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Distribution of the p-Torsion of Jacobian Groups of Regular Matroids

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Abstract

Given a regular matroid M and a map $\lambda\colon E(M) \to \mathbb{N}$, we construct a regular matroid M_λ. Then we study the distribution of the p-torsion of the Jacobian groups of the family $\{M_\lambda\}_{\lambda \in \mathbb{N}^{E(M)}}$. We approach the problem by parameterizing the Jacobian groups of this family with nontrivial p-torsion by the $\mathbb{F}_p$-rational points of the configuration hypersurface associated to M. In this way, we reduce the problem to counting points over finite fields. As a result, we obtain a closed formula for the proportion of these groups with nontrivial p-torsion as well as some estimates. In addition, we show that the Jacobian groups in this family with nontrivial p-torsion appear with frequency close to $1/p$, provided M is irreducible.

Keywords: Arithmetic statistics, regular matroid, series extension, Jacobian group, torsion, configuration polynomial, configuration hypersurface, finite field, rational point, density.

Summary for Lay Audience

Characterizing a mathematical property or object often proves to be an arduous task. A more manageable approach is to consider a collection of mathematical objects of certain type and then study the variation of a particular property within that collection. More precisely, one looks for the proportion of members in the collection having said property. These techniques have proved to be fruitful in arithmetic statistics and it is in this field that our problem lies.

In this work, we deal with matroids, which are combinatorial structures that were created to study the abstract properties of linear independence. Starting with a "base" regular matroid, we construct a collection of regular matroids. To each regular matroid, we associate a finite abelian group, which is, in particular, a finite set. Understanding the structure of this group is important for many areas of mathematics, yet, little is known. One natural step in this direction is to establish when the Jacobian group of a regular matroid has p-torsion, that is, when is its size divisible by p? In this thesis, we determine the proportion of members (in the collection under consideration) having Jacobian group with non-trivial p-torsion. We conclude that this proportion is close to $1/p$ when the base matroid is connected.

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$\mathbb{Z}$	: ring of integers
$\mathbb{N}$	: set of positive integers
$\mathbb{N}_0$	: set of non-negative integers
$\mathbb{Q}$	: field of rational numbers
$\mathbb{F}_q$	: finite field with q elements
$\mathbb{R}$	: field of real numbers
$A \sqcup B$	: disjoint union
$\#A$	: cardinality of a set
$\mathscr{P}(A)$	: power set
$G[X]$	: subgraph induced by X
$B(G)$	: incidence matrix of G
$L(G)$	: Laplacian matrix of G
$r(M)$	: rank of M
$\Lambda(M)$	: lattice of M
$\Lambda(M)^\#$	: dual lattice of M
$\mathrm{Jac}(M)$	: Jacobian of M
$\kappa(M)$	: number of bases of M
Ψ_M	: configuration polynomial of M
$\mathbb{P}^n_k$	: n-dimensional projective space over k
$\mathbb{A}^n_k$	: n-dimensional affine space over k
X_F	: hypersurface cut out by F
$\mu(S)$	: density of S

Introduction

The problem addressed in this dissertation belongs to the field of arithmetic statistics, a branch of number theory that is concerned with the distribution of invariants of arithmetic objects. Questions on the statistical behavior of ray class groups, distribution of the zeros of L-functions of number fields and function fields, distribution of the rank of elliptic curves, and distribution of the Sylow p-subgroups of Jacobian groups of curves over finite fields, among others, are of interest in this field.

The work of Henri Cohen and Hendrik W. Lenstra on heuristics on class groups of number fields [12] along with the work of Eduardo Friedman and Lawrence C. Washington [16], which addressed the function field case, marked the beginning of the field of arithmetic statistics. In these papers, heuristics were established to explain statistical observations about class groups of imaginary quadratic fields and divisor class groups of curves over a finite field. A consequence of these heuristics is that a finite abelian group appears as a class group of an imaginary quadratic field with frequency inversely proportional to the order of its automorphism group. Moreover, this set of heuristics predict that for an odd prime p the proportion of imaginary quadratic fields with class number divisible by p should be

$$1 - \prod_{n=1}^{\infty}(1 - p^{-n}).$$

One can view a finite graph as a discrete analogue of a Riemann surface (or an algebraic curve over a finite field), and a remarkable number of properties of Riemann surfaces carry over to graphs [3, 4, 20]. In particular, the Jacobian of a curve has a graph-theoretic analogue known as the Jacobian group of a graph (or sandpile group or critical group, or chip-firing group). The Jacobian group of a graph G, denoted $\mathrm{Jac}(G)$, arises in many contexts of mathematics such as arithmetic geometry (for instance, the group of components of the Néron model of a Jacobian of a curve over a local field is given as a Jacobian of a graph [10, 22]), combinatorics, and statistical physics. We refer the reader to [23] for a discussion of these and other connections. Consequently, understanding the structure of these groups is of interest; however, very little is known. So far, there are very few families of graphs for which the Jacobian groups have been described. Hence, one is naturally led to study Cohen-Lenstra Heuristics on graphs.

Some distributions of these groups have been considered. In [23], Lorenzini approaches the question "How often is $\mathrm{Jac}(G)$ cyclic?" by studying some families of random graphs. Along these lines, Clancy, Kaplan, Leake, Payne, and Wood [11] observed that the Jacobian of a random graph is cyclic with probability slightly greater than 0.7935, agreeing with the Cohen–Lenstra heuristics. In [31], Wood proved that the probability of a finite abelian group being isomorphic to the Jacobian of a random graph is zero; yet, the frequency with which a finite abelian p-group Γ appears as the Sylow p-subgroup of the Jacobian of a random graph is proportional to $1/(\#\Gamma \cdot \#\mathrm{Aut}(\Gamma))$. Only random graphs have been explored so far.

The Jacobian group of a graph admits a generalization to regular matroids. This class of matroids encompasses the class of cycle matroids, that is, the matroids associated to finite undirected graphs. Moreover, their Jacobian groups coincide with the Jacobian groups of their defining graphs. Motivated by the Cohen-Lenstra heuristics, in this dissertation, we determine for a prime number p the distribution of the p-torsion of Jacobian groups in families of regular matroids; to the best of this author's knowledge, this is the first work that addresses this problem. Starting with a "base" regular matroid M, and then applying extension operations to M parameterized by maps $\lambda\colon E(M) \to \mathbb{N}$ of $\mathbb{N}^{E(M)}$, where $E(M)$ is the ground set of M, we construct a collection of regular matroids $\{M_\lambda\}_{\lambda \in \mathbb{N}^{E(G)}}$. We describe the variation of the p-torsion of the Jacobian groups of this family.

Our first main result is the reduction of the problem to counting points over finite fields on certain hypersurfaces. More precisely, if Ψ_M is the configuration polynomial of M and X_M is the hypersurface cut out by it over $\mathbb{F}_p$, then we have the following result.

Theorem. *Suppose that M is a regular matroid with $\#E(M) = n$. If $\Psi_M \neq 1$, then*

$$\lim_{m\to\infty} \frac{\#\{\lambda \in \mathbb{N}^{E(M)}\colon\ \mathrm{ht}(\lambda) \le m, p \mid \#\,\mathrm{Jac}(M_\lambda)\}}{\#\{\lambda \in \mathbb{N}^{E(M)}\colon\ \mathrm{ht}(\lambda) \le m\}} = \frac{(p-1)\#X_M(\mathbb{F}_p)+1}{(p-1)\#\mathbb{P}^{n-1}_{\mathbb{F}_p}(\mathbb{F}_p)+1}.$$

The limit above is completely determined if we know the quantity $\#X(\mathbb{F}_p)$. So, thinking of $\#X_M(\mathbb{F}_p)$ as a function of p, it is of interest to determine whether there is a polynomial relation for the values $\#X(\mathbb{F}_p)$ as p varies. More specifically, consider the function $\#X_M\colon q \to \#X_M(\mathbb{F}_q)$ defined on the set of prime powers q, then one is interested in knowing whether $\#X_M \in \mathbb{Z}[q]$. In 1997, Kontsevich conjectured the following: for any graph G, $\#X_{M(G)} \in \mathbb{Z}[q]$, where $M(G)$ is the cycle matroid of G. In [27], Stembridge provided evidence in support of Kontsevich's conjecture by showing that the conjecture held for all graphs containing up to 12 edges. Yet, Belkale and Brosnan [6] disproved the conjecture; in fact, they showed that these functions can be rather general. In this fashion, the best one can hope for is to be able to find bounds for $\#X_M(\mathbb{F}_q)$ independent of q or describe the matroids for which $\#X_M \in \mathbb{Z}[q]$. In this thesis, we provide an estimate for $\#X_M(\mathbb{F}_q)$. More concretely, we prove:

Theorem. *If $F \in \mathbb{F}_q[T_0, \ldots, T_n]$ is an irreducible homogeneous polynomial that is linear in one of its variables, and $X_F \subseteq \mathbb{P}^n_{\mathbb{F}_q}$ is the hypersurface cut out by F, then*

$$\#X_F(\mathbb{F}_q) = q^{n-1} + O(q^{n-2}).$$

The implied constant is computable and depends only on $\deg F$ *and* $\dim \mathbb{P}^n_{\mathbb{F}_q}$.

This result improves the estimates given in [17, 19] for these types of hypersurfaces when their singular loci have codimension at most 3 (which is the case for the hypersurfaces X_M, see [14]). Finally, we establish the following result.

Theorem. *Let p be a prime number. If M is an irreducible (i.e., connected) regular matroid, then*

$$\lim_{m\to\infty} \frac{\#\{\lambda \in \mathbb{N}^{E(M)}\colon\ \mathrm{ht}(\lambda) \le m, p \mid \#\,\mathrm{Jac}(M_\lambda)\}}{\#\{\lambda \in \mathbb{N}^{E(M)}\colon\ \mathrm{ht}(\lambda) \le m\}} = \frac{1}{p} + O\left(\frac{1}{p^2}\right).$$

The implied constant is computable and depends only on $r(M)$ and $\#E(M)$.

We now discuss the structure of the document. In the first chapter we set the bases for graphs and regular matroids. The key result of this chapter is Theorem 1.5.2, which states that a regular matroid can be represented by a totally unimodular matrix. This result is fundamental for the definition of the Jacobian group.

Chapter 2 introduces the Jacobian group of a regular matroid via lattices. Thus, an overview of lattice theory is provided in the first subsection of Section 2.1. We generalize a result (Theorem 2.1.13) of [2], which allows us to deduce different presentations for the Jacobian group of a regular matroid. These presentations are used to prove the matroid version of Kirchhoff's matrix-tree theorem (Theorem 2.1.29) and that the Jacobian of a regular matroid and its dual coincide (Corollary 2.1.25). In Section 2.2, the family of matroids $\{M_\lambda\}_{\lambda\in\mathbb{N}^{E(M)}}$ is defined. We prove some properties about these new matroids, being the most relevant that these are regular matroids as well (Proposition 2.2.10). Chapter 2 ends with the definition of a configuration polynomial, which will be the defining equation of the hypersurface (called configuration hypersurface) parameterizing the members of our family having Jacobian with non-trivial p-torsion. This polynomial is homogeneous and linear in each of its variables.

Chapter 3 is the last part of the thesis. The first section is concerned with the geometry of the hypersurface X_F cut out by a homogeneous polynomial F that is linear in one of its variables. We use tools of elimination theory to establish morphisms from some distinguished subschemes of X_F and lower dimensional subschemes of projective space (Propositions 3.1.1 and 3.1.3). The second section is about estimating $\#X_F(k)$, where k is a finite field. The results from the first section imply a disjoint union decomposition for $X_F(k)$, which is recorded in Proposition 3.2.1 and Corollary 3.2.2; combined with Corollary 3.3 in [13], they yield lower and upper bounds for $\#X_F(k)$ (Theorem 3.2.4). In the third section, we define the notion of density for a subset of $\mathbb{N}^n$. Since the family $\{M_\lambda\}_{\lambda\in\mathbb{N}^{E(M)}}$ is parameterized by elements of $\mathbb{N}^n$, we show that the set of parameters for which $\mathrm{Jac}(M_\lambda)$ has non-trivial p-torsion is positive, in fact, we give a formula for this density in Theorem 3.3.4, which is given in terms of the size of the set of the $\mathbb{F}_p$-rational points on the configuration hypersurface associated to M. We conclude the thesis with an estimate for the formula given in the preceding theorem in the case that M is irreducible, from which it follows that the Jacobian groups of the family $\{M_\lambda\}_{\lambda\in\mathbb{N}^{E(M)}}$ appear with non-trivial p-torsion with frequency close to $1/p$.

Chapter 1

Matroids

In this chapter, we cover the relevant background that is required for the development of the main results in this thesis. The material presented in the first section follows [9], while the remaining sections of this chapter follow [26].

1.1 Graph Theory

Graph theory provides the groundwork for many ideas in matroid theory, and will be used frequently throughout this thesis. For the sake of completeness, in this section, we present a brief introduction to graph theory.

1.1.1 Basic definitions and examples

Definition 1.1.1. A *graph* G is given by a triple (V, E, ep) where V and E are finite sets, called respectively the *set of vertices* of G and the *set of edges* of G, and

$$\mathrm{ep} : E \to V^{(2)}$$

is a map called the *endpoint map*, where $V^{(2)}$ is the set of subsets of V whose cardinality is 1 or 2.

When there is no confusion, we simply denote the graph $G = (V, E, \mathrm{ep})$ by G and the sets V, E will always be understood as the vertex set and the edge set of G.

Definition 1.1.2. Let G be a graph.

(a) A *loop* is an edge $e \in E$ such that $\#\,\mathrm{ep}(e) = 1$.

(b) Two edges $e, f \in E$ are called *parallel* if $\mathrm{ep}(e) = \mathrm{ep}(f)$.

(c) Two vertices $v, w \in V$ are called *adjacent* if $v, w \in \mathrm{ep}(e)$ for some $e \in E$.

(d) If $e \in E$ is an edge and $v \in \mathrm{ep}(e)$, then we say that e is *incident* to v.

(e) If $v, w \in V$ are vertices, then define $E(v, w) := \{e \in E \colon v, w \in \mathrm{ep}(e)\}$. If $v = w$, then $E(v) := E(v, v)$.

(f) The *valency* of a vertex v is $\mathrm{val}(v) := \#E(v)$.

Example 1.1.3 (Empty graph)**.** The *empty graph*, denoted $\emptyset$, is the graph defined by the triple $(\emptyset, \emptyset, \emptyset \xrightarrow{\mathrm{ep}} \emptyset)$.

Example 1.1.4 (Path graph)**.** Let $n \in \mathbb{N}_0$. The *n-path graph*, denoted by Path_n, is the graph with vertex set $V = \{0, \ldots, n\}$, edge set $E = \{e_1, \ldots, e_n\}$, and endpoint map given by $\mathrm{ep}(e_i) = \{i-1, i\}$ for all $i \in \{1, \ldots, n\}$. When $n = 0$, Path_0 is a graph with a single vertex and no edges.

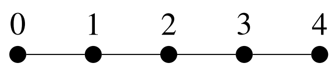


Figure 1.1: Path_4.

Example 1.1.5 (Banana graph)**.** Let $n \in \mathbb{N}$. The *banana graph* on n edges is the graph with two vertices and n edges adjacent to both.

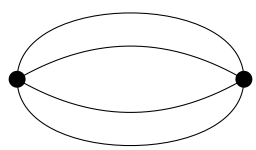


Figure 1.2: Banana graph on 4 edges.

Example 1.1.6 (Cycle graph)**.** Let $n \in \mathbb{N}$. The *n-cycle graph*, denoted by Cycle_n, is the graph with vertex set $V = \mathbb{Z}/n\mathbb{Z}$, edge set $E = \{e_1, \ldots, e_n\}$, and endpoint map defined by the rule $\mathrm{ep}(e_{i+1}) = \{i, i+1\}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. When $n = 1$, Cycle_1 is a graph with a single vertex and a single edge.

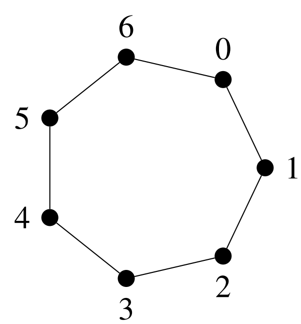


Figure 1.3: Cycle_7.

Example 1.1.7 (Complete graph)**.** Let $n \in \mathbb{N}$. The *complete graph*, denoted by K_n, is the graph with vertex set $V = \{1, \ldots, n\}$, edge set $E = \{(i, j) \in V \times V : i < j\}$, and endpoint map given by $\mathrm{ep}(i, j) = \{i, j\}$.

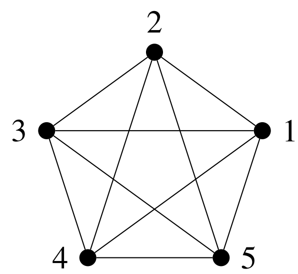


Figure 1.4: K_5.

Definition 1.1.8. Let G be a graph. A *subgraph* of G is a graph $H = (V(H), E(H), \mathrm{ep}_H)$ satisfying:

(*a*) $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$; and

(*b*) $\mathrm{ep}_H(e) = \mathrm{ep}_G(e)$, for all $e \in E(H)$.

If H is a subgraph of G, then we write $H \subseteq G$.

Definition 1.1.9. If G is a graph and $X \subseteq E$, then the *induced subgraph* by X, denoted $G[X]$, is the subgraph of G with vertex set $V(G[X]) := \{v \in V(G) \colon v \in \mathrm{ep}(e) \text{ for some } e \in X\}$ and edge set $E(G[X]) := X$.

Definition 1.1.10. Let $G = (V(G), E(G), \mathrm{ep}_G)$ and $H = (V(H), E(H), \mathrm{ep}_H)$ be two graphs. A *morphism* from G to H is a pair (f_V, f_E), where $f_V : V(G) \to V(H)$ and $f_E : E(G) \to E(H)$ are maps between the vertex sets and edge sets, respectively, such that the diagram

$$\begin{tikzcd} E(G) \arrow[r, "f_E"] \arrow[d, "\mathrm{ep}_G"'] & E(H) \arrow[d, "\mathrm{ep}_H"] \\ V_G^{(2)} \arrow[r] & V_H^{(2)} \end{tikzcd}$$

is commutative. (The bottom map is defined by $\{v, w\} \mapsto \{f_V(v), f_V(w)\}$).

If $(f_V, f_E) : G \to H$ is a morphism of graphs, we will simply refer to it as $f : G \to H$ whenever there is no danger of confusion.

Remark 1.1.11.

(*a*) We can compose morphisms of graphs in the obvious way. If $f : G \to H$ and $g : H \to K$ are morphisms of graphs, we define $g \circ f = (g_V \circ f_V, g_E \circ f_E)$, which gives a morphism from G to K.

(*b*) If G is a graph, then the pair $\mathrm{id}_G := (\mathrm{id}_V, \mathrm{id}_E)$ is a morphism from G to itself, which verifies $f \circ \mathrm{id}_G = f$ and $\mathrm{id}_G \circ g = g$ for any morphisms $f : G \to H$, $g : K \to G$.

(*c*) Part (*a*) and (*b*) show that graphs together with morphisms form a category.

Proposition 1.1.12. *Let $f : G \to H$ be a morphism of graphs. Then f is an isomorphism if and only if f_V and f_E are bijective.*

Definition 1.1.13. Let G be a graph. A subgraph $C \subseteq G$ is called a *cycle* if it is isomorphic to Cycle_n for some $n \in \mathbb{N}$.

Definition 1.1.14. Let S be a set and $\equiv_S$ be an equivalence relation on S. We say that $\equiv_S$ is *discrete* if and only if the canonical quotient map $\pi_S : S \to S/{\equiv_S}$ is bijective.

If S is a set and $\equiv_S$ is an equivalence relation on S, then we denote the quotient $S/{\equiv_S}$ by $\overline{S}$ and the equivalence class of an element s of S by $\bar{s}$.

Definition 1.1.15. Let G be a graph. Let $\equiv_V$ and $\equiv_E$ be equivalence relations on V and E, respectively. The pair $(\equiv_V, \equiv_E)$ is said to be *compatible* if and only if

$$e \equiv_E f \implies \pi_V(\mathrm{ep}(e)) = \pi_V(\mathrm{ep}(f)).$$

Definition 1.1.16. Let G be a graph and let $(\equiv_V, \equiv_E)$ be a compatible pair. The *quotient graph* of G by $(\equiv_V, \equiv_E)$ is the graph, denoted by $\overline{G}$, with vertex set $\overline{V}$, edge set $\overline{E}$, and endpoint map given by $\mathrm{ep}_{\overline{G}}(\bar{e}) = \pi_V(\mathrm{ep}(e))$.

Remark 1.1.17. The pair (π_V, π_E) gives a morphism from G to $\overline{G}$.

Definition 1.1.18. Let G be a graph and let $X \subseteq E$.

(*a*) We define the *deletion* of X from G to be the subgraph $G - X := (V(G), E(G) - X)$. When $X = \{e\}$ we write $G\backslash e := G - \{e\}$.

(*b*) We define the *contraction* of X to be the graph $G/X := (G - X)/(\equiv_V, \equiv_E)$, where $\equiv_E$ is the discrete relation on $E(G - X)$ and $\equiv_V$ is the smallest equivalence relation on $V(G - X)$ satisfying that the endpoints of each $e \in X$ are in the same equivalence class. When $X = \{e\}$ we write $G/e := G/\{e\}$.

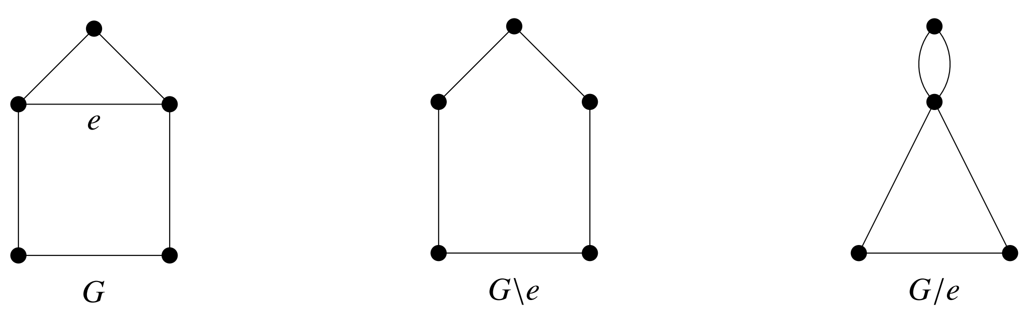


Figure 1.5: Example of deletion and contraction operations.

Definition 1.1.19. The *disjoint union* (or *coproduct*) of the graphs $\{G_i = (V_i, E_i, \mathrm{ep}_i)\}_{i=1}^n$ is the graph with vertex set $\bigsqcup_{i=1}^n V_i$, edge set $\bigsqcup_{i=1}^n E_i$, and endpoint map $\bigsqcup_{i=1}^n \mathrm{ep}_i$. We denote this graph by $\bigsqcup_{i=1}^n G_i$.

Definition 1.1.20. A *pointed graph* is a pair (G, v) where G is a graph and $v \in V(G)$.

Let (G, v) and (H, w) be two pointed graphs. Define $G \vee H := (G \sqcup H)/(\equiv_V, \equiv_E)$, where $\equiv_E$ is the discrete equivalence relation on the edge set of $G \sqcup H$ and $\equiv_V$ is the smallest equivalence relation on $V(G) \sqcup V(H)$ satisfying $v \equiv_V w$.

Definition 1.1.21. The pointed graph $(G \vee H, \bar{v})$ is called the *wedge product* of (G, v) and (H, w).

Example 1.1.22. Let $n \geq 1$ be an integer. Suppose $n = k + l$ for some nonnegative integers k, l. Then (Path_n, k) is isomorphic to $(\mathrm{Path}_k, k) \vee (\mathrm{Path}_l, 0)$.

Remark 1.1.23. If G is a graph isomorphic to a graph $H \vee K$ for some graphs H, K, then G has subgraphs H', K' isomorphic to H and K respectively, satisfying $V(H') \cup V(K') = V(G)$, $E(H') \cup E(K') = E(G)$, $V(H') \cap V(K') = \{v\}$, and $E(H') \cap E(K') = \emptyset$. In this case, we obtain $(G, v) = (K', v) \vee (H', v)$. Conversely, If G contains subgraphs H and K satisfying $V(H) \cup V(K) = V(G)$, $E(H) \cup E(K) = E(G)$, $V(H) \cap V(K) = \{v\}$, and $E(H) \cap E(K) = \emptyset$, then $(G, v) = (K, v) \vee (H, v)$.

1.1.2 Connectivity

Definition 1.1.24. Let G be a graph and let $v, w \in V$. A *path* from v to w is a morphism $f : \mathrm{Path}_n \to G$ satisfying $f(0) = v$ and $f(n) = w$.

Proposition 1.1.25. *Let G be a graph. Define the following relation on V:*

$$v \sim w \iff \text{there exists a path from } v \text{ to } w.$$

The relation $\sim$ is an equivalence relation on V.

Proof. See Section 3.1 in [9]. □

Lemma 1.1.26. *Let G be a graph. Suppose that $V_1, \ldots, V_n$ are the equivalence classes of V defined by the relation from Proposition 1.1.25. For each $i = 1, \ldots, n$, define*

$$G_i = (V_i, \{e \in E \mid \mathrm{ep}(e) \subseteq V_i\}).$$

Then G_i is a subgraph of G.

Proof. See Section 3.1 in [9]. □

Definition 1.1.27. The subgraphs $G_1, \ldots, G_n$ from Lemma 1.1.26 are called the *connected components* of G. We denote the set of connected components of G by $\pi_0(G)$. If $\#\pi_0(G) = 1$ or G is the empty graph, then G is said to be *connected*, otherwise, it is *disconnected*.

Remark 1.1.28.

(*a*) If $G_1, \ldots, G_n$ are the connected components of a graph G, then $G = \bigsqcup_{i=1}^n G_i$.

(*b*) Each connected component of a graph is connected.

To conclude this subsection, we introduce the notion of 2-connectivity which is directly linked to matroid connectivity.

Definition 1.1.29. A connected graph G with at least 3 vertices is said to be 2-*connected* if for every vertex $v \in V(G)$, the graph $G - v := (V(G) - \{v\}, E(G) - E(v))$ remains connected.

Proposition 1.1.30. *Let G be a connected on at least 3 vertices. Then G is 2-connected if and only if every pair of non-loop edges is contained in the edge set of a cycle of G.*

Proof. The proof follows directly from Theorem 5.2 in [9]. □

Remark 1.1.31. If a connected graph G with more than 3 vertices fails to be 2-connected, then $G = H \vee K$ for some subgraphs $H, K \subseteq G$, where neither H nor K is isomorphic to Cycle_1.

1.1.3 Forests

Definition 1.1.32. Let G be a graph.

(a) A graph containing no cycles is called a *forest* or *acyclic*.

(b) A connected forest is called a *tree*.

(c) A *subforest* of G is a subgraph which is a forest, if it is connected, then it is called a *subtree*. A *spanning forest* of G is a maximal subforest and a *spanning tree* of G is a maximal subtree.

Remark 1.1.33.

(a) If F is forest of G for which there is a vertex $v \in V(G) - V(F)$, then the subgraph $F' = (V(F) \cup \{v\}, E(F))$ is a forest containing F. Hence, any spanning forest F of G must satisfy $V(F) = V(G)$.

(b) If G is a connected graph, then $F \subseteq G$ is a spanning forest if and only if it is a spanning tree (see Exercise 4.1.2 in [9]).

(c) Suppose that $G_1, \dots, G_n$ are the connected components of a graph G and $F \subseteq G$ is a spanning forest. Consider the subgraphs $F \cap G_i := (V(F) \cap V(G_i), E(F) \cap E(G_i))$. Note that $F \cap G_i$ is a spanning tree of G_i since it is a spanning forest of a connected graph (see Remark 1.1.33 (b)). Hence, when studying spanning forests we may reduce to the case where G is connected.

Proposition 1.1.34. *Every acyclic subgraph of a graph G is contained in a spanning forest of G.*

Proof. We may assume that G is connected by Remark 1.1.33 (c). Let H be an acyclic subgraph of G. Consider the set

$$\mathscr{C} = \{I \subseteq G \colon H \subseteq I \text{ and } I \text{ is acyclic}\}.$$

The set $\mathscr{C}$ is nonempty since $H \in \mathscr{C}$. As $\mathscr{C}$ is finite (because G is finite), it must have a maximal element T with respect to inclusion. We claim that T is a tree.

First of all, observe that $V(T) = V(G)$. Otherwise, we could pick a vertex $v \in V(G) - V(T)$ and construct the acyclic subgraph $(V(T) \cup \{v\}, E(T))$ which contains H and it is strictly larger than T violating the maximality of T. Now we are left to show that T is connected since by assumption it is acyclic. Let C be a connected component of T. Observe that $V(C) = V(G)$. Otherwise, there is $v \in V(G) - V(C)$ and $w \in V(C)$ that are not connected by a path in T. Consequently, the graph $(V(T), E(T) \cup \{e\})$ is an acyclic subgraph of G containing H that it is strictly larger than T. However, this is not possible by maximality of T. Thus $V(C) = V(G)$. As C is a connected component of T satisfying $V(C) = V(T)$, it follows that $C = T$. This proves our claim. □

Corollary 1.1.35. *Every graph has a spanning forest.*

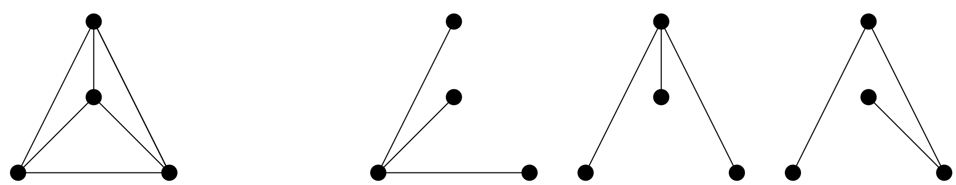


Figure 1.6: Some spanning trees of K_4.

Proposition 1.1.36. *Let T be a connected graph. T is a tree if and only if*

$$\#E(T) = \#V(T) - 1.$$

If F is a forest, then $\#E(F) = \#V(F) - \#\pi_0(F)$.

Proof. See Theorem 4.3 and Exercise 4.1.4 in [9]. □

Definition 1.1.37. A subset of edges X of a graph G is called a *bond* if it is minimal with respect to the condition $\#\pi_0(G - X) > \#\pi_0(G)$. When $X = \{e\}$, we call e a *cut-edge*.

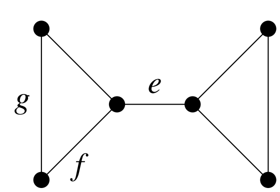


Figure 1.7: The edge e is a cut-edge and the pair $\{f, g\}$ is a bond.

Proposition 1.1.38. *Let G be a graph. The following are equivalent:*

(*a*) *$e \in E(G)$ is a cut-edge;*

(*b*) *$e \in E(F)$ for every spanning forest F of G;*

(*c*) $e \notin E(C)$ *for every cycle* $C \subseteq G$.

Proof. The equivalence $(a) \iff (c)$ corresponds to Proposition 3.2 in [9]. The equivalence $(b) \iff (c)$ follows from Proposition 1.1.34 by noting that if C is a cycle with $e \in E(C)$, then $C\backslash e$ is acyclic. □

Definition 1.1.39. The number of spanning forests of a graph G is denoted by $\kappa(G)$.

Proposition 1.1.40. *If G is a graph, then*

$$\kappa(G) = \begin{cases} \kappa(G\backslash e) = \kappa(G/e), & \text{if } e \text{ is either a loop or a cut-edge;} \\ \\ \kappa(G) = \kappa(G\backslash e) + \kappa(G/e), & \text{otherwise.} \end{cases}$$

Proof. It follows from Remark 1.1.33 (*c*) and Proposition 4.9 in [9]. □

Remark 1.1.41. If e is not a loop, then one can show that there is a one-to-one correspondence between the spanning forests of G/e and the spanning forests of G containing e (see Exercise 4.2.1 in [9]). Similarly, if e is not a cut-edge, then one can show that there is a one-to-one correspondence between the spanning forests of $G\backslash e$ and the spanning forests of G that do not contain e.

1.1.4 Special matrices

In this subsection, G will denote a graph without loops.

Definition 1.1.42. An *orientation* on G is a function $(t, h)\colon E \to V \times V$ with the property that if $(t, e)(e) = (t(e), v(e))$, then $\mathrm{ep}(e) = \{t(e), h(e)\}$ for all $e \in E$.

Definition 1.1.43. Let (t, h) be an orientation on G. The *incidence matrix* of G with respect to (t, h), denoted $B(G)$, is the $V \times E$-matrix whose entries are given by

$$b_{ve} := \begin{cases} 1, & h(e) = v; \\ -1, & t(e) = v; \\ 0, & \text{otherwise.} \end{cases}$$

If (t, h) and (t', h') are two orientations on G, and B, B' are the incidence matrices with respect to each orientation, then $B' = BP$ where P is the matrix obtained from the identity matrix by multiplying the entry (e, e) by -1 if $(t(e), h(e)) = (h'(e), t'(e))$.

Note that $B(G)$ is defined over any field. Henceforth, when a property of $B(G)$ is stated without indicating the field of definition, it will mean that we are working over an arbitrary field.

A finial observation is that if $G_1, \ldots, G_n$ are the connected components of G, then

$$B(G) = \begin{bmatrix} B(G_1) & 0 & \cdots & 0 \\ 0 & B(G_2) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & B(G_n) \end{bmatrix}. \tag{1.1}$$

Proposition 1.1.44. $\operatorname{rank} B(G) = \#V(G) - \#\pi_0(G)$.

Proof. See Theorem 2.3 in [5]. □

Proposition 1.1.45. *A set of columns of $B(G)$ are linearly independent if and only if the corresponding edges induce a forest.*

Proof. See Lemma 2.5 in [5]. □

A matrix $A \in M_{m\times n}(\mathbb{Z})$ is said to be *totally unimodular* if the determinant of any square submatrix of A is 0 or ± 1. By definition, it follows that each entry of A is 0 or ± 1. There is a partial converse, that is, if A is an integral matrix where each entry is 0 or ± 1 and each column contains at most one 1 and at most one -1, then it is totally unimodular (see Lemma 5.1.4 in [26]).

Proposition 1.1.46. *The matrix $B(G)$ viewed as an integral matrix is totally unimodular.*

Proof. See Lemma 2.6 in [5]. □

Definition 1.1.47. If G is a graph, then the $V \times V$-matrix $A(G) := (\#E(v, w))_{v,w\in V}$ is called the *adjacency matrix* of G. The $V \times V$ diagonal matrix whose diagonal entries are the valencies of the vertices of G is denoted by $D(G)$.

Definition 1.1.48. The *Laplacian matrix* of a graph G is the $V\times V$ matrix $L(G) := D(G)-A(G)$.

Proposition 1.1.49. $L(G) = B(G)B(G)^t$.

Proof. It follows from computing directly the product $B(G)B(G)^t$. □

Lastly, we state a classical result from algebraic graph theory, which counts the number of spanning forests of a graph via minors of its Laplacian.

Theorem 1.1.50 (Kirchhoff's Matrix-Tree Theorem)**.** *If G is a graph without loops with connected components $G_1, \dots, G_n$ and $v_i \in V(G_i)$, $i = 1, \dots, n$, then the determinant of the matrix obtained from $L(G)$ by deleting the rows and columns indexed by $v_1, \dots, v_n$ equals the number of spanning forests of G.*

Proof. One can reduce to the case where G is connected by noting that $L(G)$ is a block diagonal matrix with diagonal blocks $L(G_i)$, $i = 1, \dots, n$. Indeed, this is a direct consequence from (1.1) and Proposition 1.1.49. Hence, it all boils down to showing that any cofactor of $L(G)$, with G connected, equals $\kappa(G)$. A proof for this fact can be found in [5]. □

1.2 Equivalent Characterizations of Matroids

1.2.1 Independent sets and circuits

Definition 1.2.1. A *matroid* M is a pair $(E, \mathcal{I})$ consisting of a finite set E, called the *ground set* of M, and a family $\mathcal{I}$ of subsets of E, called the *independent sets* of M, satisfying:

(I1) $\emptyset \in \mathcal{I}$;

(I2) if $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then also $I_2 \in \mathcal{I}$;

(I3) if $I_1, I_2 \in \mathcal{I}$ and $\#I_1 < \#I_2$, then $I_1 \cup \{e\} \in \mathcal{I}$ for some $e \in I_2 - I_1$.

A subset of E that is not in $\mathcal{I}$ is called *dependent*.

From now on, we refer to a matroid by M and the sets E and $\mathcal{I}$ will be understood as the ground set and the set of independent sets of M, respectively. If more matroids are being considered, then we write $E(M)$ and $\mathcal{I}(M)$.

Example 1.2.2 (Vector matroids)**.** Let V be a finite dimensional vector space over a field K and $p\colon K^n \to V$ be a surjective linear map. The *vector matroid* of V over K, denoted $M(V)$, is the matroid with ground set $E(M(V)) := \{e_1, \ldots, e_n\}$ the canonical basis of K^n, and independent sets

$$\mathcal{I}(M(V)) := \{X \subseteq E(M(V))\colon\, p(X) \text{ is linearly independent}\}.$$

In particular, when $V = K^r$ the linear map p is represented by a matrix $A \in M_{r\times n}(K)$ with respect to the canonical bases. In this fashion, we write $M(A)$ for $M(V)$. The matrix A together with the labeling $e_1, \ldots, e_n$ is all we need to describe $M(A)$.

Example 1.2.3 (Uniform matroids)**.** Let $m, n \in \mathbb{N}_0$ with $m \leq n$. The *uniform matroid* of rank m on an set of n elements, denoted $U_{m,n}$, is the matroid with ground set $E(U_{m,n})$ a set of size n and independent sets

$$\mathcal{I}(U_{m,n}) := \{X \subseteq E(U_{m,n})\colon \#X \leq m\}.$$

The Uniform matroid $U_{0,0}$ is called the *empty* matroid and we denote it by $\emptyset$.

Definition 1.2.4. Two matroids M_1 and M_2 are said to be *isomorphic*, written $M_1 \cong M_2$, if there is a bijective map $\Phi\colon E(M_1) \to E(M_2)$ such that, $X \in \mathcal{I}(M_1)$ if and only $\Phi(X) \in \mathcal{I}(M_2)$. We call such a bijection Φ an *isomorphism* from M_1 to M_2.

Now we look at another characterization of matroids.

Definition 1.2.5. A minimal dependent set of a matroid M will be called a *circuit*. The set of all circuits of M is denoted by $\mathcal{C}$ or $\mathcal{C}(M)$.

Theorem 1.2.6. *Let E be a finite set and suppose $\mathcal{C} \subseteq \mathcal{P}(E)$. Then $\mathcal{C}$ is the collection of circuits of a matroid with ground set E if and only if $\mathcal{C}$ has the following properties:*

(C1) $\emptyset \notin \mathcal{C}$;

(C2) *if $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$;*

(C3) *if C_1, C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - \{e\}$.*

Proof. See Theorem 1.1.4 and Lemma 1.1.3 in [26]. □

Remark 1.2.7. By Theorem 1.2.6, a matroid M is completely determined by its set of circuits. Its independent sets will be those subsets of $E(M)$ that contain no circuit.

Example 1.2.8 (Cycle matroids)**.** Let $E(G)$ be the set of edges of a graph G and $\mathcal{C}$ be the collection of edge sets of cycles of G. Then $\mathcal{C}$ is the set of circuits of a matroid with ground set $E(G)$ (see Proposition 1.1.7 in [26]). The matroid obtained this way is called the *cycle matroid* of G. It is denoted by $M(G)$. If X is a subset of edges, then X is an independent set of $M(G)$ if and only if X does not contain the edge set of any cycle of G if and only if $G[X]$ is a forest.

Remark 1.2.9. Let G be a graph with connected components $G_1, \ldots, G_n$. If $H = G_1 \vee \cdots \vee G_n$, then H is connected and its cycle matroid is isomorphic to $M(G)$ (observe that the wedge product of two graphs preserves the cycle structure of the two graphs). Hence, we may assume that a graphic matroid is always isomorphic to the cycle matroid of a connected graph.

Example 1.2.10 (Circuits of a uniform matroid)**.** Consider the matroid $U_{m,n}$. Then

$$\mathcal{C}(U_{m,n}) = \begin{cases} \emptyset, & m = n; \\ \{X \subseteq E(U_{m,n}) \colon \#X = m + 1\}, & m < n. \end{cases}$$

Definition 1.2.11.

(a) A matroid M is said to be *linear* or *representable* if it is isomorphic to the vector matroid of a finite dimensional vector space V over a field K. In that case, we say that V *represents* M over K.

(b) A matroid that is isomorphic to the cycle matroid of a graph is called *graphic*.

The cycle matroid of a graph G is linear. Indeed, fix a vertex $v \in V(G)$ and let us suppose that G is connected (we can assume this by Remark 1.2.9). Let $B(G)[v]$ denote the incidence matrix of G without row v and let B be the matrix obtained from $B(G)[v]$ by adding an extra zero column for each loop of G. Using Propositions 1.1.44, 1.1.45, and Example 1.2.2, it is easy to see that $M(G) = M(B)$. If G has no loops, then $M(G) = M(B(G)[v])$.

1.2.2 Bases

Definition 1.2.12. A maximal independent set of a matroid is called a *basis*. The collection of all bases of a matroid M is denoted by $\mathcal{B}$ or $\mathcal{B}(M)$.

Theorem 1.2.13. *Let $\mathcal{B}$ a set of subsets of a finite set E. Then $\mathcal{B}$ is the collection of bases of a matroid with ground set E if and only if it has the following properties:*

(**B1**) $\mathcal{B} \neq \emptyset$*;*

(**B2**) *if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then there is $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.*

Proof. See Theorem 1.2.3 and Lemma 1.2.2 in [26]. □

Remark 1.2.14. By knowing the bases of a matroid M we can recover its independent sets as follows: a subset of $E(M)$ is independent if and only if it is contained in some basis of M.

Proposition 1.2.15. *Let $\mathcal{B}$ be the set of bases of a matroid M. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_2 - B_1$, then there is $y \in B_1 - B_2$ such that $(B_1 - \{y\}) \cup \{x\} \in \mathcal{B}$.*

Proof. See Lemma 2.1.2 in [26]. □

Proposition 1.2.16. *If B_1 and B_2 are two bases of a matroid M, then $\#B_1 = \#B_2$.*

Proof. See Lemma 1.2.1 in [26]. □

Example 1.2.17 (Bases of a uniform matroid)**.** The bases of the matroid $U_{m,n}$ are all the subsets of $E(U_{m,n})$ whose cardinality is m.

Example 1.2.18 (Bases of a graphic matroid)**.** Let G be a graph. In Example 1.2.8, we observed that a subset X of $E(G)$ is an independent set of $M(G)$ if and only if $G[X]$ is a forest. Thus, X is a basis if and only if $G[X]$ is a forest and $G[X \cup \{e\}]$ contains a cycle for all $e \notin X$. When G is connected, X is a basis of $M(G)$ if and only if $G[X]$ is a spanning tree.

Example 1.2.19 (Bases of a linear matroid)**.** If V is a finite dimensional vector space over a field K and $p\colon K^n \to V$ is a surjective linear map, then X is a basis of $M(V)$ if and only if $p(X)$ is a basis of V.

1.2.3 Rank

The rank of a matroid is the matroid version of the dimension of a vector space. This is achieved by the fact that all bases of a matroid have the same cardinality (see Proposition 1.2.16).

Lemma 1.2.20. *If M is a matroid and $X \subseteq E$, then the collection*

$$\mathcal{I}|X := \{I \subseteq X\colon I \in \mathcal{I}\}$$

satisfies (**I1**), (**I2**)*, and* (**I3**)*. Hence, the pair $(X, \mathcal{I}|X)$ is a matroid.*

Definition 1.2.21. The matroid $(X, \mathcal{I}|X)$ from Lemma 1.2.20 is called the *restriction* of M to X and it is denoted by $M|X$.

Remark 1.2.22. If M is a matroid and $X \subseteq E$. Then a direct computation shows that the collection of circuits of $M|X$ is precisely the collection $\{C \subseteq X\colon C \in \mathcal{C}(M)\}$.

Definition 1.2.23. Let M be a matroid and suppose $X \subseteq E$. We define the *rank* of X, denoted $r(X)$, to be the cardinality of a basis B of $M|X$.

Theorem 1.2.24. *Let E be a finite set. A function $r\colon \mathcal{P}(E) \to \mathbb{N}_0$ is the rank function of a matroid with ground set E if and only if r has the following properties:*

(**R1**) *for all $X \in \mathcal{P}(E)$, $0 \le r(X) \le \#X$;*

(**R2**) *if $X \subseteq Y \subseteq E$, then $r(X) \le r(Y)$;*

(**R3**) *if $X, Y \in \mathcal{P}(E)$, then $r(X \cup Y) + r(X \cap Y) \le r(X) + r(Y)$.*

Proof. See Theorem 1.3.2 and Lemma 1.3.1 in [26]. □

Proposition 1.2.25. *Let M be a matroid with rank function r and suppose that $X \subseteq E$. Then* (*i*) *$X \in \mathcal{I}$ if and only if $r(X) = \#X$;*

(*ii*) *$X \in \mathcal{B}$ if and only if $\#X = r(X) = r(M)$; and*

(*iii*) *$X \in \mathcal{C}$ if and only if $X \neq \emptyset$ and, for all $x \in X$, $r(X - \{x\}) = \#X - 1 = r(X)$.*

Proof. See Proposition 1.3.5 in [26]. □

Example 1.2.26 (Rank function of a graphic matroid)**.** Let G be a graph. A basis X of $M(G)$ is a subset of $E(G)$ such that $G[X]$ is a forest of G and $G[X \cup \{e\}]$ contains a cycle for all $e \notin X$. By Proposition 1.1.36, we know that $\#E(G[X]) = \#V(G) - \#\pi_0(G)$ (compare with 1.1.44). Hence, $r(M(G)) = \#V(G) - \#\pi_0(G)$. Similarly, the rank of a subset X of $E(G)$ is given by $\#V(G[X]) - \#\pi_0(G[X])$.

Example 1.2.27 (Rank function of a linear matroid)**.** Let V be a finite dimensional vector space over a field K and $p\colon K^n \to V$ be a surjective linear map. For any $X \subseteq E(M(V))$, we have $r(X) = \dim_K \operatorname{span}(p(X))$.

Example 1.2.28 (Rank function of a uniform matroid)**.** If $X \subseteq E(U_{m,n})$, then

$$r(X) = \begin{cases} \#X, & \#X < m; \\ m, & \#X \geq m. \end{cases}$$

We conclude this subsection by introducing some terminology borrowed from graphs.

Definition 1.2.29. Let M be a matroid. An element $e \in E$ is called a *loop* if $\{e\}$ is a circuit of M. If $\{f, g\} \subseteq E$ is a circuit of M, then f and g are said to be *parallel* in M.

Remark 1.2.30. Note that an element e of M is a loop if and only if $e \notin B$ for all $B \in \mathcal{B}$. Particularly, in the case of a graphic matroid $M(G)$, an edge $e \in E(G)$ is a loop of $M(G)$ if and only if e is a loop (in the sense of graphs) of G. In the case of a linear matroid $M(V)$ induced by a surjective linear map $p\colon K^n \to V$, we have that $e \in E$ is a loop if and only if $p(e) = 0$.

1.3 Matroid Operations

1.3.1 Duality

Theorem 1.3.1. *Let M be a matroid. The collection $\mathcal{B}^*(M) := \{E - B\colon B \in \mathcal{B}\}$ is the set of bases of a matroid on E.*

Proof. See Theorem 2.1.1 in [26]. □

Definition 1.3.2. Let M be a matroid. The matroid with ground set $E(M)$ and set of bases $\mathcal{B}^*(M)$, is called the *dual* of M and is denoted by M^*.

The independent sets, bases, circuits, rank, and loops of M^* are called *coindependent sets*, *cobases*, *cocircuits*, *corank*, and *coloops*, respectively. Using this terminology, we see that B^* is a cobasis of M if and only if $E - B^*$ is a basis of M. This is not necessarily true for coindependent sets and cocircuits. Also, an element e of M is a coloop if and only if $e \in B$ for every basis B of M.

Example 1.3.3. For any matroid M, $(M^*)^* = M$.

Example 1.3.4 (Dual of a uniform matroid)**.** Consider $U_{m,n}$. The collection $\mathcal{B}^*(U_{m,n})$ consists of all subsets of $E(U_{m,n})$ with $n - m$ elements, therefore $U^*_{m,n} = U_{n-m,n}$.

Example 1.3.5 (Dual of a linear matroid)**.** Let V be a finite dimensional vector space over a field K with a surjective linear map $p\colon K^n \to V$. If $W = \ker p$, then we have the following short exact sequence

$$0 \longrightarrow W \xrightarrow{i} K^n \xrightarrow{p} V \longrightarrow 0,$$

by taking duals, we obtain the short exact sequence

$$0 \longrightarrow V^* \xrightarrow{p^*} K^n \xrightarrow{i^*} W^* \longrightarrow 0.$$

Consider the matroid $M(W^*)$ induced by i^*. Observe that X is a basis of $M(V)$ if and only if $E(M(V)) - X$ is a basis of $M(W^*)$. This implies that $M(V)^* \cong M(W^*)$. In particular, we see that the dual of a linear matroid is linear as well.

If we work with coordinates, then the linear matroid $M(K^r)$ can be represented by a matrix of the form $[I_r|D]$ (see Section 1.5). Using the short exact sequences above one can show that the matrix $[-D^t|I_{n-r}]$ represents i^*. Thus $M(K^r)^* = M([-D^t|I_{n-r}])$ where the columns of $[-D^t|I_{n-r}]$ are labeled in the same order as the columns of $[I_r|D]$.

Example 1.3.6 (Dual of a graphic matroid)**.** Unlike linear matroids, the matroid $M(G)^*$ need not be graphic for a graph G. A well-known example is $M(K_5)^*$. Nonetheless, there is a class of graphs called *planar* for which $M(G)^*$ is graphic. Intuitively, a connected graph G is planar if it can be drawn in $\mathbb{R}^2$ so that no edge cross. This drawing determines a partition of $\mathbb{R}^2$ into regions bounded by the edges of G plus the outer region. From this, we can construct a graph G^* as follows: the vertices will be the regions created by G, and two vertices are connected by k edges if and only if their corresponding regions share k edges of G. It turns out that $M(G)^* \cong M(G^*)$. For a more rigorous treatment of this topic see Chapter 2 Section 3 in [26].

Even though the dual of a graphic matroid $M(G)$ is not graphic in general, we can still characterize some special sets of $M(G)^*$ in terms of G. If X be a set of edges of G, then

- X is a cobasis if and only if there is a spanning forest $F \subseteq G$ satisfying $E(F) \cap X = \emptyset$.
- X is a cocircuit if and only if it is a bond of G (see Definition 1.1.37).
- $e \in E(G)$ is a coloop if and only if it is a cut-edge (see Definition 1.1.37 and Figure 1.8).

Proposition 1.3.7. *Let M be a matroid. Denote by r^* the rank function of M^*. For all $X \subseteq E$,*

$$r^*(X) = r(E - X) + \#X - r(M).$$

In particular, $r(M) + r^*(M^*) = \#E$.

Proof. See Proposition 2.1.9 in [26]. □

Figure 1.8: Example of a graph G and its dual.

1.3.2 Deletion and contraction

These two operations are generalizations of those of graphs (see Definition 1.1.18).

Definition 1.3.8. Let M be a matroid and let $e \in E$.

(*a*) The *deletion* of e from M is the matroid $M\backslash e := M|(E - \{e\})$.

(*b*) The *contraction* of e from M is the matroid $M/e := (M^*\backslash e)^*$.

Let us describe the independent sets, bases, circuits, and rank functions of $M\backslash e$ and M/e. Both matroids have as ground set $E(M) - \{e\}$. By Definition 1.2.21, we have

$$\mathcal{I}(M\backslash e) = \{I \in \mathcal{I}(M) \colon e \notin I\}, \tag{1.2}$$

$$\mathcal{B}(M\backslash e) = \begin{cases} \{B - \{e\} \colon B \in \mathcal{B}(M)\}, & \text{if } e \text{ is a coloop;} \\ \{B \in \mathcal{B}(M) \colon e \notin B\}, & \text{if } e \text{ is not a coloop.} \end{cases} \tag{1.3}$$

$$\mathcal{C}(M\backslash e) = \{C \in \mathcal{C}(M) \colon e \notin C\}, \tag{1.4}$$

$$r_{M\backslash e}(X) = r_M(X), \quad \forall X \subseteq E(M\backslash e). \tag{1.5}$$

For the matroid M/e, following Propositions 3.1.6, 3.1.7, and Corollary 3.1.8 in [26], we have

$$\mathcal{I}(M/e) = \begin{cases} \{I - \{e\} \colon I \in \mathcal{I}(M), e \in I\}, & \text{if } e \text{ is not a loop;} \\ \mathcal{I}(M), & \text{if } e \text{ is a loop.} \end{cases} \tag{1.6}$$

$$\mathcal{B}(M/e) = \begin{cases} \{B - \{e\} \colon B \in \mathcal{B}(M), e \in B\}, & \text{if } e \text{ is not a loop;} \\ \mathcal{B}(M), & \text{if } e \text{ is a loop.} \end{cases} \tag{1.7}$$

$$r_{M/e}(X) = \begin{cases} r_M(X \cup \{e\}) - 1, & \text{if } e \text{ is not a loop;} \\ r_M(X), & \text{if } e \text{ is a loop.} \end{cases} \tag{1.8}$$

The circuits of M/e are the nonempty minimal sets of the set $\{C - \{e\} \colon C \in \mathcal{C}(M)\}$.

Example 1.3.9 (Deletion and contraction on graphic matroids)**.** If G is a graph and $e \in E(G)$, then

$$M(G)\backslash e = M(G\backslash e) \quad \text{and} \quad M(G)/e = M(G/e).$$

So that, the deletion and contraction operations on a graphic matroid result in graphic matroids again.

Example 1.3.10 (Deletion and contraction on linear matroids)**.** Consider $A \in M_{r\times n}(K)$ and let $e \in E(M(A))$. We denote by $A\backslash e$ the matrix obtained from A by removing the column corresponding to e. Then

$$M(A)\backslash e = M(A\backslash e).$$

We now describe $M(A)/e$. If e is a loop, then the column in A corresponding to e is zero (see Remark 1.2.30) and $M(A)/e = M(A)\backslash e$. Otherwise, e is not a loop and by turning the eth column of A into a standard basis vector (by performing row operations) we obtain a matrix A'. Let A'/e be the matrix obtained from A' by deleting the row and column containing the nonzero entry of e. Then, by Proposition 3.2.6 in [26], we obtain

$$M(A)/e = M(A')/e = M(A'/e).$$

So that, the deletion and contraction operations on a linear matroid result in linear matroids again.

1.3.3 Direct sum

Proposition 1.3.11. *Let M_1 and M_2 be two matroids. The set $E(M_1) \sqcup E(M_2)$ along with the collection $\{I_1 \sqcup I_2 \colon I_1 \in \mathcal{I}(M_1), I_2 \in \mathcal{I}(M_2)\}$ define a matroid.*

Proof. See Proposition 4.2.8 in [26]. ◻

Definition 1.3.12. The matroid from Proposition 1.3.11 is called the *direct sum* of M_1 and M_2. It is denoted by $M_1 \oplus M_2$.

Remark 1.3.13. By 4.2.12 and 4.2.14 from [26], we have $\mathcal{C}(M_1 \oplus M_2) = \mathcal{C}(M_1) \cup \mathcal{C}(M_2)$ and $\mathcal{B}(M_1 \oplus M_2) = \{B_1 \sqcup B_2 \colon B_1 \in \mathcal{B}(M_1), B_2 \in \mathcal{B}(M_2)\}$ for any two matroids M_1 and M_2.

Example 1.3.14. Let H, K be two graphs. If $G \in \{H \vee K, H \sqcup K\}$, then $M(G) = M(H) \oplus M(K)$.

Example 1.3.15. If V_1, V_2 are finite dimensional vectors spaces over a field K with surjective linear maps $p_1 \colon K^n \to V_1$ and $p_2 \colon K^m \to V_2$ respectively, then the matroid $M(V_1 \oplus V_2)$ induced by $p_1 \oplus p_2 \colon K^{n+m} \to V_1 \oplus V_2$ verifies $M(V_1 \oplus V_2) \cong M_1(V_2) \oplus M(V_2)$.

Proposition 1.3.16. *The classes of graphic matroids and linear matroids are closed under direct sums.*

Proof. See Proposition 4.2.11 in [26]. ◻

1.3.4 Parallel and series extensions

Definition 1.3.17. Let M and N be two matroids.

(*a*) We say that N is a *parallel extension* of M if there is a circuit $\{e, f\}$ of N such that $e \neq f$ and $N \setminus e = M$.

(*b*) We say that N is a *series extension* of M if there is a cocircuit $\{e, f\}$ of N such that $e \neq f$ and $N/e = M$.

Example 1.3.18. Let G be a graph. Suppose that $e \in E(G)$ is not a cut-edge. Let H be the graph obtained from G by replacing e with a path of length 2 (intuitively, we are subdividing the edge e). Then $M(H)$ is a series extension of $M(G)$. We stress that not all series extensions of a graphic matroid arise this way. See Figure 1.9 for an example.

Figure 1.9: Serial extension that does not come from subdivision.

Proposition 1.3.19. *Let M and N be two matroids. Then N is a parallel extension of M if and only if N^* is a series extension of M^*.*

Proof. See Chapter 5 Section 4 in [26]. □

1.4 Connectivity

The notion of matroid connectivity is inspired by graphs. We will see that this generalization to matroids corresponds to the notion of 2-connectivity of graphs rather than connectivity as one would expect. This is due to the fact that if G is a disconnected graph, then there is a connected graph H such that $M(G) \cong M(H)$.

Definition 1.4.1. A matroid is said to be *irreducible* (or *connected*) if is not the direct sum of two nonempty matroids. Otherwise, it is *reducible*.

Proposition 1.4.2. *A matroid M is irreducible if and only if for every pair of distinct elements of $E(M)$, there is a circuit containing both.*

Proof. We prove the proposition by contrapositive. Suppose $M = M_1 \oplus M_2$ for two nonempty matroids M_1 and M_2. Then there are elements $e \in E(M_1)$ and $f \in E(M_2)$. By Remark 1.3.13, $\mathcal{C}(M) = \mathcal{C}(M_1) \cup \mathcal{C}(M_2)$. Since $E(M_1) \cap E(M_2) = \emptyset$, it follows that no circuit of M contains both e and f.

Now suppose that there is a pair $\{e, f\} \subseteq E(M)$ with the property that no circuit of M contains it. Define the following relation on $E(M)$:

$$a \sim b \iff \text{there exists } C \in \mathcal{C}(M) \text{ such that } a, b \in C.$$

This relation $\sim$ is an equivalence relation (for a proof of this claim see Proposition 4.1.2 in [26]). There are at least two equivalence classes as $e \nsim f$. Let $E_1, \dots, E_n$, $n \geq 2$, be all the equivalences classes determined by $\sim$. We claim that $M = M|E_1 \oplus \cdots \oplus M|E_n$. It is clear that $E(M) = E_1 \sqcup \cdots \sqcup E_n$. If I is an independent set of M, then for all $k = 1, \dots, n$, $I \cap E_k$ is an independent set of M (by (**I2**)) contained in E_k. Furthermore, I is the disjoint union of the $I \cap E_k$, this shows that $\mathcal{I}(M) \subseteq \mathcal{I}(M|E_1 \oplus \cdots \oplus M|E_n)$. To prove the other containment, observe that the independent sets of each $M|E_k$ are independent sets of M as well by Definition 1.2.21, in addition, if $I_k \in \mathcal{I}(E_k)$, then any circuit C of M is not contained in $I_1 \sqcup \cdots \sqcup I_n$ (see Remark 1.2.7). This completes the proof. □

Definition 1.4.3. The matroids $M|E_1, \dots, M|E_n$ appearing in the proof of Proposition 1.4.2 are called the *irreducible (or connected) components* of M.

Remark 1.4.4. It can be noted in the proof of Proposition 1.4.2 that if M is a matroid and $M_1, \dots, M_n$ are its irreducible components, then $M = M_1 \oplus \cdots \oplus M_n$.

Proposition 1.4.5. *Let G be a graph without loops and without isolated vertices. If $\#V(G) \geq 3$, then $M(G)$ is irreducible if and only if G is 2-connected.*

Proof. This is a direct consequence from Propositions 1.1.30 and 1.4.2. □

Proposition 1.4.6. *A matroid M is irreducible if and only if M^* is irreducible.*

Proof. See Corollary 4.2.4 in [26]. □

Proposition 1.4.7. *If N is a parallel (series) extension of an irreducible matroid M, then N is irreducible.*

Lemma 1.4.8. *Let M be a matroid. If C_1 and C_1 are circuits of M such that $f \in C_1 \cap C_2$ and $h \in C_1 - C_2$, then M has a circuit C_3 such that $h \in C_3 \subseteq (C_1 \cup C_2) - \{f\}$.*

Proof. See Exercise 14 from Chapter 1 Section 1 in [26]. □

Proof of Proposition 1.4.7. Suppose that N is a parallel extension of M, that is, there exits a circuit $\{e, f\}$ of N such that $e \neq f$ and $N \backslash e = M$. Let $h, g \in E(N)$ be distinct. We will show that $\{h, g\}$ is contained in some circuit of N. If $h, g \neq e$, then $h, g \in E(M)$ so that there is a circuit C of M containing them. Note that C is also a circuit of N (see (1.4)). If $g = e$ and $h \neq f$, then there is a circuit C of M containing $\{f, h\}$. If we take $C_1 = C$ and $C_2 = \{e, f\}$, then we can use Lemma 1.4.8 to get a third circuit C_3 of N such that $h \in C_3 \subseteq (C - \{f\}) \cup \{e\}$. Since $C - \{f\}$ is an independent set of N, C_3 must contain e. Therefore C_3 is a circuit of N containing $\{e, h\}$. Whence, by Proposition 1.4.2, N is irreducible.

Now we assume that N is a series extension of M. By Proposition 1.4.6, M^* is irreducible and by Proposition 1.3.19, N^* is a parallel extension of M^*. Therefore N^* is irreducible and so is N. □

1.5 Regular Matroids

In this last section, we discuss some general facts of regular matroids, which constitute the class of matroids we are most interested in since the Jacobian group of a graph admits a generalization to this class.

Definition 1.5.1. A matroid that is representable over any field is called *regular*.

Before we continue the discussion on regular matroids, let us mention some facts on the representations of general linear matroids. Suppose that M is a representable matroid over a field K of rank $r > 0$. Thus there exists a matrix $A \in M_{r\times n}(K)$ of rank r such that $M \cong M(A)$. Observe that the linear dependence relations among the columns of A are preserved under row operations, multiplication of a column by a nonzero scalar, interchanging two columns (labels moving with the columns), and replacing the entries of A by their image under some field automorphism of K. As a result, if A' is a matrix obtained from A by performing some of these operations, then the identity map $\mathrm{id}\colon E(M(A)) \to E(M(A'))$ induces a matroid isomorphism $M(A) \cong M(A')$ (recall that $E(M(A)) = E(M(A'))$ is the canonical basis of K^n), whence A' is another representation of M over K. We would like to treat these two representations for M as equal. Hence we say that two representations A_1 and A_2 of M are equivalent if

- the identity map $\mathrm{id}\colon E(M(A_1)) \to E(M(A_2))$ induces an isomorphism $M(A_1) \to M(A_2)$; and

- there is a field automorphism σ of K, and there are invertible matrices $U \in GL_r(K)$ and $P \in GL_n(K)$, where P is a matrix obtained by permuting the rows of some nonsingular diagonal matrix, such that
$$A_1 = \sigma(UA_2P).$$

This defines an equivalence relation on the set of all matrices of rank r over K representing M. In particular, we see that A is equivalent to a matrix of the form $[I_r|D]$, where I_r is the $r \times r$ identity matrix and D is some $r \times (n-r)$ matrix (so that $M \cong M(A) = M([I_r|D])$). This can be achieved by computing the reduced row echelon form of A and bringing the columns containing a pivot to the first positions.

A representable matroid M over a field K is called *uniquely K-representable* if all the matrices representing M over K are equivalent. For instance, regular matroids are uniquely representable over any field.

Now we assume that $A \in M_{r\times n}(\mathbb{Z})$ is totally unimodular and $M \cong M(A)$ over $\mathbb{Q}$. If B is a subset of $E(M(A))$ and $A|_B$ denotes the matrix formed with the columns of A labeled by B, then $B \in \mathcal{B}(M(A))$ if and only if $\#B = r$ and $\det A|_B = \pm 1$. Hence if K is an arbitrary field and we regard A as a matrix over K, then the matroid $M(A)$ is represented by A over K, the same is true for M. In summary, a matroid represented by a totally unimodular matrix is regular. In fact, the converse is also true as shown below.

The incidence matrix of a graph is an example of a totally unimodular matrix (Proposition 1.1.46). Therefore, a graphic matroid is always regular (see the comment right after Definition 1.2.11).

Theorem 1.5.2. *For a matroid M the following are equivalent:*

(*a*) *M is regular;*

(*b*) *M is representable over* $\mathbb{Q}$ *by a totally unimodular matrix.*

Moreover, if M is regular and A is a totally unimodular matrix representing M over $\mathbb{Q}$*, then A is also a representation of M over a field K when A is viewed as a matrix over K.*

Proof. See Theorem 6.6.3 in [26]. □

If a linear matroid M of rank r is represented by an $r \times n$ matrix A, then A is equivalent to the matrix $[I_r|D]$ obtained from the reduced row echelon form of A by bringing the columns containing a pivot to the first positions. One can use the following procedure, which some authors called "pivoting", to get the reduced row echelon form of A; choose a nonzero entry a_{st} of A and for each $i \in \{1, \ldots, s-1, s+1, \ldots, r\}$, replace row i of A by row $i - (a_{it}/a_{st})$ row s, then multiply row s by $1/a_{st}$. As a result, we turn the tth column of A into the sth standard basis vector of K^r. It turns out that the resulting matrix is again totally unimodular by Lemma 2.2.20 in [26]. So that, the reduced row echelon form of a totally unimodular matrix is totally unimodular. Thus, the matrix $[I_r|D]$ is totally unimodular as well.

Proposition 1.5.3. *The dual of a regular matroid is regular.*

Proof. Suppose that M is a regular matroid of rank r and $\#E(M) = n$. If $r = 0$ or $n - r = 0$, the result is trivial. Hence, we may assume that r and $n - r$ are positive. Then there is a totally unimodular matrix of the form $[I_r|D]$ representing M. So, by Example 1.3.5, M^* is represented by the matrix $[-D^t|I_{n-r}]$, which is totally unimodular. □

After all discussions held so far, it becomes clear that the class of regular matroids is closed under direct sums and under deletion and contraction operations. Series and parallel extensions of a regular matroid are also regular as shown below.

Proposition 1.5.4. *Let M be a regular matroid. If N is a parallel (series) extension of M, then N is regular.*

Proof. Suppose that N is a parallel extension of M and $\#E(M) = n$. There is a circuit $\{e, f\}$ of N such that $e \neq f$ and $N \backslash e = M$. If $r(N) = 0$, then N is represented over any field, thus N is regular. If $r(N) > 0$, then $r(M) > 0$, in fact, $r(N) = r(M)$ (see (1.8)). So there is a totally unimodular matrix A representing M. Suppose that $\phi\colon M \to M(A)$ is an isomorphism. Let A' be the matrix obtained from A by duplicating the column $\phi(f)$. We label this new column with the $(n+1)$th canonical vector of K^{n+1}. It is clear that A' is totally unimodular and that $\phi'\colon E(N) \to E(M(A'))$ given by $\phi'(g) = \phi(g)$ for all $g \neq e$ and $\phi'(e) = e_{n+1}$ gives an isomorphism. Thus by Theorem 1.5.2, the result follows.

We now suppose that N is a series extension of M. By Proposition 1.3.19, N^* is a parallel extension of M^*. Being M regular, so is its dual (see Proposition 1.5.3). Thus, from what we just proved it follows that N^* is regular, consequently $(N^*)^* = N$ is regular as well. □

Chapter 2

Jacobian Groups

The Jacobian group of a graph (also known as the sandpile group, critical group, or chip-firing group) admits a generalization to regular matroids. However, little has been said about the structure of these groups. Some results along these lines may be found in [21] and [25].

2.1 Jacobian Group of a Regular Matroid

The Jacobian group of a regular matroid is an example of the determinant group of a lattice. Since many of the properties we are going to describe for the Jacobian group of a matroid are just consequences from the theory of lattices, we will develop some of the relevant ideas of this theory.

2.1.1 Theory of lattices

In what follows, all rings are commutative with 1 and all modules are unital.

Let R be a ring and M be an R-module. The dual module of M, denoted M^*, is the R-module $\mathrm{Hom}_R(M, R)$.

When M is a free R-module of rank n, M^* is also a free R-module of rank n. In fact, for each basis $\{e_1, \ldots, e_n\}$ of M there exists a unique *dual basis* $\{e_1^*, \ldots, e_n^*\}$ of M^* with the property $e_i^*(e_j) = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

A *pairing* on M is a bilinear form $\langle \cdot, \cdot \rangle \colon M \times M \to R$. If $\langle m, n \rangle = \langle n, m \rangle$ for all $m, n \in M$, then $\langle \cdot, \cdot \rangle$ is said to be *symmetric*. The pairing $\langle \cdot, \cdot \rangle$ induces an R-module homomorphism

$$\begin{aligned} \varphi \colon M &\to M^* \\ m &\mapsto (n \mapsto \langle m, n \rangle). \end{aligned} \tag{2.1}$$

If $\ker \varphi = \{0\}$ then $\langle \cdot, \cdot \rangle$ is *nondegenerate*, and if φ is an isomorphism then $\langle \cdot, \cdot \rangle$ is *perfect*.

Every perfect paring is nondegenerate, but the converse is not true in general, not even for free modules of finite rank. However, if M is a finite dimensional vector space then the converse is true.

Suppose M is a free R-module of finite rank n and $\mathcal{B} = \{e_1, \dots, e_n\}$ is a basis of M. If the pairing $\langle\cdot,\cdot\rangle$ is perfect, then (2.1) is an isomorphism of R-modules. If we let $e'_i := \varphi^{-1}(e^*_i)$ for all $i = 1, \dots, n$, then $\{e'_1, \dots, e'_n\}$ is a basis of M satisfying $\langle e'_i, e_j\rangle = \delta_{ij}$. When $\langle\cdot,\cdot\rangle$ is symmetric we can similarly recover $\{e_1, \dots, e_n\}$ from $\{e'_1, \dots, e'_n\}$.

Definition 2.1.1. If M is a free R-module of finite rank n and $\mathcal{B} = \{e_1, \dots, e_n\}$ is a basis of M, then the *Gram matrix* of M with respect to $\mathcal{B}$ is the $n \times n$ matrix $Gr_M(\mathcal{B}) := (\langle e_i, e_j\rangle)$.

Notice that $Gr_M(\mathcal{B})$ is nothing but the matrix representing $\varphi\colon M \to M^*$ in the bases $\{e_1, \dots, e_n\}$ and $\{e^*_1, \dots, e^*_n\}$. If $\mathcal{B}' = \{f_1, \dots, f_n\}$ is another basis of M and we let $f_j = \sum_{i=1}^n c_{ij} e_i$ for all $j \in \{1, \dots, n\}$, then the matrix $C = (c_{ij})$ is invertible over R and we have

$$\begin{aligned}
\langle f_i, f_j\rangle &= \left\langle \sum_{k=1}^n c_{ki} e_k, \sum_{l=1}^n c_{lj} e_l \right\rangle \\
&= \sum_{k=1}^n \sum_{l=1}^n c_{ki} c_{lj} \langle e_k, e_l\rangle \\
&= (C^t Gr_M(\mathcal{B}) C)_{ij},
\end{aligned}$$

thus $Gr_M(\mathcal{B}') = C^t Gr_M(\mathcal{B}) C$.

Example 2.1.2. Let R be a ring. Consider the free R-module $M = R^n$. Define the pairing $\langle\cdot,\cdot\rangle$ on M given by $\langle (r_1, \dots, r_n), (s_1, \dots, s_n)\rangle = r_1 s_1 + \cdots + r_n s_n$. Let $\mathcal{B} = \{e_1, \dots, e_n\}$ be a basis of M and let B be the matrix whose columns are $e_1, \dots, e_n$. Then $Gr_M(\mathcal{B}) = B^t B$.

Definition 2.1.3. Let M and N be two R-modules with pairings $\langle\cdot,\cdot\rangle_M$ and $\langle\cdot,\cdot\rangle_N$, respectively. An *isometry* from M to N is an R-module isomorphism $\phi\colon M \to N$ such that for all $m_1, m_2 \in M$, $\langle \phi(m_1), \phi(m_2)\rangle_N = \langle m_1, m_2\rangle_M$.

Proposition 2.1.4. *Two free R-modules of finite rank M and N with pairings $\langle\cdot,\cdot\rangle_M$ and $\langle\cdot,\cdot\rangle_N$, respectively, are isometric if and only if there are bases $\mathcal{B} \subseteq M$ and $\mathcal{B}' \subseteq N$ for which $Gr_M(\mathcal{B})$ and $Gr_N(\mathcal{B}')$ coincide.*

Proof. See Subsection 2.2.4 in [18]. □

Definition 2.1.5. Let R be an integral domain with fraction field K and let V be a finite dimensional vector space over K. A (*full*) *R-lattice* of V is a finitely generated R-module $\Lambda \subseteq V$ that spans V as a K-vector space.

Definition 2.1.6. Let R be a Noetherian domain with fraction field K, and let V be a finite dimensional vector space over K with a perfect pairing $\langle\cdot,\cdot\rangle$. If Λ is an R-lattice of V, its *dual lattice* (with respect to the perfect pairing $\langle\cdot,\cdot\rangle$ on V) is the R-module

$$\Lambda^\# := \{x \in V \colon \langle x, z\rangle \in R \text{ for all } z \in \Lambda\}.$$

Theorem 2.1.7. *Let R be a Noetherian domain with fraction field K, and let V be a finite dimensional vector space over K with a perfect pairing $\langle\cdot,\cdot\rangle$. If Λ is an R-lattice of V, then $\Lambda^\#$ is an R-lattice of V isomorphic to Λ^*.*

Proof. See Theorem 5.12 in [28]. □

Corollary 2.1.8. *Let R be a Noetherian domain with fraction field K, let V be a finite dimensional vector space over K with perfect pairing $\langle \cdot, \cdot \rangle$, and let Λ be a free R-lattice of V with R-basis $\{e_1, \ldots, e_n\}$. The dual lattice $\Lambda^{\#}$ is a free R-lattice of V that has a unique R-basis $\{e_1', \ldots, e_n'\}$ that satisfies $\langle e_i', e_j \rangle = \delta_{ij}$.*

Proof. See Corollary 5.14 in [28]. □

Definition 2.1.9. Let R be an integral domain with fraction field K and let V be a finite dimensional vector space over K with a pairing $\langle \cdot, \cdot \rangle$. A lattice Λ of V satisfying $\langle z_1, z_2 \rangle \in R$ for all $z_1, z_1 \in \Lambda$ is said to be *integral* over R.

Keeping the notation of Theorem 2.1.7, and assuming further that Λ is integral, we see that $\Lambda \subseteq \Lambda^{\#}$. Hence we define the *determinant group* of Λ to be the R-module

$$\mathrm{Jac}(\Lambda) := \frac{\Lambda^{\#}}{\Lambda}.$$

Recall that there is a canonical R-module homomorphism

$$\begin{aligned} \varphi \colon \Lambda &\to \Lambda^* \\ x &\mapsto (z \mapsto \langle x, z \rangle). \end{aligned}$$

Also, we have the following canonical isomorphism of R-modules (for more details see proof of Theorem 5.12 in [28])

$$\begin{aligned} \theta \colon \Lambda^{\#} &\to \Lambda^* \\ x &\mapsto (z \mapsto \langle x, z \rangle). \end{aligned}$$

Notice that $\theta(\Lambda) = \varphi(\Lambda)$. Hence the homomorphism θ induces an isomorphism of R-modules $\bar{\theta} \colon \mathrm{Jac}(\Lambda) \to \Lambda^*/\varphi(\Lambda)$.

When R is a PID, Λ is a free R-lattice of V. By the theory of finitely generated modules over a PID, there exits a basis $\{b_1, \ldots, b_n\}$ of $\Lambda^{\#}$ and nonzero elements $d_1, \ldots, d_n \in R$ such that $d_i \mid d_{i+1}$ for all $i \in \{1, \ldots, n-1\}$ and

$$\mathrm{Jac}(\Lambda) \cong R/d_1R \oplus \cdots \oplus R/d_nR.$$

Since $\mathrm{Jac}(\Lambda) \cong \Lambda^*/\varphi(\Lambda)$, one can find the sequence $d_1, \ldots, d_n$ by computing the Smith normal form of the Gram matrix of Λ with respect to any of its bases. Additionally, we also obtain the following result.

Proposition 2.1.10. *Isometric integral lattices have isomorphic determinant groups.*

Proposition 2.1.11. *Let R be a Noetherian domain with fraction field K, and let V be a finite dimensional vector space over K. Suppose that W is a subspace of V and Λ is an R-lattice of V. If $\Lambda_W := \Lambda \cap W$, then Λ_W is an R-lattice of W.*

Proof. It is easy to see that Λ_W is an R-module. Furthermore, it is finitely generated as it is a submodule of the Noetherian module Λ. Lastly, we show that Λ_W spans W. Choose a basis $\{w_1, \dots, w_k\}$ of W and complete it to a basis $\{w_1, \dots, w_k, w_{k+1}, \dots, w_n\}$ of V. Since Λ spans V and it is finitely generated over R, there is a nonzero element $r \in R$ such that $rw_i \in \Lambda$ for all $i \in \{1, \dots, n\}$. Hence $\{rw_1, \dots, rw_k\}$ is a basis of W contained in Λ_W, which shows that Λ_W spans W, as desired. □

Recall that if V is a finite dimensional vector space over a field K with a symmetric perfect pairing such that $\langle v, v\rangle \neq 0$ for all nonzero $v \in V$, then for any subspace $W \subseteq V$ we can define the subspace $W^\perp := \{v \in V : \langle v, w\rangle = 0 \text{ for all } w \in W\}$ and obtain the decomposition $V = W \oplus W^\perp$. We let $P\colon V \to W$ and $P^\perp\colon V \to W^\perp$ denote the projection maps.

The next theorem is a generalization of Lemma 1 from Section 4 in [2]. The proof given there can be easily adapted to this case after noting the following.

Lemma 2.1.12. *Let R be an integral domain with fraction field K and let V be a K-vector space. If $\Lambda \subseteq V$ is an R-module and W is a subspace of V, then Λ/Λ_W is a torsion-free R-module.*

Proof. Suppose $rz \in \Lambda_W$ for some $r \in R - \{0\}$ and some $z \in \Lambda$. Then $z = \frac{1}{r} rz \in W$, hence $z \in \Lambda_W$. □

Theorem 2.1.13. *Let R be a PID with fraction field K, and let V be a finite dimensional vector space over K with a symmetric perfect pairing $\langle\cdot,\cdot\rangle$ such that $\langle v, v\rangle \neq 0$ for all nonzero $v \in V$. Suppose that W is a subspace of V and Λ is an R-lattice of V. Then*

(a) *$P(\Lambda^\#) = \Lambda^\#_W$ and $P^\perp(\Lambda^\#) = \Lambda^\#_{W^\perp}$.*

(b) *If Λ is integral over R, then the projections P and $P^\perp$ induce the short exact sequences*

$$0 \longrightarrow \frac{\Lambda_W \oplus \Lambda^\#_{W^\perp}}{\Lambda_W \oplus \Lambda_{W^\perp}} \longrightarrow \frac{\Lambda^\#}{\Lambda_W \oplus \Lambda_{W^\perp}} \xrightarrow{\bar{P}} \frac{\Lambda^\#_W}{\Lambda_W} \longrightarrow 0$$

and

$$0 \longrightarrow \frac{\Lambda^\#_W \oplus \Lambda_{W^\perp}}{\Lambda_W \oplus \Lambda_{W^\perp}} \longrightarrow \frac{\Lambda^\#}{\Lambda_W \oplus \Lambda_{W^\perp}} \xrightarrow{\overline{P^\perp}} \frac{\Lambda^\#_{W^\perp}}{\Lambda_{W^\perp}} \longrightarrow 0.$$

(c) *If Λ is integral and $\mathrm{Jac}(\Lambda) = 0$, then $\bar{P}$ and $\overline{P^\perp}$ are R-module isomorphisms.*

Example 2.1.14. Let $R = \mathbb{Z}$, $K = \mathbb{Q}$, $V = \mathbb{Q}^4$, and $\Lambda = \mathbb{Z}^4$. We regard V with the standard inner product. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 1 \end{bmatrix},$$

and let $W = \ker(A) \subseteq V$. Then $W^\perp = \mathrm{row}(A)$. The free abelian groups Λ_W and Λ_W have bases $\{(-2, 1, 1, 0), (3, -1, 0, 1)\}$ and $\{(1, 0, 2, -3), (0, 1, -1, 1)\}$, respectively. Hence

$$\frac{\Lambda}{\Lambda_W \oplus \Lambda_{W^\perp}} = \mathrm{Coker} \begin{bmatrix} -2 & 3 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 1 \end{bmatrix} \cong \mathbb{Z}/17\mathbb{Z},$$

where the isomorphism is deduced by looking at the Smith normal form of the 4×4 matrix. By Theorem 2.1.13, we have that $\mathrm{Jac}(\Lambda_W) \cong \mathbb{Z}/17\mathbb{Z} \cong \mathrm{Jac}(\Lambda_{W^\perp})$. One can directly compute $\mathrm{Jac}(\Lambda_{W^\perp})$ by looking at the Smith normal form of the Gram matrix of $\Lambda_{W^\perp}$ with respect to the basis $\{(1,0,2,-3),(0,1,-1,1)\}$, which is AA^t. The Smith normal form is

$$\begin{bmatrix} 1 & 0 \\ 0 & 17 \end{bmatrix}.$$

2.1.2 Jacobian group of a graph

For a thorough treatment of the following exposition (and comparison with the Jacobian of a Riemann surface) we recommend the references [2] and [7].

Suppose that G is a connected graph without loops. Let $\mathbb{Q}^{E(G)}$ and $\mathbb{Q}^{V(G)}$ be the free $\mathbb{Q}$-vector spaces on the edge set and vertex set of G, respectively. Analogously, we let $\mathbb{Z}^{E(G)}$ and $\mathbb{Z}^{V(G)}$ be the free abelian groups on the edge set and vertex set of G respectively. We have canonical group homomorphisms $\mathbb{Z}^{E(G)} \hookrightarrow \mathbb{Q}^{E(G)}$ and $\mathbb{Z}^{V(G)} \hookrightarrow \mathbb{Q}^{V(G)}$. The vector space $\mathbb{Q}^{V(G)}$ has a canonical basis: for each $v \in V(G)$, let $\chi_v \colon V(G) \to \{0,1\}$ be the characteristic function of v. Hence the set $\{\chi_v \colon v \in V(G)\}$ is a $\mathbb{Q}$-basis of $\mathbb{Q}^{V(G)}$. Similarly, we construct the canonical basis $\{\chi_e \colon e \in E(G)\}$ of $\mathbb{Q}^{E(G)}$. The vector spaces $\mathbb{Q}^{E(G)}$ and $\mathbb{Q}^{V(G)}$ come with canonical inner products defined as follows:

$$\langle f, g\rangle := \sum_{e\in F} f(e)g(e), \quad \forall\, f,g \in \mathbb{Q}^F,$$

where $F \in \{E(G), V(G)\}$.

Choose an orientation (t,h) on G and consider the $\mathbb{Q}$-linear map $\partial \colon \mathbb{Q}^{E(G)} \to \mathbb{Q}^{V(G)}$ which is defined on basis elements by $\partial(\chi_e) = \chi_{h(e)} - \chi_{t(e)}$. There exists a unique $\mathbb{Q}$-linear map $\partial^* \colon \mathbb{Q}^{V(G)} \to \mathbb{Q}^{E(G)}$ satisfying $\langle \partial(f), g\rangle = \langle f, \partial^*(g)\rangle$ for all $f \in \mathbb{Q}^{E(G)}$ and $g \in \mathbb{Q}^{V(G)}$. Thus we have the decomposition $\mathbb{Q}^{E(G)} = \ker\partial \oplus \operatorname{Im}\partial^*$. The $\mathbb{Q}$-vector spaces $\ker\partial$ and $\operatorname{Im}\partial^*$ are known as the *cycle space* and *cut space* of G, respectively. Now let $\Lambda(G) := \ker\partial \cap \mathbb{Z}^{E(G)}$, which is a full rank $\mathbb{Z}$-lattice of $\ker\partial$. It is integral over $\mathbb{Z}$, and therefore we can speak of its determinant group $\mathrm{Jac}(\Lambda(G))$, which we call the *Jacobian group* of G. So that

$$\mathrm{Jac}(G) := \frac{\Lambda(G)^\#}{\Lambda(G)}.$$

We highlight that $\mathrm{Jac}(G)$ does not depend on the choice of (t,h).

Notice that $\operatorname{Im}\partial^* = (\ker\partial)^\perp$ so that $V = \mathbb{Q}^{E(G)}$, $W = \ker\partial$, and $\Lambda = \mathbb{Z}^{E(G)}$ verify the conditions of Theorem 2.1.13. Hence $\Lambda_W = \Lambda(G)$ and $\mathrm{Jac}(G) \cong \mathrm{Jac}(\Lambda_{W^\perp})$.

Let us describe the lattice $\Lambda_{W^\perp}$. This a lattice of $W^\perp = \operatorname{Im}\partial^*$, which we denote by $\Lambda^*(G)$. The matrix representing ∂ with respect to the canonical bases is precisely $B(G)$. Therefore, the rows of $B(G)$ span $\operatorname{Im}\partial^*$. By Proposition 1.1.44 and noting that the sum of the rows of $B(G)$ is zero, it follows that any set consisting of $\#V(G) - 1$ rows of $B(G)$ is a basis of $\operatorname{Im}\partial^*$. In fact, it is a $\mathbb{Z}$-basis of $\Lambda^*(G)$. If $v \in V(G)$ and $B(G)[v]$ is the matrix obtained from $B(G)$ by removing the vth row, then $B(G)[v]B(G)[v]^t = L(G)_v$ is the Gram matrix of $\Lambda^*(G)$, where $L(G)_v$ is the resulting matrix after removing the vth row and column of $L(G)$. Since the sum of the rows of $L(G)$ is zero, we then have that the invariant factors of $\mathrm{Jac}(G)$ can be read off from the Smith normal form of $L(G)$. We also have the isomorphism (see Theorem 2.1.13)

$$\operatorname{Jac}(G) \cong \frac{\mathbb{Z}^{E(G)}}{\Lambda(G) \oplus \Lambda^*(G)}.$$

To conclude this section, we compute the Jacobian groups for the families of path graphs, banana graphs, cycle graphs, and complete graphs.

Example 2.1.15. Consider the path graph Path_n, $n \in \mathbb{N}_0$. Recall that $V(\mathrm{Path}_n) = \{0, \dots, n\}$ and $E(\mathrm{Path}_n) = \{e_1, \dots, e_n\}$. Let (t, h) be given by $(t, h)(e_i) = (i-1, i)$ for all $i \in \{1, \dots, n\}$. Then ∂ is injective for if $f = \sum_{i=1}^{n} f(e_i)\chi_{e_i} \in \ker \partial$, then

$$0 = \partial(f) = f(e_n)\chi_{h(e_n)} - f(e_1)\chi_{t(e_1)} + \sum_{i=1}^{n-1} \Big(f(e_i) - f(e_{i+1})\Big)\chi_{t(e_{i+1})}.$$

The set $\{\chi_{t(e_i)} : i = 1, \dots, n\} \cup \{\chi_{h(e_n)}\}$ is the canonical basis of $\mathbb{Q}^{V(\mathrm{Path}_n)}$, thus it is linearly independent, so that $f(e_1) = f(e_2) = \cdots = f(e_n) = 0$. Hence $\operatorname{Jac}(\mathrm{Path}_n) = 0$ for all $n \in \mathbb{N}_0$.

Example 2.1.16. Consider the cycle graph Cycle_n, $n \in \mathbb{N}$. Recall that $V(\mathrm{Cycle}_n) = \mathbb{Z}/n\mathbb{Z}$ and $E(\mathrm{Cycle}_n) = \{e_1, \dots, e_n\}$. Let (t, h) be defined by $(t, h)(e_i) = (i-1, i)$. Pick $f = \sum_{i=1}^{n} f(e_i)\chi_{e_i}$ in $\ker \partial$. We have

$$0 = \partial(f) = \Big(f(e_n) - f(e_1)\Big)\chi_{t(e_1)} + \sum_{i=1}^{n-1} \Big(f(e_i) - f(e_{i+1})\Big)\chi_{t(e_{i+1})},$$

which implies that $f(e_1) = \cdots = f(e_n)$. If we set $\alpha = \chi_{e_1} + \cdots + \chi_{e_n}$, then $f = f(e_1)\alpha$. It is clear that $\alpha \in \ker \partial$, as a result, $\ker \partial = \operatorname{span}_{\mathbb{Q}}(\alpha)$. In particular, $\Lambda(\mathrm{Cycle}_n) = \mathbb{Z}\alpha$ and a direct computation shows that $\Lambda(\mathrm{Cycle}_n)^{\#} = \frac{1}{n}\Lambda(\mathrm{Cycle}_n)$. Consequently $\operatorname{Jac}(\mathrm{Cycle}_n) \cong \mathbb{Z}/n\mathbb{Z}$.

Example 2.1.17. Let us compute $\operatorname{Jac}(K_n)$, $n \geq 2$. To do so, we will compute the Smith normal form of its Laplacian.

$$\begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix} \xrightarrow[\text{are zero}]{\text{row and column sums}} \begin{bmatrix} n-1 & -1 & \cdots & -1 & 0 \\ -1 & n-1 & \cdots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & n-1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{to column 1}]{\text{add columns 2 up to } n-1} \begin{bmatrix} 1 & -1 & \cdots & -1 & 0 \\ 1 & n-1 & \cdots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -1 & \cdots & n-1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \xrightarrow[\text{through } n-1]{\text{add column 1 to column 2}} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & n & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Hence $\operatorname{Jac}(\mathrm{K}_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$.

Example 2.1.18. The Laplacian matrix of the banana graph on n edges is

$$\begin{bmatrix} n & -n \\ -n & n \end{bmatrix},$$

then its Jacobian is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

2.1.3 Jacobian group of a regular matroid

Now the aim is to be able to replicate the same constructions from the previous subsection for regular matroids. One problem immediately arises: unlike graphic matroids, there is no natural choice for the matrix representing a regular matroid. Nonetheless, recall that a matroid M is regular if and only if it can be represented by a totally unimodular matrix (see Theorem 1.5.2). Moreover, we know that the incidence matrix of a graph is totally unimodular (Proposition 1.1.46) and the Jacobian group is constructed from this matrix. Hence we will start by defining the Jacobian group for a particular totally unimodular representation of M. Later we shall show that the isomorphism type of this group is independent of the totally unimodular representation of M.

The results presented in Subsection 2.1.3 and Subsection 2.1.4 are drawn from Section 4.3 in [24] and Section 2.3 in [30]. Let M be a regular matroid of rank $r > 0$ with ground set E such that $\#E = n$. Suppose that M is represented by an $r \times n$ totally unimodular matrix A over $\mathbb{Q}$. Consider the $\mathbb{Q}$-vector space $\mathbb{Q}^n$ equipped with the canonical inner product. We define $\Lambda_A(M) := \ker A \cap \mathbb{Z}^n$. This is an integral full rank $\mathbb{Z}$-lattice of $\ker A$ so that we can speak of its dual $\Lambda_A(M)^\#$.

Definition 2.1.19. The *Jacobian group* of M with respect to the totally unimodular representation A is the determinant group

$$\mathrm{Jac}_A(M) := \frac{\Lambda_A(M)^\#}{\Lambda_A(M)}.$$

If $r = 0$, then $\mathrm{Jac}(M) := 0$.

Theorem 2.1.20. *If A and A' are two totally unimodular representations of a matroid M of rank $r > 0$, then $\mathrm{Jac}_A(M) \cong \mathrm{Jac}_{A'}(M)$.*

Before proving Theorem 2.1.20, we need the next result, which states that if we have two totally unimodular representations of M, then we can obtain one from the other by performing row and column operations over $\mathbb{Z}$. More precisely:

Lemma 2.1.21. *Let M be a regular matroid of rank $r \geq 1$ represented by two $r \times n$ totally unimodular matrices A and A'. Then there is an $r \times r$ unimodular matrix U and a signed permutation matrix P such that $UAP = A'$.*

Proof. See 4.2 in [24]. □

Proof of Theorem 2.1.20. If $r = n$, then $\ker A = \ker A' = 0$, thus $\mathrm{Jac}_A(M) = \mathrm{Jac}_{A'}(M)$. Assume $n - r \neq 0$. By Lemma 2.1.21, there is a unimodular matrix U and a signed permutation matrix P such that $UAP = A'$. Let $\mathcal{B} = \{b_1, \ldots, b_{n-r}\}$ be a basis of $\Lambda_{A'}(M)$ and let B be the matrix with columns $b_1, \ldots, b_{n-r}$. Then $A'B = 0$. Using the identity $UAP = A'$, one shows that the columns of the matrix PB form a basis of $\Lambda_A(M)$. If we look at the Gram matrices of B and PB, then $Gr(B) = B^tB$ and $Gr(PB) = ((PB)^t)(PB) = B^t(P^tP)B$ (see Example 2.1.2). Since P is orthogonal, the result follows from Propositions 2.1.4 and 2.1.10. □

We now turn to other presentations for $\mathrm{Jac}_A(M)$. Let us define $\Lambda_A^*(M) := \mathrm{row}(A) \cap \mathbb{Z}^n$. If we let $\Lambda = \mathbb{Z}^n$, then $\Lambda^\# = \Lambda$ and by Theorem 2.1.13

$$\mathrm{Jac}_A(M) \cong \frac{\mathbb{Z}^n}{\Lambda_A(M) \oplus \Lambda_A^*(M)} \cong \mathrm{Jac}(\Lambda_A^*(M)).$$

Since A is totally unimodular, its rows form a basis of $\Lambda_A^*(M)$. The Gram matrix of $\Lambda_A^*(M)$ with respect to this basis is AA^t. Hence

$$\mathrm{Jac}_A(M) \cong \mathrm{Coker}(AA^t).$$

Remark 2.1.22. Since we are interested in properties of $\mathrm{Jac}_A(M)$ that are invariant under isomorphism, we drop the subscript A and simply speak of the Jacobian of M.

Remark 2.1.23. If G is a connected graph without loops, then $\mathrm{Jac}(G) = \mathrm{Jac}_{B(G)[v]}(M(G))$. Hence we define for any graph G its Jacobian group to be $\mathrm{Jac}(G) := \mathrm{Jac}(M(G))$.

Corollary 2.1.24. *If M_1 and M_2 are regular matroids, then* $\mathrm{Jac}(M_1 \oplus M_2) \cong \mathrm{Jac}(M_1) \oplus \mathrm{Jac}(M_2)$.

Proof. If A_1 and A_2 are two totally unimodular representations of M_1 and M_2 respectively, then the block diagonal matrix A with blocks A_1 and A_2 on its diagonal is a totally unimodular representation of $M_1 \oplus M_2$. Hence $\mathrm{Coker}(AA^t) \cong \mathrm{Coker}(A_1A_1^t) \oplus \mathrm{Coker}(A_2A_2^t)$. □

In the case of graphic matroids, Corollary 2.1.24 implies that if H and K are graphs and $G \in \{H \vee K, H \sqcup K\}$, then $\mathrm{Jac}(G) = \mathrm{Jac}(H) \oplus \mathrm{Jac}(K)$. As a consequence of this fact is that any finite abelian group is isomorphic to the Jacobian of a connected graph. Indeed, $\bigoplus_{i=1}^k \mathbb{Z}/p_i^{m_i}\mathbb{Z} \cong \mathrm{Jac}(\mathrm{Cycle}_{p_1^{m_1}} \vee \cdots \vee \mathrm{Cycle}_{p_k^{m_k}})$.

Corollary 2.1.25. *If M is a regular matroid, then* $\mathrm{Jac}(M) \cong \mathrm{Jac}(M^*)$.

Proof. Suppose that M has rank r and $\#E(M) = n$. If $r = 0$, then $r(M^*) = n - r = n$, so that $\mathrm{Jac}(M) = 0 = \mathrm{Jac}(M^*)$. Now assume $r > 0$ and $n - r \neq 0$. Then $M \cong M([I_r|D])$ where $[I_r|D]$ is totally unimodular , so that $M^* \cong M([-D^t|I_{n-r}])$ by Proposition 1.5.3. Let $A = [I_r|D]$ and $B = [-D^t|I_{n-r}]$. Note that $AB^t = 0$ and B^t has rank $n - r = \dim_{\mathbb{Q}} \ker(A)$, which shows that the columns of B^t form a basis of $\ker A$, in fact, these form a basis of $\Lambda_A(M)$ for B^t being totally unimodular. Similarly, the columns of A^t form a basis of $\Lambda_B(M^*)$. Thus we have

$$\begin{aligned}
\Lambda_A(M) &= \ker(A) \cap \mathbb{Z}^n = \mathrm{Im}(B^t) \cap \mathbb{Z}^n = \mathrm{row}(B) \cap \mathbb{Z}^n = \Lambda_B^*(M^*),\\
\Lambda_A^*(M) &= \mathrm{row}(A) \cap \mathbb{Z}^n = \mathrm{Im}(A^t) \cap \mathbb{Z}^n = \ker(B) \cap \mathbb{Z}^n = \Lambda_B(M^*).
\end{aligned}$$

Consequently $\mathbb{Z}^n/(\Lambda_A(M) \oplus \Lambda_A^*(M)) = \mathbb{Z}^n/(\Lambda_B(M^*) \oplus \Lambda_B^*(M^*))$. Then the result is clear. □

Example 2.1.26. Let $n \geq 2$. Consider $G = \mathrm{Cycle}_n$. Then G^* is the banana graph on n edges. By Corollary 2.1.25, $\mathrm{Jac}(G) \cong \mathrm{Jac}(G^*)$. Compare this with Examples 2.1.16 and 2.1.18.

2.1.4 Number of bases of a regular matroid

If we consider the analogy between curves and graphs, one could think of the order of the Jacobian of a graph as the class number of an ideal class group. In fact, the order of the Jacobian of a graph appears in the analytic class number formula for graphs [20]. This number, by Kirchhoff's Matrix Theorem (see Theorem 1.1.50), corresponds to the number of spanning forests, which is the number of bases of a graphic matroid.

Definition 2.1.27. The number of bases of a matroid M is denoted by $\kappa(M)$.

The following proposition is a generalization of Proposition 1.1.40.

Proposition 2.1.28. *If M is a matroid and $e \in E(M)$, then*

$$\kappa(M) = \begin{cases} \kappa(M \backslash e) = \kappa(M/e), & \text{if } e \text{ is either a loop or a coloop;} \\ \\ \kappa(M \backslash e) + \kappa(M/e), & \text{otherwise.} \end{cases}$$

Proof. It is a direct consequence from (1.3) and (1.7). □

Theorem 2.1.29. *If M is a regular matroid, then $\#\operatorname{Jac}(M) = \kappa(M)$.*

Proof. Let $r = r(M)$. If $r = 0$, then $\kappa(M) = 1 = \#\operatorname{Jac}(M)$. Suppose $r > 0$ and let A be an $r \times n$ totally unimodular representation of M. Let $E_r = \{S \subseteq E(M) \colon \#S = r\}$. By the Cauchy-Binet formula, we have

$$\det(AA^t) = \sum_{S \in E_r} \det\left(A[\{1,\dots,r\}|S]\right) \det\left(A^t[S|\{1,\dots,r\}]\right),$$

where $A[\{1,\dots,r\}|S]$ is the matrix formed by the columns of A labeled by S, similarly, the matrix $A^t[S \mid \{1,\dots,r\}]$ is formed by the rows of A^t labeled by S. Then, it is clear that $A^t[S|\{1,\dots,r\}] = A[\{1,\dots,r\}|S]^t$. Therefore,

$$\det(AA^t) = \sum_{S \in E_r} \det(A[\{1,\dots,r\}|S])^2.$$

Now observe that $\det(A[\{1,\dots,r\}|S])^2$ is either 0 or 1 according to whether S is the set of labels for r linearly independent columns of A, whence

$$\det(AA^t) = \sum_{B \in \mathcal{B}(M)} 1 = \kappa(M).$$

As $\operatorname{Jac}(M) \cong \operatorname{Coker}(AA^t)$ and $\#\operatorname{Coker}(AA^t) = \det(AA^t)$, the result follows. □

2.2 The Matroids M_λ

2.2.1 Metric graphs

Let G be a graph. Consider the following operation on G: given an arbitrary map $\lambda \colon E(G) \to \mathbb{N}$, we let G_λ be the graph obtained from G by replacing every edge $e \in E(G)$ with a path of length $\lambda(e)$. It is called a *metric graph* and the pair (G, λ) is called a *model* of G_λ. Intuitively, the map λ assigns a length to each edge of G.

In this way, from G we obtain a one-parameter family of graphs $\{G_\lambda\}_{\lambda \in \mathbb{N}^{E(G)}}$. Note that G is a member of this family as $G = G_1$ where 1 is the constant function 1.

Figure 2.1: $\lambda(e) = \lambda(f) = 2, \lambda(g) = 1, \lambda(h) = 3$.

Example 2.2.1. If $G = \text{Cycle}_1$ and λ_n denotes the map that sends the unique edge of G to n, then $G_{\lambda_n} = \text{Cycle}_n$ for all $n \in \mathbb{N}$. Similarly, if $H = \text{Path}_1$, then $H_{\lambda_n} = \text{Path}_n$ for all $n \in \mathbb{N}$.

2.2.2 Metric matroids

Suppose that M is a matroid of rank r where $\#E(M) = n$. Pick $\lambda \in \mathbb{N}^{E(M)}$. For each $e \in E(M)$ we let $E_e(M)$ denote a set with $\lambda(e)$ elements such that $e \in E_e(M)$ and $E_e(M) \cap E_f(M) = \emptyset$ for $e \neq f$. We define

$$E(M_\lambda) := \bigsqcup_{e \in E(M)} E_e(M).$$

Let $\mathcal{B}(M_\lambda)$ be the collection of all subsets B of $E(M_\lambda)$ for which there exists a basis B' of $\mathcal{B}(M)$ and a tuple $(x_e)_{e \notin B'} \in \prod_{e \notin B'} E_e(M)$ such that $B = E(M_\lambda) - \{x_e \colon e \notin B'\}$ (equivalently, there exists a basis B'' of $\mathcal{B}(M^*)$ and a tuple $(x_e)_{e \in B''} \in \prod_{e \in B''} E_e(M)$ such that $B = E(M_\lambda) - \{x_e \colon e \in B''\}$).

Proposition 2.2.2. *The collection $\mathcal{B}(M_\lambda)$ is the set of bases of a matroid with ground set $E(M_\lambda)$.*

Proof. We must prove that $\mathcal{B}(M_\lambda)$ verifies (**B1**) and (**B2**). The first condition follows immediately as $\mathcal{B}(M^*) \neq \emptyset$. For the second condition, we pick $B_1, B_2 \in \mathcal{B}$ and suppose $x \in B_1 - B_2$.

By definition, there are bases $B_1' = \{a_1, \ldots, a_{n-r}\}$ and $B_2' = \{b_1, \ldots, b_{n-r}\}$ of the matroid M^* for which there are tuples $(c_1, \ldots, c_{n-r}) \in \prod_{i=1}^{n-r} E_{a_i}(M)$ and $(d_1, \ldots, d_{n-r}) \in \prod_{i=1}^{n-r} E_{b_i}(M)$ such that $B_1 = E(M_\lambda) - \{c_1, \ldots, c_{n-r}\}$ and $B_2 = E(M_\lambda) - \{d_1, \ldots, d_{n-r}\}$.

Observe that $B_1 - B_2 = \{d_1, \ldots, d_{n-r}\} - \{c_1, \ldots, c_{n-r}\}$. Similarly, we have the equality $B_2 - B_1 = \{c_1, \ldots, c_{n-r}\} - \{d_1, \ldots, d_{n-r}\}$. Thus, there is $j \in \{1, \ldots, n-r\}$ such that $x = d_j$. Let us consider two cases. The first one is that there exists $k \in \{1, \ldots, n-r\}$ such that $b_j = a_k$. In this case, $c_k \in E_{b_j}(M)$ so that $c_k \in B_2 - B_1$ and $(B_1 - \{d_j\}) \cup \{c_k\} = E(M_\lambda) - (\{c_1, \ldots, c_{n-r}, d_j\} - \{c_k\})$ where

$$(c_1, \ldots, c_{k-1}, d_j, c_{k+1}, \ldots, c_{n-r}) \in \prod_{i=1}^{n-r} E_{a_i}(M).$$

Hence $(B_1 - \{d_j\}) \cup \{c_k\} \in \mathcal{B}(M_\lambda)$.

The second case is that $b_j \neq a_i$ for all $i \in \{1, \ldots, n-r\}$. Then $b_j \in B_2' - B_1'$. By Proposition 1.2.15, there exists $a_k \in B_1' - B_2'$ such that $(B_1' - \{a_k\}) \cup \{b_j\} \in \mathcal{B}(M^*)$. Since $a_k \notin B_2'$, $E_{a_k}(M) \neq E_{b_i}(M)$ for all $i \in \{1, \ldots, n-r\}$. Therefore $c_k \notin E_{b_i}(M)$ for all i and consequently $c_k \in B_2 - B_1$. As before, we have that $(B_1 - \{d_j\}) \cup \{c_k\} = E(M_\lambda) - (\{c_1, \ldots, c_{n-r}, d_j\} - \{c_k\})$ where

$$(c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_{n-r}, d_j) \in \prod_{\substack{i=1 \\ i \neq k}}^{n-r} E_{a_i}(M) \times E_{b_j}(M).$$

Then $(B_1 - \{d_j\}) \cup \{c_k\} \in \mathcal{B}(M_\lambda)$ as $\{a_1, \ldots, a_{n-r}, b_j\} - \{a_k\} \in \mathcal{B}(M^*)$. This completes the proof. $\square$

Definition 2.2.3. If $\lambda \in \mathbb{N}^{E(M)}$, then the *metric matroid* M_λ is given by $(E(M_\lambda), \mathcal{B}(M_\lambda))$.

The matroid M_1 given by the constant function $e \mapsto 1$ is equal to M.

For $e \in E(M)$, let $\chi_e \colon E(M) \to \{0, 1\}$ be the characteristic function of e.

Example 2.2.4. The matroid $M_{1+\chi_e}$ has ground set $E(M_{1+\chi_e}) = E(M) \sqcup \{e'\}$ and its collection of bases is

$$\mathcal{B}(M_{1+\chi_e}) = \{B \cup \{e'\} \colon B \in \mathcal{B}(M)\} \cup \{B \cup \{e\} \colon B \in \mathcal{B}(M), e \notin B\}. \tag{2.2}$$

Proposition 2.2.5. *Let $e \in E(M)$. Then*

(*a*) *if e is not a coloop of M, then $M_{1+\chi_e}$ is a series extension of M;*

(*b*) *if e is a coloop of M, then $M_{1+\chi_e} \cong M \oplus U_{1,1}$.*

Proof. (*a*) From the description of $\mathcal{B}(M_{1+\chi_e})$ in (2.2), it is clear that the set $\{e, e'\}$ is a cocircuit of $M_{1+\chi_e}$ satisfying $e \neq e'$. In addition, from (1.7) it follows that $M_{1+\chi_e}/e' = M$.

(*b*) If e is a coloop of M, then $M_{1+\chi_e} = \big(E(M) \sqcup \{e'\}, \{B \sqcup \{e'\} \colon B \in \mathcal{B}(M)\}\big)$, which is isomorphic to $M \oplus U_{1,1}$. □

Example 2.2.6. Let us describe a procedure to compute $M_{1+\chi_e}$ for a representable matroid M. Suppose that e is not a coloop of M so that $M_{1+\chi_e}$ is a series extension of M. By Proposition 1.3.19, $M^*_{1+\chi_e}$ is a parallel extension of M^*. Assume $M^* \cong M(A)$ where $A \in M_{n-r\times n}(K)$ for some field K. Suppose that e corresponds to e_i in $M(A)$. If A' is the matrix obtained from A by duplicating column e_i and labeling this new column by e_{n+1}, then $M^*_{1+\chi_e} \cong M(A')$, hence $M_{1+\chi_e} \cong M(A')^*$. For the sake of clarity, we illustrate this with the next example.

Consider the following matrices over $\mathbb{Q}$:

$$A = \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}, \qquad B = \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{bmatrix} \end{array}.$$

Let $M = M(A)$. We shall find a representation for $M_{1+\chi_{e_1}}$ over $\mathbb{Q}$. Observe that $M^* = M(B)$. Then the matrix

$$B' = \begin{array}{c} \begin{array}{ccccc} e_1 & e_5 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ -2 & -2 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

represents the matroid $M^*_{1+\chi_{e_1}}$. Therefore, the matroid $M_{1+\chi_{e_1}}$ is represented by the matrix

$$\begin{array}{c} \begin{array}{ccccc} e_1 & e_5 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

Proposition 2.2.7. *For any $\lambda \in \mathbb{N}^{E(M)}$,*

$$r(M_\lambda) = \sum_{e \in E(M)} (\lambda(e) - 1) + r(M).$$

Proof. By Proposition 1.3.7, we have that $r(M_\lambda) + r(M^*_\lambda) = \#E(M_\lambda)$. By definition of $\mathcal{B}(M_\lambda)$, we have $r(M^*_\lambda) = r(M^*)$, then $r(M^*_\lambda) = \#E(M) - r(M)$. Hence

$$r(M_\lambda) = \#E(M_\lambda) - \#E(M) + r(M) = \sum_{e \in E(M)} (\lambda(e) - 1) + r(M).$$

□

Proposition 2.2.8. *Let M be a matroid. If $\lambda \in \mathbb{N}^{E(M)}$, then*

$$\kappa(M_\lambda) = \sum_{B \in \mathcal{B}(M)} \prod_{e \notin B} \lambda(e).$$

Proof. By definition, there is a bijection between $\mathcal{B}(M_\lambda)$ and the set

$$\bigsqcup_{B \in \mathcal{B}(M)} \prod_{e \notin B} E_e(M).$$

Hence the result follows. □

The next proposition will allow us to prove properties about M_λ by induction on λ.

Proposition 2.2.9. *Let $e \in E(M)$ and $\lambda \in \mathbb{N}^{E(M)}$. If $\lambda(e) > 1$, then $(M_{\lambda-\chi_e})_{1+\chi_e} = M_\lambda$.*

Proof. Suppose that $E(M_\lambda) = \bigsqcup_{f \in E(M)} E_f(M)$. By definition, the ground set of $M_{\lambda-\chi_e}$ can be taken to be $E(M_{\lambda-\chi_e}) = \bigsqcup_{f \in E(M)} E'_f(M)$, where $E'_f(M) = E_f(M)$ for $f \neq e$, and $E'_e(M) = E_e(M) - \{e'\}$ for some $e' \in E_e(M) - \{e\}$. So that, by (2.2), we have that $E((M_{\lambda-\chi_e})_{1+\chi_e}) = E(M_{\lambda-\chi_e}) \sqcup \{e'\}$ and

$$\mathcal{B}((M_{\lambda-\chi_e})_{1+\chi_e}) = \{B \cup \{e'\} \colon B \in \mathcal{B}(M_{\lambda-\chi_e})\} \cup \{B \cup \{e\} \colon B \in \mathcal{B}(M_{\lambda-\chi_e}), e \notin B\}. \tag{2.3}$$

We will show that $\mathcal{B}((M_{\lambda-\chi_e})_{1+\chi_e}) \subseteq \mathcal{B}(M_\lambda)$ and $\mathcal{B}(M_\lambda) \subseteq \mathcal{B}((M_{\lambda-\chi_e})_{1+\chi_e})$. To see the first containment, we pick an arbitrary basis $B' \in \mathcal{B}((M_{\lambda-\chi_e})_{1+\chi_e})$. According to (2.3), we must consider two cases for B':

(*i*) If $B' = B \cup \{e'\}$ with $B \in \mathcal{B}(M_{\lambda-\chi_e})$, then there exist a basis $B_0 \in \mathcal{B}(M)$ and a tuple $(x_f)_{f \notin B_0} \in \prod_{f \notin B_0} E'_f(M)$ such that $B = E(M_{\lambda-\chi_e}) - \{x_f \colon f \notin B_0\}$. Since $e' \notin \{x_f \colon f \notin B_0\}$ we have
$$B' = B \cup \{e'\} = E(M_\lambda) - \{x_f \colon f \notin B_0\} \in \mathcal{B}(M_\lambda).$$

(*ii*) If $B' = B \cup \{e\}$ with $B \in \mathcal{B}(M_{\lambda-\chi_e})$ and $e \notin B$, then there exist a basis $B_0 \in \mathcal{B}(M)$ and a tuple $(x_f)_{f \notin B_0} \in \prod_{f \notin B_0} E'_f(M)$ such that $B = E(M_{\lambda-\chi_e}) - \{x_f \colon f \notin B_0\}$. Since $e \notin B$, then $x_e = e$ and $e \notin B_0$. Define for $f \notin B_0$ with $f \neq e$, $y_f = x_f$ and $y_e = e'$. Then $(y_f)_{f \notin B_0} \in \prod_{f \notin B_0} E_f(M)$ and
$$B' = B \cup \{e\} = E(M_\lambda) - \{y_f \colon f \notin B_0\} \in \mathcal{B}(M_\lambda).$$

This shows that $\mathcal{B}((M_{\lambda-\chi_e})_{1+\chi_e}) \subseteq \mathcal{B}(M_\lambda)$.

Now pick $B \in \mathcal{B}(M_\lambda)$. There exist $B_0 \in \mathcal{B}(M)$ and $(x_f)_{f \notin B_0} \in \prod_{f \notin B_0} E_f(M)$ such that $B = E(M_\lambda) - \{x_f \colon f \notin B_0\}$. If $e' \in B$, then $e' \notin \{x_f \colon f \notin B_0\}$; thus

$$B = E(M_\lambda) - \{x_f \colon f \notin B_0\} = \Big(E(M_{\lambda-\chi_e}) - \{x_f \colon f \notin B_0\}\Big) \cup \{e'\} \in \mathcal{B}\Big((M_{\lambda-\chi_e})_{1+\chi_e}\Big).$$

If $e' \notin B$, then $e' \in \{x_f \colon f \notin B_0\}$, that is, $e \notin B_0$ and $x_e = e'$. We define $y_f = x_f$ if $f \neq e$ and $y_e = e$. So $(y_f)_{f \notin B_0} \in \prod_{f \notin B_0} E'_f(M)$ and

$$B = E(M_\lambda) - \{x_f \colon f \notin B_0\} = \Big(E(M_{\lambda-\chi_e}) - \{y_f \colon f \notin B_0\}\Big) \cup \{e\} \in \mathcal{B}\Big((M_{\lambda-\chi_e})_{1+\chi_e}\Big).$$

Thus $B(M_\lambda) \subseteq \mathcal{B}((M_{\lambda-\chi_e})_{1+\chi_e})$. □

Suppose $E(M) = \{e_1, \ldots, e_n\}$. We regard λ as the tuple $(\lambda(e_1), \ldots, \lambda(e_n)) \in \mathbb{N}^n$ and order $\mathbb{N}^n$ by the lexicographical order.

Proposition 2.2.10. *If M is irreducible (resp. linear or regular), so is M_λ for any $\lambda \in \mathbb{N}^{E(M)}$.*

Proof. We use induction on λ. The base case is $\lambda = (1, 1, \ldots, 1) \in \mathbb{N}^n$, then $M_\lambda = M$. Thus, the result follows by assumption. We now suppose $\lambda > (1, \ldots, 1)$ and that the proposition is true for any $\gamma < \lambda$. Let $i = \min\{j \colon \lambda(e_j) > 1\}$. Set $\gamma = \lambda - \chi_{e_i}$ so that $\gamma < \lambda$. Thus, by the induction hypothesis, the matroid M_γ is irreducible (resp. linear or regular). By Proposition 2.2.9, $(M_\gamma)_{1+\chi_{e_i}} = M_\lambda$.

If M_γ is irreducible, then e_i is not a coloop of M_γ, therefore, by Proposition 2.2.5, M_λ is a series extension of M_γ. Whence M_λ is irreducible by Proposition 1.4.7.

If M_γ is linear (resp. regular), then we consider two cases:

(*i*) If e_i is not a coloop of M_γ, then by Proposition 2.2.5, the matroid M_λ is a series extension of M_γ. Hence, by Proposition 1.5.4, the result follows.

(*ii*) If e_i is a coloop of M_γ, we have that $M_\lambda \cong M_\gamma \oplus U_{1,1}$ (see Proposition 2.2.5). Since the direct sum of linear (resp. regular) matroids is linear (resp. regular), the result is clear.

□

Proposition 2.2.11. *If G is a graph and $\lambda \in \mathbb{N}^{E(G)}$, then $M(G)_\lambda = M(G_\lambda)$.*

Proof. First of all, note that $M(G)_{1+\chi_e} = M(G_{1+\chi_e})$ for all $e \in E(G)$ (this follows from (2.2)).

We proceed by induction on λ. If $\lambda = (1, \ldots, 1)$, then there is nothing to show. Suppose that $(1, \ldots, 1) < \lambda$ and that the proposition is true for all $\gamma \in \mathbb{N}^{E(G)}$ verifying $\gamma < \lambda$. Let $i = \min\{j \colon \lambda(e_j) > 1\}$ (we are assuming $E(G) = \{e_1, \ldots, e_n\}$). Set $\gamma = \lambda - \chi_{e_i}$ so that $\gamma < \lambda$. Thus, by the induction hypothesis, $M(G)_\gamma = M(G_\gamma)$. As a result,

$$M(G)_\lambda = (M(G)_\gamma)_{1+\chi_{e_i}} = M(G_\gamma)_{1+\chi_{e_i}} = M((G_\gamma)_{1+\chi_e}) = M(G_\lambda).$$

□

2.3 Configuration Polynomials

Let M be a regular matroid. Let $\lambda := \{\lambda_e \colon e \in E(M)\}$ be a set of variables indexed by the elements of M. The *configuration polynomial* of M is

$$\Psi_M := \sum_{B \in \mathcal{B}(M)} \prod_{e \notin B} \lambda_e \in \mathbb{Z}[\lambda]. \tag{2.4}$$

By convention, a product of variables indexed by the empty set equals 1. If G is a graph, we define the configuration polynomial of G to be $\Psi_G := \Psi_{M(G)}$.

The configuration polynomial of a linear matroid can be defined by the theory of configurations of vector spaces; however, it depends on the field of definition. Under this new definition, the configuration polynomial of a regular matroid is independent of the field of definition and has exactly the form as in (2.4). For further discussion of the general case we refer the reader to [14].

Remark 2.3.1.

(a) By definition, Ψ_M is a homogeneous polynomial of degree $r(M^*)$. It is linear in each variable and its coefficients are all 1.

(b) If e is a loop of M, then λ_e divides Ψ_M. Thus, the configuration polynomial of a regular matroid is reducible in general.

(c) If e is a coloop of M, then $\partial_{\lambda_e}\Psi_M = 0$.

Example 2.3.2. Consider the diamond graph

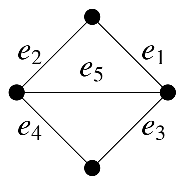


Figure 2.2: Diamond graph.

Its configuration polynomial is

$$\lambda_{e_1}\lambda_{e_3} + \lambda_{e_1}\lambda_{e_4} + \lambda_{e_2}\lambda_{e_3} + \lambda_{e_2}\lambda_{e_4} + \lambda_{e_1}\lambda_{e_5} + \lambda_{e_2}\lambda_{e_5} + \lambda_{e_3}\lambda_{e_5} + \lambda_{e_4}\lambda_{e_5}.$$

Example 2.3.3. Let G be the banana graph on n edges. Then

$$\Psi_G = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \lambda_{e_j}.$$

Example 2.3.4. Let $G = \mathrm{Cycle}_n$. Then

$$\Psi_G = \sum_{i=1}^{n} \lambda_{e_i}.$$

Lemma 2.3.5. *If $M = N \oplus K$ is a direct sum of two matroids, then $\Psi_M = \Psi_N\Psi_K$.*

Proof. By Remark 1.3.13, a basis of M is of the form $B_1 \sqcup B_2$ with $B_1 \in \mathcal{B}(N)$ and $B_2 \in \mathcal{B}(K)$. Furthermore, $E(M) - B_1 \sqcup B_2 = (E(N) - B_1) \sqcup (E(K) - B_2)$. Thus

$$\Psi_M = \sum_{\substack{B\in\mathcal{B}(M) \\ B=B_1\sqcup B_2}} \left(\prod_{e\in E(N)-B_1} \lambda_e \prod_{e\in E(K)-B_2} \lambda_e \right) = \left(\sum_{B\in\mathcal{B}(N)} \prod_{e\notin B} \lambda_e \right)\left(\sum_{B\in\mathcal{B}(K)} \prod_{e\notin B} \lambda_e \right) = \Psi_N\Psi_K.$$

□

The following proposition relates the irreducibility of a matroid and the irreducibility of its configuration polynomial. This result was proved in [14]. For the sake of completeness, we give a proof here.

Proposition 2.3.6. *Let M be a regular matroid.*

(*a*) *If $M_1, \dots, M_n$ are the irreducible components of M, then $\Psi_M = \Psi_{M_1} \cdots \Psi_{M_n}$.*

(*b*) *If M is nonempty and has no coloops, then M is irreducible if and only if Ψ_M is irreducible over any field.*

(*c*) $\Psi_M = 1$ *if and only if $r(M) = \#E(M)$ if and only if M is isomorphic to a finite direct sum of copies of $U_{1,1}$.*

Proof. (*a*) Since M is the direct sum of its irreducible components, part (*a*) follows from Lemma 2.3.5 by induction.

(*b*) We prove (*b*) by contrapositive. Assume first that M is reducible, that is, $M = N \oplus K$ where N, K are nonempty matroids. So that $\Psi_M = \Psi_N \Psi_K$. As M has no coloops, it follows that $r(N) < \#E(N)$ and $r(K) < \#E(K)$, which shows that Ψ_N and Ψ_K are nonconstant.

Now assume that Ψ_M is not irreducible, then $\Psi_M = GH$ for some nonconstant homogeneous polynomials G, H. Define $E_G = \{e \in E(M) \colon \deg_{\lambda_e} G > 0\}$ and $E_H = \{e \in E(M) \colon \deg_{\lambda_e} H > 0\}$. Consider the matroids $M_G = M|E_G$ and $M_H = M|E_H$. We claim that $M = M_G \oplus M_H$.

Since M has no coloops, $\deg_{\lambda_e} \Psi_M > 0$ for all $e \in E(M)$; moreover, as Ψ_M is linear in each variable, the polynomials G and H have no variables in common, then $E(M) = E_G \sqcup E_H$. Now we shall show that $\mathcal{B}(M) = \mathcal{B}(M_G \oplus M_H)$. Before proving this, let us note the following.

If $B_G \in \mathcal{B}(M|E_G)$, then there exists $B \in \mathcal{B}(M)$ such that $B_G = B \cap E_G$. So that

$$(E_G - B_G) \sqcup E_H = E(M) - B_G = E(M) - (B \cap E_G) = (E(M) - B) \cup E_H,$$

then

$$\begin{aligned}\#(E_G - B_G) + \#E_H &= \#(E(M) - B) + \#E_H - \#((E(M) - B) \cap E_H)\\ &= \deg \Psi_M + \#E_H - \deg H,\end{aligned}$$

from which it follows that $\#(E_G - B_G) = \deg G$. A similar argument shows that for any basis B_H of $M|E_H$, $\#(E_H - B_H) = \deg H$.

If $B \in \mathcal{B}(M)$, then $B = (B \cap E_G) \sqcup (B \cap E_H)$, then $E(M) - B = (E_G - (B \cap E_G)) \sqcup (E_H - (B \cap E_H))$. As $\#(E(M) - B) = \deg \Psi_M$, $\deg G \leq \#(E_G - (B \cap E_G))$, and $\deg H \leq \#(E_H - (B \cap E_H))$, it follows that each of these inequalities is in fact an equality. Hence $B \cap E_G \in \mathcal{B}(M|E_G)$ and $B \cap E_H \in \mathcal{B}(M|E_H)$. Consequently, $B \in \mathcal{B}(M|E_G \oplus M|E_H)$.

If $B_G \sqcup B_H \in \mathcal{B}(M_G \oplus M_H)$, then

$$\#(B_G \sqcup B_H) = \#B_G + \#B_H = (\#E_G - \deg H) + (\#E_H - \deg G) = E(M) - \deg \Psi_M = r(M).$$

Therefore $B_G \sqcup B_H$ is a basis of M by Proposition 1.2.25. Hence $M = M|E_G \oplus M|E_H$, showing that M is reducible.

(*c*) If $\Psi_M = 1$, then $E(M) - B = \emptyset$ for any $B \in \mathcal{B}(M)$. Therefore, $r(M) = \#E(M)$. If $r(M) = \#E(M)$, then $E(M)$ is the only basis of M. Then $M = \bigoplus_{e \in E(M)} M|\{e\}$. Since $M|\{e\} \cong U_{1,1}$, M is isomorphic to a finite direct sum of copies of $U_{1,1}$ (we assume that a direct sum of matroids over the empty set equals the empty matroid). Lastly, if M is isomorphic to a finite direct sum of copies of $U_{1,1}$. Then $E(M)$ is the only basis of M, hence $\Psi_M = 1$. □

Proposition 2.3.7. *If M is a regular matroid and $e \in E(M)$, then*

$$\Psi_M = \begin{cases} \Psi_{M/e}, & \textit{if } e \textit{ is a coloop;} \\ \lambda_e \Psi_{M \backslash e}, & \textit{if } e \textit{ is a loop;} \\ \lambda_e \Psi_{M \backslash e} + \Psi_{M/e}, & \textit{otherwise.} \end{cases}$$

Proof. This follows from definition of Ψ_M, Proposition 2.1.28, and the descriptions of $\mathcal{B}(M/e)$ and $\mathcal{B}(M \backslash e)$ in (1.7) and (1.3). □

Theorem 2.3.8. *If M is a regular matroid and $\lambda \in \mathbb{N}^E$, then $\Psi_M(\lambda) = \#\,\mathrm{Jac}(M_\lambda)$.*

Proof. If $\lambda \in \mathbb{N}^E$, then by Proposition 2.2.8, we have

$$\Psi_M(\lambda) = \sum_{B \in \mathcal{B}(M)} \prod_{e \notin B} \lambda(e) = \kappa(M_\lambda).$$

Since M_λ is regular (see Proposition 2.2.10), it follows that $\Psi_M(\lambda) = \#\,\mathrm{Jac}(M_\lambda)$ by Theorem 2.1.29. □

Chapter 3

Configuration Hypersurfaces

The configuration polynomial of a regular matroid is a homogeneous polynomial with the property that is linear in each variable. We will study the hypersurfaces cut out by polynomials that are linear in one of their variables.

3.1 Geometric Aspects

Throughout this section k will denote an arbitrary field . We denote respectively by $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ the affine and projective space of dimension n over k defined by $\mathbb{A}^n_k := \operatorname{Spec} k[T_1, \dots, T_n]$ and $\mathbb{P}^n_k := \operatorname{Proj} k[T_0, \dots, T_n]$. If $F \in k[T_0, \dots, T_n]$ is a homogeneous polynomial, then the hypersurface cut out by F is $X_F := \operatorname{Proj} k[T_0, \dots, T_n]/(F)$. If $\overline{G} \in k[T_0, \dots, T_n]/(F)$, then we endow the closed set $V_+(\overline{G}) := \{\mathfrak{p} \in \operatorname{Proj} k[T_0, \dots, T_n]/(F) \colon \overline{G} \in \mathfrak{p}\} \subseteq X_F$ with the structure of closed subscheme induced by the canonical closed immersion $\operatorname{Proj} k[T_0, \dots, T_n]/(F, G) \hookrightarrow X_F$. The principal open set determined by $\overline{G}$ is $D_+(\overline{G}) := \{\mathfrak{p} \in \operatorname{Proj} k[T_0, \dots, T_n]/(F) \colon \overline{G} \notin \mathfrak{p}\}$. If Z, Y are closed subschemes of $\mathbb{P}^n_k$, then $Y \cap Z$ is the scheme-theoretic intersection.

Let $F \in k[T_0, \dots, T_n]$ be a nonzero homogeneous polynomial that is linear in one of its variables; we can write (up to a permutation of the variables)

$$F = T_0 G_1 + G_0, \tag{3.1}$$

for some some homogeneous polynomials $G_0, G_1 \in k[T_1, \dots, T_n]$ with $G_1 \neq 0$. Note that $\deg G_0 = \deg F$ and $\deg G_1 = \deg F - 1$. Consider the hypersurfaces X_F, X_{G_i}, $i = 0, 1$. These come with canonical closed immersions $X_F \hookrightarrow \mathbb{P}^n_k$, $X_{G_i} \hookrightarrow \mathbb{P}^{n-1}_k$, $i = 0, 1$. Lastly, consider the subschemes $V_+(\overline{G}_1), D_+(\overline{G}_1) \subseteq X_F$.

Proposition 3.1.1. *$\mathbb{P}^{n-1}_k - X_{G_1}$ and $D_+(\overline{G}_1) \subseteq X_F$ are isomorphic as k-schemes.*

Proof. Define the following homomorphism of graded k-algebras:

$$\begin{aligned} \varphi \colon k[T_0, \dots, T_n]_{G_1} &\to k[T_1, \dots, T_n]_{G_1} \\ T_i &\mapsto T_i, \quad i = 1, \dots, n \\ T_0 &\mapsto -\frac{G_0}{G_1}. \end{aligned}$$

It is well-defined by the universal property of localization. We claim that $\ker\varphi = (F)_{G_1}$. The inclusion $(F)_{G_1} \subseteq \ker\varphi$ follows from observing that $\varphi(F) = (-G_0/G_1)G_1 + G_0 = 0$. Now let $P/G_1^m \in \ker\varphi$ and write $P = T_0^l P_1 + P_0$ for some $P_0, P_1 \in k[T_1,\dots,T_n]$ and some $l \in \mathbb{N}_0$. We have

$$0 = \varphi\left(\frac{P}{G_1^m}\right) = \frac{(-1)^l G_0^l P_1 + G_1^l P_0}{G_1^{m+l}} \implies G_1^l P_0 = (-1)^{l+1} G_0^l P_1, \tag{3.2}$$

from which it follows that

$$\frac{P}{G_1^m} = \frac{T_0^l P_1 + P_0}{G_1^m} = \frac{(T_0G_1)^l P_1}{G_1^{m+l}} + \frac{(-1)^{l+1}G_0^l P_1}{G_1^{m+l}} = \left((T_0G_1)^l + (-1)^{l+1}G_0^l\right)\frac{P_1}{G_1^{m+l}}.$$

Note that $(T_0G_1)^l + (-1)^{l+1}G_0^l = (F - G_0)^l + (-1)^{l+1}G_0^l$ and that the latter term is divisible by F (by the Binomial Theorem). Hence $P/G_1^m \in (F)_{G_1}$. This proves our claim.

It is clear that φ is surjective, hence it is an isomorphism and

$$\left(\frac{k[T_0,\dots,T_n]}{(F)}\right)_{\overline{G}_1} \cong \frac{k[T_0,\dots,T_n]_{G_1}}{(F)_{G_1}} \cong k[T_1,\dots,T_n]_{G_1},$$

where the first isomorphism is the canonical one. Since the degree zero parts of these graded algebras are preserved under these isomorphisms, we get

$$\left(\frac{k[T_0,\dots,T_n]}{(F)}\right)_{(\overline{G}_1)} \cong k[T_1,\dots,T_n]_{(G_1)}. \tag{3.3}$$

As these are the coordinate rings of the affine open sets $D_+(\overline{G}_1)$ and $\mathbb{P}_k^{n-1} - X_{G_1}$, respectively, the result is clear. □

Let $p = [1,0,\dots,0] \in \mathbb{P}_k^n$ be the rational point corresponding to the ideal $(T_1,\dots,T_n)$. Let $\pi_p\colon \mathbb{P}_k^n - \{p\} \to \mathbb{P}_k^{n-1}$ denote the projection centered at p induced by the k-algebra homomorphism $\phi\colon k[T_1,\dots,T_n] \to k[T_0,\dots,T_n]$ given by $\phi(T_j) = T_j$ for all $j = 1,\dots,n$. At the level of rational points we have $\pi_p([a_0,\dots,a_n]) = [a_1,\dots,a_n]$.

Corollary 3.1.2. *The restriction of π_p to $D_+(\overline{G}_1) \to \mathbb{P}_k^{n-1} - X_{G_1}$ is an isomorphism of k-schemes.*

Proof. Observe that $p \notin D_+(\overline{G}_1)$ and $\pi_p(D_+(\overline{G}_1)) \subseteq \mathbb{P}_k^{n-1} - X_{G_1}$. Thus π_p restricts to a morphism of k-schemes $D_+(\overline{G}_1) \to \mathbb{P}_k^{n-1} - X_{G_1}$. This morphism corresponds to the homomorphism

$$\begin{aligned} k[T_1,\dots,T_n]_{G_1} &\to \left(\frac{k[T_0,\dots,T_n]}{(F)}\right)_{(\overline{G}_1)} \\ \frac{T_i}{G_1^k} &\mapsto \frac{\overline{T}_i}{\overline{G}_1^k}, \quad i = 1,\dots,n. \end{aligned}$$

It is easy to see that this homomorphism and the one given in (3.3) are inverses of each other. □

From Corollary 3.1.2 we have the following commutative diagram:

$$\begin{tikzcd} D_+(\overline{G}_1) \arrow[r, hook] \arrow[d, "\cong"'] & X_F - \{p\} \arrow[r, hook] & \mathbb{P}^n_k - \{p\} \arrow[d, "\pi_p"] \\ \mathbb{P}^{n-1}_k - X_{G_1} \arrow[rr, hook] & & \mathbb{P}^{n-1}_k \end{tikzcd}$$

When F is irreducible it follows from Proposition 3.1.1 that X_F is birational to $\mathbb{P}^{n-1}_k$. Hence X_F is rational.

If $X = \operatorname{Proj} k[T_1, \ldots, T_n]/I$ is a projective scheme, then the scheme $\operatorname{Proj} k[T_1, \ldots, T_n][T_0]/I$ is called the *projective cone* of X with *vertex* p and it is denoted by $\operatorname{Cone}_p(X)$. It is well-known that π_p induces a surjective morphism $\theta\colon \operatorname{Cone}_p(X) - \{p\} \to X$ where the fibre of θ over $x \in X$ is isomorphic to $\mathbb{A}^1_{\kappa(x)}$ (here $\kappa(x)$ is the residue field of x).

Proposition 3.1.3. *The projection morphism π_p induces a surjective morphism of k-schemes $\theta\colon V_+(\overline{G}_1) - \{p\} \to X_{G_1} \cap X_{G_0}$, where the fibre of θ over $y \in X_{G_1} \cap X_{G_0}$ is isomorphic to $\mathbb{A}^1_{\kappa(x)}$.*

Proof. The closed immersion $X_F \hookrightarrow \mathbb{P}^n_k$ induces an isomorphism $V_+(\overline{G}_1) \cong V_+(F) \cap V_+(G_1)$ (as k-schemes) and $V_+(F) \cap V_+(G_1) = V_+(G_1, G_0) \subseteq \mathbb{P}^n_k$ as schemes. Since G_1, G_0 are independent of T_0, then $V_+(G_1, G_0) = \operatorname{Cone}_p(X_{G_1} \cap X_{G_0})$. Hence the result follows. □

Definition 3.1.4. Let M be a regular matroid and k be an arbitrary field. The scheme

$$X_M := \operatorname{Proj} k[\lambda_e\colon e \in E(M)]/(\Psi_M) \subseteq \mathbb{P}^{\#E(M)-1}_k$$

is called the *configuration hypersurface* of M.

If G is a graph, we define the configuration hypersurface of G to be $X_G := X_{M(G)}$.

3.2 Counting Points over Finite Fields

We keep the notation from the preceding section and we assume k to be a finite field with q elements.

Proposition 3.2.1. *If $F \in k[T_0, \ldots, T_n]$ is a homogeneous polynomial as in* (3.1)*, then there is a natural identification*

$$X_F(k) = \mathit{Cone}_p(X_{G_1} \cap X_{G_0})(k) \sqcup (\mathbb{P}^{n-1}_k(k) - X_{G_1}(k)).$$

Proof. We have $X_F = V_+(\overline{G}_1) \sqcup D_+(\overline{G}_1)$ as topological spaces. Since $V_+(\overline{G}_1) \cong \operatorname{Cone}_p(X_{G_1} \cap X_{G_0})$ as k-schemes, the result is a direct consequence of Propositions 3.1.1 and 3.1.3. □

Corollary 3.2.2. *If $F \in k[T_0, \ldots, T_n]$ is a homogeneous polynomial as in* (3.1)*, then*

$$\#X_F(k) = q\#(X_{G_1}(k) \cap X_{G_0}(k)) + \#\mathbb{P}^{n-1}_k(k) - \#X_{G_1}(k) + 1.$$

Proof. It is a direct consequence of Proposition 3.2.1. □

The equality in Corollary 3.2.2 was first deduced by Stembridge [27]. Nonetheless, the geometry behind this identity is concealed by the probabilistic methods used.

Theorem 3.2.3 ([13, Corollary 3.3])**.** *If $X \subseteq \mathbb{P}^n_k$ is an equidimensional closed subscheme of dimension $d < n$ and degree δ, then*

$$\#X(k) \leq \delta\Big(\#\mathbb{P}^d_k(k) - \#\mathbb{P}^{2d-n}_k(k)\Big) + \#\mathbb{P}^{2d-n}_k(k).$$

When m is a negative integer then the value of $\#\mathbb{P}^m_k(k)$ is zero by convention.

Theorem 3.2.4. *Suppose that $F \in k[T_0, \ldots, T_n]$ is a homogeneous polynomial as in* (3.1)*. If F is irreducible over k, then there are monic polynomials $f(t), g(t) \in \mathbb{Z}[t]$ of degree $n-1$ independent of k such that*

$$g(q) \leq \#X_F(k) \leq f(q).$$

The coefficients of $g(t)$ and $f(t)$ depend only on $\deg F$ *and* $\dim \mathbb{P}^n_k$.

We need the following result to prove Theorem 3.2.4. But before that, let us define what a complete intersection is. We say that $X = \operatorname{Proj} k[T_0, \ldots, T_n]/(f_1, \ldots, f_r)$ is a *complete intersection* if $\operatorname{codim}(X, \mathbb{P}^n_k) = r$ (equivalently, $\dim X = n - r$).

Proposition 3.2.5. *Suppose that $F \in k[T_0, \ldots, T_n]$ is a homogeneous polynomial as in* (3.1)*. If F is irreducible over k of* $\deg F > 1$*, then $X_{G_1} \cap X_{G_0}$ is a complete intersection.*

Proof. We need to prove that $\operatorname{codim}(X_{G_1} \cap X_{G_0}, \mathbb{P}^{n-1}_k) = 2$. It is enough to show that the height of the ideal (G_0, G_1), denoted $\operatorname{ht}(G_0, G_1)$, is 2 in $k[T_1, \ldots, T_n]$. Since this ideal is generated by two polynomials, $\operatorname{ht}(G_0, G_1) \leq 2$. Let $\mathfrak{p}$ be a prime ideal containing (G_0, G_1). Then $(G_0) \subseteq \mathfrak{p}$. We claim that $\mathfrak{p}$ is not a minimal prime ideal of (G_0). Indeed, as F is irreducible, G_1 and G_0 are coprime, therefore $\overline{G}_1$ is not a nonzero divisor of $k[T_0, \ldots, T_n]/(G_0)$, which shows that G_1 cannot be contained in any minimal prime ideal of G_0. Consequently there must be a nonzero prime ideal $\mathfrak{q}$ strictly contained in $\mathfrak{p}$. Then $\operatorname{ht}(\mathfrak{p}) \geq 2$, which implies that $\operatorname{ht}(G_0, G_1) \geq 2$ as $\operatorname{ht}(G_0, G_1) = \inf\{\operatorname{ht}(\mathfrak{p}) \colon (G_0, G_1) \subseteq \mathfrak{p}\}$. This completes the proof. □

Proof of Theorem 3.2.4. If $m := \deg F = 1$, then $X_F \cong \mathbb{P}^{n-1}_k$ so that $\#X_F(k) = (q^n - 1)/(q-1)$. Hence we take $f(t) = g(t) = t^{n-1} + t^{n-2} + \cdots + t + 1$.

Now assume $m > 1$ so that $X_{G_1} \cap X_{G_0}$ is a complete intersection by Proposition 3.2.5. Since complete intersections are equidimensional (see Section 10.135 in [29]), it follows that $X_{G_1} \cap X_{G_0}$ is equidimensional of dimension $n-3$. By Bézout's theorem (see Theorem III-71 in [15]) its degree is $m(m-1)$. We now use Theorem 3.2.3 to get

$$\#X_{G_1}(k) \cap X_{G_0}(k) \leq m(m-1)(\#\mathbb{P}^{n-3}_k(k) - \#\mathbb{P}^{n-6}_k(k)) + \#\mathbb{P}^{n-6}_k(k).$$

The polynomial

$$h(t) = m(m-1)\left(\frac{t^{n-2}-1}{t-1}\right) + (1 + m - m^2)\left(\frac{t^{n-5}-1}{t-1}\right) \in \mathbb{Z}[t]$$

has degree $n-3$ and satisfies $\#X_{G_1}(k) \cap X_{G_0}(k) \leq h(q)$. Moreover, $\#\mathbb{P}^{n-1}_k(k) - \#X_{G_1}(k) + 1$ is bounded above by $(q^n - 1)/(q-1) + 1$. Hence, if we take $f(t) = (t^n - 1)/(t-1) + th(t) \in \mathbb{Z}[t]$, then $f(t)$ is monic of degree $n-1$ and $\#X_F(k) \leq f(q)$ by Corollary 3.2.2.

On the other hand, by Proposition 3.2.1 and by Applying Theorem 3.2.3 to X_{G_1} we have

$$\begin{aligned}\#X_F(k) &\geq \#\mathbb{P}_k^{n-1}(k) - \#X_{G_1}(k) + 1\\ &\geq \#\mathbb{P}_k^{n-1}(k) - \Big((m-1)\Big(\#\mathbb{P}_k^{n-2}(k) - \#\mathbb{P}_k^{n-4}(k)\Big) + \#\mathbb{P}_k^{n-4}(k)\Big) + 1,\end{aligned}$$

so we take

$$g(t) = \frac{t^n - 1}{t-1} - (m-1)\left(\frac{t^{n-1}-1}{t-1}\right) + m\left(\frac{t^{n-3}-1}{t-1}\right) + 1 \in \mathbb{Z}[t]$$

then $g(t)$ is monic of degree $n-1$ and $g(q) \leq \#X_F(k)$. □

Theorem 3.2.6. *Suppose that $F \in k[T_0, \dots, T_n]$ is a homogeneous polynomial satisfying* (3.1). *If F is irreducible over k, then*

$$\#X_F(k) = q^{n-1} + O(q^{n-2}).$$

The implied constant is computable and depends only on $\deg F$ *and* $\dim \mathbb{P}_k^n$.

Lemma 3.2.7. *If $h(t) = a_n t^n + \cdots + a_0 \in \mathbb{R}[t]$ is a polynomial with $a_n \neq 0$ and $C = |a_n| + \cdots + |a_0|$, then $-Ct^n \leq h(t) \leq Ct^n$ for $t > 1$.*

Proof. for $t > 1$ we have that

$$\begin{aligned}|a_n t^n + \cdots + a_0| &\leq |a_n| t^n + \cdots + |a_0|\\ &= t^n\left(|a_n| + \frac{|a_{n-1}|}{t} + \cdots + \frac{|a_0|}{t^n}\right)\\ &\leq t^n(|a_n| + \cdots + |a_0|)\\ &= Ct^n.\end{aligned}$$

□

Proof of Theorem 3.2.6. Consider the polynomials $f(t), g(t)$ from the Theorem 3.2.4. Let C_1 and C_2 be the sum of the absolute values of the coefficients of the polynomials $f(t) - p^{n-1}$ and $g(t) - p^{n-1}$, respectively. If $C = \max\{C_1, C_2\}$, then by Lemma 3.2.7

$$-Cq^{n-2} \leq \#X_F(k) - q^{n-1} \leq Cq^{n-2}.$$

Whence $\#X_F(k) = q^{n-1} + O(q^{n-2})$. □

The next two propositions will be useful for some computations in Example 3.3.7.

Proposition 3.2.8. *Suppose that $F \in k[T_0, \dots, T_n]$ is a homogeneous polynomial as in* (3.1). *Let F_λ be the homogeneous polynomial obtained from F by replacing T_0 with $S_0 + \cdots + S_m$, that is, $F_\lambda = (S_0 + \cdots + S_m)G_1 + G_0 \in k[S_0, \dots, S_m, T_1, \dots, T_n]$. Then*

$$\#X_{F_\lambda}(k) = q^m \#X_F(k) + \#\mathbb{P}_k^{m-1}(k).$$

Proof. By applying the projective change of coordinates induced by

$$\begin{aligned} k[S_0,\dots,S_m,T_1,\dots,T_n] &\to k[S_0,\dots,S_m,T_1,\dots,T_n]\\ S_0 &\mapsto S_0-\sum_{i=1}^m S_i,\\ S_i &\mapsto S_i,\quad i=1,\dots,m,\\ T_i &\mapsto T_i,\quad i=1,\dots,n.\end{aligned}$$

We see that $X_{F_\lambda}\cong X_G$ (as k-schemes) where $G=S_0G_1+G_0$. Consider the linear scheme $Y=V_+(S_0,T_1,\dots,T_n)\subseteq\mathbb{P}_k^{m+n}$, then $Y\cong\mathbb{P}_k^{m-1}$. Let $\pi_Y\colon\mathbb{P}_k^{m+n}-Y\to\mathbb{P}_k^n$ be the projection from Y to $\mathbb{P}_k^n$. This morphism is induced by

$$\begin{aligned} k[T_0,\dots,T_n] &\to k[S_0,\dots,S_m,T_1,\dots,T_n]\\ T_0 &\mapsto S_0,\\ T_i &\mapsto T_i,\quad i=1,\dots,n.\end{aligned}$$

At the level of k-rational points this morphism is given by $[a_0,\dots,a_{m+n}]\mapsto[a_0,a_{m+1}\dots,a_{m+n}]$. The pullback of G is precisely F. Therefore this projection induces a surjective morphism $X_G-Y\to X_F$, where the fiber of each $x\in X_F$ is isomorphic to $\mathbb{A}^m_{k(x)}$. Lastly, note that $Y\subseteq X_G$ whence $X_G(k)=\mathbb{P}_k^{m-1}(k)\sqcup\bigsqcup_{x\in X_F(k)}\mathbb{A}_k^m(k)$. □

The next proposition is the matroid version of Lemma 1.3 and Corollary 1.4 in [1]. The proof given there works for this case too. We let Σ_n denote the union of the coordinate hyperplanes.

Proposition 3.2.9. *Suppose that M is a regular matroid with $\#E(M)=\{e_1,\dots,e_n\}$. Then*

$$\Psi_M(\lambda_{e_1},\dots,\lambda_{e_n})=\left(\prod_{i=1}^n\lambda_{e_i}\right)\Psi_{M^*}(\lambda_{e_1}^{-1},\dots,\lambda_{e_n}^{-1}).$$

Moreover, the Cremona transformation induces an isomorphism $X_M-\Sigma_n\cong X_{M^*}-\Sigma_n$ over k.

3.3 The Family $\{M_\lambda\}_{\lambda\in\mathbb{N}^{E(M)}}$

In this section, we fix a prime number p. Let M be a regular matroid and let $\lambda\in\mathbb{N}^E$, we define $\lambda_p\in\mathbb{N}^E$ to be the map given by the rule $\lambda_p(e)=k$, where $1\le k\le p$ and $\lambda(e)\equiv k \bmod p$.

Definition 3.3.1. The *height* for a map $\lambda\in\mathbb{N}^E$ is $\mathrm{ht}(\lambda):=\max\{\lambda(e)\colon e\in E\}$.

Definition 3.3.2. The *density* of a subset $S\subseteq\mathbb{N}^E$ is

$$\mu(S):=\lim_{m\to\infty}\frac{\#\Big(S\cap\{\lambda\in\mathbb{N}^E\colon\ \mathrm{ht}(\lambda)\le m\}\Big)}{\#\{\lambda\in\mathbb{N}^E\colon\ \mathrm{ht}(\lambda)\le m\}},$$

provided the limit exits.

Definition 3.3.3. We define $\mathcal{J}_p(M) := \{\lambda \in \mathbb{N}^E \colon p \mid \#\operatorname{Jac}(M_\lambda)\}$. If G is a graph, we define $\mathcal{J}_p(G) := \mathcal{J}_p(M(G))$.

Observe that for a graph G, we have that $\mathcal{J}_p(G) = \{\lambda \in \mathbb{N}^E \colon p \mid \#\operatorname{Jac}(G_\lambda)\}$ by virtue of Proposition 2.2.11.

Theorem 3.3.4. *If $\#E(M) = n$ and $\Psi_M \neq 1$, then*

$$\mu(\mathcal{J}_p(M)) = \frac{(p-1)\#X_M(\mathbb{F}_p) + 1}{(p-1)\#\mathbb{P}^{n-1}_{\mathbb{F}_p}(\mathbb{F}_p) + 1}.$$

Proof. For $m \in \mathbb{N}$, we let

$$\begin{aligned} B_m &:= \{\lambda \in \mathbb{N}^{E(M)} \colon \operatorname{ht}(\lambda) \leq m\} \\ A_m &:= \{\lambda \in B_m \colon \Psi_M(\lambda) \equiv 0 \bmod p\}. \end{aligned}$$

By Theorem 2.3.8, $\Psi_M(\lambda) = \#\operatorname{Jac}(M_\lambda)$, therefore

$$\#A_m = \#\{\lambda \in B_m \colon p \mid \#\operatorname{Jac}(M_\lambda)\}.$$

Now suppose $m \geq p$ and write $m = pt_m + l$ for some $t_m \in \mathbb{N}$ and $0 \leq l < p$. Consider the following map:

$$\begin{aligned} \theta_m \colon B_m &\to B_p \\ \lambda &\mapsto \lambda_p. \end{aligned}$$

The map θ_m is surjective as $B_p \subseteq B_m$ and $\lambda_p = \lambda$ for all $\lambda \in B_p$. We will find a lower bound and an upper bound for the size of $\#\theta_m^{-1}(\gamma)$ with $\gamma \in B_p$. A map $\lambda \colon E(M) \to \mathbb{N}$ is a preimage of γ if and only if the following two conditions hold:

(*a*) $1 \leq \lambda(e) \leq pt_m + l$ for all $e \in E(M)$ and

(*b*) for each $e \in E(M)$, there exists a nonnegative integer k_e such that $\lambda(e) = pk_e + \gamma(e)$.

According to whether $\gamma(e) \leq l$ or $\gamma(e) > l$, the possible values for k_e are:

(*i*) $0 \leq k_e \leq t_m$, if $\gamma(e) \leq l$;

(*ii*) $0 \leq k_e \leq t_m - 1$, if $\gamma(e) > l$.

It follows that

$$t_m^n \leq \#\theta_m^{-1}(\gamma) \leq (t_m + 1)^n.$$

Moreover, as $\theta_m^{-1}(A_p) = A_m$ we have

$$\#A_m = \sum_{\gamma \in A_p} \#\theta_m^{-1}(\gamma)$$

then

$$t_m^n \#A_p \leq \#A_m \leq (t_m + 1)^n \#A_p.$$

On the other hand, $\#B_m = m^n = (pt_m + l)^n$ and $\#B_p = p^n$. Using the inequalities

$$(pt_m + l)^n \leq (pt_m + p)^n = p^n(t_m + 1)^n \quad \text{and} \quad p^n t_m^n \leq (pt_m + l)^n,$$

we get

$$\frac{\#A_m}{\#B_m} \leq \frac{(t_m+1)^n \#A_p}{(pt_m + l)^n} \leq \frac{(t_m+1)^n \#A_p}{p^n t_m^n} = \frac{(t_m+1)^n}{t_m^n}\frac{\#A_p}{\#B_p},$$
$$\frac{\#A_m}{\#B_m} \geq \frac{t_m^n \#A_p}{(pt_m + l)^n} \geq \frac{t_m^n \#A_p}{(t_m+1)^n p^n} = \frac{t_m^n}{(t_m+1)^n}\frac{\#A_p}{\#B_p}.$$

So that,

$$\left(\frac{t_m}{t_m+1}\right)^n \frac{\#A_p}{\#B_p} \leq \frac{\#A_m}{\#B_m} \leq \left(\frac{t_m+1}{t_m}\right)^n \frac{\#A_p}{\#B_p}.$$

By letting $m \to \infty$ on both sides and noting that $t_m \to \infty$, we get

$$\mu(\mathcal{J}_p(M)) = \lim_{m\to\infty} \frac{\#A_m}{\#B_m} = \frac{\#A_p}{\#B_p}.$$

Finally, notice that

$$\frac{\#A_p}{\#B_p} = \frac{(p-1)\#X_M(\mathbb{F}_p) + 1}{(p-1)\#\mathbb{P}^{n-1}_{\mathbb{F}_p}(\mathbb{F}_p) + 1}$$

This follows easily by noting that there is a bijection between B_p and $\mathbb{A}^n_{\mathbb{F}_p}(\mathbb{F}_p)$, and that there is a bijection between D_p and $V(\Psi_M)(\mathbb{F}_p) \subseteq \mathbb{A}^n_{\mathbb{F}_p}(\mathbb{F}_p)$, and that each equivalence class of $\mathbb{P}^{n-1}_{\mathbb{F}_p}(\mathbb{F}_p)$ contains $p-1$ points of $\mathbb{A}^n_{\mathbb{F}_p}(\mathbb{F}_p)$. □

Example 3.3.5. Let $G = \text{Cycle}_2$.

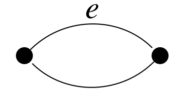


Figure 3.1: Cycle_2.

For $\lambda \in \mathbb{N}^{E(G)}$ we have $G_\lambda = \text{Cycle}_{\lambda(e)+\lambda(f)}$, whence $\text{Jac}(G_\lambda) \cong \mathbb{Z}/(\lambda(e)+\lambda(f))\mathbb{Z}$. Hence $\{\text{Jac}(G_\lambda)\}_{\lambda\in\mathbb{N}^{E(G)}}$ is the family of all finite cyclic groups.

On the other hand, $\Psi_G = \lambda_e + \lambda_f$ so that $X_G \cong \mathbb{P}^0_{\mathbb{F}_p}$. Thus, by Theorem 3.3.4 we obtain

$$\mu(\mathcal{J}_p(G)) = \frac{(p-1)\#X_G(\mathbb{F}_p) + 1}{(p-1)\#\mathbb{P}^1_{\mathbb{F}_p}(\mathbb{F}_p) + 1} = \frac{1}{p},$$

as expected.

Theorem 3.3.6. *If M is irreducible, then*

$$\mu(\mathcal{J}_p(M)) = \frac{1}{p} + O\left(\frac{1}{p^2}\right).$$

The implied constant is computable and depends only on $r(M)$ and $\#E(M)$.

Proof. If M is irreducible, then Ψ_M is irreducible and we can write $\Psi_M = \lambda_e \Psi_{M\backslash e} + \Psi_{M/e}$ for any $e \in E(M)$ (see Proposition 2.3.7) where $\Psi_{M\backslash e}, \Psi_{M/e}$ are independent of λ_e. So all the results from the previous section apply to X_M. In particular,

$$\#X_M(\mathbb{F}_p) = p^{n-1} + O(p^{n-2})$$

where $n = \#E(M) - 1$ (notice that $\#E(M) \geq 3$ as M is irreducible).

On the other hand, by Theorem 3.3.4

$$\mu(\mathcal{J}_p(M)) = \frac{(p-1)\#X_M(\mathbb{F}_p) + 1}{(p-1)\#\mathbb{P}^n_{\mathbb{F}_p}(\mathbb{F}_p) + 1}.$$

Observe that $(p-1)\#\mathbb{P}^n_{\mathbb{F}_p}(\mathbb{F}_p) + 1 = p^{n+1}$. From Theorem 3.2.6 we know that there exists a computable constant C depending only on $\deg \Psi_M (= \#E(M) - r(M))$ and $\dim \mathbb{P}^{\#E(M)-1}_k$, which shows that C depends only on $\#E(M)$ and $r(M)$, such that

$$-Cp^{n-2} \leq \#X_M(\mathbb{F}_p) - p^{n-1} \leq Cp^{n-2},$$

furthermore

$$(p-1)\#X_M(\mathbb{F}_p)+1 = (p-1)\Big((\#X_M(\mathbb{F}_p)-p^{n-1})+p^{n-1}\Big)+1 = (p-1)(\#X_M(\mathbb{F}_p)-p^{n-1})+(p-1)p^{n-1}+1,$$

from which it follows that

$$-C(p-1)p^{n-2} + (p-1)p^{n-1} + 1 \leq (p-1)\#X_M(\mathbb{F}_p) + 1 \leq C(p-1)p^{n-2} + (p-1)p^{n-1} + 1.$$

Thus,

$$\frac{1}{p} - \frac{C+1}{p^2} + \frac{C}{p^3} + \frac{1}{p^{n+1}} \leq \frac{(p-1)\#X_M(\mathbb{F}_p) + 1}{(p-1)\#\mathbb{P}^n_{\mathbb{F}_p}(\mathbb{F}_p) + 1} \leq \frac{1}{p} + \frac{C-1}{p^2} - \frac{C}{p^3} + \frac{1}{p^{n+1}}.$$

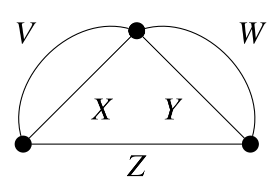
Since $C \geq 1$ (see Proof of Theorem 3.2.6), we get

$$-\frac{C+1}{p^2} \leq \frac{(p-1)\#X_M(\mathbb{F}_p) + 1}{(p-1)\#\mathbb{P}^n_{\mathbb{F}_p}(\mathbb{F}_p) + 1} - \frac{1}{p} \leq \frac{C+1}{p^2}.$$

This concludes the proof. □

To illustrate the preceding results, we propose the following example.

Example 3.3.7. Let k be a finite field with q elements. Let G be the following graph



Its configuration polynomial is given by

$$\Psi_G = VWY + WXY + VWX + VXY + VWZ + XYZ + WXZ + VYZ.$$

The diamond graph is the dual of G. Hence we have $\#(X_G(k) - \Sigma_4(k)) = \#(X_{G^*}(k) - \Sigma_4(k))$ by Proposition 3.2.9. Let us compute $\#(X_{G^*}(k) - \Sigma_4(k))$, which is equal to

$$\#X_{G^*}(k) - \#\left(\bigcup_{T\in\{V,W,X,Y,Z\}} \Big(X_{G^*}(k) \cap V(T)(k)\Big)\right).$$

Firstly, we compute $\#X_{G^*}(k)$. Observe that G^* can be obtained from the banana graph on 3 edges by subdividing two of its edges as shown below.

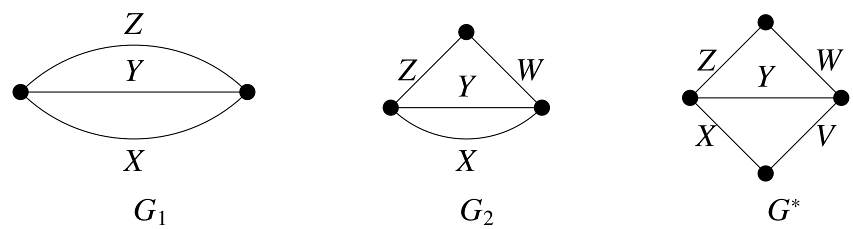


Figure 3.2: Diamond graph obtained from Banana graph by subdividing edges.

Observe that $\Psi_{G^*} = (X + V)\Psi_{G_2\setminus X} + \Psi_{G_2/X}$ and $\Psi_{G_2} = (Z + W)\Psi_{G_1\setminus Z} + \Psi_{G_1/Z}$. Then by Proposition 3.2.8, we can compute $\#X_{G^*}(k)$ as follows. It is easy to see that $X_{G_1} \cong \mathbb{P}^1_k$ so that $\#X_{G_1}(k) = q + 1$. Therefore

$$\#X_{G_2}(k) = q\#X_{G_1}(k) + \#\mathbb{P}^0_k(k),$$
$$\#X_{G^*}(k) = q\#X_{G_2}(k) + \#\mathbb{P}^0_k(k).$$

Thus we get $\#X_{G^*}(k) = q^3 + q^2 + q + 1$.

Next we calculate $\#\Big(X_{G^*}(k) \cap V(T)(k)\Big)$ for $T \in \{V, W, X, Y, Z\}$ and then we use the exclusion-inclusion principle to compute

$$\#\left(\bigcup_{T\in\{V,W,X,Y,Z\}} \Big(X_{G^*}(k) \cap V(T)(k)\Big)\right).$$

Observe that for any $T \in \{V, W, X, Z\}$ we have an isomorphism of k-schemes $X_{G^*} \cap V(T) \cong X_{G_2}$. Thus then $\#\Big(X_{G^*}(k) \cap V(T)(k)\Big) = q^2 + q + 1$ for $T \in \{V, W, X, Z\}$. In addition, the scheme $X_{G^*} \cap V(Y)$ is isomorphic to the configuration hypersurface of the graph $\mathrm{Cycle}_2 \vee \mathrm{Cycle}_2$, then $\#\Big(X_{G^*}(k) \cap V(Y)(k)\Big) = 2q^2 + q + 1$. Hence we obtain

$$\#(X_{G^*}(k) - \Sigma_4(k)) = q^3 - 5q^2 + 10q - 7. \tag{3.4}$$

A similar argument shows that

$$\#(X_G(k) \cap \Sigma_4(k)) = 7q^2 - 9q + 8. \tag{3.5}$$

Adding the expressions (3.4) and (3.5) we obtain

$$\#X_G(k) = q^3 + 2q^2 + q + 1.$$

The proof of Theorem 3.2.4 shows us how to compute $f(t)$ and $g(t)$ explicitly. These are $f(t) = t^3 + 7t^2 + 7t + 1$ and $g(t) = t^3 - t^2 - t + 3$. Also, the constants C_1 and C_2 from Theorem 3.2.6 are $C_1 = 15$ and $C_2 = 5$. Finally,

$$\mu(\mathcal{J}_p(G)) = \frac{(p-1)(p^3 + 2p^2 + p + 1) + 1}{p^5} = \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^3}.$$

Remark 3.3.8. If M is a regular matroid, then we have the family of regular matroids $\{M_\lambda\}_{\lambda\in\mathbb{N}^{E(M)}}$. By taking duals, we obtain a new family of regular matroids $\{(M_\lambda)^*\}_{\lambda\in\mathbb{N}^{E(M)}}$. In particular, the matroid $(M_{1+\chi_e})^*$ is a parallel extension of M^* when e is not a coloop of M; this follows from Proposition 1.3.19 and Proposition 2.2.5.

In addition, by Corollary 2.1.25, we know that $\operatorname{Jac}(M_\lambda) \cong \operatorname{Jac}((M_\lambda)^*)$. So that, Theorem 3.3.4 and Theorem 3.3.6 also predict the distribution of the p-torsion of the Jacobian groups for this new family. More concretely, if we let $\mathcal{J}_p^*(M) := \{\lambda \in \mathbb{N}^{E(M)} \colon\ p \mid \#\operatorname{Jac}((M_\lambda)^*)\}$, then

$$\mu(\mathcal{J}_p^*(M)) = \frac{(p-1)\#X_M(\mathbb{F}_p) + 1}{(p-1)\#\mathbb{P}^{n-1}_{\mathbb{F}_p}(\mathbb{F}_p) + 1}.$$

If M is irreducible, then

$$\mu(\mathcal{J}_p^*(M)) = \frac{1}{p} + O\left(\frac{1}{p^2}\right).$$

In general, given a regular matroid M and a map $\lambda : E(M) \to \mathbb{N}$, the Jacobian groups of M_λ and M_λ^* are not isomorphic; in fact, their orders might not have any common prime factors. Also, it is not true that $\mu(\mathcal{J}_p(M)) = \mu(\mathcal{J}_p(M^*))$. Nonetheless, Proposition 3.2.9 allows us to relate the distribution of the p-torsion of the Jacobian groups in the families $\{M_\lambda\}_{\lambda\in\mathbb{N}^{E(M)}}$ and $\{M_\lambda^*\}_{\lambda\in\mathbb{N}^{E(M^*)}}$ as follows.

Let us define $\mathcal{S}_p(M) := \{\lambda \in \mathbb{N}^{E(M)} \colon\ p \mid \#\operatorname{Jac}(M_\lambda) \text{ and } p \nmid \lambda(e) \text{ for all } e \in E(M)\}$.

Proposition 3.3.9. *If M is a regular matroid with $\#E(M) = n$, then*

$$\mu(\mathcal{S}_p(M)) = \frac{(p-1)\#\Big(X_M(\mathbb{F}_p) \cap (\mathbb{P}^{n-1}_{\mathbb{F}_p}(\mathbb{F}_p) - \Sigma_n(\mathbb{F}_p))\Big)}{(p-1)\#\mathbb{P}^{n-1}_{\mathbb{F}_p}(\mathbb{F}_p) + 1}$$

Proof. For $m \in \mathbb{N}$, we let

$$\begin{aligned} B_m &:= \{\lambda \in \mathbb{N}^{E(M)} \colon\ \operatorname{ht}(\lambda) \le m\} \\ D_m &:= \{\lambda \in B_m \colon \Psi_M(\lambda) \equiv 0 \bmod p \text{ and } p \nmid \lambda(e) \text{ for all } e \in E(M)\}. \end{aligned}$$

The argument of the proof of Theorem 3.3.4 applies to B_m and D_m in place of A_m. Hence, one obtains

$$\mu(\mathcal{S}_p(M)) = \lim_{m\to\infty} \frac{\#D_m}{\#B_m} = \frac{\#D_p}{\#B_p},$$

and

$$\frac{\#D_p}{\#B_p} = \frac{(p-1)\#\Big(X_M(\mathbb{F}_p)\cap\big(\mathbb{P}^{n-1}_{\mathbb{F}_p}(\mathbb{F}_p)-\Sigma_n(\mathbb{F}_p)\big)\Big)}{(p-1)\#\mathbb{P}^{n-1}_{\mathbb{F}_p}(\mathbb{F}_p)+1}.$$

□

Corollary 3.3.10. *If M is a regular matroid with $\#E(M) = n$, then*

$$\mu(\mathcal{S}_p(M)) = \mu(\mathcal{S}_p(M^*)).$$

Proof. By Proposition 3.2.9, $X_M - \Sigma_n \cong X_{M^*} - \Sigma_n$ over $\mathbb{F}_p$. Hence, the result follows from Proposition 3.3.9. □

Chapter 4

Conclusion

Given a regular matroid M and a map $\lambda\colon E(M) \to \mathbb{N}$, we constructed a regular matroid M_λ. To each regular matroid, we associated a finite abelian group, called Jacobian group. We studied the variation of the p-torsion of the Jacobian groups of the family $\{M_\lambda\}_{\lambda \in \mathbb{N}^{E(M)}}$.

We established a correspondence between the $\mathbb{F}_p$-rational points of the configuration hypersurface X_M of M and the maps λ for which $\#\operatorname{Jac}(M_\lambda)$ is divisible by p. Hence, we reduced the problem to counting points over finite fields. As a consequence, we obtained a closed formula for the proportion of these groups with non-trivial p-torsion as well as some estimates. In addition, we proved that the Jacobian groups in this family with non-trivial p-torsion appear with frequency close to $1/p$, provided M is irreducible.

Two open questions stem from this work. The first one is concerned with the natural generalization of the problem addressed in this thesis, that is, how often is $\#\operatorname{Jac}(M_\lambda)$ divisible by p^n? One could use a similar approach by reducing the problem to determining $\#X_M(\mathbb{Z}/p^n\mathbb{Z})$. The second question is regarding Sylow p-subgroups. If $\operatorname{Jac}(M_\lambda)_p$ denotes the unique Sylow p-subgroup of $\operatorname{Jac}(M_\lambda)$, then one would like to know the following: given a finite abelian p-group Γ, how often is $\operatorname{Jac}(M_\lambda)_p \cong \Gamma$?

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