allocation satisfies (3.17) by the previous reasoning; since $\overline{V}$ is weakly increasing by definition, this allocation belongs to $\mathscr{Q}_{\text{mon}}$. Thus the allocation is feasible. Furthermore, given this allocation, which chooses the largest nonnegative $\overline{V}(t_i)$ almost surely, the objective in problem (3.16) is equal to

$$\int_{[0,1]^n} \left(\max_{i=1,\dots,n} \overline{V}(t_i)^+\right) dF(t_1)\cdots dF(t_n),$$

which is larger than (B.5). Thus $(Q_i)_{i=1}^n = (a_i)_{i=1}^n$ (a.e.) cannot be a solution to (3.16). ∎

B.6 Details of Theorem 8

B.6.1 Proof of Lemma 4

Claim (a): Since $\mathscr{Q}$ is convex, the proof of Theorem 6 remains valid when $\mathscr{Q}_{\text{mon}}$ is replaced by $\mathscr{Q}$, so the saddle point characterization applies to any solution to problem (3.34).

Claim (b.i): Plug $\lambda = 0$ into (3.14) to see that the $\mathscr{L}(Q,0)$ is equal to $\sum_i \int_0^1 Q_i(s)V(s)dF(s)$, a linear functional on the convex domain $\mathscr{Q}$. Thus, maximization of $\mathscr{L}(\cdot,0)$ on $\mathscr{Q}$ is a linear programming, hence any solution Q^* thereof entails a hierarchical allocation of the good according to V, and a hierarchical allocation of the bad according to $-V$. Thus, for almost all t_i, t_i', $V(t_i) > V(t_i') \Rightarrow Q_i^*(t_i) > Q_i^*(t_i')$.

Claim (b.ii): Suppose, to the contrary, that $Q_i^*(s)Q_j^*(s) < 0$ for all s on a positive-measure subset of $[0,1]$. Let $Q' \in \mathscr{Q}$ be the same as Q^* except that $Q_i' = Q_j^*$ and $Q_j' = Q_i^*$. Then (B.4) holds. By $\lambda > 0$ and (3.14), it follows that, for any $\alpha \in (0,1)$,

$$\mathscr{L}\left(\alpha Q^* + (1-\alpha)Q', \lambda\right) > \alpha \mathscr{L}(Q^*,\lambda) + (1-\alpha)\mathscr{L}(Q',\lambda) = \mathscr{L}(Q^*,\lambda),$$

where the equality comes from the fact that the permutation "$Q_i' = Q_j^*$ and $Q_j' = Q_i^*$" renders $\mathscr{L}(Q', \lambda) = \mathscr{L}(Q^*, \lambda)$ since the players' types are symmetrically distributed by F and weighed by the same W. Thus, Q^* does not maximize $\mathscr{L}(\cdot, \lambda)$ on $\mathscr{Q}$, contradiction.

Claim (b.iii): Suppose, to the contrary, that $V_-^\lambda(x) > 0$ for some $x \in (0, c_i)$, with V_-^λ continuous due to (3.19), there is an interval $(a, b) \subseteq (0, c_i)$, with $a < x < b$, on which $V_-^\lambda > 0$. Perturb Q^* by the reservation operator $R_{i,T}$ such that $T = \{(t_k)_{k=1}^n \in [0,1]^n \mid t_i \in (a, b)\}$, namely, reserve both items from player i when i's type belongs to (a, b). By Lemma 3, $R_{i,T}(Q^*) \in \mathscr{Q}(Q^*, c)$ and we can apply (3.26) to both Q^* and $R_{i,T}(Q^*)$ to obtain

$$\mathscr{L}(Q^*, \lambda) - \mathscr{L}(R_{i,T}(Q^*), \lambda) = \int_a^b Q_i^*(t_i) V_-^\lambda(t_i) dF(t_i) < 0,$$

where the strictly inequality follows from $Q_i^* < 0$ on $[0, c_i) \supset (a, b)$ and $V_-^\lambda > 0$ on (a, b). Thus we have another allocation in $\mathscr{Q}$ that generates larger Lagrangian than Q^* given λ, contradicting the hypothesis that Q^* maximizes $\mathscr{L}(\cdot, \lambda)$ on $\mathscr{Q}$. ∎

B.6.2 Lemma 24 and 25: The Perturbation Method

We need to introduce two additional kinds of sign-preserving perturbations:

Reservation $R_{i,T}^A$ of the good: This is the same as $R_{i,T}$ (Section 3.4.2) except that the only modification of the original ex post allocation $(q_{kA}, q_{kB})_{k=1}^n$ is to set $\tilde{q}_{iA}(t) := 0$ for any $t \in T$ without altering the allocation of item B. That is, when T occurs, the planner keeps the good to herself if the original allocation would award the good to i.

Merge $M_{i,k,T}$: For any two distinct players i and k, any $T \subseteq [0,1]^n$, and for any $Q \in \mathscr{Q}$ that is the reduced form of an ex post allocation $(q_{lA}, q_{lB})_{l=1}^n$, define $M_{i,k,T}(Q)$ to be the reduced form of the ex post allocation $(\tilde{q}_{lA}, \tilde{q}_{lB})_{l=1}^n$ that is the same as $(q_{lA}, q_{lB})_{l=1}^n$ except that, for any $t \in T$, $\tilde{q}_{iB}(t) := q_{kB}(t) + q_{iB}(t)$ and $\tilde{q}_{kB}(t) := 0$. Namely, when T occurs, award the bad to player i whenever it is originally allocated to players i or k.

Recall that π_i denotes the projection from $\mathbb{R}^n$ onto the i^{th} dimension. The next lemma follows directly from the definitions of $R_{i,T}^A$ and $M_{i,k,T}$ and hence we omit its proof.

Lemma 23 *For any $Q \in \mathscr{Q}$ with crossing point $c := (c_l)_{l=1}^n \in [0,1]^n$, any players i and k with $i \neq k$, and any $T \subseteq [0,1]^n$ for which $\pi_i(T) \subseteq [0, c_i]$, denote $Q' := R_{i,T}^A(Q)$ and $Q'' := R_{k,T}^A(M_{i,k,T}(Q))$. Then:*

a. $Q' \in \mathscr{Q}(Q, c)$, $Q_i' = Q_i$ on $[0,1] \setminus \pi_i(T)$, and $Q_k' = Q_k$ on $[0,1]$ for all $k \neq i$;

b. if $\pi_k(T) \subseteq [0, c_k]$, then $Q'' \in \mathscr{Q}(Q, c)$, $Q_i'' = Q_i$ on $[0,1] \setminus \pi_i(T)$, $Q_k'' = Q_k$ on $[0,1] \setminus \pi_k(T)$, and $Q_l'' = Q_l$ on $[0,1]$ for all $l \notin \{i, k\}$.

Recall from Section 3.4.2 the definitions of crossing point and $\mathscr{Q}(Q^*, c)$. As explained around (3.35), any solution to the relaxed problem (3.34) is also a solution to (3.36). The next two lemmas characterize any solution to (3.36), where the $-V_-^\lambda$ corresponds to the g in the lemmas. Since the objective in the problem (3.36) is a linear functional of the reduced form Q when Q ranges in $\mathscr{Q}(Q^*, c)$, the intuition of these lemmas is to perturb Q toward the direction of $-V_-^\lambda$ (or that of g here). However, the perturbation needs to be ex post feasible—obeying (3.3)—and remain within $\mathscr{Q}(Q^*, c)$. The key of the proof is to guarantee such conditions with the sign-preserving perturbations defined earlier.

Lemma 24 *For any integrable $g : [0, 1] \to \mathbb{R}$, any $(c_i)_{i=1}^n \in [0, 1]^n$ and any $Q^* \in \mathscr{Q}$, denote*

$$E_i := \{t_i \in [0, c_i] \mid g(t_i) > 0\}$$

for each player i, and suppose:

i. $c := (c_i)_{i=1}^n$ is a crossing point of Q^*, and

ii. Q^* maximizes $\sum_i \int_0^{c_i} Q_i^-(s)g(s)dF(s)$ among all $(Q_i)_{i=1}^n \in \mathscr{Q}(Q^*, c)$.

Then each of the following sets is of zero measure for any players i and k with $i \neq k$ (where $(q_{lA}^*, q_{lB}^*)_{l=1}^n$ denotes the ex post allocation the reduced form of which is Q^*):

a. $T_i := \left\{(t_i, t_{-i}) \in E_i \times [0, 1]^{n-1} \mid q_{iA}^*(t_i, t_{-i}) > 0\right\}$;

b. $Z_i := \left\{(t_i, t_{-i}) \in E_i \times [0, 1]^{n-1} \mid \sum_{k=1}^n q_{kB}^*(t_i, t_{-i}) < 1\right\}$;

c. $O_{ik} := \left\{(t_i, t_k, t_{-(i,k)}) \in E_i \times (c_k, 1] \times [0, 1]^{n-2} \mid q_{kB}^*(t_i, t_k, t_{-(i,k)}) > 0\right\}$;

d. $S_{ik} := \left\{(t_i, t_k, t_{-(i,k)}) \in E_i \times E_k \times [0, 1]^{n-2} \mid g(t_i) > g(t_k), q_{kB}^*(t_i, t_k, t_{-(i,k)}) > 0\right\}$.

Proof Category (a): Suppose that T_i for some i is of positive measure. Then perturb Q^* by the reservation operator R_{i,T_i}^A of the good. Denote $Q := R_{i,T_i}^A(Q^*)$. By definition of R_{i,T_i}^A, Q is the same as Q^* except that, for all $t_i \in \pi_i(T_i)$,

$$Q_i(t_i) \overset{(3.3)}{=} \int_{[0,1]^{n-1}} (-q_{iB}^*(t_i, t_{-i}))\, dF_{-i}(t_{-i}) < \int_{[0,1]^{n-1}} (q_{iA}^*(t_i, t_{-i}) - q_{iB}^*(t_i, t_{-i}))\, dF_{-i}(t_{-i}) \overset{(3.3)}{=} Q_i^*(t_i).$$

Since $\pi_i(T_i) \subseteq E_i \subseteq [0, c_i]$ by definition, Q also has c as a crossing point, namely, $Q \in \mathscr{Q}(Q^*, c)$. But

$$\sum_k \int_0^{c_k} Q_k^-(s)g(s)dF(s) - \sum_k \int_0^{c_k} (Q_k^*(s))^- g(s)dF(s) = \int_{\pi_i(T_i)} (Q_i^*(s) - Q_i(s))\, g(s)dF(s)$$

is positive because the measure of $\pi_i(T_i)$ is positive and because $g > 0$ and $Q_i < Q_i^*$ on $\pi(T_i)$. This, with $Q \in \mathscr{Q}(Q^*, c)$, contradicts hypothesis (ii).

Category (b): Suppose that Z_i for some i has a positive measure. Let $(q_{kA}, q_{kB})_{k=1}^n$ be the ex post allocation that is the same as $(q_{kA}^*, q_{kB}^*)_{k=1}^n$ except that

$$q_{iB}(t) := q_{iB}^*(t) + 1 - \sum_{k=1}^n q_{kB}^*(t)$$

for all $t \in Z_i$. That is, if Z_i occurs, allocate the bad to player i if the original allocation would reserve it from all players. Denote Q for the reduced form of $(q_{kA}, q_{kB})_{k=1}^n$. Clearly Q is the same as Q^* except that (by (3.3)) $Q_i < Q_i^*$ on $\pi_i(Z_i)$. As in Category (a), $Q \in \mathscr{Q}(Q^*, c)$ and $\sum_k \int_0^{c_k} Q_k^-(s)g(s)dF(s) > \sum_k \int_0^{c_k} \left(Q_k^*(s)\right)^- g(s)dF(s)$, again a contradiction to (ii).

Category (c): Suppose that O_{ik} is of positive measure for some $i \neq k$. Let $t_k \in \pi_k(O_{ik})$. By the supposition,

$$\int_{[0,1]^{n-1}} q_{kB}^*(t_k, t_{-k})dF_{-k}(t_{-k}) \geq \int_{\pi_{-k}(O_{ik})} q_{kB}^*(t_k, t_{-k})dF_{-k}(t_{-k}) > 0,$$

where π_{-k} denotes the projection $(t_l)_{l=1}^n \mapsto (t_l)_{l \neq k}$. By definition of O_{ik}, $t_k \in (c_k, 1]$. This coupled with c being a crossing point of Q^* implies that $Q_k^*(t_k) \geq 0$. Thus, by (3.3),

$$\int_{[0,1]^{n-1}} q_{kA}^*(t_k, t_{-k})dF_{-k}(t_{-k}) \geq \int_{[0,1]^{n-1}} q_{kB}^*(t_k, t_{-k})dF_{-k}(t_{-k}) \geq \int_{\pi_{-k}(O_{ik})} q_{kB}^*(t_k, t_{-k})dF_{-k}(t_{-k}).$$

Thus there exists $O^*(t_k) \subseteq \{t_{-k} \in [0, 1]^{n-1} \mid q_{kA}^*(t_k, t_{-k}) > 0\}$ such that

$$\int_{O^*(t_k)} q_{kA}^*(t_k, t_{-k})dF_{-k}(t_{-k}) = \int_{\pi_{-k}(O_{ik})} q_{kB}^*(t_k, t_{-k})dF_{-k}(t_{-k}). \tag{B.6}$$

Denote

$$O := \bigcup_{t_k \in \pi_k(O_{ik})} (\{t_k\} \times O^*(t_k)).$$

Now perturb Q^* by the merge operator $M_{i,k,O_{ik}}$ together with the reservation operator $R_{k,O}^A$ of item A. That is, allocate the bad to player i instead of k in the event of O_{ik}, and reserve the good from player k in the event of O. Denote $Q := R_{k,O}^A(M_{i,k,O_{ik}}(Q^*))$. By (B.6), the perturbation leaves $Q_k^*(t_k)$ unchanged for every $t_k \in \pi_k(O_{ik})$, namely, $Q_k = Q_k^*$ on $\pi_k(O_{ik})$. Thus, $Q_l = Q_l^*$ on $(c_l, 1]$ for all players l, $Q_l = Q_l^*$ on $[0, c_l]$ for all $l \neq i$, $Q_i = Q_i^*$ on $[0, c_i] \setminus \pi_i(O_{ik})$, and $Q_i < Q_i^*$ on $\pi_i(O_{ik})$. Hence $Q \in \mathscr{Q}(Q^*, c)$. But

$$\sum_l \int_0^{c_l} Q_l^-(s)g(s)dF(s) - \sum_l \int_0^{c_l} (Q_l^*(s))^- g(s)dF(s) = \int_{\pi_i(O_{ik})} (Q_i^*(s) - Q_i(s))\, g(s)dF(s)$$

is positive, as in Category (a). Again we have a desired contradiction to (ii).

Category (d): Suppose that S_{ik} for some $k \neq i$ is of positive measure. Perturb Q^* by the merge operator $M_{i,k,S_{ik}}$ together with the reservation operator $R_{k,S_{ik}}^A$ of the good. Denote Q for the outcome of the perturbation, i.e., $Q := R_{k,S_{ik}}^A(M_{i,k,S_{ik}}(Q^*))$. That is, in the event S_{ik}, assign the bad to player i whenever it is originally allocated to players i or k, and reserve the good from player k. Let $(q_{lA}, q_{lB})_{l=1}^n$ be the ex post allocation the reduced form of which is Q. By definition of the perturbation, $q_{iB}(t) = q_{iB}^*(t) + q_{kB}^*(t)$ and $q_{kB}(t) = q_{kA}(t) = 0$ for all $t \in S_{ik}$. Since $\pi_i(S_{ik}) \subseteq E_i \subseteq [0, c_i]$ and $\pi_k(S_{ik}) \subseteq E_k \subseteq [0, c_k]$, Lemma 23.b implies that Q also has c as a crossing point, namely, $Q \in \mathscr{Q}(Q^*, c)$. By the definitions of $R_{k,S_{ik}}^A$ and $M_{i,k,S_{ik}}$ and the notations

$F^n(t) := F(t_1)\cdots F(t_n)$, $q_l^*(t) := q_{lA}^*(t) - q_{lB}^*(t)$ and $q_l(t) := q_{lA}(t) - q_{lB}(t)$,

$$\begin{aligned}
& \sum_l \int_0^{c_l} Q_l^-(s)g(s)dF(s) - \sum_l \int_0^{c_l} \left(Q_l^*(s)\right)^- g(s)dF(s) \\
= \; & \sum_l \int_0^{c_l} \int_{[0,1]^{n-1}} \left(-q_l(t_l, t_{-l}) + q_l^*(t_l, t_{-l})\right) g(t_l)dF_{-l}(t_{-l})dF(t_l) \\
= \; & \int_{S_{ik}} \left(\left(-q_i(t) + q_i^*(t)\right) g(t_i) + \left(-q_k(t) + q_k^*(t)\right) g(t_k)\right) dF^n(t) \\
= \; & \int_{S_{ik}} \left(q_{kB}^*(t)g(t_i) + \left(q_{kA}^*(t) - q_{kB}^*(t)\right) g(t_k)\right) dF^n(t) \\
= \; & \int_{S_{ik}} \left(q_{kB}^*(t)\left(g(t_i) - g(t_k)\right) + q_{kA}^*(t)g(t_k)\right) dF^n(t) \\
> \; & 0,
\end{aligned}$$

where the first equality is due to the definition of Q_l^- (and $(Q_l^*)^-$) and (3.3), the second and third equalities are due to the definitions of $R^A_{k,S_{ik}}$ and $M_{i,k,S_{ik}}$, and the inequality due to the fact that $g(t_i) > g(t_k) > 0$ on S_{ik} and the hypothesis that S_{ik} is of positive F^n-measure. Again we obtain a contradiction to the hypothesis (ii), as desired.

Lemma 25 *For any integrable $g : [0,1] \to \mathbb{R}$, any $(c_i)_{i=1}^n \in [0,1]^n$ and any $Q^* \in \mathcal{Q}$, denote*

$$E := \left\{ s \in \left[0, \min_{i=1,\dots,n} c_i\right] \;\middle|\; g(s) > 0 \right\}$$

and suppose:

i. $c := (c_i)_{i=1}^n$ is a crossing point of Q^*, and

ii. Q^* maximizes $\sum_i \int_0^{c_i} Q_i^-(s)g(s)dF(s)$ among all $(Q_i)_{i=1}^n \in \mathcal{Q}(Q^*, c)$.

Then for any player i and almost every $t_i, t_i' \in E$, $g(t_i) > g(t_i') \Rightarrow Q_i^*(t_i) < Q_i^*(t_i')$.

Proof For any player i and any $t_i \in E$, define:

$$\begin{aligned}
\mathcal{B}_i(t_i, \succ) &:= \left\{ (t_k)_{k\neq i} \in \prod_{k\neq i} [0, c_k] \;\middle|\; g(t_i) > \max\left\{0, \max_{k\neq i} g(t_k)\right\} \right\}, \\
\mathcal{B}_i(t_i, \sim) &:= \left\{ (t_k)_{k\neq i} \in \prod_{k\neq i} [0, c_k] \;\middle|\; g(t_i) = \max\left\{0, \max_{k\neq i} g(t_k)\right\} \right\},
\end{aligned}$$

and $\mathcal{B}_i(t_i) := \mathcal{B}_i(t_i, \succ) \cup \mathcal{B}_i(t_i, \sim)$. We have

$$\begin{aligned}
Q_i^*(t_i) &= -\int_{[0,1]^{n-1}} q_{iB}^*(t_i, t_{-i})dF_{-i}(t_{-i}) \\
&= -\int_{[0,c_*]^{n-1}} q_{iB}^*(t_i, t_{-i})dF_{-i}(t_{-i}) \\
&= -\int_{\mathcal{B}_i(t_i)} q_{iB}^*(t_i, t_{-i})dF_{-i}(t_{-i}),
\end{aligned}$$

with the first line due to (3.3) and Lemma 24.a, the second line due to Lemma 24.c, and the third line Lemma 24.d. Also observe that, for almost every $t_i \in E$,

$$q^*_{iB}(t_i, \cdot) = 1 \quad \text{a.e. on} \quad \mathscr{B}_i(t_i, \succ). \tag{B.7}$$

That is because $q^*_{iB}(t_i, t_{-i}) = \sum_{k=1}^n q^*_{kB}(t_i, t_{-i})$ for almost all $t_{-i} \in \mathscr{B}_i(t_i, \succ)$ by Lemma 24.d, and $\sum_{k=1}^n q^*_{kB}(t) = 1$ for almost all $t \in \bigcup_{t_i \in E} (\{t_i\} \times \mathscr{B}_i(t_i, \succ))$ by Lemma 24.b. Thus, for almost every $t_i, t'_i \in E$ such that $g(t_i) > g(t'_i)$, which means $\mathscr{B}_i(t'_i) \subsetneq \mathscr{B}(t_i, \succ)$ and, because F has no gap in $[0, 1]$, $\mathscr{B}(t_i, \succ) \setminus \mathscr{B}_i(t'_i)$ is of positive measure, we have

$$\begin{aligned}
\int_{\mathscr{B}_i(t_i)} q^*_{iB}(t_i, t_{-i}) dF_{-i}(t_{-i}) &\geq \int_{\mathscr{B}_i(t_i,\succ)} q^*_{iB}(t_i, t_{-i}) dF_{-i}(t_{-i}) \\
&= \int_{\mathscr{B}_i(t_i,\succ)} dF_{-i}(t_{-i}) \\
&> \int_{\mathscr{B}_i(t'_i)} dF_{-i}(t_{-i}) \\
&\geq \int_{\mathscr{B}_i(t'_i)} q^*_{iB}(t'_i, t_{-i}) dF_{-i}(t_{-i}),
\end{aligned}$$

with the second line due to (B.7), and the third line due to $\mathscr{B}(t_i, \succ) \setminus \mathscr{B}_i(t'_i)$ having positive measure. Thus, $Q^*_i(t_i) < Q^*_i(t'_i)$, as asserted.

B.6.3 Lemma 26: The Indeterminacy Case of the Relaxed Problem

Lemma 26 *For any $\lambda > 0$, $c_* \in (0, 1)$ and $Q^* \in \mathscr{Q}_{\text{mon}}$, if (Q^*, λ) is a saddle point with respect to $(\mathscr{L}, \mathscr{Q})$, $Q^*_i < 0$ on a positive-measure subset of $[0, c_*]$ for some i, and $c := (c_i)_{i=1}^n$ with $c_i := c_*$ for all i is a crossing point of Q^*, then*

$$\left\{ Q \in \mathscr{Q}(Q^*, c) \;\middle|\; \sum_i \int_0^{c_*} Q^-_i(s)\,(sf(s) + F(s))\,ds = z \right\} \tag{B.8}$$

contains a continuum.

Proof Since $\lambda > 0$ and (Q^*, λ) is a saddle point with respect to $(\mathscr{L}, \mathscr{Q})$, the constraint (3.17) in the relaxed problem (3.34) is binding for Q^*. This, combined with (3.37) and the hypothesis that c is a crossing point of Q^*, means

$$\sum_i \int_0^{c_*} (Q^*_i(s))^- \,(sf(s) + F(s))\,ds = z. \tag{B.9}$$

Note that $\mathscr{Q}(Q^*, c)$ is a convex set and the mapping

$$\phi : Q \longmapsto \sum_i \int_0^{c_*} Q^-_i(s)\,(sf(s) + F(s))\,ds$$

is linear on $\mathscr{Q}(Q^*, c)$. Denote $\mathbf{0}$ for the element of $\mathscr{Q}(Q^*, c)$ that assigns zero to $Q_i(s)$ for all $s \in [0, c_*]$ and all i. Note that $\phi(\mathbf{0}) = 0$.

First we claim $z > \min_{\mathcal{Q}(Q^*,c)} \phi$. By hypothesis, $\left(Q_i^*\right)^- > 0$ on a positive-measure subset of $[0, c_*]$ for some i, it follows from (B.9) that $z > 0$. Thus the claim follows from the fact that $0 = \phi(\mathbf{0})$ and $\mathbf{0} \in \mathcal{Q}(Q^*, c)$.

Second, we claim $z < \max_{\mathcal{Q}(Q^*,c)} \phi$. Otherwise, $z = \max_{\mathcal{Q}(Q^*,c)} \phi$. Then by (B.9) Q^* maximizes $\sum_i \int_0^{c_*} Q_i^-(s)\,(sf(s) + F(s))\,ds$ among all $Q \in \mathcal{Q}(Q^*, c)$. Let $g(s) := sf(s) + F(s)$ for all $s \in [0, 1]$ and apply Lemma 25. Note that the set E in Lemma 25 is $(0, c_*]$ here. Thus, for every player i and almost every $s, s' \in (0, c_*]$, $sf(s) + F(s) > s'f(s') + F(s') \Rightarrow Q_i^*(s) < Q_i^*(s')$. Note that $sf(s) + F(s)$ is equal to 0 when $s = 0$ and strictly positive when $s > 0$. Thus, by differentiability of f, there exists an interval $I \subseteq [0, c_*]$ on which $sf(s) + F(s)$ is strictly increasing in s. Hence $Q_i^*(s)$ is a strictly decreasing function of s almost everywhere on I. But then $Q^* \notin \mathcal{Q}_{\text{mon}}$, contradiction. Thus $z < \max_{\mathcal{Q}(Q^*,c)} \phi$.

Third, there exist at least two distinct elements of the set (B.8). Let $\overline{Q}$ be a maximizer of ϕ on $\mathcal{Q}(Q^*, c)$. Thus,

$$0 = \phi(\mathbf{0}) < z = \phi(Q^*) < \phi(\overline{Q}).$$

Since $sf(s) + F(s) > 0$ for all $s \in (0, 1]$, $\left(\overline{Q}_i\right)^- > 0$ a.e. on $(0, c_*]$ for all i. For any $\theta \in [0, c_*]$ and for any player i, let $T_i^\theta := \{(t_k)_{k=1}^n \in [0, 1]^n \mid t_i \in [0, \theta]\}$ and perturb $\overline{Q}$ iteratively by $R_{1,T_1^\theta}, R_{2,T_2^\theta}, \ldots, R_{n,T_n^\theta}$. That is, let

$$Q^{>\theta} := R_{n,T_n^\theta}\left(\cdots\left(R_{2,T_2^\theta}\left(R_{1,T_1^\theta}(Q^*)\right)\right)\cdots\right),$$

which results from modifying Q^* by reserving both items from any player whose type is in $[0, \theta]$. Then $Q^{>0} = \overline{Q}$, $Q^{>c_*} = \mathbf{0}$, and

$$\begin{aligned} \phi(Q^{>\theta}) &= \textstyle\sum_i \int_\theta^{c_*} \left(\overline{Q}_i(s)\right)^- (sf(s) + F(s))\,ds, \\ 0 = \phi(Q^{>c_*}) &< z = \phi(Q^*) < \phi(Q^{>0}). \end{aligned} \tag{B.10}$$

By continuity of the integration operator, there exists a $\theta_* \in [0, c_*]$ for which $\phi(Q^{>\theta_*}) = \phi(Q^*)$. Furthermore, for any i, since $\left(\overline{Q}_i\right)^- > 0$ a.e. on $(0, c_*]$, Ineq. (B.10) implies $0 < \theta_* < c_*$. Note, for any i, that $\left(Q_i^{>\theta_*}\right)^- = 0$ on $[0, \theta_*]$ and $\left(Q_i^{>\theta_*}\right)^- > 0$ on $(\theta_*, c_*]$.

Analogously, for any $\tau \in [0, c_*]$ and any i, perturb $\overline{Q}$ iteratively by the reservation operators that reserve both items from any player whose type belongs to $[\tau, c_*]$. By the same reasoning as above, there exists a $\tau_* \in (0, c_*)$ and a $Q^{<\tau_*} \in \mathcal{Q}(Q^*, c)$ for which $\phi(Q^{<\tau_*}) = \phi(Q^*)$ and, for all i, $\left(Q_i^{<\tau_*}\right)^- > 0$ on $[0, \tau_*)$ and $\left(Q_i^{<\tau_*}\right)^- = 0$ on $[\tau_*, c_*]$.

Finally, note that $Q^{>\theta_*}$ and $Q^{<\tau_*}$ are two distinct elements of the set (B.8). Since the set is convex, any convex combination between the two elements also belongs to the set. Thus, the set (B.8) contains a continuum of elements, as asserted.

B.6.4 Proof of Corollary 8

Given any (f, w) such that the bad is needed, according to the proof of Theorem 8, the only case where the relaxed problem (3.34) admits an optimal mechanism as a solution is $V_-^\lambda = 0$ on $(0, c_*)$ for some $c_* \in (0, 1)$ and some $\lambda > 0$. By (3.19), that means $\frac{W(s)}{F(s)+sf(s)} = 1 + \lambda$ for all

$s \in (0, c_*)$, with W derived from w by (3.8). Thus, this case means

$$\exists c_* \in (0,1) : \forall s \in (0, c_*) : \frac{d}{ds} \ln\left(\frac{W(s)}{F(s) + sf(s)}\right) = 0. \tag{B.11}$$

This condition can be violated with slight perturbations of f or w at points near 0. Thus one can formalize the space of (f, w) such that the contrary of (B.11) is generic. ■

B.7 Generalization to ex ante Asymmetric Players

Here we sketch how to generalize the saddle point characterization (Theorem 6), and briefly indicate generalization of the other two theorems, to the asymmetric-player model: each player i's type is independently drawn according to a commonly known, possibly player-specific, cdf F_i with density f_i positive on its support $[0, 1]$; and player i is weighed in the social welfare function according to a possibly player-specific *welfare distribution* $W_i : \mathbb{R} \to \mathbb{R}_+$, which is a weakly increasing function generated by a Radon measure, such that the social welfare from a mechanism (Q, P) is equal to

$$\sum_i \int_0^1 U_i(t_i \mid Q, P) dW_i(t_i). \tag{B.12}$$

There is no loss of generality to assume (B.12) as the social welfare, because any interim Pareto optimal mechanism in this environment is a maximizer of (B.12) subject to IC, IR and BB, for some profile $(W_i)_{i=1}^n$ of distribution functions across players.[1]

Given the general model, it is easy to generalize (3.9) to

$$\begin{aligned} \int_0^1 P_i(t_i) dW_i(t_i) \quad = \quad & \int_0^1 Q_i(t_i) t_i dW_i(t_i) + \int_0^{t_i^0} Q_i(t_i) W_i(t_i) dt_i - \int_{t_i^0}^1 Q_i(t_i) \left(W_i(1) - W_i(t_i)\right) dt_i \\ & - W_i(1) U_i(t_i^0 | Q, P), \end{aligned}$$

where $W_i(1)$ need not be equal to one for all i, because the welfare distributions $(W_i)_{i=1}^n$ may assign different average weights to different players. It is also easy to generalize (3.17) to

$$\sum_i \int_0^1 Q_i(t_i) t_i \, dF_i(t_i) + \sum_i \langle Q_i : \rho(F_i)| \geq 0. \tag{B.13}$$

The first nontrivial difference due to the generalization is that the optimal social welfare (3.16) becomes the following nonlinear functional of the allocation Q:

$$\sum_i \int_0^1 Q_i(t_i) t_i \, d\left(\omega F_i(t_i)\right) + \sum_i \langle Q_i : \rho(\omega F_i) - \rho(W_i)| \,, \tag{B.14}$$

where $\omega := \max_i W_i(1)$.

[1] The proof is in Zheng [24], available upon request.

First,. we observe nonlinearity of (B.14). By the definition of two-part operators, (B.14) is linear in Q if and only if $\langle Q_i : \rho(\omega F_i) - \rho(W_i)|$ is linear in Q_i for each i, and the latter is linear if and only if $\rho_+(\omega F_i) - \rho_+(W_i) = \rho_-(\omega F_i) - \rho_-(W_i)$. By (3.11), that means $W_i(1) = 1$ for all i, which is not necessarily true when W_i is a *distribution* but not a cdf.

Second, we explain why (B.14) is true for the generalization of Lemma 21. Mimicking the proof of Lemma 21, one readily sees that the social welfare (B.12) generated by any IC mechanism (Q, P) is equal to

$$\sum_i W_i(1) \min_{[0,1]} U_i(\cdot|Q, P) - \sum_i \langle Q_i : \rho(W_i)|\,, \tag{B.15}$$

and BB implies

$$\begin{aligned}\sum_i W_i(1) \min_{[0,1]} U_i(\cdot|Q, P) &\leq \left(\max_i W_i(1)\right) \sum_i \min_{[0,1]} U_i(\cdot|Q, P) \\ &\leq \left(\max_i W_i(1)\right)\left(\sum_i \int_0^1 Q_i(t_i)t_i \, dF_i(t_i) + \sum_i \langle Q_i : \rho(F_i)|\right).\end{aligned}$$

By the reason analogous to the proof of Lemma 21, the above-displayed weak inequality holds as equality when P is optimally chosen among those that implement Q: Let $(P_i^Q)_{i=1}^n$ be the payment rule that implements Q with $\min_{[0,1]} U_i(\cdot|Q, P) = 0$ for all i. The ex ante expected revenue generated by $(P_i^Q)_{i=1}^n$ is equal to $\sum_i \int_0^1 Q_i(t_i)t_i \, dF_i(t_i) + \sum_i \langle Q_i : \rho(F_i)|$. Thus, combining $(P_i^Q)_{i=1}^n$ with distributing $\sum_i \int_0^1 Q_i(t_i)t_i \, dF_i(t_i) + \sum_i \langle Q_i : \rho(F_i)|$ to any member of $\arg\max_i W_i(1)$ as lump sums, we obtain a payment rule with which

$$\sum_i W_i(1) \min_{[0,1]} U_i(\cdot|Q, P) = \left(\max_i W_i(1)\right)\left(\sum_i \int_0^1 Q_i(t_i)t_i \, dF_i(t_i) + \sum_i \langle Q_i : \rho(F_i)|\right).$$

Plug this equation into (B.15) and set $\omega := \max_i W_i(1)$ to get (B.14).

Thus, due to asymmetric welfare weights across players, the lump sum transfer in an optimal mechanism is not rebated to players indiscriminately, but rather distributed only to those players whose ex ante expected welfare weights, $W_i(1)$, are largest among all.

Based on the reasoning sketched above, any interim Pareto optimal mechanism is a solution of maximizing (B.14) among $Q \in \mathscr{Q}_{\text{mon}}$ subject to (B.13). As in the proof of Theorem 6, the set of $Q \in \mathscr{Q}_{\text{mon}}$ subject to (B.13) is convex and contains an interior point. The only difference from that proof is that the objective (B.14) is nonlinear in general. However, the objective (B.14) one can prove is a concave functional on $\mathscr{Q}_{\text{mon}}$, hence the conditions corresponding to those in Luenberger [15, Corollary 1, p219] are met, and so the saddle point condition is necessary and sufficient for any solution to this constrained optimization problem.

To prove concavity of the objective (B.14), it suffices to prove that $\langle Q_i : \rho(\omega F_i) - \rho(W_i)|$ is a concave functional of Q_i for each i. The proof is similar to that of Lemma 22. By the definition of two-part operators, we need only to show $\rho_+(\omega F_i) - \rho_+(W_i) \leq \rho_-(\omega F_i) - \rho_-(W_i)$

on [0, 1] for all i: for any $t_i \in [0,1]$, by (3.11),

$$\begin{aligned}(\rho_+(\omega F_i))(t_i) - (\rho_+(W_i))(t_i) &= \omega(-1 + F_i(t_i)) - (-W_i(1) + W_i(t_i)) \\ &= \omega F_i(t_i) - W_i(t_i) - (\omega - W_i(1)) \\ &\leq \omega F_i(t_i) - W_i(t_i) \\ &= \omega(\rho_-(F_i))(t_i) - (\rho_-(W_i))(t_i),\end{aligned}$$

with the inequality due to $\omega = \max_i W_i(1)$.

In sum, in the general asymmetric model, any interim Pareto optimal mechanism is a solution of maximizing (B.14) among $Q \in \mathscr{Q}_{\text{mon}}$ subject to (B.13), and hence satisfies the saddle point condition with respect to the Lagrangian

$$\begin{aligned}\mathscr{L}(Q,\lambda) &:= \sum_i \int_0^1 Q_i(t_i)t_i\, d(\omega F_i(t_i)) + \sum_i \langle Q_i : \rho(\omega F_i) - \rho(W_i)| \\ &\quad +\lambda\left(\sum_i \int_0^1 Q_i(t_i)t_i\, dF_i(t_i) + \sum_i \langle Q_i : \rho(F_i)|\right) \\ &= \sum_i \int_0^1 Q_i(t_i)t_i\, d((\omega+\lambda)F_i(t_i)) + \sum_i \langle Q_i : \rho((\omega+\lambda)F_i) - \rho(W_i)|, \quad \text{(B.16)}\end{aligned}$$

defined for all $Q \in \mathscr{Q}$ and all $\lambda \in \mathbb{R}_+$.

Remark *This saddle point characterization is also a necessary and sufficient condition for any interim Pareto optimal mechanism in the partnership dissolution IPV environment where player i's initial share is θ_i. To see that, interpret the $x_{iA} - x_{iB}$ in (3.1) as player i's net gain in i's share of the partnership, and hence (3.1) is i's* net *payoff from acquiring a net amount $x_{iA} - x_{iB}$ of shares and paying an amount y_i of money. (This payoff is net in the sense that if player i vetoes the dissolution plan then i keeps i's initial share θ_i thereby getting the payoff $\theta_i t_i$.). The only modification on the model is to define an ex post allocation as a function $(q_i)_{i=1}^n : [0,1]^n \to \prod_i[-\theta_i, 1-\theta_i]$ such that $q_i(t)$ is player i's net gain in shares given realized type profile t, with the feasibility condition (3.2) replaced by $\sum_i q_i(t) = 0$ to reflect the market clearance condition on the net trades of shares. This modification, however, has no effect on the saddle point characterization.*

The "if" part of Theorem 7 can also be generalized. The reasoning is analogous to that in Section 3.4.1. For simplicity of exposition, assume that the welfare distributions W_i are all absolutely continuous in F_i with density w_i so that $W_i(1) = 1$ for all i. Suppose that the bad is not allocated at all in an optimal mechanism. Then the generalized saddle point characterization implies that $\lambda = 0$ and so the Lagrangian (B.16) is reduced to (B.14). As noted previously, the assumption $W_i(1) = 1$ for all i implies that (B.14) is a linear functional of Q and hence the proof of the "if" part of Theorem 7 can be easily extended. Thus, any optimal mechanism allocates the bad with a strictly positive probability if

$$\int_0^{t_i}\left(s - \frac{W_i(s) - F_i(s)}{f_i(s)}\right) dF_i(s) < 0$$

for some $t_i \in (0, 1)$ and some player i. One can see that this condition is satisfied if $w_i(0) > 2$ for some player i, which is hence sufficient for the bad to be allocated sometimes given asymmetric players. Thus Corollary 6 is generalized. In the more general case where $W_i(1)$ need not be equal to one for all i, the Lagrangian (B.16) remains to be a nonlinear functional of Q. The argument in that case is much more involved. Nevertheless, one can obtain in that case a sufficient condition for the bad to be allocated sometimes by any optimal mechanism: $w_i(0) > 2 \max_k W_k(1)$ for some player i.

The "only if" part of Theorem 7, as well as Theorem 8, relies on conditions necessary for all—rather than only for some—optimal mechanisms. These conditions we obtain through the perturbation method presented in Appendix B.6.2. There, Lemmas 24 and 25 allow for reduced-form allocations whose cutoffs c_i between positive and negative domains to be different across players i. In addition, one can generalize the two lemmas so that the function g there is player-specific. Thus it is possible that both theorems are generalizable.

Appendix C

Appendices to Chapter 4

C.1 Proof of Lemma 6

Proof When $x_i \geq y_0$, Ineq. (4.3) is $w_i - p_i > x_i$, which guarantees the purchase of i. It thus suffices to show when $y_0 > x_i$, i is purchased if and only if

$$v_i + \min\{z_i, z_i^* + y_0 - x_i\} - p_i > y_0. \tag{C.1}$$

Sufficiency: By Theorem 9, when $y_0 > x_i$, the consumer purchases i if and only if:

1) $w_i - p_i > x_i$, so i is the candidate product in block 1.

2) $v_i + z_i - p_i > y_0$, so the consumer cannot find a product better than i in block 2.

Let's call the two conditions condition 1 and condition 2 respectively.

- Suppose $z_i \leq z_i^* + y_0 - x_i$. By (C.1), condition 2 is satisfied.
  - If $z_i \geq z_i^*$, $w_i - p_i = v_i + z_i^* - p_i \geq \underbrace{v_i + z_i - p_i - y_0}_{>0 \text{ by (C.1)}} + x_i > x_i$. So condition 1 is satisfied.
  - If $z_i < z_i^*$, Ineq. (C.1) implies $w_i - p_i = v_i + z_i - p_0 > y_0 > x_i$. So condition 1 is satisfied.
- Suppose $z_i > z_i^* + y_0 - x_i$. This combined with $y_0 > x_i$ implies $z_i > z_i^*$. Ineq. (C.1) implies $w_i - p_i = v_i + z_i^* - p_i > x_i$. So condition 1 is satisfied. When $z_i > z_i^* + y_0 - x_i$, $v_i + z_i - p_i > v_i + \left(z_i^* + y_0 - x_i\right) - p_i > y_0$, where the second inequality follows from (C.1). So condition 2 is satisfied.

Necessity: Since $y_0 > x_i$, if $v_i + \min\{z_i, z_i^* + y_0 - x_i\} - p_i < y_0$, either $z_i \leq z_i^* + y_0 - x_i$, so $v_i + z_i - p_i < y_0$ (condition 2 is violated), or $z_i > z_i^* + y_0 - x_i$ (thus, $z_i > z_i^*$ by $y_0 > x_i$), so $w_i - p_i = v_i + z_i^* - p_i < x_i$ (condition 1 is violated). That is, if (C.1) is violated, the consumer does not purchase product i in block 1.

C.2 Derivation of (4.9)

The probability of $x_0 \in [u_0, s]$ is:

$$\begin{aligned} H_*^1(s) - H_*^1(u_0) &= \prod_{i \in N_1} H_i(s+p_i) - \prod_{i \in N_1} H_i(u_0+p_i) \\ &= \int_{u_0}^{s} d \prod_{i \in N_1} H_i(w+p_i) \\ &= \sum_{i \in N_1} \int_{u_0}^{s} \prod_{j \in N_1 \backslash \{i\}} H_j(w+p_j) dH_i(w+p_i), \\ &= \sum_{i \in N_1} \left[H_*^1(s) - H_*^1(u_0) - \int_{u_0}^{s} H_i(w+p_i) d \prod_{j \in N_1 \backslash \{i\}} H_j(w+p_j) \right], \quad \text{(C.2)} \\ &\underset{(x_i = w)}{=} \sum_{i \in N_1} \left[\int_{u_0}^{s} H_i(s+p_i) - H_i(x_i+p_i) d\widetilde{H}_i^1(x_i) \right], \end{aligned}$$

where the third equality follows from factorial decomposition, the fourth equality comes from integration by parts and the last line is follow by the definition of $\widetilde{H}_i^1$. Namely, $\widetilde{H}_i^1(x_i)$ have a mass at $x_i = u_0$: for any function f, $\int_{u_0}^{u} f(x) d\widetilde{H}_i^1(x) = \int_{u_0}^{u} f(x) d \prod_{j \in N_1 \backslash \{i\}} H_j(x+p_j)) + f(u_0)\widetilde{H}_i^1(u_0)$.

C.3 Proof of Theorem 10

Under Assumption 1, Theorem 1 and 3 in Bagnoli and Bergstrom (2005) [3] imply for all i, $F_i(\cdot)$, $G_i(\cdot)$, $1 - F_i(\cdot)$ and $1 - G_i(\cdot)$ are log-concave. With Assumption 1 and 2, Proposition 2 in CDK shows both $1 - H_i(\cdot)$ and $H_i(\cdot)$ are log-concave. Since for $k \in \{1, 2\}$, $i \in N_k$ and $x \geq u_0$,

$$\widetilde{H}_i^k(x) = \prod_{j \in N_k \backslash \{i\}} H_j(x+p_j), \quad \text{(C.3)}$$

and log-concave is preserved under product operations, both $\widetilde{H}_i^k(\cdot)$ and $H_*^k(\cdot)$ are log-concave. That is, the distribution of x_i, y_i, x_0 and y_0 are log-concave.

Demand in block 1 By Lemma 6, $D_i^1(\boldsymbol{p})$ is the probability of

$$V_i + \min\{Z_i, z_i^* + (Y_0 - X_i)^+\} - p_i > \max\{X_i, Y_0\}.$$

If two independent random variables have distributions with increasing hazard rates, so does their sum. Since V_i, Z_i, X_i and Y_0 are independent from each other, a sufficient condition of log-concavity of $D_i^1(\boldsymbol{p})$ in p_i is that $\Pr(\max\{X_i, Y_0\} - \min\{Z_i, z_i^* - (Y_0 - X_i)^+\} < t)$ is log-concave in t. Under assumption 2, for any $t > u_0$, the probability can be transferred to:

$$\begin{aligned} & \Pr\left(\max\{X_i, Y_0\} - \min\{Z_i, z_i^* + (Y_0 - X_i)^+\} < t\right) \\ = & \Pr\left(\max\{X_i - z_i^*, X_i - Z_i, Y_0 - Z_i\} < t\right), \\ = & \Pr\left(\{X_i < t + \min\{Z_i, z_i^*\}\} \cap \{Y_0 < t + Z_i\}\right), \\ = & \int_{\underline{z}_i}^{\bar{z}_i} \widetilde{H}_i^1\left(t + \min\{z_i, z_i^*\}\right) H_*^2\left(t + z_i\right) dG_i(z_i), \\ = & \int_{\underline{z}_i}^{\bar{z}_i} \prod_{j \in N_1 \setminus \{i\}} H_j\left(t + \min\{z_i, z_i^*\} + p_j\right) \prod_{j \in N_2} H_j\left(t + z_i + p_j\right) dG_i(z_i), \end{aligned} \tag{C.4}$$

where the first equality follows from exploring the cases that $X_i \geq Y_0$ and $X_i < Y_0$ separately, and the last equality comes from the definition of $\widetilde{H}_i^1$ and H_*^2. Since $\min\{z_i, z_i^*\}$ is a concave function in z_i, and log-concavity is preserved under multiplication, the integrand of (C.4) is log-concave in t, z_i and p_j. By Prékopa-Leinder inequality, log-concavity is preserved under integration. This, coupled with g_i is log-concave (Assumption 1), implies (C.4) is log-concave and log-supermodular in t and p_j for any $j \neq i$, as both t and p_j are additive seperatble terms in H_j in (C.4).[1] Quint (2014) [18]'s Theorem 1 applies and $D_i^1(\boldsymbol{p})$ is log-supermodular in p_i and p_j for any $i \neq j$. [2]

Demand in block 2 $D_i^2(\boldsymbol{p})$ is the probability of

$$W_i - p_i > \max\{U_1, Y_i\}$$

I show $J(\cdot)$ is log-concave under Assumption 1, 2 and 3, the result then follows by the same arguments as that for block 1 demand. By (4.10),

$$J(u) = H_*^1(u) - \mathcal{K}(u).$$

Since $H_*^1(u) = \prod_{i \in N_1} H_i(u + p_i)$ is log-concave in both u and p_i, it suffices to prove the sign of $(\log(H_*^1(u) - \mathcal{K}(u)))''$ is the same as the sign of $(\log H_*^1(u))''$ when the variance of V_i is large enough.

Lemma 27 *For any $u \geq u_0$, $\mathcal{K}(u)$ defined in (4.10) can be written as:*

$$\mathcal{K}(u) = \sum_{i \in N_1} \int_{u_0}^{u} \widetilde{H}_i^1(x_i)(1 - G_i(z_i^* + u - x_i)) dF_i(x_i + p_i - z_i^*). \tag{C.5}$$

And the derivative is

$$\begin{aligned} \frac{d\mathcal{K}(u)}{du} = & \sum_{i \in N_1} \widetilde{H}_i^1(u)\left(1 - G_i(z_i^*)\right) f_i(u + p_i - z_i^*) \\ & - \sum_{i \in N_1} \int_{u_0}^{u} \widetilde{H}_i^1(x_i) g_i(z_i^* + u - x_i) dF_i(x_i + p_i - z_i^*). \end{aligned} \tag{C.6}$$

[1] By Quint's definition, a function is log-supermodular if its log is supermodular.

[2] In specific, one can replace $G_2(t)$ in the proof of Quint's Theorem 1 by (C.4).

Proof By (4.10), and the definition of $\widetilde{H}_i^1$,

$$\mathcal{K}(u) = \sum_{i\in N_1}\left[\int_{u_0}^{u} H_i(u+p_i) - \widehat{H}_i(u+p_i, u-x_i)\right] d\prod_{j\in N_1\setminus\{i\}} H_j(x_i+p_j)$$
$$+\sum_{i\in N_1}\left[H_i(u_0+p_i) - \widehat{H}_i(u+p_i, u-u_0)\right]\widetilde{H}_i^1(u_0),$$

which by integration by parts and $\widetilde{H}_i^1(x_i) = \prod_{j\in N_1\setminus\{i\}} H_j(x_i+p_j)$ on $[u_0, u]$, equals

$$\begin{aligned}\mathcal{K}(u) &= -\sum_{i\in N_1}\int_{u_0}^{u}\widetilde{H}_i^1(x_i)d_x\left[\widehat{H}_i(u+p_i, u-x_i) - H_i(u+p_i)\right],\\ &= -\sum_{i\in N_1}\int_{u_0}^{u}\widetilde{H}_i^1(x_i)d_x\widehat{H}_i(u+p_i, u-x_i),\end{aligned}$$

where $d_x\widehat{H}_i(u+p_i, u-x_i)$ is the derivative with respect to x_i, which is

$$\begin{aligned}\frac{d}{dx_i}\widehat{H}_i(u+p_i, u-x_i) &= \frac{d}{dx_i}\left[(1-G_i(z_i^*+u-x_i))F_i(x_i+p_i-z_i^*)\right]\\ &\quad+\frac{d}{dx_i}\int_{\underline{z}_i}^{z_i^*+u-x_i} F_i(u+p_i-z_i)dG_i(z_i),\\ &= (1-G_i(z_i^*+u-x_i))f_i(x_i+p_i-z_i^*),\end{aligned}$$

if $z_i^*+u-x_i \le \bar{z}_i$, and equals zero otherwise. Thus

$$\mathcal{K}(u) = \sum_{i\in N_1}\int_{u_0}^{u}\widetilde{H}_i^1(x_i)(1-G_i(z_i^*+u-x_i))dF_i(x_i+p_i-z_i^*).$$

The derivative, by Leibniz rule, is exactly (C.6).

Let $V_i^\sigma = \sigma V_i$ with distribution $F_i^\sigma(v) = F_i(v/\sigma)$, and $W_i^\sigma = V_i^\sigma + \min\{Z_i, z_i^*\}$. Denote the distribution of $\max_{j\in N_1\setminus\{i\}} W_j^\sigma$ as $(\widetilde{H}_i^1)^\sigma$. By Lemma 27, $\mathcal{K}$ with V_i^σ can be written as:

$$\mathcal{K}^\sigma(u) = \sum_i\int_{u_0}^{u}(\widetilde{H}_i^1)^\sigma(x_i)\left(1-G_i(z_i^*+u-x_i)\right)dF_i^\sigma(x_i+p_i-z_i^*).$$

Denote $b_i := F_i^\sigma(u+p_i-z_i^*)$, $a_i := F_i^\sigma(u_0+p_i-z_i^*)$, and $r_i := F_i^\sigma(x_i+p_i-z_i^*)$, by change of variable:

$$\mathcal{K}^\sigma(u) = \sum_i\int_{a_i}^{b_i}(\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(r_i)+z_i^*-p_i)\left(1-G_i(z_i^*+(F_i^\sigma)^{-1}(b_i)-(F_i^\sigma)^{-1}(r_i))\right)dr_i.$$

With $F_i^\sigma(v_i) = F_i(v_i/\sigma)$, we have $(F_i^\sigma)^{-1}(r) = \sigma F_i^{-1}(r)$, $f_i^\sigma((F_i^\sigma)^{-1}(r)) = f_i(F_i^{-1}(r))/\sigma$ and $(f_i^\sigma)'(F_i^{-1}(r)) = f_i(F_i^{-1}(r))/\sigma^2$. Thus,

$$\mathcal{K}^\sigma(u) = \sum_i\int_{a_i}^{b_i}(\widetilde{H}_i^1)^\sigma(\sigma F_i^{-1}(r_i)+z_i^*-p_i)\left(1-G_i(z_i^*+\sigma(F_i^{-1}(b_i)-F_i^{-1}(r_i))\right)dr_i,$$

Since $(\widetilde{H}_i^1)^\sigma$ is bounded between zero and one, given $F_i^{-1}(b_i) > F_i^{-1}(r_i)$, $\mathcal{K}^\sigma$ converges to zero when σ explodes as $\left(1 - G_i(z_i^* + \sigma(F_i^{-1}(b_i) - F_i^{-1}(r_i))\right)$ goes to zero.

The derivative of $\mathcal{K}^\sigma(u)$ is

$$\begin{aligned}\frac{d\mathcal{K}^\sigma(u)}{du} &= \sum_{i\in N_1} (\widetilde{H}_i^1)^\sigma(u)\,(1 - G_i(z_i^*))\, f_i^\sigma(u + p_i - z_i^*) \\ &\quad - \sum_{i\in N_1} \int_{u_0}^{u} (\widetilde{H}_i^1)^\sigma(x_i) g_i(z_i^* + u - x_i) dF_i^\sigma(x_i + p_i - z_i^*). \end{aligned} \tag{C.7}$$

Rewrite the above expression in terms of a_i, b_i and r_i:

$$\begin{aligned} &\sum_{i\in N_1} (\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(b_i) + z_i^* - p_i)\,(1 - G_i(z_i^*))\, f_i^\sigma\left((F_i^\sigma)^{-1}(b_i)\right) \\ &\qquad - \sum_{i\in N_1} \int_{a_i}^{b_i} (\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(r_i) + z_i^* - p_i) g_i(z_i^* + (F_i^\sigma)^{-1}(b_i) - (F_i^\sigma)^{-1}(r_i)) dr_i, \end{aligned}$$

where the first term converges to zero when σ explodes since $f_i^\sigma((F_i^\sigma)^{-1}(b_i)) = f_i(F_i^{-1}(b_i))/\sigma$ and $(\widetilde{H}_i^1)^\sigma$ is bounded between zero and one. The second term is negative as all terms in its integrand has non-negative value. That is, the first order derivative of $\mathcal{K}^\sigma$ is non-positive when σ is large enough.

Similarly, we can express the second order derivative of $\mathcal{K}^\sigma$ as:

$$\begin{aligned}\frac{d^2}{du^2}\mathcal{K}^\sigma(u) &= \sum_{i\in N_1} \left((\widetilde{H}_i^1)^\sigma(u)\right)' (1 - G_i(z_i^*)) f_i^\sigma(u + p_i - z_i^*) \\ &\quad + \sum_{i\in N_1} (\widetilde{H}_i^1)^\sigma(u)\,(1 - G_i(z_i^*))\,(f_i^\sigma)'(u + p_i - z_i^*) \\ &\quad - \sum_{i\in N_1} (\widetilde{H}_i^1)^\sigma(u) g_i(z_i^*) f_i^\sigma(u + p_i - z_i^*) \\ &\quad - \sum_{i\in N_1} \int_{u_0}^{u} (\widetilde{H}_i^1)^\sigma(x_i) g_i'(z_i^* + u - x_i) dF_i^\sigma(x_i + p_i - z_i^*), \end{aligned}$$

which, in terms of a_i, b_i and r_i is:

$$\begin{aligned}\frac{d^2}{du^2}\mathcal{K}^\sigma(u) &= \sum_{i\in N_1} \left((\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(b_i) + z_i^* - p_i)\right)' (1 - G_i(z_i^*)) f_i^\sigma((F_i^\sigma)^{-1}(b_i)) \\ &\quad + \sum_{i\in N_1} (\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(b_i) + z_i^* - p_i)\,(1 - G_i(z_i^*))\,(f_i^\sigma)'((F_i^\sigma)^{-1}(b_i)) \\ &\quad - \sum_{i\in N_1} (\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(b_i) + z_i^* - p_i) g_i(z_i^*) f_i^\sigma((F_i^\sigma)^{-1}(b_i)) \\ &\quad - \sum_{i\in N_1} \int_{u_0}^{u} (\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(r_i) + z_i^* - p_i) g_i'(z_i^* + (F_i^\sigma)^{-1}(b_i) - (F_i^\sigma)^{-1}(r_i)) dr_i, \end{aligned}$$

where the second and third term converges to zero when σ is large enough as $f_i^\sigma((F_i^\sigma)^{-1}(b)) = f_i(F_i^{-1}(b))/\sigma$ and $(f_i^\sigma)'(F_i^{-1}(b)) = f_i(F_i^{-1}(b))/\sigma^2$ and $(\widetilde{H}_i^1)^\sigma$ is bounded between zero and one.

The first term is zero as $H_i^\sigma(w_i + p_i)$ converges to either zero or $F_i^\sigma(w_i - z_i^* + p_i)$ (shown by CDK in the supplementary material). So $\left((\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(b_i) + z_i^* - p_i)\right)'$ is either converges to zero or

$$\sum_{j\in N_1\setminus\{i\}} \prod_{k\in N_1\setminus\{i,j\}} F_k^\sigma((F_i^\sigma)^{-1}(b_i) + z_i^* - p_i - z_k^* + p_k) f_j^\sigma((F_i^\sigma)^{-1}(b_i) + z_i^* - p_i - z_j^* + p_j),$$

which converges to zero since for any $q \in \mathbb{R}$,

$$F_k^\sigma((F_i^\sigma)^{-1}(b_i) + q) = F_k\left(\frac{\sigma F_i^{-1}(b_i) + x}{\sigma}\right) \underset{\sigma\to\infty}{\longrightarrow} F_k(F_i^{-1}(b_i)),$$

which is finite, and

$$f_j^\sigma((F_i^\sigma)^{-1}(b_i) + q) = \frac{1}{\sigma} f_i\left(\frac{\sigma F_i^{-1}(b_i) + x}{\sigma}\right) \underset{\sigma\to\infty}{\longrightarrow} 0.$$

The last term is non-negative as $g_i'(z_i^* + (F_i^\sigma)^{-1}(b_i) - (F_i^\sigma)^{-1}(r_i))$ converges to $g_i'(\overline{z}_i)$ when σ explodes by Assumption 3 (either $g_i'(\overline{z}_i) \le 0$ or $\overline{z}_i = \infty$).

Putting all things together, we have $\mathcal{K}^\sigma(u)$ converges to zero, $\mathcal{K}^\sigma(u)'$ is non-positive and $\mathcal{K}^\sigma(u)''$ is non-negative when σ is large enough. Thus,

$$\begin{aligned}
\lim_{\sigma\to\infty}(\log J(u))'' &= \lim_{\sigma\to\infty} \frac{((H_*^1)^\sigma(u) - \mathcal{K}^\sigma(u))''((H_*^1)^\sigma(u) - \mathcal{K}^\sigma(u)) - \left((H_*^1)^\sigma(u)' - \mathcal{K}^\sigma(u)'\right)^2}{((H_*^1)^\sigma(u) - \mathcal{K}^\sigma(u))^2}, \\
&\le \lim_{\sigma\to\infty} \frac{(H_*^1)^\sigma(u)''(H_*^1)^\sigma(u) - \left((H_*^1)^\sigma(u)'\right)^2}{((H_*^1)^\sigma(u))^2}, \\
&= \lim_{\sigma\to\infty}(\log H_*^1(u))'',
\end{aligned}$$

which is negative. Thus,

$$\begin{aligned}
\Pr(\max\{u_1, Y_i\} < t) &= J(t)\widetilde{H}_i^2(t), \\
&= J(t) \prod_{j\in N_2\setminus\{i\}} H_j(t + p_j)
\end{aligned}$$

is a product of two log-concave functions, and is thus log-concave. Log-supermodular of demand follows the same arguments before.

C.4 Proof of Theorem 11

Proof To prove the theorem, I provide a tâtonnement process that reaches an equilibrium:

1. Seller $j \in N_2^0$ submits bid b_j. Fix any $j' \in \arg\max_{j\in N_2^0} b_j$.
2. Seller $i \in N_1^0$ submits bid $b_i^{j'}$.
   - If $\max_i b_i^{j'} \ge b_{j'}$, the auction ends. The stable position is (N_1^0, N_2^0).

- Else, proceed to step 3:

3. Seller $j \in N_2^0 \setminus \{j'\}$ submits $b_j^{j'}$.

   - If $b_{j'} \geq b_j^{j'}$ for all $j \in N_2^0$, the auction ends. The stable position is $(N_1^0 + j' - l(j'), N_2^0 - j' + l(j'))$.
   - Else, let $j'' := \arg\max_{j \in N_2^0 \setminus \{j'\}} b_j^{j'}$ and repeat step 2 and 3 by replacing j' with j''.

The above process eventually reaches an equilibrium since it stops if and only if the winner's bid exceeds all other sellers' willingness to pay for the sponsored link position under the current position. The equilibrium exits because whenever $b_j^{j'} > b_{j'}$ in step 3, the current highest bid strictly increases (from $b_{j'}$ to $b_{j''}^{j'}$). So it is impossible to have an infinite loop in steps 2-3, as all sellers have finite profits.

C.5 Proof of Theorem 12

Proof The bidding equilibrium exists if there exits the winner r and the runner-up j, and the winning bid b, such that (4.13) and (4.14) are satisfied. The proof is constructive to find r, j and b. First, there exists $r \in N_1^0$ and $j \in N_2^0$ such that $r = l(j)$ and (4.14) holds with sufficiently small $b > 0$. This is because r's position is moved from block 1 into block 2 if j wins. And in the extreme case where $b \to 0$, seller r can get the sponsored link for free and is strictly better off to win the auction. Now, with (4.14) holds for some r, j and b, the equilibrium is constructed by the following process:

1. If (4.13) also holds for r and b then the equilibrium exists in which r wins with bid b.

2. Else, violation of (4.13) means there exists another j' such that

$$\pi_{j'}^{j'}(b) - \pi_{j'}^{r}(b) > 0, \tag{C.8}$$

   which is (4.14) under strict inequality. Since the left-hand side of (C.8) is decreasing in b, there exists $b' > b$ such that (C.8) holds with equality.

3. Since $\pi_{j'}^{j'}(b') - \pi_{j'}^{i}(b') = 0$. Let j' be the current winner who submit a bid b' and r be the runner-up: replace j by r, r by j' and b by b'. So (4.14) holds with equality. Then go back to step 1.

An equilibrium is constructed following the 3-step procedure above. Every time step 3 is activated, the current standing bid increases. Whenever the process stops, we find r, j and b such that (4.13) and (4.14) are satisfied, so an bidding equilibrium exists.

The only case that an equilibrium does not exist is that the three-step process above forms a loop that never ends. That is, there is a sequence of sellers $(i_1, i_2, \ldots, i_m)$ and a corresponding sequence of bids $(b_1, b_2, \ldots, b_m)$ such that $\pi_{i_s}^{i_s}(b_{s-1}) > \pi_{i_s}^{i_{s-1}}(b_{s-1})$ for any $2 \leq s \leq m$, and $\pi_{i_1}^{i_1}(b_m) > \pi_{i_1}^{i_m}(b_m)$, so there is a cyclic relation in (C.8).

However, since b_s are adjusted in the way that $\pi_{i_s}^{i_s}(b_s) = \pi_{i_s}^{i_{s-1}}(b_s)$ for any $s \in \{2, \ldots, m\}$, and $\pi_{i_1}^{i_1}(b_1) = \pi_{i_1}^{i_m}(b_1)$, monotonicity of the left-hand side of (C.8) in b implies $b_s > b_{s-1}$ for $s \in \{2, \ldots, m\}$ and $b_1 > b_m$. A contradiction.

C.6 Pricing Equilibrium with Symmetric Sellers

Under the fixed payment, the price equilibrium in Section 4.6 is that all sellers in block 1 set an identical price at p_1^f and all sellers in block 2 set an identical price at p_2^f.

Suppose not. There exists seller $j \in N_1$ such that $p_j \neq p_1^f$ and all other sellers following the equilibrium price. Let $\boldsymbol{p}$ be the corresponding vector of prices, then Theorem 10 implies

$$\frac{1}{p_1^f - c} = -\left.\frac{dD_i^1(p, \boldsymbol{p}_{-i})/dp}{D_i^1(p, \boldsymbol{p}_{-i})}\right|_{p=p_1^f}, \tag{C.9}$$

for any $i \in N_1 \setminus \{j\}$ and

$$\frac{1}{p_j - c} = -\left.\frac{dD_j^1(p, \boldsymbol{p}_{-j})/dp}{D_j^1(p, \boldsymbol{p}_{-j})}\right|_{p=p_j}. \tag{C.10}$$

However, (C.9) and (C.10) cannot hold simultaneously. This is because by Theorem 10, for any $i \neq j$ and $k \in \{1, 2\}$, $\partial^2 \log D_i^k(\boldsymbol{p})/\partial p_i \partial p_j > 0$. If $p_j > p_1^f$ and $p_i = p_1^f$ for any $i \in N_1 \setminus \{j\}$, then

$$-\frac{dD_i^1(p, \boldsymbol{p}_{-i})/dp}{D_i^1(p, \boldsymbol{p}_{-i})} < -\frac{dD_j^1(p, \boldsymbol{p}_{-j})/dp}{D_j^1(p, \boldsymbol{p}_{-j})},$$

for any p, since $\boldsymbol{p}_{-i}$ contains $p_j > p_1^f$, while $\boldsymbol{p}_{-j}$ only contains p_1^f. Since the right-hand sides of both FOCs are increasing in p and $1/(p - c)$ is decreasing in p, the solution of the FOCs implies $p_j < p_1^f$. A contradiction.

Similarly, $p_j > p_1^f$ can never hold. So all sellers in block 1 charge the same price. Sellers in block 2 charge the same price by the same reason. The uniqueness of the pricing equilibrium under per-transaction follows from the same arguments and is thus omitted.

C.7 Proof of Theorem 14

Theorem 14 is proved by separating consumer surplus in (4.18) into two parts. One is independent of product position and the other depends on product position. The latter one turns out to be (4.20).

Step 1: reformulate the first sum in (4.18): I use $\wedge$ to denote the intersection of sets and $\vee$ to denote the union of sets. By definition of z_i^* in (4.1), $s_i = \mathbb{E}[\left(Z_i - z_i^*\right)^+]$. The first sum in (4.18) can be rewritten as:

$$
\begin{aligned}
& \sum_{i\in N_1} \mathbb{E}\left[\mathbb{1}_{\{\{X_i<W_i-p_i\}\wedge\{Y_0<V_i+Z_i-p_i\}\}}(V_i+Z_i-p_i) - \mathbb{1}_{\{X_i<V_i+z_i^*-p_i\}}\left(Z_i-z_i^*\right)^+\right] \\
= & \sum_{i\in N_1} \mathbb{E}\Big[\Big(\mathbb{1}_{\{\{\max\{X_i,Y_0\}<W_i-p_i\}} + \mathbb{1}_{\{X_i<W_i-p_i\leq Y_0<V_i+Z_i-p_i\}}\Big)(V_i+Z_i-p_i) \\
& - \mathbb{1}_{\{X_i<W_i-p_i\}}\left(Z_i-z_i^*\right)^+\Big], \\
= & \sum_{i\in N_1} \mathbb{E}\Big[\Big(\mathbb{1}_{\{\{\max\{X_i,Y_0\}<W_i-p_i\}} + \mathbb{1}_{\{X_i<W_i-p_i\leq Y_0<V_i+Z_i-p_i\}}\Big)(V_i+Z_i-p_i) \\
& - \Big(\mathbb{1}_{\{\max\{X_i,Y_0\}<W_i-p_i\}} + \mathbb{1}_{\{\{X_i<W_i-p_i\}\wedge\{Y_0\geq W_i-p_i\}\}}\Big)\left(Z_i-z_i^*\right)^+\Big], \\
= & \sum_{i\in N_1} \mathbb{E}\left[\mathbb{1}_{\{\{\max\{X_i,Y_0\}<W_i-p_i\}}(W_i-p_i)\right] + \sum_{i\in N_1} \mathbb{E}\left[\mathbb{1}_{\{X_i<W_i-p_i\leq Y_0<V_i+Z_i-p_i\}}(V_i+Z_i-p_i)\right] \\
& - \sum_{i\in N_1} \mathbb{E}\left[\mathbb{1}_{\{X_i<W_i-p_i\leq Y_0\}}\left(Z_i-z_i^*\right)^+\right],
\end{aligned}
\tag{C.11}
$$

where the first equality comes from:

$$
\begin{aligned}
& \{X_i < W_i - p_i\} \wedge \{Y_0 < V_i + Z_i - p_i\} \\
= & \{\max\{X_i, Y_0\} < W_i - p_i\} \vee \{X_i < W_i - p_i \leq Y_0 < V_i + Z_i - p_i\},
\end{aligned}
$$

and $\mathbb{1}_{\{X_i<V_i+z_i^*-p_i\}}\left(Z_i-z_i^*\right)^+ = \mathbb{1}_{\{X_i<W_i-p_i\}}\left(Z_i-z_i^*\right)^+$.[3]

Step 2: reformulate the second sum in (4.18): Similarly, by $s_i = \mathbb{E}[\max\{0, z_i - z_i^*\}]$, the second sum in (4.18) can be rewritten as:

$$
\begin{aligned}
& \sum_{i\in N_2} \mathbb{E}\left[\mathbb{1}_{\{\max\{U_1,Y_i\}<W_i-p_i\}}(V_i+Z_i-p_i) - \mathbb{1}_{\{\max\{U_1,Y_i\}<V_i+z_i^*-p_i\}}\left(Z_i-z_i^*\right)^+\right] \\
= & \sum_{i\in N_2} \mathbb{E}\left[\mathbb{1}_{\{\max\{U_1,Y_i\}<W_i-p_i\}}(W_i-p_i)\right] \\
= & \sum_{i\in N_2} \mathbb{E}\Big[\Big(\mathbb{1}_{\{\max\{X_0,Y_i\}<W_i-p_i\}} - \mathbb{1}_{\{\{X_0\leq W_i-p_i\leq U_1\}\wedge\{Y_i<W_i-p_i\}\}}\Big)(W_i-p_i)\Big],
\end{aligned}
\tag{C.12}
$$

where the first equality uses the fact that

$$
\mathbb{1}_{\{\max\{U_1,Y_i\}<V_i+z_i^*-p_i\}}\left(Z_i-z_i^*\right)^+ = \mathbb{1}_{\{\{\max\{U_1,Y_i\}<W_i-p_i\}}\left(Z_i-z_i^*\right)^+,
$$

and the second equality uses the fact that $u_1 \geq x_0$ for any realization of $(\boldsymbol{v}_k, \boldsymbol{z}_k)_{k=1}^2$ and $\boldsymbol{p}$.

Step 3: rearrangement: Since (4.18) is equal to $\mathbb{E}\left[\mathbb{1}_{\{\max\{X_0,Y_0\}<u_0\}}\right] u_0$ plus (C.11) plus (C.12),

[3] If $z_i \leq z_i^*$, both terms equal zero. If $z_i > z_i^*$, $w_i = v_i + z_i^*$, the indicators are the same. The logic is borrowed from CDK (2018).

consumer surplus can be expressed as $W = \overline{W} - W^L$, where

$$
\begin{aligned}
W^L &:= -\sum_{i\in N_1}\mathbb{E}\left[\mathbb{1}_{\{X_i<W_i-p_i\le Y_0<V_i+Z_i-p_i\}}(V_i+Z_i-p_i)\right] \\
&\quad +\sum_{i\in N_1}\mathbb{E}\left[\mathbb{1}_{\{X_i<W_i-p_i\le Y_0\}}\left(Z_i-z_i^*\right)^+\right]+\sum_{i\in N_2}\mathbb{E}\left[\mathbb{1}_{\{\max\{X_0,Y_i\}<W_i-p_i\le U_1\}}(W_i-p_i)\right], \\
\overline{W} &:= \mathbb{E}\left[\mathbb{1}_{\{\max\{X_0,Y_0\}<u_0\}}\right]u_0+\sum_{i\in N_1}\mathbb{E}\left[\mathbb{1}_{\{\{\max\{X_i,Y_0\}<W_i-p_i\}}(W_i-p_i)\right], \\
&\quad +\sum_{i\in N_2}\mathbb{E}\left[\mathbb{1}_{\{\max\{X_0,Y_i\}<W_i-p_i\}}(W_i-p_i)\right] \\
&= \mathbb{E}\left[\max\{u_0,X_0,Y_0\}\right].
\end{aligned}
$$

We can further simplify W^L into an expression of X_0, Y_0 and U_1:

$$
\begin{aligned}
W^L &= -\sum_{i\in N_1}\mathbb{E}\left[\left(\mathbb{1}_{\{X_i<W_i-p_i\le Y_0\}}-\mathbb{1}_{\{X_i<W_i-p_i\le V_i+Z_i-p_i\le Y_0\}}\right)(V_i+Z_i-p_i)\right] \\
&\quad +\sum_{i\in N_1}\mathbb{E}\left[\mathbb{1}_{\{X_i<W_i-p_i\le Y_0\}}\left(Z_i-z_i^*\right)^+\right]+\sum_{i\in N_2}\mathbb{E}\left[\mathbb{1}_{\{\max\{X_0,Y_i\}<W_i-p_i\le U_1\}}(W_i-p_i)\right], \\
&= \sum_{i\in N_1}\mathbb{E}\left[\mathbb{1}_{\{X_i<W_i-p_i\le V_i+Z_i-p_i\le Y_0\}}(V_i+Z_i-p_i)\right]-\sum_{i\in N_1}\mathbb{E}\left[\mathbb{1}_{\{X_i<W_i-p_i\le Y_0\}}(W_i-p_i)\right] \\
&\quad -\left(\sum_{i\in N_2}\mathbb{E}\left[\mathbb{1}_{\{\max\{U_1,Y_i\}<W_i-p_i\}}(W_i-p_i)\right]-\sum_{i\in N_2}\mathbb{E}\left[\mathbb{1}_{\{\max\{X_0,Y_i\}<W_i-p_i\}}(W_i-p_i)\right]\right), \\
&= \mathbb{E}[\mathbb{1}_{\{U_1\le Y_0\}}U_1]-\mathbb{E}[\mathbb{1}_{\{X_0\le Y_0\}}X_0]-\left(\mathbb{E}[\mathbb{1}_{\{U_1\le Y_0\}}Y_0]-\mathbb{E}[\mathbb{1}_{\{X_0\le Y_0\}}Y_0]\right), \\
&= \mathbb{E}[\mathbb{1}_{\{X_0\le Y_0\}}(Y_0-X_0)]-\mathbb{E}[\mathbb{1}_{\{U_1\le Y_0\}}(Y_0-U_1)], \\
&= \mathbb{E}\left[(Y_0-X_0)^+\right]-\mathbb{E}\left[(Y_0-U_1)^+\right], \qquad \text{(C.13)}
\end{aligned}
$$

where the first equality comes from $W_i - p_i = V_i + Z_i - p_i - \left(Z_i - z_i^*\right)^+$ and the fact that

$$\{\max\{X_0, Y_i\} < W_i - p_i\} = \{\max\{U_1, Y_i\} < W_i - p_i\} \wedge \{\max\{X_0, Y_i\} < W_i - p_i \le U_1\},$$

as $u_1 > x_0$ under any realization, and the third equality follows from interchanging sum and expectation in a linear operator and the definitions of X_0, Y_0 and U_1.

Curriculum Vitae

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