Non-Circular Hydraulic Jumps Due to Inclined Jets

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Abstract

When a laminar inclined circular jet impinges on a horizontal surface, it forms a non-circular hydraulic jump governed by a non-axisymmetric flow. In this thesis, we use the boundary-layer and thin-film approaches in the three dimensions to theoretically analyse such flow and the hydraulic jumps produced in such cases. We particularly explore the interplay among inertia, gravity, and the effective inclination angle on the non-axisymmetric flow. The boundary-layer height is found to show an azimuthal dependence at strong gravity level only; however, the thin film thickness as well as the hydraulic jump profile showed a strong non-axisymmetric behaviour at all gravity levels. Interestingly, at the free surface, radial and azimuthal velocity components are found prior to the jump location. The presence of an azimuthal velocity component is unique to non-circular jumps. Finally, a comparison between the current work and the existing literature is presented.

Keywords

inclined impinging jet, non-circular hydraulic jumps, non-axisymmetric flow, three-dimensional boundary layers, three-dimensional thin films, free surface flow, interfacial flow.
Summary for Lay Audience

When a vertical liquid column impacts a solid horizontal surface, a thin film of the liquid spreads on the surface in a symmetric manner with respect to the liquid column. At a certain distance from the point of impact on the horizontal surface, the thickness of this film rises abruptly, this is known as a “hydraulic jump”. It is also referred as a “circular hydraulic jump” due to the circular form it takes with respect to the point of liquid impact on the surface. Circular hydraulic jumps are commonly found in kitchen sinks when water discharges from the tap. On the other hand, when the liquid column is tilted by a certain angle, different geometries, other than the circular one, are expected to be produced, such as an ellipse. The underlying fluid flow dynamics that caused a shift in the hydraulic jump shape has been a challenge for scientists and engineers to theoretically explain, due to the complexity of the flow in this case.

In many engineering applications, jet impingement is used for cooling hot surfaces, where a jet of fluid impacts a hot surface to cool it. It was noticed that the occurrence of hydraulic jumps in such applications negatively affects the performance of cooling. Although circular hydraulic jumps have been extensively studied in the literature, non-circular jumps produced due to inclined jets is have not been given enough attention.

In this thesis, we develop the mathematical and physical formulations needed to analyse the phenomenon of the “non-circular” hydraulic jumps. We study the effect of several parameters on the shape of the formed hydraulic jumps, such as the gravity effect and the inclination degree of the liquid column as it impacts the solid surface. At the edge of the thin film, surface velocities with different directions are observed. Also, it has been found that the thickness of the thin film shows a different behaviour as perceived from different angles with respect to the point of fluid impact on the surface. These observations are found to characterise the flow of liquid in cases where non-circular hydraulic jump occurs.
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Nomenclature

a  Radius of the jet, m
a/b  Hydraulic jump profile aspect ratio
A  Azimuthal shear stress on the disk surface, dimensionless
\( \bar{A} \)  Rescaled azimuthal shear stress on the disk surface, dimensionless
F  Function of free surface, \( F = z - h \)
Fr  Froude number, \( W/\sqrt{ag} \)
Frh  Depth Froude number, \( W/\sqrt{hg} \)
g  Gravitational acceleration, m/s\(^2\)
h*  Film thickness, dimensional
h  Film thickness, dimensionless
\( \bar{h} \)  Rescaled film thickness, dimensionless
hj  Film thickness immediately before the jump, dimensionless
H  Fluid depth downstream of the jump, dimensionless
Hj  Film thickness immediately downstream the jump, dimensionless
Ha  Nozzle to disk drop height, m
Ho  Final fluid depth at the edge of the disk, dimensionless
L  Length scale chosen in the horizontal direction, m
m  Ellipse minor axis, m
M  Ellipse major axis, m
\( \vec{n} \)  Normal vector to a chosen surface
P  Pressure, dimensionless
P*  Pressure, N/m\(^2\)
q  Volume flux, \( Q/2\pi \), m\(^3\)/s
Q  Jet volumetric flow rate, m\(^3\)/s
R  Inclined jet cross section profile, dimensionless
r  Radial coordinate, m
r*  Radial coordinate, dimensionless
\bar{r}  Rescaled radial coordinate, dimensionless
\bar{\bar{r}}  Radial coordinate based on Kate et al. (2007) scaling, dimensionless
r_0  Transition point location, dimensionless
r_{e*}  Impingement region profile, m
r_e  Impingement region profile, dimensionless
r_j  Hydraulic jump location, dimensionless
\bar{r}_j  Rescaled hydraulic jump location, dimensionless
\bar{\bar{r}}_j  Hydraulic jump location based on Kate et al. (2007) scaling, dimensionless
r_\infty  Disk radius, dimensionless
Re  Reynolds number, wa/ν
S  Stagnation point
S_\delta  Stagnation point shift, m
u  Radial velocity component, dimensionless
u*  Radial velocity, m/s
V  Jet velocity based on Kate et al. (2007) scaling, m
v  Azimuthal velocity, dimensionless
v*  Azimuthal velocity, m/s
W  Jet velocity, m/s
W_r  Radial component of jet velocity, m/s
w  Vertical velocity, dimensionless
w*  Vertical velocity, m/s
z  Vertical coordinate, dimensionless
z*  Vertical coordinate, m
Greek symbols

\(\alpha\) Dimensionless parameter, \(\text{Re}^{1/3}\text{Fr}^2\)

\(\delta\) Boundary layer thickness, dimensionless

\(\bar{\delta}\) Rescaled boundary layer thickness, dimensionless

\(\epsilon\) Perturbation parameter, dimensionless

\(\phi\) Effective jet inclination angle

\(\phi_p\) Jet nozzle inclination angle

\(\eta\) Scaled vertical coordinate, \(z/h\)

\(\mu\) Dynamic viscosity, Kg/m.s

\(\nu\) Kinematic viscosity \(m^2/s\)

\(\theta\) Azimuthal coordinate

\(\rho\) Fluid density, kg/m\(^3\)

\(\sigma^*\) Normal stress, N/m\(^2\)

\(\tau^*\) Tangential stress, N/m\(^2\)

\(\tau_{rz}\) Radial shear stress on the disk surface, dimensionless

\(\tau_{\theta z}\) Rescaled radial shear stress on the disk surface, dimensionless

\(\bar{\tau}_{\theta z}\) Azimuthal shear stress on the disk surface, dimensionless


1 Introduction

1.1 Introduction

This chapter provides a simple description of hydraulic jumps and their importance in the design of relevant engineering applications. The different types and classifications of hydraulic jumps are then presented. Afterwards, the main theories that underlie the study of hydraulic jumps are discussed. A thorough literature review is then presented before concluding this chapter with the thesis objectives and outline.

1.2 Background and applications

1.2.1 Background

When a jet discharges from a normal circular nozzle and impinges on a flat disk, a radial flow is observed forming a thin film. The thickness of the film keeps decreasing gradually until a sudden increase takes place, this is known as a “circular hydraulic jump” (Figure 1-1). The upstream region of the jump is usually referred to as the “supercritical region” in which the surface velocity of the fluid is relatively high. On the other hand, the downstream region of the jump is usually referred to as the “subcritical region” where the velocity of the fluid drops significantly, and the liquid depth increases noticeably.

Figure 1-1. Circular hydraulic jump of a normal impinging jet.
When the inclination of the jet angle (ϕ) changes from being 90°, where ϕ is the angle that the jet makes with the horizontal, the shape of the formed hydraulic jump deviates from a circular to a non-circular one (Figure 1-2). According to experimental work done by Kate et al. (2007a) and Kate et al. (2008), it has been noticed that based on the jet inclination angle, different shapes of the hydraulic jump are formed. For example, at 25° < ϕ ≤ 90°, smooth elliptical shapes are formed, meanwhile at ϕ = 90° a circular hydraulic jump profile is obtained (Figure 1-2).

Figure 1-2. A non-circular water hydraulic jump produced at a jet inclination angle, ϕ.

At angles where ϕ ≤ 25°, the profile of the hydraulic jump formed can no longer be described as a smooth curve, due to the formation of sharp corners (Figure 1-3). At such low jet inclination angles, the jet interacts with the jump profile causing an indefinite jump profile shape.

Throughout this thesis the focus is on studying the non-circular hydraulic jumps produced by oblique jets at the range of 25° < ϕ ≤ 90°. The study of hydraulic jumps formed due to inclined jet at ϕ ≤ 25° is out of the scope of the current thesis because the surface tension has a major rule at these cases, this factor will be neglected thought out the study. From this perspective, circular hydraulic jumps are considered as a special case where the flow discharges from a normal impinging jet on flat surface (ϕ = 90°).
1.2.2 Types of hydraulic jumps:

In this section, planar hydraulic jumps are briefly presented, and then radial hydraulic jumps are discussed in detail which are the focus of this thesis.

1.2.2.1 Planar hydraulic jumps:

Planar hydraulic jumps take place in open channel flows, such as rivers and dams. It occur when a flow with a high speed meets a slower stream of flow or an obstacle. It is accompanied by a sudden increase in the liquid level, then there is a sharp drop in the flow speed. In such flows the dimensionless Froude number is used as a good parameter to predict this phenomenon, so the depth Froude number is defined as as:

\[ F_{r_h} = \frac{V}{\sqrt{gh}}, \tag{1.2.1} \]

where \( V \) is the flow speed, \( g \) is the acceleration due to gravity and \( h \) is the fluid depth. It resembles the ratio between the speed of the flow to a wave propagating through the surface of the flow under the influence of gravity. In conditions where \( F_{r_h} > 1 \) the flow is called supercritical flow, \( F_{r_h} < 1 \) is subcritical flow and when \( F_{r_h} = 1 \) the flow is called critical flow. as shown in Figure 1-4. Planar hydraulic jumps are best described as a transition of flow from being supercritical to subcritical.

Figure 1-3 At low inclination angles, \( \phi \leq 25^\circ \), jumps with corners are formed.
There are numerous engineering applications of planar hydraulic jumps, such as in dams and spillways. In these applications, it is important to dissipate the flow energy to slow down the flow to protect the spillway bed from being eroded by the effect of high-speed flows. In addition, in wastewater and water treatment plants (White, 2011), planar hydraulic jumps are deployed to enhance mixing between two different fluids (for example water and water/sludge mixture).

1.2.2.2 Radial hydraulic jumps:

Craik et al. (1981) and Liu & Lienhard (1993) observed the different types of circular hydraulic jumps, yet their attempts were merely a description of the flow structures noticed for different jumps regimes. Ellegaard et al. (1996) were the first to describe and categorise the tangible differences between the two main types of circular hydraulic jumps (type I & II).

Type I jumps are “standard” circular hydraulic jumps based on Bush et al. (2006) description. These jumps are characterized by a flow separation taking place downstream of the jump which is noticed in the form of a recirculatory motion (Tani, 1949). Due to the absence of an obstacle downstream of the jump, the flow drains off under the effect of gravity at the edge of the impingement surface where there a decrease in the fluid height (Figure 1-5(a)). The flow downstream from the jump is believed to be unidirectional due
to the presence of only one eddy (separation roller) downstream which is a main character of type I jumps (Tani, 1949).

![Figure 1-5 The difference between Type I, Type IIa, and Type IIb.](image)

The main character of type-II jumps is the presence of an obstacle downstream of the jump (Figure 1-5(b,c)). which makes it considered as “phenomenological” jumps. The transition from type I to II can be generally achieved by increasing the fluid depth downstream of the jump using an obstacle. Another main character of type II jumps is the formation of a secondary eddy near the free surface which causes a pronounced flow reversal on the free surface downstream of the jump.

The location of the secondary eddy is used to distinguish type IIa from type IIb jumps. In the former case, the secondary eddy takes place downstream of the jump (Figure 1-5(b,c)). As for type IIb jumps, the secondary eddy takes place at the jump location. For that reason, type II-b jumps are sometimes called “double jumps. Due to the strong flow separation in type II jumps, the surface speed may not witness a deceleration past the jump (Bush & Aristoff, 2003). The non-circular flow patterns that were reported by Ellegaard et al. (1998) and Bush et al. (2006) are exclusive to type II jumps. More details on their approaches will be covered in the literature review section.
1.2.3 Classification of hydraulic jumps

Another way of categorising the hydraulic jumps can be found in Figure 1-6. To begin with, planar hydraulic jumps have been a major interest in the design of open channel flow facilities; their study is beyond the scope of this thesis. On the other hand, the term ‘radial’ is used to refer to the jumps produced by jet impingement on a substrate, where a thin film appears followed by the occurrence of the hydraulic jump. As will be found in the next section, major interest in the literature has been given to circular jumps where it was often described as ‘radial’ jumps as well, yet it should be emphasized that the term radial is a general description of circular and non-circular jumps all together.

The study of non-circular hydraulic jumps can be classified into two main branches. The first branch is the study of non-circular jumps produced as a result of the instability of circular ones. The second branch is the study of non-circular jumps produced due to non-circular jets. Because of the non-circular jet cross section produced by a jet inclination, the current theory is mainly based on the second branch of study.

![Figure 1-6 Classification of hydraulic jumps](image)

1.2.4 Applications:

Radially spreading thin films produced by the impingement of liquid jets play an important role in many industries, such as in metal cutting processes, reciprocating engines cooling, electronic cooling (Jörg et al., 2018), thermal management (Alimohammadi et al., 2014) and hot rolling processes (Zumbrunnen et al., 1992). Jet
impingement cooling provides a reliable way to remove heat from localized hot area, normally by directing the flow towards it. The high rate of mass and heat transfer on these hot surfaces provides an effective way to cool these surfaces with a great response (Baghel et al., 2020). It is worth mentioning that high-viscosity liquids, such as oil, are frequently used in many industries to achieve the desired rate of heat transfer (Jörg et al., 2018).

The first challenge when it comes to cooling hot surfaces using jet impingement is the presence of hydraulic jumps, which deteriorates the cooling performance on such surfaces (Wang & Khayat, 2020). The reason for that is the decay of the bulk fluid velocity downstream of the hydraulic jumps which impacts the heat transfer process. From the design perspective, sometimes it is impossible to provide normal jets on hot surfaces due to the spacing constraints, in these cases the jet must be inclined to accommodate the allowed spacing (Sparrow et al., 1980). In addition, sometimes it is of an interest to create an uneven cooling effect on the targeted surfaces, or to control the cooling process by providing more cooling in certain directions along the hot surface by enforcing an inclined jet. Consequently, the study of the hydrodynamics of non-circular hydraulic jumps formed due to the jet obliquity is of importance.

1.3 Modeling the flow field of a spreading jet of circular and non-circular hydraulic jump

The major theories and important dimensionless groups for describing the non-circular hydraulic jumps are discussed in this section.

1.3.1 Dimensionless numbers:

In this section, the dimensionless groups that are used throughout this thesis are discussed. Starting by the Reynolds number, which is defined in the current work as:

$$ Re = \frac{W a}{\nu} $$

(1.3.1)

Where W is the jet velocity, a is the jet nozzle nominal radius in the case of non-circular and for a circular jet it is simply the nozzle radius. Here \( \nu \) is the kinematic viscosity of the
fluid under study. Reynolds number represents the ratio between the inertial to the viscous forces in a given flow.

Another dimensionless number is the Froude number, which can be expressed as:

\[ Fr = \frac{W}{\sqrt{ag}} , \]  

where \( g \) is the acceleration due to gravity. In general, the Froude number represents the ratio of flow speed to a wave speed propagating in that flow. Another perspective of the significance of the Froude number is that it represents the ratio between the inertial forces to the gravity forces (Cengel & Cimbala, 2018).

1.3.2 Thin film theory:

When a viscous fluid flows on a surface where the length of the film created is much larger than its thickness, the thin film theory can be applied, which aims to predict the dynamics of the thin film created. As depicted in Figure 1-7, \( \epsilon = \frac{h}{L} \) where \( \epsilon \ll 1 \). This theory provides a way to approximate the full Navier-Stokes equations in order to capture the behaviour of the flow where thin films are involved.

Thin films exhibit strong sensitivity to the boundary conditions imposed on it, for example if a thin film of liquid is bounded by two stationary surfaces, the no slip condition becomes imperative to apply at both boundaries. In this case, the Navier-Stokes equations reduce to pressure terms balancing out viscous terms. This type of thin-film flow was discovered by Reynold (1886). Reynolds lubrication theory plays an important role in engineering practices such as in lubrication bearings (Batchelor, 2000). On the other hand, when the fluid becomes bounded by only one solid surface from one side while the other boundary becomes in contact with another fluid, a liquid–liquid or a liquid–gas interface takes place, in which it is expected that the thin film would exhibit a different behaviour. By applying the appropriate conditions on the upper boundary, we experience a strong contribution from the inertial terms (Reynolds number becomes relatively high), along with the contribution from the pressure and viscous terms. It is
worth mentioning that at the liquid-gas interface other factors might come into play such as the interfacial shear stress if the gas is of significant viscosity and surface tension might be important to incorporate if there is a strong curvature along the thin film interface. *(See Figure 1-7).*

![Figure 1-7](image)

**Figure 1-7** Configuration of a thin-film flow where the no-slip condition applies to the lower boundary while liquid-gas interface on the upper boundary is applied. The length of the solid boundary is infinitely long compared to the thin film thickness $h$ which is exaggerated for illustration purpose.

The flow in which radial hydraulic jumps occur can be modeled as an inertia dominated thin film flow upstream of the jump, and downstream of the jump a pure lubrication flow takes place where the inertia approximately vanishes. In this analogy the hydraulic jump acts as a transition between the two thin film flow regimes (Rojas et al., 2010). The flow of radial hydraulic jumps can be thought as a superposition of boundary layer flows with thin film approximation due to the presence of liquid-air interface.

### 1.3.3 Boundary layer theory:

When a potential flow passes over a solid body, two different flow regions are formed. The outer flow region where the viscous effects are negligible and the only terms that balance each other in the momentum equation are the inertial and pressure gradient terms (Euler equation). The inner flow region is where the viscous effects play an important
rule and no longer can be ignored; this imposes the no slip condition which means that the flow velocity on a stationary body is necessarily zero.

The study of hydraulic jumps relies on the concept of the Prandtl (1904) boundary layer theory, where the velocity gradient with respect to the normal direction is higher than with respect to a direction parallel to the surface of the body on which the boundary layer is formed. The edge of the boundary layer, $\delta$, is defined as the thickness at which the velocity inside the boundary layer reaches 99% of the outer flow velocity. Due to the axisymmetric nature of the circular hydraulic jumps, its study has been based on the two-dimensional boundary layer flow formed upstream of the jump. In this study, a new method of analysing the hydraulic jumps is established by introducing the three-dimensional boundary layer (Figure 1-8) equations; this will enable us to tackle the non-circular hydraulic jumps. The governing equations for a boundary layer non-axisymmetric-laminar flow shall be expressed.

The continuity equation in the dimensional form reads:

$$u^* + \frac{u^*}{r} + \frac{1}{r} v^* + w^* = 0. \quad (1.3.3)$$

The dimensional conservation of momentum equation in the radial direction reads:

$$\rho \left( u^* u_r^* + \frac{v^*}{r} u_\theta^* - \frac{v^* v^*}{r} + w^* u_z^* \right) = -p_r^* + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r u_r^* \right) - \frac{u^*}{r} + \frac{1}{r^2} u_\theta^* - \frac{2}{r} v^* \right) + \rho^*.$$

(1.3.4)

The dimensional conservation of momentum equation in the $\theta$ direction reads:

$$\rho \left( u^* v_r^* + \frac{v^*}{r} u_\theta^* + \frac{u^*}{r} v^* + w^* v_z^* \right) = -\frac{1}{r} \rho^* + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r v_r^* \right) - \frac{v^*}{r^2} + \frac{1}{r^2} v_\theta^* + \frac{2}{r^2} w^* \right).$$

(1.3.5)

The dimensional conservation of momentum equation in the $z$ direction reads:

$$\rho \left( u^* w_r^* + \frac{v^*}{r} w_\theta^* + w^* w_z^* \right) = -p_z^* - \rho g + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r w_r^* \right) + \frac{1}{r^2} w_\theta^* + \frac{2}{r} w_z^* \right).$$

(1.3.6)
The star superscript denotes a dimensional quantity. The physical quantities \( \rho \), \( \mu \) and \( g \) denote fluid density, dynamic viscosity, and acceleration due to gravity, respectively. The subscripts indicate the partial first and second derivates.

The governing equations are scaled using a length scale of “a” in the vertical direction, “L” as a length scale in the radial direction. “W” is the jet velocity which is used as a velocity scale. The ratio between the length scales is defined by “\( \varepsilon \)”, such that \( \varepsilon = \frac{a}{L} \) where \( \varepsilon \ll \ll 1 \). The radial coordinate \( r^* = \frac{r}{\varepsilon} \). The radial (u), azimuthal (v) and vertical velocity (w) components are scaled as \( Z^* = za \), \( u^* = uW \), and \( w^* = \varepsilon wW \). Due to the hydrostatic nature of the pressure term in thin films, the pressure term is scaled as \( p^* = (pga)p \). After using the scaling mentioned, the continuity and the momentum equations in the dimensionless form become:

\[
\frac{\varepsilon W}{a} \left( u_r + \frac{u_\theta}{r} + \frac{v_\theta}{r} + w_z \right) = 0. \tag{1.3.7}
\]

\[
\varepsilon Re \left( uu_r + \frac{v}{r} u_\theta - \frac{v^2}{r} + wu_z \right) = -p_r \frac{\varepsilon Re}{Fr^2} + \varepsilon^2 \left( \frac{1}{r} \frac{\partial}{\partial r} (ru_r) - \frac{u}{r^2} + \frac{1}{r^2} u_\theta \right) + \varepsilon u_{zz}. \tag{1.3.8}
\]

\[
\varepsilon Re \left( uv_r + \frac{v}{r} v_\theta + \frac{uv}{r} + wv_z \right) = -p_\theta \frac{\varepsilon Re}{Fr^2} + \varepsilon^2 \left( \frac{1}{r} \frac{\partial}{\partial r} (ru_r) - \frac{v}{r^2} + \frac{1}{r^2} v_\theta \right) + \varepsilon v_{zz}. \tag{1.3.9}
\]

\[
\varepsilon^2 Re \left( uw_r + \frac{v}{r} w_\theta + ww_z \right) = -p_z \frac{Re}{Fr^2} + \varepsilon^3 \left( \frac{1}{r} \frac{\partial}{\partial r} (rw_r) + \frac{1}{r^2} w_\theta \right) + \varepsilon w_{zz}. \tag{1.3.10}
\]

These equations are subject to no-slip and no-penetration at the disk surface:

\[
u(r, \theta, z = 0) = v(r, \theta, z = 0) = w(r, \theta, z = 0). \tag{1.3.11}
\]

In this thesis, the boundary-layer theory and the thin-film approximation are employed. From the above equations, in order to have the inertial terms, gravity terms and viscous
terms as leading order terms, it is imperative to assume $Re = O\left(e^{-1}\right)$ and $Fr = O(1)$, so the governing equations become as follows:

The continuity equation:
\[
ur + \frac{1}{r} v_\theta + w_z = 0.
\] (1.3.12)

The r-momentum equation:
\[
Re \left( uu_r + \frac{v}{r} u_\theta - \frac{v^2}{r} + wu_z \right) = -\frac{Re}{Fr^2} \left( p_r \right) + u_{zz}.
\] (1.3.13)

The $\theta$-momentum equation:
\[
Re \left( uv_r + \frac{v}{r} v_\theta + \frac{uv}{r} + wv_z \right) = -\frac{Re}{Fr^2} \left( \frac{1}{r} p_\theta \right) + v_{zz}.
\] (1.3.14)

The z-momentum equation:
\[
p_z = -1.
\] (1.3.15)

The above variables represent dimensionless quantities. The components $u$, $v$, $w$ denote the velocity of the fluid inside the boundary layer with respect to $r$, $\theta$ and $z$ respectively. The boundary layer equations will be used in the formulation to model the flow in the pre-jump region.

Figure 1-8 In 3-D boundary layers the azimuthal velocity component and the $\theta$ dependence can no longer be neglected.
1.3.4 The Kinematic conditions for a thin film.

This section starts by assuming a thin film of liquid flowing on a disk as shown in Figure 1-9. The free surface of the thin film, for a steady flow can be expressed as:

\[ F(r, \theta, z) = z - h(r, \theta). \]  

(1.3.16)

![Figure 1-9 The thin film employed in the current derivation](image)

At the interface, there are two different fluids where there is necessarily no exchange between particles of both fluids. Thus, the condition for a point to stay on the surface of the thin-film interface can be shown as:

\[ \frac{\partial F(r, \theta, z)}{\partial t} = 0, \]  

(1.3.17)

where \( \frac{\partial}{\partial t} \) is the material derivate of the function representing the interface, \( F(r, \theta, z) \). By expanding the condition (1.3.17) and by using the chain rule:

\[
\frac{\partial F}{\partial t} = \frac{\partial F}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial F}{\partial \theta} \frac{\partial \theta}{\partial t} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial t} = 0, 
\]

(1.3.18)

where \( \frac{\partial r}{\partial t} = u \), \( \frac{\partial \theta}{\partial t} = \frac{v}{r} \), \( \frac{\partial z}{\partial t} = w \). For a steady flow, the condition (1.3.18) can be expressed as:

\[
\frac{\partial F}{\partial t} = w - \frac{\partial h}{\partial r} u - \frac{1}{r} \frac{\partial h}{\partial \theta} v = 0. 
\]

(1.3.19)
This leads to the dimensionless kinematic condition at the liquid/gas interface, or in other terms at the surface of the thin film:

$$w(r, \theta, z = h) = u(r, \theta, z = h) \frac{\partial h}{\partial r} + v(r, \theta, z = h) \frac{\partial h}{\partial \theta}. \quad (1.3.20)$$

The dynamic and kinematic conditions derived in this section are essential for establishing the formulation of non-axisymmetric thin-film flow as will be shown in the next chapter.

### 1.3.5 The Dynamic conditions for a thin film

In this section, the state of stress at the interface between the thin film and air is studied. By assuming that the mean curvature at the surface of the interface to be negligible, the stress on the interface vanishes. Thus, it is concluded that \( n \cdot \sigma^* = 0 \), where \( \sigma^* \) is the stress tensor at the interface. The stress tensor \( \sigma \) can be expressed as \( \sigma = -p\bar{I} + \tau \), here there are contributions from the pressure, \( p \), and tangential stress \( \tau \), meanwhile \( \bar{I} \) is an identity tensor.

In other terms, the zero-stress condition on the interface can be expressed as

$$\begin{pmatrix} n \cdot \sigma_i^* \end{pmatrix} = n_r \sigma_{ri}^* + n_\theta \sigma_{\theta i}^* + n_z \sigma_{zi}^* = 0. \quad \text{Here the star superscript indicates dimensional quantities. An index “i” corresponds to the r, } \theta \text{ and z coordinates. Expanding the stress tensor, } \begin{pmatrix} n \cdot \sigma_i^* \end{pmatrix}, \text{ in } r, \theta \text{ and z directions yields:}$$

$$\begin{pmatrix} n \cdot \sigma_r^* \end{pmatrix} = n_r \sigma_{rr}^* + n_\theta \sigma_{r\theta}^* + n_z \sigma_{rz}^* = n_r \tau_{rr}^* - n_r p^* + n_\theta \tau_{r\theta}^* + n_z \tau_{rz}^*. \quad (1.3.21)$$

$$\begin{pmatrix} n \cdot \sigma_\theta^* \end{pmatrix} = n_r \sigma_{r\theta}^* + n_\theta \sigma_{\theta\theta}^* + n_z \sigma_{\theta z}^* = n_r \tau_{r\theta}^* - n_\theta p^* + n_\theta \tau_{\theta\theta}^* + n_z \tau_{\theta z}^*. \quad (1.3.22)$$

$$\begin{pmatrix} n \cdot \sigma_z^* \end{pmatrix} = n_r \sigma_{rz}^* + n_\theta \sigma_{\theta z}^* + n_z \sigma_{zz}^* = n_r \tau_{rz}^* + n_\theta \tau_{\theta z}^* + n_z \left( -p^* + \tau_{zz}^* \right) \approx n_r \tau_{rz}^* + n_\theta \tau_{\theta z}^* - n_z p^*. \quad (1.3.23)$$
It should be indicated here that the stress contribution coming from the vertical velocity component, \( w \), is small compared to the other directions, therefore all the terms that contain \( w \) are ignored.

Using the same function \((F)\) used for representing the film free surface for a steady flow, in terms of dimensional quantities, \( F^*(r, \theta, z) = z^* - h^*(r, \theta) = 0 \). Then the unit normal on the interface becomes:

\[
\vec{n} = \frac{VF^*(r, \theta, z)}{|VF^*(r, \theta, z)|} = \frac{e_z - h^*_r e_r - r^* - h^*_\theta e_\theta}{\left(1 + h^*_r + r^* - 2 h^*_\theta\right)^{1/2}} \approx \vec{e}_z - h^*_r e_r - r^* - h^*_\theta e_\theta. \quad (1.3.24)
\]

The components of the normal vector to the interface can be expressed in the coordinates \( r, \theta \) and \( z \) respectively as:

\[
\vec{n} \cdot \vec{e}_r = n_r = -\frac{-h^*_r}{\left(1 + h^*_r + r^* - 2 h^*_\theta\right)^{1/2}} \approx -h^*_r. \quad (1.3.25)
\]

\[
\vec{n} \cdot \vec{e}_\theta = n_\theta = -\frac{-r^* - h^*_\theta}{\left(1 + h^*_r + r^* - 2 h^*_\theta\right)^{1/2}} \approx -r^* - h^*_\theta. \quad (1.3.26)
\]

\[
n_z = \frac{1}{\left(1 + h^*_r + r^* - 2 h^*_\theta\right)^{1/2}} \approx 1. \quad (1.3.27)
\]

Substituting for \( n_r, n_\theta \) and \( n_z \) into the stress tensor after expanding the stress and pressure terms in \( r, \theta \) and \( z \) directions:

\[
\begin{align*}
(\vec{n} \cdot \sigma^*)_r &= -2\mu \frac{ru^*_r + h^*_r p^* - \frac{h^*_\theta}{r^*} \mu \left(\frac{u^*_\theta}{r^*} + \frac{\nu^*_r}{r^*}\right)}{r^*} + \mu u^*_z = 0. \quad (1.3.28) \\
(\vec{n} \cdot \sigma^*)_\theta &= -h^*_r \mu \left(\frac{u^*_\theta}{r^*} + \nu^*_r\right) + \frac{h^*_\theta}{r^*} \mu + 2\mu \frac{h^*_r}{r^*} \left(\frac{\nu^*_\theta}{r^*}\right) + \mu v^*_z = 0. \quad (1.3.29)
\end{align*}
\]
\[
\left( \mathbf{n} \cdot \sigma^* \right)_z = -\mu h^*_r \left( u^*_z \right) - \mu \frac{h^*_\theta}{r} \left( v^*_z \right) - p^* = 0.
\] (1.3.30)

By using the same scaling in section 1.3.3, the dimensionless stress tensor at the film surface in the expanded form in the \( r, \theta \) and \( z \) directions reads:

\[
\left( \mathbf{n} \cdot \sigma \right)_r = -2 \varepsilon^2 h^*_r u^*_r + \varepsilon \frac{\text{Re}}{\text{Fr}^2} \left( h^*_r p \right) - \varepsilon^2 \frac{h^*_\theta}{r} \left( u^*_r + v^*_r \right) + u^*_z = 0
\] (1.3.31)

\[
\left( \mathbf{n} \cdot \sigma \right)_\theta = -h^*_r \varepsilon^2 \left( \frac{u^*_\theta}{r} + v^*_r \right) + \varepsilon \frac{\text{Re}}{\text{Fr}^2} \left( p \frac{h^*_\theta}{r} \right) + 2 \varepsilon^2 \frac{h^*_\theta}{r} \left( u^*_r \right) + v^*_z + v^*_z = 0
\] (1.3.32)

\[
\left( \mathbf{n} \cdot \sigma \right)_z = -h^*_r \left( \varepsilon \right) \left( u^*_z \right) - \frac{h^*_\theta}{r} \varepsilon^2 \left( v^*_z \right) - \frac{\text{Re}}{\text{Fr}^2} = 0
\] (1.3.33)

From the argument used in the previous section, where gravity and inertia effects balance each other for thin film flows. In this case, \( \text{Re} = O\left( \varepsilon^{-1} \right) \) and \( \text{Fr} = O(1) \). Therefore from (1.3.31), dynamic conditions on the thin film surface are found:

From the only term of a leading order in (1.3.33), shows that:

\[
p(r, \theta, z = h) = 0.
\] (1.3.34)

\[
u_z (r, \theta, z = h) = 0.
\] (1.3.35)

Similarly, from (1.3.32):

\[
v_z (r, \theta, z = h) = 0.
\] (1.3.36)

The dynamic conditions (1.3.35), (1.3.36) and (1.3.34) are important for analysing the flow for thin films. As it will be discussed in the next chapter, these conditions will be used in our formulation for flow in the pre-jump region.

1.3.6 Kármán-Pohlhausen integral approach:

Kármán-Pohlhausen (K-P) approach presents an effective approximate method for solving the boundary layer equations. It is impossible to get an exact solution (Blasius solution) of the boundary layer equations in some cases, such as when the pressure
gradient due to gravity inside the boundary layer is to be included; in this case the K-P method is useful. The K-P method is based on an ansatz polynomial self-similar velocity profile of the flow inside the boundary layer, this enables generation of an integral boundary layer equation. By solving the momentum equation along the boundary layer, properties of the boundary layer can be found such as the boundary thickness at any location, the shear stress and the separation point (Schlichting & Gersten, 2000). The K-P method will be deployed in this thesis to solve the boundary layer equations upstream of the hydraulic jump.

1.4 Literature review:
In this section, a literature study of laminar steady circular hydraulic jumps is presented since the study of non-circular jumps is based on the earlier theories of circular jumps. In the second part of this literature study, the focus is given to the work that dealt with non-circular jumps.

Rayleigh (1914) was one of the earliest to attempt establish a theory predicting the radius of hydraulic jumps. His theory was mainly focused on river bores with large tidal variations, yet he proposed that this theory could be extended to model radial hydraulic jumps as well. His inviscid theory was established using planar coordinates with the assumption that mass and momentum are conserved, which is assumed to hold for both planar and radial hydraulic jumps. Moreover, he assumed that the energy across the jump is not conserved. The assumption that mass and momentum are conserved is indeed valid for radial jumps; however, neglecting the viscous effects leads to misidentification the surface velocity. That explains why the surface velocity is constant upstream of the jump as per Rayleigh (1914) conclusions. Rayleigh’s theory also overlooks the fluid layer thinning prior to the jump which causes viscous stresses. Another point is that planar jumps such as in channel and river flows are non-stationary, which means that the location of the jumps keeps moving downstream. This does not hold for radial jumps that can be kept stationary as long as some of the parameters are kept constant (such as the jet flow rate and the impingement disk geometry). Rayleigh’s shock condition, which states that upstream of the jump, the flow is supercritical and subcritical downstream of the
jump is necessary, yet it is not sufficient to determine the location of the radial jump using an inviscid approach.

Tani (1949) proposed a model to predict the radius of a circular hydraulic jump. He incorporated viscosity and gravity, which led to the thin-film boundary layer approximation. By assuming a parabolic velocity profile for the flow upstream of the jump and using K-P approach, he provided an ordinary differential equation that expressed the thin film behaviour with respect to the radial location upstream of the jump. He showed the integral curves of his thin film equation, which were mainly based on artificially chosen initial conditions. These curves showed one spiral - critical point, which suggested the existence of a singularity in the governing equation downstream of the critical point.

As per Tani (1949), the formation of the jump was mainly attributed to a reverse flow and flow separation of the thin film from the surface of the impingement disk. Tani referred to the presence of a thin film thinning that takes place due to a high velocity gradient. On the other hand, a hydrostatic pressure due to gravity downstream of the flow perpetuates the reverse flow which causes the circular hydraulic jump to happen.

Tani’s explanation of the occurrence of the circular hydraulic jump has been mainly argued by Watson (1964). Watson developed one of the pioneering theories to describe the occurrence of the hydraulic jumps due to radially spreading water by a normal laminar and turbulent impinging jets. His method was based on using the boundary layer equations including viscosity, along with neglecting the gravity and the surface tension effects. A similarity solution was used to solve his equations. Given a predetermined fluid thickness downstream of the jump, a mass and momentum flux balance was applied across the jump to determine its location. In his work, Watson showed experimental results that corroborated his conclusions. He stated that gravity has no influence on the jump radius, which is merely contingent upon the liquid height downstream of the jump.

Watson’s theory matched his experiment results, yet Craik et al. (1981) and Liu & Lienhard (1993) showed that it is valid only under certain conditions. For small jump radius, where the surface tension has a major rule on water hydraulic jumps, Watson’s
theory could not predict the location of the jump. Bush & Aristoff (2003) extensively studied Watson’s model theoretically and experimentally. They modified his theory by adding the effect of surface tension. They succeeded in showing the important role of surface tension in the formation circular hydraulic jumps.

Watson’s observations about the flow downstream of the jump led to further investigation. He stated that in the absence of surface tension effects, the flow is unidirectional downstream of the jump, where flow separation takes place. This was confirmed by the experimental work of Ishigi et al. (1977), Nakoryakov et al. (1978), Craik et al. (1981), Liu & Lienhard (1993) and Bohr et al. (1996). Albeit, the main difference between Watson’s experiment and the other ones is the location where the separation takes place. Watson’s experiment represented a type I jump, where the separation occurs downstream of the jump and the flow is unidirectional and therefore his theory matched his results (Craik et al., 1981). Watson’s observations do not apply to type II jumps that are marked by eddies and flow separation accompanying the jump with a flow reversal on the surface. Bush & Aristoff (2003A) showed that a transition from type I to type II jumps occurs by increasing the fluid depth downstream of the jump. They noted that for type II jumps the effect of surface tension must be accounted for, as opposed to type I jumps.

On the other hand, as a continuation to Tani (1949) work which described type I jumps, Bohr et al. (1993) developed the shallow water approach which deals with the momentum equation upstream of the jump. Starting with the thin film equation, then they expressed it in terms of the average surface velocity prior to the jump. They confirmed the presence of a spiral critical point as a confirmation to Tani’s conclusions, but they confirmed its independence on the initial conditions and the averaging process carried out. Also, they predicted the advent of a singularity near the critical point close to the location of the jump. Bohr et al. (1993) approach was based on dealing with an inner solution prior to the jump and an outer solution after the jump, then connecting both solutions through a force balance at the jump. They argued that the hydraulic jump location scales as $Q^{5/8}v^{-3/8}g^{-1/8}$, where $Q$ is the volume flow rate, $v$ is the kinematic viscosity and $g$ is
the acceleration, in addition to a scaling constant which merely depends on the ansatz velocity profile.

The general trend of the scaling form introduced by Bohr et al. (1993) has matched numerous theoretical work (Tani, 1949; Kate et al., 2007a; Kasimov, 2008; Wang & Khayat, 2019) and experimental work (Bohr et al., 1996; Ellegaard et al., 1996; Brechet & Néda, 1999). On the contrary, the scaling constant has been argued to be inconsistent and does not express the exact location of the jump. Kasimov (2008) argued that the scaling constant must be a function of the surface tension and the boundary conditions downstream of the jump.

Fernandez-Feria et al. (2019) compared Bohr’s shallow water approach, with the depth averaged model. The latter model is based on the shallow water approach, yet it expresses the film equation upstream of the jump in terms of the fluid depth rather than the average velocity. They also compared the two models with the numerical solution of the full Navier-Stokes equations to identify the jump location. The main aim of their study was to identify the influence of the surface tension of the jump. Comparing against their numerical results, Fernandez-Feria et al. (2019) affirmed that the location of the jump in Bohr et al. (1993) model lies at the location of the singularity, which is located downstream of the critical point. They also noted that there is no global solution of the film equation upstream of the jump for all initial conditions which contrasts with Bohr et al. (1993) findings. Fernandez-Feria et al. (2019) concluded the important role of gravity on the location of the hydraulic jump. They claimed that the location of the jump does not depend on the surface tension upstream of the jump.

Wang & Khayat (2021) have concluded recently, in their numerical study that the location of the jump indeed depends on the surface tension of the fluid as far as low viscosity flow is concerned (such as water). Yet, they argued that the jump in highly viscous flows mainly depends on gravity due to the reverse hydrostatic pressure caused by the fluid accumulation downstream of the jump. Wang & Khayat (2021) findings matched with their earlier theoretical formulation to determine the location of the circular hydraulic jumps of highly viscous (10 or more times more viscous than water) fluid under
the influence of gravity (Wang & Khayat, 2019). Wang & Khayat (2019) theory predicted the location of the jump location found by Duchesne et al. (2014) where they used silicon oil. Wang & Khayat (2019) showed that the location of the hydraulic jump coincides on the singularity location of the thin film equation for high viscous flow, and it is independent of the flow properties downstream (type I jumps).

Some other parameters affecting the location of circular hydraulic jumps such as the effect of slip on the impingement disk. This was studied by Prince et al. (2012, 2014) and Khayat (2016). It has been shown that the surface slip maintains the flow inertia, leading to pushing the jump location further downstream. A close behaviour was found by imposing a rotation on the impingement disk (Wang & Khayat, 2018). The nozzle to disk influence on the location on the hydraulic jump was studied by Brechet & Néda (1999). They proposed that the distance of the jet to the disk has no influence on the jump location. Yet Kuraan et al. (2017) argued that this does not hold when the ratio between the nozzle to disk distance to the radius of the nozzle is less than 0.4.

Non-circular hydraulic jumps were for the first time observed by Ellegaard et al. (1998) in their experimental work. The experiment mainly dealt with the impingement of a normal jet on a horizontal disk with a weir on the rim of the disk. Its purpose was to control the downstream height of the type II jumps produced. The main aim of the study was to use a highly viscous liquid of 10 times more viscous than water (Ethylene glycol) to capture stationary type II jumps. Surprisingly, spontaneous polygonal jumps appeared after breaking the symmetry of the type II circular jump they had produced earlier. By changing some of the experiment parameters, such as the flow rate of the jet, the nozzle to disk distance, and the kinematic viscosity of the liquid, which cased the number of the sides of the polygonal jump profile increased up to 14. The breaking of the axisymmetric jump was attributed to instabilities taking place for type II circular jumps. They suggested that the surface tension has a major role in the analysis of non-circular jumps they created.

More experimental work on the non-circular jumps was done by Bush et al. (2006), where it was confirmed that breaking the symmetry could take place only for type II
jumps. Using a highly viscous fluid (30 times more viscous than water), they found new shapes of non-circular structures of non-circular jumps. To mention, clover shapes, butterflies, cat eyes and bowties, which emerged from type IIb jumps exclusively. Bush et al. (2006) established a parametric study to show the influence of parameters, such as the jet flow rate, nozzle radius, fluid kinematic viscosity, surface tension, and the fluid downstream depth (and the dimensionless groups pertaining to it) on the structure of the jumps formed. They suggested that the main reason of the breaking of the axisymmetric jumps is due to the surface tension instabilities. They also explained the role of viscosity on decelerating the flow, which forces the hydraulic jump to occur at smaller radius for high viscosity fluids. At small jump radius, they noted that the surface tension effect becomes strong forming instabilities for type II hydraulic jumps. Also, fluid viscosity was attributed to maintain the non-circular structures of the non-circular jumps formed.

Teymourtash & Mokhlesi (2015) classified the jump profiles found by Ellegaard et al. (1998) and Bush et al. (2006) as stable jumps, while they observed the unstable type. They characterised unstable jumps by the presence of rotational surface waves upstream of the jump, that were noticed to emerge on the surface of the jump in clockwise and anticlockwise directions. It was also noted that a transition of an unstable jump to a stable jump can be achieved by increasing the liquid downstream depth of the jump.

In his study on the stability of circular hydraulic jumps, Kasimov (2008) modified the shallow water model by Bohr et al. (1993) by accounting for the surface tension and the shape of the disk. He noted that there are no stable circular hydraulic jumps that exist above a critical value of surface tension. This critical value of the surface tension decreases as the viscosity of the liquid increases, which explains breaking the axisymmetric jumps for highly viscous liquids.

Martens et al. (2012) established a theoretical model to study the instabilities found in Ellegaard et al. (1998) and Bush et al. (2006) work that resulted in non-circular jumps. Their phenomenological model handled type II jumps which are prone to instabilities. Their theory was based on the separation eddy ‘roller’ that takes place at the front of type II jumps. They used the conservation of mass, radial force balance on the roller surface
between shear stress and the hydrostatic pressure. In the azimuthal direction they drew a force balance between viscosity, surface tension and pressure. By analyzing the linear stability of their model in addition to handling a truncated version of the full nonlinear version of that model they were able to capture the same noncircular jumps found in the earlier noted experiments by confirming the role played by surface tension in such instabilities. Another theoretical and numerical work was done by Rojas & Tirapegui (2015). Using the small perturbation theory, they identified the threshold that render the instability in type II jumps highlighting the role played by radial shear force, azimuthal surface tension and hydrostatic azimuthal force in such instabilities.

Kate et al. (2007a) studied experimentally and theoretically the formation of non-circular hydraulic jumps due to an inclined jet on a horizontal surface. They provided geometric relations expressing the shape of the impingement region of such flows, which was elliptical in nature. They concluded that the stagnation point shifts to the upstream focus of impingement region for inclined jets. Based on their experimental work using water as the working liquid, they categorized oblique-formed hydraulic jumps based on the jet inclination angle into smooth ‘elliptical’ jumps and jumps with distinct ‘corners’, which was supplemented later by Kate et al. (2008) experimental work. In their theoretical work to model smooth-profile jumps, they used Bohr et al. (1993) shallow water approach. To eliminate the azimuthal dependence from the conservation of mass and momentum equations, they scaled their governing equations using the $\theta$ dependent variables, thus reproducing the same governing equations for the circular dimensionless case. Kate et al. (2007a) theoretical work matched their experimental parameters, such as the aspect ratio of the jump profiles formed, yet in the next chapter it will be shown that their approach does not lead to the exact location of the non-circular hydraulic jumps they showed experimentally.

Kate et al. (2007b) attributed the formation of jumps with corners that was formed at low-jet inclination angles to ‘jet-jump’ interaction and ‘jump -jump’ interactions, it was suggested that there is an analogy between these jumps with shock waves interaction. They did not provide a theoretical model that captures the exact parameters influencing such jumps, such as the effect of surface tension effect.
Other experimental studies concerning oblique jet impingements on various substrates were covered, such as impingement on moving surfaces (Kate et al., 2009) and on vertical hydrophobic surfaces (Kibar & Süleyman, 2018).

1.5 Impingement region due to an inclined jet (Kate et al., 2007a):

In this section, Kate et al. (2007a) derivations for determining the profile of the impingement region formed due to an inclined jet on a horizontal disk are reproduced. Due to the jet inclination, it is predicted that an elliptical impingement region will be produced. The elliptical behaviour of the impingement region is expected to affect the behaviour of the entire flow downstream. In this section, the relation that predicts the shape of the impingement region “r_e” that is formed due to the discharge of a jet of velocity ‘W’ by a nozzle inclined by angle $\phi_n$ is redrived.

In the first subsection, the difference between the nozzle inclination angle, and the effective jet inclination angle is shown. Followed by an argument for applying a mass balance across two distinct regions in the flow domain for an inclined jet flow. Also, a momentum balance in the x-direction at the jet exit and a location downstream of the flow is drew. Finally, an expression that describes the polar form of the impingement region as a function of the jet inclination angle and the azimuthal direction will be found.

1.5.1 Relationship between $\phi_n$ and $\phi$:

When a jet discharges from an inclined circular nozzle by angle $\phi_n$ (Figure 1-10) at an elevation $H_n$, the inclination angle of the jet immediately at the impingement tends to change from being $\phi_n$ to $\phi$, due to the effect of gravity which tends to diverge the jet angle. Indeed, the inclination of the jet causes a non-circular impingement region that depends on the effective jet inclination angle, $\phi$. It has been shown by Kate et al. (2007a) that in this case $\phi$ can be expressed as:
\[ \phi = \tan^{-1} \left( \frac{\left( W^2 \sin^2 \phi_n + 2gH_n \right)^{1/2}}{W \cos \phi_n} \right) \], \tag{1.4.1} \]

\( \phi \) is the effective jet inclination angle. Here, \( W \) is the jet velocity (m/s), \( \phi_n \) is the nozzle inclination angle with respect to the horizontal, \( g \) is the acceleration due to gravity (m/s\(^2\)) and \( H_n \) is the drop height from the nozzle centre to the surface of the impingement disk (m) (Kate et al., 2007a). Equation (1.4.1) shows that \( W, H_n \) and \( g \) have a direct effect on \( \phi \), for example as \( W \) increases it is expected to have a stronger jet inclination. Also, it should be noticed that \( H_n \) has a major contribution on \( \phi \).

In the current thesis, the main parameter that entails the flow configuration is the effective jet inclination, \( \phi \), thus that the combination of the other factors that produce it will not be considered.
Figure 1-10 The impingement of an inclined circular jet with an effective inclination angle $\phi$. Note the stagnation point has shifted from the geometric center of the nozzle by $S_s$.

1.5.2 Applying mass balance across the flow created by an inclined jet:

The aim of this section is to use the conservation of mass concept by drawing a mass balance at two different locations in the flow field. The first location is the normal flow reaching the disk surface just at the impingement, then the flow downstream of the impingement region is focused on.

As shown in Figure 1-11, The vertical jet velocity component is responsible for causing the impinging on the surface of the disk, hence at the impingement region the flow is strictly two dimensional. The mass flow rate ($\dot{m}$) crossing the impingement region due to the normal component of the velocity is $\dot{m} = \rho W \sin \phi dA$, where $dA = \int_0^{2\pi} r_e^* dr_e^* d\theta$. $r_e^*$ is the impingent region profile and the star superscript here indicates a dimensional quantity.
\[ \dot{m} = \rho W \sin \phi \int_0^{2\pi} \int_0^{r_e} r_e^* d\theta \]  
\[ = \rho W \sin \phi \int_0^{2\pi} \frac{r_e^*}{2} d\theta \]  
(Kate et al., 2007a). \hfill (1.4.2)

Figure 1-11 Mass flow rate in the impingement region produced due to the normal component of the jet velocity.

Further downstream at location \( r_p \), the mass flow rate crossing the surface is \( dA = h^* r_p^* \)

where \( h^* \) is the film thickness at that location, as shown in Figure 1-12. The conservation of mass across the blue area can be written as:

\[ \dot{m} = \rho W \int_0^{2\pi} r_p^* h^* d\theta. \]  
\hfill (1.4.3)
Figure 1-12 Mass flow rate at location \( r_p \) where \( S \) is the stagnation point of the flow and the origin of our assigned coordinates. The radial jet velocity \( W_r \) is also shown.

By setting the terms (1.4.2) and (1.4.3) equal:

\[
W \sin \phi \int_0^{2\pi} \frac{r_e^*}{2} \, d\theta = W_r \int_0^{2\pi} r_p^* h^* \, d\theta. \tag{1.4.4}
\]

In the inviscid region after the jet impinges the disk, two arbitrary points along any streamline (Figure 1-10) can be chosen, the first point where the flow velocity is \( W \) and the other point where the velocity is \( W_r \). By applying an energy balance at the two chosen points after neglecting the elevation difference and knowing that the pressure is constant everywhere, it could be found that \( \frac{W^2}{2} = \frac{W_r^2}{2} \) (Kate et al., 2007a). In this case (1.4.4) becomes:

\[
r_p^* h^* = \frac{r_e^*}{2} \sin \phi. \tag{1.4.5}
\]

Where \( r_e^* (\phi, 0) \) is the polar radius of the envelope of the impingement region, and \( r_p^* \) is the radial distance of a point located on the plate (relative to the point \( S \)), and \( h^* \) is the film thickness, \( \alpha \) is the jet radius (dimensional) and \( W \) is the jet velocity (dimensional). Again, the star superscript denotes dimensional quantities.
1.5.3 Applying momentum balance in the x-direction:

In this part, a momentum balance in the x-direction at radial location $r_p^*$ and immediately at the jet exit is applied. The x-momentum generated due to $\overline{W}_r$ passing through a control surface at location $r^* = r_p^*$, where the film thickness is $h$ (as shown in Figure 1-13):

$$\int_{cs} \rho \overline{W}_r (\overline{W}_r \cdot \overline{n}) \, dA.$$  \hfill (1.4.6)

Here, $\overline{n} = 1$ while $\overline{W}_r$ is directed inside the surface; $\overline{W}_r \cdot \overline{n} = -\overline{W}_r$ and $dA = h \, r_p^* d\theta$.

The x-momentum passing into the chosen control surface reads:

$$-\int_{0}^{2\pi} \rho \overline{W}_r \overline{W}_r \left( h \, r_p^* \right) \, d\theta.$$ \hfill (1.4.7)

On the other hand, as for the x-momentum generated at the jet exit:

$$\rho \pi a^2 W \overline{W} \cos \phi.$$ \hfill (1.4.8)

By balancing the x-momentum at $r_p^*$ and at the jet discharge, from (1.4.7) and (1.4.8):

$$\rho \pi a^2 W^2 \cos \phi = -\int_{0}^{2\pi} \rho \overline{W}_r^2 \left( h \, r_p^* \right) \, d\theta.$$ \hfill (1.4.9)

Recalling that $W = \overline{W}_r$ and eq. (1.4.5):

$$\pi \cot \phi = -\int_{0}^{\pi} \left( \frac{r_e^*}{a} \right)^2 d\theta.$$ \hfill (1.4.10)

Equation (1.4.10) is an integral equation that represents a relation between the impingement region $r_e^*$, the jet radius ‘a’ and the inclination angle of the jet.
A control volume is chosen to determine the x-momentum produced by the radial jet velocity $W_r$.

1.5.4 The profile of the impingement region created by the flow from an inclined jet:

As depicted in Figure 1-14, the impingement region can be interpreted as the horizontal projection of an inclined circular jet with radius ‘$a$’ and inclination angle of ‘$\phi$’ on the horizontal plane. The elliptical impingement region has a major axis of ‘$2a/sin\phi$’ and minor axis of ‘$a$’. The origin of the impingement region is chosen to coincide with the stagnation point $S$, which is accounted for by a shift in the x coordinate of the origin of the formed elliptical impingement region. The equation representing the shape of the impingement region can be given as:

$$\frac{(x - S_s)^2}{M^2} + \frac{y^2}{m^2} = 1.$$  \hspace{1cm} (1.4.11)

$S_s$ is the distance between the origin and $S$, $M$ is the major axis, and is the minor axis of the ellipse. Recalling that $M = \frac{2a}{\sin \phi}$ and $m=a$, the impingement profile in the polar form reads:
\[
\left( \frac{r_e^*}{a} \right)^2 + \left( \frac{r_e^*}{a} \cos \theta - \frac{S_s}{a} \right)^2 \sin^2 \phi = 1 .
\] (1.4.12)

In this part, geometrical relations to estimate the profile of the impingement region in the polar coordinates are derived:

From equations (1.4.12) and (1.4.10), Kate et al. (2007a) showed that:

\[
S_s = a \cot \phi ,
\] (1.4.13)

\[
r_e^*(\phi, \theta) = a \left( \frac{\sin \phi}{1 + \cos \phi \cos \theta} \right) .
\] (1.4.14)
$S_s$ is the distance the stagnation point shifts from the geometric centre of the elliptical impingement region produced. $r_e^* (\phi, \theta)$ represents the impingement region profile produced due to the impingement of a jet inclined with angle $\phi$ with a horizontal plane parallel to the disk surface.

1.6 The objective and thesis outline

1.6.1 Research gap:

It is clear from the literature review section that there has been a lot of attention devoted to circular hydraulic jumps. The effect of factors such as surface tension and effect of gravity on the formation of the jumps has been extensively studied. Based on the work in the literature, for low viscosity fluids, such as water, the effect of surface tension must be accounted for (Wang & Khayat, 2021). Also, it can be agreed that gravity has a major rule on the formation of highly viscous jumps (Fernandez-Feria et al., 2019; Wang & Khayat, 2019).

On the other hand, there has not been enough attention to non-circular jumps. A major portion of the work that dealt with non-circular jumps was mainly focused on type II jumps. When it comes to type I jumps that are formed due to the impingement of inclined jets, only a single work (Kate et al., 2007a) can be found in the literature. The reason for that is indeed the complexity the momentum equations in the three-dimensional case possess, which makes it a burdensome to solve. Kate et al. (2007a) work was based on ignoring the momentum equation in the $\theta$ direction. It can be argued that they have totally overlooked an important factor in predicting the behaviour of non-circular jumps since the flow is non-axisymmetric in nature (Teymourtash & Mokhlesi, 2015).

1.6.2 Thesis objective:

The main objective of this thesis is to establish a theoretical model to determine the location of the non-circular hydraulic jumps for highly viscous liquids that are produced due to the impingement of an inclined jet on a horizontal surface. In the current thesis, the flow of three-dimensional boundary layers is studied, along with applying the thin-film
approximation under the influence of gravity. By the end of this thesis, the aim is to have a better understanding the physics of such complex flows where non-circular jumps arise.

1.6.3 Thesis outline:

In chapter 2, a theoretical model for highly viscous non-circular jumps produced by an inclined jet impingement will be presented. The effect of various parameters on the non-axisymmetric flow produced will also be discussed in that chapter. Chapter 3 includes concluding remarks and suggestions for future work based on the results presented in chapter 2.

1.7 Summary

In this chapter, an overview of hydraulic jump types and classification is presented, also applications where the study of hydraulic jumps is of importance has been discussed. The theoretical foundation for boundary-layer and thin-film flow is then illustrated before a literature review for hydraulic jumps is covered. Then trigonometric relations for inclined jet impingement are redrived from Kate et al. (2007a).
2 Flow in the developing boundary layer region and fully viscous region due to the inclined jet impingement for high viscous liquid.

2.1 Introduction

The main objective of this chapter is to establish a theoretical formulation to predict the location of the non-circular hydraulic jumps and to model the behaviour of the three-dimensional flow accompanying it. The effect of inertia, viscosity, and gravity on a non-axisymmetric thin-film flow as a result of the impingement of an inclined jet on a stationary – solid disk is studied.

In subsection 2.2, the physical domain of the problem is illustrated. Then, the general formulation of the problem is presented.

In subsection 2.3, the developing boundary layer, the azimuthal shear stress on the disk and the free surface height in the developing boundary layer region are determined using K-P approach. The $\theta$ dependent transition point is located, which is a point after which the flow becomes entirely fully viscous.

In subsection 2.4, The fully viscous layer region is examined by K-P approach, the influence of gravity and the jet inclination angle on the thin-film flow and the jump location is assessed. Also, a comparison with a previous theoretical and experimental work found in the literature is drawn.

2.2 Physical domain and problem statement

Consider a steady laminar incompressible flow of a Newtonian liquid jet discharging from a nozzle of characteristic radius $a$, impinging at a volume flow rate $Q$ on a flat disk. The jet may be impinging on a flat horizontal disk normally or obliquely (at an angle $\phi$). The jet cross section is then assumed to be non-circular in the latter case with a profile of $R(\theta)$, and circular in the former case. The flow configuration is depicted schematically in Figure 2-1, where dimensionless variables and parameters are used. The problem is formulated in the $(r, \theta, z)$ fixed coordinates, with the origin coinciding with the flow
stagnation point. The impinging flow is assumed to be dependent on θ, thus accounting for non-axisymmetric flow on the plate and the formation of an elliptical hydraulic jump due to the jet obliquity. In this case, \( u(r, \theta, z) \), \( v(r, \theta, z) \) and \( w(r, \theta, z) \) are denoted by the corresponding dimensionless velocity components in the radial, azimuthal and vertical direction, respectively. The r-axis is taken along the disk radius and the z-axis is taken normal to the disk. The length and the velocity scales are conveniently taken to be the characteristic radius of the jet, \( a \), and average velocity \( W = Q/\pi a^2 \) respectively. Since the pressure is expected to be predominantly hydrostatic for a thin film, it will be scaled by \( \rho g a \). Two main dimensionless groups emerge in this case: the Reynolds number \( Re = Wa/\nu \), where \( \nu \) is the kinematic viscosity, the Froude number \( Fr = W/\sqrt{ag} \), \( g \) being the acceleration due to gravity. It will be shown that the problem can be reduced to a one-parameter problem.

### 2.2.1 The physical domain

There are four distinct flow regions produced on the surface of the impinged disk, region (i) is the impingement region, when the jet is normal the impingement region is non-circular, meanwhile the inclination of the jet forms an elliptical impingement region. A point of major importance is the location of the stagnation point in which the flow velocity reaches zero, due to the obliquity of the jet its location shifts from the geometric center of the elliptical impingement region formed (Kate et al., 2007a). In this case, the origin of the used coordinates is assumed to coincide with the stagnation point. In the current study, the impingement region is small to be accounted for since it is expected to be at the order of the jet radius (Kate et al., 2007a). On the other hand, the non-circularity of the impingement region’s shape due to the jet inclination is accounted for by using the effective jet cross section, \( R(\theta) \).
Downstream of the stagnation point, the velocity of the flow increases from being zero while a developing boundary layer $\delta(r, \theta)$ starts to emerge which is defined as the developing boundary layer region ‘region (ii)’. Outside the developing boundary layer, the flow is inviscid and uniform with a velocity similar to that of the jet speed, $W$ (Wang & Khayat, 2019). Region (ii) can be defined at $r \ll r_0(\theta)$, where $r_0(\theta)$ is location of the transition point at which the viscous effects become considerable right up to the free surface.

Region (iii), $r_0 \leq r \ll r_j$, is the fully viscous region where the entire flow is of a boundary layer type and the inviscid flow ceases to exist. At this stage, a thin film is created with a height of $z = h(r, \theta)$.

Further downstream of the transition point the hydraulic jump occurs at a location $r = r_j(\theta)$. The flow beyond this region transforms from being supercritical to subcritical where the height of the liquid film becomes $H(r, \theta)$ where the region (iv) commences.

Figure 2-1 Schematic illustration of a non-axisymmetric jet flow impinging on a flat stationary disk and the hydraulic jump of type I with one vortex downstream. Shown are the impingement region (i), the developing boundary-layer region (ii), the fully developed layer region (iii) and the jump region (iv). $R(\theta)$ represents the non-circular jet cross section.
The jump region is characterized by the formation of a separation eddy. As per Figure 2-1, $h_j(\theta)$ is the liquid height just upstream of the jump, while $H_j(\theta)$ is the height just downstream of the jump. At $r = r_\infty$, the liquid drains by the gravity effect from the edge of the disk at a height of $H_\infty$. In the current thesis, the focus is on the flow in region (ii) and (iii).

### 2.2.2 Governing equations and boundary conditions

The Reynolds number is assumed to be moderately large corresponding to the flow of a laminar regime that would hold in the entire flow upstream of the jump. Therefore, for a steady non-axisymmetric thin-film flow where gravity is accounted for, the mass and momentum conservation equations are formulated using a thin-film and Prandtl boundary-layer approach, in the three dimensions. By letting a subscript with respect to $r$, $\theta$ or $z$ denote partial differentiation, after recalling the reduced dimensionless conservation equations (1.3.12), (1.3.13), (1.3.14) and (1.3.15):

\[
    u_r + \frac{u}{r} + \frac{1}{r}v_\theta + w_z = 0,
\]

(2.2.1a)

\[
    \text{Re} \left( uu_r + \frac{v}{r}u_\theta + wu_z - \frac{v^2}{r} \right) = -\frac{\text{Re}}{Fr^2} p_r + u_{zz}.
\]

(2.2.1b)

\[
    \text{Re} \left( uv_r + \frac{v}{r}v_\theta + wv_z + \frac{uv}{r} \right) = -\frac{\text{Re}}{Fr^2} \frac{1}{r} p_\theta + v_{zz},
\]

(2.2.1c)

\[
    p_z = -1.
\]

(2.2.1d)

This is the 3-D form of the thin-film equations used to model the spreading liquid flow (Tani, 1949; Bohr et al., 1993, 1996; Kate et al., 2007a). It is observed that the hydrostatic pressure for a thin film vanishes on the surface due to the absence of the curvature effect (surface tension) as shown in the dynamic conditions in the previous chapter (condition (1.3.34)). In addition, upstream of the jump, the variation of the film thickness is expected to be smooth and gradual both radially and azimuthally. Nevertheless, due to the nature of the flow, it is expected the radial variation in the hydrostatic pressure to be more significant than in the azimuthal direction. At the surface
of the disk, the no-slip and no-penetration conditions are assumed everywhere. In this case:

\[
    u(r, \theta, z = 0) = v(r, \theta, z = 0) = w(r, \theta, z = 0) = 0, \quad (2.2.2)
\]

At the free surface, recall (1.3.20), (1.3.35), (1.3.36) and (1.3.34) the kinematic and dynamic conditions for steady flow take the form:

\[
    w(r, \theta, z = h) = u(r, \theta, z = h)h_r(r, \theta) + \frac{v(r, \theta, z = h)}{r}h_\theta(r, \theta), \quad (2.2.3a)
\]

\[
    u_z(r, \theta, z = h) = v_z(r, \theta, z = h) = p(r, \theta, z = h) = 0. \quad (2.2.3b, c, d)
\]

Next, consider the conservation of mass at any location caused by the impingement of the inviscid flow in the non-circular jet of uniform velocity. Recall that the jet non-circularity is the result of (circular) jet inclination or a jet issuing from a non-circular equivalent duct. The dimensionless jet profile (scaled by \(a\)) is \(R(\theta)\), then the dimensionless flux is given by

\[
    \frac{1}{2\pi} \int_0^{2\pi} R(\theta) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} R^2 \, d\theta = 1. \quad (2.2.4)
\]

Conservation of mass for a flow through a wedge of width \(r'd\theta\) yields:

\[
    \int_0^h u(r < r_j, \theta, z) \, dz = \frac{R^2(\theta)}{2r}. \quad (2.2.4)
\]

It is noted that (2.2.4) conserves the same form. The flow field is sought separately in the developing boundary-layer region and in the fully developed viscous boundary-layer region.

Integrating (2.2.1d) subject to (2.2.3d) the pressure becomes \(p(r, \theta, z) = -z + h(r, \theta)\) which is eliminated so equations (2.2.1b) and (2.2.2c) become:

\[
    \text{Re} \left( \frac{uu_r + v}{r} u_\theta + wu_z - \frac{v^2}{r} \right) = -\frac{\text{Re}}{\text{Fr}^2} h_r + u_{zz}, \quad (2.2.5a)
\]
The inclusion of gravity causes the flow to be non-self-similar in character. Therefore, approximate solutions are sought in each region using K-P integral approach (Schlichting & Gersten, 2000). A cubic profile is used for the velocity as will be discussed in the next section. The cubic profile leads to an agreement with Watson (1964) as shown by Wang & Khayat (2019). It was used earlier by Khayat (2016), it was found to yield an agreement against his numerical solution, and for a flow on a rotating disk Wang & Khayat (2018) producing good agreement with experimental results.

Finally, it is useful to discuss briefly how the vertical velocity component is eliminated when the K-P approach is implemented in the developing and viscous layer regions. First, from (2.2.1a) it could be found that

\[
W(r,\theta,z) = -\frac{1}{r}\frac{\partial}{\partial r}\left(\int_0^z u(r,\theta,z)dz\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\int_0^z v(r,\theta,z)dz.
\]

When this expression is used, and after a lengthy derivation, the convective terms in (2.2.1b) and (2.2.1c) become:

\[
\begin{align*}
(2.2.6a) \quad u_r + \frac{v}{r} - u_\theta + wu_z - \frac{v^2}{r} &= \frac{1}{r} \left( (ru_\theta) + \frac{1}{r} (uv)_{\theta} \left( \int_0^z (ru)_r + v_\theta \right) dz \right) - \frac{v^2}{r}, \\
(2.2.6b) \quad uv_r + \frac{v}{r} - v_\theta + vw_z + \frac{uv}{r} &= (uv)_r + 2 \frac{v}{r} v_\theta + \frac{uv}{r} - \left( \int_0^z \left( u_r + u_\theta + v_\theta \right) dz \right) + \frac{uv}{r}.
\end{align*}
\]

### 2.2.3 Symmetry breaking impinging jet configurations

It shall be mentioned that the symmetry-breaking configuration due to a circular jet impinging obliquely to the disk is the main aspect of this study. Although the cases of a non-circular nozzle or a non-circular inclined jet can be covered by the present formulation, it will not be considered here.
For an inclined jet impinging at angle $\phi$, Kate et al. (2007a) showed the relation between the product of the radial distance and the film thickness with respect to the impingement region profile, as it was shown in the previous chapter.

By recalling (2.2.4), at small radial distance where the flow is almost entirely inviscid ($u(r, \theta) \sim 1$) it reduces to $rh = \frac{R^2(\theta)}{2}$. Also, recall equation (1.4.5), $rh = \frac{r_e^2}{2} \sin \phi$, where the impingement profile in the dimensionless form is $r_e = \frac{\sin \phi}{1 + \cos \phi \cos \theta}$. Thus, from both expressions, the jet cross section in the dimensionless form can be expressed as:

$$R(\theta) = \frac{\sin^{3/2} \phi}{1 + \cos \phi \cos \theta}.$$  \hspace{1cm} (2.2.7)

**Figure 2-2** $R(\theta)$ represents the jet cross section for an inclined jet, the effective inclination angle $\phi = 65^\circ$. 
As shown in Figure 2-3, \( R(\theta) \) represents the jet cross section produced by an inclined circular jet at an elevation of \( H_n \) which produces an effective inclination angle \( \phi \), where gravity tend to diverge the jet from being at the same angle as the nozzle inclination angle \( (\phi_n) \).

2.3 The flow in the developing boundary layer region and the influence of gravity

Throughout this study, the stagnation region (i) under the impinging jet will not be considered. The velocity in the outer flow region grows rapidly from 0 at the stagnation point to the jet velocity. Close to the stagnation point (on the order of the jet radius), the flow in the inviscid region becomes uniform and reaches the same jet speed (Lienhard, 2005). The flow at the boundary-layer edge corresponds to the potential flow near the stagnation point with the same velocity of the jet, knowing that the boundary-layer leading edge coincides with the stagnation point Liu et al. (1993). A uniform horizontal flow outside the boundary layer can be assumed (as illustrated Figure 2-1).

It is possible to solve the flow assuming a similarity character, yet it is only possible at the absence of gravity (Watson, 1964). In case where the gravity is included, a non-self-
similar solution should be sought subject to initial conditions at the boundary layer edge. By examining the flow in the boundary-layer region, where the outer flow region is inviscid (Figure 2-1), namely at \( \delta(r, \theta) \leq z < h(r, \theta) \). The velocity in the outer flow region remains radial corresponding to the jet speed. The boundary-layer height \( \delta(r, \theta) \) is determined by considering the mass and momentum balance over the boundary-layer region (ii). The following conditions at the outer edge of the boundary layer must hold:

\[
\begin{align*}
 u(r < r_0, \theta, z = \delta) &= 1, \\
 v(r < r_0, \theta, z = \delta) &= u_z(r < r_0, \theta, z = \delta) = v_z(r < r_0, \theta, z = \delta) = 0.
\end{align*}
\]  

(2.3.1a, b-d)

Therefore, for \( r \leq r_0 \), the use of (2.3.1a) in (2.2.4) leads the following form for the conservation of mass:

\[
\delta \int_{0}^{\delta} u \, dz + h - \delta = \frac{R^2}{2r}.
\]  

(2.3.2)

From (2.2.1a), \( w(r, \theta, z) = \frac{-1}{r \partial r} \left( \int_{0}^{z} u(r, \theta, z) \, dz \right) - \frac{1}{r \partial \theta} \int_{0}^{z} v(r, \theta, z) \, dz \). Upon integrating equations (2.2.5b-c) over the boundary-layer thickness and using (2.27a,b) subject to (2.3.1a-d), the integral form of the radial and azimuthal momentum equations are obtained:

\[
\begin{align*}
 \text{Re} \left( \frac{\partial}{\partial r} \int_{0}^{\delta} ru \, dz + \frac{\partial}{\partial \theta} \int_{0}^{\delta} u v dz - \frac{\partial}{\partial \theta} \int_{0}^{\delta} v dz - \int_{0}^{\delta} v^2 \, dz \right) &= -\frac{\text{Re}}{\text{Fr}^2} h_0 \delta - ru_z(r, \theta, z = 0), \\
 \text{Re} \left( \frac{\partial}{\partial r} \int_{0}^{\delta} u v dz + \frac{\partial}{\partial \theta} \int_{0}^{\delta} v^2 \, dz + \int_{0}^{\delta} u v dz \right) &= -\frac{\text{Re}}{\text{Fr}^2} h_0 \delta - rv_z(r, \theta, z = 0).
\end{align*}
\]  

(2.3.3a, b)

The boundary layer grows with radial distance, eventually invading the entire film width, reaching the jet free surface at \( r = r_0(\theta) \). For \( r < r_0(\theta) \) and above the boundary-layer
outer edge, at some height \( z = h(r, \theta) > \delta(r, \theta) \), lies the free surface. The height of the free surface in region (ii) is then determined from mass conservation inside and outside the boundary layer.

For simplicity, a cubic profile for the velocity components is chosen, which satisfy conditions (2.3.1a,b). Thus, it is assumed that:

\[
\begin{align*}
    u(r < r_0, \theta, z \leq \delta) &= \frac{1}{2} \left[ 3 \left( \frac{z}{\delta} \right) - \left( \frac{z}{\delta} \right)^3 \right] = \frac{1}{2} \left( 3\eta - \eta^3 \right) = f(\eta). \\
    v(r < r_0, \theta, z \leq \delta) &= A \left( z - 2 \frac{z^2}{\delta} + \frac{z^3}{\delta^2} \right) = A\delta \left( \eta - 2\eta^2 + \eta^3 \right).
\end{align*}
\] (2.3.4) (2.3.5)

Where \( \eta = \frac{z}{\delta} \). Here \( A(r, \theta) = v_z(r, \theta, z = 0) \) is the dimensionless azimuthal shear stress on the disk. A cubic profile will be used like (2.3.4), which will also be modified and implemented in the fully viscous region (iii). Different velocity profiles were used in the literature, for example a parabolic profile used by Bohr et al. (1993) and Kasimov (2008). Cubic velocity profiles for a flow on a disk with isotropic and anisotropic slip were adopted by Prince et al. (2012, 2014). Watson (1964) and Dressaire et al. (2010) used similarity profile to simulate non-axisymmetric hydraulic jump patterns. Upon inserting (2.3.4) into (2.3.2), the film height in terms of the boundary-layer thickness can be expressed as:

\[
    h(r, \theta) = \frac{3}{8} \delta(r, \theta) + \frac{R^2(\theta)}{2r}. \tag{2.3.6}
\]

Further, substituting for \( u \) and \( v \) from (2.3.5) and (2.3.6), and eliminating \( h \) through (2.3.7), also by introducing the following transformation:

\[
    (r, A) = Re^{1/3} (\bar{r}, \bar{A}), \quad (h, \delta) = Re^{-1/3} (h, \delta), \quad \alpha = Re^{1/3} Fr^2. \tag{2.3.7a-c}
\]
Thus, coupled equations for $\bar{\delta}(\bar{r}, \theta)$ and $\bar{A}(\bar{r}, \theta)$ in the developing boundary layer region ($\bar{r} \leq \bar{r}_0$) are obtained as follows:

\[
\left( \frac{39}{280} - \frac{3}{8} \delta \right) \frac{\partial}{\partial \bar{r}} \bar{\delta}^2 + \frac{4}{105} \bar{A} \frac{\partial^2 \bar{\delta}}{\partial \theta^2} = -\frac{39}{280} \frac{\bar{\delta}^2}{\bar{r}} - \frac{\delta^2}{2\alpha} + \frac{3}{2}, \tag{2.3.8a}
\]

\[
\frac{19}{420} \left( \bar{r} \bar{A} \bar{\delta}^2 \right) + \frac{1}{105} \left( \bar{A}^2 \bar{\delta}^3 \right) + \frac{19}{420} \bar{R} \bar{A} \delta^2 = -\delta \left( \frac{\bar{R}}{\bar{r}} \frac{d\bar{R}}{d\theta} + \frac{3}{8} \delta \right) - \bar{r} \bar{A}, \tag{2.3.8b}
\]

\[
\bar{h}(r, \theta) = \frac{3}{8} \bar{\delta}(r, \theta) + \frac{R^2(\theta)}{2\bar{r}}. \tag{2.3.8c}
\]

Equations (2.3.8a, b) are solved subject to the azimuthal periodicity of $\bar{\delta}$, $\bar{\delta}_\theta$, $\bar{A}$ and $\bar{A}_\theta$. At the origin, $\bar{\delta}(\bar{r}=0, \theta) = \bar{A}(\bar{r}, \theta) = 0$. The solution is obtained by discretizing the derivatives for the azimuthal direction and integrating the resulting ODEs in the radial direction using Matlab Runge-Kutta scheme for a stiff system. Noting that the free surface height at any location is determined using (2.3.8c). This integration is carried out until the transition point $\bar{r}_0(\theta)$ is reached, that is when $\bar{\delta}(\bar{r}_0, \theta) = 4R^2 / 5\bar{r}_0$ is satisfied.

2.3.1 The developing boundary layer the Azimuthal shear stress on the plate in region (ii)

*Figure 2-4* illustrates the profiles of the boundary-layer and film heights against the radial distance at different $\theta$, for an inclined jet at an angle $\phi = 65^0$. In this case, $R(\theta)$ is given in (2.2.7). It is observed that the BL does not grow following the classical radical behaviour, (Watson, 1964; Schlichting & Gersten, 2000) which is recovered only in the absence of gravity.
Figure 2-4 The behaviour of the developing boundary-layer in region (ii). The rescaled boundary-layer height, $\delta$, and rescaled film thickness, $h$, are plotted against the rescaled radial distance. The rescaled transition location $r_0(\theta)$ coincides with the intersection of the two heights. The results are given at $\alpha = 10$ and $\phi = 65^\circ$.

Indeed, as in *Figure 2-5* in the limit of infinite Froude number ($\alpha \to \infty$), equation (2.3.8b) indicates that $\bar{A}(\bar{r},\theta) \to 0$ and the solution of (2.3.8a) reduces to $\bar{\delta}(\bar{r},\theta) \sim 2\sqrt{\frac{70}{39}} \bar{r}$. Interestingly, although the BL height is independent of $\theta$, the film thickness reflects the non-circular character through $h(r,\theta) \sim \frac{3}{4} \sqrt{\frac{70}{39}} r + \frac{R^2(\theta)}{2r}$.

*Figure 2-6* shows the significant influence of gravity on the boundary layer thickness. At the radial location, $\bar{r} = 0.1$, $\bar{\delta}$ shows a strong non-circular behaviour at strong gravity level. It is noticed that as the gravity magnitude decreases, $\bar{\delta}$ shows weak dependence on $\theta$ at the same radial location, followed by a complete azimuthal independence at $\alpha = \infty$. 
Figure 2-5 At the absence of gravity ($\alpha \to \infty$), the rescaled boundary layer thickness, $\delta$, loses its $\theta$ dependence, yet the azimuthal dependence still is shown in the rescaled film height, $h$, and the rescaled translation point $r_0(\theta)$ response (where both heights intersect).

A significant departure from the classical $\sqrt{\bar{r}}$ behaviour for the BL height is observed, especially close to impingement. In fact, this behaviour is inferred from the asymptotic solution of equation (2.3.8a) for small $r$. In this case, the last two terms on the right-hand side become dominant, suggesting a linear growth for the boundary-layer height near impingement. More precisely, $\tilde{\delta}(\bar{r}, \theta) \sim \sqrt{3\alpha \frac{\bar{r}}{R(\theta)}}$, a behaviour clearly reflected in Figure 2-4 and Figure 2-6 with respect to both the radial distance and azimuthal angle. This expression also explains the drop in boundary-layer height and the diminishing slope as $\theta$ increases (see (2.2.7)). From (2.3.8c), since the boundary-layer height is small near
impingement, the film height decays like 
\[ h(r, \theta) \sim \frac{R^2(\theta)}{2r} \]
regardless of the level of gravity, at a rate that depends strongly on \( \theta \). This behaviour is also reflected by the \( \tilde{h} \)
curves in Figure 2-4, showing a wider spread with respect to \( \theta \) compared to \( \bar{\delta} \).

Figure 2-6 The influence of gravity, \( \alpha \), on the dependence of the rescaled boundary layer thickness, \( \delta \), on \( \theta \). The rescaled radial location \( r=0.1 \) holds for all the cases presented, the jet inclination angle at all cases is \( \phi=65^0 \).

The behaviour of \( \bar{A} \) is much more difficult to assess analytically due to the presence of the two distinct regions as depicted from Figure 2-7. In case \( \alpha = \text{Re}^{1/3} \text{Fr}^2 \) is large, a solution to
\[ O(\varepsilon) = O\left(\frac{1}{\alpha}\right) \]
for \( \bar{A} \) can be sought based on the leading order solution for \( \bar{\delta} \) from (2.3.8a):

\[
\bar{\delta}(r, \theta) = 2 \sqrt[3]{\frac{70}{39} r}, \quad h(r, \theta) = \frac{3}{4} \sqrt[3]{\frac{70}{39}} r + \frac{R^2}{2r}, \quad \bar{\delta}(\theta) = \left(\frac{78}{875}\right)^{1/3} R^{4/3} \quad (3.9a-c)
\]
Substituting (3.9a-c) in equation (2.3.8b), the equation and solution become for the terms of $O(\varepsilon)$:

$$\frac{19}{78} \left( \frac{r^2 A}{r} \right) + \frac{97}{78} r A = -\frac{140}{13} \frac{RR'}{\sqrt{r}}, A(r, \theta) = -\frac{1}{\alpha} \frac{104}{41} \sqrt{\frac{35}{13}} \frac{RR'}{r^{3/2}}$$

(3.10a-b)

Thus, to the leading order terms in (2.3.8a), it can be concluded that the BL height remains unaffected by jet inclination or shape. The film thickness $\tilde{h} \sim R^2$ near impingement, otherwise it is unaffected further downstream (Figure 2-5). Interestingly, the transition location exhibits an azimuthal dependence following $\tilde{h}_\theta \sim R^{4/3}$. The azimuthal shear stress on the disk follows $A \sim \frac{1}{\alpha} \frac{RR'}{r^{3/2}}$ confirming the decrease with radial distance, which holds true even at small values of $\alpha$ (Figure 2-8).

*Figure 2-7* shows the behaviour of $\tilde{A}(\tilde{r}, \theta)$ with respect to the radial location at different angles. $\tilde{A}(\tilde{r}, \theta)$ exhibits a sharp surge near the impingement region, the negative sign denotes that $\tilde{A}(\tilde{r}, \theta)$ acts in the clockwise direction. $\tilde{A}(\tilde{r}, \theta)$ decreases asymptotically until it vanishes again far downstream.

From *Figure 2-8*, the horizontal symmetry of $\tilde{A}$ confirms the behaviour predicted by (3.10b). At $\theta = 140^0$, $RR'$ is maximum (for $\phi = 65^0$), which corresponds to highest value of $\tilde{A}$. Also, by examining the $\theta$-momentum equation along the line of symmetry, it is evident that $\tilde{A}_\theta = 0$ everywhere along that line. It is worth mentioning that the main reason the presence of azimuthal influence in boundary layer region is the azimuthal pressure term which is created by the nozzle’s non-circular behaviour (elliptical cross section as the jet inclines). From the expression of $\tilde{A}$, $\tilde{A}(\tilde{r}, \theta) = -\frac{1}{\alpha} \frac{104}{41} \sqrt{\frac{35}{13}} \frac{RR'}{\tilde{r}^{3/2}}$, at $\theta = 0$ and $\theta = \pi$ the azimuthal shear stress does not exist, which is observed to coincide with $\tilde{\delta}_0 = 0$ at the same angles.
Figure 2-7 The rescaled azimuthal shear stress, $A$, on the disk surface versus the rescaled radial locations in region (ii). The results are produced at $\alpha=10$ and $\phi=65^0$. For the different angles chosen, the end of each curve indicates the location of the rescaled transition point $r_0(\theta)$ at the given angle.

The contours in Figure 2-9 show the azimuthal shear stress on the disk surface, $A$, at different alpha levels in the in the developing boundary region. Gravity is shown to have a significant influence on $A$. At stronger gravity, $A$ is generally higher on the disk surface. By increasing $\alpha$, $A$ shows a drop until it almost completely vanishes at the absence of gravity. Close to the impingement location, $A$ is highest compared with at downstream, which is a common behaviour at all gravity levels. It is worth mentioning that the negative sign in the magnitude of $A$ indicates that $\overline{\tau}_{0z}(\overline{r}, 0 \leq \theta \leq \pi, z = 0)$ is in the clockwise direction, while $\overline{\tau}_{0z}(\overline{r}, \pi \leq \theta \leq 2\pi, z = 0)$ is in the opposite direction, meanwhile $A = 0$ along the line of symmetry.
The azimuthal behaviour of the rescaled azimuthal shear stress, $A$, on the impingement disk surface at different radial locations at $\alpha=10$ and $\phi=65^\circ$.

The effect of $\phi$ on $\delta(\bar{r}, \theta=0)$ is shown in Figure 2-10. At steep inclination, the inertia is found to be minimal in the direction of $\theta = 0$, this is observed from the thickness of the boundary layer and the location of $\bar{\eta}(\theta = 0)$. As the jet inclination is decreased ($\phi$ gets closer to $90^\circ$) inertia is enhanced leading to a thinning of $\delta$, and $\bar{\eta}(\theta = 0)$ is pushed further downstream. $\phi$ has a direct effect on enhancing or limiting the level of inertia for any given direction, thus the momentum transfer is controlled by the jet inclination,
Figure 2-9 The rescaled azimuthal shear stress, A, on the surface on the impingement disk at $\phi=65^0$. The outer edge of the contours indicates the location of the rescaled transition point location $r_0(\theta)$. Different values of $\alpha$ are shown (a) $\alpha = 10$, (b) $\alpha = 30$, (c) $\alpha = 50$ and (d) $\alpha = 70$. 
Figure 2-10 The effect of the jet inclination angle on the rescaled boundary layer thickness, $\delta$, along the rescaled radial location at $\alpha = 20$ and $\theta = 0^\circ$. The vertical line indicates the location of the rescaled transition point $r_0(\theta)$.

In Figure 2-11, the behaviour of $\overline{v}_0(\theta)$ with respect to $\phi$ is shown. At a steep jet inclination, $\overline{v}_0(\theta)$ exhibits an elongated elliptical shape since most of the mass and momentum is directed towards $\theta = 180^\circ$. As $\phi$ increases, the shape of $\overline{v}_0(\theta)$ gets closer to being circular which shows a non-linear response with respect to $\phi$, this has been confirmed at the absence of gravity as showed in (3.9c).
2.4 The fully viscous region and prediction of the location of the jump

2.4.1 The equation for the film thickness and azimuthal velocity

In the fully-viscous region, the potential flow in the radial ceases to exist, and the radial and azimuthal velocity components $U(r, \theta) = u(r, \theta, z = h)$ and $V(r, \theta) = v(r, \theta, z = h)$ at the free surface become dependent on $r$ and $\Theta$. The integral form of the conservation equations is found by integrating it over the film thickness. The equation for the conservation of mass is already given in (2.2.4). The integral form of the momentum conservation equations is derived by integrating (2.2.5a) and (2.2.5b) over the film thickness along with using (2.2.6a,b). which read:

Figure 2-11 The effect of the jet inclination angle $\phi$ on the location of the rescaled transition point $r_0(\theta)$. 
Again, a cubic velocity profile subject to conditions (2.2.3a) and (2.2.3b-d) is assumed. Noting that \( \eta = \frac{z}{h(r, \theta)} \), the radial and azimuthal velocity profiles are of a similar general form and are given in terms of the surface velocity components as:

\[
\begin{align*}
  u(r_0 < r < r_1, \theta, z) = U(r, \theta) f(\eta), \\
  v(r_0 < r < r_1, \theta, z) = V(r, \theta) f(\eta).
\end{align*}
\]  

(2.4.2a-b)

Here, it should be noted \( f(\eta) = \frac{1}{2} \left( 3\eta - \eta^3 \right) \) is the same as given as in (2.3.5).

Substituting for \( u \) from (2.4.2a) the mass conservation equation (2.2.4) yields the following expression of \( U \) in terms of \( h \) and \( R \):

\[
U(r, \theta) = \frac{4}{5} \frac{R^2(\theta)}{r h(r, \theta)}.
\]  

(2.4.3)

This equation agrees with equation (15) of Prince et al. (2012) in the limit of an axisymmetric flow \( (R = 1) \) when setting their slip parameter equal to zero. Equation (2.4.2a-b) is substituted into (2.4.1a) and (2.4.1b), \( U(r, \theta) \) is eliminated using (2.4.3). Also, using the transformation relations in (2.3.7a-c) along with \( V \to \tilde{V} \), the equations for the film thickness \( h \) and azimuthal surface velocity \( V \) in the fully-viscous region are obtained as follows:

\[
\left( 1 - \frac{272}{875} \frac{R^4}{r^2 h^3} \right) \tilde{h}_r + \frac{68}{175} \frac{1}{r^2 h} \left( R^2 \tilde{V}_\theta \right) = \frac{272}{875} \frac{R^4}{r^3 h^2} + \frac{17}{35} \frac{\tilde{V}^2}{r} - \frac{6}{5} \frac{R^2}{r h^3}.
\]  

(4.4a)
\[
\frac{17}{35} \left[ 4R^2 \frac{\partial (\bar{r}V)}{\partial \bar{r}} + \frac{\partial}{\partial \theta} \left( \bar{h} \bar{V}^2 \right) \right] = -\frac{\bar{h}}{\alpha} \bar{h} \theta - \frac{3}{2} \bar{r}^2 \bar{V} \cdot
\]

(2.4.4b)

which are solved simultaneously for \( \bar{r} \geq \bar{r}_0 \) subject to \( \bar{h}(\bar{r} = \bar{r}_0, \theta) = \delta(\bar{r} = \bar{r}_0, \theta) \), \( \bar{V}(\bar{r} = \bar{r}_0, \theta) = 0 \) in addition to the azimuthal periodicity of \( \bar{h}(\bar{r}, \theta) \) and \( \bar{V}(\bar{r}, \theta) \). Similar to region (ii), the solution is obtained by discretizing the derivatives in the azimuthal direction to the first order and integrating the resulting ODEs in the radial direction. The resulting system of equations is solved until a singularity is reached, which is an indication of the location of the hydraulic jump.

2.4.2 The supercritical flow and the location of the hydraulic jump

Figure 2-12 shows how the film thickness behaves in region (iii) at different angles, a general trend of the film thickness reaching a minimum downstream of the transition point is found. The azimuthal dependence is also observed in the way the film thickness behaves after reaching the minimum. For example, at \( \theta = 180^0 \), the film thickness shows a smooth increase before reaching the jump location, however at smaller angles more abrupt jumps are taking place, as indicated by the vertical lines which express the location of the jump.
Figure 2-12 The behaviour of the rescaled developing boundary-layer height, δ, and the rescaled film thickness, h (in region (ii) and (iii)) at different angles with respect to the rescaled radial direction. Also indicated in vertical lines are the locations of the rescaled hydraulic jump location \( r_j \) for different \( \theta \) values.

The three-dimensional behaviour of the film thickness in region (ii) and region (iii) all the way to the hydraulic jump location is shown in Figure 2-13. Due to symmetry, the flow from \( \theta = 0 \) to \( \theta = 180^0 \) is shown. Generally, it is noticeable that the jet inclination enhances the asymmetry of the film height as well as the jump location (Figure 2-12). In directions where inertia is enhanced, the jump is expected to take place at locations where the liquid thickness abruptly increases, as shown in Figure 2-13.
The effect of the jet inclination angle on the rescaled film thickness, $\tilde{h}$, behaviour with respect to the projected radial location on the x and y axes at $\alpha = 20$.

The behaviour of the azimuthal surface velocity, along the radial direction in the fully viscous region is shown in Figure 2-14. Upstream of the fully viscous region, the azimuthal surface velocity is zero everywhere since the velocity at the free surface is essentially the same as the jet velocity where potential flow exists. In the fully viscous region, $\tilde{V}$ grows almost linearly with respect to $\tilde{r}$.
Figure 2-14 The azimuthal surface velocity $\vec{V}$ versus the rescaled radial location at different angles. The figure shows results at $\alpha = 10$ and $\phi = 65^0$.

Figure 2-15 shows the behaviour of $\vec{V}$ in the entire domain, it is evident that flow shows a symmetry along the horizontal axis. In region (ii) where the potential flow exists, $\vec{V} = 0$. Interestingly, as the viscous effects become significant and the flow becomes fully viscous (the vertical line indicating $\bar{r}_0(\theta)$), $\vec{V}$ exhibit non-zero values. Along the horizontal axis, the line joining $\theta = 0$ and $\theta = 180^0$, $\vec{V}$ completely vanishes. The arrows indicate the direction of $\vec{V}$, which is assessed based on the value of $\vec{V}$, when $\vec{V}$ is a negative that indicates that $\vec{V}$ is in the clockwise direction, and the opposite is true. The symmetry of the flow domain can be observed from the behaviour of (2.4.4a-b) where the periodicity is imposed by $R(\theta)$. $\vec{V}$ shows a steady fluctuating behaviour closer to the jump location, due to the complexity of the governing equation for $\vec{V}$, it cannot be
explained analytically. This increase of $\vec{V}$ can be attributed to the conservation of mass, where the flow speeds up in the azimuthal direction to compensate a drop in the radial surface velocity (as will be shown later in this chapter).

**Figure 2-15** The flow field for a symmetric flow, where the behaviour of the azimuthal surface velocity, $\vec{V}$, mirrors about the line of symmetry. The arrows indicate the direction of $\vec{V}$, on the upper half ($0 \leq \theta \leq 180^0$) $\vec{V}$ is negative with a clockwise sense whereas the opposite is true for the lower half ($180 \leq \theta \leq 360^0$).

Figure 2-16, shows the behaviour of $\vec{V}$ for a three-dimensional flow. $\vec{V}$ emerges at the highest elevation of $\vec{h}$ (at $\theta = 180^0$, where the arrows are indicated) following the direction of the downslope of $\vec{h}$ . This indicates that $\vec{h}_\theta$ as well as the gravity have a major influence on driving $\vec{V}$. The line bands represent the azimuthal increment taken from $0$ to $180^0$, where $\Delta \theta = 5^0$.

The role of gravity on enhancing $\vec{V}$ is shown in Figure 2-17. The low values of $\alpha$ indicate strong gravity effect, which has a direct influence on the magnitude of $\vec{V}$. The
presence of rotational surface waves with clockwise and counterclockwise senses was noticed by Teymourtash & Mokhlesi (2014) in their experimental work on unstable non-circular jumps, which indicates that the presence of an azimuthal component for the free surface velocity in the super critical flow region is a sign of the non-circularity of the jump.

Figure 2-16 Three-dimensional representation of the rescaled jet height, h, with respect to the projected radial location on the x and y axes. The color map represents the magnitude of the azimuthal surface velocity \( \vec{V} \). The region where \( \vec{V} = 0 \) in the developing boundary layer region.

The location of the hydraulic jump is highly influenced by the gravity level applied, as is depicted in Figure 2-18. The stronger gravity effect tends to cause a smaller jump radius, which is attributed to the enhanced inertia caused. The location of the singularity that
indicates the jump location appears in the term \[ \frac{1}{\alpha} \left( \frac{272}{875} \frac{R^4}{r^2 h^3} \right) \] in equation (2.4.4a). From the relation \[ \alpha = 875 \tilde{r}_j^2 \tilde{h}_j^3 / 272 R^4 \], the location of the jump and the film thickness just at the jump are expected to behave as \[ \tilde{r}(\theta) \approx \alpha^{1/8} R(\theta)^{5/4} \] and \[ \tilde{h}_j(\theta) \approx \alpha^{1/4} R(\theta)^{1/2} \]. In the limiting of an axisymmetric flow (R=1), this agrees with Kasimov (2008) and Wang & Khayat (2019).
Figure 2-17 The influence of gravity on the azimuthal free surface velocity, $V_r(\theta)$. $r_0(\theta)$ indicates the rescaled transition point location and $r_j(\theta)$ is the rescaled jump location. All the contours are given at $\phi=65^0$. Different values of $\alpha$ are shown (a) $\alpha = 10$, (b) $\alpha = 30$, (c) $\alpha = 50$ and (d) $\alpha = 70$. 
Figure 2-18 The influence of gravity on the rescaled jump location $r_j(\theta)$ different values of $\alpha$ are selected at jet inclination angle $\phi=65^0$. The series of profiles produced by inclining the jet with different angles is shown in Figure 2-19. At severe inclination, where $\phi$ is small, there is a noticeable elongation in the direction of $\theta=180^0$. It is worth recalling that the current theory does not predict the jumps produced at cases $\phi \leq 25^0$ where jumps with corners are formed (Kate et al., 2007b). The severity of elongation of the produced elliptical profile of the jump is reduced significantly as $\phi$ is set to be at a mild inclination, whereas a circular jump is recovered at $\phi=90^0$. 
The influence of the jet inclination angle, $\phi$, on the location of the rescaled hydraulic jump location $r_j(\theta)$ at $\alpha = 20$.

As noticed from Figure 2-20, the surface velocity in the developing boundary layer region is strictly equal to the jet velocity, which is attributed to the presence of an outer potential flow beyond the boundary layer edge, which maintains the equivalence of the surface velocity to the jet velocity. At the transition point, as the viscous forces overtake the whole film thickness, the magnitude of the radial velocity is highly impacted showing a significant decay. It is apparent that the azimuthal dependence affects $U$ in the fully viscous region, which can be identified from expression (2.4.3). At the limit of $R=1$, it is expected to have a uniform flow field of $U$ where an axi-symmetric radial velocity distribution is recovered matching the results of Wang & Khayat (2019). In our case, the jet inclination is the main cause of the inclusion of the $\theta$ dependence for $U$, yet another method of obtaining an azimuthally influenced radial surface velocity can be achieved by imposing an azimuthally varying slip surface (Prince et al., 2014).
Figure 2-20 The behaviour of the free surface radial velocity in the pre-jump region at different θ under jet inclination angle φ=65° and α = 10. \( r_0(\theta) \) indicates the location of the rescaled transition point, and \( r_j(\theta) \) is the rescaled jump location.

In reference to Figure 2-20 and Figure 2-21 collectively, the decay of \( U \) with respect to the \( \bar{r} \) shows different behaviour with respect to θ. There is a sharp decrease in \( U \) at lower angles than 90°, which is a sign of a diminishing inertia. At \( \theta > 90^0 \), the drop of \( U \) becomes milder as result of enhanced inertia which tends to counteract the viscous effect. This is comparable to the flow enhanced by the centrifugal momentum transfer achieved by rotating the impingement disk (Wang & Khayat, 2018).
Figure 2-21 Three-dimensional representation of the rescaled jet height, h, with respect to the projected radial location on the x and y axes, the colour map represents the magnitude of the radial free surface velocity, U, at $\phi=65^0$ and $\alpha=20$.

The influence of gravity on the radial surface velocity in the region (ii) and (iii) is presented in Figure 2-22. Near the transition point, the behaviour is the same under all magnitudes of $\alpha$, moving downstream closer to the jump, the supercritical flow region shows a strong dependence on the strength of gravity applied. At strong gravity, for example at $\alpha=10$, higher radial surface velocity is sustained in the supercritical flow region, yet a sharp declination of radial surface velocity takes place ahead of the jump as gravity effect is attenuated.
Figure 2-2 The influence of gravity on the radial free surface velocity. $r_0(\theta)$ indicates the rescaled transition point location and $r_j(\theta)$ is the rescaled jump location. All the contours are given at $\phi=65^0$. Different values of $\alpha$ are shown (a) $\alpha = 10$, (b) $\alpha = 30$, (c) $\alpha = 50$ and (d) $\alpha = 70$.

Figure 2-23 shows the development of the dimensionless radial wall shear stress (skin friction) along the surface of the impingement disk. In the developing boundary layer region, generally there is a significant decay of the radial shear stress, yet the rate of
decay depends mainly on the azimuthal direction. At any $\theta$, the radial shear is generally less at angles close to $\theta=0$ which is due to the low shear rate. By getting closer to $\theta=180^\circ$, the rapid decrease of the thin film provokes a high shear rate which can be sensed in at the sudden increase of the radial shear stress at the onset of the fully viscous region. Further downstream, as the thin film thickness is slightly recovered (Figure 2-12), the radial shear stress on the disk surface decreases. The effect of gravity on the $\tau_{rz}$ on the disk surface is shown in Figure 2-24. The strong gravity level has a direct effect on enhancing the inertia, which is accompanied by a high shear rate on the thin film cause higher radial shear stress on the disk.

![Figure 2-23](image)

**Figure 2-23** The rescaled radial shear stress $\tau_{rz}$ on the surface of the disk is plotted against the rescaled radial location at different angles, at $\phi = 65^\circ$ and $\alpha = 10$.

From **Figure 2-25**, the azimuthal shear stress ($\tau_{\theta z}$) on the surface of the disk is maximum near the impingement region, which has been already shown in the developing
boundary layer region (*Figure 2-7*). As for the influence of gravity on $\zeta_0z$ in the fully viscous region ($t_0 \leq r \leq t_j$), $\zeta_0z$ generally follows a close behaviour as $V$ (*Figure 2-17*) with respect to different values of $\alpha$. From the insets given in *Figure 2-25*, from $t_0 < r < t_j$ at all gravity levels, $\zeta_0z$ keeps growing until it reaches the location of the jump, this behaviour is completely opposite to that of $\zeta_0z$ in *Figure 2-24*. 
Figure 2-24 The influence of gravity on the rescaled radial shear stress $\tau_{rz}$ on the surface of the disk. $r_0(\theta)$ indicates the rescaled transition point location and $r_j(\theta)$ is the rescaled jump location. All the contours are given at $\phi = 65^0$. Different values of $\alpha$ are shown (a) $\alpha = 10$, (b) $\alpha = 30$, (c) $\alpha = 50$ and (d) $\alpha = 70$. 
Figure 2-25 The rescaled azimuthal shear stress on the surface of the disk, $\tau_{\theta z}$, with respect to the projected rescaled radial location in the pre-jump region with respect to the x and y axes. The inset shows the rescaled $\tau_{\theta z}$ in the fully viscous region. The figure shows the influence of gravity on the rescaled $\tau_{\theta z}$, Different values of $\alpha$ are shown (a) $\alpha = 10$, (b) $\alpha = 30$, (c) $\alpha = 50$ and (d) $\alpha = 70$. 
2.4.3 Relation to Kate et al. (2007a)

It is worth highlighting the difference between the formulation of Kate et al. (2007a) and the present formulation. In addition to neglecting the BL region, Kate et al. (2007a) also neglected the azimuthal component of the flow. Indeed, on setting $V = 0$ and neglecting surface tension, and introducing the rescaling:

$$
\bar{r} \rightarrow \alpha^{1/8} R^{5/4} \bar{r} \\
\bar{h} \rightarrow \alpha^{1/4} R^{1/2} \bar{h}
$$

(2.4.5a-b)

By eliminating $\alpha$ and $R$, equation (2.4.4a) reduces to:

$$
\left(1 - \frac{272}{875} \frac{1}{\bar{r}^2 \bar{h}^3}\right)\bar{h} = \frac{272}{875} \frac{1}{\bar{r}^3 \bar{h}^2} - \frac{6}{5} \frac{1}{\bar{r}^5 \bar{h}^3}, \text{ at } 0 \leq \bar{r} \leq \bar{r}_j
$$

(2.4.6)

Also, by scaling (2.3.8c), the initial condition is obtained:

$$
\bar{h} = \frac{R^{1/4}}{2\alpha^{3/8} \bar{r}}
$$

(2.4.6a)

From (2.4.6), it can be clearly seen that the factors $\alpha$ and $R(\theta)$ are eliminated from the thin film equations, yet they reappear in the initial condition (2.4.6a). Thus, it is inevitable to solve (2.4.6) subject to (2.4.6a) at each direction separately. This is a confirmation that the location of the singularity depends on the azimuthal location at the angle of interest. This is a point that has been overlooked by Kate et al. (2007a), as they proposed that the $\theta$ dependence has no influence on the location where the film thickness height shows a singularity.

Kate et al. (2007a) scaled the momentum equations using their equations (3.10a-d), where $q(\theta) = \frac{V a^2}{2} R(\theta)$, $a$ and $V$ being jet radius and jet velocity respectively. On the other hand, they scaled the conservation of mass using a scale of a non-azimuthally dependent flux “q” which let them assume that the radial flux to be independent of $\theta$, see equation (3.12d) of Kate et al. (2007a). This approximation is not physically accurate, based on the conservation of mass equation (2.2.4). It is evident that the main cause of breaking the symmetry in the flow domain is the term $R(\theta)$, which entrains the azimuthal
dependence in all the governing equations. The usage of two different assumptions about the flow flux \( q \) led to an inconsistent initial condition for their equations.

The intrinsic difference between the current theory and Kate et al. (2007a) is solving the thin film equation. Recall that the current theory is mainly based on the theory presented by Wang & Khayat (2019), also noting that Kate et al. (2007a) theory is based on Bohr et al. (1993). To show the similarity between both theories, the axisymmetric case is recovered and region (ii) from the current theory is ignored. On the other hand, by assuming a cubic velocity profile for Kate et al. (2007a) / Bohr et al. (1993), then recasting the scaling equation (21) at Bohr et al. (1993) in terms of \( \alpha \). This will lead to their thin film equation \( \left( 1 - \frac{1088}{875} \frac{1}{h^3r^2} \right) \hat{h}_F = \frac{1}{h^2r^2} \left( \frac{1088}{875h^2} - \frac{12}{5h} \right) \), which is subject initially to \( \hat{h} = \frac{1}{\alpha^{3/8}r} \). After recalling equation (2.4.6) and the initial condition (2.4.6a) by setting \( R = 1 \) and \( V = 0 \), it can be noticed that both approaches are equivalent, yet the difference lies in the constants in the thin film equations which are compensated for in the initial conditions.

Another way to establish the equivalence with the formulation of Kate et al. (2007a), substitute their average velocity from their equation (3.23) using their equation (3.16) to obtain the thin film equation as proposed by them:

\[
\left( 1 - \frac{6}{5} \frac{1}{r^2h^3} \right) h_r = \frac{6}{5} \frac{1}{r^3h^2} - \frac{3}{rh^3}. \tag{2.4.7}
\]

Aside from the different velocity profiles used in both theories, equations (2.4.7) and (2.4.6) are essentially equivalent. The rescaling in (2.4.5a-b) is equivalent to the scaling (3.10) in Kate et al. (2007a), except for an additional factor of \( \frac{1}{2} \) used in the latter approach as showed in the previous section. Their method of scaling is based on the scaling of the jump radius suggested originally by Bohr et al. (1993).

According to Bohr et al. (1993) & Kate et al. (2007a), (2.4.7) was further expressed in a general form of \( C_1 \frac{1}{hr} \frac{\partial}{\partial r} \frac{1}{h_{r1}} = -h' - C_2 \frac{1}{h^3r} \) where \( C_1 = 6/5 \) and \( C_2 = 3 \). After eliminating the coefficients using \( r \to C_1^{1/2} C_2^{-3/8} r \) and \( h \to C_2^{1/4} h \) and solving the thin
film equation using the same initial condition used by Bohr et al. (1993). It is found that the location of singularity take place at \( r \approx 1.38 \), as opposed to the value reported by Bohr et al. (1993) and Kate et al. (2007a) which is \( r \approx 1 \). The lower value of the estimated singularity location gave rise to an underestimation of the location of the hydraulic jump; however, it showed more agreement with experiments conducted using water. After using the correct values, the location of the hydraulic jump can be expressed as \( R_j(\theta) \approx 1.38Cq^{5/8}v^{-3/8}g^{-1/8} \) where \( q(\theta) = \frac{Q(\theta)}{2\pi} \) and \( C \approx 0.72 \), \( C \) is a factor used for a parabolic velocity profile. For the current theory, the estimated hydraulic jump location behaves as \( R_j(\theta) \approx C(\theta) \frac{Q^{5/8}}{\pi}v^{-3/8}g^{-1/8} \), which is the same behaviour as reported by Kate et al. (2007a) and Bohr et al. (1993), except for the values of the constants.

To identify the importance of including the boundary layer region and the azimuthal velocity components, (2.4.6) is solved subject to (2.4.6a) at each azimuthal direction and recasting the radius of the jump as \( \bar{R}_j(\theta) \). Figure 2-26 shows the location of the jump found from section (2.2.4) and in the case where the developing boundary layer region and \( V = 0 \). Ignoring the effect of region (ii) and \( V \) on the flow led to overestimate the location of the hydraulic jump by about 8% at \( \phi = 65^0 \). As the inclination increased, the error increased to 11% at \( \phi = 35^0 \).

Kate et al. (2007a) conducted an experiment to determine the jump radius produced by an inclined water jet on a horizontal surface. As far as the aspect ratio (a/b) of the produced jump profile is concerned, the current theory showed a close agreement with their results as shown in Figure 2-27. At \( \phi \leq 30^0 \), there is a deviation from the experimental results, yet the current theory shows a slight enhancement (approximately 3%) in predicting the aspect ratio. The deviation is expected to be due to the commencement of formation of jet-jump interaction in some regions in the flow field, in which the current theory does not account for. At \( \phi \leq 25^0 \), Kate et al. (2007a) noted that the jet-jump interaction becomes dominant, which causes the formation of jump with corners.
Figure 2-26 Comparison between the estimated location of rescaled hydraulic jump location, $r_j(\theta)$, in the case of ignoring the azimuthal velocity component, $V$, and the developing boundary layer region. The results are produced at $\alpha=20$ and (a) $\phi = 65^0$ (b) $\phi = 35^0$
The aspect ratio of the hydraulic jump profile with respect to the change of the jet inclination angle ($\phi$). The current theory is compared with the theoretical and experimental work of Kate et al. (2007a). The experiment parameters are $Q = 5.83 \times 10^{-5}$ m$^3$/s, $a = 3.5$ mm, $H = 56$ mm (Jet to disk distance). The vertical line indicates that the current theory does not hold at $\phi \leq 25^0$.

2.5 Summary

In the current chapter, a theoretical formulation to predict the location of the non-circular hydraulic jumps produced due to the impingement of an inclined jet is established. The regions governing the non-axisymmetric flow produced are closely studied. The effect of some of parameters (such as gravity and jet inclination angle) on the behaviour of the boundary layer and film thickness formed in both regions is presented. In addition, a comparison has been made between the current work and the previous work in the literature, a close agreement has been shown.
3 Conclusions and Future Work

In this chapter, the conclusions of the current thesis are discussed, followed by some recommendations for future work.

3.1 Conclusions:

In this study, the flow of an inclined jet impinging on a stationary disk is examined theoretically. The current study exclusively dealt with high-viscosity liquids (10 times more viscous than water). The main assumption made is that the effective inclination angle of the jet (\(\phi\)) follows \(\phi \geq 25^0\). The present study focused on the three-dimensional behaviour of the impingement flow produced by the jet inclination, where the main aim was to predict the location of the non-circular hydraulic jumps created. The current theory is restricted to laminar non-circular steady jumps of type (I), which is marked by a sharp increase in the fluid thickness at the jump location. (Figure 2-1)

Despite the previous findings by Kate et al. (2007a), their prediction remains somewhat questionable due to the excessive approximations they made to model the non-axisymmetric flow. The scarcity of theoretical models in the literature that focused on non-circular jumps is attributed to the complexity faced upon including the azimuthal effects in the equations governing the flow.

The current model dealt with steady non-axisymmetric flow of a Newtonian fluid. The approach presented is based on the Kármán–Pohlhausen integral method. This method was helpful to model the behaviour of the boundary layer flow in the developing boundary-layer region (ii) and the fully developed viscous region (iii). The integral form of the continuity and the three-dimensional momentum equation in regions (i) and (ii) are treated numerically at each region. The flow from both regions was matched at the transition point. It was noted that the flow in the presence of gravity, the problem cannot be solved using a similarity solution (Watson, 1964).

In region (ii), both the radial and azimuthal velocity components were assumed to exhibit a cubic velocity profile. Both profiles were assumed to have different coefficients,
due to the different boundary conditions for both components. In region (iii) both velocity profiles were assumed to hold a self-similar cubic profile with a similar general form.

Throughout the formulation of the problem, the Reynolds number (Re) and the Froude number (Fr), were both defined in terms of the jet radius and the jet velocity. Using the parameter \( \alpha = \frac{Re^{\frac{1}{3}} Fr^2}{2} \), the problem has been casted in terms of only one parameter.

It was found that high gravity level tends to enhance the non-axisymmetric behaviour of the developing boundary layer thickness. Interestingly, in the absence of gravity the developing boundary layer became axisymmetric while the non-axisymmetric behaviour was noticed only on the film thickness. At strong gravity level, it was noticed that the boundary layer thickness depended strongly on the azimuthal location. The transition point showed a \( \theta \) dependence at all gravity levels, which moved towards the edge of the impingement disk as gravity became stronger. The transition point showed an eccentric - elliptical profile as a response to the jet inclination.

In the fully viscous region (iii), the two thin film equations, in the radial and azimuthal coordinates were formulated. The thin-film equations were solved until an azimuthally-dependent singularity was reached. The occurrence of the singularity depends on the radial-thin-film equation, where it coincides with the jump location. At a given radial location on the disk, the inertia gets weaker, and a balance occurs between the inertia and gravity terms (equation 2.4.4a) occurs. This coincides with the singularity in the thin-film equations, indicating the location of the jump. It has been found that in non-axisymmetric flow, the strength of inertia depends on the azimuthal location, as a response to the jet inclination. At each azimuthal location, the balance between the dominant effects (inertia, gravity and viscous) occurs at a different radial location, thus leading to the non-circular jump.

Overall, the jump profile is elliptical in all cases of an inclined jet at different gravity strengths, yet the elongation of the jump shape produced mainly depends on the degree of the inclination of the jet. It was observed that by increasing the gravity level, the jump radius gets smaller.
The increase of gravity level enhances the inertia, which leads to a strong thinning in the film. This film thinning effect evokes a stronger viscous friction due to the high shear rate, which is explained by the increase of the radial shear stress at the onset of region (iii). It was indicated that the film thickness decreases downstream of the stagnation point at all levels of gravity and it exhibits a minimum before the occurrence of the jump. This behaviour holds true regardless of the orientation, however, the severity of the thin-film thinning depends on orientation.

The behaviour of the radial surface velocity with respect to the radial as well as the azimuthal location was shown. The skin friction on the surface of the disk in the radial direction followed the same trend of the radial surface velocity. It showed a decay downstream from the impingement location.

It was proved in our formulation that the presence of the azimuthal surface velocity in the fully viscous region is essential for the occurrence of non-circular jumps. The azimuthal shear stress was found to be strongest near the stagnation point. In region (ii) it showed a decay downstream; however, in region (iii) it followed the behaviour of the azimuthal surface velocity as it grew closer to the jump location.

### 3.2 Future work and recommendations:

As is presented in the current thesis, non-circular hydraulic jumps and the non-axisymmetric flow accompanying it are not given enough attention in the literature. This includes experimental, theoretical, and numerical work.

Although experimental work has been done by Kate et al. (2007a) to study the effect of jet inclination on the location of the jump. Their work was mainly conducted using a low viscosity fluid (water). Hydraulic jumps for water are difficult to study without relying on some information about the flow downstream of the jump (Bohr et al., 1993). Also, it has been discussed that our current theory does not apply for flow produced by highly inclined jets ($\phi \leq 25^0$) where jumps with corners were formed.
To address the issues stated above, considering the surface tension effect in the current theory will produce more interesting results. It is expected to capture the flow of fluids with strong surface tension, such as water. In addition, it is expected to predict jumps with corners that are produced by jet-jump interaction. It is worth mentioning that the inclusion of the surface tension could cause some numerical difficulty regarding solving the governing equations. Especially when the concavity of the interface is severe, this is an issue that might need some deliberate approximations. Another rich addition to the literature would be an experimental investigation of the hydraulic jumps caused by jet inclination for highly viscous liquids.

The current study could be investigated further using a Computational Fluid Dynamics (CFD) numerical model, namely using volume of fluid / multi-phase flow model. Although meshing the current problem might be a burdensome, making use of the periodicity of the flow domain could ease tackling that problem. Recent CFD work for hydraulic jumps have shown promising results, as it dealt with some of the uncertainties experienced in theoretical models (Wang & Khayat, 2021; Fernandez-Feria et al., 2019).

3.3 Summary

This chapter started by presenting conclusions drawn from the current study. The effect of some parameters on the hydraulic jump and the thin-film flow produced by an inclined jet impingement is discussed. Then recommendations for future work are covered.
References


Appendices

Appendix A1: Derivation of the conservation of mass relation for three-dimensional flow.

In this appendix, the conservation of mass of across a non-circular flow region is derived. For an impinging inclined jet, we expect that the impingement region and the flow downstream to exhibit \( \theta \) dependence. As in Figure A-1, we define a control surface or a rubber band “S” that lies at a distance \( r \) from the origin. We assume that \( S \) is the control surface surrounding a control volume \( V \). This volume is the closed region of thickness \( dz \).

Figure A-1. For a three-dimensional flow, the shaded grey portion expresses the control surface under study, \( dS \). The control volume, \( V \), is the space enclosed by the non-circular band shown.

Note that the velocity vector is 3D: \( \bar{\mathbf{v}} = u^* \mathbf{e}_r + v^* \mathbf{e}_\theta + w^* \mathbf{e}_z \) and \( \bar{n} \) is a vector normal to \( dS \), where \( dS = r^* d\theta dz^* \). Thus, \( \mathbf{n} = n_r \mathbf{e}_r + n_\theta \mathbf{e}_\theta \) and \( \bar{\mathbf{v}} \cdot \bar{n} = u^* n_r + v^* n_\theta \). For an infinitesimal control volume portion, \( dQ \), which the flow of velocity \( \bar{\mathbf{v}} \) enters:

\[
\begin{align*}
dQ &= \bar{\mathbf{v}} \cdot \bar{n} dS = \left( u^* n_r + v^* n_\theta \right) r^* d\theta dz^*. 
\end{align*}
\]

(A1-1)
Consequently, $Q = \iiint_{S} \mathbf{v} \cdot \mathbf{n} dS = \int_{0}^{2\pi} \int_{0}^{h(r^*, \theta)} \left( u^* n_r + v^* n_\theta \right) r^* d\theta^* dz^*$. Now, since $V$ and $S$ are arbitrary, let us choose $S$ to be the interior of a cylinder of radius $r$ and height $dz$. In this case, $n_r = 1$, $n_\theta = 0$, so that:

$$Q = \iiint_{S} \mathbf{v} \cdot \mathbf{n} dS = r \int_{0}^{2\pi} \int_{0}^{u^* (r^*, \theta, z^*)} d\theta^* dz^*. \quad (A1-2)$$

This shows that $v$ is not involved in the conservation of mass.

We can verify this result by integrating the continuity equation:

$$\iiint_{0}^{2\pi h^*(r^*, \theta)} \left( \frac{\partial (r^* u^*)}{\partial \theta^*} + \frac{\partial v^*}{\partial \theta^*} + \frac{\partial (r^* w^*)}{\partial z^*} \right) dz^* d\theta^* = \iiint_{0}^{2\pi h^*(r^*, \theta)} \frac{\partial (r^* u^*)}{\partial \theta^*} dz^* d\theta^* + \iiint_{0}^{2\pi h^*(r^*, \theta)} \frac{\partial v^*}{\partial \theta^*} dz^* d\theta^* \quad (A1-3)$$

Using Leibniz rule for the first two terms on the left-hand side:

$$\iiint_{0}^{2\pi h^*(r^*, \theta)} \frac{\partial (r^* u^*)}{\partial \theta^*} dz^* d\theta^* = \iiint_{0}^{2\pi} \frac{\partial h^*(r^*, \theta)}{\partial \theta^*} \int_{0}^{u^* (r^*, \theta, z^*)} dz^* - r^* \frac{\partial h^*}{\partial \theta^*} u^* \left( r^*, \theta, z^* = h^* \right) d\theta. \quad (A1-4)$$

$$\iiint_{0}^{2\pi h^*(r^*, \theta)} \frac{\partial v^*}{\partial \theta^*} dz^* d\theta^* = \iiint_{0}^{2\pi} \frac{\partial h^*(r^*, \theta)}{\partial \theta^*} \int_{0}^{v^* (r^*, \theta, z^*)} dz^* - \frac{\partial h^*}{\partial \theta^*} v^* \left( r^*, \theta, z^* = h^* \right) d\theta. \quad (A1-5)$$

By substituting these two terms, the continuity equation becomes:
Using the kinematic condition from (1.3.20), substituting that term in the continuity equation yields:

\[
\int_0^{2\pi} \frac{\partial}{\partial r^*} \int_0^r r^* u^* dz^* - r^* \frac{\partial h^*}{\partial r^*} u^* \left( r^* , \theta , z^* = h^* \right) d\theta + \int_0^{2\pi} \frac{\partial}{\partial \theta} \int_0^r v^* dz^* - \frac{\partial h^*}{\partial \theta} v^* \left( r^* , \theta , z^* = h^* \right) d\theta \\
+ r^* \int_0^{2\pi} w^* \left( r^* , \theta , z^* = h^* \right) d\theta = 0.
\]

(A1-6)

After canceling similar terms with opposite signs, reduces the continuity equation to:

\[
\int_0^{2\pi} \frac{\partial}{\partial r^*} \int_0^r r^* u^* dz^* d\theta + \int_0^{2\pi} \frac{\partial}{\partial \theta} \int_0^r v^* dz^* d\theta.
\]

(A1-8)

Due to the periodicity of the flow, the above term is expected to vanish.

Therefore, we are left with:

\[
\int_0^{2\pi} \frac{\partial}{\partial r^*} \int_0^r r^* u^* dz^* d\theta = \int_0^{2\pi} \frac{d}{dr^*} \int_0^r r^* u^* dz^* d\theta = 0 \\
= \int_0^{2\pi} \int_0 r^* u^* dz^* d\theta = Q.
\]

(A1-10)

Q denotes the jet volumetric flow rate.
Now, considering the flow issuing from an inclined nozzle that produces a jet with an inclination angle $\phi$. Assuming the radius of the nozzle is $a^*$ with a jet velocity $W$*, the volumetric flow rate across the jet cross section $R^* (\theta)$ can be expressed as:

$$Q = W \int_0^{2\pi} \int_0^{R^* (\theta)} r^* \, dr \, d\theta = W \int_0^{2\pi} \frac{R^* (\theta)^2}{2} \, d\theta.$$

(A1-11)

From (A1-10) and (A1-11):

$$Q = W \int_0^{2\pi} \frac{R^* (\theta)^2}{2} \, d\theta = r^* \int_0^{2\pi} u^* \, dz \, d\theta.$$

(A1-12)

By non-dimensioning the above equations, by using the scales $r^* = ar$, $u^* = uW$ and $z^* = az$, we have:

$$\frac{R^* (\theta)^2}{2} = r \int_0^h u^* \, dz.$$

(A1-13)

This equation is the same as equation (2.2.4) which is used for assessing the conservation of mass across the non-axisymmetric flow.
Appendix B2: Solving the thin-film equations in the fully viscous region.

In this appendix we show the methodology followed to solve the thin-film equations in region (iii). First, we introduce the following scaling:

\[
\bar{r} = r R^{5/4}, \quad \left(\bar{h}, \bar{\delta}\right) = \left(h, \delta\right) R^{1/4}, \quad V = \bar{V} R^{1/4}.
\] (A2-1)

The main purpose of the scaling is to eliminate the \( \theta \) dependence (comes from the term \( R \)) from the coefficient of \( h_r \) where the singularity is expected to occur.

Using scaling in (A2-1), equations (2.4.4a) and (2.4.4b) become:

\[
\left(\frac{1}{\alpha} - \frac{272}{875} \frac{1}{r^2 h^3}\right) h_{\bar{r}} = \frac{272}{875} \frac{1}{r^2 h^2} - \frac{6}{5} \frac{1}{r h^3} + \frac{17}{35} \frac{V^2}{r} - \frac{68}{175} \frac{R^{3/4}}{r^2 h} \left(\bar{V} R^{-3/4}\right). \tag{A2-2}
\]

\[
\left(\frac{68}{175}\right) \frac{\partial \bar{V}}{\partial \bar{r}} = \left(- \frac{3\bar{r}}{2h} - \frac{68}{175 r}\right) \partial \bar{h} - \frac{\bar{h}}{\alpha} \partial \theta - \frac{17}{35} \frac{1}{R^{9/4}} \frac{\partial}{\partial \theta} \left(\bar{V}^2 \bar{h} R^{9/4}\right). \tag{A2-3}
\]

(A2-2) and (A2-3) are subject to \( \bar{h}(\bar{r}_0(\theta), \theta) = \bar{\delta}(\bar{r}_0(\theta), \theta) \) and \( \bar{V}(\bar{r}_0(\theta), \theta) = 0 \). These are the thin film equations in the radial and azimuthal directions at \( \bar{r}_0(\theta) \leq \bar{r} \leq \bar{r}_j(\theta) \). For a system of equations that deals with (A2-2) and (A2-3) at each \( \theta \), each pair of these equations is defined at a different location, based on \( \bar{r}_0(\theta) \). To eliminate this issue, we introduce a coordinate mapping transformation that follows:

\[
\begin{align*}
x(\bar{r}, \theta) &= \bar{r} - \bar{r}_0(\theta), \quad y(\bar{r}, \theta) = 0. \tag{A2-4}
\end{align*}
\]

\( x \) and \( y \) are two coordinates that are defined at \( 0 \leq x \leq x_j, \quad 0 \leq y \leq 2\pi \) respectively. Using the chain rule:

\[
\begin{align*}
\frac{\partial f}{\partial \bar{r}} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{r}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{r}} = \frac{\partial f}{\partial x}, \\
\frac{\partial f}{\partial \theta} &= -\frac{\partial \bar{r}_0}{\partial \theta} \frac{\partial f}{\partial \bar{r}} + \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}. \tag{A2-5}
\end{align*}
\]
We note that \( f \) holds for the variables \( \bar{h} \) and \( \bar{V} \).

By applying the transformation (A2-4) to (A2-2) and (A2-3). Also using (A2-5), for substituting for the derivatives of \( \bar{h} \) and \( \bar{V} \) in terms of \( x \) and \( y \), the film equation in the \( r \) direction becomes:

\[
\left( \frac{1}{\alpha} - \frac{272}{875(x - \bar{r}_0)^2 h^3} \right) \ddot{h}_x - \left( \frac{68}{175(x - \bar{r}_0)^2 h \theta} \right) \frac{\partial \bar{V}}{\partial x} = \left( \frac{136}{525(x - \bar{r}_0)^2 h} + \frac{17 \bar{V}}{70(x - \bar{r}_0) R \theta} \right) \frac{1}{h^2} \frac{\partial \bar{V}}{\partial x} + \frac{6}{5(x - \bar{r}_0)} \frac{17 \bar{V}^2}{35(x - \bar{r}_0)^2 h} - \frac{68}{175} \frac{\partial \bar{V}}{\partial y}.
\]

(A2-6)

Also, for the film equation in the \( \theta \) direction:

\[
- \left( \frac{\bar{h}}{\alpha} + \frac{17 \bar{V}^2}{35} \right) \frac{d\bar{r}_0}{d\theta} \frac{\partial \bar{V}}{\partial x} + \left( \frac{68}{175} - \frac{34 \bar{V}}{35 h} \frac{d\bar{r}_0}{d\theta} \right) \frac{\partial \bar{V}}{\partial x} = \left( \frac{3(x - \bar{r}_0)}{2h} + \frac{68}{175} \frac{1}{(x - \bar{r}_0)} + \frac{153 \bar{V} \bar{h}}{140 R} \frac{dR}{d\theta} \right) \frac{\partial \bar{V}}{\partial x} - \frac{34 \bar{V}}{35} \frac{\partial \bar{V}}{\partial y} - \left( \frac{\bar{h}}{\alpha} + \frac{17 \bar{V}^2}{35} \right) \frac{\partial \bar{h}}{\partial y}.
\]

(A2-7)

(A2-6) and (A2-7) are subject to \( \bar{h}(x = 0, y) = \bar{\delta}(x = 0, y) \) and \( \bar{V}(x = 0, y) = 0 \). It is clear from (A2-6) and (A2-7) that we replaced the \( r \) and \( \theta \) axes by \( x \) and \( y \) respectively, which are 2 dependent axes on both \( r \) and \( \theta \). We note that (A2-6) and (A2-7) take the following forms:

A1\( \ddot{h}_x \) + A2\( \ddot{V}_x \) = B11,

(A2-8a)

A2\( \ddot{h}_x \) + A22\( \ddot{V}_x \) = B21.

(A2-8b)

Such that the coefficient A11, A12, A21, A22, B11 and B21 are defined as:

\[
A11 = \frac{1}{\alpha} - \frac{272}{875} \frac{1}{(x - \bar{r}_0)^2 h^3},
\]

(A2-9a)
A12 = \frac{68}{175} \frac{1}{(x - \bar{r}_1)^2} \frac{d\tilde{r}_0}{d\theta}, \quad (A2-9b)

B11 = \left( \frac{136}{525} \frac{1}{(x - \bar{r}_0)^2} \tilde{h} + \frac{17\tilde{V}}{50} \frac{dR}{d\theta} - \frac{1}{h^2} \right) \left( \frac{6}{5} \frac{1}{(x - \bar{r}_1)\tilde{h}} + \frac{17\tilde{V}^2}{35} \frac{d\tilde{V}}{d\theta} - \frac{68}{175} \frac{\partial\tilde{V}}{\partial\tilde{y}} \right).

(A2-9c)

A21 = -\left( \frac{\tilde{h} + \frac{17\tilde{V}^2}{35}}{\alpha} \right) \frac{d\tilde{r}_0}{d\theta}, \quad (A2-10a)

A22 = \frac{68}{175} \frac{34}{35} \tilde{h} \tilde{V} \frac{d\tilde{r}_0}{d\theta}, \quad (A2-11b)

B21 = -\left( \frac{3(x - \bar{r}_1)}{2\tilde{h}} + \frac{68}{175} \frac{1}{(x - \bar{r}_0)} + \frac{153}{140} \frac{\tilde{V}h}{d\theta} \right) \tilde{V} \left( \frac{34}{35} \frac{\tilde{h}\tilde{V}}{\partial\tilde{y}} \frac{\partial\tilde{V}}{\partial\tilde{y}} - \left( \frac{\tilde{h} + \frac{17\tilde{V}^2}{35}}{\alpha} \right) \right).

(A2-11c)

Now, (A2-8a) and (A2-8b) can be expressed in the following matrix form:

\[
\begin{bmatrix}
B1(x, y, \tilde{h}, \tilde{V}, \tilde{h}_y, \tilde{V}_y) \\
B2(x, y, \tilde{h}, \tilde{V}, \tilde{h}_y, \tilde{V}_y)
\end{bmatrix} = \begin{bmatrix}
A11(x, y, \tilde{h}, \tilde{V}) & A12(x, y, \tilde{h}, \tilde{V}) \\
A21(x, y, \tilde{h}, \tilde{V}) & A22(x, y, \tilde{h}, \tilde{V})
\end{bmatrix} \begin{bmatrix}
\tilde{h}_x \\
\tilde{V}_x
\end{bmatrix}. \quad (A2-12)
\]

After isolating the vector \begin{bmatrix} \tilde{h}_x \\ \tilde{V}_x \end{bmatrix} and finding the inverse of the matrix of coefficients A11, A12, A21 and A22, (A2-12) becomes:

\[
\frac{1}{\text{DET}} \begin{bmatrix}
B1A22 - A12B2 \\
A11B2 - A21B1
\end{bmatrix} = \begin{bmatrix}
\tilde{h}_x \\
\tilde{V}_x
\end{bmatrix}. \quad (A2-13)
\]

After expanding (A2-13), the equations for \( \tilde{h}_x \) and \( \tilde{V}_x \) respectively read:
\[ \tilde{h}_x = \frac{D_1}{D_{E}} = P(x, y, \tilde{h}, \tilde{V}, \tilde{h}_y, \tilde{V}_y). \]  
(A2-14)

\[ \tilde{V}_x = \frac{D_2}{D_{E}} = Q(x, y, \tilde{h}, \tilde{V}, \tilde{h}_y, \tilde{V}_y). \]  
(A2-15)

We define:

\[ D_{E} = A_{11}A_{22} - A_{12}A_{21}, \]  
(A2-16a)

\[ D_1 = B_{11}A_{22} - A_{12}B_{2}, \]  
(A2-16b)

\[ D_2 = A_{11}B_{2} - A_{21}B_{1}. \]  
(A2-16c)

Equations (A2-14) and (A2-15) are subject to:

\[ \tilde{h}(x = 0, y) = \bar{h}(x = 0, y) \text{ and } \tilde{V}(x = 0, y) = 0, \]  
(A2-17a, b)

\[ \tilde{h}(x, y = 0) = \tilde{h}(x, y = 2\pi), \]  
(A2-17c)

\[ \tilde{V}(x, y = 0) = \tilde{V}(x, y = 2\pi). \]  
(A2-17d)

Equations (A2-14) and (A2-15) can be solved by discretizing it in the \( \theta \) direction and solving the resulting ODE following \( 0 \leq x \leq x_j \). We denote the location of the \( \theta \)-independent singularity as \( x_j \). The \( \theta \)-dependent jump location can be found as:

\[ \tilde{r}_j(\theta) = x_j + \tilde{r}_0(\theta). \]  
(A2-18)

From the scaling in (A2-1), the location of the rescaled hydraulic jump can be using the following:

\[ \bar{r}_j(\theta) = \tilde{r}_j(\theta) R^{5/4}. \]  
(A2-19)
Appendix C3: Numerical approach for solving the boundary layer equations in section (2.3)

In this appendix we show the numerical approach used to solve the boundary layer equations in section (2.3).

As for equations (2.3.8b) and (2.3.8b), the main aim is to transform the given PDEs into a system of ODEs to solve. The given equations are discretized in the θ direction, all the derivatives with respect to θ are evaluated using the finite difference backward scheme. The discretized (2.3.8b) and (2.3.8b) equations respectively become:

\[
\frac{d\bar{\delta}_i}{d\bar{r}} = \frac{3}{2} \left( \frac{1}{105} \bar{A}_i \frac{2\bar{\delta}_i^2}{\bar{r}^2} + \frac{R_i^2}{2\alpha \bar{r}} \right) \left( \frac{\bar{\delta}_i^2}{\bar{r}} \right) - \frac{4}{105} \frac{\bar{A}_i \bar{\delta}_i^2 - \bar{A}_{i-1} \bar{\delta}_{i-1}^2}{\Delta \theta}
\]

\[
\left( \frac{39}{280} - \frac{3 \bar{\delta}_i}{8 \alpha} \right) \bar{\delta}_i
\]

\[-\left( 1 + \left( \frac{d\bar{\delta}_i}{d\bar{r}} + \frac{\bar{\delta}_i}{\bar{r}} \right) \frac{38}{420} \frac{d\bar{A}_i}{d\bar{r}} \right) \bar{A}_i - \frac{1}{\bar{r}} \frac{1}{105} \left( \frac{\bar{A}_i 2\bar{\delta}_i^3 - \bar{A}_{i-1} 2\bar{\delta}_{i-1}^3}{\Delta \theta} \right)
\]

\[
\frac{d\bar{A}_i}{d\bar{r}} = \frac{3}{8\alpha} \frac{\bar{\delta}_i}{\bar{r}} \frac{\bar{\delta}_i - \bar{\delta}_{i-1}}{\Delta \theta} - \frac{1}{\alpha} \frac{2R_i}{\bar{r}} \frac{dR}{d\theta}
\]

These equations are solved from \(0 < \bar{r} < \bar{r}_0\) using Runge–Kutta method. The index \(i = 1, \ldots, N\), where \(N\) is the total number of nodes chosen in the θ axis. \(\Delta \theta\) is the azimuthal increment along the θ axis. (A3-1) and (A3-2) are subject to:

\[
\bar{\delta}(r = 0, \theta) = 0, \bar{A}(r = 0, \theta) = 0.
\]

Also, by distributing the nodes from 0 to \(2\pi\), based on the periodicity of the domain we have the condition:

\[
\bar{A}(r, i = 1) = \bar{A}(r, i = N), \bar{\delta}(r, i = 1) = \bar{\delta}(r, i = N).
\]
Its worth mentioning that the term \( \frac{dR}{d\theta} \), shall be evaluated directly based on the expression on \( R(\theta) \).
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