7-30-2021 3:15 PM

Cubical Models of Higher Categories

Brandon Doherty, *The University of Western Ontario*

Supervisor: Kapulkin, Krzysztof R., *The University of Western Ontario*

A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Mathematics

Abstract

This thesis concerns model structures on presheaf categories, modeling the theory of ∞-categories. We introduce the categories of simplicial and cubical sets, and review established examples of model structures on these categories for ∞-groupoids and $(\infty, 1)$-categories, including the Quillen and Joyal model structures on simplicial sets, and the Grothendieck model structure on cubical sets. We also review the complicial model structure on marked simplicial sets, which presents the theory of (∞, n)-categories. We then construct a model structure on the category of cubical sets whose cofibrations are the monomorphisms and whose fibrant objects are defined by the right lifting property with respect to inner open boxes, the cubical analogue of inner horns. We show that this model structure is Quillen equivalent to the Joyal model structure on simplicial sets via the triangulation functor. To do this, we develop a theory of cones in cubical sets. As an application, we show that cubical quasicategories admit a convenient notion of a mapping space, which we use to characterize the weak equivalences between fibrant objects in our model structure. We also develop model structures for (∞, n)-categories on marked cubical sets, and show that these are equivalent to the complicial model structures on marked simplicial sets.

Keywords: higher categories, homotopy theory, model categories, simplicial sets, cubical sets

Summary for lay audience

The field of higher category theory, which studies abstract mathematical objects known as higher categories or ∞-categories, has applications to a wide range of mathematical disciplines. While many of its key motivating examples come from the study of topological spaces, higher category theory has also found applications to areas of pure mathematics such as formal logic, algebra, and geometry, as well as to theoretical physics and computer science.

Many of the most successful frameworks for the study of higher category theory make use of simplices, higher-dimensional shapes analogous to the triangle and the tetrahedron. In recent years, there has been significant interest in developing a *cubical* framework for higher category theory – one which would make use of higher-dimensional analogues of the square and the familiar three-dimensional cube. It is expected that such a framework will have many useful applications to the above-mentioned scientific areas.

In this thesis, we begin by reviewing some of the basic theory of higher categories, and the established simplicial models for two specific types of higher categories, known as ∞-groupoids and $(\infty, 1)$-categories, as well as an established cubical model for the theory of ∞-groupoids. Building on this previous work, we then construct and study a cubical model for the theory of $(\infty, 1)$-categories. We show that this model is equivalent, in a suitable sense, to the previously-established simplicial model of $(\infty, 1)$-categories, thereby showing that they do indeed model the same kinds of higher categories. To prove this equivalence, we develop a theory of *cubical cones*, shapes which are intermediate, in a suitable sense, between simplices and cubes. As an application of our work, we use our cubical framework to construct certain ∞-groupoids known as *mapping spaces* from a given $(\infty, 1)$-category, and show how this construction is simplified compared to its traditional simplicial analogue. We also adapt the theory of cubical cones to more general types of objects, called (∞, n)-categories.

Co-Authorship statement

Chapters 1 through 4 are primarily concerned with exposition of background material, as opposed to original work, though parts of chapters 2, 3, and 4 are adapted from co-authored material appearing in [DKLS20] and [DKM21]. In particular, Theorem 2.2.18 and its proof are joint work with Kapulkin and Maehara, and first appeared in [DKM21].

Chapters 5 through 8 contain joint work with Kapulkin, Lindsey and Sattler, which also appears in [DKLS20].

Chapter 9 contains joint work with Kapulkin and Maehara, which also appears in [DKM21].

Acknowledgements

I would like to thank my supervisor, Chris Kapulkin, for going above and beyond to provide advice, insight, guidance and mentorship throughout the past four years. I would also like to thank Zach Lindsey, Yuki Maehara, and Christian Sattler for their valuable contributions to our joint projects and our many illuminating mathematical discussions.

Contents

Chapter 1

Introduction

1.1 Higher categories

In recent years, the field of higher category theory has attracted substantial mathematical interest. In contrast to traditional category theory, this discipline is concerned with the study of *n-categories*, having not only objects and morphisms, but also *higher morphisms* between morphisms. To be precise, if we refer to the morphisms between objects as 1-morphisms, then an n-category, for $n \geq 2$ also contains 2-morphisms between its 1-morphisms, and more generally, $(m+1)$-morphisms between its m-morphisms for every $1 \leq m \leq n-1$. An *∞-category* contains morphisms of arbitrarily high degree; for brevity, we will sometimes simply refer to n-categories with the understanding that we may have $n = \infty$.

The composition operation in a higher category may be strictly associative and unital, as in traditional category theory, or these properties may hold only up to an invertible higher morphism – for instance, given a composable triple of n-morphisms f, g, h, the composites $h \circ (g \circ f)$ and $(h \circ g) \circ f$ may not be equal, but may be related by an invertible $(n+1)$-morphism $H\colon h\circ(g\circ f)\to(h\circ g)\circ f$. In fact, composition in a higher category may not even be uniquely defined; rather, given a composable pair of n-morphisms there may be a family of n-morphisms which can be regarded as their composite, with any two such composites related by an invertible $(n+1)$-morphism.

In particular, we are often interested in n-categories in which all morphisms above degree k, for some $0 \le k \le n$, are invertible. In general these are called (n,k)-*categories*; $(n,0)$-categories are more commonly called n-groupoids. When $n = \infty$ we allow the variable k to take the value ∞ as well; an (∞,∞)-*category* is an ∞-category for which we make no assumptions about invertibility of morphisms of any degree.

Given an n-category $\mathcal{C}$ and a pair of objects $X, Y \in \mathcal{C}$, we have an $(n-1)$-category $\mathcal{C}(X,Y)$, with objects given by 1-morphisms $X \to Y$ in $\mathcal{C}$, and m-morphisms for $m \ge 1$ given by $(m+1)$-morphisms of $\mathcal{C}$. If $\mathcal{C}$ is an n-groupoid, then $\mathcal{C}(X,Y)$ is an $(n-1)$-groupoid; if $\mathcal{C}$ is an (n,k)-category for $n \ge 1$ then $\mathcal{C}(X,Y)$ is an $(n-1,k-1)$-category.

Let us consider some standard examples to better understand the concept of a higher category. The usual category Top of topological spaces and continuous functions can be extended to an ∞-category as follows:

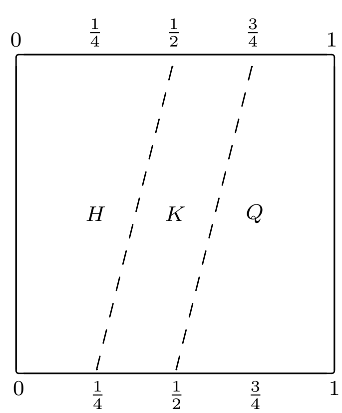
- objects are topological spaces;
- 1-morphisms from X to Y are continuous functions $X \to Y$, with composition and identities given by the usual composition and identity functions;
- for $n \ge 2$, given a pair of parallel $(n-1)$-morphisms f, g between X and Y, an n-morphism from f to g is a homotopy $H\colon X \times [0,1]^{n-1} \to Y$ such

that $H|_{X\times[0,1]^{n-2}\times\{0\}} = f$ and $H|_{X\times[0,1]^{n-2}\times\{1\}} = g$. Composition is then given by the usual composition of homotopies, and identities are given by constant homotopies.

For concreteness, suppose that by "the usual composition of homotopies" in the definition above, we mean the definition given in [Hat02], i.e. that for homotopies $H\colon f \sim g$ and $K\colon g \sim h$ given by functions $X \times [0,1] \to Y$, the composite homotopy $KH\colon f \sim h$ is given by:

$$KH(x,\varepsilon) = \begin{cases} H(x,2\varepsilon) & 0 \le \varepsilon \le \frac{1}{2} \\ K(x,2\varepsilon-1) & \frac{1}{2} \le \varepsilon \le 1 \end{cases}$$

This composition operation is easily seen not to be strictly associative or unital; for instance, given a composable triple of homotopies H, K, Q, writing out the definitions of $Q(KH)$ and $(QK)H$ shows that they are not equal. It is, however, associative and unital *up to homotopy*; for instance, in the situation described above we have a homotopy $Q(KH) \sim (QK)H$ given by reparametrization, as illustrated below. (The top edge of the square represents $(QK)H$, the bottom edge represents $Q(KH)$, and the interior represents a homotopy between them.)



Moreover, there is no reason to favour the definition above as the unique composite of H with K; for any $\alpha \in (0,1)$ we can define a composite homotopy $(KH)_\alpha$ as follows:

$$(KH)_\alpha(x,\varepsilon) = \begin{cases} H(x, \frac{1}{\alpha}\varepsilon) & 0 \le \varepsilon \le \alpha \\ K(x, \frac{1}{1-\alpha}\varepsilon - \frac{\alpha}{1-\alpha}) & \alpha \le \varepsilon \le 1 \end{cases}$$

For any α, α' we have a homotopy $(KH)_\alpha \sim (KH)_{\alpha'}$; thus composition is defined *up to homotopy*.

We may note that Top is, in fact, an $(\infty,1)$-category, as every homotopy $H\colon f \sim g$ has an inverse homotopy $H^{-1}\colon g \sim f$, such that the composites HH^{-1} and $H^{-1}H$ are homotopic to the constant homotopies on g and f, respectively. Once again, we see that these inverses are unique only up to homotopy.

For another example, let X be any topological space; the *fundamental ∞-groupoid* of X is the ∞-groupoid $\Pi X = \mathsf{Top}(*, X)$, where $*$ denotes the one-point space. Unwinding the definitions above, we can characterize ΠX as follows

- Objects are points of X;
- For $n \ge 1$, n-morphisms are continuous functions $I^n \to X$, where I denotes the interval $[0,1]$, with the domain and codomain operations given by restriction to $[0,1]^{n-1} \times \{0\}$ and $[0,1]^{n-1} \times \{1\}$, respectively.

Thus 1-morphisms in ΠX are paths in X, 2-morphisms are homotopies of paths, 3-morphisms are homotopies of homotopies, and so on. Furthermore, if we identify homotopic paths and forget about the higher morphisms of ΠX, the resulting ordinary category is the familiar fundamental groupoid of X.

1.2 Applications of higher categories

Our interest in higher category theory is not purely for its own sake; it has many applications in other areas of mathematics. We will briefly discuss some of these applications in the interest of placing our work in a broader context. Though the details of these applications are beyond the scope of this thesis, we will provide brief overviews, together with references for further reading.

One application is to the algebraic discipline of rewriting theory, which concerns the word problem in monoids presented by generators and relations. In higher-dimensional rewriting theory, a monoid M presented by a set of generators S and relations R (each consisting of a pair of words in S) may be regarded as a $(2,1)$-category with a single object $*$. The 1-morphisms of this 2-category are all words in S, with composition given by concatenation; note that the empty word, the identity on $*$, is thus the only invertible 1-morphism. The 2-morphisms are then freely generated by the elements of R, in the following sense: for any pair $(w,w') \in R$ we have an invertible 2-morphism $w \to w'$. This construction can be extended yet further, identifying presentations of monoids with $(\infty,1)$-categories in which higher morphisms encode more subtle combinatorial data. See [Luc18] for an introduction to this topic.

Another application of higher categories is to topological quantum field theory, which uses (∞,n)-categories to study the structure of n-dimensional manifolds. For an introduction to this topic, see [Lur09b]. Higher categories also feature prominently in the work of Gaitsgory and Rozenblyum on derived algebraic geometry [GR17a], [GR17b].

1.3 Models of higher categories

Up to this point, we have discussed higher categories only informally, without stating a concrete definition. In fact, there are many different models of higher category theory – many ways of realizing the intuitive notion of a higher category described above. Some of the most prominent and well-established models, which will play a key role in this thesis, involve combinatorial objects called *simplicial sets*. Formally, simplicial sets are contravariant functors from the category Δ of non-empty finite posets to the category of sets; intuitively, they may be thought of as spaces pieced together from (oriented) simplices in arbitrary dimensions, joined along common faces. Though this description may call to mind the familiar simplicial complexes of algebraic topology, simplicial sets have a great deal more flexibility; for instance, a simplex need not be uniquely determined by its faces, and an n-simplex may be regarded as a *degenerate* $(n+1)$-simplex, i.e. one which has collapsed down to a lower dimension.

When viewing a simplicial set as a higher category, its vertices correspond to objects, and its n-simplices for $n \geq 1$ correspond to n-morphisms. Identities are given by degenerate simplices; degenerate edges, in particular, will be represented in diagrams with the symbol $=$. A composite for a composable pair of edges $f\colon x \to y, g\colon y \to z$ consists of an edge $h\colon x \to z$, together with a 2-simplex α as depicted below:

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\alpha"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z \end{tikzcd}$$

Note that there may, in general, be many such composites for any given composable pair, and any given composite 1-simplex h may be witnessed by many different 2-simplices α. Furthermore, in an arbitrary simplicial set, these composites may not exist at all. Thus we may say more precisely that higher categories are modeled by simplicial sets having certain *horn-filling* properties.

For $n \geq 0$, the *standard n-simplex*, denoted Δ^n, is the simplicial set consisting of a single n-simplex and all of its faces; n-simplices in a simplicial set X may be identified with maps $\Delta^n \to X$. (When these objects are defined rigourously as contravariant functors, this follows from the Yoneda lemma.) For $n \geq 1$ and $0 \leq i \leq n$, the *n-dimensional i-horn*, denoted Λ^n_i, is the simplicial set consisting of all faces of the n-simplex, except for the face opposite its i^{th} vertex; a horn in a simplicial set X is a map $\Lambda^n_i \to X$. A *filler* for such a horn is an extension of the corresponding map to Δ^n.

From this description, we can see that composable pairs of edges in X correspond to horns $\Lambda^1_1 \to X$, and a 2-simplex α witnessing a composition of f and g as shown above is precisely a filler for the corresponding horn. Taking this concept further, we may define composition of higher morphisms via filling of higher-dimensional horns. Thus we may model ∞-groupoids as *Kan complexes*, simplicial sets having fillers for all horns, defined by Kan [Kan57]. Likewise, $(\infty, 1)$-categories are modeled by *quasicategories*, simplicial sets having fillers for all horns Λ^n_i with $0 < i < n$, defined by Boardman and Vogt [BV73]. To model (∞, n)-categories for $n \geq 2$, we make use of *marked simplicial sets*, simplicial sets in which certain simplices are designated as "marked", and thought of as equivalences. These higher categories are then identified with *complicial sets*, marked simplicial sets having fillers for horns with certain specified faces marked, defined by Verity [Ver08b].

These horn-filling conditions not only allow us to define all necessary composites, but also ensure that composition is well-defined, associative and unital, up to homotopy. They also ensure that all morphisms of degree at least 1 (in the case of Kan complexes) or 2 (in the case of quasicategories) are invertible in a suitable sense.

There exist many other models for higher category theory; we briefly describe a few below, though this list is not exhaustive.

- Topological spaces (or more precisely, retracts of CW complexes) can be viewed as a model of ∞-groupoids, as shown by Quillen [Qui67].

- Rezk's framework of *complete Segal spaces* [Rez01], models $(\infty, 1)$-categories using *bisimplicial sets*, i.e. contravariant functors from $\Delta \times \Delta$ to Set.

- Simplicial categories, i.e. categories enriched over simplicial sets, model the theory of $(\infty, 1)$-categories, as shown by Bergner [Ber07]. Specifically, $(\infty, 1)$-categories are represented by simplicial categories $\mathcal{C}$ which are *locally Kan*, i.e. those in which, given any pair of objects X, Y, the simplicial set $\mathcal{C}(X, Y)$ is a Kan complex.

- The framework of ∞-cosmoi, developed by Riehl and Verity [RV21], allows for "model-independent" study of $(\infty, 1)$-categories. Essentially, an ∞-cosmos is a category with additional structure whose objects represent $(\infty, 1)$-categories; constructions and results established in one ∞-cosmos can be transferred to others along suitably defined equivalences.

For more on models of $(\infty, 1)$-categories specifically, see [Ber10].

1.4 Comparing models of higher categories

The wide variety of models of higher category theory raises a natural question: in what sense are these models equivalent? To put it another way, what does it mean to say that, for instance, quasicategories and complete Segal spaces both model the theory of $(\infty, 1)$-categories? A precise answer is given by the formalism of *model categories*. Originally described by Quillen [Qui67], a model category is a category $\mathcal{C}$ equipped with additional structure which allows for the development of a *homotopy category* $\mathsf{Ho}\mathcal{C}$. The objects of $\mathsf{Ho}\mathcal{C}$ are certain nicely-behaved objects of $\mathcal{C}$ (the *fibrant and cofibrant* objects), and the morphisms are equivalence classes of maps under a suitably-defined homotopy relation. Equivalences between model categories are *Quillen equivalences*, adjunctions which are compatible with the model structure in a suitable way and which induce adjoint equivalences of categories on the homotopy categories; thus two Quillen-equivalent model categories can be said to model "the same homotopy theory".

All of the models of higher category theory described above, with the exception of those coming from ∞-cosmoi, arise as the fibrant and cofibrant objects of some model category. To establish that two models arising in this way define the homotopy theory of the same kind of higher category, we can establish a Quillen equivalence between the relevant model categories. For instance, recall from the discussion above that Quillen established retracts of CW complexes as a model for the homotopy theory of ∞-groupoids. To be more precise, this involves establishing model structures on the categories of topological spaces and simplicial sets, having Kan complexes and retracts of CW complexes, respectively, as their fibrant and cofibrant objects, and exhibiting a Quillen equivalence between them. (For a more modern presentation of

Quillen's model structure on simplicial sets and its equivalence with spaces, see [GJ99].)

Likewise, quasicategories are the fibrant and cofibrant objects of the *Joyal model structure* on simplicial sets, constructed by Joyal [Joy09], while complete Segal spaces are the fibrant and cofibrant objects of the *Rezk model structure* on bisimplicial sets, constructed by Rezk [Rez01]. These model structures were shown to be Quillen equivalent by Joyal and Tierney [JT07]. Similarly, the Bergner model structure on the category of simplicial categories, constructed by Bergner [Ber07], has as its cofibrant and fibrant objects locally Kan simplicial categories (satisfying certain additional conditions); this model structure was shown to be Quillen-equivalent to the Joyal model structure by Joyal [Joy07] and Lurie [Lur09a].

1.5 Cubical models of higher categories

Each model of higher category theory has its own advantages and disadvantages, and different models may be more or less suitable for different purposes. For instance, the join construction is easy and convenient to formulate in the setting of simplicial sets, as for any $m, n \geq 0$ the join $\Delta^m \star \Delta^n$ is equal to Δ^{m+n+1}. In particular, for each $n \geq 1$ we have $\Delta^n = \Delta^0 \star \Delta^{n-1} = \Delta^{n-1} \star \Delta^0$ – in other words, simplices can be constructed inductively by taking cones on simplices of lower dimension. Thus quasicategories are a convenient model to use in formulating the theory of limits and colimts in $(\infty, 1)$-categories, for example, as cones play a fundamental role in their definition.

Products of simplices, on the other hand, are much more difficult to work with. Even in the simplest non-trivial case, we see that the product $\Delta^1 \times \Delta^1$

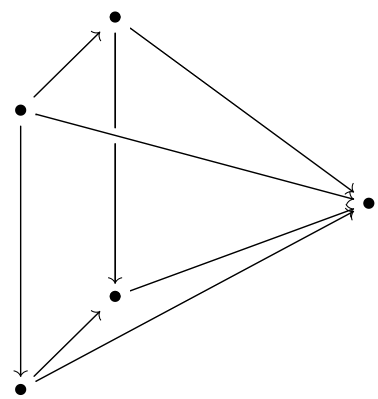
consists of a pair of non-degenerate 2-simplices, as illustrated below.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Products of higher-dimensional simplices quickly increase in complexity. In contrast, products of hypercubes (viewed as posets) are much more nicely behaved. Much like how the join of simplices is again a simplex, the product of hypercubes is again a hypercube; much like how simplices can be inductively constructed by taking joins with the point, hypercubes can be inductively constructed by taking products with the interval. Thus a model for higher category theory based on cubical rather than simplicial shapes may be more suitable for applications involving products. For instance, homotopies are typically defined by taking products (or more generally, some form of monoidal product) with a suitable interval object, and so cubical models often make it much easier to define homotopies explicitly. As another example, the Gray tensor product, a natural monoidal product of higher categories which is in many ways better-behaved than the cartesian product, is also easier to formulate using cubes; see [CKM20] for more on this.

Cisinski [Cis06], [Cis14] established a model for ∞-groupoids based on *cubical sets*. Similar in concept to simplicial sets, these can be thought of as complexes pieced together from cubes in different dimensions, joined along common faces. Analogously to the horns which feature prominently in simplicial higher category theory, we can define an *open box* to be a subcomplex of a cube consisting of all but one of its faces. This allows us to model ∞-groupoids as *cubical Kan complexes*, cubical sets having fillers for all open boxes, first described by Kan [Kan55]. This is done by constructing a model structure on the category of cubical sets, the *Grothendieck model structure*, and showing it is Quillen-equivalent to the model structure for ∞-groupoids on the category of simplicial sets. The left adjoint in this Quillen equivalence, the *triangulation functor*, constructs a simplicial set from a cubical set by breaking up its cubes into simplices. (For an alternate approach to the construction of this model structure and its Quillen equivalence with Quillen's model structure on simplicial sets, see [Jar06].)

The primary goal of the new research described in this thesis is to establish cubical sets as a model for the theory of $(\infty, 1)$-categories. Specifically, following [DKLS20], we construct another model structure on cubical sets, the *cubical Joyal model structure*, and show that it is Quillen equivalent to the Joyal model structure on simplicial sets, once again via the triangulation functor. As a tool to help establish this Quillen equivalence, we will first develop a theory of *cubical cones*, encompassing both the construction of a cone on a cubical set, and the identification of cones on cubes within a cubical set. This theory allows us to use cones on cubes to mediate between simplicial and cubical shapes, allowing for a comparison of the simplicial and cubical Joyal model structures. For example, the diagram below depicts a cone on a 2-cube; a key intuition behind the proof of the Quillen equivalence involves recognizing this shape as being "more cubical" than a 3-simplex, yet also "more simplicial" than a 3-cube.



The fibrant and cofibrant objects in this model structure are *cubical quasi-categories*, cubical sets having fillers for all *inner open boxes* (open boxes with a specified edge degenerate). The proof of the Quillen equivalence involves showing that any cubical quasicategory can be built up from its "maximal simplicial subcomplex" (the maximal subcomplex consisting of iterated cones on vertices) by a series of inner open box fillings, which do not change the object's homotopy type.

This theory of cubical cones is expected to have wide applications beyond the present work, allowing for the development of cubical analogues of many different model structures. For instance, in Chapter 9, following [DKM21], model structures for (∞, n)-categories are developed on the category of marked cubical sets. Cubical cones are used to show that these are Quillen equivalent to analogous model structures on marked simplicial sets, developed by Ozornova and Rovelli [OR20].

The following table summarizes the main model structures of interest which use simplicial and cubical sets to model higher categories.

Model ∖ Theory	∞-groupoids	$(\infty,1)$-categories	(∞,n)-categories
Simplicial sets	[Qui67]	[Joy09]	[OR20]
Cubical sets	[Cis06], [Cis14]	[DKLS20]	[DKM21]

The cubical approach to higher category theory has a great many advantages. As previously mentioned, cubical sets have the potential to simplify computations involving products of higher categories and explicit construction of homotopies. The potential for simplification of proofs extends further; for instance, Kapulkin and Voevodsky [KV20] developed a simpler and more explicit approach to *straightening*, a widely-used construction involving simplicial categories, which instead makes use of categories enriched over cubical sets.

Cubical sets and their homotopy theory are of interest in many other areas of mathematics as well. For instance, Krishnan [Kri15] applied cubical methods to the study of directed topological spaces, which have applications to the modeling of space-time in theoretical physics. Cubical sets also play a key role in the logical system of cubical type theory, introduced by Cohen, Coquand, Huber and Mörtberg [CCHM18]. The cubical Joyal model structure thus has a key role to play in understanding how $(\infty,1)$-categories relate to these areas. In particular, it may have applications to establishing the foundations of the theory of $(\infty,1)$-categories in cubical type theory, building on the work of Riehl and Shulman [RS17].

1.6 Outline of the thesis

The structure of this thesis is as follows. Chapter 2 covers essential background information about model categories, including their basic theory, as well as techniques for constructing them. Chapter 3 introduces the category of simplicial sets, as well as the category of marked simplicial sets, and discusses their homotopy theory. In particular, this includes the construction of the Quillen and Joyal model structures on simplicial sets, as well as the complicial model structure on marked simplicial sets. Along the same lines, Chapter 4 introduces the categories of cubical sets and marked cubical sets, defines essential constructions such as the triangulation functor, and describes the Grothendieck model structure on the category of cubical sets.

With this background material established, Chapters 5 through 8 focus on the cubical Joyal model structure, largely following [DKLS20]. In chapter 5, we construct a model structure for $(\infty, 1)$-categories on *cubical sets with weak equivalences* (cubical sets having markings on their edges, but not on cubes of higher dimensions). In Chapter 6, we construct the cubical Joyal model structure on the category of cubical sets, using the model structure on cubical sets with weak equivalences as a tool. In Chapter 7, we develop the theory of cubical cones and use it to prove that the triangulation functor and its right adjoint form a Quillen equivalence between the Joyal and cubical Joyal model structures. As this proof involves many routine calculations, some of these are relegated to Appendix A for the sake of readability. Chapter 8 concerns the construction of mapping spaces in cubical quasicategories, and how these can be used to characterize equivalences of cubical quasicategories. Finally, Chapter 9, which follows [DKM21], concerns the construction of model structures for (∞, n)-categories on marked cubical sets, and the proof that these model structures are Quillen-equivalent to the complicial model structures on marked simplicial sets.

Chapter 2

Model categories

In this chapter we review some of the general theory of model categories, which we will use to study the homotopy theory of ∞-categories in later chapters. In Section 2.1, we recall the definition of a model category and related concepts, as well as fundamental aspects of model category theory, such as the construction of the homotopy category and comparing model categories via Quillen adjunctions. Though this theory was originally developed by Quillen [Qui67], our exposition will largely follow that of Hovey [Hov99], and will focus primarily on those results which are most relevant to the new results discussed in later chapters.

In Section 2.2, we review techniques for constructing model categories. These techniques include Cisinski-Olschok theory, which allows for the easy construction of model structures on locally presentable categories (see Theorem 2.2.14), as well as a technique for transfering a model structure along an adjunction (see Theorem 2.2.28).

2.1 Definitions and basic results

We begin by defining some basic category-theoretic concepts which are fundamental to the theory of model categories. Throughout this thesis we assume that all categories under discussion are locally small, unless otherwise noted.

Definition 2.1.1. Let $f\colon A \to B$ and $g\colon X \to Y$ be maps in a category $\mathcal{C}$. The map f has the *left lifting property* with respect to g (or equivalently, g has the *right lifting property* with respect to f) if, for every diagram of the form

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "q"] & Y \end{tikzcd}$$

there exists a *lift*, i.e. a map $r\colon B \to X$ such that $rf = p$ and $gr = q$.

Definition 2.1.2. Given a class of maps W in a category $\mathcal{C}$, we denote the class of maps having the left lifting property with respect to W by $l(W)$, and the class of maps having the right lifting property with respect to W by $r(W)$.

Definition 2.1.3. A *weak factorization system* on a category $\mathcal{C}$ is a pair (L, R) of classes of morphisms in $\mathcal{C}$, such that:

- $L = l(R)$;
- $R = r(L)$;
- every map in $\mathcal{C}$ can be factored as gf for some $f \in L, g \in R$.

We refer to L and R as the *left class* and *right class* of the weak factorization system, respectively.

Throughout what follows, we will assume that all weak factorization systems under discussion are *functorial*, i.e. that the choice of factorizations defines a functor from the morphism categoy $\mathcal{C}^{[1]}$ to the category of composable pairs $\mathcal{C}^{[2]}$. Much of the theory that follows can be developed without assuming functoriality, albeit less conveniently, but functorial factorizations are available in all the specific cases that will be of interest here.

Definition 2.1.4. Let L be a class of maps in a category $\mathcal{C}$. The class L is *saturated* if it is closed under (transfinite) composition, retracts, and pushout. The *saturation* of a class of maps M is the smallest saturated class containing M.

Lemma 2.1.5. *The left class of any weak factorization system is saturated.*

Proof. This follows from the fact that transfinite composition, pushout, and retracts all preserve left lifting properties. □

Definition 2.1.6. A weak factorization system (L, R) is *cofibrantly generated* if there is a set of maps $M \subseteq L$ such that $R = r(M)$. In this case, we refer to M as a *cellular model* for L.

Definition 2.1.7. Let (L, R) be a weak factorization system on a category $\mathcal{C}$ with initial object $\varnothing$. An object $c \in \mathcal{C}$ is *cofibrant* if the unique map $\varnothing \to C$ is in L.

The following lemmas about weak factorization systems have useful applications to model categories.

Lemma 2.1.8. *If (L, R) defines a weak factorization system on a category $\mathcal{C}$, then (R, L) defines a weak factorization system on $\mathcal{C}^{\mathrm{op}}$.* □

Lemma 2.1.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories equipped with weak factorization systems (L, R) and (L', R'), respectively. Given an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$, we have $FL \subseteq L'$ if and only if $UR' \subseteq R$.*

Proof. Let $f \in L, g \in R'$. By adjointness, there is a bijection between diagrams of the following two forms:

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "f"'] & UX \arrow[d, "Ug"] \\ B \arrow[r, "q"] & UY \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "\overline{p}"] \arrow[d, "Ff"'] & X \arrow[d, "g"] \\ FB \arrow[r, "\overline{q}"] & Y \end{tikzcd}$$

such that the diagram in $\mathcal{C}$ on the left admits a lift if and only if the diagram in $\mathcal{D}$ on the right admits a lift. The stated result thus follows. □

Definition 2.1.10. A class of maps W in a category $\mathcal{C}$ satisfies the *two-out-of-three property* if, given a composable pair of maps f, g in W, if any two of f, g, and gf are in W then so is the third.

With these concepts established, we can now define our basic objects of study.

Definition 2.1.11. Let $\mathcal{C}$ be a complete and co-complete category. A *model structure* on $\mathcal{C}$ consists of three classes of maps:

- W, the *weak equivalences*;
- C, the *cofibrations*;
- F, the *fibrations*;

such that:

- the pairs $(L \cap W, R)$ and $(L, R \cap W)$ both define weak factorization systems;
- W satisfies the two-out-of-three property.

A *model category* is a category equipped with a specified model structure. We now introduce some basic terminology in the theory of model categories.

- A fibration or cofibration in a model category is *trivial* if it is also a weak equivalence.
- An object X in a model category is *cofibrant* if the unique map from the initial object to X is a cofibration, and *fibrant* if the unique map from X to the terminal object is a fibration.
- Given an object X in a model category $\mathcal{C}$, a *cofibrant replacement* of X is a cofibrant object QX equipped with a trivial fibration $QX \to X$.
- Given an object X in a model category $\mathcal{C}$, a *fibrant replacement* of X is a fibrant object RX equipped with a trivial cofibration $X \to RX$.
- A model category is *cofibrantly generated* if the weak factorization systems $(C, F \cap W)$ and $(C \cap W, F)$ are cofibrantly generated.
- A *pseudo-generating* set of trivial cofibrations in a model category is a set of trivial cofibrations S such that a map between fibrant objects is a fibration if and only if it has the right lifting property with respect to S.

Every object X in a model category $\mathcal{C}$ admits fibrant and cofibrant replacements, by suitably factoring the maps $\varnothing \to X$ and $X \to *$, where $\varnothing$ and $*$ denote the initial and terminal objects of $\mathcal{C}$, respectively. In fact, by applying functorial factorization we obtain cofibrant and fibrant replacement functors $Q, R\colon \mathcal{C} \to \mathcal{C}$, with a natural trivial fibration $Q \Rightarrow \mathrm{id}_{\mathcal{C}}$ and a natural trivial cofibration $\mathrm{id}_{\mathcal{C}} \Rightarrow R$. Furthermore, we may observe that a cofibrant replacement of a fibrant object is fibrant, and dually, a fibrant replacement of a cofibrant object is cofibrant.

Next we record some miscellaneous results on model categories which will be of use in later chapters.

Lemma 2.1.12. *If (W, C, F) defines a model structure on $\mathcal{C}$, then (W, F, C) defines a model structure on $\mathcal{C}^{\mathrm{op}}$.*

Proof. Immediate from Lemma 2.1.8. □

Lemma 2.1.13. *In any model category, the classes of weak equivalences, cofibrations and fibrations are closed under composition and retracts. Furthermore, the classes of cofibrations and trivial cofibrations are closed under pushout and transfinite composition, while the classes of fibrations and trivial fibrations are closed under pullback and transfinite precomposition.*

Proof. The closure of weak equivalences under composition is immediate from the two-out-of-three property; the closure of weak equivalences under retracts is [Joy09, Prop. E.1.3]. The remaining assertions follow from Lemma 2.1.5. □

In general, weak equivalences are not closed under pushout or pullback. In many cases of interest, however, they are closed under pushout along cofibrations or pullback along fibrations.

Definition 2.1.14. A model category $\mathcal{C}$ is *left proper* if any pushout in $\mathcal{C}$ of a weak equivalence along a cofibration is a weak equivalence, and *right proper* if any pullback in $\mathcal{C}$ of a weak equivalence along a fibration is a weak equivalence.

Proposition 2.1.15 ([Ree21, Thm. B]). *Let $A \to B$ be a weak equivalence and $A \to C$ a cofibration in a model category $\mathcal{C}$, with A, B, and C cofibrant. Then the pushout map $C \to B \cup_A C$ is a weak equivalence. Dually, if $X \to Z$ is a weak equivalence and $Y \to Z$ a fibration, with X, Y, and Z fibrant, then the pullback map $X \times_Z Y \to Y$ is a weak equivalence.* □

Corollary 2.1.16. *Any model category with all objects cofibrant is left proper. Dually, any model category with all objects fibrant is right proper.* □

Lemma 2.1.17 (Ken Brown's Lemma, [Hov99, Lem. 1.1.12]). *Let $\mathcal{C}$ be a model category, and $\mathcal{D}$ a category equipped with a class of weak equivalences satisfying the two-out-of-three property. If a functor $F: \mathcal{C} \to \mathcal{D}$ sends trivial cofibrations between cofibrant objects to weak equivalences, then F sends all weak equivalences between cofibrant objects to weak equivalences. Dually, if F sends trivial fibrations between fibrant objects to weak equivalences, then F sends all weak equivalences between fibrant objects to weak equivalences.* □

The purpose of defining a model structure on a category is to study the homotopy theory of certain nicely-behaved objects, namely those which are both cofibrant and fibrant. We do so by means of the following concepts.

Definition 2.1.18. Let X be an object in a model category $\mathcal{C}$. A *cylinder object* for X is a factorization of the co-diagonal map $X \sqcup X \to X$ as $X \sqcup X \to IX \to X$, where $X \sqcup X \to IX$ is a cofibration and $IX \to X$ is a weak equivalence. Dually, a *path object* for X is a factorization of the diagonal map $X \to X \times X$ as $X \to PX \to X \times X$, where $X \to PX$ is a weak equivalence and $PX \to X \times X$ is a fibration.

Let $f, g: X \to Y$ be maps in $\mathcal{C}$. The maps f and g are *left homotopic* with respect to a given cylinder object for X if there exists a map $H: IX \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
X \sqcup X \arrow[r, "(f,g)"] \arrow[d] & Y \\
IX \arrow[ur, "H"'] &
\end{tikzcd}$$

Dually, f and g are *right homotopic* with respect to a given path object for Y if there exists a map $K: X \to PY$ such that following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "(f,g)"] \arrow[d, "K"'] & Y \times Y \\
PY \arrow[ur] &
\end{tikzcd}$$

The factorization axioms ensure that every object in a model category admits a cylinder object and a path object; moreover, our assumption of functorial factorization ensures that these constructions can be made functorial.

Proposition 2.1.19 (Quillen, [Hov99, Cors. 1.2.6 & 1.2.7])**.** *Let X and Y be objects in a model category $\mathcal{C}$ with X cofibrant and Y fibrant. Then the relations of left and right homotopy on $\mathcal{C}(X,Y)$ are independent of the choice of cylinder object or path object in their definitions, and these relations coincide. Moreover, they define an equivalence relation on $\mathcal{C}(X,Y)$, which is compatible with composition.* $\square$

We refer to the equivalence relation of Proposition 2.1.19 as *homotopy*, and write $f \sim g$ to indicate that f is homotopic to g. Given a pair of cofibrant and fibrant objects X, Y in a model category $\mathcal{C}$, we let $[X,Y]$ denote the set of maps from X to Y modulo this equivalence relation.

Definition 2.1.20. Let X and Y be objects in a model category $\mathcal{C}$ with both X and Y cofibrant and fibrant. A map $f: X \to Y$ is a *homotopy equivalence* if it admits a *homotopy inverse*, i.e. there exists a map $g{:}Y \to X$ such that $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$.

Lemma 2.1.21. *Let $f{:}X \to Y$ be a homotopy equivalence in a model category $\mathcal{C}$, with homotopy inverse $g{:}Y \to X$. A map $g'{:}Y \to X$ is a homotopy inverse to f if and only if $g \sim g'$.*

Proof. This follows from the fact that the homotopy relation is compatible with composition. □

Proposition 2.1.22 (Quillen, [Hov99, Prop. 1.2.8])**.** *A map between objects of a model category which are both cofibrant and fibrant is a weak equivalence if and only if it is a homotopy equivalence.* □

Let $\mathcal{C}$ be a model category. The category $\mathcal{C}[W^{-1}]$ is obtained from $\mathcal{C}$ by formally inverting the weak equivalences. Let $\mathcal{C}_c, \mathcal{C}_f$, and $\mathcal{C}_{cf}$ denote the full subcategories of $\mathcal{C}$ on objects which are cofibrant, fibrant, and both cofibrant and fibrant, respectively; by formally inverting the weak equivalences in these full subcategories we obtain categories $\mathcal{C}_c[W^{-1}], \mathcal{C}_f[W^{-1}]$, and $\mathcal{C}_{cf}[W^{-1}]$.

The following result shows that the homotopy theory defined by a model category is fully captured by those objects which are fibrant, cofibrant, or both.

Proposition 2.1.23 (Quillen, [Hov99, Prop. 1.2.3])**.** *The inclusions $\mathcal{C}_{cf} \hookrightarrow \mathcal{C}_c \hookrightarrow \mathcal{C}$ and $\mathcal{C}_{cf} \hookrightarrow \mathcal{C}_f \hookrightarrow \mathcal{C}$ induce equivalences of categories $\mathcal{C}_{cf}[W^{-1}] \hookrightarrow \mathcal{C}_c[W^{-1}] \hookrightarrow \mathcal{C}[W^{-1}]$, with inverse equivalences induced by the cofibrant and fibrant replacement functors Q and R.* □

Although $\mathcal{C}[W^{-1}]$ is a natural definition of the homotopy category of $\mathcal{C}$, it is often difficult to work with in practice; for instance, it is not clear from the definition that $\mathcal{C}[W^{-1}]$ remains locally small, as its morphisms are zig-zags of morphisms in $\mathcal{C}$.

Definition 2.1.24. Let $\mathcal{C}$ be a model category. The *homotopy category* $\mathsf{Ho}\mathcal{C}$ is defined as follows:

- The objects are the cofibrant and fibrant objects of $\mathcal{C}$;
- $\mathsf{Ho}\mathcal{C}(X, Y) = [X, Y]$, with composition induced by that of $\mathcal{C}$.

That these data define a category follows from Proposition 2.1.19.

Theorem 2.1.25 (Quillen, [Hov99, Thm. 1.2.1])**.** *For any model category $\mathcal{C}$, there is an isomorphism of categories $\mathcal{C}_{cf}[W^{-1}] \to \mathsf{Ho}\mathcal{C}$, acting as the identity on objects. For a map $f\colon X \to Y$, this isomorphism sends f to the homotopy class $[f]$; if f is a weak equivalence then the formal inverse $f^{-1}\colon Y \to X$ is mapped to the homotopy class of homotopy inverses of f.* □

Corollary 2.1.26. *The natural functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ sends a map in $\mathcal{C}$ to an isomorphism if and only if f is a weak equivalence.*

Proof. Proposition 2.1.22 shows that the isomorphisms in $\mathsf{Ho}\mathcal{C}$ are precisely the homotopy classes of weak equivalences between cofibrant and fibrant objects. The stated result thus follows from Proposition 2.1.23 and Theorem 2.1.25, together with the fact that the cofibrant and fibrant replacement functors preserve weak equivalences. □

In view of Proposition 2.1.23 and Theorem 2.1.25, we will also refer to $\mathcal{C}[W^{-1}]$ as the homotopy category of $\mathcal{C}$, distinguishing between equivalent definitions only when necessary.

We compare model categories by means of adjunctions which are compatible with the defining weak factorization systems.

Lemma 2.1.27. *For an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ between model categories, the following conditions are equivalent:*

- *F preserves cofibrations and trivial cofibrations;*
- *F preserves cofibrations and U preserves fibrations;*
- *F preserves trivial cofibrations and U preserves trivial fibrations;*
- *U preserves fibrations and trivial fibrations.*

Proof. Immediate from Lemma 2.1.9. □

Definition 2.1.28. An adjunction between model categories is a *Quillen adjunction* if it satisfies the equivalent conditions of Lemma 2.1.27.

A left adjoint is a *left Quillen functor*, and its right adjoint is a *right Quillen functor*, if the adjunction is Quillen.

Lemma 2.1.29. *Left Quillen functors preserve weak equivalences between cofibrant objects. Dually, right Quillen functors preserve weak equivalences between fibrant objects.*

Proof. Immediate from Lemma 2.1.17. □

Lemma 2.1.29 shows that a left Quillen functor $F{:}\mathcal{C} \to \mathcal{D}$ induces a functor $F{:}\mathcal{C}_c[W^{-1}] \to \mathcal{C}[W^{-1}]$, and dually a right Quillen functor U induces $U{:}\mathcal{C}_f[W^{-1}] \to \mathcal{C}[W^{-1}]$. Moreover, as the cofibrant and fibrant replacement functors preserve all weak equivalences, they define functors $Q{:}\mathcal{C}[W^{-1}] \to \mathcal{C}_c[W^{-1}]$, $R{:}\mathcal{C}[W^{-1}] \to \mathcal{C}_f[W^{-1}]$. This allows us to define an action of a Quillen adjunction on homotopy categories.

Definition 2.1.30. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a Quillen adjunction. The *left derived functor* of F, denoted LF, is the composite:

$$\mathcal{C}[W^{-1}] \xrightarrow{Q} \mathcal{C}_c[W^{-1}] \xrightarrow{F} \mathcal{D}[W^{-1}]$$

Similarly, the *right derived functor* of U, denoted RU, is the composite:

$$\mathcal{D}[W^{-1}] \xrightarrow{R} \mathcal{D}_f[W^{-1}] \xrightarrow{U} \mathcal{C}[W^{-1}]$$

This definition may seem less than ideal, as it is not strictly functorial and appears to depend on the choice of cofibrant and fibrant replacement functors in $\mathcal{C}$ and $\mathcal{D}$. However, the following results show that the derived functor construction is well-defined and functorial up to natural isomorphism.

Proposition 2.1.31. *Let $\mathcal{C}$ be a model category with cofibrant replacement functors Q, Q'. For any left Quillen functor $F{:}\mathcal{C} \to \mathcal{D}$, the left derived functors defined with respect to Q and Q' are naturally isomorphic. Likewise, for any right Quillen functor $U{:}\mathcal{D} \to \mathcal{C}$, the right derived functors with respect to a pair of fibrant replacement functors R, R' are naturally isomorphic.*

Proof. Observe that by Theorem 2.1.25, for any pair of cofibrant replacement functors Q, Q', the induced functors $\mathcal{C}[W^{-1}] \to \mathcal{C}_c[W^{-1}]$ are naturally isomorphic, as they are inverses to the equivalence of categories $\mathcal{C}_c[W^{-1}] \to \mathcal{C}[W^{-1}]$. It thus follows that their composites with any given functor will be naturally isomorphic. A similar result holds for fibrant replacement. □

Proposition 2.1.32 ([Hov99, Thm. 1.3.7])**.** *Let $\mathcal{C}$ be a model category. Then there are natural isomorphisms $L(\mathrm{id}_{\mathcal{C}}) \cong \mathrm{id}_{\mathcal{C}[W^{-1}]} \cong R(\mathrm{id}_{\mathcal{C}})$.*

Moreover, let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U, F' : \mathcal{D} \rightleftarrows \mathcal{E} : U'$ be a pair of Quillen adjunctions. Then there are natural isomorphisms $L(F'F) \cong L(F') \circ L(F), R(UU') \cong R(U) \circ$

$R(U')$. □

Proposition 2.1.33 ([Hov99, Lem. 1.3.10]). *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a Quillen adjunction. Then we have an adjunction $LF : \mathcal{C}[W^{-1}] \rightleftarrows \mathcal{D}[W^{-1}] : RU$.* □

We refer to the adjunction of Proposition 2.1.33 as the *derived adjunction*.

The appropriate notion of equivalence between model categories is that of a Quillen equivalence, which we now define.

Definition 2.1.34. A Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a *Quillen equivalence* if the derived adjunction is an adjoint equivalence of categories.

We conclude this section by reviewing some results which allow us to easily recognize Quillen adjunctions and Quillen equivalences.

Proposition 2.1.35 ([JT07, Prop. 7.15]). *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be an adjunction between model categories. If F preserves cofibrations and U preserves fibrations between fibrant objects, then the adjunction is Quillen.* □

This statement has an immediate corollary, which we will apply in practice:

Corollary 2.1.36. *Let $F : \mathcal{C} \to \mathcal{D}$ be a left adjoint between model categories such that $\mathcal{C}$ has a pseudo-generating set of trivial cofibrations S. If F preserves cofibrations and sends S to trivial cofibrations, then F is a left Quillen functor.*

□

Proposition 2.1.37 ([Hov99, Cor. 1.3.16]). *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a Quillen adjunction between model categories. Then the following are equivalent.*

(i) *$F \dashv U$ is a Quillen equivalence.*

(ii) *F reflects weak equivalences between cofibrant objects and, for every fibrant Y, the derived counit $FQUY \to FUY \to Y$ is a weak equivalence.*

(iii) U reflects weak equivalences between fibrant objects and, for every cofibrant X, the derived unit $X \to UFX \to URFX$ is a weak equivalence.

Again, in practice we will often apply the following corollary:

Corollary 2.1.38. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a Quillen adjunction between model categories.*

(i) If U preserves and reflects weak equivalences, then the adjunction is a Quillen equivalence if and only if, for all cofibrant $X \in \mathcal{C}$, the unit $X \to UFX$ is a weak equivalence.

(ii) If F preserves and reflects weak equivalences, then the adjunction is a Quillen equivalence if and only if, for all fibrant $Y \in \mathcal{D}$, the counit $FUY \to Y$ is a weak equivalence. □

We will also have some use for the following consequence of this result, which concerns involutions of model categories. Recall that any involution of a category is self-adjoint, with the identity natural transformation as both unit and counit.

Corollary 2.1.39. *Let $\mathcal{C}$ be a model category, and $F{:}\mathcal{C} \to \mathcal{C}$ an involution. If the adjunction $F \dashv F$ is Quillen, then it is a Quillen equivalence.*

Proof. If $F \dashv F$ is Quillen, then F preserves trivial cofibrations and trivial fibrations, hence all weak equivalences. The fact that F is an involution thus implies that it reflects weak equivalences as well. Both the unit and counit of the adjunction are the identity natural transformation on $\mathcal{C}$, thus we may apply either statement of Corollary 2.1.38 to conclude that the adjunction is a Quillen equivalence. □

Though it will not be a primary focus of this thesis, we will have some use for the theory of monoidal model categories.

Definition 2.1.40. Let $f\colon A \to B, g\colon X \to Y$ be maps in a cocomplete category $\mathcal{C}$ equipped with a monoidal product $\otimes$. The *pushout product* $f \hat{\otimes} g$ is the canonical map $A \otimes Y \cup_{A \otimes X} B \otimes X \to B \otimes Y$.

Definition 2.1.41. A *monoidal model category* is a model category $\mathcal{C}$ equipped with a closed monoidal product $\otimes$ with unit object S, satisfying the following axioms:

(i) For any pair of cofibrations i, j in $\mathcal{C}$, the pushout product $i \mathbin{\widehat{\otimes}} j$ is a cofibration. Moreover, if either i or j is trivial then so is $i \mathbin{\widehat{\otimes}} j$.

(ii) For any cofibrant object X, the canonical maps $QS \otimes X \to S \otimes X$ and $X \otimes QS \to X \otimes S$ are weak equivalences.

The following lemma shows that when the unit object of $\mathcal{C}$ is cofibrant, it suffcies to consider the first condition above; this hypothesis will hold in all of the examples we consider in this thesis.

Lemma 2.1.42. *Let $\mathcal{C}$ be a model category equipped with a monoidal product $\otimes$ with unit object S. If $\mathcal{C}$ satisfies condition (i) of Definition 2.1.41, and S is cofibrant, then $\mathcal{C}$ satisfies condition (ii) of Definition 2.1.41 as well.*

Proof. Let $\varnothing$ denote the initial object of $\mathcal{C}$. For any object X and any map $A \to B$, the pushout product $(\varnothing \to X) \mathbin{\widehat{\otimes}} (A \to B)$ is the map $X \otimes A \to X \otimes B$. Condition (i) thus implies that for all cofibrant X, the functor $X \otimes (-)$ preserves trivial cofibrations. Thus $X \otimes (-)$ preserves weak equivalences between cofibrant objects by Lemma 2.1.17. A similar proof holds for $(-) \otimes X$. □

2.2 Constructing model categories

In this section, we review some standard techniques for constructing model categories, which will be applied throughout subsequent chapters. We begin by reviewing the machinery of Cisinski-Olschok theory, which allows for the construction of model structures on locally presentable categories. Subsequently, we consider situations in which a model structure on a category $\mathcal{C}$ can be defined by declaring the weak equivalences and cofibrations (resp. fibrations) to be created by a left adjoint (resp. right adjoint) functor into a model category $\mathcal{D}$.

We begin by defining some basic category-theoretic concepts.

Definition 2.2.1. Let λ be a regular cardinal.

(i) An object X in a category $\mathcal{C}$ is λ-*small* if the functor $\mathcal{C}(X,-)\colon\mathcal{C}\to\mathsf{Set}$ preserves λ-directed colimits.

(ii) A category $\mathcal{C}$ is λ-*accessible* if it has λ-directed colimits and there is a set $\mathcal{A}$ of λ-small objects of $\mathcal{C}$ such that every object of $\mathcal{C}$ is a λ-directed colimit of objects of $\mathcal{A}$.

(iii) A category $\mathcal{C}$ is λ-*locally presentable* if it is λ-accessible and cocomplete, and *locally presentable* if it is λ-locally presentable for some regular cardinal λ.

(iv) A functor $F\colon\mathcal{C}\to\mathcal{D}$ is λ-*accessible* if both $\mathcal{C}$ and $\mathcal{D}$ are λ-accessible and F preserves λ-directed colimits, and *accessible* if it is λ-accessible for some regular cardinal λ.

(v) Given a category $\mathcal{C}$, a full subcategory $\mathcal{D}\subseteq\mathcal{C}$ is λ-*accessibly embedded* if it is closed under λ-directed colimits, and *accessibly embedded* if it is

λ-accessibly embedded for some regular cardinal λ.

Lemma 2.2.2 ([AR94, Cor. 1.28])**.** *Every locally presentable category is complete.* $\square$

The following standard result is frequently of use in constructing model structures on locally presentable categories.

Theorem 2.2.3 (Small object argument, [Cis19, Prop. 2.1.9])**.** *Let $\mathcal{C}$ be a cocomplete category, and M a set of maps in $\mathcal{C}$. Suppose that there exists some regular cardinal λ such that the domains of all maps in M are λ-small. Then $(l(r(M)), r(M))$ defines a weak factorization system on $\mathcal{C}$. Moreover, $l(r(M))$ is the saturation of M.* $\square$

Remark 2.2.4. Although Theorem 2.2.3 is sufficient for our present purposes, the statement remains true with weaker hypotheses; see [Hov99, 2.1.14] or [Rie14, Thm. 12.2.2].

Definition 2.2.5. Given a category $\mathcal{C}$, a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ is *reflective* if the inclusion functor $\mathcal{D} \hookrightarrow \mathcal{C}$ has a left adjoint.

The following lemma is useful when dealing with reflective subcategories.

Lemma 2.2.6 ([GZ67, Prop. 1.3])**.** *In an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$, the right adjoint U is fully faithful if and only if the counit $FU \Rightarrow \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism.* $\square$

Throughout subsequent chapters, we will construct model structures on reflective subcategories of presheaf categories. Thus we will have considerable use for the following result.

Proposition 2.2.7 ([AR94, Thm. 1.46]). *A category $\mathcal{C}$ is locally presentable if and only if it is equivalent to an accessibly embedded reflective subcategory of the presheaf category $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ for some small category $\mathcal{C}$.* □

Lemma 2.2.8. *Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ a reflective subcategory of the presheaf category $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$. Let $F\colon \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{D}$ denote the left adjoint of the inclusion $\mathcal{D} \hookrightarrow \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$. Then every object of $\mathcal{D}$ is the colimit of a diagram of objects of the form $F\mathcal{C}(-,c)$ for $c \in \mathcal{C}$.*

Proof. Let $d \in \mathcal{D}$. By a standard result about presheaf categories, in $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ we have $d = \operatorname*{colim}_{\mathcal{C}(-,c)\to d} \mathcal{C}(-,c)$. The stated result thus follows from Lemma 2.2.6 and the fact that F preserves colimits as a left adjoint. □

The key result used to construct model structures on locally presentable categories is the following.

Theorem 2.2.9 (Jeff Smith's Theorem, [Bek00, Thm. 1.7, Prop. 1.15, 1.19]). *Let $\mathcal{C}$ be a locally presentable category. Let W be a class of morphisms forming an accessibly embedded, accessible subcategory of $\mathcal{C}^{[1]}$, and I a set of morphisms in $\mathcal{C}$. Suppose that the following conditions are satisfied.*

- *W satisfies the two-out-of-three axiom.*
- *$r(I) \subseteq W$.*
- *The intersection $W \cap l(r(I))$ is closed under pushouts and transfinite composition.*

Then $\mathcal{C}$ admits a cofibrantly generated model structure with weak equivalences W and generating cofibrations I.

Theorem 2.2.9 is the foundation of Cisinski-Olschok theory, which allows for the easy construction of model structures on locally presentable categories having specified classes of cofibrations and pseudo-generating trivial cofibrations. This theory was first developed by Cisinski [Cis06] in the special case of model structures on presheaf categories with monomorphisms as the cofibrations, and was later generalized by Olschok [Ols11]. For convenience, we will typically use this theory in practice rather than applying Theorem 2.2.9 directly.

Throughout what follows, let $\mathcal{C}$ denote a locally presentable category equipped with a weak factorisation system (L, R) which is cofibrantly generated, with a cellular model $M \subseteq L$, and in which all objects are cofibrant.

Definition 2.2.10. A *cylinder functor* on $\mathcal{C}$ consists of an endofunctor $I\colon \mathcal{C} \to \mathcal{C}$, together with natural transformations $\partial^0, \partial^1\colon \mathrm{id} \to I$, $\sigma\colon I \to \mathrm{id}$, such that:

- ∂^0 and ∂^1 are sections of σ;
- for all $X \in \mathcal{C}$, the map $(\partial^0, \partial^1)\colon X \sqcup X \to IX$ is in L.

Fix a locally presentable category $\mathcal{C}$ and a cylinder functor I. For a map $X \to Y$ in $\mathcal{C}$ and $\varepsilon \in \{0, 1\}$, let $IX \cup_\varepsilon Y$ and $IX \cup (Y \sqcup Y)$ be defined by the following pushout squares:

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "\partial^\varepsilon"'] & Y \arrow[d] \\
IX \arrow[r] & IX \cup_\varepsilon Y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \sqcup X \arrow[r] \arrow[d] & Y \sqcup Y \arrow[d] \\
IX \arrow[r] & IX \cup (Y \sqcup Y) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

Definition 2.2.11. A cylinder functor on $\mathcal{C}$ is *cartesian* if the following two properties hold:

(i) the endofunctor I is a left adjoint;

(ii) for all maps $X \to Y$ in L, the maps $IX \cup_\varepsilon Y \to IY$ and $IX \cup (Y \sqcup Y) \to IY$ are in L.

Remark 2.2.12. The assumption that $\mathcal{C}$ is locally presentable and the adjoint functor theorem imply that condition (i) above is equivalent to the statement that I preserves colimits.

Definition 2.2.13. Let $f, g: X \to Y$ be maps of in $\mathcal{C}$. An *elementary homotopy* from f to g is a map $H: IX \to Y$ such that $H\partial^0 = f, H\partial^1 = g$. A *homotopy* is a zig-zag of elementary homotopies. The set $[X, Y]$ is the set of maps from X to Y modulo the relation of homotopy.

It is easy to see that pre- and post-composition by a fixed map preserve the relation of homotopy; thus a map $X \to Y$ induces maps $[Z, X] \to [Z, Y]$ and $[Y, Z] \to [X, Z]$ for any Z.

From here on, we will assume that our choice of cylinder functor I is cartesian.

Let S be an arbitrary set of maps in L. The set of morphisms $\Lambda(S)$ is defined by the following inductive construction. We begin by setting:

$$\Lambda^0(S) = S \cup \{IX \cup_\varepsilon Y \to IY | X \to Y \in M, \varepsilon \in \{0,1\}\}$$

Now, given $\Lambda^n(S)$, we define:

$$\Lambda^{n+1}(S) = \{IX \cup (Y \sqcup Y) \to IY | X \to Y \in \Lambda^n(S)\}$$

Finally, we let $\Lambda(S) = \bigcup_{n \geq 0} \Lambda^n(S)$. We now define several distinguished classes of maps and objects in $\mathcal{C}$.

- A *cofibration* is a map in L; a *trivial fibration* is a map in R.

- An *anodyne map* is a map in the saturation of $\Lambda(S)$; a *naive fibration* is a map having the right lifting property with respect to the anodyne maps.

- A *fibrant object* is an object X such that the map from X to the terminal object is a naive fibration.

- A *weak equivalence* is a map $X \to Y$ such that the induced map $[Y, Z] \to [X, Z]$ is a bijection for any fibrant Z.

- A *trivial cofibration* is a map which is both a cofibration and a weak equivalence; a *fibration* is a map having the right lifting property with respect to the trivial cofibrations.

Theorem 2.2.14. *Let $\mathcal{C}$ be a locally presentable category equipped with the following data:*

- *a weak factorization system (L, R), cofibrantly generated by a set of maps $M \subseteq L$, with all objects cofibrant;*

- *a cartesian cylinder functor $(I, \partial^0, \partial^1, \sigma)$;*

- *a set of maps $S \subseteq L$.*

Then the classes above define a left proper, cofibrantly generated model structure on $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$, in which a map between fibrant objects is a fibration if and only if it is a naive fibration.

Proof. The existence of the model structure is [Ols11, Thm. 3.16]; the characterization of fibrations between fibrant objects is [Ols11, Lem. 3.30]. Left properness follows from Corollary 2.1.16. □

Note that, in all of the examples of model structures we will construct using this result, the set S itself forms a set of pseudo-generating trivial cofibrations, without the need for the larger set $\Lambda(S)$. This is not immediate from Theorem 2.2.14, but in each case it follows from a more detailed analysis of the model structure in question, such as those which apppear in the references that will be given for further reading on each model structure.

Corollary 2.2.15. *The homotopy category of $\mathcal{C}$ with the model structure of Theorem 2.2.14 can be described as follows:*

- *its objects are the fibrant presheaves;*
- *the maps from X to Y are given by $[X, Y]$.* □

Proposition 2.2.16. *The model structure of Theorem 2.2.14 is independent of the choice of generating cofibrations M.*

Proof. This is immediate from [Ols11, Lem. 3.6] and the fact that a model structure is determined by any two of the classes of weak equivalences, fibrations and cofibrations. □

In certain special cases, we can use Theorem 2.2.14 to obtain model structures for which the set S itself forms a pseudo-generating set of trivial cofibrations.

Definition 2.2.17. For a class of maps M in a category $\mathcal{C}$, the *cellular closure* of M, denoted $\mathsf{cell}(M)$, is the closure of M under pushout and (transfinite) composition.

Note that in general, the cellular closure of a class of maps differs from its saturation, as the cellular closure need not be closed under retracts.

Theorem 2.2.18. *Let $\mathcal{C}$ be a locally presentable category equipped with:*

- *sets M, S of monomorphisms;*
- *a biclosed monoidal structure $\otimes$ whose unit is the terminal object 1; and*
- *a bipointed object $\partial_0, \partial_1 : 1 \to I$*

such that

(i) $\mathsf{cell}(M)$ *is the class of all monomorphisms of $\mathcal{C}$;*

(ii) $(\partial_0, \partial_1) : 1 \sqcup 1 \to I$ *is a monomorphism;*

(iii) $\partial_0, \partial_1 \in \mathsf{cell}(S)$*;*

(iv) $M \mathbin{\widehat{\otimes}} M \subset \mathsf{cell}(M))$*;*

(v) $M \hat{\otimes} S \subset \mathsf{cell}(S)$*; and*

(vi) $S \hat{\otimes} M \subset \mathsf{cell}(S)$*.*

Then there exists a left proper, cofibrantly generated model structure on $\mathcal{C}$ such that:

(i) the cofibrations are precisely the monomorphisms; and

(ii) a map into a fibrant object is a fibration if and only if it has the right lifting property with respect to each member of S.

Moreover this model structure is monoidal with respect to $\otimes$.

Proof. We will apply Theorem 2.2.14 (with L taken to be all monomorphisms). Our functorial cylinder is given by

$$X \sqcup X \cong X \otimes (1 \sqcup 1) \xrightarrow{X \otimes (\partial_0, \partial_1)} X \otimes I \xrightarrow{X \otimes !} X \otimes 1 \cong X.$$

Note that (1), (2) and (4) imply that $X \otimes (\partial_0, \partial_1)$ is a monomorphism for each X since it can be written as $i_X \hat{\otimes} (\partial_0, \partial_1)$ where $i_X : \varnothing \to X$ is the unique map from the initial object; this map is a monomorphism by Proposition 2.2.7 and the corresponding result in presheaf categories. Similarly, since any map from a terminal object (and in particular ∂_0, ∂_1) is a monomorphism, we can deduce that $X \otimes \partial_0$ and $X \otimes \partial_1$ are always monomorphisms. Moreover the biclosedness implies that $(-) \otimes I$ is cocontinuous, so this cylinder is cartesian. Thus, by Theorem 2.2.14, we obtain a model structure on $\mathcal{C}$ such that:

(i) the cofibrations are precisely the monomorphisms; and

(ii') a map into a fibrant object is a fibration if and only if it has the right lifting property with respect to each member of $\Lambda(S)$

where $\Lambda(S)$ is the closure of

$$S \cup \left\{ f \hat{\otimes} \partial_\epsilon : f \in M, \epsilon \in \{0,1\} \right\}$$

under the operation $(-) \hat{\otimes} (\partial_0, \partial_1)$. We wish to reduce (ii') to (ii). Indeed, it follows from (3) and (5) that the above generating set of Λ is contained in $\mathsf{cell}(S)$. Moreover $\mathsf{cell}(S)$ is closed under the operation $(-) \hat{\otimes} (\partial_0, \partial_1)$ by (1), (2) and (6), which implies $\Lambda \subset \mathsf{cell}(S)$.

That this model structure is monoidal with respect to $\otimes$ follows from [Mae21, Proposition A.4] and (4-6). □

The theory of EZ-Reedy categories is often useful in producing examples of weak factorization systems satisfying the hypotheses of Theorem 2.2.14. We now describe some of this theory, primarily following the exposition of [BR13]. Note that some other references use different terminology; for instance, the definition of EZ-Reedy categories in [BM11] is more general than that which is presented here.

Definition 2.2.19. A *Reedy category* is a category $\mathcal{C}$ equipped with wide subcategories $\mathcal{C}_+, \mathcal{C}_-$ and a degree function $d\colon \mathsf{Ob}\mathcal{C} \to \mathbb{N}$, such that:

- Every morphism f in $\mathcal{C}$ has a unique factorization of the form $f_+ f_-$, where f_+ is in $\mathcal{C}_+$ and f_- is in $\mathcal{C}_-$.
- For every morphism $f\colon c \to c'$ in $\mathcal{C}_+$ we have $d(c) \leq d(c')$. Likewise, for every morphism $f\colon c \to c'$ in $\mathcal{C}_-$ we have $d(c) \geq d(c')$. Moreover, if equality holds in either of these inequalities, then $c = c'$ and f is the identity.

We now record some basic consequences of this definition.

Proposition 2.2.20. *In a Reedy category $\mathcal{C}$, the intersection $\mathcal{C}_+ \cap \mathcal{C}_-$ consists of all objects and their identity maps. Moreover, the identity maps are the only isomorphisms in $\mathcal{C}$.*

Proof. The characterization of $\mathcal{C}_+ \cap \mathcal{C}_-$ is immediate from the definition. To see that $\mathcal{C}$ has no non-identity isomorphisms, let f be an isomorphism of $\mathcal{C}$, and let $f_+ f_-$ be the unique factorization of f from Definition 2.2.19. Now let $g = f_- f^{-1}$, and let $g_+ g_-$ be its unique factorization. Then $f_+ g_+ g_- = f_+ f_- f^{-1} = f f^{-1} = \mathrm{id}$. Uniqueness of factorizations thus implies that $f_+ g_+ = \mathrm{id}$. The inequalities of Definition 2.2.19 thus imply that the domain and codomain of f_+ have equal degree, implying that f_+ is an identity. A similar proof involving a factorization of $f^{-1} f_+$ shows that f_- is an identity. □

Definition 2.2.21. An *Eilenberg-Zilber Reedy category*, or *EZ-Reedy category*, is a Reedy category $\mathcal{C}$ satisfying the following additional axioms:

- Every map in $\mathcal{C}_-$ has at least one section;
- If a pair of maps in $\mathcal{C}_-$ have the same set of sections, then they are equal.

We now consider presheaves over EZ-Reedy categories, and present a series of results showing that such presheaves are particularly nicely behaved. Throughout what follows, for $f\colon c \to c'$ in $\mathcal{C}$, $X\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$, and $x \in X(c')$, we denote the element $Xf(x) \in X(c)$ by xf.

Definition 2.2.22. Let $\mathcal{C}$ be an EZ-Reedy category, $X\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ a presheaf over $\mathcal{C}$, and $c \in \mathcal{C}$. An element $x \in X(c)$ is *degenerate* if there exists a map $\sigma\colon c \to c'$ in $\mathcal{C}_-$ and an element $x' \in c'$ such that $x = x'\sigma$, and *non-degenerate* otherwise.

Proposition 2.2.23 ([BR13, Prop. 4.2])**.** *Let $\mathcal{C}$ be an EZ-Reedy category and $X\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ a presheaf over $\mathcal{C}$. For every $c \in \mathcal{C}$ and $x \in X(c)$, there exists a unique pair consisting of a map $\sigma\colon c \to c'$ in $\mathcal{C}_-$ and a non-degenerate element $x' \in X(c')$ such that $x = x'\sigma$.* □

Remark 2.2.24. In the case where x is non-degenerate, the pair (σ, x') of Proposition 2.2.23 is simply (id_c, x).

For an object c in an EZ-Reedy category $\mathcal{C}$, we let $\partial\mathcal{C}(-,c)$ denote the subobject of the representable presheaf $\mathcal{C}(-,c)$ given by the union of the images of the maps $\mathcal{C}(-,c') \to \mathcal{C}(-,c)$ for all non-identity maps $c' \to c$ in $\mathcal{C}_+$. There is a canonical inclusion $\partial\mathcal{C}(-,c) \to \mathcal{C}(-,c)$.

Proposition 2.2.25 ([Ara14, Prop. 1.5])**.** *Let $\mathcal{C}$ be an EZ-Reedy category, and let M denote the class of inclusions $\partial\mathcal{C}(-,c) \to \mathcal{C}(-,c)$ for all $c \in \mathcal{C}$. Then $l(r(M))$ is the class of all monomorphisms in $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$.* □

Next we review a theorem which allows us to induce one model structure from another using an adjunction between their respective categories.

Definition 2.2.26. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be an adjunction between model categories. The model structure on $\mathcal{C}$ is *left induced* by F if F preserves and reflects cofibrations and weak equivalences. Likewise, the model structure on $\mathcal{D}$ is *right induced* by U if U preserves and reflects weak equivalences and fibrations.

Remark 2.2.27. Note that for a given adjunction $\mathcal{C} \rightleftarrows \mathcal{D}$ and a given model structure on $\mathcal{D}$, the left-induced model structure is unique, if one exists, since the definition determines the cofibrations and weak equivalences of $\mathcal{C}$. Likewise, for a given model structure on $\mathcal{C}$, the right-induced model structure is unique, if one exists.

Theorem 2.2.28 ([HKRS17, Thm. 2.2.1])**.** *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be an adjunction between locally presentable categories such that $\mathcal{D}$ carries a cofibrantly generated model structure with all objects cofibrant. If, for every object $X \in \mathcal{C}$, the co-diagonal map admits a factorization $X \sqcup X \xrightarrow{i_X} IX \xrightarrow{p_X} X$, such that Fi_X is a cofibration and Fp_X is a weak equivalence, then $\mathcal{C}$ admits a model structure left-induced by F from that of $\mathcal{D}$.* □

Chapter 3

Model structures on simplicial sets

We now turn our attention to some specific examples of model structures, on the category of *simplicial sets* and related categories. Informally, simplicial sets may be thought of as complexes built out of simplices glued along common faces; formally they are defined as presheaves on a category Δ. In this chapter we will review some of the established theory of simplicial sets, with a focus on model structures presenting higher categories; in later chapters we will develop cubical analogues of these model structures and compare them with the simplicial versions described here.

Section 3.1 reviews the basic theory of simplicial sets, including the join construction and the nerve functor $N\colon \mathsf{Cat} \to \mathsf{sSet}$. Section 3.2 describes two model structures on the category of simplicial sets: the *Quillen model structure*, which models ∞-groupoids, and the *Joyal model structure*, which models $(\infty, 1)$-categories. Finally, in Section 3.3, we describe *simplicial sets with weak equivalences* and *marked simplicial sets*, simplicial sets equipped with distinguished sets of "marked" simplices, which model the theories of $(\infty,1)$- and (∞,n)-categories via the *marked model structure* and the *complicial model structures*, respectively.

3.1 The simplex category and simplicial sets

For $n \geq 0$, let $[n]$ denote the totally ordered set with $n+1$ elements, i.e. the poset $\{0 \leq \cdots \leq n\}$. Let Δ denote the full subcategory of the category of posets on these objects. The maps in Δ are generated (under composition in the category of posets) by two distinguished classes:

- *faces* $\partial_i^n \colon [n-1] \to [n]$ for $n \geq 1, 0 \leq i \leq n$, given by:

$$\partial_i^n(a) = \begin{cases} a & a \leq i-1 \\ a+1 & a \geq i \end{cases}$$

- *degeneracies* $\sigma_i^n \colon [n+1] \to [n]$ for $n \geq 0, 0 \leq i \leq n$, given by:

$$\sigma_i^n(a) = \begin{cases} a & a \leq i \\ a-1 & a \geq i+1 \end{cases}$$

For simplicity of notation, we will typically omit the superscript n when discussing these maps.

These maps obey the following *simplicial identities*:

$\partial_j \partial_i = \partial_{i+1} \partial_j$ for $j \leq i$;

$\sigma_i \sigma_j = \sigma_j \sigma_{i+1}$ for $j \leq i$;

$$\sigma_j \partial_i = \begin{cases} \partial_{i-1}\sigma_j & \text{for } j \leq i-2; \\ \mathrm{id} & \text{for } j \in \{i-1, i\}; \\ \partial_i \sigma_{j-1} & \text{for } j \geq i+1. \end{cases}$$

The following standard result can be proven using basic combinatorics.

Theorem 3.1.1. *Every map in the category Δ can be factored uniquely as a composite*

$$(\partial_{b_1} \cdots \partial_{b_q})(\sigma_{a_1} \cdots \sigma_{a_p}),$$

where $0 \leq a_1 < \cdots < a_p$ and $b_1 > \cdots > b_q \geq 0$. □

Corollary 3.1.2. *The category Δ admits the structure of an EZ Reedy category, in which:*

- $\deg([n]) = n$*;*
- Δ_+ *is generated under composition by the face maps;*
- Δ_- *is generated under composition by the degeneracy maps.* □

The category of simplicial sets, i.e. contravariant functors $\Delta^{\mathrm{op}} \to \mathsf{Set}$, will be denoted by sSet. The image of the object $[n]$ under a simplicial set X will be denoted X_n. We will write Δ^n for the representable simplicial set represented by $[n]$; by the Yoneda lemma, elements of X_n correspond to maps $\Delta^n \to X$.

We adopt the convention of writing the action of simplicial operators on the right; for instance, the 0-face of an n-simplex $x \colon \Delta^n \to X$ will be denoted $x\partial_0$. Simplices in the image of a degeneracy operator will be referred to as *degenerate*.

Intuitively, we think of a simplicial set $X \colon \Delta^{\mathrm{op}} \to \mathsf{Set}$ as a complex made up of simplices joined along common faces, with the set $X_n = X([n])$ consisting of all n-simplices in this complex. Each face map $\partial_i \colon X_n \to X_{n-1}$ sends an n-simplex x to one of its $(n-1)$-dimensional faces; each degeneracy map $\sigma_i \colon X_n \to$

X_{n+1} sends an n-simplex x to an $(n+1)$-simplex obtained by viewing x as an $(n+1)$-simplex collapsed to a lower dimension.

Viewed from this perspective, Δ^n consists of a single n-simplex and all of its faces (plus degenerate simplices). We write $\partial\Delta^n \to \Delta^n$ for the maximal proper subobject of Δ^n, i.e., the union of all faces of the n-simplex. We will refer to Δ^n and $\partial\Delta^n$ as the *n-simplex* and the *boundary* of the n-simplex, respectively. The subobject of Δ^n given by the union of all faces except ∂_i will be denoted Λ^n_i and referred to as an i-horn. The horn Λ^n_i is *inner* if $0 < i < n$, and *outer* otherwise.

The *critical edge* of the n-simplex Δ^n with respect to a face ∂_i, where $i \in \{0, n\}$ is the edge $0 \to 1$ for $i = 0$ or $(n-1) \to n$ for $i = n$. In the standard form of Theorem 3.1.1, these respectively correspond to the maps $\partial_n\partial_{n-1}\cdots\partial_2$ and $\partial_{n-2}\partial_{n-3}\cdots\partial_0$. The critical edge of an outer horn Λ^n_i refers to the critical edge with respect to the face ∂_i.

For a simplicial set X, a horn in X is a map $x\colon \Lambda^n_i \to X$. A *filler* for such a horn is a simplex $\Delta^n \to X$ restricting to x on Λ^n_i; in other words, a lift in the diagram

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "x"] \arrow[d] & X \arrow[d] \\
\Delta^n \arrow[r] & \Delta^0
\end{tikzcd}$$

Bonudaries in simplicial sets, and fillers for boundaries, are defined similarly.

From Theorem 3.1.1 and Corollary 3.1.2, we obtain the following:

Proposition 3.1.3. *Given a simplicial set X, for any simplex $x\colon \Delta^n \to X$ there exists a unique (possibly empty) sequence $a_1 < \cdots < a_p$, and a unique non-degenerate simplex $y\colon \Delta^{n-p} \to X$ such that $x = y\sigma_{a_1}\cdots\sigma_{a_p}$.* □

This factorization is called the *standard form* of x.

Corollary 3.1.4. *A map $X \to Y$ in* sSet *is determined by its action on the non-degenerate simplices of X.* □

Corollary 3.1.5. *A map $X \to Y$ in* sSet *is a monomorphism if and only if it maps non-degenerate simplices of X to non-degenerate simplices of Y, and does so injectively.* □

We may observe that each poset $[n]$ is isomorphic to its opposite poset $[n]^{\mathrm{op}}$, with the isomporphism sending $i \in [n]$ to $n - i$. Thus we obtain an involution $(-)^{\mathrm{op}} \colon \Delta \to \Delta$, defined as follows:

- $[n]^{\mathrm{op}} = [n]$;
- $(\partial^n_i)^{\mathrm{op}} = \partial^n_{n-i}$;
- $(\sigma^n_i)^{\mathrm{op}} = \sigma^n_{n-i}$.

By left Kan extension, we obtain an endomorphism $(-)^{\mathrm{op}} \colon \mathsf{sSet} \to \mathsf{sSet}$.

$$\begin{tikzcd}
\Delta \arrow[r, "(-)^{\mathrm{op}}"] \arrow[d] & \Delta \arrow[r] & \mathsf{sSet} \\
\mathsf{sSet} \arrow[urr, "(-)^{\mathrm{op}}"'] & &
\end{tikzcd}$$

Some simple computations show:

Lemma 3.1.6. *The functor $(-)^{\mathrm{op}}$ is an involution of* sSet. □

In particular, for $X \in \mathsf{sSet}$, the simplices of X are in bijection with those of X^{op}; given $x \colon \Delta^n \to X$ we have a corresponding simplex $x^{\mathrm{op}} \colon \Delta^n = (\Delta^n)^{\mathrm{op}} \to X^{\mathrm{op}}$.

Another important construction in the theory of simplicial sets is the *join*. To define this construction, we will require some discussion of augmented simplicial sets.

The *augmented simplex category* Δ_a is the full subcategory of the category of posets on the objects $[n], n \geq -1$, where $[-1]$ denotes the empty poset $\varnothing$. Combinatorially, Δ_a can be obtained by adjoining an initial object to the simplex category Δ. The category of *augmented simplicial sets*, denoted sSet_a, is the presheaf category $\mathsf{Set}^{\Delta_a^{\mathrm{op}}}$. An augmented simplicial set can be viewed as a simplicial set X together with a set X_{-1} and a structure map $\partial_0 \colon X_{-1} \to X_0$, such that $x\partial_0\partial_0 = x\partial_1\partial_0$ for all $x \in X_1$.

Pre-composition with the inclusion $\Delta \hookrightarrow \Delta_a$ induces a forgetful functor $\mathsf{sSet}_a \to \mathsf{sSet}$. Moreover, we have an inclusion $\mathsf{sSet} \hookrightarrow \mathsf{sSet}_a$, regarding any simplicial set X as an augmented simplicial set by defining X_{-1} to be the one-element set. We thus obtain an adjunction $\mathsf{sSet}_a \rightleftarrows \mathsf{sSet}$.

Given any two posets P, Q, we can define the *join* $P \star Q$ as follows.

- $\mathsf{Ob}(P \star Q) = (\mathsf{Ob}P) \sqcup (\mathsf{Ob}Q)$.
- For $x, y \in \mathsf{Ob}(P \star Q)$, $x \leq y$ if and only if one of the following conditons holds:
  - $x, y \in P$ and $x \leq y$;
  - $x, y \in Q$ and $x \leq y$;
  - $x \in P$ and $y \in Q$.

This definition is functorial in P and Q. Moreover, for $m, n \geq -1$ we have $[m] \star [n] = [m + n + 1]$; thus this restricts to a bifunctor $\star \colon \Delta_a \times \Delta_a \to \Delta_a$. By left Kan extension, we obtain a bifunctor $\star \colon \mathsf{sSet}_a \times \mathsf{sSet}_a \to \mathsf{sSet}_a$, as depicted below.

$$\begin{tikzcd} \Delta_a \times \Delta_a \arrow[r] \arrow[d, hook] & \mathsf{sSet}_a \\ \mathsf{sSet}_a \times \mathsf{sSet}_a \arrow[ur, "\star"'] & \end{tikzcd}$$

Analyzing this construction, we obtain the following concrete description of the join.

Proposition 3.1.7. *For $X, Y \in \mathsf{sSet}_a$, the join $X \star Y$ may be described as follows. For $n \geq -1$ we have:*

$$(X \star Y)_n = \coprod_{i=-1}^{n} (X_i \times Y_{n-i-1})$$

Given a pair of simplices $x\colon \Delta^m \to X$, $y\colon \Delta^n \to Y$, where $m + n \geq 0$, the faces of the $(m+n+1)$-simplex (x, y) of $X \star Y$ are computed as follows. For $0 \leq i \leq m+n+1$ we have:

$$(x, y)\partial_i = \begin{cases} (x\partial_i, y) & \text{for } 0 \leq i \leq m; \\ (x, y\partial_{i-m-1}) & \text{for } m+1 \leq i \leq m+n+1; \end{cases}$$

Degeneracy maps are computed similarly. □

From the description above, we may observe that the join of simplicial sets, regarded as augmented simpilcial sets via the inclusion $\mathsf{sSet} \hookrightarrow \mathsf{sSet}_a$ described above, is again in the image of this inclusion. The join operation thus defines a bifunctor $\star\colon \mathsf{sSet} \times \mathsf{sSet} \to \mathsf{sSet}$. More precisely, this is the composite:

$$\mathsf{sSet} \times \mathsf{sSet} \to \mathsf{sSet}_a \times \mathsf{sSet}_a \to \mathsf{sSet}_a \to \mathsf{sSet}$$

From here on, we concern ourselves only with the join of simplicial sets, rather than augmented simplicial sets; in this context it should be understood that Δ^{-1} denotes $\varnothing$ and X_{-1} denotes the one-element set for any $X \in \mathsf{sSet}$.

For any $X \in \mathsf{sSet}$, the functor $X \star -\colon \mathsf{sSet} \to \mathsf{sSet}$ admits a natural transformation from the identity, sending an n-simplex $y \in Y_n$ to $(*, y) \in X \star Y$;

likewise, $- \star X$ admits a natural transformation from the identity, defined similarly.

We note some basic facts about the join operation, which follow easily from its definition.

Lemma 3.1.8. *The join construction satisfies the following properties.*

(i) A join of simplices is again a simplex: for any $m, n \geq 0$ we have $\Delta^m \star \Delta^n \cong \Delta^{m+n+1}$.

(ii) Taking the join with the empty simplicial set is the identity: for any $X \in \mathsf{sSet}$ we have $\varnothing \star X \cong X \cong X \star \varnothing$.

(iii) The join construction is not symmetric: in general, for $X, Y \in \mathsf{sSet}$ we do not have $X \star Y \cong Y \star X$. □

(iv) For any $X, Y \in \mathsf{sSet}$ we have $(X \star Y)^{\mathrm{op}} \cong Y^{\mathrm{op}} \star X^{\mathrm{op}}$.

We conclude this section by defining the nerve functor, one of the key tools in the theory of simplicial sets.

The inclusion $\Delta \hookrightarrow \mathsf{Cat}$ defines a cosimplicial object in Cat. Taking the left Kan extension along the Yoneda embedding $\Delta \hookrightarrow \mathsf{sSet}$, we obtain a functor $\tau_1 \colon \mathsf{sSet} \to \mathsf{Cat}$, as illustrated below.

$$\begin{tikzcd}
\Delta \arrow[r] \arrow[d, hook] & \mathsf{Cat} \\
\mathsf{sSet} \arrow[ur, "\tau_1"'] &
\end{tikzcd}$$

The functor τ_1 takes a simplicial set X to its *fundamental category*, which is obtained as the quotient of the free category on the directed graph $X_1 \rightrightarrows X_0$

modulo the relations: $x\sigma_0 = \mathrm{id}_x$ for $x \in X_0$ and $gf = h$ for every 2-simplex

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"] & & \bullet \end{tikzcd}$$

This functor has a right adjoint $N\colon \mathsf{Cat} \to \mathsf{sSet}$, given by $(N\mathcal{C})_n = \mathsf{Cat}([n], \mathcal{C})$, with simplicial structure maps induced by pre-composition with the corresponding maps in Δ. We refer to N as the *nerve functor*.

Lemma 3.1.9. *We have natural isomorphisms* $\tau_1 \circ (-)^{\mathrm{op}} \cong (-)^{\mathrm{op}} \circ \tau_1$ *in* Cat *and* $N \circ (-)^{\mathrm{op}} \cong (-)^{\mathrm{op}} \circ N$ *in* sSet.

Proof. By adjointness, it suffices to prove the assertion for $\tau_1 \circ (-)^{\mathrm{op}}$ and $(-)^{\mathrm{op}} \circ \tau_1$. For this, in turn, it suffices to prove that we have the desired natural isomorphism for representable simplicial sets; this follows from the existence of the natural isomorphism $[n] \cong [n]^{\mathrm{op}}$ described above. $\square$

3.2 Homotopy theory of simplicial sets

Next we consider two model structures on sSet which are of fundamental importance in higher category theory. We will construct both of these model structures using Cisinski-Olschok theory, although both were originally constructed by other means.

Definition 3.2.1. Let Mono denote the class of monomorphisms in sSet, and let Tfib denote the class of maps in sSet having the right lifting property with respect to the monomorphisms.

Both of these model structures will have Mono and Tfib as their cofibrations and trivial fibrations, respectively; thus we begin with the following general results.

Lemma 3.2.2. *The classes* $(\mathsf{Mono}, \mathsf{Tfib})$ *form a cofibrantly generated weak factorization system on* sSet*, with the set of boundary inclusions as a cellular model and all objects cofibrant.*

Proof. This follows from Theorem 2.2.3, Proposition 2.2.25, and Corollary 3.1.2. □

When viewing simplicial sets (or more precisely, those which are fibrant in the model structures to be constructed) as ∞-categories, we think of vertices as objects and higher-dimensional simplices as higher cells. Composition of cells corresponds to filling inner horns. For instance, consider a composable pair of arrows $x \xrightarrow{f} y \xrightarrow{g} z$ in a simplicial set X; this corresponds to an inner horn $\Lambda^2_1 \to X$. A filler for this horn is a 2-simplex $\alpha \colon \Delta^2 \to X$ of the form:

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\alpha"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z \end{tikzcd}$$

Viewing this 2-simplex as a diagram commuting up to homotopy in an ∞-category, it witnesses h as a (not necessarily unique) composite of f with g.

Degenerate simplices correspond to identities in this framework; in particular, for $x \colon \Delta^0 \to X$, the degenerate edge $x\sigma_0$ corresponds to the identity 1-cell on the objet x. Note that for any edge $x \xrightarrow{f} y$ in X, the degenerate 2-simplices $f\sigma_0$ and $f\sigma_1$ witness f as a composite of $x\sigma_0$ with f and f with $y\sigma_0$, respectively. In view of this correspondence, when illustrating simplicial sets, we will represent degenerate edges with = symbols.

Definition 3.2.3. Let J denote the simplicial set depicted below:

$$\begin{tikzcd}
1 \arrow[r] \arrow[dr, equal] & 0 \arrow[d] \arrow[dr, equal] & \\
& 1 \arrow[r] & 0
\end{tikzcd}$$

An *equivalence* in a simplicial set X is an edge $\Delta^1 \to X$ which factors through the inclusion of the middle edge $\Delta^1 \to J$. Viewing degenerate edges as identities, such an edge has left and right inverses witnessed by the images in X of the two non-degenerate 2-simplices of J; thus these are the edges which correspond to invertible 1-cells.

With this motivation in mind, we now construct the desired model structures.

Example 3.2.4. Taking the product with the 1-simplex Δ^1 defines a cylinder functor on sSet, with the natural transformations ∂^0, ∂^1 given by taking the product with the corresponding face maps $\Delta^0 \to \Delta^1$ (note that this means each ∂_ε is given by the inclusion of the vertex $1-\varepsilon$), and the natural transformation σ given by taking the product with $\sigma_0 \colon \Delta^1 \to \Delta^0$. Applying Theorem 2.2.14 with this cylinder functor, the weak factorization system $(\mathsf{Mono}, \mathsf{Tfib})$, the cellular model of Lemma 3.2.2, and $S = \{\Lambda^n_i \hookrightarrow \Delta^n | n \geq 1, 1 \leq i \leq n\}$ (the set of all horn inclusions), we obtain the *Quillen model structure* on sSet, characterized as follows:

- Cofibrations are monomorphisms;
- Fibrant objects are *Kan complexes*, simplicial sets having fillers for all horns;

- Fibrations are characterized by the right lifting property with respect to the horn inclusions;
- Weak equivalences are *weak homotopy equivalences*, maps $X \to Y$ inducing bijections $[Y, Z] \to [X, Z]$ for all Kan complexes Z.

Note that the characterization of fibrations is not immediate from Theorem 2.2.14. This model structure, and the characterization of its fibrations, are due to Quillen [Qui67]. The details of the construction using Cisinski-Olschok theory can be found in [Cis19, Sec. 3.1], in which the characterization of the fibrations appears as [Cis19, Thm. 3.1.29].

Proposition 3.2.5 (Quillen, [Hov99, 4.2.8])**.** *The Quillen model structure is monoidal with respect to the cartesian monoidal structure on* sSet. □

Example 3.2.6. Taking the product with J defines a cylinder functor on sSet, with the natural transformations ∂^0, ∂^1 given by taking the product with the endpoint inclusions $\{1\} \hookrightarrow J, \{0\} \hookrightarrow J$ (i.e., the composites of the corresponding face maps $\Delta^0 \to \Delta^1$ with the middle edge inclusion $\Delta^1 \to J$), and the natural transformation σ given by taking the product with the map $J \to \Delta^0$. Applying Theorem 2.2.14 with this cylinder functor, the weak factorization system $(\mathsf{Mono}, \mathsf{Tfib})$, the cellular model of Lemma 3.2.2, and $S = \{\Lambda^n_i \hookrightarrow \Delta^n | n \geq 2, 1 < i < n\}$ (the set of inner horn inclusions), we obtain the *Joyal model structure* on sSet, characterized as follows:

- Cofibrations are monomorphisms;
- Fibrant objects are *quasicategories*, simplicial sets having fillers for all inner horns;

- Fibrations between fibrant objects are characterized by the right lifting property with respect to the inner horn inclusions and the endpoint inclusions $\{\varepsilon\} \hookrightarrow J, \varepsilon \in \{0,1\}$;
- Weak equivalences are *weak categorical equivalences*, maps $X \to Y$ inducing bijections $[Y,Z] \to [X,Z]$ for all quasicategories Z.

This model structure, and the characterization of fibrations between fibrant objects, are due to Joyal [Joy09]; for the details of its construction via Cisinski-Olschok theory, see [Cis19, Sec. 3.3], in which the characterization of fibrations between fibrant objects appears as [Cis19, Thm. 3.6.1].

Definition 3.2.7. We will refer to homotopy equivalences between fibrant objects in the Quillen and Joyal model structures as *homotopy equivalences* and *categorical equivalences*, respectively.

Proposition 3.2.8 ([Joy09, Thm. 6.12])**.** *The Joyal model structure is monoidal with respect to the cartesian monoidal structure on* sSet. □

Before considering applications of these model structures to higher category theory, we state a basic result showing that they are compatible with the involution $(-)^{\mathrm{op}}$; we will see analogues of this result in many other model structures in later sections.

Proposition 3.2.9. *The adjunction* $(-)^{\mathrm{op}} : \mathsf{sSet} \rightleftarrows \mathsf{sSet} : (-)^{\mathrm{op}}$ *defines a Quillen self-equivalence of both the Quillen and Joyal model structures.*

Proof. By Corollary 2.1.39, it suffices to show that the adjunction is Quillen when the domain and codomain are both equipped with either the Quillen or the Joyal model structure. It is clear that $(-)^{\mathrm{op}}$ preserves monomorphisms, which are the cofibrations in both model structures. That the adjunction is

Quillen thus follows from Corollary 2.1.36, together with the fact that $(-)^{\mathrm{op}}$ sends (inner) horn inclusions to (inner) horn inclusions. $\square$

The Quillen model structure models the homotopy theory of ∞-groupoids, while the Joyal model structure models the homotopy theory of $(\infty, 1)$-categories. That is to say, Kan complexes (resp. quasicategories) can be thought of as ∞-groupoids (resp. $(\infty, 1)$-categories), with vertices corresponding to objects and higher-dimensional simplices corresponding to higher cells, as described above.

The following results further illustrate this correspondence.

Proposition 3.2.10 ([Joy02, Thm. 1.3])**.** *Let X be a quasicategory, and $x\colon \Lambda^n_i \to X$ an outer horn in X. If the critical edge of x is an equivalence, then x admits a filler.* $\square$

Corollary 3.2.11. *A quasicategory X is a Kan complex if and only if all of its edges are equivalences.*

Proof. To see that every edge in a Kan complex is an equivalence, observe that the inclusion of the middle edge into J is a composite of (outer) horn-fillings, hence a trivial cofibration in the Quillen model structure; thus every edge of a Kan complex factors through J. On the other hand, a quasicategory in which all edges are equivalences admits fillers for all horns by Proposition 3.2.10. $\square$

We can further study the homotopy theory of quasicategories by means of the following constructions.

Definition 3.2.12. Let $X \in \mathsf{sSet}$ be a quasicategory. For $f, g\colon \Delta^1 \to X$, the edges f and g are *homotopic*, or $f \sim g$, if there exists 2-simplex in X of the form:

$$\begin{tikzcd} & y \arrow[dr, equal] & \\ x \arrow[ur, "f"] \arrow[rr, "g"] & & y \end{tikzcd}$$

We now record some basic results on this homotopy relation, which follow from elementary exercises in horn-filling. These results can also be found in [Cis19, Sec. 1.6]. Our first such result shows that, while the conditions on the simplex used in the definition of the homotopy relation may seem arbitrary, the other natural choices produce the same relation.

Lemma 3.2.13. *Let X be a quasicategory with edges $f, g\colon \Delta^1 \to X$, and consider the following boundaries in X:*

$$\begin{tikzcd} & \bullet \arrow[dr, equal] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "g"] & & \bullet \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[dr, equal] & \\ \bullet \arrow[ur, "g"] \arrow[rr, "f"] & & \bullet \end{tikzcd}$$

$$\begin{tikzcd} & \bullet \arrow[dr, "f"] & \\ \bullet \arrow[ur, equal] \arrow[rr, "g"] & & \bullet \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, equal] \arrow[rr, "f"] & & \bullet \end{tikzcd}$$

Each of these boundaries admits a filler if and only if all of the others do. □

Lemma 3.2.14. *The relation $\sim$ on the edges of a quasicategory is an equivalence relation.* □

Lemma 3.2.15. *Composition of edges in a quasicategory X is well-defined up to homotopy. That is, if $x, x'\colon \Delta^2 \to x$ are fillers for a horn $\Lambda^2_1 \to X$, then $x\partial_1 \sim x'\partial_1$.* □

Lemma 3.2.16. *Composition of edges in a quasicategory X is associative up to homotopy. That is, given a string of edges $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$, for any choice of composites we have $(h \circ g) \circ f \sim h \circ (g \circ f)$.* □

Lemma 3.2.17. *The homotopy relation on edges in a quasicategory X is compatible with composition. That is, given edges $f \sim g \colon x \to y$, $p \colon w \to x$, and $q \colon y \to z$ in X, for any choice of composites we have $q \circ f \circ p \sim q \circ g \circ p$.* □

These results allow us to define the homotopy category of a quasicategory as follows.

Definition 3.2.18. Let $X \in \mathsf{sSet}$ be a quasicategory. The *homotopy category* of X, denoted $\mathsf{Ho}X$, is defined as follows:

- the objects of $\mathsf{Ho}X$ are the 0-simplices of X;
- the morphisms from x to y in $\mathsf{Ho}X$ are the homotopy classes of edges in X.
- the identity map on $x \in X_0$ is given by $x\sigma_0$;
- the composition of $f \colon x \to y$ and $g \colon y \to z$ is given by filling the horn

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dotted, "gf"] & & z \end{tikzcd}$$

By choosing composites for strings of edges in a quasicategory and applying Lemmas 3.2.15 to 3.2.17, we obtain the following result.

Lemma 3.2.19. *Let X be a quasicategory. Then we have an isomorphism $\tau_1 X \cong \mathsf{Ho}X$, natural in X.* □

Lemma 3.2.20. *Let X be a quasicategory and $f\colon \Delta^1 \to X$ and edge of X. The homotopy class of f is an isomorphism in $\mathsf{Ho}X$ if and only if f is an equivalence.*

Proof. Immediate from the definition of $\mathsf{Ho}X$. □

Corollary 3.2.21. *A quasicategory X is a Kan complex if and only if $\mathsf{Ho}X$ is a groupoid.*

Proof. Immediate from Corollary 3.2.11 and Lemma 3.2.20. □

Definition 3.2.22. Let x_0 and x_1 be 0-simplices in a simplicial set X. The *mapping space* from x_0 to x_1 is the simplicial set $\mathrm{Hom}_X(x_0, x_1)$ given by the following pullback.

$$\begin{tikzcd} \mathrm{Hom}_X(x_0,x_1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{\Delta^1} \arrow[d] \\ \Delta^0 \arrow[r, "(x_0,x_1)"] & X^{\partial\Delta^1} \end{tikzcd}$$

The mapping space admits the following concrete description:

$$\mathrm{Hom}_X(x_0,x_1)_n = \left\{ \Delta^n \times \Delta^1 \xrightarrow{s} X \;\middle|\; s \circ (\Delta^n \times \partial_{1-\varepsilon}) = x_\varepsilon \right\},$$

with simplicial operations induced by those of X.

From this description we can see that the simplices of $\mathrm{Hom}_X(x_0, x_1)$ are not simplices of X, but rather, maps from products of simplices into X. Thus it is often preferable to work with the *left* and *right mapping spaces* in a simplicial set, defined below.

Definition 3.2.23. Let $X \in \mathsf{sSet}, x_0, x_1\colon \Delta^0 \to X$. The *left mapping space* $\mathrm{Hom}^L_X(x_0, x_1)$ is defined by:

$$\mathrm{Hom}^L_X(x_0,x_1)_n = \left\{\Delta^{n+1} \xrightarrow{s} X \;\middle|\; s|_{\Delta^{\{0\}}} = x_0,\ s\partial_0 = x_1\right\},$$

with simplicial operations induced by those of X, meaning that the face map ∂_i of $\mathrm{Hom}^L_X(x_0,x_1)$ corresponds to the face map ∂_{i+1} of X, and similarly for degeneracies.

Similarly, the *right mapping space* $\mathrm{Hom}^R_X(x_0,x_1)$ is defined by:

$$\mathrm{Hom}^R_X(x_0,x_1)_n = \left\{\Delta^{n+1} \xrightarrow{s} X \;\middle|\; s\partial_{n+1} = x_0,\ s|_{\Delta^{\{n+1\}}} = x_1\right\},$$

with simplicial operations induced by those of X, meaning that the face map ∂_i of $\mathrm{Hom}^R_X(x_0,x_1)$ corresponds to the face map ∂_i of X, and similarly for degeneracies.

Some routine calculations show:

Lemma 3.2.24. *For $X \in \mathsf{sSet}, x_0, x_1 \colon \square^0 \to X$, we have natural isomorphisms:*

- $\mathrm{Hom}_X(x_0,x_1)^{\mathrm{op}} \cong \mathrm{Hom}_{X^{\mathrm{op}}}(x_1,x_0)$*;*
- $\mathrm{Hom}^L_X(x_0,x_1)^{\mathrm{op}} \cong \mathrm{Hom}^R_{X^{\mathrm{op}}}(x_1,x_0)$*;*
- $\mathrm{Hom}^R_X(x_0,x_1)^{\mathrm{op}} \cong \mathrm{Hom}^L_{X^{\mathrm{op}}}(x_1,x_0)$*.* □

We now consider results which clarify the relationship between the Quillen and Joyal model structures.

Proposition 3.2.25 ([Lur09a, Prop. 1.2.2.3], [Rez20, Prop. 34.2])**.** *Let X be a quasicategory, with vertices $x_0, x_1 \colon \Delta^0 \to X$. Then the mapping spaces $\mathrm{Hom}_X(x_0,x_1), \mathrm{Hom}^L_X(x_0,x_1), \mathrm{Hom}^R_X(x_0,x_1)$ are Kan complexes.* □

One of the key results in the homotopy theory of quasicategories is the following, which characterizes categorical equivalences as those maps between quasicategories which are fully faithful and essentially surjective up to homotopy, in a sense defined by the constructions we have just established.

Theorem 3.2.26 (Fundamental theorem of quasicategories, [Rez20, Props. 34.2 and 43.2])**.** *Let $f\colon X \to Y$ be a map between quasicategories. Then f is a categorical equivalence if and only if the following two conditions are satisfied:*

- $\mathsf{Ho} f\colon \mathsf{Ho} X \to \mathsf{Ho} Y$ *is an equivalence of categories;*
- *for $x_0, x_1\colon \Delta^0 \to X$, the induced map $\mathrm{Hom}_X(x_0, x_1) \to \mathrm{Hom}_Y(fx_0, fx_1)$ is a homotopy equivalence in the Quillen model structure.*

In Section 8.1, we will exhibit a new proof of this result using cubical sets.

3.3 Marked simplicial sets

To define marked cubical sets, we need to introduce a new category Δ^+, an enlargement of Δ. The category Δ^+ consists of objects of the form $[n]$ for $n \geq 0$, as well as objects $[n]_e$ for $n \geq 1$. The maps of Δ^+ are generated by the usual generating maps of Δ along with the following:

- $\varphi^n\colon [n] \to [n]_e$ for $n \geq 1$;
- $\zeta^n_i\colon [n+1]_e \to [n]$ for $n \geq 1, 0 \leq i \leq n$;

subject to the usual simplicial identities, plus the following:

$\zeta_i \varphi = \sigma_i$;

$\sigma_i \zeta_j = \sigma_j \zeta_{i+1}$ for $j \leq i$.

A *structurally marked simplicial set* is a contravariant functor $X\colon (\Delta^+)^{\mathrm{op}} \to \mathsf{Set}$ and a morphism of structurally marked simplicial sets is a natural transformation of such functors. We will write sSet^{++} for the category of structurally marked simplicial sets. When working with the category of structurally marked simplicial sets, we will write X_n for the value of X at $[n]$ and eX_n for the value of X at $[n]_e$. The representable presheaf at the object $[n]$ will be denoted Δ^n, while the representable presheaf at the object $[n]_e$ will be denoted $\widetilde{\Delta}^n$.

Structurally marked simplicial sets should be thought of as simplicial sets with (possibly multiple) labels on their simplices of positive dimension, such that each degenerate simplex $x\sigma_i$ has, in particular, the distinguished label $x\zeta_i$. For $n \geq 1$, $\widetilde{\Delta}^n$ has Δ^n as its underlying simplicial set, with a unique marking on the unique non-degenerate n-simplex, while all other non-degenerate simplices are unmarked.

A *marked simplicial set* is a structurally marked simplicial set for which each map $eX_n \to X_n$ is a monomorphism. We write sSet^+ for the category of marked simplicial sets. Alternatively, we may view a marked simplicial set as a pair (X, eX) consisting of a simplicial set X together with a subset $eX \subseteq \bigcup_{n\geq 1} X_n$ of simplices of positive dimension that includes all degenerate simplices, with a morphism of marked simplicial sets being a map of simplicial sets that preserves marked simplices.

Let Δ' denote the full subcategory of Δ^+ on the objects $[n]$ for $n \geq 0$ and $[1]_e$. A *simplicial set with weak equivalence structure* is a contravariant functor $X\colon (\Delta')^{\mathrm{op}} \to \mathsf{Set}$ and a morphism of simplicial sets with weak equivalence structure is a natural transformation of such functors. We will write sSet'' for the category of simplicial sets with weak equivalence structure. A *simplicial set with weak equivalences* is a simplicial set with weak equivalence structure for which the map $eX_1 \to X_1$ is a monomorphism. We will write sSet' for the category of simplicial sets with weak equivalences. Similarly to the above description of marked simplicial sets, we may think of a simplicial set with weak equivalences as a simplicial set X together with a subset of X_1, consisting of those edges considered to be marked, which includes all degenerate edges.

Note that the definition of a marked simplicial set given above does not coincide with that given by Lurie [Lur09a]; Lurie's marked simplicial sets are what we refer to here as simplicial sets with weak equivalences.

The forgetful functor taking a (structurally) marked simplicial set to its underlying simplicial set admits both a left and a right adjoint, given by the minimal and maximal marking respectively. The minimal marking on a simplicial set X, denoted $X^\flat$, marks exactly the degenerate simplices, whereas the maximal marking, denoted $X^\sharp$, marks all positive-dimensional simplices of X. If considered as structurally marked simplicial sets, the marked simplices of $X^\flat$ and $X^\sharp$ are marked exactly once.

The forgetful functor and its adjoints factor through sSet' in a natural way: for every (structurally) marked simplicial set X we can obtain an "underlying simplicial set with weak equivalence (structure)" whose underlying simplicial set is the same as that of X by forgetting about the markings on simplices of dimension greater than 1. The minimal marking of a simplicial set can be obtained by first marking degenerate edges to obtain a simplicial set with weak equivalences, and then marking all degenerate simplices to obtain a marked simplicial set; similarly, the maximal marking can be obtained by first marking all edges, and then marking all simplices of dimension greter than 1.

There is moreover an inclusion $\mathsf{sSet}^+ \to \mathsf{sSet}^{++}$. This inclusion admits a left adjoint taking $X \in \mathsf{sSet}^{++}$ to $\mathrm{Im}X$ given by $(\mathrm{Im}X)_n = X_n$ and $e(\mathrm{Im}X)_n = \varphi^*(eX_n)$, i.e., the image of eX_n under $\varphi^* = X(\varphi^n)$. The inclusion is easily seen to not have a right adjoint, since it fails to preserve the pushout of $\Delta^1 \to \widetilde{\Delta}^1$ against itself. Likewise, there is an inclusion $\mathsf{sSet}' \to \mathsf{sSet}''$ which admits a similarly-defined left adjoint $\mathrm{Im}\colon \mathsf{sSet}'' \to \mathsf{sSet}'$.

Altogether we obtain the following diagram of adjunctions:

$$
\begin{tikzcd}
\mathsf{sSet}^{++} \arrow[rr, bend left, "\mathrm{Im}"] \arrow[dr] \arrow[dd] & & \mathsf{sSet}^{+} \arrow[ll] \arrow[dl] \arrow[dd] \\
& \mathsf{sSet} \arrow[ul, bend left, "(-)^\flat"] \arrow[ul, bend right, "(-)^\sharp"'] \arrow[ur, bend right, "(-)^\flat"'] \arrow[ur, bend left, "(-)^\sharp"] \arrow[dl, bend right, "(-)^\flat"'] \arrow[dl, bend left, "(-)^\sharp"] \arrow[dr, bend left, "(-)^\flat"] \arrow[dr, bend right, "(-)^\sharp"'] & \\
\mathsf{sSet}'' \arrow[uu, bend left, "(-)^\flat"] \arrow[uu, bend right, "(-)^\sharp"'] \arrow[ur] \arrow[rr, "\mathrm{Im}"] & & \mathsf{sSet}' \arrow[ll, bend left] \arrow[ul] \arrow[uu, bend left, "(-)^\flat"] \arrow[uu, bend right, "(-)^\sharp"']
\end{tikzcd}
\tag{3.3.1}
$$

In the context of (structurally) marked simplicial sets, we regard a simplicial set with its minimal marking by default, writing X for $X^\flat$.

Definition 3.3.1. For each $n \geq 0$, we have a functor $\tau_n \colon \mathsf{sSet}^+ \to \mathsf{sSet}^+$, the *$n$-trivialization functor*, defined as follows: $X \in \mathsf{sSet}^+$, $\tau_n X$ is obtained from X by marking all simplices of dimension greater than n.

From the definition above, it is clear that τ_0 is the composite of the underlying simplicial set functor with the maximal marking functor. Moreover, for $m \leq n$ we have a commuting triangle of natural transformations

$$
\begin{tikzcd}
& \tau_n \arrow[dr, Rightarrow] & \\
\mathrm{id} \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] & & \tau_m
\end{tikzcd}
$$

Many of our constructions and results will be equally valid for (structurally) marked simplicial sets and simplicial sets with weak equivalence (structure); for the sake of efficiency, in such cases we will use the notations $\Delta^\bullet$ to denote either Δ' or Δ^+, $\mathsf{sSet}^\bullet$ to denote either sSet' or sSet^+, and $\mathsf{sSet}^{\bullet\bullet}$ to denote either sSet'' or sSet^{++}. Likewise, many of our constructions and results will be equally valid in both sSet' or sSet^+ and its structural counterpart; in these cases we will use the notation $\mathsf{sSet}'^{(\prime)}$, $\mathsf{sSet}^{+(+)}$, or $\mathsf{sSet}^{\bullet(\bullet)}$, as appropriate. (Of course, the interpretation of any of these ambiguous notations must be consistent within any given statement and its proof.)

Definition 3.3.2. Let $X \to Y$ be a map in $\mathsf{sSet}^\bullet$. This map is:

- *regular* if it creates markings, i.e. a simplex of X is marked if and only if its image in Y is marked;
- *entire* if the underlying simplicial set map is an isomorphism, i.e. Y is obtained from X by marking a (possibly empty) set of its unmarked simplices.

Proposition 3.3.3 ([OR20, Prop. C.4])**.** *The category Δ^+ is an EZ Reedy category with the Reedy structure defined as follows:*

- $\deg([0]) = 0$*,* $\deg([n]) = 2n - 1$ *for* $n \geq 1$*, and* $\deg([n]_e) = 2n$ *for* $n \geq 1$*;*
- $(\Delta^+)_+$ *is generated by the maps* ∂_i^n *and* φ^n *under composition;*
- $(\Delta^+)_-$ *is generated by the maps* σ_i^n *and* ζ_i^n *under composition.* □

Corollary 3.3.4. *The category Δ' is an EZ Reedy category with the Reedy structure defined by restricting that of Δ^+.*

Proof. In view of Proposition 3.3.3, it suffices to show the following:

- for any map f in Δ', the two maps in the factorization $f_+ f_-$ in Δ^+ are both contained in Δ';

- for any map σ in Δ'_{-}, all sections of σ are contained in Δ'.

Both of these statements are easily verified. □

Lemma 3.3.5. *The categories* sSet' *and* sSet^+ *are locally presentable.*

Proof. This follows from Proposition 2.2.7, together with the fact that the inclusions $\mathsf{sSet}^{\bullet} \hookrightarrow \mathsf{sSet}^{\bullet\bullet}$ admit left adjoints. □

Observe that we may extend the functor $(-)^{\mathrm{op}}\colon \Delta \to \Delta$ of Section 3.1 to obtain an involution $(-)^{\mathrm{op}}\colon \Delta^+ \to \Delta^+$, by having this functor act as the identity on the objects $[n]_e$, and on the additional generating morphisms of Δ^+ as follows:

- $(\varphi^n)^{\mathrm{co}} = \varphi^n$;
- $(\zeta^n_i)^{\mathrm{co}} = \zeta^n_{n-i}$;

It is clear that these functors restrict to involutions of Δ'. By left Kan extension we obtain an involution $(-)^{\mathrm{op}}\colon \mathsf{sSet}^{\bullet\bullet} \to \mathsf{sSet}^{\bullet\bullet}$, which restricts to an involution of $\mathsf{sSet}^{\bullet}$. Given $X^{\bullet} \in \mathsf{sSet}^{\bullet}$ with underlying simplicial set X, the underlying simplicial set of $(X^{\bullet})^{\mathrm{op}}$ is X^{op}, with a simplex $x^{\mathrm{op}}\colon \Delta^n \to (X^{\bullet})^{\mathrm{op}}$ marked if and only if x is marked in $X^{\bullet}$.

We may also extend the join operation to marked simplicial sets and simplicial sets with weak equivalences.

Definition 3.3.6. For $X, Y \in \mathsf{sSet}^{\bullet}$, the *join* of X and Y is the object $X \star Y \in \mathsf{sSet}^{\bullet}$ defined as follows:

- the underlying simplicial set of $X \star Y$ is the join of the underlying simplicial sets of X and Y;

- a simplex (x,y) of appropriate dimension is marked in $X \star Y$ if and only if either x is marked in X or y is marked in Y.

Once again, we obtain a bifunctor $\star\colon \mathsf{sSet}^{\bullet} \times \mathsf{sSet}^{\bullet} \to \mathsf{sSet}^{\bullet}$, and the basic properties of Lemma 3.1.8 hold in the marked setting as well. Furthermore, the natural inclusions $X \hookrightarrow X \star Y \hookleftarrow Y$ are regular.

Next we consider the construction of model structures on sSet' and sSet^{+}. As with the model structures of Section 3.2, we will construct these model structures using Cisinski-Olschok theory, although they were originally constructed by other means.

We begin by establishing the necessary weak factorization systems. As in the case of (unmarked) simplicial sets, we let Mono refer to the class of monomorphisms in $\mathsf{sSet}^{\bullet(\bullet)}$, and $\mathsf{Tfib} = r(\mathsf{Mono})$, relying on context to distinguish between the analogous classes in different categories.

Lemma 3.3.7. *The classes* $(\mathsf{Mono}, \mathsf{Tfib})$ *form a cofibrantly generated weak factorization system on* $\mathsf{sSet}^{\bullet(\bullet)}$*, with a cellular model given by the set:*

$$M = \{\partial\Delta^n \to \Delta^n \mid n \geq 0\} \cup \{\varphi\colon \Delta^n \to \widetilde{\Delta}^n | [n]_e \in \Delta^{\bullet}\}$$

Proof. For $\mathsf{sSet}^{\bullet\bullet}$, this follows from Theorem 2.2.3 and Proposition 2.2.25 together with either Proposition 3.3.3 or Corollary 3.3.4.

For $\mathsf{sSet}^{\bullet}$, it follows from Theorem 2.2.3 that $(l(r(M)), r(M))$ is a weak factorization system with cellular model M, and that $l(r(M))$ is the saturation of M; thus we must show that $l(r(M)) = \mathsf{Mono}$. To see this, first note that the class of monomorphisms is closed under pushouts, retracts, and transfinite composition, as this is true in sSet and a map in $\mathsf{sSet}^{\bullet}$ is a monomorphism if and only if its underlying simplicial set map is a monomorphism. Thus $l(r(M)) \subseteq \mathsf{Mono}$.

Furthermore, every monomorphism is in $l(r(M))$. To see this, let $X^\bullet \to Y^\bullet$ be a monomorphism in $\mathsf{sSet}^\bullet$; then the underlying simplicial set map $X \to Y$ is a monomorphism, hence a transfinite composite of pushouts of boundary inclusions. Thus $X^\flat \to Y^\flat$ is a transfinite composite of pushouts of boundary inclusions in $\mathsf{sSet}^\bullet$. Now consider the pushout of this map along the inclusion $X^\flat \to X^\bullet$.

$$\begin{tikzcd} X^\flat \arrow[r] \arrow[d] & X^\bullet \arrow[d] \\ Y^\flat \arrow[r] & X^\bullet \cup_{X^\flat} Y^\flat \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The map $X^\bullet \hookrightarrow X^\bullet \cup_{X^\flat} Y^\flat$ adds to $X^\bullet$ all additional simplices of $Y^\bullet$, without marking any of them. We can then mark all marked simplices of $Y^\bullet$ which are either not present or not marked in $X^\bullet$ by taking pushouts of the relevant maps $\varphi\colon \Delta^n \to \widetilde{\Delta}^n$. $\square$

Next we define certain maps in sSet' which will be among the pseudo-generating trivial cofibrations of the model structure on this category.

Definition 3.3.8. For $n \geq 1$ and $i \in \{0, n\}$, the n-dimensional *i-marked horn inclusion* is the morphism of simplicial sets with weak equivalences whose underlying simplicial set map is $\Lambda^n_i \hookrightarrow \Delta_n$, with the critical edge with respect to ∂_i marked in both the domain and codomain, and all other non-degenerate edges unmarked.

Definition 3.3.9. The *saturation map* is the inclusion $J \to J'$, where J' denotes the simplicial set J with the middle edge marked and all other edges unmarked, illustrated below.

$$\begin{tikzcd}
1 \arrow[r] \arrow[dr, equal] & 0 \arrow[d, "\sim"] \arrow[dr, equal] & \\
& 1 \arrow[r] & 0
\end{tikzcd}$$

Definition 3.3.10. For $i \in \{0,1,2\}$, the *i-two out of three map* is the inclusion $\Delta_i^{2\prime} \to (\Delta^2)^\sharp$, where $\Delta_i^{2\prime}$ denotes the simplicial set with weak equivalences whose underlying simplicial set is Δ^2, with ∂_i as its only unmarked edge. For instance, $\Delta_1^{2\prime}$ is illustrated below.

$$\begin{tikzcd}
& 1 \arrow[dr, "\sim"] & \\
0 \arrow[ur, "\sim"] \arrow[rr] & & 2
\end{tikzcd}$$

We are now able to construct the desired model structure on sSet'.

Example 3.3.11. Taking the product with the marked 1-simplex $\widetilde{\Delta}^1$ defines a cylinder functor on sSet', with the natural transformations ∂^0, ∂^1 given by taking the product with the endpoint inclusions $\{0\} \hookrightarrow \widetilde{\Delta}^1, \{1\} \hookrightarrow \widetilde{\Delta}^1$. We may apply Theorem 2.2.14 with this cylinder functor, the weak factorization system of Lemma 3.3.7, and the set S consisting of the inner horn inclusions, the marked outer horn inclusions, the saturation map, and the two-out-of-three maps, we obtain the *marked model structure* on sSet', characterized as follows:

- Cofibrations are monomorphisms;
- Fibrant objects are *marked quasicategories*, simplicial sets having fillers for all inner and marked outer horns;

- Fibrations between fibrant objects are characterized by the right lifting property with respect to the inner and marked outer horn inclusions, the saturation map, and the two-out-of-three maps;
- Weak equivalences are maps $X \to Y$ inducing bijections $[Y, Z] \to [X, Z]$ for all marked quasicategories Z.

For more on this model structure, see [Lur09a, Sec. 3.1], in which it appears as a special case of a model structure on $\mathsf{sSet}' \downarrow X$ for an arbitrary simplicial set with weak equivalences X.

The following result exhibits the marked model structure as a model for the theory of $(\infty, 1)$-categories.

Theorem 3.3.12. *The minimal marking and underlying cubical set functors define a Quillen equivalence between the Joyal model structure on* sSet *and the marked model structure on* sSet'.

Proof. This is a special case of [Lur09a, Thm. 3.1.5.1(A0)], taking $S = \Delta^0$ in the statement of that result. ☐

Proposition 3.3.13. *The marked model structure is monoidal with respect to the cartesian monoidal structure on* sSet'.

Proof. This is a special case of [Lur09a, Cor. 3.1.4.3], taking $S = T = \Delta^0$ in the statement of that result. ☐

Next we will construct a family of model structures on sSet^+ which model the theory of (∞, n)-categories for each $n \in \{0, \ldots, \infty\}$. We begin by defining the horns which will model composition of higher morphisms.

Definition 3.3.14. For $n \geq 1$ and $0 \leq i \leq n$, the *i-complicial simplex* in dimension n, denoted Δ^n_i, is the marked simplicial set defined as follows:

- the underlying simplicial set of Δ^n_i is Δ^n;
- a non-degenerate m-simplex of Δ^n is marked in Δ^n_i if and only if its standard form, when viewed as a composite face map $\Delta^m \to \Delta^n$, does not contain any map ∂_j with $j \in \{i-1, i, i+1\}$.

The *i-complicial horn* in dimension n, denoted Λ^n_i, is the regular subcomplex of Δ^n_i whose underlying simplicial set is the n-dimensional i-horn. The marked simplicial set $(\Delta^n_i)'$ is obtained from Δ^n_i by marking all $(n-1)$-simplices except for ∂_i.

The *i-complicial horn inclusion* is the inclusion $\Lambda^n_i \hookrightarrow \Delta^n_i$. For $n \geq 2$, the *elementary i-complicial marking extension* is the entire map $(\Delta^n_i)' \to \tau_{n-2}\Delta^n_i$.

We let Δ^3_{eq} denote the marked simplicial set whose underlying simplicial set is Δ^3, and whose non-degenerate marked simplices consist of all non-degenerate 2-simplices, together with the 1-simplices $0 \to 2$ and $1 \to 3$. The *elementary saturation map* is the entire map $\Delta^3_{\mathrm{eq}} \to (\Delta^3)^\sharp$. In general, a *saturation map* is any map of the form $\Delta^n \star \Delta^3_{\mathrm{eq}} \to \Delta^n \star (\Delta^3)^\sharp$ for $n \geq -1$. (Here Δ^{-1} denotes $\varnothing$, so that the elementary saturation map is a saturation map.)

We are now ready to construct the desired model structures on sSet^+. Before doing so, however, we will define some terminology that will allow us to describe their fibrant objects efficiently.

Definition 3.3.15. A *complicial set* is a marked simplicial set having the right lifting property with respect to all complicial horn-fillings and elementary complicial marking extensions. A complicial set is:

- *saturated* if it has the right lifting property with respect to all saturation maps;

- *n-trivial*, for $n \geq 0$, if it has the right lifting property with respect to all markings $\Delta^m \to \widetilde{\Delta}^m$ for $m > n$ (in other words, if all of its simplices of dimension greater than n are marked).

Example 3.3.16. Taking the product with the marked 1-simplex $\widetilde{\Delta}^1$ defines a cylinder functor on sSet^+, with the natural transformations ∂^0, ∂^1 given by taking the product with the endpoint inclusions $\{0\} \hookrightarrow \widetilde{\Delta}^1, \{1\} \hookrightarrow \widetilde{\Delta}^1$. We will apply Theorem 2.2.14 with this cylinder functor and the weak factorization system of Lemma 3.3.7. The set S will consist of the following classes of maps:

(i) the complicial horn-fillings $\Lambda^n_i \hookrightarrow \Delta^n_i$;

(ii) the elementary complicial marking extensions $(\Delta^n_i)' \to \tau_{n-2}\Delta^n$;

together with either, both, or neither of the following:

(iii) the saturation maps $\Delta^n \star \Delta^3_{\mathrm{eq}} \to \Delta^n \star (\Delta^3)^\sharp$;

(iv) the marking maps $\Delta^m \to \widetilde{\Delta}^m$ for all m greater than some fixed $n \geq 0$.

Taking only (i) and (ii), we obtain the *complicial model structure* on sSet^+; if in addition we include (iii), (iv), or both in S, we obtain the *saturated*, *n-trivial*, or *saturated n-trivial* complicial model structures.

These model structures can be characterized as follows:

- Cofibrations are monomorphisms;

- Fibrant objects are (saturated, n-trivial) complicial sets;

- Fibrations between fibrant objects are characterized by the right lifting property with respect to the maps in S;

- Weak equivalences are maps $X \to Y$ inducing bijections $[Y, Z] \to [X, Z]$ for all (saturated, n-trivial) complicial sets Z.

For more on these model structures, see [OR20].

Although the definition of the saturation maps may seem somewhat arbitrary in that we take the join with Δ^n only on the left and not on the right, the following result shows that a more general construction would still give the same model structure.

Proposition 3.3.17 ([OR20, Rmk. 1.20])**.** *For all $m, n \geq -1$ (interpreting Δ^{-1} as $\varnothing$), the map $\Delta^m \star \Delta^3_{\mathrm{eq}} \star \Delta^n \to \Delta^m \star (\Delta^3)^\sharp \star \Delta^n$ is a trivial cofibration in the (n-trivial) saturated complicial model structure on sSet^+.* □

For $n \geq 0$, the n-trivial complicial model structures (saturated and unsaturated) model the theory of (∞, n)-categories, while the complicial model structures with no triviality properties model the theory of (∞, ∞)-categories. The key difference between the saturated and unsaturated model structures is this: in any complicial set, saturated or otherwise, every marked simplex corresponds to an invertible morphism, but we need saturation to ensure that every simplex corresponding to an invertible morphism is marked.

Proposition 3.3.18. *The adjunction $(-)^{\mathrm{op}} : \mathsf{sSet}^+ \rightleftarrows \mathsf{sSet}^+ : (-)^{\mathrm{op}}$ is a Quillen self-equivalence of each of the model structures of Example 3.3.16.*

Proof. By Corollary 2.1.39, it suffices to show that $(-)^{\mathrm{op}}$ is a left Quillen functor. For this, it suffices to show that it preserves the classes of complicial horn inclusions, complicial marking extensions, saturation maps, and markers. For saturation maps, this follows from Proposition 3.3.17 and Lemma 3.1.8 (iv), together with the fact that $L^{\mathrm{op}} \to (L')^{\mathrm{op}}$ is isomorphic to $L \to L'$. For the other three classes, it is immediate from the definitions. □

For our purposes, it will often be more convenient to work with an alternative to the saturation maps.

Definition 3.3.19. Let $L \subset \Delta^3_{\mathrm{eq}}$ denote the regular subset of Δ^3_{eq} whose underlying simplicial set consists of the faces ∂_0 and ∂_3 of Δ^3. More conretely, L is the marked simplicial set illustrated below:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[dr, "\sim"'] \arrow[dr, phantom, "\sim", shift left=3] & \bullet \arrow[d] \arrow[dr, "\sim"] \arrow[dr, phantom, "\sim"', shift right=3] & \\
& \bullet \arrow[r] & \bullet
\end{tikzcd}$$

Let $L' = \tau_0 L$, i.e. the simplicial set obtained by marking the three umarked 1-simplices of L. The *elementary Rezk map* is the entire map $L \to L'$. In general, a *Rezk map* is any map of the form $\Delta^n \star L \to \Delta^n \star L'$ for $n \geq -1$.

Lemma 3.3.20. *In each diagram of inclusions*

$$\begin{tikzcd}
\Delta^n \star L \arrow[r] \arrow[d] & \Delta^n \star \Delta^3_{\mathrm{eq}} \arrow[d] \\
\Delta^n \star L' \arrow[r] & \Delta^n \star (\Delta^3)^\sharp
\end{tikzcd}$$

for $n \geq -1$, the horizontal maps are complicial.

Proof. Both inclusions $L \hookrightarrow \Delta^3_{\mathrm{eq}}$ and $L' \hookrightarrow (\Delta^3)^\sharp$ can be written as composites of pushouts of complicial horn inclusions and elementary complicial marking extensions, proving the statement for $n = -1$. The statement thus follows from [Ver08b, Lemma 39], which shows that taking the join with a fixed object preserves complicial maps. □

Corollary 3.3.21. *Every Rezk map is a trivial cofibration in the model structures for (n-trivial) saturated complicial sets.*

Proof. This is immediate from Lemma 3.3.20 and the two-out-of-three property. □

Definition 3.3.22. Let $[n] \in \Delta$ and let $0 \leq p, q \leq n$ be such that $p+q = n$. Then we write $\amalg_1^{p,q}\colon [p] \to [n]$ for the simplicial operator $i \mapsto i$, and $\amalg_2^{p,q}\colon [q] \to [n]$ for the operator $i \mapsto p + i$.

Definition 3.3.23. Let $X, Y \in \mathsf{sSet}^+$, let $(x, y) \in X_n \times Y_n$ be a simplex of $X \times Y$, and let $0 \leq i \leq n$. We say that (x, y) is *i-cloven* if *either* $x\, \amalg_1^{i,n-i}$ is marked in X or $y\, \amalg_2^{i,n-i}$ is marked in Y. We say that (x, y) is *fully cloven* if it is i-cloven for all $0 \leq i \leq n$.

The *Gray tensor product* of X and Y, denoted $X \otimes Y$, is defined to be the marked simplicial set with underlying simplicial set $X \times Y$, where a simplex $(x, y) \in X_n \times Y_n$ is marked if and only if it is fully cloven.

Theorem 3.3.24 ([Ver08a, Lem. 131])**.** *The Gray tensor product endows* sSet^+ *with a (nonsymmetric) monoidal structure, such that the forgetful functor* $(\mathsf{sSet}^+, \otimes) \to (\mathsf{sSet}, \times)$ *is strict monoidal.*

Although the monoidal structure described above is not closed, and therefore cannot be used to define monoidal model categories, we nevertheless have the following result.

Proposition 3.3.25. *In any of the model structures of Example 3.3.16, given a pair of cofibrations $i\colon A \to B, j\colon X \to Y$ in* sSet*, the pushout Gray tensor product $i \hat{\otimes} j\colon A \otimes Y \cup_{A\otimes X} B \otimes X \to B \otimes Y$ is a cofibration. Moreover, if either i or j is trivial then so is $i \hat{\otimes} j$.*

Proof. That the pushout Gray tensor product of cofibrations is a cofibration follows from the corresponding result for the cartesian product on sSet. The remainder of the statement is immediate from [ORV20, Cor. 2.3]. □

Definition 3.3.26. A *pre-complicial set* is a marked simplicial set X with the right lifting property with respect to the complicial marking extensions. These form a reflective subcategory of sSet^+ which we will denote $\mathsf{PreComp}$. We will denote the localization functor $X \mapsto X^{\mathrm{pre}}$; for $X \in \mathsf{sSet}^+$, the pre-complicial set X^{pre} will be referred to as the *pre-complicial reflection* of X.

Proposition 3.3.27 ([CKM20, Thm. 1.31])**.** *For every $X \in \mathsf{sSet}^+$, the unit map $X \to X^{\mathrm{pre}}$ is a trivial cofibration in all of the model structures of Example 3.3.16.* □

Chapter 4

Cubical sets

We now concern ourselves with the category cSet of *cubical sets*, structures analogous to simplicial sets which may be viewed as complexes assembled from cubes rather than simplices. Like simplicial sets, cubical sets are defined as presheaves on a certain category – in this case, the *box category*, denoted $\square$. In this chapter we will review the established theory of cubical sets, building towards original material which will be introduced in later chapters.

Section 4.1 reviews the basic theory of cubical sets, including the *geometric product*, a monoidal product on cubical sets which is often used in place of the cartesian product, and the *triangulation* adjunction $T : \mathsf{cSet} \rightleftarrows \mathsf{sSet} : U$, which is used to compare model structures on categories of cubical and simplicial sets. Section 4.2 describes the *Grothendieck model structure* on cSet, a cubical analogue of the Quillen model structure, due to Cisinski, which models the theory of ∞-groupoids. Finally, in Section 4.3, we describe *cubical sets with weak equivalences* and *marked cubical sets*. Much like simplicial sets with weak equivalences and marked simplicial sets, these are cubical sets in which some cubes are considered to be "marked", and they will later be used to model the theory of $(\infty, 1)$- and (∞, n)-categories.

4.1 The box category and cubical sets

We begin by defining the box category $\square$. The objects of $\square$ are posets of the form $[1]^n$ and the maps are generated (inside the category of posets) under composition by the following four special classes:

- *faces* $\partial^n_{i,\varepsilon} \colon [1]^{n-1} \to [1]^n$ for $i = 1, \ldots, n$ and $\varepsilon = 0, 1$ given by:

$$\partial^n_{i,\varepsilon}(x_1, x_2, \ldots, x_{n-1}) = (x_1, x_2, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-1});$$

- *degeneracies* $\sigma^n_i \colon [1]^n \to [1]^{n-1}$ for $i = 1, 2, \ldots, n$ given by:

$$\sigma^n_i(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n);$$

- *negative connections* $\gamma^n_{i,0} \colon [1]^n \to [1]^{n-1}$ for $i = 1, 2, \ldots, n-1$ given by:

$$\gamma^n_{i,0}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, \max\{x_i, x_{i+1}\}, x_{i+2}, \ldots, x_n).$$

- *positive connections* $\gamma^n_{i,1} \colon [1]^n \to [1]^{n-1}$ for $i = 1, 2, \ldots, n-1$ given by:

$$\gamma^n_{i,1}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, \min\{x_i, x_{i+1}\}, x_{i+2}, \ldots, x_n).$$

These maps obey the following *cubical identities*:

$$\partial_{j,\varepsilon'}\partial_{i,\varepsilon} = \partial_{i+1,\varepsilon}\partial_{j,\varepsilon'} \text{ for } j \leq i;$$

$$\sigma_j\partial_{i,\varepsilon} = \begin{cases} \partial_{i-1,\varepsilon}\sigma_j & \text{for } j < i; \\ \mathrm{id} & \text{for } j = i; \\ \partial_{i,\varepsilon}\sigma_{j-1} & \text{for } j > i; \end{cases}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_{i+1} \quad \text{for } j \leq i;$$

$$\gamma_{j,\varepsilon'}\gamma_{i,\varepsilon} = \begin{cases} \gamma_{i,\varepsilon}\gamma_{j+1,\varepsilon'} & \text{for } j > i; \\ \gamma_{i,\varepsilon}\gamma_{i+1,\varepsilon} & \text{for } j = i \\ & \text{and } \varepsilon' = \varepsilon; \end{cases} \qquad \sigma_j\gamma_{i,\varepsilon} = \begin{cases} \gamma_{i-1,\varepsilon}\sigma_j & \text{for } j < i; \\ \sigma_i\sigma_i & \text{for } j = i; \\ \gamma_{i,\varepsilon}\sigma_{j+1} & \text{for } j > i. \end{cases}$$

$$\gamma_{j,\varepsilon'}\partial_{i,\varepsilon} = \begin{cases} \partial_{i-1,\varepsilon}\gamma_{j,\varepsilon'} & \text{for } j < i-1; \\ \mathrm{id} & \text{for } j = i-1,\, i \\ & \text{and } \varepsilon = \varepsilon'; \\ \partial_{i,\varepsilon}\sigma_i & \text{for } j = i-1,\, i \\ & \text{and } \varepsilon' = 1-\varepsilon; \\ \partial_{i,\varepsilon}\gamma_{j-1,\varepsilon'} & \text{for } j > i; \end{cases}$$

Theorem 4.1.1 ([GM03, Thm. 5.1])**.** *Every map in the category $\square$ can be factored uniquely as a composite*

$$(\partial_{c_1,\varepsilon'_1}\cdots\partial_{c_r,\varepsilon'_r})(\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,\varepsilon_q})(\sigma_{a_1}\cdots\sigma_{a_p}),$$

where $1 \le a_1 < \cdots < a_p$, $1 \le b_1 \le \cdots \le b_q$, $b_i < b_{i+1}$ if $\varepsilon_i = \varepsilon_{i+1}$, and $c_1 > \cdots > c_r \ge 1$. □

Corollary 4.1.2. *The category $\square$ is an EZ Reedy category with the Reedy structure defined as follows:*

- $\deg([1]^n) = n$*;*
- $\square_+$ *is generated under composition by the face maps;*
- $\square_-$ *is generated under composition by the degeneracy and connection maps.* □

The category of cubical sets, i.e. contravariant functors $\square^{\mathrm{op}} \to \mathsf{Set}$, will be denoted by cSet. We will write $\square^n$ for the representable cubical set represented by $[1]^n$. As with simplicial operators, we adopt the convention of writing the action of cubical operators on the right. For instance, the $(1,0)$-face of an n-cube $x\colon \square^n \to X$ will be denoted $x\partial_{1,0}$. Cubes in the image of a degeneracy or connection operator will be referred to as *degenerate*.

We write $\partial\square^n \to \square^n$ for the maximal proper subobject of $\square^n$, i.e., the union of all of its faces. We will refer to these as the *n-box* and the *boundary* of the n-box, respectively. The subobject of $\square^n$ given by the union of all faces except $\partial_{i,\varepsilon}$ will be denoted $\sqcap^n_{i,\varepsilon}$ and referred to as an (i,ε)-open box.

We will occasionally represent cubical sets using pictures. In that, 0-cubes are represented as vertices, 1-cubes as arrows, 2-cubes as squares, and 3-cubes as cubes.

For a 1-cube f, we draw

$$x \xrightarrow{f} y$$

to indicate $x = f\partial_{1,0}$ and $y = f\partial_{1,1}$. For a 2-cube s, we draw

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[d, "f"'] & y \arrow[d, "g"] \\ z \arrow[r, "k"] & w \end{tikzcd}$$

to indicate $s\partial_{1,0} = f$, $s\partial_{1,1} = g$, $s\partial_{2,0} = h$, and $s\partial_{2,1} = k$. As for the convention when drawing 3-dimensional boxes, we use the following ordering of axes:

$$\begin{tikzcd} \cdot \arrow[rr, "1"] \arrow[dr, "3"] \arrow[dd, "2"'] & & \cdot \\ & \cdot & \\ \cdot & & \end{tikzcd}$$

For readability, we do not label 2- and 3-cubes. Similarly, if a specific 0-cube is irrelevant for the argument or can be inferred from the context, we represent it by $\bullet$, and we omit labels on edges whenever the label is not relevant for the argument.

Lastly, a degenerate 1-cube $x\sigma_1$ on x is represented by

$$x = x,$$

while a 2- or 3-cube whose boundary agrees with that of a degenerate cube is assumed to be degenerate unless indicated otherwise. For instance, a 2-cube depicted as

$$\begin{tikzcd} x \arrow[r, equal] \arrow[d, "f"'] & x \arrow[d, "f"] \\ y \arrow[r, equal] & y \end{tikzcd}$$

represents $f\sigma_1$.

We write $\partial\Box^n \to \Box^n$ for the maximal proper subobject of $\Box^n$, i.e., the union of all of its faces. We will refer to these as the *n-cube* and the *boundary* of the n-cube, respectively. The subobject of $\Box^n$ given by the union of all faces except $\partial_{i,\varepsilon}$ will be denoted $\sqcap^n_{i,\varepsilon}$ and referred to as an (i,ε)-open box.

In many cases, we will construct cubes in a cubical set X by *filling* open boxes, i.e. extending a map $\sqcap^n_{i,\varepsilon} \to X$ to $\Box^n$. When illustrating the filling of a 2-dimensional open box, the new edge obtained from the filling will be indicated with a dashed line. For instance, the diagram below illustrates the filling of a $(1,0)$-open box.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, dashed] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

From Theorem 4.1.1 and Corollary 4.1.2, we obtain the following:

Proposition 4.1.3. *Given a cubical set X, for any cube $x \colon \Box^n \to X$ there exist unique (possibly empty) sequences $a_1 < \cdots < a_p, b_1 \le \cdots \le b_q, \varepsilon_1, \ldots, \varepsilon_q \in \{0,1\}$,* *where* $b_i < b_{i+1}$ *if* $\varepsilon_i = \varepsilon_{i+1}$*, and a unique non-degenerate cube* $y\colon \square^{n-p-q} \to X$ *such that* $x = y\gamma_{b_1,\varepsilon_1} \cdots \gamma_{b_q,\varepsilon_q} \sigma_{a_1} \cdots \sigma_{a_p}$*.* $\square$

This factorization is called the *standard form* of x.

Corollary 4.1.4. *A map* $X \to Y$ *in* cSet *is determined by its action on the non-degenerate cubes of* X*.* $\square$

Corollary 4.1.5. *A map* $X \to Y$ *in* cSet *is a monomorphism if and only if it maps non-degenerate cubes of* X *to non-degenerate cubes of* Y*, and does so injectively.* $\square$

For brevity, we will often say that the standard form of a cube x is zf, or "ends with f", where f is some map in $\square$; this is understood to mean that f is the rightmost map in the standard form of x. For instance, if the standard form of x is $z\sigma_{a_p}$, then $z = y\gamma_{b_1,\varepsilon_1} \cdots \gamma_{b_q,\varepsilon_q} \sigma_{a_1} \cdots \sigma_{a_{p-1}}$ in the notation of Proposition 4.1.3.

Definition 4.1.6. The *critical edge* of $\square^n$ with respect to a face $\partial_{i,\varepsilon}$ is the unique edge of $\square^n$ which is adjacent to $\partial_{i,\varepsilon}$ and which, together with $\partial_{i,\varepsilon}$, contains both of the vertices $(0, \dots, 0)$ and $(1, \dots, 1)$.

More explicitly, the critical edge with respect to $\partial_{i,\varepsilon}$ corresponds to the map $f\colon [1] \to [1]^n$ given by $f_i = \mathrm{id}_{[1]}, f_j = \mathrm{const}_{1-\varepsilon}$ for $j \neq i$.

The assignment $([1]^m, [1]^n) \mapsto [1]^{m+n}$ defines a functor $\square \times \square \to \square$. Post-composing it with the Yoneda embedding and left Kan extending, we obtain the *geometric product* functor

$$\begin{tikzcd} \square \times \square \arrow[r] \arrow[d, hook] & \mathsf{cSet} \\ \mathsf{cSet} \times \mathsf{cSet} \arrow[ur, "\otimes"'] & \end{tikzcd}$$

The standard formula for left Kan extensions gives us the following formula for the geometric product:

$$X \otimes Y = \operatorname*{colim}_{\substack{x:\square^m \to X \\ y:\square^n \to Y}} \square^{m+n}$$

Note that the geometric product of cubical sets does not coincide with the cartesian product. However, the geometric product implements the correct homotopy type, and is better behaved than the cartesian product – for instance, for $m, n \geq 0$ we have $\square^m \otimes \square^n = \square^{m+n}$. Furthermore, the geometric product is taken to the cartesian product by the geometric realization functor to spaces.

Proposition 4.1.7. *The geometric product $\otimes$ defines a monoidal structure on the category of cubical sets, with the unit given by $\square^0$.* □

This monoidal structure is however not symmetric. Indeed, the existence of a symmetry natural transformation would in particular imply that there is a non-identity bijection $[1]^2 \to [1]^2$ in $\square$.

In particular, for any $X, Y \in \mathsf{cSet}$, the unique maps from X and Y to $\square^0$ induce maps $\pi_X : X \otimes Y \to X, \pi_Y : X \otimes Y \to Y$.

Given a cubical set X, we form two non-isomorphic functors $\mathsf{cSet} \to \mathsf{cSet}$: the left tensor $- \otimes X$ and the right tensor $X \otimes -$. As they are both co-continuous, they admit right adjoints and we write $\underline{\hom}_L(X, -)$ for the right adjoint of the left tensor and $\underline{\hom}_R(X, -)$ for the right adjoint of the right tensor. Explicitly, these functors are given by $\underline{\hom}_L(X, Y)_n = \mathsf{cSet}(\square^n \otimes X, Y)$, $\underline{\hom}_R(X, Y)_n = \mathsf{cSet}(X \otimes \square^n, Y)$. Thus the monoidal structure on cSet given by the geometric product is closed, but non-symmetric.

The standard construction of an arbitrary small colimit as a coequalizer of coproducts gives us the following lemma about colimts in presheaf categories.

Lemma 4.1.8. *Let $\mathcal{C}$ be a category and D a small diagram in $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$. Then any map $\mathcal{C}(-,c) \to \operatorname{colim} D$ factors through some map in the colimit cone.* $\square$

This lemma allows us to describe the geometric product of cubical sets explicitly.

Proposition 4.1.9. *For $X, Y \in \mathsf{cSet}$, the geometric product $X \otimes Y$ admits the following description.*

- *For $k \geq 0$, the k-cubes of $X \otimes Y$ consist of all pairs $(x\colon \square^m \to X, y\colon \square^n \to Y)$ such that $m + n = k$, subject to the identification $(x\sigma_{m_1+1}, y) = (x, y\sigma_1)$.*
- *For $x\colon \square^m \to X, y\colon \square^n \to Y$, the faces, degeneracies, and connections of the $(m+n)$-cube (x,y) are computed as follows:*
  - $(x,y)\partial_{i,\epsilon} = \begin{cases} (x\partial_{i,\epsilon}, y) & 1 \leq i \leq m \\ (x, y\partial_{i-m,\epsilon}) & m+1 \leq i \leq m+n \end{cases}$
  - $(x,y)\sigma_i = \begin{cases} (x\sigma_i, y) & 1 \leq i \leq n_1 + 1 \\ (x, y\sigma_{i-m}) & m+1 \leq i \leq m+n+1 \end{cases}$
  - $(x,y)\gamma_{i,\varepsilon} = \begin{cases} (x\gamma_{i,\varepsilon}, y) & 1 \leq i \leq m \\ (x, y\gamma_{i-m,\varepsilon}) & m+1 \leq i \leq m+n \end{cases}$

Proof. We begin by noting that for every pair $(x\colon \square^m \to X, y\colon \square^n \to Y)$ there is a corresponding $(m+n)$-cube $(x,y)\colon \square^{m+n} \to X \otimes Y$ given by the colimit cone. Next we will show that faces, degeneracies and connections of these cones are computed as described in the statement.

For such an $(m+n)$-cube (x,y), consider a face $(x,y)\partial_{i,\varepsilon}$ for $1 \leq i \leq m$. We can express the face map $\partial^{m+n}_{i,\varepsilon}$ as $\partial^m_{i,\varepsilon} \otimes \square^n$; thus $(x,y)\partial_{i,\varepsilon} = (x\partial_{i,\varepsilon}, y)$ by the naturality of the colimit cone.

$$\begin{tikzcd}
\Box^{m-1} \otimes \Box^n \arrow[r, "\partial_{i,\varepsilon} \otimes \Box^n"] \arrow[dr, "(x\partial_{i,\varepsilon}, y)"'] & \Box^m \otimes \Box^n \arrow[d, "(x,y)"] \\
& X \otimes Y
\end{tikzcd}$$

Likewise, for $m+1 \le i \le m+n$ we have $\partial^{m+n}_{i,\varepsilon} = \Box^m \otimes \partial^n_{i-m,\varepsilon}$, implying $(x,y)\partial_{i,\varepsilon} = (x, y\partial_{i-m,\varepsilon})$. Similar proofs hold for degeneracies and connections. In particular, this implies that for any (x,y) we have $(x\sigma_{m+1}, y) = (x, y\sigma_1)$, as both are equal to $(x,y)\sigma_{m+1}$.

To see that all cubes in $X \otimes Y$ are of this form, note that by Lemma 4.1.8, every cube of $X \otimes Y$ is equal to $(x,y)\psi$ for some such pair (x,y) and some map ψ in $\Box$. We have shown that the set of cubes arising from pairs is closed under faces, degeneracies and connections; since these classes generate all maps in $\Box$, this proves our claim.

Finally, we must show that the cubes of $X \otimes Y$ are not subject to any additional identifications, beyond the identification $(x\sigma_{m+1}, y) = (x, y\sigma_1)$ mentioned above. In other words, we must show that for each $k \ge 0$, $(X \otimes Y)_k$ is the quotient of the set $\{(x \colon \Box^m \to X, y \colon \Box^n \to Y) | m + n = k\}$ under the smallest equivalence relation $\sim$ such that $(x'\sigma_{m+1}, y') \sim (x', y'\sigma_1)$ for all $x' \colon \Box^{m'} \to X, y' \colon \Box^{n'} \to Y$ such that $m' + n' = k - 1$.

To that end, let $x \colon \Box^m \to X, y \colon \Box^n \to Y, x' \colon \Box^{m'} \to X, y' \colon \Box^{n'} \to Y$, such that $m + n = m' + n'$ and $(x,y) = (x',y')$ in $(X \otimes Y)$. Without loss of generality, assume $m \ge m'$. We compute the image of this cube under the map $\pi_X \colon X \otimes Y \to$

X.

$$\pi_X(x,y) = \pi_X(x',y)$$
$$\therefore x\sigma_{m+1}\sigma_{m+2}\cdots\sigma_{m+n} = x'\sigma_{m'+1}\cdots\sigma_{m'+n'}$$

(If n or n' is equal to 0, we interpret the corresponding string of degeneracies to be empty.) We can apply face maps to both sides of this equation to reduce the left-hand side to x. If $m = m'$ then this gives the equation $x = x'$, and a similar calculation shows $y = y'$. Otherwise, we have $x = x'\sigma_{m'+1}\cdots\sigma_m$. In this case, a similar calculation shows $y' = y\sigma_1\cdots\sigma_1$, where σ_1 is applied $m - m'$ times on the right-hand side of the equation. From this we can see that $(x,y) \sim (x',y')$. Thus we see that quotienting the set of pairs (x,y) of appropriate dimensions by $\sim$ does indeed suffice to obtain $(X \otimes Y)_k$. □

Corollary 4.1.10. *For cubical sets X and Y, we have $(X \otimes Y)_1 \cong (X_1 \times Y_0) \cup_{(X_0 \times Y_0)} (X_0 \times Y_1)$.* □

The following lemma, which can be verified by simple computation, allows us to express boundary inclusions and open box inclusions as pushout products with respect to this monoidal structure.

Lemma 4.1.11.

(i) For $m, n \geq 0$, we have

$$(\partial\square^m \to \square^m) \,\widehat{\otimes}\, (\partial\square^n \to \square^n) = (\partial\square^{m+n} \to \square^{m+n}).$$

(ii) For $1 \leq i \leq m$ and $\varepsilon \in \{0,1\}$, the open-box inclusion $\sqcap^n_{i,\varepsilon} \hookrightarrow \square^n$ is the

pushout product

$$(\partial\Box^{i-1} \hookrightarrow \Box^{i-1}) \mathbin{\widehat{\otimes}} (\{1-\varepsilon\} \hookrightarrow \Box^1) \mathbin{\widehat{\otimes}} (\partial\Box^{m-i} \hookrightarrow \Box^{m-i}). \qquad \Box$$

We conclude this section by defining certain functors which we will use to compare model structures.

We define two endofunctors $(-)^{\mathrm{co}}, (-)^{\mathrm{co-op}} \colon \Box \to \Box$ as follows:

- Both $(-)^{\mathrm{co-op}}$ and $(-)^{\mathrm{co}}$ act as the identity on objects;
- $(-)^{\mathrm{co}}$ acts on generating morphisms as follows:
  - $(\partial^n_{i,\varepsilon})^{\mathrm{co}} = \partial^n_{n-i+1,\varepsilon}$;
  - $(\sigma^n_i)^{\mathrm{co}} = \sigma^n_{n-i+1}$;
  - $(\gamma^n_{i,\varepsilon})^{\mathrm{co}} = \gamma^n_{(n-1)-i+1,\varepsilon}$
- $(-)^{\mathrm{co-op}}$ acts on generating morphisms as follows:
  - $(\partial^n_{i,\varepsilon})^{\mathrm{co-op}} = \partial^n_{i,1-\varepsilon}$;
  - $(\sigma^n_i)^{\mathrm{co-op}} = \sigma^n_i$;
  - $(\gamma^n_{i,\varepsilon})^{\mathrm{co-op}} = \gamma^n_{i,1-\varepsilon}$.

From the definition we can see that the endofunctors $(-)^{\mathrm{co}}$ and $(-)^{\mathrm{co-op}}$ commute; we denote their composite by $(-)^{\mathrm{op}}$.

By left Kan extension, we obtain functors $(-)^{\mathrm{co}}, (-)^{\mathrm{co-op}}, (-)^{\mathrm{op}} \colon \mathsf{cSet} \to \mathsf{cSet}$.

$$\begin{tikzcd}
\Box \arrow[r] \arrow[d, hook] & \Box \arrow[r, hook] & \mathsf{cSet} \\
\mathsf{cSet} \arrow[urr] & &
\end{tikzcd}$$

Some simple computations show:

Lemma 4.1.12. *The functors* $(-)^{\mathrm{co}}, (-)^{\mathrm{co-op}}, (-)^{\mathrm{op}}$ *are involutions of* cSet. $\square$

In particular, for $X \in \mathsf{cSet}$, the cubes of X are in bijection with those of X^{co}, $X^{\mathrm{co-op}}$, and X^{op}; given $x\colon \square^n \to X$ we have corresponding cubes $x^{\mathrm{co}}\colon \square^n = (\square^n)^{\mathrm{co}} \to X^{\mathrm{co}}$, $x^{\mathrm{co-op}}\colon \square^n = (\square^n)^{\mathrm{co-op}} \to X^{\mathrm{co-op}}$, $x^{\mathrm{op}}\colon \square^n = (\square^n)^{\mathrm{op}} \to X^{\mathrm{op}}$.

Proposition 4.1.13 ([CKM20, Prop. 1.17])**.** *The endofunctors* $(-)^{\mathrm{co}}$, $(-)^{\mathrm{co-op}}$, $(-)^{\mathrm{op}}$ *on* cSet *interact with the geometric product as follows:*

- *The functor* $(-)^{\mathrm{co}}$ *is strong anti-monoidal, i.e.* $(X \otimes Y)^{\mathrm{co}} \cong Y^{\mathrm{co}} \otimes X^{\mathrm{co}}$*;*
- *The functor* $(-)^{\mathrm{co-op}}$ *is strong monoidal, i.e.* $(X \otimes Y)^{\mathrm{co-op}} \cong X^{\mathrm{co-op}} \otimes Y^{\mathrm{co-op}}$*;*
- *The functor* $(-)^{\mathrm{op}}$ *is strong anti-monoidal, i.e.* $(X \otimes Y)^{\mathrm{op}} \cong Y^{\mathrm{op}} \otimes X^{\mathrm{op}}$. $\square$

Let $\square_0$ denote the subcategory of $\square$ generated by the face, degeneracy, and negative connection maps, and let cSet_0 denote the presheaf category $\mathsf{Set}^{\square_0^{\mathrm{op}}}$. This is the category of cubical sets studied in [KLW19].

By pre-composition, the inclusion $i\colon \square_0 \hookrightarrow \square$ defines a functor $i^*\colon \mathsf{cSet} \to \mathsf{cSet}_0$. Left and right Kan extension define left and right adjoints of this functor, respectively denoted $i_!, i_*\colon \mathsf{cSet}_0 \to \mathsf{cSet}$.

We may characterize the functors $i^*, i_*, i_!$ as follows:

- For $X \in \mathsf{cSet}$, $n \geq 0$ we have $(i^*X)_n = X_n$, with structure maps computed as in X. However, the cubes of X whose standard forms end with positive connections become non-degenerate in i^*X.
- For $X \in \mathsf{cSet}_0$, we have $(i_*X)_n = \mathsf{cSet}_0(i^*\square^n, X)$.

- The cubes of $i_!X$ consist of those of X, together with freely added positive connections and their degeneracies. Given a map $f\colon X \to Y$ in cSet_0, $i_!f$ acts identically to f on the non-degenerate cubes of $i_!X$; by Corollary 4.1.4 this is enough to determine $i_!f$.

Given a map $f\colon i_!X \to Y$ in cSet, f and the adjunct map $\overline{f}\colon X \to i^*Y$ act identically on non-degenerate cubes. From this we obtain the following:

Lemma 4.1.14. *For $X \in cSet_0$, a map $f\colon i_!X \to Y$ is a monomorphism if and only if the adjunct map $\overline{f}\colon X \to i^*Y$ is a monomorphism.*

Proof. By Proposition 4.1.3 and its analogue in cSet_0, a map in either category is a monomorphism if and only if it acts injectively on non-degenerate cubes. Since X and $i_!X$ have the same non-degenerate cubes, and f and $\overline{f}$ act identically on non-degenerate cubes, this proves the claim. $\square$

The restriction of the nerve functor defines a functor $\square \to \mathsf{sSet}$; taking the left Kan extension of this functor along the Yoneda embedding, we obtain the *triangulation* functor $T\colon \mathsf{cSet} \to \mathsf{sSet}$.

$$\begin{tikzcd}
\square \arrow[r] \arrow[d, hook] & \mathsf{sSet} \\
\mathsf{cSet} \arrow[ur, "T"'] &
\end{tikzcd}$$

The triangulation functor has a right adjoint $U\colon \mathsf{sSet} \to \mathsf{cSet}$ given by $(UX)_n = \mathsf{sSet}((\Delta^1)^n, X)$. Intuitively, we think of triangulation as creating a simplicial set TX from a cubical set X by subdividing the cubes of X into simplices.

We now record some basic facts about triangulation. In the given references, these results are proven using a different definition of the category $\square$, lacking connection maps, but the proofs apply equally well to the cubical sets under consideration here.

Proposition 4.1.15 ([Cis06, Ex. 8.4.24])**.** *The triangulation functor sends geometric products to cartesian products; that is, for cubical sets X and Y, there is a natural isomorphism $T(X \otimes Y) \cong TX \times TY$.* □

Corollary 4.1.16. *Triangulation preserves pushout products; that is, for maps f, g in cSet there is a natural isomorphism $T(f \mathbin{\widehat{\otimes}} g) \cong Tf \mathbin{\widehat{\times}} Tg$.*

Proof. Immediate by Proposition 4.1.15 and the fact that T preserves colimits as a left adjoint. □

Proposition 4.1.17 ([Cis06, Lem. 8.4.29])**.** *The triangulation functor preserves monomorphisms.* □

Finally, we relate the adjunction $T \dashv U$ to the involutions $(-)^{\mathrm{co}}$, $(-)^{\mathrm{co-op}}$, and $(-)^{\mathrm{op}}$ of cSet and the involution $(-)^{\mathrm{op}}$ of sSet.

Proposition 4.1.18. *We have the following natural isomorphisms in sSet and cSet:*

(i) $T \circ (-)^{\mathrm{co}} \cong T$;

(ii) $T \circ (-)^{\mathrm{co-op}} \cong (-)^{\mathrm{op}} \circ T$;

(iii) $T \circ (-)^{\mathrm{op}} \cong (-)^{\mathrm{op}} \circ T$;

(iv) $(-)^{\mathrm{co}} \circ U \cong U$;

(v) $(-)^{\mathrm{co-op}} \circ U \cong U \circ (-)^{\mathrm{op}}$;

(vi) $(-)^{\mathrm{op}} \circ U \cong U \circ (-)^{\mathrm{op}}$.

Proof. It suffices to prove (i) and (ii). As T and the involutions preserve colimits, it suffices to establish the desired natural isomorphisms on the objects $\square^n$. For this, observe that the maps between these objects are generated, under composition and the geometric product, by the maps $\partial_{1,\varepsilon} \colon [0] \to [1]$,

$\sigma_1 \colon [1] \to [0]$, and $\gamma_{1,\varepsilon} \colon [1]^2 \to [1]$. By Propositions 4.1.13 and 4.1.15, it thus suffices to show that $T \circ (-)^{\mathrm{co}}$ and T (resp. $T \circ (-)^{\mathrm{co-op}}$ and $(-)^{\mathrm{op}} \circ T$) agree on these maps; this can easily be verified. □

4.2 Homotopy theory of cubical sets

Here we consider our first example of a model structure on cSet.

Lemma 4.2.1. *The boundary inclusions $\partial\square^n \to \square^n$ generate all monomorphisms of cSet under pushout and transfinite composition.*

Proof. This follows from Corollary 4.1.2. □

Definition 4.2.2. A map of cubical sets is a *Kan fibration* if it has the right lifting property with respect to all open box fillings. A cubical set X is a *cubical Kan complex* if the map $X \to \square^0$ is a Kan fibration.

The functor $\square^1 \otimes - \colon \mathsf{cSet} \to \mathsf{cSet}$, together with the natural transformations $\partial^1_{1,0} \otimes -, \partial^1_{1,1} \otimes - \colon \mathrm{id} \to \square^1 \otimes -$, and $\pi \colon \square^1 \otimes - \to \mathrm{id}$, defines a cylinder functor on cSet in the sense of Definition 2.2.10. Thus, for any $X, Y \in \mathsf{cSet}$ we have a set $[X, Y]$ of homotopy classes of maps from X to Y defined by this cylinder functor.

Theorem 4.2.3 (Cisinski)**.** *The category cSet carries a cofibrantly generated model structure, referred to as the Grothendieck model structure, in which*

- *cofibrations are the monomorphisms;*
- *weak equivalences are, maps $X \to Y$ inducing bijections $[Y, Z] \to [X, Z]$ for all cubical Kan complexes Z;*
- *fibrations are the Kan fibrations.*

Proof. The existence of the model structure and characterization of the cofibrations, weak equivalences, and fibrant objects follows from applying Theorem 2.2.14 with the cylinder functor $I = \square^1 \otimes -$, cellular model $M = \{\partial\square^n \to \square^n | n \geq 0\}$, and $S = \varnothing$. The characterization of the fibrations is given in [Cis14, Thm. 1.7]. $\square$

Proposition 4.2.4 ([Cis14, Thm. 1.7])**.** *The Grothendieck model structure on* cSet *is monoidal with respect to the geometric product of cubical sets.* $\square$

As in the case of simplicial sets, the canonical inclusion $\square \to \mathsf{Cat}$ induces the adjoint pair $\tau_1 \colon \mathsf{cSet} \rightleftarrows \mathsf{Cat} : \mathsf{N}_\square$ via hom-out and the left Kan extension. In particular, $\mathsf{N}_\square(\mathcal{C})_n = \mathsf{Cat}([1]^n, \mathcal{C})$. The functor τ_1 takes a cubical set X to its *fundamental category*, which is obtained as the quotient of the free category on the directed graph $X_1 \rightrightarrows X_0$ modulo the relations: $x\sigma_1 = \mathrm{id}_x$ and $gf = qp$ for every 2-cube

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "p"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "q"] & \bullet \end{tikzcd}$$

4.3 Marked cubical sets

In this section we define marked cubical sets, analogous to the marked simplicial sets of Section 3.3. To do this, we need to introduce a new category $\square^+$, an enlargement of $\square$. The category $\square^+$ consists of objects of the form $[1]^n$ for $n \geq 0$, as well as objects $[1]^n_e$ for $n \geq 1$. The maps of $\square^+$ are generated by the usual generating maps of $\square$ along with the following:

- $\varphi^n \colon [1]^n \to [1]^n_e$ for $n \geq 1$;
- $\zeta^n_i \colon [1]^n_e \to [1]^{n-1}$ for $n \geq 1, 1 \leq i \leq n$;

- $\xi^n_{i,\varepsilon}\colon [1]^n_e \to [1]^{n-1}$ for $n \geq 2, 1 \leq i \leq n-1, \varepsilon \in \{0,1\}$

subject to the usual cubical identities, plus the following:

$$\zeta_i \varphi = \sigma_i;$$
$$\xi_{i,\varepsilon}\varphi = \gamma_{i,\varepsilon};$$
$$\sigma_i \zeta_j = \sigma_j \zeta_{i+1} \text{ for } j \leq i;$$
$$\gamma_{j,\varepsilon}\xi_{i,\delta} = \begin{cases} \gamma_{i,\varepsilon'}\xi_{j,\varepsilon} & \text{for } j > i; \\ \gamma_{i,\varepsilon'}\xi_{i+1,\delta} & \text{for } j = i, \\ & \varepsilon' = \varepsilon; \end{cases}$$
$$\sigma_j \xi_{i,\varepsilon} = \begin{cases} \gamma_{i-1,\varepsilon}\zeta_j & \text{for } j < i; \\ \sigma_i \zeta_i & \text{for } j = i; \\ \gamma_{i,\varepsilon}\zeta_{j+1} & \text{for } j > i. \end{cases}$$

Proposition 4.3.1 ([CKM20, Prop. 2.1])**.** *The category $\square^+$ is an EZ Reedy category with the Reedy structure defined as follows:*

- $\deg([1]^0) = 0$*,* $\deg([1]^n) = 2n - 1$ *for $n \geq 1$, and* $\deg([1]^n_e) = 2n$ *for $n \geq 1$;*
- $(\square^+)_+$ *is generated by the maps $\partial^n_{i,\varepsilon}$ and φ^n under composition;*
- $(\square^+)_-$ *is generated by the maps $\sigma^n_i, \gamma^n_{i,\varepsilon}, \zeta^n_i$, and $\xi^n_{i,\varepsilon}$ under composition.*

□

A *structurally marked cubical set* is a contravariant functor $X\colon (\square^+)^{\mathrm{op}} \to \mathsf{Set}$ and a morphism of structurally marked cubical sets is a natural transformation of such functors. We will write cSet^{++} for the category of structurally marked cubical sets. When working with the category of structurally marked cubical sets, we will write X_n for the value of X at $[1]^n$ and eX_n for the value of X at $[1]^n_e$. As in cSet, the representable presheaf at the object $[1]^n$ will be denoted $\square^n$, while the representable presheaf at the object $[1]^n_e$ will be denoted $\widetilde{\square}^n$.

As in the simplicial case, structurally marked cubical sets should be thought of as cubical sets with (possibly multiple) labels on their cubes of positive dimension, such that each degenerate cube has, in particular, one distinguished label: for a cube of the form $x\sigma_i$ this is $x\zeta_i$, while for a cube of the form $x\gamma_{i,\varepsilon}$ this is $x\xi_{i,\varepsilon}$. For $n \geq 1$, the underlying cubical set of $\widetilde{\Box}^n$ has a unique marking on the unique non-degenerate n-cube, while all other non-degenerate cubes are unmarked.

A *marked cubical set* is a structurally marked cubical set for which each map $eX_n \to X_n$ is a monomorphism. We write cSet^+ for the category of marked cubical sets. Alternatively, we may view a marked cubical set as a pair (X, eX) consisting of a cubical set X together with a subset $eX \subseteq \bigcup\limits_{n\geq 1} X_n$ of cubes of positive dimension that includes all degenerate cubes, with a morphism of marked cubical sets being a map of cubical sets that preserves marked cubes.

Let $\Box'$ denote the full subcategory of $\Box^+$ on the objects $[1]^n$ for $n \geq 0$ and $[1]_e$. A *cubical set with weak equivalence structure* is a contravariant functor $X \colon (\Box')^{\mathrm{op}} \to \mathsf{Set}$ and a morphism of cubical sets with weak equivalence structure is a natural transformation of such functors. We will write cSet'' for the category of cubical sets with weak equivalence structure. A *cubical set with weak equivalences* is a cubical set with weak equivalence structure for which the map $eX_1 \to X_1$ is a monomorphism. We will write cSet' for the category of cubical sets with weak equivalences. Similarly to the above description of marked cubical sets, we may think of a cubical set with weak equivalences as a cubical set X together with a subset of X_1, consisting of those edges considered to be marked, which includes all degenerate edges.

As in the simplicial case, the forgetful functor from (structurally) marked cubical sets to cubical sets admits both a left and a right adjoint, given by the minimal and maximal marking respectively. The minimal marking on a cubical set X, denoted $X^\flat$, marks exactly the degenerate cubes, whereas the maximal marking, denoted $X^\sharp$, marks all positive-dimensional cubes of X. As we would expect, all of these functors factor through cSet'.

There is moreover an inclusion $\mathsf{cSet}^{+} \to \mathsf{cSet}^{++}$. This inclusion admits a left adjoint taking $X \in \mathsf{cSet}^{++}$ to $\mathrm{Im}X$ given by $(\mathrm{Im}X)_n = X_n$ and $e(\mathrm{Im}X)_n = \varphi^*(eX_n)$, i.e., the image of eX_n under $\varphi^* = X(\varphi^n)$. The inclusion is easily seen to not have a right adjoint, since it fails to preserve the pushout of $\square^1 \to \widetilde{\square}^1$ against itself. Likewise, there is an inclusion $\mathsf{cSet}' \to \mathsf{cSet}''$ which admits a similarly-defined left adjoint $\mathrm{Im}\colon \mathsf{cSet}'' \to \mathsf{cSet}'$.

Altogether we obtain the following diagram of adjunctions:

$$\begin{tikzcd}
\mathsf{cSet}^{++} \arrow[rr, bend left, "\mathrm{Im}"] \arrow[dd, bend right, "(-)^\flat"'] \arrow[dd] \arrow[dd, bend left, "(-)^\sharp"] \arrow[dr, bend right, "(-)^\sharp"'] \arrow[dr] & & \mathsf{cSet}^{+} \arrow[ll, hook'] \arrow[dd, bend right, "(-)^\flat"'] \arrow[dd] \arrow[dd, bend left, "(-)^\sharp"] \arrow[dl, bend left, "(-)^\sharp"] \arrow[dl] \\
& \mathsf{cSet} \arrow[ul, bend right, "(-)^\flat"'] \arrow[ur, bend left, "(-)^\flat"] \arrow[dl, bend left, "(-)^\flat"] \arrow[dr, bend right, "(-)^\flat"'] & \\
\mathsf{cSet}'' \arrow[rr, bend right, "\mathrm{Im}"'] \arrow[ur, bend right, "(-)^\sharp"'] \arrow[ur] \arrow[uu] & & \mathsf{cSet}' \arrow[ll, hook'] \arrow[ul, bend left, "(-)^\sharp"] \arrow[ul] \arrow[uu]
\end{tikzcd} \tag{$*$}$$

As in Section 3.3, we will use the ambiguous notations $\mathsf{cSet}^{+(+)}$, $\mathsf{cSet}'^{(\prime)}$, $\mathsf{cSet}^{\bullet}$, $\mathsf{cSet}^{\bullet\bullet}$, and $\mathsf{cSet}^{\bullet(\bullet)}$ to indicate that a construction or result is applicable to more than one of the categories under discussion.

As before, we define terminology to describe certain distinguished kinds of maps in $\mathsf{cSet}^{\bullet}$.

Definition 4.3.2. Let $X \to Y$ be a map in $\mathsf{cSet}^{\bullet}$. This map is:

- *regular* if it creates markings, i.e. a simplex of X is marked if and only if its image in Y is marked;
- *entire* if the underlying cubical set map is an isomorphism, i.e. Y is

obtained from X by marking a (possibly empty) set of its unmarked simplices.

Monoidal products analogous to that of Section 4.1 exist for marked cubical sets and cubical sets with weak equivalences.

Definition 4.3.3. For $X, Y \in \mathsf{cSet}^\bullet$, the *geometric product* $X \otimes Y$ is a marked cubical set defined as follows:

- The underlying cubical set of $X \otimes Y$ is the geometric product of the underlying cubical sets of X and Y;
- For $[1]^n_e \in \square^\bullet$, $e(X \otimes Y)_n$ is the set of cubes $(x, y) \in (X \otimes Y)_n$ such that either x is a marked cube of X or y is a marked cube of Y.

When dealing with cSet^+ alone, as in Chapter 9, the geometric product on that category will be referred to as the *lax Gray tensor product*.

Regular and entire morphisms satisfy the following closure properties under pushout products with respect to the geometric product.

Lemma 4.3.4 (cf. [CKM20, Lem. 2.17]). *Let f and g be monomorphisms in* $\mathsf{cSet}^\bullet$.

(i) If both f and g are regular, then so is $f \hat{\odot} g$.

(ii) If either f or g is entire, then so is $f \hat{\odot} g$.

(iii) If both f and g are entire, then $f \hat{\odot} g$ is an isomorphism. $\square$

Proposition 4.3.5. *The forgetful functors* $\mathsf{cSet}^+ \to \mathsf{cSet}'$ *and* $\mathsf{cSet}' \to \mathsf{cSet}$, *as well as the minimal and maximal marking functors* $(-)^\flat, (-)^\sharp \colon \mathsf{cSet} \to \mathsf{cSet}^\bullet$, *are strong monoidal with respect to the geometric product.*

Proof. That the forgetful functors are monoidal is immediate from the definitions of the geometric products on cSet' and cSet^+. To see that the minimal and maximal marking functors are monoidal, let $X, Y \in \mathsf{cSet}$. The marked cubes of $X^\flat \otimes Y^\flat$ are those pairs (x, y) for x and y of appropriate dimensions for which either x is marked in $X^\flat$ or y is marked in $Y^\flat$ – in other words, those for which either x or y is degenerate. As these are precisely the degenerate cubes of $X \otimes Y$, we see that $X^\flat \otimes Y^\flat \cong (X \otimes Y)^\flat$. Similarly, since all cubes of $X^\sharp$ and $Y^\sharp$ of the appropriate dimensions are marked, we see that the same holds for $X^\sharp \otimes Y^\sharp$, thus $X^\sharp \otimes Y^\sharp \cong (X \otimes Y)^\sharp$. □

Remark 4.3.6. In contrast to Proposition 4.3.5, the minimal and maximal marking functors $(-)^\flat, (-)^\sharp \colon \mathsf{cSet}' \to \mathsf{cSet}^+$ are not strong monoidal. To see this, let $X, Y \in \mathsf{cSet}'$, and consider a 2-cube in the geometric product of their underlying cubical sets corresponding to a pair $(x \colon \Box^1 \to X, y \colon \Box^1 \to Y)$. In both $X^\flat \otimes Y^\flat$ and $X^\sharp \otimes Y^\sharp$, this 2-cube is marked if and only if either x is marked in X or y is marked in Y. In contrast, it is necessarily marked in $(X \otimes Y)^\sharp$, and is marked in $(X \otimes Y)^\flat$ if and only if either x or y is degenerate. However, we do have natural transformations $(X \otimes Y)^\flat \to (X)^\flat \otimes (Y)^\flat$ and $X^\sharp \otimes Y^\sharp \to (X \otimes Y)^\sharp$ acting as the identity on underlying cubical sets, showing that $(-)^\flat$ is oplax monoidal while $(-)^\sharp$ is lax monoidal.

As in the case of cubical sets, given $X \in \mathsf{cSet}^\bullet$, we form two non-isomorphic functors $\mathsf{cSet}^\bullet \to \mathsf{cSet}^\bullet$: the left tensor $- \otimes X$ and the right tensor $X \otimes -$. As they are both co-continuous, they admit right adjoints; we write $\underline{\mathrm{hom}}_L(X, -)$ for the right adjoint of the left tensor $- \otimes X$ and $\underline{\mathrm{hom}}_R(X, -)$ for the right adjoint of the right tensor $X \otimes -$.

Observe that we may extend the functors $(-)^{\mathrm{co}}, (-)^{\mathrm{co-op}}, (-)^{\mathrm{op}} \colon \Box \to \Box$ of Section 4.1 to obtain involutions $(-)^{\mathrm{co}}, (-)^{\mathrm{co-op}}, (-)^{\mathrm{op}} \colon \Box^+ \to \Box^+$, by having these functors act as the identity on the objects $[1]^n_e$, and having $(-)^{\mathrm{co}}$ and $(-)^{\mathrm{co-op}}$ act on the additional generating morphisms of $\square^+$ as follows:

- $(\varphi^n)^{\mathrm{co}} = \varphi^n$;
- $(\zeta^n_i)^{\mathrm{co}} = \zeta^n_{n-i+1}$;
- $(\xi^n_{i,\varepsilon})^{\mathrm{co}} = \xi^n_{n-i,\varepsilon}$;
- $(\varphi^n)^{\mathrm{co-op}} = \varphi^n$;
- $(\zeta^n_i)^{\mathrm{co-op}} = \zeta^n_i$;
- $(\xi^n_{i,\varepsilon})^{\mathrm{co-op}} = \xi^n_{i,1-\varepsilon}$.

It is clear that these functors restrict to involutions of $\square'$. By left Kan extension we obtain involutions $(-)^{\mathrm{co}}, (-)^{\mathrm{co-op}} \colon \mathsf{cSet}^{\bullet\bullet} \to \mathsf{cSet}^{\bullet\bullet}$, which restrict to involutions of $\mathsf{cSet}^{\bullet}$. Given $X^\bullet \in \mathsf{cSet}^{\bullet}$ with underlying cubical set X, the underlying cubical set of $(X^\bullet)^{\mathrm{co}}$ is X^{co}, with a cube $x^{\mathrm{co}} \colon \square^n \to (X^\bullet)^{\mathrm{co}}$ marked if and only if x is marked in $X^\bullet$, and similarly for $(X^\bullet)^{\mathrm{co-op}}$.

Proposition 4.1.13 extends easily to the marked setting.

Proposition 4.3.7. *The endofunctors* $(-)^{\mathrm{co}}$, $(-)^{\mathrm{co-op}}$, $(-)^{\mathrm{op}}$ *on* $\mathsf{cSet}^{\bullet}$ *interact with the geometric product as follows:*

- *The functor* $(-)^{\mathrm{co}}$ *is strong anti-monoidal, i.e.* $(X \otimes Y)^{\mathrm{co}} \cong Y^{\mathrm{co}} \otimes X^{\mathrm{co}}$*;*
- *The functor* $(-)^{\mathrm{co-op}}$ *is strong monoidal, i.e.* $(X \otimes Y)^{\mathrm{co-op}} \cong X^{\mathrm{co-op}} \otimes Y^{\mathrm{co-op}}$*;*
- *The functor* $(-)^{\mathrm{op}}$ *is strong anti-monoidal, i.e.* $(X \otimes Y)^{\mathrm{op}} \cong Y^{\mathrm{op}} \otimes X^{\mathrm{op}}$*.* $\square$

Using Propositions 4.1.13 and 4.3.7 and the adjunctions $(-)^{\mathrm{co}} \dashv (-)^{\mathrm{co}}, (-)^{\mathrm{co-op}} \dashv (-)^{\mathrm{co-op}}$, we obtain:

Corollary 4.3.8. *For X, Y in cSet or $\mathsf{cSet}^\bullet$, we have isomorphisms, natural in X and Y:*

- $\underline{\hom}_L(X,Y)^{\mathrm{co}} \cong \underline{\hom}_R(X^{\mathrm{co}}, Y^{\mathrm{co}})$, $\underline{\hom}_R(X,Y)^{\mathrm{co}} \cong \underline{\hom}_L(X^{\mathrm{co}}, Y^{\mathrm{co}})$;
- $\underline{\hom}_L(X,Y)^{\mathrm{co-op}} \cong \underline{\hom}_L(X^{\mathrm{co-op}}, Y^{\mathrm{co-op}})$, $\underline{\hom}_R(X,Y)^{\mathrm{co-op}} \cong \underline{\hom}_R(X^{\mathrm{co-op}}, Y^{\mathrm{co-op}})$;
- $\underline{\hom}_L(X,Y)^{\mathrm{op}} \cong \underline{\hom}_R(X^{\mathrm{op}}, Y^{\mathrm{op}})$, $\underline{\hom}_R(X,Y)^{\mathrm{op}} \cong \underline{\hom}_L(X^{\mathrm{op}}, Y^{\mathrm{op}})$.

□

We now consider marked versions of the adjunction $T \dashv U$, as developed in [CKM20]. The definition for cubical sets with weak equivalences is easy, as for any cubical set X, edges of TX are in bijection with those of X.

Definition 4.3.9. For $X \in \mathsf{cSet}'$, we define $TX \in \mathsf{sSet}'$ as follows:

- The underlying simplicial set of TX is the triangulation of the underlying simplicial set of X;
- An edge of TX is marked if and only if the corresponding edge of X is marked. That is, $e(TX)_1 = eX_1$ and the structure maps ϕ and ζ_1 act identically to the corresponding maps in X.

This definition extends to morphisms in the natural way, and implies an analogous definition for the right adjoint $U \colon \mathsf{sSet}' \to \mathsf{cSet}'$.

Proposition 4.3.10. *The natural isomorphisms of Proposition 4.1.18 hold for the adjunction $T : \mathsf{cSet}' \rightleftarrows \mathsf{sSet}' : U$.*

Proof. As in the proof of Proposition 4.1.18, it suffices to prove items (i) and (ii), and for this it suffices show that we have the desired isomorphisms on representable cubical sets with weak equivalences. For maps between representables of the form $\square^n$ this is immediate from Proposition 4.1.18. Thus we only need to show that $T \circ (-)^{\mathrm{co}}$ and T (resp. $T \circ (-)^{\mathrm{co-op}}$ and $(-)^{\mathrm{op}} \circ T$) agree on the maps $\varphi \colon \square^1 \to \widetilde{\square}^1$ and $\zeta_1 \colon \widetilde{\square}^1 \to \square^0$; this is immediate from the definitions of T and the involutions. $\square$

To extend triangulation to marked cubical sets, we first need an explicit description of the simplices of $T\square^n = (\Delta^1)^n = N[1]^n$. For $r \geq 0$, observe that since $\Delta^r = N[r]$ and the nerve functor is fully faithful, r-simplices $\Delta^r \to (\Delta^1)^n$ can be identified with order-preserving maps $\phi \colon [r] \to [1]^n$. Such a map ϕ can be identified with a unique function $\{1, \ldots, n\} \to \{1, \ldots, r, \pm\infty\}$, defined as follows:

$$i \mapsto \begin{cases} +\infty, & \pi_i \circ \phi(r) = 0, \\ p, & \pi_i \circ \phi(p-1) = 0 \text{ and } \pi_i \circ \phi(p) = 1, \\ -\infty, & \pi_i \circ \phi(0) = 1. \end{cases}$$

Under this identification, a simplicial operator $\alpha \colon [q] \to [r]$ sends an r-simplex ϕ to the q-simplex α defined as follows:

$$(\phi\alpha)(i) = \begin{cases} +\infty, & \phi(i) > \alpha(q), \\ p, & \alpha(p-1) < \phi(i) \leq \alpha(p), \\ -\infty, & \phi(i) \leq \alpha(0). \end{cases}$$

It will typically be convenient to represent such functions as strings of length n with entries drawn from the set $\{1, \ldots, m, \pm\infty\}$ (for brevity, we will write $+$ for $+\infty$ and $-$ for $-\infty$). We let ι_n denote the inclusion $\{1, \ldots, n\} \to \{1, \ldots, n, \pm\infty\}$, viewed as an n-simplex of Δ^n; represented as a string, this is $1 \ldots n$.

Under this identification, a simplicial operator $\alpha\colon [q] \to [r]$ sends an r-simplex ϕ to the q-simplex α defined as follows:

$$(\phi\alpha)(i) = \begin{cases} +\infty, & \phi(i) > \alpha(q), \\ p, & \alpha(p-1) < \phi(i) \leq \alpha(p), \\ -\infty, & \phi(i) \leq \alpha(0). \end{cases}$$

In particular, when representing simplices as strings, face maps of an m-simplex ϕ can be computed as follows:

- The face $\phi\partial_0$ is computed by replacing every 1 in ϕ by $-$, and reducing all other entries by 1. For instance, $(1\,2\,3 + -)\partial_0 = -1\,2 + -$.
- For $0 < i < m$, the face $\phi\partial_i$ is computed by reducing every entry of ϕ which is greater than i by 1. For instance, $(1\,2\,3 + -)\partial_1 = 1\,1\,2 + -$, while $(1\,2\,3 + -)\partial_2 = 1\,2\,2 + -$.
- The face $\phi\partial_n$ is computed by replacing every n in ϕ by $+$. For instance, $(1\,2\,3 + -)\partial_3 = 1\,2 + + -$.

Alternatively, we may view every face map ∂_i as being computed by reducing all entries of ϕ which are greater than i by 1, identifying entries less than 1 with $-$ and entries greater than $n-1$ with $+$ when dealing with $(n-1)$-simplices.

Likewise, the degeneracy $\phi\sigma_i$ can be computed by raising all entries of ϕ greater than i by 1; from this we obtain the following result.

Lemma 4.3.11. *An r-simplex $\phi\colon \{1, \ldots, n\} \to \{1, \ldots, r, \pm\infty\}$ of $(\Delta^1)^n$ is degenerate if and only if there is some $i \in \{1, \ldots, r\}$ for which $\phi^{-1}(i) = \varnothing$.* □

Definition 4.3.12. We define the functor $T\colon \square^+ \to \mathsf{sSet}^+$ as follows:

- $T[1]^n$ has $(\Delta^1)^n$ as its underlying simplicial set, with an r-simplex $\phi\colon \{1,\ldots,n\} \to \{1,\ldots,r,\pm\infty\}$ unmarked if and only if there exists a sequence $i_1 < \cdots < i_r$ in $\{1,\ldots,n\}$ such that $\phi(i_p) = p$ for all $p \in \{1,\ldots,r\}$;
- $T[1]^n_e$ is obtained from $T[1]^n$ by marking the n-simplex ι_n.

By left Kan extension, this definition extends to a colimit-preserving functor $T\colon \mathsf{cSet}^+ \to \mathsf{sSet}^+$, with a right adjoint $U\colon \mathsf{sSet}^+ \to \mathsf{cSet}^+$. Once again, it is clear that these functors restrict to an adjunction $T : \mathsf{cSet}^+ \rightleftarrows \mathsf{sSet}^+ : U$. From the definition, we can see that the only unmarked n-simplex of $T\square^n$ is ι_n, while all n-simplices of $T\widetilde{\square}^n$ are marked.

We will analyze the marked triangulation functor further in Chapter 9, when we consider model structures on cSet^+. For now, we state a couple of preliminary results involving the constructions of this section, which will be of use in developing these model structures.

Items (i), (ii), (iv), and (v) of Proposition 4.1.18 do not hold in the marked setting, as can be verified by considering the case of a 2-cube with a unique marked edge. However, we do have the following result.

Proposition 4.3.13 ([CKM20, Prop. 5.8]). *There exist natural isomorphisms $T \circ (-)^{\mathrm{op}} \cong (-)^{\mathrm{op}} \circ T$ and $U \circ (-)^{\mathrm{op}} \cong (-)^{\mathrm{op}} \circ U$ in sSet^+ and cSet^+.* □

We also have a natural analogue of Lemma 3.3.7, whose proof is essentially identical.

Lemma 4.3.14. *The classes $(\mathsf{Mono}, \mathsf{Tfib})$ form a cofibrantly generated weak factorization system on $\mathsf{cSet}^{\bullet(\bullet)}$, with a cellular model given by the set:*

$$M = \{\partial\square^n \to \square^n \mid n \geq 0\} \cup \{\varphi\colon \square^n \to \widetilde{\square}^n \mid [n]_e \in \square^\bullet\}$$

□

Finally, we have a lemma concerning triangulation and the complicial model structures.

Proposition 4.3.15. *Let i and j be cofibrations in cSet^+, and let sSet^+ be equipped with any of the model structures of Example 3.3.16. If either Ti or Tj is a trivial cofibration, then T sends the pushout lax Gray tensor product $i \hat{\otimes} j$ to a trivial cofibration as well.*

Proof. This follows from Proposition 3.3.25 together with [CKM20, Thm. 6.5]. $\square$

We conclude this chapter with a note on terminology. In [DKLS20], the term (structurally) marked cubical sets refers to the objects of $\mathsf{cSet}'^{(\prime)}$; we have chosen to refer to these as cubical sets with weak equivalence (structure) to avoid confusion with the objects of $\mathsf{cSet}^{+(+)}$. Our model structure on cSet', however, will still be called the cubical marked model structure, for consistency with the established name of the marked model structure on sSet'.

Chapter 5

The cubical marked model structure

The goal of this chapter is to construct a model category structure on the category cSet' of cubical sets with weak equivalences. Although this could be done using Cisinski-Olschok theory, as described in Section 2.2, we instead choose to construct the model structure via direct application of Theorem 2.2.9, in order to obtain greater insight into the associated homotopy theory.

In Section 5.1, we define the distinguished classes of maps in this model structure, as well as its fibrant objects, the *marked cubical quasicategories*, and prove some basic lemmas about them. In Sections 5.2 and 5.3, we study the homotopy theory of marked cubical quasicategories, including their homotopy categories and homotopy equivalences between them. Finally, in Sections 5.4 to 5.6, we prove the existence of the desired model structure.

5.1 Classes of maps

To begin, we lay out the definitions of the classes of maps that will comprise our model structure on cSet'. Recall from Section 4.3 that the minimal and maximal markings of a cubical set X are denoted $X^\flat$ and $X^\sharp$, respectively, and that a cubical set is understood to be equipped with its minimal marking unless otherwise noted.

The *cofibrations* are the monomorphisms. The *trivial fibrations* are the maps with the right lifting property with respect to the cofibrations.

Using Lemma 4.2.1, one obtains:

Lemma 5.1.1. *The cofibrations are the saturation of the set consisting of the boundary inclusions $\partial\Box^n \to \Box^n$ for $n \geq 0$ and the inclusion $\Box^1 \to \widetilde{\Box}^1$.* $\square$

By Lemma 5.1.1, we have a cofibrantly generated weak factorization system (cofibrations, trivial fibrations).

Definition 5.1.2. We introduce three classes of maps in cSet'.

(i) Let the *marked open box inclusions* $\iota^n_{i,\varepsilon}$ be the marked cubical set maps whose underlying cubical set maps are the open box inclusions $\sqcap^n_{i,\varepsilon} \to \Box^n$, with the critical edge marked in each (except for the domain of $\iota^1_{i,\varepsilon}$, i.e. $\Box^0$, in which the critical edge is not present).

(ii) Let K be the cubical set depicted as:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r, equal] \arrow[d] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, equal] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

Let K' be the marked cubical set that has the middle edge in the above marked. Define the *saturation* map to be the inclusion $K \subseteq K'$.

(iii) For each of the four faces of the square, let the *3-out-of-4 map* associated to that face be the inclusion of $\square^2$ with all but that face marked into $(\square^2)^\sharp$.

The *anodyne maps* are defined as the saturation of the set of maps consisting of the marked open box inclusions, the saturation map, and the 3-out-of-4 maps. The *naive fibrations* are those maps that have the right lifting property against anodyne maps. Call an object X of cSet' a *marked cubical quasicategory* if the map $X \to \square^0$ is a naive fibration.

Note that the definition of a marked open box inclusion combines the intuition behind both the inner and the special outer horns from the theory of marked simplicial sets. For instance, filling 2-dimensional marked open box amounts to composing two edges with an inverse of an equivalence, as can be seen in the following diagrams:

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[d] \\ \bullet \arrow[r, "\sim"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "\sim"] \arrow[d] & \bullet \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[d, "\sim"] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "\sim"'] & \bullet \arrow[d] \\ \bullet & \bullet \end{tikzcd}$$

$$\iota^2_{1,0} \qquad\qquad \iota^2_{1,1} \qquad\qquad \iota^2_{2,0} \qquad\qquad \iota^2_{2,1}$$

Remark 5.1.3. Viewing marked cubical quasicategories as $(\infty, 1)$-categories, the marked edges represent equivalences. The generating anodyne maps have the following $(\infty, 1)$-categorical meanings.

- The n-dimensional marked open box fillings for $n \geq 2$ correspond to composition of maps and homotopies, analogous to filling inner and marked horns in quasicategories. They also ensure that every morphism presented by a marked edge has a left and right inverse, i.e., is an equivalences.

- The 1-dimensional marked open box fillings, $\iota^1_{1,\varepsilon} \colon \square^0 \to (\square^1)^\sharp$, are the inclusions of endpoints into the marked interval; thus marked edges may be lifted along naive fibrations, analogous to the lifting of isomorphisms along isofibrations in 1-category theory.

- The saturation map ensures that equivalences, having both left and right inverses, are marked.

- The 3-out-of-4 maps represent the principle that if three maps in a commuting square are equivalences, then so is the fourth. They encode a condition analogous to the two-out-of-three property.

Remark 5.1.4. For $n \geq 1$, the representable marked cubical set $\square^n$ is not a marked cubical quasicategory, as it lacks fillers for certain marked open boxes. This stands in constrast to the case of simplicial sets, in which the representables Δ^n are quasicategories.

Lemma 5.1.5. *Let X be a marked cubical quasicategory, and $x \colon \square^1 \to X$ an edge of X. Then x is marked if and only if it factors through the inclusion of the middle edge $\square^1 \to K$.*

Proof. The inclusions $K \to K'$ and $(\square^1)^\sharp \to K'$ are both anodyne (the latter as a composite of marked open box fillings). The stated result thus follows from the fact that $X \to \square^0$ has the right lifting property with respect to both of these maps. $\square$

Lemma 5.1.6. *For a marked cubical set X to be a marked cubical quasicategory, it suffices for the map $X \to \square^0$ to have the right lifting property with respect to marked open box fillings and the saturation map.*

Proof. Assume that X has the right lifting property with respect to marked open box inclusions and the saturation map. The proof of Lemma 5.1.5 only requires lifting with respect to these maps, so the marked edges of X are precisely those which factor through K.

To show that $X \to \square^0$ lifts against the 3-out-of-4 maps, we must show that, if three sides of a 2-cube in X are marked, then so is the fourth. Using the fact that the three marked sides factor through K, we can show that the fourth does as well by an exercise in filling three-dimensional marked open boxes. We illustrate this argument for the case where the $(1,0)$-face is unmarked; the other three cases are similar.

Consider the following 2-cube in X:

$$\begin{tikzcd}
x \arrow[r, "p", "\sim"'] \arrow[d, "f"'] & y \arrow[d, "g", "\sim"'] \\
w \arrow[r, "q", "\sim"'] & z
\end{tikzcd}$$

To show that f factors through K, we must construct a pair of 2-cubes as depicted below:

$$\begin{tikzcd}
w \arrow[r, "f_R^{-1}"] \arrow[d, equal] & x \arrow[r, equal] \arrow[d, "f"] & x \arrow[d, equal] \\
w \arrow[r, equal] & w \arrow[r, "f_L^{-1}"] & x
\end{tikzcd}$$

As we have shown that marked 1-cubes factor through K, we assume the existence of similar 2-cubes for g, p, and q, with their left and right inverses denoted similarly.

We construct the left inverse f_L^{-1} by marked open box filling, as depicted below.

$$\begin{tikzcd}
w \arrow[r, dashed, "f_L^{-1}"] \arrow[d, "q"'] & x \arrow[d, "p"] \\
z \arrow[r, "g_R^{-1}"] & y
\end{tikzcd}$$

To obtain the 2-cube witnessing f_L^{-1} as a left inverse for f, we fill the following $(2,0)$-marked open box.

$$\begin{tikzcd}
x \arrow[rr, equal] \arrow[dr, "f"] \arrow[dd, "p"'] & & x \arrow[dr, equal] \arrow[dd, equal] & \\
& w \arrow[rr, "f_L^{-1}", crossing over] & & x \arrow[dd, "p"] \\
y \arrow[rr, equal] \arrow[dr, "g"'] & & y \arrow[dr, equal] & \\
& z \arrow[rr, "g_R^{-1}"] \arrow[from=uu, "q"', crossing over] & & y
\end{tikzcd}$$

Similarly, we construct f_R^{-1} by marked open box filling as follows.

$$\begin{tikzcd}
w \arrow[r, dashed, "f_R^{-1}"] \arrow[d, "q"'] & x \arrow[d, "p"] \\
z \arrow[r, "g_R^{-1}"] & y
\end{tikzcd}$$

To obtain the 2-cube witnessing f_R^{-1} as a right inverse for f, we fill the following $(2,0)$-marked open box.

$$\begin{tikzcd}
w \arrow[rr, "f_R^{-1}"] \arrow[dr, equal] \arrow[dd, "q"'] & & x \arrow[dr, "f"] \arrow[dd, "p", near start] & \\
& w \arrow[rr, equal, crossing over] \arrow[dd, "q", near end] & & w \arrow[dd, "q"] \\
z \arrow[rr, "g_R^{-1}", near start] \arrow[dr, equal] & & y \arrow[dr, "g"] & \\
& z \arrow[rr, equal] & & z
\end{tikzcd}$$

Thus we see that f factors through K, and is therefore marked. $\square$

Remark 5.1.7. In view of Lemma 5.1.6, it is natural to wonder whether omitting the 3-out-of-4 maps as generators would change the class of anodyne maps. To see that it would, observe that, using the small object argument, we can factor any three-out-of-four map as a composite of a map in the saturation of the marked open box fillings and the saturation map, followed by a map having the right lifting property with respect to these maps. Examining the details of this construction, we can see that the second of these maps will not have the right lifting property with respect to the 3-out-of-4 maps. Thus the 3-out-of-4 maps are not in the saturation of the other two classes of generating anodynes.

One may further note that, without the 3-out-of-4 maps as generators, anodyne maps would not be closed under pushout product with cofibrations, e.g., $\iota^1_{1,0} \mathbin{\widehat{\otimes}} (\partial\square^1 \to (\square^1)^\sharp)$ is a 3-out-of-4 map. . This makes them crucial for our development.

Definition 5.1.8. Given a map $f\colon X \to Y$ of marked cubical sets, a *naive fibrant replacement* of f consists of a diagram as depicted below, with $\overline{X}$ and

$\overline{Y}$ marked cubical quasicategories, ι_X and ι_Y anodyne, and $\overline{f}$ a naive fibration.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota_X"'] & Y \arrow[d, "\iota_Y"] \\ \overline{X} \arrow[r, "\overline{f}"] & \overline{Y} \end{tikzcd}$$

We have a cofibrantly generated weak factorization system (anodyne maps, naive fibrations). This induces a functorial factorization of any map $X \to Y$ as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\eta_f"', "\mathsf{anod}"] & & Y \\ & Mf \arrow[ur, "Qf"', "\mathsf{n.f.}"] & \end{tikzcd}$$

where Q is an endofunctor on $(\mathsf{cSet}')^{\to}$ sending objects to naive fibrations and $\eta\colon \mathrm{Id} \to Q$ is pointwise anodyne. Where f is the unique map $X \to \square^0$, we write η_X for η_f. Given $f\colon X \to Y$, we can use this factorization to obtain a canonical naive fibrant replacement of f:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_{\eta_Y f}"'] & Y \arrow[d, "\eta_Y"] \\ \overline{X} \arrow[r, "Q(\eta_Y f)"] & \overline{Y}. \end{tikzcd}$$

We declare f to be a *weak equivalence* if $Q(\eta_Y f)$ is a trivial fibration. A *trivial cofibration* is a map that is a cofibration and weak equivalence, and a *fibration* is a map that has the right lifting property against trivial cofibrations.

We now want to show that if Y is a marked cubical quasicategory, so is $\underline{\hom}_L(X, Y)$. The following lemma on pushout-products helps with the proof of this fact.

Lemma 5.1.9. *The pushout product of two cofibrations is a cofibration. Fur-*

thermore, the pushout product of an anodyne map and a cofibration is anodyne.

Proof. Since $\otimes$ preserves colimits in each variable and anodynes are stable under pushouts and transfinite compositions, we can use induction on skeleta to show that if $S \to T$ is one of the generating cofibrations (resp. anodynes), then $(S \to T)\hat{\otimes}(\partial\square^n \to \square^n)$ and $(S \to T)\hat{\otimes}(\square^1 \to (\square^1)^\sharp)$ are cofibrations (resp. anodyne). This will show that if i and j are cofibrations, and i is anodyne, then $i \mathbin{\widehat{\otimes}} j$ is anodyne; the proof for the case where j is anodyne is entirely analogous.

Several cases can be taken care of by the following fact: If $f\colon A \to B$ is an inclusion which is a surjection on vertices and $p\colon X \to Y$ is an isomorphism of underlying cubical sets, then $f \mathbin{\widehat{\otimes}} p$ is an isomorphism. This follows because the pushout-product is an isomorphism of underlying cubical sets, and so we need only consider what edges are marked. But the marked edges of $(B \otimes Y)_e = (B_e \times Y_0) \cup_{B_0 \times Y_0} (B_0 \times Y_e)$, and since each map is a bijection on vertices, all of these edges appear in $(B \otimes X) \cup_{A \otimes X} (A \otimes Y)$.

This claim, along with the fact that taking the pushout-product with $\varnothing \to \square^0$ is the identity, handles all but the following pushout products:

- $(\partial\square^m \to \square^m) \mathbin{\widehat{\otimes}} (\partial\square^n \to \square^n)$: this is the map $\partial\square^{m+n} \to \square^{m+n}$. This completes the proof of the first statement, concerning the pushout product of two cofibrations; the remaining cases complete the second statement, concerning the pushout product of a cofibration and an anodyne map.

- $\iota^m_{i,\varepsilon} \mathbin{\widehat{\otimes}} (\partial\square^n \to \square^n)$: the underlying cubical set map is the open box inclusion $\sqcap^{m+n}_{i,\varepsilon} \to \square^{m+n}$, with edges in the codomain being marked if and only if they are present and marked in the domain. The critical edge is marked, so this is anodyne as a pushout of a marked open box filling.

- $\iota^1_{i,\varepsilon} \mathbin{\widehat{\otimes}} (\partial\square^1 \to (\square^1)^\sharp)$: this is the 3-out-of-4 map associated to the face $(1, 1-\varepsilon)$. □

Corollary 5.1.10. *If $f\colon A \to B$ is a cofibration and $g\colon X \to Y$ is a naive fibration, then the pullback exponential $f \rhd g\colon \underline{\mathrm{hom}}(A,Y) \to \underline{\mathrm{hom}}(A,X) \times_{\underline{\mathrm{hom}}(A,Y)} \underline{\mathrm{hom}}(B,Y)$ (where $\underline{\mathrm{hom}}$ may designate either $\underline{\mathrm{hom}}_L$ or $\underline{\mathrm{hom}}_R$) is a naive fibration. Furthermore, if f is anodyne or g is a trivial fibration, then $f \rhd g$ is a trivial fibration.*

In particular, if Y is a marked cubical quasicategory, then for any X, $\underline{\mathrm{hom}}(X,Y)$ is a marked cubical quasicategory.

Proof. Let $i\colon C \to D$ be anodyne; we wish to show that $f \rhd g$ has the right lifting property with respect to i. By a standard duality, it suffices to show that g has the right lifting property with respect to $i \hat{\otimes} f$. This map is anodyne by Lemma 5.1.9, so the first statement holds.

For the second statement, we can apply the same result with i an arbitrary cofibration. Then g has the right lifting property with respect to $i \mathbin{\widehat{\otimes}} f$, either because f, and hence also $i \mathbin{\widehat{\otimes}} f$, are anodyne, or because $i \mathbin{\widehat{\otimes}} f$ is a cofibration and g is a trivial fibration.

The third statement follows from the first by the fact that $\underline{\mathrm{hom}}(X,Y) \to \square^0$ is the pullback exponential of the cofibration $\varnothing \to X$ with the naive fibration $Y \to \square^0$. □

5.2 Homotopies

Next we define the closely-related concepts of connected components in a marked cubical set, and homotopies of maps between cubical sets.

Definition 5.2.1. For a marked cubical set X, let $\sim_0$ denote the relation on X_0, the set of vertices of X, given by $x \sim_0 y$ if there is a marked edge from x to y in X. Let $\sim$ denote the smallest equivalence relation on X_0 containing $\sim_0$.

Remark 5.2.2. For $x, y \in X_0$, one can easily see that $x \sim y$ if and only if x and y are connected by a zigzag of marked edges.

Definition 5.2.3. For a marked cubical set X, the *set of connected components* $\pi_0(X)$ is $X_0/\sim$.

We may observe that the construction of $\pi_0(X)$ is functorial, since maps of marked cubical sets preserve marked edges, and hence preserve the equivalence relation $\sim$.

Definition 5.2.4. An *elementary left homotopy* $h\colon f \sim g$ between maps $f, g\colon A \to B$ is a map $h\colon (\square^1)^\sharp \otimes A \to B$ such that $h|_{\{0\}\otimes A} = f$ and $h|_{\{1\}\otimes A} = g$. Note that the elementary left homotopy h corresponds to an edge $(\square^1)^\sharp \to \underline{\hom}_L(A, B)$ between the vertices corresponding to f and g. A *left homotopy* between f and g is a zig-zag of elementary left homotopies.

A left homotopy from f to g corresponds to a zig-zag of marked edges in $\underline{\hom}_L(A, B)$ and so maps from A to B are left homotopic exactly if they are in the same connected component of $\underline{\hom}_L(A, B)$. We write $[A, B]$ for the set of left homotopy classes of maps $A \to B$.

These induce notions of *elementary left homotopy equivalence* and *left homotopy equivalence*. Each of these notions has a "right" variant using $A \otimes (\square^1)^\sharp$ and $\underline{\hom}_R(A, B)$. Unless the potential for confusion arises or a statement depends on the choice, we will drop the use of "left" and "right".

Lemma 5.2.5. *In a marked cubical quasicategory X, the relations $\sim_0$ and $\sim$ coincide.*

Proof. Using 2-dimensional open box fillers with certain edges degenerate, and the 3-out-of-4 property, we can reduce any zigzag of marked edges connecting x and y in X to a single marked edge from x to y. $\square$

By adjointness, we obtain the following corollary.

Corollary 5.2.6. *If $f, g\colon A \to B$ are homotopic and B is a marked cubical quasicategory, then f and g are elementarily homotopic. Hence, between marked cubical quasicategories homotopy equivalences coincide with elementary homotopy equivalences.*

Proof. By Corollary 5.1.10, $\underline{\hom}(A, B)$ is a marked cubical quasicategory, and so $\sim_0$ is an equivalence relation on $\underline{\hom}(A, B)_0$ by Lemma 5.2.5. Translating what this means for homotopies gives the result. $\square$

Lemma 5.2.7. *If $f, g\colon X \to Y$ are left homotopic, then for any Z, then the induced maps $\underline{\hom}_L(Y, Z) \to \underline{\hom}_L(X, Z)$ are right homotopic.*

Proof. We consider the case of elementary homotopies; the general result follows from this. An elementary left homotopy $f \sim g$ is given by a map $H\colon (\square^1)^\sharp \otimes X \to Y$. Pre-composition with H induces a map $\underline{\hom}_L(Y, Z) \to \underline{\hom}_L((\square^1)^\sharp \otimes X, Z)$. Under the adjunction defining $\underline{\hom}_L$, this corresponds to a map $\underline{\hom}_L(Y, Z) \otimes (\square^1)^\sharp \otimes X \to Z$, which in turn corresponds to a map $\underline{\hom}_L(Y, Z) \otimes (\square^1)^\sharp \to \underline{\hom}_L(X, Z)$. This defines an elementary right homotopy between the pre-composition maps induced by f and g. $\square$

5.3 Category theory in a marked cubical quasicategory

Let X be a marked cubical quasicategory and $x, y \in X_0$. We will write $X_1(x, y)$ for the subset of X_1 consisting of 1-cubes f with $f\partial_{1,0} = x$ and $f\partial_{1,1} = y$. Define an equivalence relation relation $\sim_X$ on the set $X_1(x, y)$ of edges from x to y as follows: $f \sim_X g$ if and only if there is a 2-cube in X of the form

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] & y \arrow[d, equal] \\ x \arrow[r, "g"] & y \end{tikzcd}$$

It is straightforward to verify that this is indeed an equivalence relation: reflexivity follows from degeneracies, whereas symmetry and transitivity are given by filling 3-dimensional open boxes.

We now define three increasingly strong refinements of the concept of a homotopy equivalence.

Definition 5.3.1. Let $f : X \to Y$ be a map in cSet. Then:

- f is a *semi-adjoint equivalence* if there exist $g : Y \to X$ and homotopies $H : gf \sim \mathrm{id}_X$, $K : fg \sim \mathrm{id}_Y$ such that $fH \sim Kf$ as edges of $\underline{\mathrm{hom}}(X, Y)$;
- f is a *strong homotopy equivalence* if there exist g, H, K as above with $fH = Kf$;
- a map $g : Y \to X$ is a *strong deformation section* of f if $fg = \mathrm{id}_Y$ and there exists a homotopy $H : gf \sim \mathrm{id}_X$ such that $fH = \mathrm{id}_f$.

Our next goal will be two show the following:

Lemma 5.3.2. *Let $f\colon X \to Y$ be a map of marked cubical quasicategories. The following are equivalent:*

(i) f is a homotopy equivalence;

(ii) f is a semi-adjoint equivalence.

Furthermore, if f is a naive fibration, then these are equivalent to:

(iii) f is a strong homotopy equivalence.

We will prove this by means of a 2-categorical argument.

We define the *homotopy category* $\mathsf{Ho}X$ of a marked cubical quasicategory X as follows:

- the objects of $\mathsf{Ho}X$ are the 0-cubes of X;
- the morphisms from x to y in $\mathsf{Ho}X$ are the equivalence classes of edges $X_1(x,y)/\sim_X$;
- the identity map on $x \in X_0$ is given by $x\sigma_1$;
- the composition of $f\colon x \to y$ and $g\colon y \to z$ is given by filling the open box

$$\begin{tikzcd} x \arrow[r, dashed, "gf"] \arrow[d, "f"'] & z \arrow[d, equal] \\ y \arrow[r, "g"] & z \end{tikzcd}$$

Using standard arguments about open box fillings, one verifies the following lemma.

Lemma 5.3.3. *The above data define a category.* □

Proof. We must show that composition is well-defined, associative, and unital with the given identities.

To see that it is well-defined, suppose that $f \sim f'$, $g \sim g'$, $h \sim h'$, and $gf = h$. Then we can construct the following $(3,1)$-open box:

$$\begin{tikzcd}
x \arrow[rr, "h"] \arrow[dr, equal] \arrow[dd, "f"'] & & z \arrow[dr, equal] \arrow[dd, equal] & \\
& x \arrow[rr, "h'"] \arrow[dd, "f'"'] & & z \arrow[dd, equal] \\
y \arrow[rr, "g"] \arrow[dr, equal] & & z \arrow[dr, equal] & \\
& y \arrow[rr, "g'"] & & z
\end{tikzcd}$$

As the critical edge is degenerate, this open box admits a filler; the $(3,1)$-face of this filler witnesses $g'f' = h'$.

To see that composition is associative, consider a composable triple of edges f, g, h. We can construct the following $(3,0)$-open box:

$$\begin{tikzcd}
x \arrow[rr, "(hg)f"] \arrow[dr, "f"] \arrow[dd, "gf"'] & & w \arrow[dr, equal] \arrow[dd, equal] & \\
& y \arrow[rr, "hg"] \arrow[dd, "g"'] & & w \arrow[dd, equal] \\
z \arrow[rr, "h"] \arrow[dr, equal] & & w \arrow[dr, equal] & \\
& z \arrow[rr, "h"] & & w
\end{tikzcd}$$

This open box admits a filler, since the critical edge is degenerate; the $(3,0)$-face of this filler witnesses $h(gf) = (hg)f$.

Finally, for an edge f from x to y, the equalities $(\sigma_1 y)f = f$ and $f(\sigma_1 x) = f$ are witnessed by the 2-cubes $f\gamma_{1,0}$ and $f\sigma_2$, respectively. □

Lemma 5.3.4. *Let X be a marked cubical quasicategory. There is a 2-cube of the form*

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "p"'] & y \arrow[d, "g"] \\
z \arrow[r, "q"] & w
\end{tikzcd}$$

if and only if $gf = qp$ in $\mathsf{Ho}X$.

Proof. Consider the following 3-cube:

$$\begin{tikzcd}
x \arrow[rr, "qp"] \arrow[dr, "p"] \arrow[dd, "f"'] & & w \arrow[dd, equal] \arrow[dr, equal] & \\
& z \arrow[rr, "q"] \arrow[dd, "q"] & & w \arrow[dd, equal] \\
y \arrow[rr, "g"] \arrow[dr, "g"'] & & w \arrow[dr, equal] & \\
& w \arrow[rr, equal] & & w
\end{tikzcd}$$

The equality $gf = qp$ in $\mathsf{Ho}X$ is equivalent to the existence of a filler for the back face of this cube, using the fact that composition in $\mathsf{Ho}X$ is well-defined. Thus we want to show that there is a filler for the back face if and only if there is a filler for the left face. If we assume that either of these 2-cubes exists, then together with the remaining faces of the cube depicted above, it forms a marked open box in X, with critical edge $w\sigma_1$. Thus we can fill this open box to obtain a filler for the missing face. □

Lemma 5.3.5. *Let X be the underlying cubical set of a marked cubical quasicategory X'. The categories $\mathsf{Ho}X'$ and $\tau_1 X$ are equivalent.*

Proof. There is a natural inclusion $\mathsf{Ho}X' \to \tau_1 X$, which is the identity on objects and takes a 1-cube f to a string of length 1 consisting of f. This is clearly faithful and essentially surjective. To see that it is full, we simply fill in 2-dimensional open boxes with one degenerate edge to reduce a sequence of arbitrary length to a sequence of length 1. $\square$

The assignment $X \mapsto \mathsf{Ho}X$ extends in a straightforward manner to a functor taking a marked cubical quasicategory to its homotopy category. Postcomposing this functor with $\mathsf{core}\colon \mathsf{Cat} \to \mathsf{Gpd}$, we obtain a groupoid $\mathsf{Ho}^\sharp X$.

Lemma 5.3.6. *The groupoid $\mathsf{Ho}^\sharp X$ can be constructed directly as follows:*

- *Objects are 0-cubes of X;*
- *Morphisms from x to y are equivalence classes of marked edges from x to y;*
- *Composition and identities are defined as in $\mathsf{Ho}X$.*

Proof. Let X be a marked cubical quasicategory. By definition, an edge $f\colon \square^1 \to X$ is invertible in $\mathsf{Ho}X$ if and only if it factors through the map $\square^1 \to K$ which picks out the middle edge. Since the inclusions $(\square^1)^\sharp \to K'$ and $K \to K'$ are anodyne, this holds if and only if f is marked. $\square$

Definition 5.3.7. Define a strict 2-category $\mathsf{Ho_2cSet}'$ whose objects are the marked cubical quasicategories and whose mapping category from X to Y is

$$\mathsf{Ho_2cSet}'(X,Y) := \mathsf{Ho}\underline{\mathrm{hom}}_L(X,Y).$$

This means the 1-morphisms are the usual 1-morphisms $X \to Y$, and the 2-morphisms are maps $X \otimes \square^1 \to Y$, modulo an equivalence relation. Denote the (vertical) composition in $\mathsf{Ho}\underline{\mathrm{hom}}_L(X,Y)$ with $\circ$. The (horizontal) composition

$$\mathsf{Ho}\underline{\mathrm{hom}}_L(Y,Z) \times \mathsf{Ho}\underline{\mathrm{hom}}_L(X,Y) \to \mathsf{Ho}\underline{\mathrm{hom}}_L(X,Z)$$

(which will be written by concatenation) is defined on objects by the usual composition. If $H\colon Y \otimes \square^1 \to Z$ and $K\colon X \otimes \square^1 \to Y$ are morphisms $K\colon g \to g'$ and $H\colon f \to f'$, respectively, define the morphism $KH\colon gf \to g'f'$ by choosing a fill for the open box of $\underline{\mathrm{hom}}_L(X,Z)$ depicted by

$$\begin{tikzcd} gf \arrow[r, "Kf"] \arrow[d, equal] & g'f \arrow[d, "g'H"] \\ gf \arrow[r, dotted, "KH"] & g'f' \end{tikzcd}$$

where the top edge is induced by the composite $X \otimes \square^1 \to Y \otimes \square^1 \to Z$ and the right edge by $X \otimes \square^1 \to Y \to Z$. The fact that the $\underline{\mathrm{hom}}_L(X,Y)$ are marked cubical quasicategories ensures this defines a well-defined, associative, unital, and functorial operation. For functoriality, note that the morphism $X \otimes \square^1 \otimes \square^1 \overset{H\otimes\square^1}{\to} Y \otimes \square^1 \overset{K}{\to} Z$ yields a 2-cube $\square^2 \to \underline{\mathrm{hom}}_L(X,Z)$ which can be depicted as

$$\begin{tikzcd} gf \arrow[r, "Kf"] \arrow[d, "gH"'] & g'f \arrow[d, "g'H"] \\ gf' \arrow[r, "Kf'"] & g'f' \end{tikzcd}$$

and so by Lemma 5.3.4, we have $(g'H) \circ (Kf) = (Kf') \circ (gH)$, which implies the interchange law.

Definition 5.3.8. Let $\mathsf{Ho}_2^\sharp\mathsf{cSet}'$ denote the maximal $(2,1)$-category contained in $\mathsf{Ho}_2\mathsf{cSet}'$, i.e. the 2-category whose objects are marked cubical sets, with $\mathsf{Ho}_2^\sharp\mathsf{cSet}'(X,Y) = \mathsf{Ho}^\sharp\underline{\mathrm{hom}}_L(X,Y)$, and the 2-categorical operations induced by those of $\mathsf{Ho}_2\mathsf{cSet}'$.

The $\mathsf{Ho}^\sharp$ construction, together with the following general results about $(2,1)$-categories, give us the desired result about compatibility of homotopies.

Lemma 5.3.9 (Undergraduate Lemma)**.** *Let X be an object in a $(2,1)$-category C, and let $H: p \sim id_X$ be a morphism in $\mathsf{C}(X,X)$. Then $pH = Hp$.*

Proof. By the interchange law,

$$H \circ (pH) = (H\mathrm{id}_X) \circ (pH) = (\mathrm{id}_X H) \circ (Hp) = H \circ (Hp).$$

Since $\mathsf{C}(X,X)$ is a groupoid, we can cancel H. □

Lemma 5.3.10 (Graduate Lemma)**.** *Let X, Y be objects in a $(2,1)$-category C, $f: X \leftrightarrows Y : g$ two morphisms between them, and $H: gf \to id_X$ and $K: fg \to id_Y$ two 2-cells. Then there is a 2-cell $K': fg \to id_Y$ for which $K'f = fH$.*

Proof. Define $K' := K \circ (fHg) \circ (Kfg)^{-1}$. Now, we compute:

$$\begin{aligned} K'f &= Kf \circ (fHgf) \circ (Kfgf)^{-1} \\ &= Kf \circ (fgfH) \circ (Kfgf)^{-1} && \text{(by 5.3.9)} \\ &= fH && \text{(by naturality/interchange)} \end{aligned}$$

□

Proof of Lemma 5.3.2. The implications $(iii) \Rightarrow (ii) \Rightarrow (i)$ are clear. The implication $(i) \Rightarrow (ii)$ follows from applying Lemma 5.3.10 to the $(2,1)$-category $\mathsf{Ho}^\sharp_2\mathsf{cSet}'$.

Now let f be a naive fibration and a semi-adjoint equivalence. By Corollary 5.1.10, the map $\underline{\mathrm{hom}}(X,X) \to \underline{\mathrm{hom}}(X,Y)$ is a naive fibration. A simple exercise in 2-dimensional marked open box filling, using this fact and the definition of a semi-adjoint equivalence, shows that there exists a homotopy $H' \colon gf \sim \mathrm{id}_X$ such that $fH' = Kf$. ◻

5.4 Fibration category of marked cubical quasicategories

Lemma 5.4.1. *Every anodyne map between marked cubical quasicategories is a homotopy equivalence.*

Proof. Now let $f \colon X \to Y$ be anodyne, with X and Y marked cubical quasicategories. We can obtain a retraction $r \colon Y \to X$ as a lift in the following diagram:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d] \\ Y \arrow[r] & \square^0 \end{tikzcd}$$

We can then obtain a left homotopy $fr \sim \mathrm{id}_Y$ as a lift in the following diagram:

$$\begin{tikzcd} (\partial \square^1 \otimes Y) \cup ((\square^1)^\sharp \otimes X) \arrow[r, "{[[fr, \mathrm{id}_Y], f\pi_1]}"] \arrow[d] & Y \arrow[d] \\ (\square^1)^\sharp \otimes Y \arrow[r] & \square^0 \end{tikzcd}$$

The lift exists since the left-hand map is anodyne by Lemma 5.1.9.

An analogous proof shows that f is a right homotopy equivalence. ◻

Lemma 5.4.2. *Let $f \colon X \to Y$ be a naive fibration. The following are equivalent:*

(i) f is a trivial fibration;

(ii) f has a strong deformation section;

(iii) f is a strong homotopy equivalence.

Proof. If $f\colon X \to Y$ is a trivial fibration, then we can obtain a section $g\colon Y \to X$ as a lift of the following diagram:

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

We can then obtain a left homotopy $H\colon gf \sim \mathrm{id}_X$ satisfying $fH = \mathrm{id}_f$ as a lift in the following diagram:

$$\begin{tikzcd}
X \sqcup X \arrow[r, "{[sf, \mathrm{id}_X]}"] \arrow[d, tail] & X \arrow[d, "f"] \\
(\square^1)^\sharp \otimes X \arrow[r, "f\pi_X"] & Y
\end{tikzcd}$$

This shows (i) $\Rightarrow$ (ii), and the implication (ii) $\Rightarrow$ (iii) is trivial. To show that (iii) $\Rightarrow$ (i), we first show that (iii) implies the following condition:

(iii)' the canonical map $\iota^1_{1,0} \rhd f \to f$ in $(\mathsf{cSet}')^\to$ admits a section.

To see (iii) $\Rightarrow$ (iii)', suppose f is a strong homotopy equivalence with homotopy inverse $g\colon Y \to X$ and homotopies $H\colon gf \sim \mathrm{id}_X, K\colon fg \sim \mathrm{id}_Y$ satisfying $fH = Kf$. Then we have the following commuting diagram in cSet':

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & \underline{\hom}((\square^1)^\sharp, X) \arrow[r] \arrow[d, "\iota^1_{1,0} \rhd f"] & X \arrow[d, "f"] \\
Y \arrow[r] & X \times_Y \underline{\hom}((\square^1)^\sharp, Y) \arrow[r] & Y
\end{tikzcd}$$

The top-left map is the adjunct of H, while the bottom-left map is induced by g and the adjunct of K; the right-hand square is as in the statement of (iii)', and hence the composite square is simply the identity square on f.

Finally, note that $\iota^1_{1,1} \hat{\rhd} f$ is a trivial fibration by Corollary 5.1.10. Therefore, if the square given in the statement of (iii)' has a section, then f is a trivial fibration as a retract of a trivial fibration. Thus (iii)' ⇒ (i). □

Corollary 5.4.3. *A map $f\colon X \to Y$ between marked cubical quasicategories is a trivial fibration exactly if it is a homotopy equivalence and a naive fibration.*

Proof. This follows from Lemmas 5.3.2 and 5.4.2, together with the fact that every trivial fibration is a naive fibration since all anodyne maps are cofibrations. □

Proposition 5.4.4. *The category of marked cubical quasicategories forms a fibration category, with naive fibrations as the fibrations and homotopy equivalences as the weak equivalences.*

Proof. The class of homotopy equivalences is closed under 2-out-of-3. Corollary 5.4.3 shows that the maps between marked cubical quasicategories which are naive fibrations and homotopy equivalences are exactly the trivial fibrations; both fibrations and trivial fibrations are defined via a right lifting property, and hence they are stable under pullback. By Lemma 5.4.1, each anodyne map between marked cubical quasicategories is a homotopy equivalence, and so the (anodyne, naive fibration)-factorization gives the factorization axiom. □

Lemma 5.4.5. *Let $f\colon X \to Y$ be a map between marked cubical quasicategories. Then the following conditions are equivalent:*

(i) f is a weak equivalence;

(ii) f is a left homotopy equivalence;

(iii) f is a right homotopy equivalence.

Proof. Consider the canonical naive fibrant replacement of f used in the definition of the weak equivalences:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota_X"'] & Y \arrow[d, "\iota_Y"] \\ \overline{X} \arrow[r, "\overline{f}"] & \overline{Y} \end{tikzcd}$$

(here $\iota_Y = \eta_Y, \overline{f} = Q(\eta_Y f), \iota_X = \eta_{\eta_Y f}$).

By Lemma 5.4.1, ι_X and ι_Y are left homotopy equivalences. Since left homotopy equivalences satisfy the two-out-of-three property, f is a left homotopy equivalence if and only if $\overline{f}$ is one. By Corollary 5.4.3, $\overline{f}$ is a left homotopy equivalence if and only if it is a trivial fibration, i.e. if and only if f is a weak equivalence. So (i) $\Leftrightarrow$ (ii); an analogous argument shows (i) $\Leftrightarrow$ (iii). □

5.5 Cofibration category of marked cubical sets

Our next result shows that the definition of the weak equivalences is not sensitive to the choice of naive fibrant replacement.

Lemma 5.5.1. *Let $f\colon X \to Y$ be a map of marked cubical sets. The following are equivalent:*

(i) f is a weak equivalence.

(ii) there exists a naive fibrant replacement of f by a trivial fibration;

(iii) any naive fibrant replacement of f is a trivial fibration.

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are immediate from the definition of the weak equivalences. To prove (ii) $\Rightarrow$ (iii), consider a map $f\colon X \to Y$ having a naive fibrant replacement by a trivial fibration $\overline{f}\colon \overline{X} \to \overline{Y}$, and an arbitrary naive fibrant replacement $\overline{f}'\colon \overline{X}' \to \overline{Y}'$ of f. As depicted below, let $\overline{f}''\colon \overline{X}'' \to \overline{Y}''$ be a naive fibrant replacement of the induced map between the pushouts $\overline{X} \cup_X \overline{X}' \to \overline{Y} \cup_Y \overline{Y}'$.

$$
\begin{tikzcd}
X \arrow[rr] \arrow[dr, "f"] \arrow[dd] & & \overline{X}' \arrow[dr, "\overline{f}'"] \arrow[dd] & & & & \\
& Y \arrow[rr] \arrow[dd] & & \overline{Y}' \arrow[dd] & & & \\
\overline{X} \arrow[rr] \arrow[dr, "\overline{f}"] & & \overline{X} \cup_X \overline{X}' \arrow[rrr] \arrow[dr] & & & \overline{X}'' \arrow[dr, "\overline{f}''"] & \\
& \overline{Y} \arrow[rr] & & \overline{Y} \cup_Y \overline{Y}' \arrow[rrr] & & & \overline{Y}''
\end{tikzcd}
$$

The maps $\overline{X} \to \overline{X}'', \overline{Y} \to \overline{Y}'', \overline{X}' \to \overline{X}'', \overline{Y}' \to \overline{Y}''$ are anodyne, as anodyne maps are closed under pushout and composition. Furthermore, $\overline{f}$ is a trivial fibration by assumption. Thus all of these maps are homotopy equivalences by Lemma 5.4.1 and Corollary 5.4.3. So we can apply the two-out-of-three property to see that $\overline{f}''$ is a homotopy equivalence; applying it again, we see that $\overline{f}'$ is a homotopy equivalence. Thus $\overline{f}'$ is a trivial fibration by Corollary 5.4.3. Since $\overline{f}'$ was arbitrary, we have shown that f satisfies (iii). $\square$

Corollary 5.5.2. *Every anodyne map is a weak equivalence.*

Proof. Let $f\colon X \to Y$ be anodyne. The following diagram gives a naive fibrant replacement of f:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_Y f"'] & Y \arrow[d, "\eta_Y"] \\ \overline{Y} \arrow[r, equal] & \overline{Y} \end{tikzcd}$$

Since $\mathrm{id}_{\overline{Y}}$ is a trivial fibration, f is a weak equivalence by Lemma 5.5.1. $\square$

Proposition 5.5.3. *The following are equivalent for a marked cubical map $A \to B$:*

(i) $A \to B$ is a weak equivalence;

(ii) for any marked cubical quasicategory X, the map $\underline{\mathrm{hom}}(B, X) \to \underline{\mathrm{hom}}(A, X)$ is a homotopy equivalence;

(iii) for any marked cubical quasicategory X, the map $\pi_0(\underline{\mathrm{hom}}(B, X)) \to \pi_0(\underline{\mathrm{hom}}(A, X))$ is a bijection.

Proof. First, suppose that $A \to B$ is a weak equivalence. Thus, there is a square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ \overline{A} \arrow[r] & \overline{B} \end{tikzcd}$$

with $A \to \overline{A}$ and $B \to \overline{B}$ anodyne, and $\overline{A} \to \overline{B}$ a trivial fibration. By Corollary 5.4.3, $\overline{A} \to \overline{B}$ is a left homotopy equivalence.

Applying $\underline{\mathrm{hom}}_L(-, X)$ to the diagram above, we obtain a diagram in which all objects are marked cubical quasicategories by Corollary 5.1.10:

$$\begin{tikzcd} \underline{\mathrm{hom}}_L(A, X) & \underline{\mathrm{hom}}_L(B, X) \arrow[l] \\ \underline{\mathrm{hom}}_L(\overline{A}, X) \arrow[u] & \underline{\mathrm{hom}}_L(\overline{B}, X) \arrow[l] \arrow[u] \end{tikzcd}$$

The vertical maps are trivial fibrations by Corollary 5.1.10, hence homotopy equivalences by Corollary 5.4.3. By Lemma 5.2.7, the bottom horizontal map is a right homotopy equivalence, since $\overline{A} \to \overline{B}$ is a left homotopy equivalence. Hence so is the upper horizontal map by 2-out-of-3. Thus we have proven (i) $\Rightarrow$ (ii).

The implication (ii) $\Rightarrow$ (iii) is clear, so it remains to show (iii) $\Rightarrow$ (i). For that, we first observe that it suffices to consider A and B marked cubical quasicategories. To see this, consider the canonical naive fibrant replacement $\overline{f}\colon \overline{A} \to \overline{B}$ of a map $f\colon A \to B$. By definition, f is a weak equivalence if and only if $\overline{f}$ is a trivial fibration; by Corollary 5.4.3 and Lemma 5.4.5, this holds if and only if $\overline{f}$ is a weak equivalence. Furthermore, the anodyne maps ι_X, ι_Y are weak equivalences by Corollary 5.5.2, and therefore satisfy (iii); hence f satisfies (iii) if and only if $\overline{f}$ does, by the 2-out-of-3 property for bijections.

Hence we can assume A and B are marked cubical quasicategories. Now take $X := A$ and set $g := (\pi_0 f^*)^{-1}[\mathrm{id}_A]$. The verification that a representative of the class $g \in \pi_0 \underline{\mathrm{hom}}_L(B, A)$ defines a homotopy inverse of f is straightforward; thus f is a weak equivalence by Lemma 5.4.5. □

Corollary 5.5.4. *The weak equivalences satisfy the 2-out-of-6 property (and hence the 2-out-of-3 property).*

Proof. This is immediate from condition (iii) of Proposition 5.5.3. □

Corollary 5.5.5. *The endpoint inclusions $\square^0 \to K$ are trivial cofibrations.*

Proof. The maps in question are clearly cofibrations. To see that they are weak equivalences, consider the following commuting diagram:

$$\begin{tikzcd} \square^0 \arrow[r] \arrow[d] & K \arrow[d] \\ (\square^1)^\sharp \arrow[r] & K' \end{tikzcd}$$

The left, right, and bottom maps are anodyne, hence weak equivalences by Corollary 5.5.2. Thus the top map is a weak equivalence by Corollary 5.5.4. □

Lemma 5.5.6. *Trivial fibrations are weak equivalences.*

Proof. If $A \to B$ is a trivial fibration, then it is a homotopy equivalence by Corollary 5.4.3. Hence $\underline{\mathrm{hom}}(B, X) \to \underline{\mathrm{hom}}(A, X)$ is a homotopy equivalence for all marked cubical quasicategories X by Lemma 5.2.7, and hence $A \to B$ a weak equivalence by Proposition 5.5.3. □

Proposition 5.5.7. *The category of marked cubical sets forms a cofibration category with the above classes of weak equivalences and cofibrations.*

Proof. The class of weak equivalences is closed under 2-out-of-3 by Corollary 5.5.4. The category clearly has an initial object and pushouts. Cofibrations are the left class in a weak factorization system, hence stable under pushout. Using the characterization of weak equivalences given by item (ii) of Proposition 5.5.3, stability of cofibrations that are weak equivalences under pushout reduces to stability of trivial fibrations under pullback. By Lemma 5.5.6, trivial fibrations are weak equivalences, so the (cofibration, trivial fibration)-factorization gives the factorization axiom. □

5.6 Model structure for marked cubical quasicategories

Definition 5.6.1. A marked cubical set is *finite* (resp. *countable*) if it has only finitely (resp. countably) many non-degenerate cubes. The *cardinality* of a finite marked cubical set is its total number of non-degenerate cubes, in all dimensions.

Lemma 5.6.2. *The trivial fibrations form an ω_1-accessible, ω_1-accessibly embedded subcategory of* $(\mathsf{cSet}')^{\to}$.

Proof. It suffices to show two things: that filtered colimits (and hence in particular ω_1-filtered colimits) in cSet' preserve trivial fibrations, and that any trivial fibration can be expressed as an ω_1-filtered colimit in cSet' of trivial fibrations between countable marked cubical sets. The first statement follows from the fact that the domains and codomains of the generating cofibrations are finite.

For the second statement, consider a trivial fibration $f: X \to Y$. Let P denote the poset of countable subcomplexes of X; note that we consider edges of subcomplexes of X to be marked if and only if they are marked in X. This category is ω_1-filtered since any countable union of countable subcomplexes is countable.

Let i denote the inclusion $P \hookrightarrow \mathsf{cSet}'$; the colimit of this diagram is X. The images under f of the countable subcomplexes of X, with the natural inclusions, also define a diagram $fi: P \to \mathsf{cSet}'$. One can easily show that trivial fibrations are surjective on underlying cubical sets; thus every cube of Y appears in fS for some countable subcomplex $S \subseteq X$. So fi is a filtered diagram of subcomplexes of Y, in which the maps are inclusions and each cube of Y is contained in some object of the diagram, with every marked edge of Y being marked in some subcomplex in the diagram. From this, one can show that the colimit of fi is Y. The map f induces a natural transformation from i to fi, whose induced map on the colimits is f itself.

However, it may not be the case that for every component of this natural transformation is a trivial fibration. Thus we will replace i by a different diagram, still having colimit X, with a natural transformation to fi which does satisfy this property. For each countable subcomplex $S \subseteq X$, we will define a new countable subcomplex $\overline{S} \subseteq X$, such that $f\overline{S} = fS$, $f|_{\overline{S}} \colon \overline{S} \to f\overline{S}$ is a trivial fibration, and for $S' \subseteq S$, we have $\overline{S'} \subseteq \overline{S}$.

We first define $\overline{S}$ for finite S, proceeding by induction on cardinality. For $S = \varnothing$, we can simply set $\overline{S} = \varnothing$. Now assume that we have defined $\overline{S}$ for $|S| \leq m$, and consider a subcomplex S of cardinality $m+1$. We will inductively define a family of subcomplexes $\overline{S}^i$ for $i \geq 0$, each countable and satisfying $f\overline{S}^i = fS$. Begin by setting $\overline{S}^0 = S \cup \bigcup\limits_{S' \subsetneq S} \overline{S'}$. Then $\overline{S}^0$ is countable, $f\overline{S}^0 = fS$, and for $S' \subseteq S$ we have $\overline{S'} \subseteq \overline{S}^0$.

Now assume that we have defined $\overline{S}^i$ for some $i \geq 0$, and let $\mathcal{D}$ be the set of all diagrams D of the form:

$$\begin{tikzcd}
\partial\square^n \arrow[r, "\partial x_D"] \arrow[d] & \overline{S}^i \arrow[d] \\
\square^n \arrow[r, "y_D"] & fS
\end{tikzcd}$$

Because $\overline{S}^i$ and fS are countable, while $\partial\square^n$ and $\square^n$ are finite for any given n, there are countably many such diagrams. Because f is a trivial fibration, for each such diagram we may choose a filler in X, i.e. an n-cube $x_D \colon \square^n \to X$ whose boundary is ∂x_D, such that $fx_D = y_D$. Let $\overline{S}^{i+1} = \overline{S}^i \cup \bigcup\limits_{D \in \mathcal{D}} \{x_D\}$. Then

$\overline{S}^{i+1}$ is still countable, since we have added at most countably many cubes to $\overline{S}^{i}$, and its image under f is still fS, since each x_D was chosen to map to a specific $y_D \in fS$.

Now let $\overline{S} = \bigcup\limits_{i\geq 0} \overline{S}^{i}$. This is countable, its image is fS, and for any $S' \subseteq S$ we have $\overline{S'} \subseteq \overline{S}$. Now consider a diagram:

$$\begin{tikzcd}
\partial\square^n \arrow[r, "\partial x"] \arrow[d] & \overline{S} \arrow[d] \\
\square^n \arrow[r, "y"] & fS
\end{tikzcd}$$

Because $\square^n$ is finite, the image of ∂x is contained in some finite subcomplex of $\overline{S}$, hence in some $\overline{S}^{i}$, so it has a filler in $\overline{S}^{i+1}$ which maps to y. Furthermore, $f|_{\overline{S}}$ has the right lifting property with respect to the map $\square^1 \to (\square^1)^\sharp$, i.e. an edge $x\colon \square^1 \to \overline{S}$ is marked if and only fx is marked, since this is true of edges in X. Thus $f|_{\overline{S}}\colon \overline{S} \to fS$ is a trivial fibration.

For a countably infinite $S \subseteq X$ we let $\overline{S} = \bigcup \overline{S'}$, where the union is taken over all finite subcomplexes $S' \subseteq S$. Then $f|_{\overline{S}}$ is the filtered colimit of the trivial fibrations $f|_{\overline{S'}}$, hence it is a trivial fibration.

The subcomplexes $\overline{S}$ with the natural inclusions define a diagram $\bar{i}\colon P \to \mathsf{cSet}'$, and f induces a natural trivial fibration $\bar{i} \implies fi$. Observe that $\bar{i}$ is a filtered diagram of subcomplexes of X, in which the maps are inclusions and edges in the objects are marked if and only if they are marked in X; furthermore, every cube of X is contained in some finite subcomplex S, and hence in $\overline{S}$. From this we can deduce that the colimit of $\bar{i}$ is X, by the same argument we used to show that the colimit of fi is Y. The induced map between colimits is f; thus we have expressed f as an ω_1-filtered colimit of trivial fibrations between countable marked cubical sets. $\square$

Lemma 5.6.3. *The weak equivalences form an ω_1-accessible, ω_1-accessibly embedded subcategory of* $(\mathsf{cSet}')^{\to}$.

Proof. The (anodyne, naive fibration) factorization gives us a naive fibrant replacement functor $F\colon (\mathsf{cSet}')^{\to} \to (\mathsf{cSet}')^{\to}$. By [Joy09, Prop. D.2.10], this functor is ω_1-accessible, since the domains and codomains of the generating anodyne maps are all countable. By definition, the category of weak equivalences we is given by the following pullback in Cat:

$$\begin{tikzcd} \mathsf{we} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathsf{cSet}')^{\to} \arrow[d, "F"] \\ \mathsf{tfib} \arrow[r, tail] & (\mathsf{cSet}')^{\to} \end{tikzcd}$$

By Lemma 5.6.2, tfib is an ω_1-accessible category, and its embedding into $(\mathsf{cSet}')^{\to}$ is an ω_1-accessible functor. By [MP89, Thm. 5.1.6], the category of ω_1-accessible categories and ω_1-accesible functors has finite limits, and these are computed in Cat. Thus we is ω_1-accessible, and its embedding into $(\mathsf{cSet}')^{\to}$ is an ω_1-accessible functor. $\square$

Theorem 5.6.4 (Analogue of model structure on marked simplicial sets)**.** *The above classes of weak equivalences, cofibrations, and fibrations define a model structure on* cSet'.

Proof. We verify the assumptions of Theorem 2.2.9.

The category of marked cubical sets is locally finitely presentable. Weak equivalences are an ω_1-accessibly embedded, ω_1-accessible subcategory of $(\mathsf{cSet}')^{\to}$ by Lemma 5.6.3. Cofibrations have a small set of generators by Lemma 5.1.1.

Weak equivalences are closed under 2-out-of-3 and weak equivalences that are cofibrations are closed under pushout by Proposition 5.5.7. Weak equivalences are closed under transfinite composition by Lemma 5.6.3, implying that the same holds for trivial cofibrations. Every map lifting against cofibrations is a weak equivalence by Lemma 5.5.6. □

We refer to the model structure constructed above as the *cubical marked model structure*. We will now analyze this model structure, beginning with a strengthening of Lemma 5.1.9 and Corollary 5.1.10.

Lemma 5.6.5. *If $X \to Y$ is a weak equivalence, then so is $A \otimes X \to A \otimes Y$ for any $A \in \mathsf{cSet}'$.*

Proof. By the adjunction $A \otimes - \dashv \underline{\mathrm{hom}}_R(A, -)$, for $Z \in \mathsf{cSet}'$ we have a natural isomorphism $\underline{\mathrm{hom}}_R(A \otimes X, Z) \cong \underline{\mathrm{hom}}_R(X, \underline{\mathrm{hom}}_R(A, Z))$. Let Z be a marked cubical quasicategory; then we have a commuting diagram

$$\begin{tikzcd}
\underline{\mathrm{hom}}_R(A \otimes Y, Z) \arrow[r] \arrow[d, "\cong"'] & \underline{\mathrm{hom}}_R(A \otimes X, Z) \arrow[d, "\cong"] \\
\underline{\mathrm{hom}}_R(Y, \underline{\mathrm{hom}}_R(A, Z)) \arrow[r] & \underline{\mathrm{hom}}_R(X, \underline{\mathrm{hom}}_R(A, Z))
\end{tikzcd}$$

By Corollary 5.1.10, $\underline{\mathrm{hom}}_R(A, Z)$ is a marked cubical quasicategory, so the bottom map is a homotopy equivalence by Proposition 5.5.3. Hence the top map is a homotopy equivalence; thus we see that $A \otimes X \to A \otimes Y$ is a weak equivalence by Proposition 5.5.3. □

Lemma 5.6.6. *The pushout product of a cofibration and a weak equivalence is a weak equivalence.*

Proof. Let $i \colon A \to B$ be a cofibration and $f \colon X \to Y$ a weak equivalence; we will show that $i \mathbin{\widehat{\otimes}} f$ is a weak equivalence (the case of $f \mathbin{\widehat{\otimes}} i$ is similar). Consider the diagram which defines $i \mathbin{\widehat{\otimes}} f$:

$$\begin{tikzcd}
A \otimes X \arrow[r] \arrow[d] & B \otimes X \arrow[d] \arrow[ddr, bend left] & \\
A \otimes Y \arrow[r] \arrow[drr, bend right] & A \otimes Y \cup_{A \otimes X} B \otimes X \arrow[dr, "i \widehat{\otimes} f"] & \\
& & B \otimes Y
\end{tikzcd}$$

The maps $A \otimes X \to A \otimes Y$ and $B \otimes X \to B \otimes Y$ are weak equivalences by Lemma 5.6.5. The map $A \otimes X \to B \otimes X$ is a cofibration by Lemma 5.1.9. The model structure is left proper, since all objects are cofibrant; thus the map from $B \otimes X$ into the pushout is a weak equivalence. Hence $i \mathbin{\widehat{\otimes}} f$ is a weak equivalence by 2-out-of-3. $\square$

Corollary 5.6.7. *Let $i \colon A \to B, j \colon A' \to B'$ be cofibrations. If either i or j is trivial, then so is the pushout product $i \mathbin{\widehat{\otimes}} j$.*

Proof. This is immediate from Lemmas 5.1.9 and 5.6.6. $\square$

Corollary 5.6.8. *If i is a cofibration and f is a fibration, then the pullback exponential $i \rhd f$ is a fibration, which is trivial if i or f is trivial.* $\square$

Corollary 5.6.9. *The category cSet', equipped with the cubical marked model structure and the geometric product, is a monoidal model category.* $\square$

Next we will characterize the fibrant objects, and fibrations between fibrant objects, of this model structure.

Proposition 5.6.10. *A map between marked cubical quasicategories is a fibration if and only if it is a naive fibration. In particular, the fibrant objects of the cubical marked model structure are precisely the marked cubical quasicategories.*

Proof. It is clear that every fibration is a naive fibration. Now let $f\colon X \to Y$ be a naive fibration between marked cubical quasicategories, and $i\colon A \to B$ a trivial cofibration. We wish to show that f has the right lifting property with respect to i; for this it suffices to show that $i \rhd f$ has the right lifting property with respect to the map $\varnothing \to \square^0$. For this, in turn, it suffices to show that $i \rhd f$ is a trivial fibration.

First, note that $i \rhd f$ is a naive fibration between marked cubical quasicategories by Corollary 5.1.10. Therefore, by Corollary 5.4.3, it is a trivial fibration if and only if it is a homotopy equivalence. Now consider the diagram which defines $i \rhd f$:

$$\begin{tikzcd}
\underline{\hom}(B,X) \arrow[dr, "i \rhd f"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \underline{\hom}(A,X) \times_{\underline{\hom}(A,Y)} \underline{\hom}(B,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\hom}(A,X) \arrow[d] \\
& \underline{\hom}(B,Y) \arrow[r] & \underline{\hom}(A,Y)
\end{tikzcd}$$

The maps $\underline{\hom}(B,X) \to \underline{\hom}(A,X)$ and $\underline{\hom}(B,Y) \to \underline{\hom}(A,Y)$ are trivial fibrations by Corollary 5.6.8; the map from the pullback to $\underline{\hom}(A,X)$ is a trivial fibration as a pullback of a trivial fibration. Thus $i \rhd f$ is a weak equivalence by 2-out-of-3, hence a homotopy equivalence by Lemma 5.4.5. $\square$

Proposition 5.6.11. *The adjunctions* $(-)^{\mathrm{co}} \dashv (-)^{\mathrm{co}}, (-)^{\mathrm{co-op}} \dashv (-)^{\mathrm{co-op}}$ *are Quillen self-equivalences of* cSet'.

Proof. By Corollary 2.1.39, it suffices to show that the adjunctions are Quillen. To do this, we apply Corollary 2.1.36. It is clear that both $(-)^{\mathrm{co}}$ and $(-)^{\mathrm{co-op}}$ preserve cofibrations. Now we consider the images of the generating anodyne maps under these functors. It is easy to see that both functors preserve marked open box inclusions and three-out-of-four maps; thus it remains to consider only the saturation map.

The image of the saturation map under $(-)^{\mathrm{co-op}}$ is isomorphic to the saturation map itself, and is therefore a trivial cofibration. Now consider the map $K^{\mathrm{co}} \to (K')^{\mathrm{co}}$. To show that this is a trivial cofibration, it suffices to show that it has the left lifting property with respect to fibrations between marked cubical quasicategories. If X is a marked cubical quasicategory, then $K^{\mathrm{co}} \to (K')^{\mathrm{co}}$ has the left lifting property against $X \to \square^0$ by the fact that the marked edges in X are precisely those which are invertible in $\mathsf{Ho}X$. Since $K^{\mathrm{co}} \to (K')^{\mathrm{co}}$ is an epimorphism, it therefore has the left lifting property against all maps between marked cubical quasicategories. $\square$

Chapter 6

The cubical Joyal model structure

The focus of this chapter is on constructing and studying the *cubical Joyal model structure* on cSet, a cubical analogue of the Joyal model structure on sSet which models the theory of $(\infty, 1)$-categories. In Section 6.1, we construct the cubical Joyal model structure by applying Theorem 2.2.28 and prove some of its basic properties. Section 6.2 provides a characterization of the model structure's fibrant objects, the *cubical quasicategories*, via a cubical analogue of Proposition 3.2.10. Finally, in Section 6.3, we prove further results about the cubical Joyal model structure, including a characterization of its weak equivalences, and show that triangulation defines a Quillen adjunction between the Joyal and cubical Joyal model structures.

6.1 Construction and basic analysis

Recall the adjunction $\mathsf{cSet} \rightleftarrows \mathsf{cSet}'$ of Section 4.3, in which the left adjoint is the minimal marking functor and the right adjoint is the forgetful functor. In this section we will use this adjunction to induce a model structure on cSet from the model structure on cSet' of Theorem 5.6.4.

Lemma 6.1.1. *For $X \in \mathsf{cSet}$, the image of the factorizations $X \sqcup X \to K \otimes X \to X$ and $X \sqcup X \to X \otimes K \to X$ under the minimal marking functor, where K denotes the invertible interval object of Definition 5.1.2, define cylinder objects for $X^\flat$ in cSet'.*

Proof. That the minimal marking functor sends the first map in each of these factorizations to a cofibration, i.e. a monomorphism, is clear; that it sends the second to a weak equivalence follows from Corollaries 5.5.5 and 5.6.7. □

Theorem 6.1.2 (Analogue of Joyal model structure)**.** *The category cSet of cubical sets carries a model structure in which:*

- *the cofibrations are the monomorphisms,*
- *the weak equivalences are created by the minimal marking functor,*
- *the fibrations are right orthogonal to trivial cofibrations.*

Proof. Apply Theorem 2.2.28 to the adjunction $\mathsf{cSet} \rightleftarrows \mathsf{cSet}'$ and the cubical marked model structure, with the factorization $X \sqcup X \to K \otimes X \to X$. Lemma 6.1.1 shows that this factorization satisfies the hypotheses of Theorem 2.2.28. □

We refer to the model structure constructed above as the *cubical Joyal model structure*. Its weak equivalences and homotopy equivalences will be referred to as *weak categorical equivalences* and *categorical equvialences*, respectively.

Proposition 6.1.3. *The adjunction* $\mathsf{cSet} \rightleftarrows \mathsf{cSet}'$ *is a Quillen equivalence.*

Proof. The minimal marking functor preserves and reflects weak equivalences by definition, thus we may apply Corollary 2.1.38 (ii). Let X be a marked cubical quasicategory; abusing notation slightly, let $X^\flat$ denote the minimal marking of the underlying cubical set of X. We must show that the inclusion $X^\flat \to X$ is a weak equivalence.

The marked edges of $X^\flat$ are precisely the degenerate edges; by Lemma 5.1.5, the marked edges of X are precisely those edges $\square^1 \to X$ which factor through K. Thus $X^\flat \to X$ is a pushout of a coproduct of saturation maps, hence a trivial cofibration. □

We define some terminology which will be used in the analysis of this model structure.

- For $n \geq 2$, $1 \leq i \leq n, \varepsilon \in \{0,1\}$, the (i,ε)-*inner open box*, denoted $\widehat{\sqcap}^n_{i,\varepsilon}$, is the quotient of an open box with the critical edge quotiented to a point. The (i,ε)-*inner cube*, denoted $\widehat{\square}^n_{i,\varepsilon}$, is defined similarly. The (i,ε)-*inner open box inclusion* is the inclusion $\widehat{\sqcap}^n_{i,\varepsilon} \hookrightarrow \widehat{\square}^n_{i,\varepsilon}$.

- An *inner fibration* is a map having the right lifting property with respect to the inner open box inclusions.

- An *isofibration* is a map having the right lifting property with respect to the endpoint inclusions $\square^0 \hookrightarrow K$.

- A *cubical quasicategory* is a cubical set X such that the map $X \to \square^0$ is an inner fibration.

- An *equivalence* in a cubical set X is an edge $\square^1 \to X$ which factors through the inclusion of the middle edge $\square^1 \to K$.
- For $n \geq 2, 1 \leq i \leq n, \varepsilon \in \{0,1\}$, a *special open box* in a cubical set X is a map $\sqcap^n_{i,\varepsilon} \to X$ which sends the critical edge to an equivalence.

The concept of homotopy developed in Chapter 5 adapts naturally to this setting, using equivalences in place of marked edges.

Definition 6.1.4. For a cubical set X, let $\sim_0$ denote the relation on X_0, the set of vertices of X, given by $x \sim_0 y$ if there is an equivalence from x to y in X. Let $\sim$ denote the smallest equivalence relation on X_0 containing $\sim_0$.

Remark 6.1.5. For $x, y \in X_0$, one can easily see that $x \sim y$ if and only if x and y are connected by a zigzag of equivalences.

Definition 6.1.6. For a cubical set X, the *set of connected components* $\pi_0(X)$ is $X_0/\sim$.

Definition 6.1.7. An *elementary left homotopy* $h\colon f \sim g$ between maps $f, g\colon A \to B$ is a map $h\colon K \otimes A \to B$ such that $h|_{\{0\}\otimes A} = f$ and $h|_{\{1\}\otimes A} = g$. Note that the elementary left homotopy h corresponds to an edge $K \to \underline{\hom}_L(A,B)$ between the vertices corresponding to f and g. A *left homotopy* between f and g is a zig-zag of elementary left homotopies.

A left homotopy from f to g corresponds to a zig-zag of equivalences in $\underline{\hom}_L(A,B)$ and so maps from A to B are left homotopic exactly if $\pi_0(f) = \pi_0(g)$, where the set of connected components is taken in $\underline{\hom}_L(A,B)$.

These induce notions of *elementary left homotopy equivalence* and *left homotopy equivalence*. Each of these notions has a "right" variant using $A \otimes K$ and $\underline{\hom}_R(A,B)$. As in Chapter 5, unless the potential for confusion arises or a statement depends on the choice, we will drop the use of "left" and "right". Homotopy equivalences between cubical quasicategories will be referred to as *categorical equivalences*.

Definition 6.1.8. Let X be a cubical set. The *natural marking* on X is a marked cubical set $X^\natural$ whose underlying cubical set is X, with edges marked if and only if they are equivalences.

It is easy to see that this defines a functor $(-)^\natural \colon \mathsf{cSet} \to \mathsf{cSet}'$, as maps of cubical sets preserve equivalences.

Many results about the cubical Joyal model structure follow easily from the corresponding results about the cubical marked model structure.

Lemma 6.1.9. *If i, j are cofibrations in cSet, then the pushout product $i \mathbin{\widehat{\otimes}} j$ is a cofibration. Moreover, if either i or j is trivial then so is $i \mathbin{\widehat{\otimes}} j$.*

Proof. This is immediate from Proposition 4.3.5, Lemma 5.1.9, and Corollary 5.6.7. □

Corollary 6.1.10. *Let i, f be maps in cSet. If i is a cofibration and f is a fibration, then the pullback exponential $i \rhd f$ is a fibration.* □

Corollary 6.1.11. *The category cSet, equipped with the cubical Joyal model structure and the geometric product, is a monoidal model category.* □

6.2 Cubical quasicategories

Next we will characterize the fibrant objects, and fibrations between fibrant objects, in the cubical Joyal model structure.

Lemma 6.2.1. *The inner open box inclusions $\widehat{\sqcap}^n_{i,\varepsilon} \to \widehat{\square}^n_{i,\epsilon}$, and the endpoint inclusions $\square^0 \to K$, are trivial cofibrations.*

Proof. The minimal marking of an inner open box inclusion is a pushout of a marked open box inclusion in cSet'. The minimal marking of $\square^0 \to K$ is a trivial cofibration by Corollary 5.5.5. $\square$

Lemma 6.2.2. *Cubical quasicategories have fillers for special open boxes.*

Proof. We only consider positive filling problems; the negative case is dual. We argue by induction on the dimension of the filling problem.

For a special open box of dimension 2, one can explicitly construct a filler by extending the given open box to an inner open box of dimension 3. We illustrate this construction for the case of a $(1,0)$-open box; the case of a $(2,0)$-open box is similar.

Consider the following open box, where the edge e is an equivalence:

$$\begin{tikzcd} x \arrow[r, "f"] & y \arrow[d, "g"] \\ w \arrow[r, "e"] & z \end{tikzcd}$$

Our assumption that e is an equivalence means that there exist a pair of 2-cubes as follows:

$$\begin{tikzcd} z \arrow[r, "e_R^{-1}"] \arrow[d, equal] & w \arrow[r, equal] \arrow[d, "e"] & w \arrow[d, equal] \\ z \arrow[r, equal] & z \arrow[r, "e_L^{-1}"] & w \end{tikzcd}$$

We extend this to a 3-dimensional $(1,1)$-open box, as depicted below. Here the front face is that which witnesses e_R^{-1} as a right inverse to e, while the top and bottom faces are obtained by two-dimensional inner open box filling.

$$
\begin{tikzcd}
x \arrow[rr, equal] \arrow[dd, equal] \arrow[dr, "gf"] & & x \arrow[dd, "f" near start] \arrow[dr, "e_R^{-1} gf"] & \\
& z \arrow[rr, "e_R^{-1}" near start] \arrow[dd, equal] & & w \arrow[dd, "e"] \\
x \arrow[rr, "f" near start] \arrow[dr, "gf"] & & y \arrow[dr, "g"] & \\
& z \arrow[rr, equal] & & z
\end{tikzcd}
$$

This open box is inner, hence it has a filler; the $(1,1)$-face of this 3-cube is a filler for the original 2-dimenisonal open box.

Now let X be a cubical quasicategory, and suppose that X has fillers for all special open boxes of dimension less than n. Consider a filling problem in X of dimension n:

$$
\begin{tikzcd}
(\partial \square^a \otimes \square^1 \otimes \square^b) \cup (\square^a \otimes \{0\} \otimes \square^b) \cup (\square^a \otimes \square^1 \otimes \partial \square^b) \arrow[r] \arrow[d, tail] & X \\
\square^a \otimes \square^1 \otimes \square^b \arrow[ur, dotted] &
\end{tikzcd}
$$

We regard the codomain of the left map as a negative face of a larger cube via the map

$$\square^a \otimes \square^1 \otimes \square^b \rightarrowtail \square^a \otimes \square^1 \otimes \{0\} \otimes \square^b$$

and the domain as the corresponding subobject. The original filling problem then becomes a filling problem in X of the form

$$\begin{aligned}
&(\partial \square^a \otimes \square^1 \otimes \{0\} \otimes \square^b) \\
\cup&(\square^a \otimes \{0\} \otimes \{0\} \otimes \square^b) \\
\cup&(\square^a \otimes \square^1 \otimes \{0\} \otimes \partial\square^b) \\
\to&\ \square^a \otimes \square^1 \otimes \{0\} \otimes \square^b
\end{aligned}$$

where the critical edge is

$$0^a000^b \to 0^a100^b.$$

We will solve this problem by extending the given partial data to the whole of

$$\square^a \otimes \square^1 \otimes \square^1 \otimes \square^b.$$

For $n \geq 0$, let $\Gamma^n \subseteq \square^n$ denote the union of the positive faces. We use degeneracies in the new direction to fill

$$\begin{array}{ccc}
(\Gamma^a \otimes \square^1 \otimes \{0\} \otimes \square^b) & & (\Gamma^a \otimes \square^1 \otimes \square^1 \otimes \square^b) \\
\cup(\square^a \otimes \{0\} \otimes \{0\} \otimes \square^b) & \to & \cup(\square^a \otimes \{0\} \otimes \square^1 \otimes \square^b) \\
\cup(\square^a \otimes \square^1 \otimes \{0\} \otimes \Gamma^b) & & \cup(\square^a \otimes \square^1 \otimes \square^1 \otimes \Gamma^b).
\end{array}$$

Since the critical edge is an equivalence, we can fill the square

$$\begin{tikzcd}
0^a000^b \arrow[r, equal] \arrow[d] & 0^a010^b \arrow[d, equal] \\
0^a100^b \arrow[r, dotted] & 0^a110^b
\end{tikzcd} \tag{6.2.1}$$

where the dotted edge is again an equivalence.

In the following, we will indicate the filling direction of (generalized) open boxes by underlining the appropriate factor in the pushout monoidal product. What this means is that we can factor the given generalized open box inclusion as a series of open box fillings in different dimensions, each of which fills in the specified direction. We now fill the generalized open box

$$\{0^a\} \otimes (\{0\} \to \square^1) \mathbin{\widehat{\otimes}} \underline{(\{0\} \to \square^1)} \mathbin{\widehat{\otimes}} (\{0^b\} \to \square^b)$$

if $a, b \geq 1$. Here, the critical edges are of the form $uv0w \to uv1w$ where u, v, w are certain vertices of $\square^a, \square^1, \square^b$, respectively. All of these edges are degenerate except for the bottom edge in (6.2.1), which is an equivalence. Moreover, this edge only appears as a critical edge in filling problems of lower dimension. So we may indeed fill this generalized open box using the fact that X is a cubical quasicategory and the induction hypothesis. Dually, we fill the generalized open box

$$(\{0^a\} \to \square^a) \mathbin{\widehat{\otimes}} (\{0\} \to \square^1) \mathbin{\widehat{\otimes}} \underline{(\{0\} \to \square^1)} \otimes \{0^b\}$$

if $a, b \geq 1$.

We now fill the generalized open box

$$(\{0^a\} \cup \Gamma^a \to \partial\square^a) \mathbin{\widehat{\otimes}} (\{0\} \to \square^1) \mathbin{\widehat{\otimes}} \underline{(\{0\} \to \square^1)} \mathbin{\widehat{\otimes}} (\partial\square^b \to \square^b)$$

if $a \geq 1$. Again, the critical edges are of the form as above and we may argue as before. Dually, we fill the generalized open box

$$(\partial\square^a \to \square^a) \mathbin{\widehat{\otimes}} (\{0\} \to \square^1) \mathbin{\widehat{\otimes}} \underline{(\{0\} \to \square^1)} \mathbin{\widehat{\otimes}} (\{0^b\} \cup \Gamma^b \to \partial\square^b)$$

if $b \geq 1$.

At this stage, we have defined the cube on

$$\begin{aligned}&(\partial \square^a \otimes \square^1 \otimes \square^1 \otimes \square^b)\\ \cup&(\square^a \otimes \{0\} \otimes \square^1 \otimes \square^b)\\ \cup&(\square^a \otimes \square^1 \otimes \square^1 \otimes \partial\square^b).\end{aligned}$$

We now fill the open box

$$(\partial\square^a \to \square^a)\,\widehat{\otimes}\,\underline{(\{0\} \to \square^1)}\,\widehat{\otimes}\,(\varnothing \to \{1\})\,\widehat{\otimes}\,(\partial\square^b \to \square^b),$$

noting that the critical edge $0^a000^b \to 0^a100^b$ is degenerate. We then fill the open box

$$(\partial\square^a \to \square^a)\,\widehat{\otimes}\,(\varnothing \to \{1\})\,\widehat{\otimes}\,\underline{(\{0\} \to \square^1)}\,\widehat{\otimes}\,(\partial\square^b \to \square^b),$$

noting that the critical edge $0^a000^b \to 0^a010^b$ is degenerate. We finally fill the open box

$$(\partial\square^a \to \square^a)\,\widehat{\otimes}\,(\partial\square^1 \to \square^1)\,\widehat{\otimes}\,\underline{(\{0\} \to \square^1)}\,\widehat{\otimes}\,(\partial\square^b \to \square^b),$$

noting that the critical edge $0^a000^b \to 0^a010^b$ is degenerate. This defines the entire cube. $\square$

Lemma 6.2.3. *Let $X \to Y$ be an inner fibration between cubical quasicategories. Then a lift exists for any diagram of the form*

$$\begin{tikzcd} \sqcap^n_{i,\varepsilon} \arrow[r] \arrow[d, tail] & X \arrow[d] \\ \square^n \arrow[r] & Y \end{tikzcd}$$

in which $\sqcap^n_{i,\varepsilon}$ is a special open box in X.

Proof. Again we only consider positive filling problems; the negative case is dual. Again we argue by induction on the dimension of the filling problem, with the case for dimension 2 being an exercise in filling three-dimensional open boxes, analogous to the base case of the previous proof. Once again, we will illustrate the argument for the case of a $(1,0)$-open box, with the case of a $(2,0)$-open box being similar.

Consider a $(1,0)$-open box in X whose image in Y admits a filler α, as depicted below on the left and right, respectively:

$$\begin{tikzcd} x \arrow[r, "f"] & y \arrow[d, "g"] \\ w \arrow[r, "e"] & z \end{tikzcd} \qquad \begin{tikzcd} \overline{x} \arrow[r, "\overline{f}"] \arrow[d, "h"'] & \overline{y} \arrow[d, "\overline{g}"] \\ \overline{w} \arrow[r, "\overline{e}"] & \overline{z} \end{tikzcd}$$

Once again, we assume that the critical edge e is an equivalence in X.

We begin by filling a 2-dimensional inner open box to obtain the following 2-cube in X:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, dashed, "gf"'] & y \arrow[d, "g"] \\ z \arrow[r, equal] & z \end{tikzcd}$$

We may extend the 2-cube α to a 3-dimensional $(2,0)$-open box in Y, as depicted below; here the right face is α, the front is the image in Y of the 2-cube in X which witnesses e_R^{-1} as a right inverse to e, and the bottom is the image in Y of the 2-cube constructed above.

$$
\begin{tikzcd}
\overline{x} \arrow[rrr, equal] \arrow[ddd, equal] \arrow[dr, "\overline{gf}"] & & & \overline{x} \arrow[dr, "h"] \arrow[ddd, "\overline{f}"] & \\
& \overline{z} \arrow[rrr, "\overline{e}_R^{-1}"] \arrow[ddd, equal] & & & \overline{w} \arrow[ddd, "\overline{e}"] \\
& & & & \\
\overline{x} \arrow[rrr, "\overline{f}"] \arrow[dr, "\overline{gf}"] & & & \overline{y} \arrow[dr, "\overline{g}"] & \\
& \overline{z} \arrow[rrr, equal] & & & \overline{z}
\end{tikzcd}
$$

The critical edge of this open box is the equivalence e, hence it admits a filler by Lemma 6.2.2. The top face of the cube thus obtained is a filler for an inner open box in X, hence it can be lifted along the inner fibration $X \to Y$; in particular we obtain an edge $\widetilde{h}$ mapping to h. We thus obtain the following $(1,1)$-inner open box in X:

$$
\begin{tikzcd}
x \arrow[rrr, equal] \arrow[ddd, equal] \arrow[dr, "gf"] & & & x \arrow[dr, "\widetilde{h}"] \arrow[ddd, "f"] & \\
& z \arrow[rrr, "e_R^{-1}"] \arrow[ddd, equal] & & & w \arrow[ddd, "e"] \\
& & & & \\
x \arrow[rrr, "f"] \arrow[dr, "gf"] & & & y \arrow[dr, "g"] & \\
& z \arrow[rrr, equal] & & & z
\end{tikzcd}
$$

The critical edge of this open box is degenerate, and the previously constructed 3-cube in Y is a filler for its image. Thus we may lift this filler along $X \to Y$; in particular, we obtain a filler for its right face which maps to α.

Now assume that $X \to Y$ lifts against all special open box fillings of dimension less than or equal to n, and consider a lifting problem

$$\begin{tikzcd}
(\partial \Box^a \otimes \Box^1 \otimes \Box^b) \cup (\Box^a \otimes \{0\} \otimes \Box^b) \cup (\Box^a \otimes \Box^1 \otimes \partial \Box^b) \arrow[r] \arrow[d, tail] & X \arrow[d] \\
\Box^a \otimes \Box^1 \otimes \Box^b \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

where $a + b = n$. As before, we regard the codomain of the left map as a negative face of a larger cube via the map

$$\Box^a \otimes \Box^1 \otimes \Box^b \rightarrowtail \Box^a \otimes \Box^1 \otimes \{0\} \otimes \Box^b$$

and the domain as the corresponding subobject H. The critical edge is once again $0^a 000^b \to 0^a 100^b$. Let H' be the union of H with the subobjects

$$\begin{gathered}
(\Gamma^a \otimes \Box^1 \otimes \Box^1 \otimes \Box^b) \\
\cup (\Box^a \otimes \{0\} \otimes \Box^1 \otimes \Box^b) \\
\cup (\Box^a \otimes \Box^1 \otimes \Box^1 \otimes \Gamma^b)
\end{gathered}$$

and H'' be the union of H' with the square

$$\{0^a\} \otimes \Box^1 \otimes \Box^1 \otimes \{0^b\}.$$

We use degeneracies in the new direction to extend the map to X from H to H':

$$\begin{tikzcd}
H \arrow[r] \arrow[d, hook] & X. \\
H' \arrow[ur, dotted] &
\end{tikzcd}$$

Since the critical edge is an equivalence in X, we extend the map to X from

H' to H'' by filling the square

$$\begin{tikzcd}
0^a000^b \arrow[r, equal] \arrow[d] & 0^a010^b \arrow[d, equal] \\
0^a100^b \arrow[r, dotted] & 0^a110^b
\end{tikzcd}$$

where the dotted edge is again an equivalence in X. Note that the map $X \to Y$ preserves equivalences.

We construct the dotted arrow in the diagram

$$\begin{tikzcd}
H \arrow[r] \arrow[d, tail] & H'' \arrow[r] \arrow[d] & X \arrow[d] \\
\square^a \otimes \square^1 \otimes \{0\} \otimes \square^b \arrow[r] \arrow[rr, bend right] & \square^a \otimes \square^1 \otimes \square^1 \otimes \square^b \arrow[r, dotted] & Y
\end{tikzcd}$$

by solving a filling problem

$$\begin{tikzcd}
(\square^a \otimes \square^1 \otimes \{0\} \otimes \square^b) \cup H'' \arrow[r] \arrow[d] & Y \\
\square^a \otimes \square^1 \otimes \square^1 \otimes \square^b \arrow[ur, dotted] &
\end{tikzcd}$$

as follows: the left map factors as a finite composite of open box inclusions of the form

$$(\partial\square^{a'} \to \square^{a'}) \mathbin{\widehat{\otimes}} (\{0\} \to \square^1) \mathbin{\widehat{\otimes}} \underline{(\{0\} \to \square^1)} \mathbin{\widehat{\otimes}} (\partial\square^{b'} \to \square^{b'})$$

where $\square^{a'}$ and $\square^{b'}$ are faces of $\square^a$ and $\square^b$, respectively. All critical edges are of the form $uv0w \to uv1w$ where u, v, w are certain points of $\square^a, \square^1, \square^b$, respectively. All of these edges are degenerate in Y except for the bottom edge in (6.2.1), which is an equivalence. We can thus fill these open boxes using the fact that Y is a cubical quasicategory and Lemma 6.2.2.

It remains to construct a lift

$$\begin{tikzcd} H'' \arrow[r] \arrow[d] & X \arrow[d] \\ \square^a \otimes \square^1 \otimes \square^1 \otimes \square^b \arrow[r] \arrow[ur, dotted] & Y, \end{tikzcd}$$

which is done exactly as in the proof of Lemma 6.2.2 using that $X \to Y$ is an inner fibration. $\square$

Lemma 6.2.4. *If X is a cubical quasicategory, then $X^\natural$ is a marked cubical quasicategory.*

Proof. Given a cubical quasicategory X, we have fillers for special open boxes in X by Lemma 6.2.2. This implies that $X^\natural$ has fillers for marked open boxes. Furthermore, the definition of the natural marking implies that $X^\natural$ has the right lifting property with respect to the saturation map for any cubical set X. By Lemma 5.1.6, this suffices to show that $X^\natural$ is a marked cubical quasicategory. $\square$

Theorem 6.2.5. *The fibrant objects of the the cubical Joyal model structure are given by cubical quasicategories. The fibrations between fibrant objects are characterized by lifting against inner open box inclusions and endpoint inclusions $\square^0 \hookrightarrow K$.*

Proof. By Lemma 6.2.1, every fibrant object is a cubical quasicategory and every fibration is an inner isofibration.

If X is a cubical quasicategory, then $X^\natural$ is a marked cubical quasicategory by Lemma 6.2.4. The forgetful functor $\mathsf{cSet}' \to \mathsf{cSet}$ preserves fibrant objects as a right Quillen adjoint, and the underlying cubical set of $X^\natural$ is X, thus X is fibrant.

The case of fibrations between fibrant objects proceeds in an analogous way. Let $f\colon X \to Y$ be an inner isofibration between cubical quasicategories; we will show that $f^\natural$ is a fibration in cSet'. Lifting against one-dimensional marked open box inclusions follows from the isofibration property; lifting against higher-dimensional marked open box inclusions follows from Lemma 6.2.3. To see that $f^\natural$ has the right lifting property with respect to the saturation and 3-out-of-4 maps, observe that any marked cubical quasicategory has the right lifting property with respect to these maps, hence so does any map between marked cubical quasicategories since the maps in question are epimorphisms. Since $X^\natural$ and $Y^\natural$ are marked cubical quasicategories, this implies that $f^\natural$ is a fibration by Proposition 5.6.10. $\square$

Corollary 6.2.6. *Let $f\colon X \to Y$ be a map between cubical quasicategories. Then f is a weak categorical equivalence if and only if it is a categorical equivalence.* $\square$

Proof. This follows from Lemma 6.1.1 and Theorem 6.2.5. $\square$

Corollary 6.2.7. *Let $X, Y \in \mathsf{cSet}$, with Y a cubical quasicategory. Then $\underline{\mathrm{hom}}(X, Y)$ is a cubical quasicategory.*

Proof. This follows from Corollary 6.1.10 and Theorem 6.2.5. $\square$

Using Theorem 6.2.5, we can see that this model structure can also be constructed using the Cisinski-Olschok theory of Section 2.2.

Proposition 6.2.8. *Let cSet_K denote the model structure given by applying Theorem 2.2.14 to cSet with the following data:*

- *$I = - \otimes K$, with natural transformations ∂_ε and σ induced by the endpoint inclusions and the map $K \to \square^0$;*

- $M = \{\partial\Box^n \hookrightarrow \Box^n \mid n \geq 0\}$;
- $S = \{\widehat{\sqcap}^n_{i,\varepsilon} \hookrightarrow \widehat{\Box}^n_{i,\varepsilon} \mid n \geq 2, 1 \leq i \leq n, \varepsilon = 0, 1\}$.

Then cSet_K *coincides with the cubical Joyal model structure.*

Proof. The cofibrations in both model structures are the monomorphisms. Therefore, to show that the model structures coincide, it suffices to show that they have the same fibrant objects, i.e. that the objects having the right lifting property with respect to all maps in $\Lambda(S)$ are precisely the cubical quasicategories. For this, observe that all fibrant objects of cSet_K are cubical quasicategories, since $S \subseteq \Lambda(S)$ is precisely the set of inner open box inclusions. Furthermore, an inductive argument involving Lemma 6.1.9 shows that all maps in $\Lambda(S)$ are trivial cofibrations in the cubical Joyal model structure, so all cubical quasicategories are fibrant in cSet_K. □

6.3 Further analysis of the cubical Joyal model structure

Our next goal will be to characterize the weak categorical equivalences in a manner similar to Proposition 5.5.3.

Lemma 6.3.1. *The following triangle of functors commutes:*

$$\begin{tikzcd} \mathsf{cSet} \arrow[rr, "(-)^\natural"] \arrow[dr, "\pi_0"'] & & \mathsf{cSet}' \arrow[dl, "\pi_0"] \\ & \mathsf{Set} & \end{tikzcd}$$

Proof. For $X \in \mathsf{cSet}$, X and $X^\natural$ have the same set of vertices, and the equivalence relations defining $\pi_0 X$ and $\pi_0 X^\natural$ coincide. □

Lemma 6.3.2. *Let $X, Y \in \mathsf{cSet}$, and let Y' be a marked cubical set whose underlying cubical set is Y. The underlying cubical set of $\underline{\mathrm{hom}}(X^\flat, Y')$ is isomorphic to $\underline{\mathrm{hom}}(X, Y)$, and this isomorphism is natural in both X and Y.*

Proof. We will prove the statement for $\underline{\mathrm{hom}}_R$; the proof for $\underline{\mathrm{hom}}_L$ is similar. The n-cubes in the underlying cubical set of $\underline{\mathrm{hom}}_R(X^\flat, Y')$ are maps $X^\flat \otimes \square^n \cong (X \otimes \square^n)^\flat \to Y'$ (the isomorphism follows from Proposition 4.3.5). Under the adjunction $\mathsf{cSet} \rightleftarrows \mathsf{cSet}'$, these correspond to maps $X \otimes \square^n \to Y$. □

Proposition 6.3.3. *The following are equivalent for a cubical map $A \to B$:*

(i) *$A \to B$ is a weak categorical equivalence;*

(ii) *for any cubical quasicategory X, the induced map $\underline{\mathrm{hom}}(B, X) \to \underline{\mathrm{hom}}(A, X)$ is a categorical equivalence;*

(iii) *for any cubical quasicategory X, the induced map $\pi_0(\underline{\mathrm{hom}}(B, X)) \to \pi_0(\underline{\mathrm{hom}}(A, X))$ is a bijection.*

Proof. To see that (i) $\Rightarrow$ (ii), let $A \to B$ be a weak categorical equivalence in cSet, and X a cubical quasicategory. Then $X^\natural$ is a marked cubical quasicategory by Lemma 6.2.4, so $\underline{\mathrm{hom}}(B^\flat, X^\natural) \to \underline{\mathrm{hom}}(A^\flat, X^\natural)$ is a homotopy equivalence by Proposition 5.5.3. The underlying cubical set functor preserves weak equivalences between fibrant objects by Ken Brown's lemma, so $\underline{\mathrm{hom}}(B, X) \to \underline{\mathrm{hom}}(A, X)$ is a weak categorical equivalence by Lemma 6.3.2. Hence it is a categorical equivalence by Corollaries 6.2.6 and 6.2.7.

The implication (ii) $\Rightarrow$ (iii) is clear, so now we consider (iii) $\Rightarrow$ (i). For this, let X be the underlying cubical set of a marked cubical quasicategory X', and note that by Lemmas 6.3.1 and 6.3.2, we have the following commuting diagram in Set:

$$\begin{tikzcd}
\pi_0\underline{\hom}(B, X) \arrow[r] \arrow[d, "\cong"'] & \pi_0\underline{\hom}(A, X) \arrow[d, "\cong"] \\
\pi_0\underline{\hom}(B^\flat, X') \arrow[r] & \pi_0\underline{\hom}(A^\flat, X')
\end{tikzcd}$$

Since the underlying cubical set functor preserves fibrant objects, X is a cubical quasicategory. So if (iii) holds then the top map is an isomorphism, hence so is the bottom map. Thus $A^\flat \to B^\flat$ is a weak equivalence in cSet' by Proposition 5.5.3, meaning that $A \to B$ is a weak categorical equivalence. □

We now state two straightforward properties of the cubical Joyal model structure.

Proposition 6.3.4.

(i) The Grothendieck model structure on cSet *of Theorem 4.2.3 is a localization of the cubical Joyal model structure.*

(ii) The adjunction $\tau_1 \colon \mathsf{cSet} \rightleftarrows \mathsf{Cat} : \mathsf{N}_\square$ *is a Quillen adjunction between the canonical model structure on* Cat *and the cubical Joyal model structure.*

□

The cubical Joyal model structure is clearly left proper, since all objects are cofibrant. However, it is not right proper. The proof of this fact is similar to the standard proof of the corresponding result for the Joyal model structure on sSet, but requires an additional step due to the fact that inner cubes, unlike representable simplicial sets, are generally not fibrant.

Proposition 6.3.5. *The cubical Joyal model structure is not right proper.*

Proof. We will exhibit a fibration $X \to Z$ and a weak equivalence $Y \to Z$ such that the pullback map $X \times_Z Y \to X$ is not a weak equivalence.

First consider the map $[1] \to [2]$ in Cat which picks out the morphism $0 \to 2$. This is an isofibration, hence its image under $N_\square$ is a fibration by Proposition 6.3.4 (ii).

We have a map $\widehat{\square}^2_{2,0} \to N_\square[2]$ given by the following 2-cube in $N_\square[2]$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 2 \arrow[d, equal] \\
1 \arrow[r] & 2
\end{tikzcd}$$

Now consider the following commuting diagram in cSet:

$$\begin{tikzcd}
\partial\square^1 \arrow[r, "\{0{,}2\}"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \widehat{\sqcap}^2_{2,0} \arrow[d] \\
\square^1 \arrow[r, "0\to 2"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \widehat{\square}^2_{2,0} \arrow[d] \\
N_\square[1] \arrow[r, "N_\square(0\to 2)"] & N_\square[2]
\end{tikzcd}$$

Pullbacks of two monomorphisms in cSet are given by intersections; this is immediate from the corresponding result in Set. From this, it follows that both of the squares in the diagram above are pullbacks.

The middle horizontal map is a fibration, as a pullback of a fibration. So the inclusion $\partial\square^1 \to \square^1$ is the pullback of the trivial cofibration $\widehat{\sqcap}^2_{2,0} \to \widehat{\square}^2_{2,0}$ along a fibration. However, it is not a weak equivalence by Proposition 6.3.4 (ii), since its image under τ is not an equivalence of categories. $\square$

Next we will study the interactions of the functors $(-)^{\mathrm{co}}$ and $(-)^{\mathrm{co-op}}$ of Section 4.1 with the cubical Joyal model structure.

Proposition 6.3.6. *The adjunctions $(-)^{\mathrm{co}} \dashv (-)^{\mathrm{co}}$ and $(-)^{\mathrm{co-op}} \dashv (-)^{\mathrm{co-op}}$ are Quillen self-equivalences of the cubical Joyal model structure.*

Proof. By Corollary 2.1.39, it suffices to show that the adjunctions are Quillen. We will prove the statement for $(-)^{\mathrm{co}}$; the proof for $(-)^{\mathrm{co-op}}$ is identical.

To show that the adjunction $(-)^{\mathrm{co}} \dashv (-)^{\mathrm{co}}$ is Quillen, we must show that $(-)^{\mathrm{co}}$ preserves cofibrations and trivial cofibrations. Unwinding the definitions, we must show that, given a map f in cSet, if $f^\flat$ is a (trivial) cofibration in cSet' then so is $(f^{\mathrm{co}})^\flat$. We have the following commuting diagram:

$$\begin{tikzcd}
\mathsf{cSet} \arrow[r, "(-)^{\mathrm{co}}"] \arrow[d, "(-)^\flat"'] & \mathsf{cSet} \arrow[d, "(-)^\flat"] \\
\mathsf{cSet}' \arrow[r, "(-)^{\mathrm{co}}"] & \mathsf{cSet}'
\end{tikzcd}$$

The result thus follows from the fact that the map $(-)^{\mathrm{co}} \colon \mathsf{cSet}' \to \mathsf{cSet}'$ preserves (trivial) cofibrations by Proposition 5.6.11. $\square$

The result above allows us to show that our set of pseudo-generating trivial cofibrations do not form a set of generating trivial cofirbations for the cubical Joyal model structure.

Proposition 6.3.7. *The endpoint inclusions $\square^0 \to K^{\mathrm{co}}$ have the right lifting property against all anodyne maps, but they are not fibrations.*

Proof. Fix an endpoint inclusion $\square^0 \to K^{\mathrm{co}}$; we must show that this map has the right lifting property against the inner open box inclusions and the endpoint inclusions $\square^0 \to K$. Consider the following diagram in cSet:

$$\begin{tikzcd}
\widehat{\sqcap}^n_{i,\varepsilon} \arrow[r] \arrow[d] & \square^0 \arrow[d] \\
\widehat{\square}^n_{i,\varepsilon} \arrow[r] & K^{\mathrm{co}}
\end{tikzcd}$$

We may note that constant open boxes $\widehat{\sqcap}^n_{i,\varepsilon} \to K^{\mathrm{co}}$ for $n \geq 2$ have only constant fillers; thus the map $\widehat{\square}^n_{i,\varepsilon} \to K^{\mathrm{co}}$ in this diagram factors through the unique map $\widehat{\square}^n_{i,\varepsilon} \to \square^0$, implying that the diagram admits a lift. Similarly, any map $K \to K^{\mathrm{co}}$ is constant, implying that $\square^0 \to K^{\mathrm{co}}$ has the right lifting property against the maps $\square^0 \to K$.

To see that $\square^0 \to K^{\mathrm{co}}$ is not a fibration, observe that it is the image under $(-)^{\mathrm{co}}$ of one of the anodyne maps $\square^0 \to K$, hence it is a trivial cofibration by Proposition 6.3.6. Thus it cannot be a fibration, as it is not an isomorphism and therefore does not have the right lifting property against itself. $\square$

We conclude this section with a proof of the following result, relating the cubical Joyal model structure to the Joyal model structure on simplicial sets via the triangulation functor.

Proposition 6.3.8. *The adjunction $T : \mathsf{cSet} \rightleftarrows \mathsf{sSet} : U$ is a Quillen adjunction between the cubical Joyal model structure and the Joyal model structure on* sSet.

Conceptually, this adjunction might be best understood at the level of marked simplicial and marked cubical sets. However, in order to avoid the burden of relying on the model structure on marked simplicial sets, we will compare the model structures on cSet and sSet directly.

Lemma 6.3.9. *T sends the endpoint inclusions $\square^0 \to K$ to trivial cofibrations in the Joyal model structure.*

Proof. We will construct a weak categorical equivalence from TK to the simplicial set J. It is easy to see that TK is the simplicial set depicted below:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, equal] \arrow[dr] & \bullet \arrow[r, equal] \arrow[d] \arrow[dr] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, equal] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

Let Z denote the simplicial set defined by the following pushout:

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, tail] & \Delta^0 \arrow[d] \\ \Delta^2 \arrow[r] & Z \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The map $\Delta^0 \to Z$ is a trivial cofibration, as a pushout of an inner horn inclusion; thus Z is contractible. We have a pair of cofibrations $Z \hookrightarrow TK$, picking out the bottom-left and top-right simplices in the illustration above; the induced map $Z \sqcup Z \to TK$ is a cofibration since these two simplices have no faces in common. We obtain J as a quotient of TK by contracting each of these two simplices to a point; in other words, we have the following pushout diagram:

$$\begin{tikzcd} Z \sqcup Z \arrow[r, tail] \arrow[d] & TK \arrow[d] \\ \Delta^0 \sqcup \Delta^0 \arrow[r] & J \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The left map is a weak equivalence since coproducts preserve weak equivalences in the Joyal model structure. Thus $TK \to J$ is a weak equivalence as a pushout of a weak equivalence along a cofibration. The composite of $\Delta^0 \to TK$ with this quotient map is an endpoint inclusion $\Delta^0 \to J$, hence a weak equivalence; thus $\Delta^0 \to TK$ is a weak equivalence by 2-out-of-3. $\square$

Lemma 6.3.10. *T sends inner open box inclusions to trivial cofibrations.*

Proof. By Lemma 4.1.11, Corollary 4.1.16, and the symmetry of the cartesian product in sSet, the triangulation of an open box inclusion $\sqcap^m_{i,\varepsilon} \hookrightarrow \square^m$ is the pushout product $(T\partial\square^{m-1} \hookrightarrow (\Delta^1)^{m-1})\hat{\times}(\{\varepsilon\} \hookrightarrow \Delta^1)$. Therefore, since T preserves colimits, the triangulation of $\widehat{\sqcap}^m_{i,\varepsilon} \hookrightarrow \widehat{\square}^m$ is the inclusion of the quotients of these simplicial sets in which the edge corresponding to the critical edge of

$\sqcap^m_{i,\varepsilon}$ is collapsed to a vertex.

Since $T\partial\square^{m-1} \hookrightarrow (\Delta^1)^{m-1}$ is a monomorphism of simplicial sets, it can be written as a composite of boundary fillings. Since pushout products commute with composition, we can thus rewrite $T\sqcap^m_{i,\varepsilon} \hookrightarrow (\Delta^1)^m$ as a composite of pushouts of maps of the form $(\partial\Delta^n \to \Delta^n)\hat{\times}(\{\varepsilon\} \hookrightarrow \Delta^1)$, i.e. open prism fillings. We can obtain $T\widehat{\square}^n$ from $T\sqcap^n_{i,\varepsilon}$, therefore, by filling the corresponding open prisms in $T\sqcap^n_{i,\varepsilon}$.

Each open prism filling can be explicitly written as a composite of horn fillings. Each of these horn fillings but one will be inner, and hence a trivial cofibration. However, the critical edge of the unique outer horn, i.e. the unique non-degenerate edge containing either the initial or the terminal vertices of both the horn and its missing face, corresponds to the critical edge of $\sqcap^m_{i,\varepsilon}$, hence it is degenerate. Thus this horn-filling is also a trivial cofibration by [Joy02, Thm. 2.2]. □

Proof of Proposition 6.3.8. This follows from Corollary 2.1.36, together with Proposition 4.1.17 and Lemmas 6.3.9 and 6.3.10. □

Corollary 6.3.11. *The triangulation functor preserves weak equivalences.*

Proof. Since all cubical sets are cofibrant, this is immediate from Proposition 6.3.8 and Ken Brown's lemma. □

Chapter 7

Comparison of model structures

In this chapter, we will prove that the triangulation adjunction $T : \mathsf{cSet} \rightleftarrows \mathsf{sSet} : U$ is a Quillen equivalence between the cubical Joyal and the Joyal model structures. Working directly with the this adjunction, however, is difficult, as it is hard to describe the counit $TU \Rightarrow \mathrm{id}$ explicitly.

To remedy this issue, we introduce a different adjunction $Q : \mathsf{sSet} \rightleftarrows \mathsf{cSet} : \int$, coming from the straightening-over-the-point functor, as studied in [KLW19, KV20], also closely related to Lurie's straightening construction [Lur09a, Ch. 2]. We then show that $Q \dashv \int$ is a Quillen equivalence and construct a natural weak equivalence $TQ \Rightarrow \mathrm{id}$, from which we derive our conclusion.

As in the case of triangulation, the key difficulty in proving that $Q \dashv \int$ is a Quillen equivalence lies in understanding the counit map $Q \int X \to X$. This however is a much more tractable problem as it was for instance shown in [KLW19] that it is a monomorphism. Intuitively, $Q \int X$ is the subcomplex of X which is built out of cubes with sufficiently degenerate faces that they may be regarded as simplices, e.g., a square with a single edge collapsed to a point or a cube with one face collapsed to a point and another face collapsed to an edge.

For a cubical quasicategory X, we write the inclusion $Q \int X \to X$ as a transfinite composite of pushouts of inner open box fillings, thus establishing it as an anodyne map in the cubical Joyal model structure. To determine its decomposition into individual open box fillings, we develop a theory of cones in cubical sets. Roughly speaking, the decomposition proceeds by induction on the dimension of the base of a cone contained in X. In particular, $Q \int X$ consists of cubes obtained by repeatedly taking cones only on the vertices of X (rather than on cubes of arbitrary dimension). To identify the open boxes needed to build X from $Q \int X$, we introduce the notion of a coherent family of composites, a technical construction that picks out a distinguished cone on each cube of X.

The main purpose of Section 7.1 is to set up the technical machinery needed for the proof that $Q \dashv \int$ is a Quillen equivalence, including the theory of cubical cones. Then, in Section 7.2, we define Q using the theory of cones and prove that it is a left Quillen equivalence. Finally, as indicated above, we construct in Section 7.2 a natural weak equivalence $TQ \Rightarrow \mathrm{id}$, and conclude that T is also a left Quillen equivalence.

One can imagine an alternative approach that would instead proceed by establishing that the other composite, QT, is naturally weakly equivalent to the identity, bypassing the technical proof that Q is a left Quillen equivalence. While we considered this approach, we could not see a direct natural transformation $QT \Rightarrow \mathrm{id}$ and any zigzag we could think of would involve objects similar to coherent families of composites. As a result, we opted for the approach presented in this chapter as it is both the most straightforward and provides insight into how a cubical quasicategory is built out of its maximal simplicial subcomplex.

7.1 Cones in cubical sets

We begin this section in Definition 7.1.1 by defining a cone on a cubical set and showing in Proposition 7.1.6 that taking cones defines a monad. We then proceed to analyze the faces and subcomplexes of iterated cones on standard cubes in Definition 7.1.7 through Proposition 7.1.23. In Definition 7.1.24, we define coherent families of composites and show in Theorem 7.1.25 that every cubical quasicategory admits such a family.

Definition 7.1.1. For $X \in \mathsf{cSet}$, the *cone on* X, denoted CX, is defined by the following pushout diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\partial_{1,1} \otimes X"'] & \square^0 \arrow[d] \\ \square^1 \otimes X \arrow[r] & CX \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

As this construction is functorial, we obtain the *cone functor* $C \colon \mathsf{cSet} \to \mathsf{cSet}$. For $m, n \geq 0$, the *standard* (m,n)*-cone* is the object $C^{m,n} = C^n \square^m$, i.e. the object obtained by applying C to $\square^m$ n times. We refer to the natural map $\square^0 \to CX$ appearing in this diagram as the *cone point*.

A simple computation shows that $C\varnothing \cong \square^0$, while $C\square^0 \cong \square^1$. We thus obtain the following result:

Lemma 7.1.2. *For all $n \geq 1$, $C^n \varnothing \cong C^{0,n-1}$ and $C^{0,n} \cong C^{1,n-1}$.* $\square$

To develop our understanding of the cone construction, we consider certain examples of cones CX for $X \in \mathsf{cSet}$. In all of our illustrations, we will denote the cone point of CX by c. For our simplest example, as described above, we may observe that $C\square^0 \cong \square^1$:

$$0 \longrightarrow c$$

$C\square^1$ is the quotient of $\square^2$ depicted below:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & c \arrow[d, equal] \\
1 \arrow[r] & c
\end{tikzcd}$$

For our final example, let X denote the cubical set $0 \to 1 \to 2$. Then CX is the cubical set depicted below:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & c \arrow[d, equal] \\
1 \arrow[r] \arrow[d] & c \arrow[d, equal] \\
2 \arrow[r] & c
\end{tikzcd}$$

We define the natural transformation $\eta\colon \mathrm{id} \Rightarrow C$ to be the composite of $\partial_{1,0} \otimes -\colon \mathrm{id} \to \square^1 \otimes -$ with the quotient map $\square^1 \otimes - \Rightarrow C$. We also define a natural transformation $\mu\colon C^2 \Rightarrow C$ as follows. By the universal property of the pushout, such a natural transformation corresponds to a diagram of the form depicted below:

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "\partial_{1,1} \otimes C"'] & \square^0 \arrow[d] \\
\square^1 \otimes C \arrow[r] & C
\end{tikzcd}$$

The only natural transformation $\square^0 \to C$ is the cone point. Now note that $\square^1 \otimes -$ preserves pushouts as a left adjoint. Thus we may define the map $\square^1 \otimes C \to C$

as the map between pushouts induced by the following map between diagrams:

$$\begin{tikzcd}
\Box^1 \otimes \Box^1 \otimes - \arrow[d, "\gamma_{1,0}\otimes -"'] & \Box^1 \otimes - \arrow[l, "\partial_{2,1}\otimes -"'] \arrow[r, "\pi_{\Box^1}"] \arrow[d, "\sigma_1 \otimes -"] & \Box^1 \arrow[d] \\
\Box^1 \otimes - & \mathrm{id} \arrow[l, "\partial_{1,1}\otimes -"'] \arrow[r] & \Box^0
\end{tikzcd}$$

The commutativity of the left-hand square follows from the cubical identities.

We can also view the natural transformation μ more concretely, using Proposition 4.1.9. For $X \in \mathsf{cSet}, k \geq 0$, a k-cube of $\Box^1 \otimes X$ consists of a pair $(f \colon \Box^a \to \Box^1, x \colon \Box^b \to X)$ such that $a + b = k$. The quotient CX is then obtained by identifying cubes (f, x) and (f', x') if $f = f' = \mathrm{const}_1$. Similarly, cubes of C^2X consist of pairs (f_1, f_2, x), with (f_1, f_2, x) and (f_1', f_2', x') identified if $f_1 = f_1' = \mathrm{const}_1$ or $f_1 = f_1'$ and $f_2 = f_2' = \mathrm{const}_1$. It is clear that $\gamma_{1,0} \otimes X$ respects these identifications, thus it descends to a map $\mu \colon C^2X \to CX$.

Proposition 7.1.3. *The triple (C, η, μ) defines a monad on* cSet*.*

Proof. The monad laws follow from a straightforward calculation using the cubical identities. □

Given a cubical set X, the natural way to form a cone on X is to take its geometric product with the interval $\Box^1$, and quotient one end of the cylinder to a vertex, as was done in Definition 7.1.1. This definition, however, involved certain choices: we chose to tensor on the left rather than on the right, and to quotient the subcomplex $\{1\} \otimes X$ rather than $\{0\} \otimes X$. Considering the alternative choices, we obtain four distinct cone functors. In general, we will work with the functor C of Definition 7.1.1; when the potential for ambiguity arises, we will refer to this functor as $C_{L,1}$.

Definition 7.1.4. We define the *left negative, left positive, right negative,* and *right positive cone functors*, denoted $C_{L,0}, C_{L,1}, C_{R,0}, C_{R,1} \colon \mathsf{cSet} \to \mathsf{cSet}$, respectively, by the following pushout diagrams in cSet, where X denotes an arbitrary cubical set.

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "\partial_{1,0}\otimes X"'] & \square^0 \arrow[d] \\
\square^1 \otimes X \arrow[r] & C_{L,0}X \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r] \arrow[d, "\partial_{1,1}\otimes X"'] & \square^0 \arrow[d] \\
\square^1 \otimes X \arrow[r] & C_{L,1}X \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "X\otimes\partial_{1,0}"'] & \square^0 \arrow[d] \\
X \otimes \square^1 \arrow[r] & C_{R,0}X \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r] \arrow[d, "X\otimes\partial_{1,1}"'] & \square^0 \arrow[d] \\
X \otimes \square^1 \arrow[r] & C_{R,1}X \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

To understand the differences between these definitions, we illustrate the cubical sets $C_{W,\varepsilon}\square^1$ for $W \in \{L,R\}, \varepsilon \in \{0,1\}$. These are the four quotients which can be obtained from $\square^2$ by collapsing one of its faces to a vertex.

$$
\begin{tikzcd}
c \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\
c \arrow[r] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] \arrow[d] & c \arrow[d, equal] \\
1 \arrow[r] & c
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, equal] \arrow[d] & c \arrow[d] \\
0 \arrow[r] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \arrow[d] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

$C_{L,0}\square^1$ $\qquad$ $C_{L,1}\square^1$ $\qquad$ $C_{R,0}\square^1$ $\qquad$ $C_{R,1}\square^1$

Applying the involutions $(-)^{\mathrm{co}}, (-)^{\mathrm{co-op}}, (-)^{\mathrm{op}}$ to the pushout diagrams of Definition 7.1.4, and using Proposition 4.1.13, we obtain the following result, which shows that any one of these cone concepts suffices to describe all the others.

Lemma 7.1.5. *The functors $C_{W,\varepsilon}$ for $W \in \{L,R\}, \varepsilon \in \{0,1\}$ are related by the following formulas:*

- $C_{L,0} = (-)^{\mathrm{co-op}} \circ C_{L,1} \circ (-)^{\mathrm{co-op}}$;
- $C_{R,0} = (-)^{\mathrm{op}} \circ C_{L,1} \circ (-)^{\mathrm{op}}$;

- $C_{R,1} = (-)^{\mathrm{co}} \circ C_{L,1} \circ (-)^{\mathrm{co}}$. □

Proposition 7.1.6. *For $W \in \{L, R\}, \varepsilon \in \{0, 1\}$, the functor $C_{W,\varepsilon} \colon \mathsf{cSet} \to \mathsf{cSet}$ admits the structure of a monad, with the unit η and multiplication μ induced by natural transformations* $\mathrm{id} \Rightarrow I_W$ *and $I_W^2 \Rightarrow I_W$, where $I_L, I_R \colon \mathsf{cSet} \to \mathsf{cSet}$ are functors given by $I_L X = \square^1 \otimes X$ and $I_R X = X \otimes \square^1$, as follows:*

endofunctor	unit	multiplication
$C_{L,0}$	$\partial_{1,1} \otimes -$	$\gamma_{1,1} \otimes -$
$C_{L,1}$	$\partial_{1,0} \otimes -$	$\gamma_{1,0} \otimes -$
$C_{R,0}$	$- \otimes \partial_{1,1}$	$- \otimes \gamma_{1,1}$
$C_{R,1}$	$- \otimes \partial_{1,0}$	$- \otimes \gamma_{1,0}$

Proof. This follows from Proposition 7.1.3 and Lemma 7.1.5. □

In order to express the counit $Q \int X \to X$ (for a cubical quasicategory X) as a transfinite composite of anodyne maps, we will need to analyze the images of standard cones in cubical quasicategories.

For the remainder of this section, we will work exclusively with left positive cones, with the understanding that our results may be adapted to any of the other three varieties of cones using the formulas of Lemma 7.1.5.

Definition 7.1.7. For $m, n \geq 0$, an (m, n)*-cone* in a cubical set X is a map $C^{m,n} \to X$.

Observe that each cone $C^{m,n} \to X$ corresponds to a unique $(m+n)$-cube of X by pre-composition with the quotient map $\square^{m+n} \to C^{m,n}$. Thus we will also use the term "(m, n)-cone" to refer to a map $\square^{m+n} \to X$ which factors through this quotient map. In particular, when we refer to the (i,ε)-face of a cone x, this means the (i,ε)-face of the corresponding cube: $\square^{m+n-1} \xrightarrow{\partial_{i,\varepsilon}} \square^{m+n} \to C^{m,n} \xrightarrow{x} X$.

For $m,n,k \geq 0$, recall that $\square^{m+n}_k$ is the set of maps $[1]^k \to [1]^{m+n}$ in the box category $\square$; thus we may write such a k-cube f as $(f_1,\ldots,f_{m+n})$ where each f_i is a map $[1]^k \to [1]$. This allows us to describe $C^{m,n}$ explicitly as a quotient of $\square^{m+n}$.

Lemma 7.1.8. *For all $m,n \geq 0, C^{m,n}$ is the quotient of $\square^{m+n}$ obtained by identifying two k-cubes f,g if there exists j with $1 \leq j \leq n$ such that $f_i = g_i$ for $i \leq j$ and $f_j = g_j = \mathrm{const}_1$ (the constant map $[1]^k \to [1]$ with value 1).*

Proof. We fix m and proceed by induction on n. For the base case $n = 0$, there cannot exist any j satisfying the given criteria, thus no identifications are to be made; and indeed we have $C^{m,0} = \square^m$ by definition.

Now suppose that the given description holds for $C^{m,n}$, and let q denote the quotient map $\square^{m+n} \to C^{m,n}$. Then because the functor $\square^1 \otimes -$ preserves colimits, $\square^1 \otimes C^{m,n}$ is a quotient of $\square^{1+m+n}$ with quotient map $\square^1 \otimes q$. From this description we see that $\square^1 \otimes C^{m,n}$ is obtained from $\square^{1+m+n}$ by identifying two k-cubes f,g whenever $f_1 = g_1$ and the cubes $(f_2,\ldots,f_{n+1})$ and $(g_2,\ldots,g_{n+1})$ are identified in $C^{m,n}$. In other words, we obtain $\square^1 \otimes C^{m,n}$ from $\square^{1+m+n}$ by identifying f and g if there exists j with $2 \leq j \leq n+1$ such that $f_i = g_i$ for all $i \leq j$ and $f_j = g_j = \mathrm{const}_1$. Taking the pushout of the inclusion $\partial_{1,1} \otimes C^{m,n} \colon C^{m,n} \hookrightarrow \square^1 \otimes C^{m,n}$ along the unique map $C^{m,n} \to \square^0$, we then see that $C^{m,n+1}$ is the quotient of $\square^1 \otimes C^{m,n}$ obtained by identifying cubes f,g whenever $f_1 = g_1 = \mathrm{const}_1$. Thus the description holds for $C^{m,n+1}$. $\square$

Corollary 7.1.9. *For $k \leq n$, the quotient map $\square^{m+n} \to C^{m,n}$ factors through $C^{m+k,n-k}$. In particular, if $x \colon \square^{m+n} \to X$ is an (m,n)-cone, then x is also an*

$(m+k, n-k)$-cone for all $k \leq n$. $\square$

Lemma 7.1.10. *A face $\delta \colon \square^k \to \square^{m+n}$ is mapped to a degenerate cube by the quotient map $\square^{m+n} \to C^{m,n}$ if and only if there is some $1 \leq i \leq n$ such that:*

- *the standard form of δ contains $\partial_{i,1}$;*
- *for some $i < j \leq m+n$, the standard form of δ does not contain any map $\partial_{j,\varepsilon}$.*

Proof. We fix m, and proceed by induction on n. For $n = 0$, we have $C^{m,0} = \square^m$, and the quotient map is the identity; as there are no values i for which the statement holds, it is therefore vacuously true.

Now suppose that the statement is proven for $C^{m,n}$ and consider $C^{m,n+1}$. This cubical set is constructed via the following pushout:

$$\begin{tikzcd} C^{m,n} \arrow[r] \arrow[d, "\partial_{1,1}"] & \square^0 \arrow[d] \\ \square^1 \otimes C^{m,n} \arrow[r] & C^{m,n+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Because the functor $\square^1 \otimes -$ preserves colimits, $\square^1 \otimes C^{m,n}$ is a quotient of $\square^1 \otimes \square^{m+n} \cong \square^{m+n+1}$, with the quotient map given by applying $\square^1 \otimes -$ to the quotient map $\square^{m+n} \to C^{m,n}$.

The degenerate cubes of $\square^1 \otimes C^{m,n}$ are those corresponding to pairs (x, y) where either x is a degenerate cube of $\square^1$ or y is a degenerate cube of $C^{m,n}$. Therefore the non-degenerate cubes of $\square^1 \otimes \square^{m+n}$ which are mapped to degenerate cubes of $\square^1 \otimes C^{m,n}$ are those of the form (x, y), where both x and y are non-degenerate, and y satisfies the criteria given in the statement. Under the isomorphism $\square^1 \otimes \square^{m+n}$, such cubes correspond to faces δ of $\square^{m+n}$ such that the conditions in the statement of the theorem are satisfied for some $j \geq 2$.

The quotient map $\square^1 \otimes C^{m,n}$ then maps the $(1,1)$-face onto the cone point; thus every cube of $\square^{m+n+1}$ of positive dimension whose standard form contains

$\partial_{1,1}$ is mapped to a degenerate cube by the composite quotient map $\square^{m+n+1} \to \square^1 \otimes C^{m,n} \to C^{m,n+1}$. These are precisely the faces satisfying the criteria of the statement for $i = 1$. $\square$

Using the characterization of cones given above, we can show that any face of a given cone is a cone of a specified degree.

Lemma 7.1.11. *For $i \leq n$, the image of the composite map $\square^{m+n-1} \xrightarrow{\partial_{i,0}} \square^{m+n} \to C^{m,n}$ is isomorphic to $C^{m,n-1}$. For $i \geq n+1$, $\varepsilon \in \{0,1\}$, the image of the composite map $\square^{m+n-1} \xrightarrow{\partial_{i,\varepsilon}} \square^{m+n} \to C^{m,n}$ is isomorphic to $C^{m-1,n}$.*

Proof. First consider the composite map $\square^{m+n-1} \xrightarrow{\partial_{i,0}} \square^{m+n} \to C^{m,n}$. Let $f = (f_1, \ldots, f_{m+n-1})$ denote a k-cube of $\square^{m+n-1}$, as in the proof of Lemma 7.1.8. We denote the image of this cube under $\partial_{i,0}$ by $f' = (f'_1, \ldots, f'_{m+n-1})$, where $f'_j = f_j$ for $j < i$, $f'_i = \text{const}_0$, and $f'_j = f_{j-1}$ for $j > i$. By Lemma 7.1.8, given two k-cubes f and g in $\square^{m+n-1}$, their images under $\partial_{i,0}$ will be identified in the quotient $C^{m,n}$ if and only if there exists $j \leq n$ such that $f'_l = g'_l$ for $l \leq j$ and $f'_j = g'_j = \text{const}_1$ – in other words, if there exists $j \leq n-1$ such that $f_l = g_l$ for $l \leq j$ and $f_j = g_j = \text{const}_1$. The desired isomorphism thus follows from Lemma 7.1.8.

The analysis of $\partial_{i,\varepsilon}$ where $i \geq n+1$, $\varepsilon \in \{0,1\}$ is similar, except that in that case we have $f'_j = f_j$ for all $j \leq i$. Thus we conclude that the images of f and g in the quotient $C^{m,n}$ are equal if and only if there exists $j \leq n$ such that $f_l = g_l$ for $l \leq j$ and $f_j = g_j = \text{const}_1$. $\square$

Using Lemma 7.1.11 and further computations, we can analyze the effect of cubical structure maps on cones.

Lemma 7.1.12. *Let x be an (m,n)-cone in a cubical set X. Then:*

(i) If $n \geq 1$, then for $i \leq n$, $x\partial_{i,0}$ is an $(m, n-1)$-cone;

(ii) If $m \geq 1$, then for $i \geq n+1$, $x\partial_{i,0}$ is an $(m-1, n)$-cone;

(iii) If $m \geq 1$, then for all i, $x\partial_{i,1}$ is an $(m-1, n)$-cone;

(iv) for $i \geq n+1$, $x\sigma_i$ is an $(m+1, n)$-cone;

(v) if $n \geq 1$ then for $i \leq n$, $x\gamma_{i,0}$ is an $(m, n+1)$-cone;

(vi) for $i \geq n+1$, $x\gamma_{i,\varepsilon}$ is an $(m+1, n)$-cone.

Proof. First consider item (i). By Lemma 7.1.11, we have a commuting diagram as shown below:

$$\begin{tikzcd}
\square^{m+n-1} \arrow[r, hook, "\partial_{i,0}"] \arrow[d] & \square^{m+n} \arrow[d] \\
C^{m,n-1} \arrow[r, hook] & C^{m,n}
\end{tikzcd}$$

Now, for an $(m+n)$-cube $x \in X_{m+n}$ to be an (m,n)-cone means precisely that the corresponding map $x \colon \square^{m+n} \to X$ factors through $C^{m,n}$. So the face $x\partial_{i,0}$ can be written as $\square^{m+n-1} \xrightarrow{\partial_{i,0}} \square^{m+n} \to C^{m,n} \xrightarrow{x} X$; by the diagram above we can rewrite this as $\square^{m+n-1} \to C^{m,n-1} \to C^{m,n} \xrightarrow{x} X$. So $x\partial_{i,0}$ factors through $C^{m,n-1}$, meaning that it is an $(m, n-1)$-cone.

Similar commuting diagrams can be used to prove the remaining statements. For item (ii) we may again apply Lemma 7.1.11; the other statements require new computations. We will show these computations for item (iii); the others are similar.

Let $m \geq 1, i \leq n$ and consider the composite $\square^{m+n-1} \xrightarrow{\partial_{i,1}} \square^{m+n} \to C^{m,n}$. As in the proof of Lemma 7.1.11, we let f denote an arbitrary k-cube of $\square^{m+n-1}$ and let f' denote its image under $\partial_{i,1}$; then once again we have $f'_j = f_j$ for $j \leq i-1$, but now $f'_i = \mathrm{const}_1$. So let f and g be two k-cubes of $\square^{m+n-1}$, and suppose that there exists $j \leq n$ such that $f_l = g_l$ for $l \leq j$ and $f_j = g_j = \text{const}_1$. Then there exists $j' \leq n$ such that $f'_l = g'_l$ for $l \leq j'$ and $f'_{j'} = g'_{j'} = \text{const}_1$: if $j < i$ then $j' = j$, while if $j \geq i$ then $j' = i$. So f' and g' are identified in $C^{m,n}$. Thus the composite map factors through $C^{m-1,n}$, i.e. we have a commuting diagram:

$$\begin{tikzcd}
\square^{m+n-1} \arrow[r, hook, "\partial_{i,1}"] \arrow[d] & \square^{m+n} \arrow[d] \\
C^{m-1,n} \arrow[r, hook] & C^{m,n}
\end{tikzcd}$$

So for any (m,n)-cone x, $x\partial_{i,1}$ is an $(m-1,n)$-cone. □

Corollary 7.1.13. *For $n \geq 1$, every face of a $(0,n)$-cone is a $(0,n-1)$-cone.*

Proof. Let $x: \square^n \to X$ be a $(0,n)$-cone, and consider a face $x\partial_{i,\varepsilon}$. If $i \leq n, \varepsilon = 0$, then $x\partial_{i,\varepsilon}$ is a $(0,n-1)$-cone by Lemma 7.1.12 (i). Otherwise, we may note that x is a $(1,n-1)$-cone by Corollary 7.1.9, and therefore $x\partial_{i,\varepsilon}$ is a $(0,n-1)$-cone by Lemma 7.1.12 (ii) or (iii). □

Corollary 7.1.14. *For $m \geq 1, n \geq 0$, let x be an $(m+n-1)$-cube in a cubical set X. If $x\gamma_{n,0}$ is an (m,n)-cone, then it is also an $(m-1,n+1)$-cone.*

Proof. By Lemma 7.1.12 (ii), $x\gamma_{n,0}\partial_{n+1,0} = x$ is an $(m-1,n)$-cone. Therefore, $x\gamma_{n,0}$ is an $(m-1,n+1)$-cone by Lemma 7.1.12 (v). □

In some cases it will be more convenient to characterize cones in a cubical set by a set of conditions on their faces. By a direct analysis of the cubes of $C^{m,n}$, or by an inductive argument similar to that used in the proof of Lemma 7.1.8, we have the following characterization of (m,n)-cones in X.

Lemma 7.1.15. *For m,n with $n \geq 1$, and $X \in cSet$, a cube $x: \square^{m+n} \to X$ is an (m,n)-cone if and only if for all i such that $1 \leq i \leq n$ we have*

$$x\partial_{i,1} = x\partial_{m+n,0}\partial_{m+n-1,0}\cdots\partial_{i+1,0}\partial_{i,1}\sigma_i\sigma_{i+1}\cdots\sigma_{m+n-2}\sigma_{m+n-1}$$

(In the case $m = 0, i = n$ we interpret this statement as the tautology $x\partial_{n,1} = x\partial_{n,1}$). □

We will also have use for the following result, which shows that the standard cones contain many inner open boxes.

Lemma 7.1.16. *For $n \geq 1, 2 \leq i \leq m+n$, the quotient map $\square^{m+n} \to C^{m,n}$ sends the critical edge with respect to the face $\partial_{i,0}$ to a degenerate edge.*

Proof. The critical edge in question corresponds to the function $f\colon [1] \to [1]^{m+n}$ with $f_i = \mathrm{id}_{[1]}$, $f_j = \mathrm{const}_1$ for $j \neq i$. In particular, $f_1 = \mathrm{const}_1$, so f is equivalent, under the equivalence relation of Lemma 7.1.8, to the map $[1] \to [1]^{m+n}$ which is constant at $(1, \ldots, 1)$. □

We now prove a lemma regarding the standard forms of cones.

Lemma 7.1.17. *Let $m \geq 1$, and let $x\colon C^{m,n} \to X$ be a degenerate (m,n)-cone.*

(i) If the standard form of x is $z\sigma_{a_p}$, then $a_p \geq n+1$.

(ii) If the standard form of x is $z\gamma_{b_q,1}$, then $b_q \geq n+1$.

Proof. For $n = 0$ these statements are trivial, so assume $n \geq 1$. We will prove item (i); the proof for item (ii) is similar.

Towards a contradiction, suppose that $a_p \leq n$, and let

$$z = y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,\varepsilon_q}\sigma_{a_1}\cdots\sigma_{a_{p-1}}$$

so that $z\sigma_{a_p} = x$. Taking the $(a_p, 1)$-faces of both sides of this equation, and applying Lemma 7.1.15, we see that:

$$
\begin{aligned}
z &= x\partial_{m+n,0}\cdots\partial_{a_p+1,0}\partial_{a_p,1}\sigma_{a_p}\cdots\sigma_{m+n-1} \\
\therefore z\sigma_{a_p} &= x\partial_{m+n,0}\cdots\partial_{a_p+1,0}\partial_{a_p,1}\sigma_{a_p}\cdots\sigma_{m+n-1}\sigma_{a_p} \\
\therefore x &= x\partial_{m+n,0}\cdots\partial_{a_p+1,0}\partial_{a_p,1}\sigma_{a_p}\cdots\sigma_{m+n}
\end{aligned}
$$

In the last step, we have repeatedly used the identity $\sigma_j\sigma_i = \sigma_i\sigma_{j+1}$ for $i \leq j$ to rearrange the string $\sigma_{a_p}\cdots\sigma_{m+n-1,1}\sigma_{a_p}$ into one whose indices are in strictly increasing order. (We can do this because, by our assumption on m, $m + n - 1 \geq n \geq a_p$.) Now let $y'\gamma_{b'_1,\varepsilon'_1}\cdots\gamma_{b'_{q'},\varepsilon'_{q'}}\sigma_{a'_1}\cdots\sigma_{a'_{p'}}$ be the standard form of $x\partial_{m+n,0}\cdots\partial_{a_p+1,0}\partial_{a_p,1}$; then we have:

$$
x = y'\gamma_{b'_1,\varepsilon'_1}\cdots\gamma_{b'_{q'},\varepsilon'_{q'}}\sigma_{a'_1}\cdots\sigma_{a'_{p'}}\sigma_{a_p}\cdots\sigma_{m+n}
$$

We can apply further identities to re-order the maps on the right-hand side of this equation, obtaining a standard form for x in which the rightmost degeneracy map has index greater than or equal to $m+n$. But as the standard form of x is unique, this contradicts our assumption that $a_p \leq n$. □

Corollary 7.1.18. *Let $x : \square^n \to X$. If x is a $(0, n)$-cone, then the standard form of x contains no positive connection maps.*

Proof. Let $x = y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,\varepsilon_q}\sigma_{a_1}\cdots\sigma_{a_p}$ in standard form. Towards a contradiction, suppose that there exists $1 \leq i \leq q$ such that $\varepsilon_i = 1$. By repeatedly applying face maps and using Corollary 7.1.13, we see that $y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_i,1}$ is a $(0, n-p-q+i)$-cone. Lemma 7.1.17 (ii) thus implies that $b_i \geq n-p-q+i+1$. But $\gamma_{b_i,1}$ is a map $[1]^{n-p-q+i} \to [1]^{n-p-q+i-1}$, implying $b_i \leq n-p-q+i-1$. □

Before turning our attention to coherent families of composites, we introduce certain subcomplexes of the standard cones, which will be useful in constructing coherent families of composites.

Definition 7.1.19. For $m, n \geq 0, n \leq k \leq m+n-1$, $B^{m,n,k}$ is the subcomplex of $C^{m,n}$ consisting of the images of the faces $\partial_{1,0}$ through $\partial_{k,0}$, as well as all all faces $\partial_{i,1}$, under the quotient map $\square^{m+n} \to C^{m,n}$.

In order to characterize maps out of $B^{m,n,k}$, we will need to prove a couple of lemmas concerning the faces of $C^{m,n}$.

Lemma 7.1.20. *For $m, n \geq 0, 1 \leq i_1 < i_2 \leq m+n, \varepsilon_1, \varepsilon_2 \in \{0,1\}$, where $i_j \geq n+1$ if $\varepsilon_j = 1$, the intersection of the images of the faces $\partial_{i_1,\varepsilon_1}$ and $\partial_{i_2,\varepsilon_2}$ of $\square^{m+n}$ under the quotient map $\square^{m+n} \to C^{m,n}$ is exactly the image of the face $\partial_{i_2,\varepsilon_2}\partial_{i_1,\varepsilon_1} = \partial_{i_1,\varepsilon_1}\partial_{i_2-1,\varepsilon_2}$.*

Proof. That the intersection of the images of $\partial_{i_1,\varepsilon_1}$ and $\partial_{i_2,\varepsilon_2}$ contains the image of $\partial_{i_2,\varepsilon_2}\partial_{i_1,\varepsilon_1}$ is clear, as this face is the intersection of $\partial_{i_1,\varepsilon_1}$ and $\partial_{i_2,\varepsilon_2}$ in $\square^{m+n}$. Now we will verify the opposite containment, using description of $C^{m,n}$ from Lemma 7.1.8.

To this end, consider a map $f\colon [1]^k \to [1]^{m+n}$ such that the equivalence class $[f] \in C^{m,n}_k$ is contained in the images of faces (i_1, ε_1) and (i_2, ε_2). We will construct $f'\colon [1]^k \to [1]^{m+n}$ such that $f \sim f'$ and f' is contained in the intersection of faces (i_1, ε_1) and (i_2, ε_2), thereby showing that $[f] = [f']$ is contained in the image of this intersection under the quotient map.

Since f is in the image of face (i_1, ε_1), $f \sim g$ for some $g\colon [1]^k \to [1]^{m+n}$ such that $g_{i_1} = \mathrm{const}_{\varepsilon_1}$. Therefore, at least one of the following holds:

(i) $f_{i_1} = \mathrm{const}_{\varepsilon_1}$;

(ii) $f_j = g_j = \mathrm{const}_1$ for some $j \leq \min(i_1 - 1, n)$.

If (ii) holds, then f is equivalent to any f' such that $f'_l = f_l$ for $l \leq j$; in particular, we can choose such an f' satisfying $f'_{i_1} = \mathrm{const}_{\varepsilon_1}, f'_{i_2} = \mathrm{const}_{\varepsilon_2}$.

Now suppose that (i) holds, but (ii) does not. Then because f is in the image of face (i_2, ε_2), $f \sim h$ for some $h\colon [1]^k \to [1]^{m+n}$ such that $h_{i_2} = \mathrm{const}_{\varepsilon_2}$. Therefore, at least one of the following holds:

(i) $f_{i_2} = \mathrm{const}_{\varepsilon_2}$;

(ii) $f_j = h_j = \mathrm{const}_1$ for some $i_1 + 1 \leq j \leq \min(i_2 - 1, n)$.

In case (i), we have $f_{i_1} = \mathrm{const}_{\varepsilon_1}, f_{i_2} = \mathrm{const}_{\varepsilon_2}$, so we can simply choose $f' = f$. In case (ii), f is equivalent to any f' such that $f'_l = f_l$ for $l \leq j$ (which implies $f'_{i_1} = \mathrm{const}_{\varepsilon_1}$); in particular, we can choose such an f' satisfying $f'_{i_2} = \mathrm{const}_{\varepsilon_2}$. □

Lemma 7.1.21. *For $i \leq n$, the image of the face $\partial_{i,1}$ under the quotient map $\square^{m+n} \to C^{m,n}$ is contained in the image of $\partial_{m+n,1}$.*

Proof. Let $f\colon [1]^k \to [1]^{m+n}$ be a k-cube of $\square^{m+n}$ which factors through $\partial_{i,1}$. Then $f_i = \mathrm{const}_1$. Thus f is equivalent to any $f'\colon [1]^k \to [1]^{m+n}$ such that $f'_j = f_j$ for all $j \leq i$; in particular, we may choose such an f' with $f'_{m+n} = \mathrm{const}_1$. So f' factors through $\partial_{m+n,1}$; thus $[f] = [f']$ is contained in the image of $\partial_{m+n,1}$ under the quotient map. □

Lemma 7.1.22. *For a cubical set X, a map $x\colon B^{m,n,n} \to X$ is determined by a set of $(m, n-1)$-cones $x_{i,0}\colon C^{m,n-1} \to X$ for $1 \leq i \leq n$ and a set of $(m-1, n)$-cones $x_{i,1}$ for $n+1 \leq i \leq m+n$ such that for all $i_1 < i_2, \varepsilon_1, \varepsilon_2 \in \{0,1\}, x_{i_2,\varepsilon_2}\partial_{i_1,\varepsilon_1} = x_{i_1,\varepsilon_1}\partial_{i_2-1,\varepsilon_2}$, with $x_{i,\varepsilon}$ being the image of $\partial_{i,\varepsilon}$ under x.* □

Proof. To define a map $x\colon B^{m,n,k} \to X$, it suffices to assign the values of x on the faces $[\partial_{i,\varepsilon}]$ of $C^{m,n}$ for which $i \le k$ or $\varepsilon = 1$, provided that these choices are consistent on the intersections of faces. By Lemma 7.1.21, it suffices to consider only those faces for which $i \le k, \varepsilon = 0$ or $i \ge n+1, \varepsilon = 1$. These faces are isomorphic to $C^{m,n-1}$ or $C^{m-1,n}$, respectively, by Lemma 7.1.11. By Lemma 7.1.20, to show that these choices are consistent on the intersections of faces, it suffices to show that they satisfy the cubical identity for composites of face maps. □

Proposition 7.1.23. *For all $m, n \ge 1, n \le k \le m+n-1$, the inclusion $B^{m,n,k} \hookrightarrow C^{m,n}$ is a trivial cofibration.*

Proof. We proceed by induction on m. In the base case $m = 1$, the only relevant value of k is $k = n$. The only face of $C^{1,n}$ which is missing from $B^{1,n,n}$ is $[\partial_{n+1,0}]$, so the inclusion $B^{1,n,n} \hookrightarrow C^{1,n}$ is an $(n+1,0)$-open box filling. By Lemma 7.1.16, the critical edge for this open box filling is degenerate, so the inclusion is a trivial cofibration.

Now let $m \ge 2$, and suppose the statement holds for $m-1$. For $n \le k \le m+n-2$, consider the intersection of the $(k+1,0)$-face of $C^{m,n}$, $[\partial_{k,0}]$, with the subcomplex $B^{m,n,k}$. By Lemma 7.1.20 and Lemma 7.1.21, this intersection consists of faces $(1,0)$ through $(k,0)$ and $(1,1)$ through $(m+n-1,1)$ of $[\partial_{k+1,0}]$. By Lemma 7.1.11, it is thus isomorphic to $B^{m-1,n,k}$.

Thus we can express $B^{m,n,k+1}$ as the following pushout:

$$\begin{tikzcd}
B^{m-1,n,k} \arrow[r, hook] \arrow[d, hook] & B^{m,n,k} \arrow[d, hook] \\
C^{m-1,n} \arrow[r, hook] & B^{m,n,k+1} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

By the induction hypothesis, $B^{m,n,k} \hookrightarrow C^{m-1,n}$ is a trivial cofibration, since

$n \le k \le m+n-2$. Thus $B^{m,n,k} \hookrightarrow B^{m,n,k+1}$ is a trivial cofibration, as a pushout of a trivial cofibration. From this we can see that for any $n \le k \le m+n-2$, the composite inclusion $B^{m,n,k} \hookrightarrow B^{m,n,k+1} \hookrightarrow \cdots \hookrightarrow B^{m,n,m+n-1}$ is a trivial cofibration.

Thus it suffices to prove that $B^{m,n,m+n-1} \hookrightarrow C^{m,n}$ is a trivial cofibration. Here, as in the base case, the subcomplex $B^{m,n,m+n-1}$ is only missing the face $[\partial_{m+n,0}]$, so the inclusion is an $(m+n,0)$-open box filling. The critical edge of this open box is degenerate by Lemma 7.1.16, so the inclusion is indeed a trivial cofibration.

Thus we see that the inclusion $B^{m,n,k} \hookrightarrow C^{m,n}$ is a trivial cofibration for any m, n, k satisfying the constraints given in the statement. □

We now turn our attention to coherent families of composites, a technical tool needed to build a cubical quasicategory out of its maximal simplicial subcomplex via inner open box fillings. To this end, we begin by defining coherent families of composites and then show that every cubical quasicategory admits such a family.

Definition 7.1.24. A *coherent family of composites* θ in a cubical quasicategory X consists of a family of functions $\theta^{m,n} \colon \mathsf{cSet}(C^{m,n}, X) \to \mathsf{cSet}(C^{m,n+1}, X)$, such that for any (m,n)-cone $x \colon C^{m,n} \to X$, the following identities hold:

(Θ1) For $i \le n$, $\theta^{m,n}(x)\partial_{i,0} = \theta^{m,n-1}(x\partial_{i,0})$;

(Θ2) $\theta^{m,n}(x)\partial_{n+1,0} = x$;

(Θ3) For $i \ge n+2$, $\theta^{m,n}(x)\partial_{i,1} = \theta^{m-1,n}(x\partial_{i-1,1})$;

(Θ4) If $x\sigma_i$ is an (m,n)-cone for $i \ge n+1$, then $\theta^{m,n}(x\sigma_i) = \theta^{m-1,n}(x)\sigma_{i+1}$;

(Θ5) If $x\gamma_{i,0}$ is an (m,n)-cone for $i \le n-1$, then $\theta^{m,n}(x\gamma_{i,0}) = \theta^{m,n-1}(x)\gamma_{i,0}$;

(Θ6) If $x\gamma_{i,\varepsilon}$ is an (m, n)-cone for $i \geq n+1$, then $\theta^{m,n}(x\gamma_{i,\varepsilon}) = \theta^{m-1,n}(x)\gamma_{i+1,\varepsilon}$;

(Θ7) $\theta^{m,n}(\theta^{m,n-1}(x)) = \theta^{m,n-1}(x)\gamma_{n,0}$;

(Θ8) For $m \geq 1$, if x is an $(m-1, n+1)$-cone, then $\theta^{m,n}(x) = x\gamma_{n+1,0}$.

The rough intuition behind Definition 7.1.24 is this: thinking of cubes in a cubical quasicategory X as representing diagrams commuting up to homotopy, constructing a coherent family of composites on X amounts to coherently choosing a specific composite edge for each $x \colon \square^n \to X$. For instance, consider a 2-cube x as depicted below, witnessing $gf \sim qp$:

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "p"'] & y \arrow[d, "g"] \\
z \arrow[r, "q"] & w
\end{tikzcd}$$

Then the identities (Θ1) through (Θ8) imply that $\theta^{2,0}(x)$ is a 3-cube of the form depicted below:

$$\begin{tikzcd}
x \arrow[rr, "s"] \arrow[dr, "p"] \arrow[dd, "f"'] & & w \arrow[dr, equal] \arrow[dd, equal] & \\
& z \arrow[rr, "q"] \arrow[dd, "q"] & & w \arrow[dd, equal] \\
y \arrow[rr, "g"] \arrow[dr, "g"'] & & w \arrow[dr, equal] & \\
& w \arrow[rr, equal] & & w
\end{tikzcd}$$

The edge s from x to w is homotopic to both composites gf and qp.

The remainder of this section is dedicated to proving the following theorem.

Theorem 7.1.25. *Every cubical quasicategory admits a coherent family of composites.*

To prove this, we will construct the family of functions $\theta^{m,n}$ by induction on m and n.

Definition 7.1.26 (Base case)**.** For a cubical quasicategory X and $x\colon C^{0,n} \to X$, let $\theta^{0,n}(x) = x\sigma_{n+1}$. For $x\colon C^{1,n} \to X$, let $\theta^{1,n}(x) = x\gamma_{n+1,0}$.

These define $(0, n+1)$-cones and $(1, n+1)$-cones, respectively, by Lemma 7.1.12.

Remark 7.1.27. While it may appear that these definitions of $\theta^{0,n}$ and $\theta^{1,n}$ were chosen arbitrarily, in fact they are implied by the identities of Definition 7.1.24. Specifically, the given definition of $\theta^{1,n}$ is implied by (Θ8) and Lemma 7.1.2. This, together with (Θ3) and Lemma 7.1.12 (iv), then implies the given definition of $\theta^{0,n}$.

Lemma 7.1.28. *For a cubical quasicategory X, the families of functions $\theta^{0,n}$ and $\theta^{1,n}$ satisfy the identities of Definition 7.1.24.*

Proof. We first verify the identities for $\theta^{0,n}$. The hypotheses of (Θ3), (Θ4) and (Θ6) are vacuous here, as there are no cubical structure maps satisfying the given constraints on their indices; (Θ8) similarly does not apply in this case. The remaining identities follow easily from the cubical identities:

- For (Θ1), let $i \leq n$. Then $\theta^{0,n}(x)\partial_{i,0} = x\sigma_{n+1}\partial_{i,0} = x\partial_{i,0}\sigma_n = \theta^{0,n-1}(x)\sigma_n$.
- For (Θ2), we have $\theta^{0,n}(x)\partial_{n+1,0} = x\gamma_{n+1,0}\partial_{n+1,0} = x$.
- For (Θ5), let $1 \leq i \leq n-1$. Then $\theta^{0,n}(x\gamma_{i,0}) = x\gamma_{i,0}\sigma_{n+1} = x\sigma_n\gamma_{i,0} = \theta^{0,n-1}(x)\gamma_{i,0}$.

- For (Θ7), we have $\theta^{0,n+1}(\theta^{0,n}(x)) = x\sigma_{n+1}\sigma_{n+2} = x\sigma_{n+1}\gamma_{n+1,0} = \theta^{0,n}(x)\gamma_{n+1,0}$.

Next we will verify the identities for $\theta^{1,n}$. Here (Θ8) holds by definition, while the hypothesis of (Θ6) is still vacuous, as there are no connection maps $\gamma_{i,\varepsilon}\colon [1]^n \to [1]^{n-1}$ with $i \geq n+1$. Once again, we can verify the remaining identities using the cubical identities:

- For (Θ1), let $i \leq n$. Then $\theta^{1,n}(x)\partial_{i,0} = x\gamma_{n+1,0}\partial_{i,0} = x\partial_{i,0}\gamma_{n,0} = \theta^{1,n-1}(x\partial_{i,0})$.

- For (Θ2), we have $\theta^{1,n}(x)\partial_{n+1,0} = x\gamma_{n+1,0}\partial_{n+1,0} = x$.

- For (Θ3), we need only consider the case $m' = 1, i = n+2$. For this case we have $\theta^{1,n}(x)\partial_{n+2,1} = x\gamma_{n+1,0}\partial_{n+2,1} = x\partial_{n+1,1}\sigma_{n+1} = \theta^{0,n}(x\partial_{n+1,1})$.

- For (Θ4), the only relevant degeneracy is σ_{n+1}, and we have $\theta^{1,n}(x\sigma_{n+1}) = x\sigma_{n+1}\gamma_{n+1,0} = x\sigma_{n+1}\sigma_{n+2} = \theta^{0,n}(x)\sigma_{n+2}$.

- For (Θ5), let $1 \leq i \leq n-1$. Then $\theta^{1,n}(x\gamma_{i,0}) = x\gamma_{i,0}\gamma_{n+1,0} = x\gamma_{n,0}\gamma_{i,0} = \theta^{1,n-1}(x)\gamma_{i,0}$.

- For (Θ7), we have $\theta^{1,n+1}(\theta^{1,n}(x)) = x\gamma_{n+1,0}\gamma_{n+2,0} = x\gamma_{n+1,0}\gamma_{n+1,0} = \theta^{1,n}(x)\gamma_{n+1,0}$. □

The following lemma will be used in defining $\theta^{m,n}$ in the inductive case.

Lemma 7.1.29. *Let $m \geq 2, n \geq 0$, and let X be a cubical quasicategory equipped with functions $\theta^{m,n}$ satisfying the identities of Definition 7.1.24 for all pairs (m', n') such that $m' \leq m$, $n' \leq n$, and at least one of these two inequalities is strict. Then for any $x\colon C^{m,n} \to X$, there exists an $(m, n+1)$-cone $\widetilde{\theta}^{m,n}(x)\colon C^{m,n+1} \to X$ satisfying* (Θ1), (Θ2), *and* (Θ3).

Proof. For each $i \leq n$, the face $x\partial_{i,0}$ is an $(m, n-1)$-cone by Lemma 7.1.12 (i); thus X contains an (m, n)-cone $\theta^{m,n-1}(x\partial_{i,0})$. Similarly, for each $i \geq n+2$, the face $x\partial_{i-1,1}$ is an $(m-1, n)$-cone, and so X contains an $(m-1, n+1)$-cone $\theta^{m-1,n}(x\partial_{i-1,1})$, and these cones satisfy the identities of Definition 7.1.24. Using Lemma 7.1.22, we will define a map $y\colon B^{m,n+1,n+1} \to X$ with $y_{i,0} = \theta^{m,n-1}(x\partial_{i,0})$ for $1 \leq i \leq n$, $y_{n+1,0} = x$, and $y_{i,1} = \theta^{m-1,n}(x\partial_{i-1,1})$ for $i \geq n+2$.

To show that we can define such a map, we must verify that our choices of $y_{i,\varepsilon}$ satisfy the cubical identity for composing face maps.

For $i_1 < i_2 \leq n, \varepsilon_1 = \varepsilon_2 = 0$, we have:

$$\begin{aligned}
y_{i_2,0}\partial_{i_1,0} &= \theta^{m,n-1}(x\partial_{i_2,0})\partial_{i_1,0} \\
&= \theta^{m,n-2}(x\partial_{i_2,0}\partial_{i_1,0}) \\
&= \theta^{m,n-2}(x\partial_{i_1,0}\partial_{i_2-1,0}) \\
&= \theta^{m,n-1}(x\partial_{i_1,0})\partial_{i_2-1,0} \\
&= y_{i_1,0}\partial_{i_2-1,0}
\end{aligned}$$

For $i_1 < i_2 = n+1$, we have:

$$\begin{aligned}
y_{n+1,0}\partial_{i_1,0} &= x\partial_{i_1,0} \\
&= \theta^{m,n-1}(x\partial_{i_1,0})\partial_{n,0} \\
&= y_{i_1,0}\partial_{n,0}
\end{aligned}$$

For $n+1 = i_1 < i_2$ we have:

$$\begin{aligned} y_{i_2,1}\partial_{n+1,0} &= \theta^{m-1,n}(x\partial_{i_2-1,1})\partial_{n+1,0} \\ &= x\partial_{i_2-1,1} \\ &= y_{n+1,0}\partial_{i_2-1,1} \end{aligned}$$

Finally, for $n+2 \leq i_1 < i_2$, we have:

$$\begin{aligned} y_{i_2,1}\partial_{i_1,1} &= \theta^{m-1,n}(x\partial_{i_2-1,1})\partial_{i_1,1} \\ &= \theta^{m-2,n}(x\partial_{i_2-1,1}\partial_{i_1-1,1}) \\ &= \theta^{m-2,n}(x\partial_{i_1-1,1}\partial_{i_2-2,1}) \\ &= \theta^{m-1,n}(x\partial_{i_1-1,1})\partial_{i_2-1,1} \\ &= y_{i_1,1}\partial_{i_2-1,1} \end{aligned}$$

Thus the $(n+1)$-tuple y does indeed define a map $B^{m,n+1,n+1} \to X$. Now consider the following commuting diagram:

$$\begin{tikzcd} B^{m,n+1,n+1} \arrow[r, "y"] \arrow[d, hook, "\sim"'] & X \arrow[d] \\ C^{m,n+1} \arrow[r] & \square^0 \end{tikzcd}$$

The left-hand map is a trivial cofibration by Proposition 7.1.23, while the right-hand map is a fibration by assumption. Thus there exists a lift of this diagram, i.e. an $(m, n+1)$-cone $\widetilde{\theta}^{m,n}(x) \colon C^{m,n+1} \to X$ such that for $i \leq n, \widetilde{\theta}^{m,n}(x)\partial_{i,0} = \theta^{m,n-1}(x\partial_{i,0})$, $\widetilde{\theta}^{m,n}(x)\partial_{n+1,0} = x$, and for $i \geq n+2$, $\widetilde{\theta}^{m,n}(x)\partial_{i,1} = \theta^{m-1,n}(x\partial_{i-1,1})$. $\square$

Although Lemma 7.1.29 applies for an arbitrary (m, n)-cone x with $m \geq 2$, we will not use it to construct $\theta^{m,n}$ for all such cones, as the arbitrary lift used in its proof may not satisfy (Θ4) through (Θ8). Instead, we define $\theta^{m,n}$ for $m \geq 2, n \geq 0$ by the following case analysis.

Definition 7.1.30 (Inductive case)**.** Let $m \geq 2, n \geq 0$, and let X be a cubical quasicategory equipped with functions $\theta^{m,n}$ satisfying the identities of Definition 7.1.24 for all pairs (m', n') such that $m' \leq m$, $n' \leq n$, and at least one of these two inequalities is strict. Let $x \colon C^{m,n} \to X$ be an (m, n)-cone. Then $\theta^{m,n}(x) \colon \square^{m+n+1} \to X^{m,n}$ is defined as follows:

(1) If the standard form of x is $z\sigma_{a_p}$ for some $a_p \geq n+1$, then $\theta^{m,n}(x) = \theta^{m-1,n}(z)\sigma_{a_p+1}$;

(2) If the standard form of x is $z\gamma_{b_q,0}$ for some $b_q \leq n-1$, then $\theta^{m,n}(x) = \theta^{m,n-1}(z)\gamma_{b_q,0}$;

(3) If the standard form of x is $z\gamma_{b_q,\varepsilon}$ for some $b_q \geq n+1$, then $\theta^{m,n}(x) = \theta^{m-1,n}(z)\gamma_{b_q+1,\varepsilon}$;

(4) If x is an $(m-1, n+1)$-cone not covered under any of cases (1) through (3), then $\theta^{m,n}(x) = x\gamma_{n+1,0}$;

(5) If $x = \theta^{m,n-1}(x')$ for some $x' \colon C^{m,n-1} \to X$ and x is not covered under any of cases (1) through (4) then $\theta^{m,n}(x) = x\gamma_{n,0}$;

(6) If x is not convered under any of cases (1) through (5), then $\theta^{m,n}(x)$ is the cone $\widetilde{\theta}^{m,n}(x)$ constructed in Lemma 7.1.29.

That each of the constructions of Definition 7.1.30 produces an $(m, n+1)$-cone can be seen from Corollary 7.1.9 and Lemmas 7.1.12 and 7.1.29.

Before proving that this definition satisfies the identities of Definition 7.1.24, we prove some simple lemmas about its cases.

Lemma 7.1.31. *Every degenerate cone in a cubical quasicategory X falls under one of cases (1) to (4) of Definition 7.1.30.*

Proof. This follows from Corollary 7.1.14 and Lemma 7.1.17. □

Corollary 7.1.32. *Case (6) of Definition 7.1.30 consists precisely of those (m, n)-cones of X which are:*

- *Non-degenerate;*
- *Not $(m-1, n+1)$-cones;*
- *Not equal to $\theta^{m,n-1}(x)$ for any $x \colon C^{m,n-1} \to X$.* □

Lemma 7.1.33. *Let X be a cubical quasicategory, and let $m, n \geq 0$ for which we have defined $\theta^{m,n}$ satisfying the identities of Definition 7.1.24. Then $x \colon C^{m,n} \to X$ is covered under case (6) of Definition 7.1.30, i.e.:*

- *x is non-degenerate;*
- *x is not an $(m-1, n+1)$-cone;*
- *x is not equal to $\theta^{m,n-1}(x')$ for any $x' \colon C^{m,n-1} \to X$;*

if and only if $\theta^{m,n}(x)$ is covered under case (5), i.e. it is non-degenerate and is not an $(m-1, n+2)$-cone.

Proof. First suppose x is covered under case (6). The cubical identities show that if a degenerate cube y has a non-degenerate face z, then z appears as at least two distinct faces of y. We have $\theta^{m,n}(x)\partial_{n+1,0} = x$, and x is non-degenerate by assumption, so if $\theta^{m,n}(x)$ is degenerate, then x must appear as at least one other face of $\theta^{m,n}(x)$. However, for $i \leq n$ we have $\theta^{m,n}(x)\partial_{i,0} = \theta^{m,n-1}(x\partial_{i,0})$, while for $i \geq n+2$ or $\varepsilon = 1$, $\theta^{m,n}(x)\partial_{i,\varepsilon}$ is an $(m-1, n+1)$-cone by Lemma 7.1.12. Thus none of these faces are equal to x, showing that $\theta^{m,n}(x)$ is non-degenerate. Furthermore, $\theta^{m,n}(x)$ is not an $(m-1, n+2)$-cone, as this would imply that $\theta^{m,n}(x)\partial_{n+1,0} = x$ was an $(m-1, n+1)$-cone by Lemma 7.1.12 (i).

On the other hand, if x is not covered under case (6), then $\theta^{m,n}(x)$ is degenerate, hence covered under one of cases (1) to (4) by Lemma 7.1.31. $\square$

The proof that the construction θ of Definition 7.1.30 satisfies all of the identities of Definition 7.1.24 involves many elaborate case analyses; for brevity, these calculations have been relegated to appendix A.

Proof of Theorem 7.1.25. The functions $\theta^{m,n}$ are defined inductively by Definitions 7.1.26 and 7.1.30. That this definition satisfies all the given identities is proven in Propositions A.0.1 to A.0.5. $\square$

The following lemma will be useful in various proofs involving coherent families of composites.

Lemma 7.1.34. *Let X be a cubical quasicategory equipped with a coherent family of composites θ. For $m \geq 0$ and $x \colon \square^m \to X$, the critical edge of $\theta^{m,0}(x) \colon \square^{m+1} \to X$ with respect to its $(1,0)$-face is degenerate.*

Proof. We proceed by induction on m. For $m = 0$, we have $\theta^{0,0}(x) = x\sigma_1$; so $\theta^{0,0}(x)$ is a degeneracy of a vertex, thus its unique edge is degenerate.

Now let $m \geq 1$, and suppose that the statement holds for $m-1$. The edge in question may be written as $\theta^{m,0}(x)\partial_{m+1,1}\cdots\partial_{3,1}\partial_{2,1}$. By ($\Theta$3), this is equal to $\theta^{m-1,0}(x\partial_{m,1})\partial_{m,1}\cdots\partial_{2,1}$, which is degenerate by the induction hypothesis. □

7.2 Comparison with the Joyal model structure

In this section we use the theory of cones developed in Section 7.1 to compare the cubical Joyal model structure with the Joyal model structure on sSet, showing that the model structures constructed in Chapters 5 and 6 present the theory of $(\infty, 1)$-categories. Our main goal is to prove the following:

Theorem 7.2.1. *The adjunction $T : \mathsf{cSet} \rightleftarrows \mathsf{sSet} : U$ is a Quillen equivalence between the cubical Joyal model structure on cSet and the Joyal model structure on sSet.*

Throughout this section, sSet and cSet will be equipped with the Joyal and cubical Joyal model structures, respectively, unless otherwise noted.

Due to the difficulty of working directly with the triangulation functor, we first establish a second Quillen adjunction $Q : \mathsf{sSet} \rightleftarrows \mathsf{cSet} : \int$; this adjunction was previously studied in [KLW19] for cubical sets having only negative connections, but here we will construct it using the theory of cones developed in Section 7.1. We will prove that $Q \dashv \int$ is a Quillen equivalence, and that the left derived functor of Q is an inverse to that of T. In order to define Q, we first recall a folklore result about constructing cosimplicial objects out of monads.

Proposition 7.2.2. *Let M be a monad on a category $\mathcal{C}$. Then M induces an augmented cosimplicial object $\Delta_{\text{aug}} \to \operatorname{End}\mathcal{C}$, defined as follows:*

- *For* $n \geq -1$, $[n] \mapsto M^{n+1}$;
- $(\partial_i \colon [n-1] \to [n]) \mapsto M^{n-i}\eta_{M^i}$;
- $(\sigma_i \colon [n] \to [n-1]) \mapsto M^{n-i-1}\mu_{M^i}$.

In particular, for any $c \in \mathcal{C}$ *there is an augmented cosimplicial object* $\Delta_{\text{aug}} \to \mathcal{C}$ *given by instantiating this construction at* c.

Proof. This follows from the characterization of Δ_{aug} as the universal monoidal category equipped with a monoid, together with the characterization of monads on a category $\mathcal{C}$ as monoids in $\operatorname{End}\mathcal{C}$. □

For $n \geq 0$, let Q^n denote the cubical set $C^{n+1}\varnothing = C^{0,n}$. Likewise, for $W \in \{L, R\}, \varepsilon \in \{0, 1\}$, let $Q^n_{W,\varepsilon} = C^{n+1}_{W,\varepsilon}\varnothing$.

Proposition 7.2.3. *The assignment* $[n] \mapsto Q^n$ *extends to a cosimplicial object* $Q \colon \Delta \to \mathsf{cSet}$, *with simplicial structure maps defined as follows:*

a map $Q^{n-1} \to Q^n$	0^{th} *face*	1^{st} *face*	2^{nd} *face*	$\cdots$	j^{th} *face*	$\cdots$	n^{th} *face*
is induced by a map $\square^{n-1} \to \square^n$	$\partial_{n,1}$	$\partial_{n,0}$	$\partial_{n-1,0}$	$\cdots$	$\partial_{n-j+1,0}$	$\cdots$	$\partial_{1,0}$

a map $Q^n \to Q^{n-1}$	0^{th} *deg.*	1^{st} *deg.*	2^{nd} *deg.*	$\cdots$	j^{th} *deg.*	$\cdots$	$(n-1)^{st}$ *deg.*
is induced by a map $\square^n \to \square^{n-1}$	σ_n	$\gamma_{n-1,0}$	$\gamma_{n-2,0}$	$\cdots$	$\gamma_{n-j,0}$	$\cdots$	$\gamma_{1,0}$

Proof. This follows from applying Proposition 7.2.2 to the monad of Proposition 7.1.3 and the object $\varnothing \in \mathsf{cSet}$. For a direct construction, see [KLW19, Prop. 2.3]. □

Taking the left Kan extension of this cosimplicial object along the Yoneda embedding, we obtain a functor $Q \colon \mathsf{sSet} \to \mathsf{cSet}$.

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d, hook] & \mathsf{cSet} \\ \mathsf{sSet} \arrow[ur, "Q"'] & \end{tikzcd}$$

This functor has a right adjoint $\int \colon \mathsf{cSet} \to \mathsf{sSet}$, given by $(\int X)_n = \mathsf{cSet}(Q^n, X)$.

Remark 7.2.4. Viewing sSet as the slice category $\mathsf{sSet} \downarrow \Delta^0$ and cSet as the functor category $\mathsf{cSet}^{[0]}$, the adjunction $Q \dashv \int$ coincides with the cubical straightening-unstraightening adjunction developed in [KV20].

The alternative cone monads described in Proposition 7.1.6 admit similar constructions.

Proposition 7.2.5. *For $W \in \{L, R\}$, $\varepsilon \in \{0,1\}$, the assignment $[n] \mapsto Q^n_{W,\varepsilon}$ extends to a cosimplicial object $Q_{W,\varepsilon} \colon \Delta \to \mathsf{cSet}$. For $(W,\varepsilon) \neq (L,1)$ the simplicial structure maps are defined as follows:*

a map $Q^{n-1}_{L,0} \to Q^n_{L,0}$	0^{th} *face*	1^{st} *face*	2^{nd} *face*	$\cdots$	j^{th} *face*	$\cdots$	n^{th} *face*
is induced by a map $\square^{n-1} \to \square^n$	$\partial_{n,0}$	$\partial_{n,1}$	$\partial_{n-1,1}$	$\cdots$	$\partial_{n-j+1,1}$	$\cdots$	$\partial_{1,1}$
a map $Q^n_{L,0} \to Q^{n-1}_{L,0}$	0^{th} *deg.*	1^{st} *deg.*	2^{nd} *deg.*	$\cdots$	j^{th} *deg.*	$\cdots$	$(n-1)^{st}$ *deg.*
is induced by a map $\square^n \to \square^{n-1}$	σ_n	$\gamma_{n-1,1}$	$\gamma_{n-2,1}$	$\cdots$	$\gamma_{n-j,1}$	$\cdots$	$\gamma_{1,1}$

a map $Q^{n-1}_{R,0} \to Q^n_{R,0}$	0^{th} *face*	1^{st} *face*	2^{nd} *face*	$\cdots$	j^{th} *face*	$\cdots$	n^{th} *face*
is induced by a map $\square^{n-1} \to \square^n$	$\partial_{1,0}$	$\partial_{1,1}$	$\partial_{2,1}$	$\cdots$	$\partial_{j,1}$	$\cdots$	$\partial_{n,1}$
a map $Q^n_{R,0} \to Q^{n-1}_{R,0}$	0^{th} *deg.*	1^{st} *deg.*	2^{nd} *deg.*	$\cdots$	j^{th} *deg.*	$\cdots$	$(n-1)^{st}$ *deg.*
is induced by a map $\square^n \to \square^{n-1}$	σ_1	$\gamma_{1,1}$	$\gamma_{2,1}$	$\cdots$	$\gamma_{j,1}$	$\cdots$	$\gamma_{n-1,1}$

a map $Q^{n-1}_{R,1} \to Q^n_{R,1}$	0^{th} *face*	1^{st} *face*	2^{nd} *face*	$\cdots$	j^{th} *face*	$\cdots$	n^{th} *face*
is induced by a map $\square^{n-1} \to \square^n$	$\partial_{1,1}$	$\partial_{1,0}$	$\partial_{2,0}$	$\cdots$	$\partial_{j,0}$	$\cdots$	$\partial_{n,0}$
a map $Q^n_{R,1} \to Q^{n-1}_{R,1}$	0^{th} *deg.*	1^{st} *deg.*	2^{nd} *deg.*	$\cdots$	j^{th} *deg.*	$\cdots$	$(n-1)^{st}$ *deg.*
is induced by a map $\square^n \to \square^{n-1}$	σ_1	$\gamma_{1,0}$	$\gamma_{2,0}$	$\cdots$	$\gamma_{j,0}$	$\cdots$	$\gamma_{n-1,0}$

Proof. This follows from applying Proposition 7.2.2 to the monads of Proposition 7.1.6. $\square$

Remark 7.2.6. We could instead have chosen to define the cosimplicial object of Proposition 7.2.2 by $\partial_i \mapsto M^i \eta_{M^{n-i}}$, $\sigma_i \mapsto M^i \mu_{M^{n-i-1}}$; this amounts to precomposing the cosimplicial object as we have defined it with the involution $(-)^{\mathrm{op}} \colon \mathsf{sSet} \to \mathsf{sSet}$. If we had made this choice, we would have obtained a different set of cosimplicial objects $Q_{W,\varepsilon}$.

In fact, in Chapter 9 we will make use of these alternative definitions of Q to define a functor $\mathsf{sSet}^+ \to \mathsf{cSet}^+$; specifically, we will extend $Q_{L,0} \circ (-)^{\mathrm{op}}$ to the marked case.

As with the cosimplicial object constructed using left positive cones, each of these Kan extends to a functor $Q_{W,\varepsilon} \colon \mathsf{sSet} \to \mathsf{cSet}$ having a right adjoint $\int\limits_{W,\varepsilon} \colon \mathsf{cSet} \to \mathsf{sSet}$.

Lemma 7.2.7. *The functors $Q_{W,\varepsilon}$ and $\int\limits_{W,\varepsilon}$, for $W \in \{L, R\}, \varepsilon \in \{0,1\}$, are related by the following formulas:*

- $Q_{L,0} = (-)^{\text{co-op}} \circ Q_{L,1}$;
- $Q_{R,0} = (-)^{\mathrm{op}} \circ Q_{L,1}$;
- $Q_{R,1} = (-)^{\mathrm{co}} \circ Q_{L,1}$;
- $\int\limits_{L,0} = \int\limits_{L,1} \circ (-)^{\text{co-op}}$;
- $\int\limits_{R,0} = \int\limits_{L,1} \circ (-)^{\mathrm{op}}$;
- $\int\limits_{R,1} = \int\limits_{L,1} \circ (-)^{\mathrm{co}}$;

Proof. It suffices to prove the first three items, which follow from Lemma 7.1.5. $\square$

As we did in Section 7.1, from here on we will work exclusively with left positive cones except where noted, using the subscript $(L, 1)$ only where the potential for ambiguity arises.

The analogous functor $Q \colon \mathsf{sSet} \to \mathsf{cSet}_0$, which we will denote Q_0, was previously studied in [KLW19]. In that paper, the objects Q_0^n were described as quotients of $\square_0^n$ under a certain equivalence relation; this relation is precisely that of Lemma 7.1.8 in the case $m = 0$. We begin by recalling some of the theory developed in that paper, and adapting it to the present setting.

Lemma 7.2.8. *We have a commuting triangle of adjunctions:*

$$\begin{tikzcd}
\mathsf{sSet} \arrow[rr, shift left, "Q"] \arrow[rr, phantom, "\perp"] \arrow[dr, shift left, "Q_0"] \arrow[dr, phantom, "\perp"] & & \mathsf{cSet} \arrow[ll, shift left, "\int"] \arrow[dl, shift left, "i^*"] \arrow[dl, phantom, "\perp"] \\
& \mathsf{cSet}_0 \arrow[ul, shift left, "\int_0"] \arrow[ur, shift left, "i_!"] &
\end{tikzcd}$$

Proof. It is easy to show that $i_! Q_0^n \cong Q^n$; the general result follows from this, using the fact that $i_!$ preserves colimits as a left adjoint. ☐

Lemma 7.2.9. *For any $X \in \mathsf{cSet}$, the counit $Q \int X \to X$ is a monomorphism.* ☐

Proof. The corresponding result for $Q_0 \dashv \int_0$ was proven as [KLW19, Lem. 4.2]. By Lemma 7.2.8, the counit of $Q \dashv \int$ is adjunct to that of $Q_0 \dashv \int_0$ under the adjunction $i_! \dashv i^*$. The result thus follows from Lemma 4.1.14. ☐

This lemma shows that for any cubical set X, $Q \int X$ is a subcomplex of X. Specifically, it is the subcomplex whose non-degenerate n-cubes, for each n, are those which factor through Q^n – in other words, they are the non-degenerate $(0, n)$-cones in X.

Theorem 7.2.10. *The functor $Q \colon \mathsf{sSet} \to \mathsf{cSet}$ is fully faithful.*

Proof. That Q_0 is fully faithful follows from [KLW19, Thm. 3.9]. Since $i_!$ is faithful, it follows from Lemma 7.2.8 that Q is faithful as well. In general, $i_!$ is not full; we will show, however, that it is full on the image of Q_0, which suffices to show that the composite Q is fully faithful.

Let $X, Y \in \mathsf{sSet}$, and consider a map $f \colon QX \to QY$. By Corollary 4.1.4, f is determined by its action on the non-degenerate cubes of QX. Let $x \colon \square^n \to QX$

be non-degenerate; then x is a $(0, n)$-cone, hence so is fx. Therefore, by Corollary 7.1.18, the standard form of fx contains no positive connection maps; thus fx corresponds to a cube of $Q_0 Y$.

Thus we see that the action of f on the non-degenerate cubes of QX (which coincide with those of $Q_0 X$) defines a map $Q_0 X \to Q_0 Y$; the image of this map under $i_!$ is precisely f. $\square$

Our next goal is to show the following:

Proposition 7.2.11. *The adjunction $Q \dashv \int$ is Quillen.*

To prove this, we will show that this adjunction satisfies the hypotheses of Corollary 2.1.36.

Lemma 7.2.12. *Q preserves monomorphisms.*

Proof. Q_0 preserves monomorphisms by [KLW19, Lem. 4.5]. The stated result thus follows from Lemma 7.2.8 and the fact that $i_!$ preserves monomorphisms. $\square$

Lemma 7.2.13. *The image under Q of an inner horn inclusion $\Lambda^n_i \subseteq \Delta^n$ is a trivial cofibration.*

Proof. Because Q preserves colimits, $Q\Lambda^n_i$ is the subcomplex of Q^n consisting of the images of the maps $Q\partial_j \colon Q^{n-1} \to Q^n$ for which $j \neq i$. By Proposition 7.2.3 we can see that this subcomplex is the image of $\sqcap^n_{n-i+1,0}$ under the quotient map $\square^n \to Q^n$. We thus have the following commuting square:

$$\begin{tikzcd}
\sqcap^n_{n-i+1,0} \arrow[r] \arrow[d] & Q\Lambda^n_i \arrow[d] \\
\square^n \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & Q^n
\end{tikzcd}$$

It is easy to see that this square is a pushout. Furthermore, the critical edge of the open box $\sqcap^n_{n-i+1,0} \to Q\Lambda^n_i$ is degenerate by Lemma 7.1.16. Thus $Q\Lambda^n_i \hookrightarrow Q^n$ is a trivial cofibration, as an inner open box filling. □

Lemma 7.2.14. *$QJ \cong K$.* □

Proof of Proposition 7.2.11. By Lemma 7.2.12, Q preserves cofibrations. By Lemma 7.2.13, the image under Q of an inner-horn inclusion is a trivial cofibration. By Lemma 7.2.14, the image under Q of an endpoint inclusion $\Delta^0 \to J$ is an endpoint inclusion $\square^0 \to K$, hence a trivial cofibration by Lemma 6.2.1. Thus the adjunction is Quillen by Corollary 2.1.36. □

Corollary 7.2.15. *Q preserves weak equivalences.*

Proof. Since all simplicial sets are cofibrant in the Quillen model structure, this follows from Proposition 7.2.11 and Ken Brown's lemma. □

Next we will concern ourselves with the relationship between Q and the triangulation functor. Our goal will be to develop a natural weak categorical equivalence $TQ \Rightarrow \mathrm{id}_{\mathsf{sSet}}$.

For $n \geq 0$, we have a poset map $G \colon [n] \to [1]^n$ defined by:

$$(Ga)_i = \begin{cases} 0 & i \leq n-a \\ 1 & i \geq n-a+1 \end{cases}$$

For a given $n \geq 0$, let $F \colon [1]^n \to [n]$ be defined via $Fb = n - i + 1$,where $i \in \{1, ..., n+1\}$ is maximal such that $b_j = 0$ for all $j < i$.

Lemma 7.2.16. *For any $n \geq 0$, the functors $F : [1]^n \rightleftarrows [n] : G$ are adjoint.*

Proof. Let $a \in [n]$ and $b \in [1]^n$. We have that $b \leq Ga$ if and only if $b_j = 0$ for all $j \leq n-a$ – in other words, if and only if $i \geq n-a+1$. Rearranging, we obtain that this is equivalent to $n-i+1 \leq a$, i.e., $Fb \leq a$. □

Similarly, one can show that G also has a right adjoint, although it will not play any role in this thesis.

Proposition 7.2.17. *For all n, F induces a map of simplicial sets $TQ^n \to \Delta^n$.*

Proof. First, observe that by applying the nerve functor $N\colon \mathsf{Cat} \to \mathsf{sSet}$, we get an induced map $NF\colon (\Delta^1)^n \to \Delta^n$.

The simplicial set TQ^n is a quotient of $T\Box^n = (\Delta^1)^n$. Specifically, since N is fully faithful, we may regard n-simplices $\Delta^n \to (\Delta^1)^n$ as poset maps $[n] \to [1]^n$. Then by an argument analogous to the proof of Lemma 7.1.8, using Proposition 4.1.15 and the fact that T preserves colimits, TQ^n is obtained by identifying two such maps f, g if there exists i such that $f_j = g_j$ for $j \leq i$ and $f_i = g_i = \mathrm{const}_1$. NF then acts on such maps by post-composition with F. By Lemma 7.2.16, F depends only on the position of the first 1 in an object of $[1]^n$; therefore, maps which are identified in TQ^n agree after post-composition with F. Thus NF factors through the quotient TQ^n. □

Let $\bar{F}\colon TQ^n \to \Delta^n$ denote the map constructed above. Then we can show:

Lemma 7.2.18. *The maps $\bar{F}\colon TQ^n \to \Delta^n$ form a natural transformation of co-simplicial objects in sSet. That is, for any map $\phi\colon [m] \to [n]$ in Δ, we have a commuting diagram:*

$$\begin{array}{ccc} TQ^m & \xrightarrow{TQ\phi} & TQ^n \\ \downarrow{\scriptstyle \bar{F}} & & \downarrow{\scriptstyle \bar{F}} \\ \Delta^m & \xrightarrow{\phi} & \Delta^n \end{array}$$

Proof. It suffices to show that this holds for the generating morphisms of Δ, namely the face and degeneracy maps. For each such map $\phi\colon [m] \to [n]$ we have a corresponding map $\phi'\colon [1]^m \to [1]^n$ in $\Box$, as described in Proposition 7.2.3:

- For $\partial_0\colon [n-1] \to [n]$, $\partial_0' = \partial_{n,1}$;

- For $i \geq 1, \partial_i \colon [n-1] \to [n]$, $\partial'_i = \partial_{n-i+1,0}$;
- For $\sigma_0 \colon [n] \to [n-1]$, $\sigma'_0 = \sigma_n$;
- For $\sigma_i \colon [n] \to [n-1]$, $\sigma'_i = \gamma_{n-i,0}$.

For every such ϕ we have a commuting diagram in cSet, where the vertical maps $\square^m \to Q^m$ are the quotient maps:

$$\begin{tikzcd} \square^m \arrow[r, "\phi'"] \arrow[d] & \square^n \arrow[d] \\ Q^m \arrow[r, "Q\phi"] & Q^n \end{tikzcd} \tag{7.2.1}$$

Furthermore, by direct computation using Lemma 7.2.16, we have commuting diagrams in Cat:

$$\begin{tikzcd} {[1]^m} \arrow[r, "\phi'"] \arrow[d, "F"'] & {[1]^n} \arrow[d, "F"] \\ {[m]} \arrow[r, "\phi"] & {[n]} \end{tikzcd} \tag{7.2.2}$$

Now consider the following diagram in sSet:

$$\begin{tikzcd} (\Delta^1)^m \arrow[r, "T\phi'"] \arrow[d] & (\Delta^1)^n \arrow[d] \\ TQ^m \arrow[r, "TQ\phi"] \arrow[d, "\bar{F}"'] & TQ^n \arrow[d, "\bar{F}"] \\ \Delta^m \arrow[r, "\phi"] & \Delta^n \end{tikzcd}$$

The top square commutes, as it is obtained by applying T to Diagram 7.2.1; the outer rectangle also commutes, as it is obtained by applying N to Diagram 7.2.2. We wish to show that the bottom square commutes, i.e. that $\phi \circ \bar{F} = \bar{F} \circ TQ\phi$. Since the quotient map $(\Delta^1)^m \to TQ^m$ is an epimorphism, we can show the desired equality by pre-composing with this map and performing a simple diagram chase. □

Corollary 7.2.19. *$\bar{F}$ extends to a natural transformation $\bar{F} \colon TQ \Rightarrow \mathrm{id}_{\mathsf{sSet}}$.*

Proof. Immediate from Lemma 7.2.18 and the fact that T and Q preserve colimits. □

Note that this is precisely the map considered by Lurie in [Lur09a, Prop. 2.2.2.7], since Lurie's straightening construction can be recovered from its cubical analogue by composing with triangulation [KV20, Thm. 3.8].

Proposition 7.2.20. *For every simplicial set X, the map $\bar{F} \colon TQX \to X$ is a weak categorical equivalence.*

Proof. We begin by proving the statement for the case where X is m-skeletal for some $m \geq 0$, proceeding by induction on m. For $m = 0$ or $m = 1$, the map in question is an isomorphism.

Now let $m \geq 2$, and suppose that the statement holds for any $(m-1)$-skeletal X. Then in particular, it holds for any horn Λ^m_i. For any $0 < i < n$, consider the following commuting diagram:

$$\begin{tikzcd}
TQ\Lambda^m_i \arrow[r, hook, "\sim"] \arrow[d, "\sim"'] & TQ^m \arrow[d] \\
\Lambda^m_i \arrow[r, hook, "\sim"] & \Delta^m
\end{tikzcd}$$

The left-hand map is a weak equivalence by the induction hypothesis; the bottom map is a trivial cofibration as an inner horn inclusion; and the top map is a trivial cofibration by Proposition 7.2.11 and Proposition 6.3.8. Thus $\bar{F} \colon TQ^m \to \Delta^m$ is a weak equivalence by the two-out-of-three property. Extending this result to an arbitrary m-skeletal simplicial set X is a straightforward application of the gluing lemma, using the fact that both T and Q preserve colimits.

Now let X be an arbitrary simplicial set; then $\bar{F}$ is a weak equivalence on the n-skeleton of X for each $n \geq 0$. Thus $\bar{F}\colon TQX \to X$ is a weak equivalence, using the fact that sequential colimits of cofibrations preserve weak equivalences. □

Proposition 7.2.21. *Q reflects weak categorical equivalences.*

Proof. Let $f\colon X \to Y$ be a map of simplicial sets, such that Qf is a weak categorical equivalence. We have a commuting diagram:

$$\begin{tikzcd} TQX \arrow[r, "TQf"] \arrow[d, "\bar{F}"'] & TQY \arrow[d, "\bar{F}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

The top horizontal map is a weak categorical equivalence by Proposition 4.1.17, as are the vertical maps by Proposition 7.2.20. Thus f is a weak categorical equivalence by the 2-out-of-3 property. □

We have shown that the adjunction $Q \dashv \int$ satisfies the hypotheses of Corollary 2.1.38 (ii). To show that it is a Quillen equivalence, therefore, we must prove the following:

Proposition 7.2.22. *For any cubical quasicategory X, the counit $\varepsilon\colon Q \int X \hookrightarrow X$ is a trivial cofibration.*

Proof. Let X be a cubical quasicategory. By Theorem 7.1.25, we may equip X with a coherent family of composites θ. We will build X from $Q \int X$ via successive inner open box fillings, thereby showing that the inclusion of $Q \int X$ into X is a trivial cofibration.

For $m \geq 2, n \geq -1$, let $X^{m,n}$ denote the smallest subcomplex of X containing all (m', n')-cones of X, as well as all cones of the form $\theta^{m',n'}(x)$, for $m' < m$ or $m' = m, n' \leq n$. In particular, this means $X^{2,-1} = Q \int X$, since all cubes in the image of $\theta^{0,n}$ or $\theta^{1,n}$ are degenerate.

For $m < m'$ or $m = m', n \leq n'$, we have $X^{m,n} \subseteq X^{m',n'}$. Thus we obtain a sequence of inclusions:

$$Q \int X = X^{2,-1} \hookrightarrow X^{2,0} \hookrightarrow \cdots \hookrightarrow X^{3,-1} \hookrightarrow X^{3,0} \hookrightarrow \cdots \hookrightarrow X^{m,n} \hookrightarrow \cdots$$

So to show that $Q \int X \hookrightarrow X$ is a trivial cofibration, it suffices to show that $Q \int X \hookrightarrow X^{m,n}$ is a trivial cofibration for every $X^{m,n}$. We proceed by transfinite induction. For the case $n = -1$, we may observe that $X^{m,-1}$ is the union of all subcomplexes $X^{m',n}$ for $m' < m$, i.e. the colimit of the sequence of inclusions $Q \int X \hookrightarrow \cdots \hookrightarrow X^{m',n} \hookrightarrow \cdots$. By the induction hypothesis, each of these inclusions is a trivial cofibration, hence so is $Q \int X \hookrightarrow X^{m,-1}$.

Now let $n \geq 0$, and suppose $Q \int X \hookrightarrow X^{m,n-1}$ is a trivial cofibration. Then to show that the composite $Q \int X \hookrightarrow X^{m,n-1} \hookrightarrow X^{m,n}$ is a trivial cofibration, it suffices to show that $X^{m,n-1} \hookrightarrow X^{m,n}$ is a trivial cofibration.

Let S denote the set of non-degenerate (m,n)-cones of X which are not $(m-1, n+1)$-cones, and are not in the image of $\theta^{m,n-1}$ – in other words, those (m,n)-cones which fall under case (6) of Definition 7.1.30. To construct $X^{m,n}$ from $X^{m,n-1}$, we must adjoin to $X^{m,n-1}$ all (m,n)-cones of X, and images of such cones under $\theta^{m,n}$, which are not already present in $X^{m,n}$. Using Lemmas 7.1.12, 7.1.17 and 7.1.33, and the identities (Θ1) to (Θ8), we can see that these are precisely the cones in S and their images under $\theta^{m,n}$.

Let $x \in S$; we will analyze the faces of $\theta^{m,n}(x)$ to determine which of them are contained in $X^{m,n-1}$. For $i \leq n$ we have $\theta^{m,n}(x)\partial_{i,0} = \theta^{m,n-1}(x\partial_{i,0})$ by (Θ1), while for $i \geq n+2$ or $\varepsilon = 1$, $\theta^{m,n}(x)\partial_{i,\varepsilon}$ is an $(m-1, n+1)$-cone by Lemma 7.1.12. Thus we see that the only face of $\theta^{m,n}(x)$ which is not contained in $X^{m,n-1}$ is $\theta^{m,n}(x)\partial_{n+1,0} = x$. Furthermore, the critical edge of $\theta^{m,n}(x)$ with respect to the $(n+1, 0)$-face is degenerate; for $n = 0$ this follows from Lemma 7.1.34, while for $n \geq 1$ it follows from Lemma 7.1.16. Thus the faces of $\theta^{m,n}(x)$ which are contained in $X^{m,n-1}$ form an $(m, n+1)$-inner open box.

Constructing $X^{m,n}$ from $X^{m,n-1}$ amounts to filling all of these inner open boxes; in other words, we have a pushout diagram:

$$\begin{tikzcd}
\coprod\limits_S \widehat{\sqcap}^{m+n+1}_{n+1,0} \arrow[r] \arrow[d, hook] & X^{m,n-1} \arrow[d, hook] \\
\coprod\limits_S \widehat{\square}^{m+n+1} \arrow[r] & X^{m,n} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

The left-hand map is a trivial cofibration, as a coproduct of trivial cofibrations. Hence, so is its pushout $X^{m,n-1} \hookrightarrow X^{m,n}$.

So each inclusion $Q \int X \hookrightarrow X^{m,n}$ is a trivial cofibration, hence so is $Q \int X \hookrightarrow X$. $\square$

Theorem 7.2.23. *The adjunction $Q : \mathsf{sSet} \rightleftarrows \mathsf{cSet} : \int$ is a Quillen equivalence.*

Proof. The adjunction is Quillen by Proposition 7.2.11. Q preserves and reflects the weak equivalences of the Quillen model structure on sSet by Corollary 7.2.15 and Proposition 7.2.21. Thus $Q \dashv \int$ satisfies the hypotheses of Corollary 2.1.38, item (ii) and we can apply Proposition 7.2.22 to conclude that it is a Quillen equivalence. $\square$

Corollary 7.2.24. *For all $W \in \{L, R\}, \varepsilon \in \{0, 1\}$, the adjunction $Q_{W,\varepsilon} \dashv \int\limits_{W,\varepsilon}$ is a Quillen equivalence.*

Proof. Immediate from Proposition 6.3.6, Lemma 7.2.7, and Theorem 7.2.23. □

Proof of Theorem 7.2.1. The adjunction $T \dashv U$ is Quillen by Proposition 6.3.8. To see that it is a Quillen equivalence, note that because all objects in both cSet and sSet are cofibrant, the left derived functor $L(TQ)$ is the composite of the left derived functors LT and LQ, while the left derived functor of the identity is the identity. By Proposition 7.2.20, we have a natural weak equivalence $TQ \Rightarrow \mathrm{id}_{\mathsf{sSet}}$. In the homotopy category $\mathsf{Ho}\,\mathsf{sSet}$, this natural weak equivalence becomes a natural isomorphism $LT \circ LQ \cong \mathrm{id}_{\mathsf{Ho}\,\mathsf{sSet}}$. By Theorem 7.2.23, LQ is an equivalence of categories, thus LT is an equivalence of categories as well. □

The proofs in this section can easily be adapted to show that $Q \dashv \int$ is a Quillen equivalence between the standard model structures for ∞-groupoids on sSet and cSet. (The analogue of this result for cSet_0 was essentially stated as [KLW19, Prop. 5.3], but the proof supplied there only shows that Q_0 and $\int_0$ form a Quillen adjunction.)

Proposition 7.2.25. *The adjunction $Q : \mathsf{sSet} \rightleftarrows \mathsf{cSet} : \int$ is a Quillen equivalence between the Quillen model structure on sSet and the Grothendieck model structure on cSet.*

Proof. Proposition 6.3.8 and Proposition 7.2.11 both have natural analogues, showing that $T \dashv U$ and $Q \dashv \int$ are Quillen adjunctions between these model structures (implying in particular that Q preserves weak equivalences). Since every weak equivalence in the Joyal model structure is also a weak equivalence in the Quillen model structure, $\overline{F}$ is a natural weak equivalence in the Quillen model structure as well. Thus the proof of Proposition 7.2.21 adapts to show that Q reflects the weak equivalences of the Quillen model structure. Corollary 2.1.38, item (ii) and Proposition 7.2.22 then imply the analogue of Theorem 7.2.23, since every cubical Kan complex is a cubical quasicategory and every weak equivalence in the cubical Joyal model structure is a weak equivalence in the Grothendieck model structure. $\square$

We thus obtain a new proof of the following result, previously shown in [Cis06, Prop. 8.4.30] for cubical sets without connections:

Theorem 7.2.26. *$T \dashv U$ is a Quillen equivalence between the Grothendieck model structure on cSet and the Quillen model structure on sSet.* $\square$

Chapter 8

Cubical mapping spaces

In this chapter we will describe the construction of mapping spaces in cubical sets, and prove cubical analogues of two established results in the simplicial theory of quasicategories. In Section 8.1, we introduce the definition of cubical mapping spaces, as well as the homotopy category of a cubical quasicategory, and prove a cubical analogue of Theorem 3.2.26, showing that mapping spaces and homotopy categories characterize equivalences of quasicategories. This theorem follows from a fairly straightforward computation involving explicit constructions of homotopies, demonstrating the convenience and simplifying power of the cubical approach. We then obtain a new proof of the simplicial version of this theorem using the Quillen equivalence established in chapter 7.

In Section 8.2, we define (up to homotopy) a composition operation on cubical mapping spaces. Using this composition, we can define a functor from quasicategories to categories enriched over the homotopy category of the Grothendieck model structure, and show that the equivalences of quasicategories are precisely the maps that this functor sends to equivalences of enriched categories. Although this proof is in many respects more complex than its simplicial analogue [Rez20, Sec. 34.7], it is nevertheless a significant step in the development of cubical $(\infty, 1)$-category theory.

8.1 The fundamental theorem of cubical quasicategories

Definition 8.1.1. For $X \in \mathsf{cSet}$ and $x_0, x_1 \colon \square^0 \to X$, we define the *mapping space* $\mathrm{Map}_X(x_0, x_1)$ by the following pullback:

$$\begin{tikzcd}
\mathrm{Map}_X(x_0, x_1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\hom}_L(\square^1, X) \arrow[d] \\
\square^0 \arrow[r, "{(x_0, x_1)}"] & \underline{\hom}_L(\partial\square^1, X)
\end{tikzcd}$$

From this definition, we can derive a more concrete description of the cubical mapping space. For $X \in \mathsf{cSet}$, $x_0, x_1 \colon \square^0 \to X$, we have:

$$\mathrm{Map}_X(x_0, x_1)_n = \left\{ \square^{n+1} \xrightarrow{s} X \;\middle|\; s\partial_{n+1,\varepsilon} = x_\varepsilon \right\},$$

with the cubical operations given by those of X. Note that x_ε here refers to the degeneracy of the vertex x_ε in the appropriate dimension.

There is a clear geometric intuition behind this definition, as the example below shows.

Example 8.1.2. Given a cubical set X and 0-cubes $x_0, x_1 \colon \square^0 \to X$, we have that:

- a 0-cube in $\mathrm{Map}_X(x_0, x_1)$ is a 1-cube from x_0 to x_1 in X;

- a 1-cube in $\mathrm{Map}_X(x_0, x_1)$ is a 2-cube in X of the form

$$\begin{tikzcd} x_0 \arrow[r, equal] \arrow[d] & x_0 \arrow[d] \\ x_1 \arrow[r, equal] & x_1 \end{tikzcd}$$

Given a cubical set map $f\colon X \to Y$, for any $x_0, x_1\colon \Box^0 \to X$ there is a natural map $f_*\colon \mathrm{Map}_X(x_0, x_1) \to \mathrm{Map}_Y(fx_0, fx_1)$ induced by a natural map between the pullbacks of Definition 8.1.1. Thus the mapping space construction defines a functor $\mathrm{Map}\colon \partial\Box^1 \downarrow \mathsf{cSet} \to \mathsf{cSet}$. In fact, this functor has a left adjoint, which we will now describe.

Definition 8.1.3. For $X \in \mathsf{cSet}$, the *suspension* of X is the bi-pointed cubical set $\Sigma X \in \partial\Box^1 \downarrow \mathsf{cSet}$ defined by the following pushout diagram:

$$\begin{tikzcd} X \sqcup X \arrow[r] \arrow[d] & \partial\Box^1 \arrow[d] \\ X \otimes \Box^1 \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The chosen map $\partial\Box^1 \to \Sigma X$ is that which appears in the diagram above. We denote the basepoints of ΣX, i.e. the images under this map of the vertices $0, 1 \in \partial\Box^1$, by 0 and 1, respectively. For $f\colon X \to Y$, we define $\Sigma f\colon \Sigma X \to \Sigma Y$ to be the natural map between pushouts induced by f.

Proposition 8.1.4. *The functor* $\Sigma\colon \mathsf{cSet} \to \partial\Box^1 \downarrow \mathsf{cSet}$ *is left adjoint to* $\mathrm{Map}\colon \partial\Box^1 \downarrow \mathsf{cSet} \to \mathsf{cSet}$.

Proof. Let $X, Y \in \mathsf{cSet}, y_0, y_1\colon \Box^0 \to Y$. By the universal property of the pullback, maps $X \to \mathrm{Map}_Y(y_0, y_1)$ correspond to diagrams of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & \underline{\mathrm{hom}}_L(\square^1, Y) \arrow[d] \\ \square^0 \arrow[r, "(y_0,y_1)"] & \underline{\mathrm{hom}}_L(\partial\square^1, Y) \end{tikzcd}$$

The map $\underline{\mathrm{hom}}_R(\square^1, Y) \to \underline{\mathrm{hom}}_R(\partial\square^1, Y)$ is the pullback exponential $(\partial\square^1 \hookrightarrow \square^1) \triangleright (Y \to \square^0)$. Using the duality between pushout products and pullback exponentials, and observing that the pushout object $X \otimes \square^1 \cup_{X\otimes\partial\square^1} \square^0 \otimes \partial\square^1$ is precisely ΣX, we have a natural bijection between such diagrams and diagrams of the form

$$\begin{tikzcd} \Sigma X \arrow[r, "\overline{f}"] \arrow[d] & Y \arrow[d] \\ \square^1 \arrow[r] & \square^0 \end{tikzcd}$$

in which $\overline{f}$ maps the basepoints $0 \mapsto y_0, 1 \mapsto y_1$. In other words, cubical set maps $X \to \mathrm{Map}_Y(y_0, y_1)$ are in natural bijection with bi-pointed cubical set maps $(\Sigma X, 0, 1) \to (Y, y_0, y_1)$. $\square$

By using $\underline{\mathrm{hom}}_R$ rather than $\underline{\mathrm{hom}}_L$ in the pullback diagram of Definition 8.1.1, we obtain the *left mapping space* functor $\mathrm{Map}^L \colon \partial\square^1 \downarrow \mathsf{cSet} \to \mathsf{cSet}$.

$$\begin{tikzcd} \mathrm{Map}^L_X(x_0, x_1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{hom}}_R(\square^1, X) \arrow[d] \\ \square^0 \arrow[r, "(x_0,x_1)"] & \underline{\mathrm{hom}}_R(\partial\square^1, X) \end{tikzcd}$$

This functor admits the following explicit description:

$$\mathrm{Map}^L_X(x_0, x_1)_n = \left\{ \square^{n+1} \xrightarrow{s} X \;\middle|\; s\partial_{1,\varepsilon} = x_\varepsilon \right\},$$

Once again, cubical operations are given by those of X. In this case, that means that each face map $\partial_{i,\varepsilon}$ of $\mathrm{Map}^L_X(x_0, x_1)$ is induced by the face map $\partial_{1+i,\varepsilon}$ of X, and similarly for degeneracies and connections.

Map^L also has a left adjoint, the *left suspension* $\Sigma_L \colon \mathsf{cSet} \to \partial\Box^1 \downarrow \mathsf{cSet}$, with $\Sigma_L X$ defined as a quotient of $\Box^1 \otimes X$. Where the potential for confusion may arise, we will refer to Map and Σ as the *right mapping space* and *right suspension*, denoting them by Map^R and Σ_R.

Lemma 8.1.5. *The left and right mapping space constructions are related by the following formulas:*

- $\mathrm{Map}^L_X(x_0, x_1)^{\mathrm{co}} \cong \mathrm{Map}^R_{X^{\mathrm{co}}}(x_0, x_1)$;
- $\mathrm{Map}^R_X(x_0, x_1)^{\mathrm{co}} \cong \mathrm{Map}^L_{X^{\mathrm{co}}}(x_0, x_1)$;
- $\mathrm{Map}^L_X(x_0, x_1)^{\mathrm{co-op}} \cong \mathrm{Map}^L_{X^{\mathrm{co-op}}}(x_1, x_0)$;
- $\mathrm{Map}^R_X(x_0, x_1)^{\mathrm{co-op}} \cong \mathrm{Map}^R_{X^{\mathrm{co-op}}}(x_1, x_0)$;
- $\mathrm{Map}^L_X(x_0, x_1)^{\mathrm{op}} \cong \mathrm{Map}^R_{X^{\mathrm{op}}}(x_1, x_0)$;
- $\mathrm{Map}^R_X(x_0, x_1)^{\mathrm{op}} \cong \mathrm{Map}^L_{X^{\mathrm{op}}}(x_1, x_0)$.

Proof. This follows from applying the involutions $(-)^{\mathrm{co}}$, $(-)^{\mathrm{co-op}}$, and $(-)^{\mathrm{op}}$ to the pullback diagrams defining Map^L and Map^R, and applying Corollary 4.3.8. $\square$

From here on, we will work exclusively with right mapping spaces unless otherwise noted, omitting the superscript R, with the understanding that our results may be adapted to left mapping spaces using the formulas of Lemma 8.1.5.

Proposition 8.1.6. $\Sigma \dashv \mathrm{Map}$ *is a Quillen adjunction between the Grothendieck model structure on* cSet *and the cubical Joyal model structure on* $\partial\square^1 \downarrow \mathsf{cSet}$*.*

Proof. That Σ preserves cofibrations follows from the description of the geometric product in Proposition 4.1.9. To show that Σ preserves trivial cofibrations, it suffices to show that Σ sends all open box inclusions to trivial cofibrations in the cubical Joyal model structure. To see this, observe that $\Sigma\square^n$ is the quotient of $\square^{n+1}$ in which the faces $\partial_{n+1,0}, \partial_{n+1,1}$ are quotiented down to vertices, while $\Sigma\sqcap^n_{i,\varepsilon}$ is the corresponding quotient of $\sqcap^{n+1}_{i,\varepsilon}$. For $i \leq n, \varepsilon \in \{0,1\}$, the critical edge of $\square^{n+1}$ with respect to the face $\partial_{i,\varepsilon}$ is an edge of the face $\partial_{n+1,1-\varepsilon}$, hence its image in $\Sigma\square^{n+1}$ is degenerate. Thus the inclusion $\Sigma\sqcap^n_{i,\varepsilon} \to \Sigma\square^n$ is a trivial cofibration. □

Corollary 8.1.7. *If* $f\colon X \to Y$ *is a (trivial) fibration in the cubical Joyal model structure, then each induced map* $f_*\colon \mathrm{Map}_X(x_0, x_1) \to \mathrm{Map}_Y(fx_0, fx_1)$ *is a (trivial) fibration in the Grothendieck model structure.*

In particular, if X *is a cubical quasicategory then all mapping spaces* $\mathrm{Map}_X(x_0, x_1)$ *are cubical Kan complexes.* □

We can characterize categorical equivalences in terms of these mapping spaces and the homotopy categories defined in Section 5.3.

Definition 8.1.8. Let X be a cubical quasicategory. We define the *homotopy category* $\mathsf{Ho}X$ to be the homotopy category of the marked cubical quasicategory $X^\natural$.

Lemma 8.1.9. *For a cubical quasicategory* X*, we have the following natural isomorphisms:*

- $\mathsf{Ho}\,X^{\mathrm{co}} \cong \mathsf{Ho}\,X$;

- $\mathsf{Ho}\, X^{\text{co-op}} \cong (\mathsf{Ho}\, X)^{\text{op}}$;
- $\mathsf{Ho}\, X^{\text{op}} \cong (\mathsf{Ho}\, X)^{\text{op}}$. $\square$

Our next goal is to prove the following:

Theorem 8.1.10. *Let $f\colon X \to Y$ be a map between cubical quasicategories. Then f is a categorical equivalence if and only if the following two conditions are satisfied:*

- $\mathsf{Ho} f\colon \mathsf{Ho} X \to \mathsf{Ho} Y$ *is an equivalence of categories;*
- *for all $x_0, x_1\colon \square^0 \to X$, the map $\mathrm{Map}_X(x_0, x_1) \to \mathrm{Map}_Y(fx_0, fx_1)$ is a homotopy equivalence in the Grothendieck model structure.*

The proof of this statement will require several steps. We begin by defining certain quotients of standard cubes which will be used in the proof.

Definition 8.1.11. For $n \geq 0$, we define K^n to be the quotient of $\square^{n+2}$ in which:

- $\square^1 \otimes \{0\} \otimes \square^n$, i.e. the $(2,0)$-face, is degenerate in the first dimension, i.e. the corresponding map $\square^{1+n} \to K^n$ factors through σ_1;
- $\{1\} \otimes \{1\} \otimes \square^n$ is a degeneracy of a vertex;
- the edge $\square^1 \otimes \{1, \ldots, 1\}$ is degenerate.

Let $\overline{K}^n$ denote the image of $\square^1 \otimes \partial \square^{n+1}$ in $\overline{K}^n$. For $\varepsilon \in \{0,1\}$, let K^n_ε denote the image of $\{\varepsilon\} \otimes \square^{n+1}$ in K^n. Let $\overline{K}^n_\varepsilon$ denote the intersection of $\overline{K}^n$ and K^n_ε, i.e. the image in K^n of the boundary of the $(1,\varepsilon)$-face.

Note that the inclusion $\overline{K}^n_0 \hookrightarrow K^n_0$ is isomorphic to $\partial\square^{n+1} \hookrightarrow \square^{n+1}$. Similarly, $\overline{K}^n_1 \hookrightarrow K^n_1$ is isomorphic to the quotient of $\partial\square^{n+1} \hookrightarrow \square^{n+1}$ where the $(1, 1-\varepsilon)$-face is a degeneracy of a vertex.

Lemma 8.1.12. *For all $n \geq 0$, the inclusion $\overline{K}^n_0 \hookrightarrow \overline{K}^n$ is anodyne.*

Proof. Let E denote the union of $\overline{K}^n_0$ with the image of $\square^1 \otimes \{1\} \otimes \square^n$ in $\overline{K}^n$. We first show that the inclusion $\overline{K}^n_0 \hookrightarrow E$ is anodyne. To see this, observe that the intersection of $\overline{K}^n_0$ with E is the image in K^n of $\{0\} \otimes \{1\} \otimes \square^n$. This coincides with the image in K^n of $\partial\square^1 \otimes \{1\} \otimes \square^n \cup \square^1 \otimes \{1\} \otimes \{(1, \ldots, 1)\}$. Thus $\overline{K}^n_0 \hookrightarrow E$ is a pushout of the image in K^n of the inclusion $\partial\square^1 \otimes \{1\} \otimes \square^n \cup \square^1 \otimes \{1\} \otimes \{(1, \ldots, 1)\} \hookrightarrow \square^1 \otimes \{1\} \otimes \square^n$. This map can be written as a composite of open box fillings; in K^n, the critical edges of each of these open boxes will be degenerate, hence the map will be anodyne.

To see that $E \hookrightarrow \overline{K}^n$ is anodyne, observe that E consists of the images in K^n of the boundary of the $(1, 0)$-face together with the $(2, \varepsilon)$-faces. For $2 \leq i \leq n + 2$, let E_i consist of the images in K^n of the boundary of the $(1, 0)$-face together with the (j, ε)-faces for $j \leq i$; thus $E_2 = E$ while $E_{n+1} = \overline{K}^n$. So it suffices to show that each map $E_i \hookrightarrow E_{i+1}$ is anodyne. To see this, observe that for $i \geq 2, \varepsilon \in \{0, 1\}$, the intersection of the image in K^n of the $(i+1, \varepsilon)$-face with E_i consists of the images of its $(1, 0)$-face and its (j, ε')-faces for $j \leq i$ (this can be seen from the cubical identities). In other words, this intersection is the image in K^n of $\sqcap^i_{1,1} \otimes \{\varepsilon\} \otimes \square^{(n+2)-(i+1)}$. Thus the inclusion $E_i \hookrightarrow E_{i+1}$ is a pushout of $\sqcap^i_{1,1} \otimes \partial\square^1 \otimes \square^{(n+2)-(i+1)} \hookrightarrow \square^i \otimes \partial\square^1 \otimes \square^{(n+2)-(i+1)}$. Moreover, the image in K^n of $\square^1 \otimes \{0, \ldots, 0\} \otimes \partial\square^1 \otimes \square^{(n+2)-(i+1)}$ is degenerate in the first dimension. Thus this map is a pushout of the anodyne map $\widehat{\sqcap}^i_{1,1} \otimes \partial\square^1 \otimes \square^{(n+2)-(i+1)} \hookrightarrow \widehat{\square}^i_{1,1} \otimes \partial\square^1 \otimes \square^{(n+2)-(i+1)}$. $\square$

Lemma 8.1.13. *For $n \geq 0$, the inclusion $K^n_0 \hookrightarrow K^n$ is anodyne.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
\overline{K}_0^n \arrow[r, "\sim"] \arrow[d] & \overline{K}^n \arrow[d] \arrow[ddr, bend left] & \\
K_0^n \arrow[r, "\sim"] \arrow[drr, bend right] & K_0^n \cup_{\overline{K}_0^n} \overline{K}^n \arrow[dr] \arrow[ul, phantom, "\ulcorner", very near start] & \\
 & & K^n
\end{tikzcd}$$

The inclusion $K_0^n \hookrightarrow K_0^n \cup_{\overline{K}_0^n} \overline{K}^n$ is anodyne as a pushout of an anodyne map. The inclusion $K_0^n \cup_{\overline{K}_0^n} \overline{K}^n \hookrightarrow K^n$ is a $(1,1)$-open box filling; as the critical edge is the degenerate edge $\square^1 \otimes \{0, \ldots, 0\}$, this is an inner open box filling. $\square$

Lemma 8.1.14. *Let $X \to Y$ be a fibration between cubical quasicategories. Let $x : \square^n \to X$, for $n \geq 0$, and $\varepsilon \in \{0,1\}$. Consider all diagrams of the form:*

$$\begin{tikzcd}
\partial\square^{n+1} \arrow[r] \arrow[d] & X \arrow[d] \\
\square^{n+1} \arrow[r] & Y
\end{tikzcd}$$

for which the $(1,\varepsilon)$-face of the boundary $\square^{n+1} \to X$ is x. A lift exists in every such diagram if and only if a lift exists in every such diagram for which the $(1,1-\varepsilon)$-face of $\partial\square^{n+1} \to X$ is a degeneracy of a vertex.

Proof. Fix x and a diagram of the form depicted above; we will obtain a lift in the given diagram under the assumption that a lift exists for all such diagrams in which the $(1,1-\varepsilon)$-face of $\partial\square^{n+1} \to X$ is a degeneracy of a vertex. By duality, it suffices to consider the case $\varepsilon = 0$.

By Lemmas 8.1.12 and 8.1.13, we have an injective trivial cofibration from $\overline{K}_0^n \hookrightarrow K_0^n$ to $\overline{K}^n \hookrightarrow K^n$, regarding these maps as objects in the morphism category $\mathsf{cSet}^{\to}$. (Note that the injective model structure on $\mathsf{cSet}^{\to}$ coincides with the Reedy model structure by Corollary 4.1.2.) Furthermore, the map $X \to Y$ is injective fibrant, as a fibration between fibrant objects. Therefore, identifying $\partial\square^{n+1} \hookrightarrow \square^{n+1}$ with $\overline{K}_0^n \hookrightarrow K_0^n$, the given diagram factors as:

$$\begin{tikzcd}
\overline{K}_0^n \arrow[r, hook] \arrow[d, hook] & \overline{K}^n \arrow[r] \arrow[d, hook] & X \arrow[d] \\
K_0^n \arrow[r, hook] & K^n \arrow[r] & Y
\end{tikzcd}$$

Thus, to obtain a lift in the original diagram, it suffices to obtain a lift in the right-hand diagram above. For this, observe that the inclusion $K_1^n \cup_{\overline{K}_1^n} \overline{K}^n \hookrightarrow K^n$ is a $(1,0)$-inner open box filling, whose critical edge is the degenerate edge $\square^1 \otimes \{1, \ldots, 1\}$; thus this map is anodyne. Therefore, it suffices to obtain a lift in the diagram:

$$\begin{tikzcd}
\overline{K}^n \arrow[rr] \arrow[d, hook] & & X \arrow[d] \\
K_1^n \cup_{\overline{K}_1^n} \overline{K}^n \arrow[r] & K^n \arrow[r] & Y
\end{tikzcd}$$

For this, in turn, it suffices to obtain a lift in the diagram:

$$\begin{tikzcd}
\overline{K}_1^n \arrow[r] \arrow[d, hook] & X \arrow[d] \\
K_1^n \arrow[r] & Y
\end{tikzcd}$$

The boundary $\partial\square^{n+1} \to \overline{K}_1^n \to X$ has x as its $(1,0)$-face; this follows from the fact that the image in K^n of the $(2,0)$-face is degenerate in the first dimension. Similarly, the $(1,1)$-face of this boundary is a degeneracy of a vertex, as it is precisely the image of $\{1\} \otimes \{1\} \otimes \square^n$. Thus this diagram admits a lift by assumption. $\square$

Corollary 8.1.15. *For $n \geq 0$, a fibration between cubical quasicategories has the right lifting property against $\partial\Box^{n+1} \hookrightarrow \Box^{n+1}$ if and only if it has the right lifting property against $\Sigma\partial\Box^n \hookrightarrow \Sigma\Box^n$.*

Proof. The forward implication follows from the fact that $\Sigma\partial\Box^n \hookrightarrow \Sigma\Box^{n+1}$ is a pushout of $\partial\Box^{n+1} \hookrightarrow \Box^{n+1}$. For the reverse implication, observe that $\Sigma\partial\Box^n$ (resp. $\Sigma\Box^n$) is precisely the quotient of $\partial\Box^{n+1}$ (resp. $\Box^{n+1}$) in which the $(1,0)$ and $(1,1)$-faces are degeneracies of vertices. The result then follows from applying Lemma 8.1.14 twice, once with $\varepsilon = 0$ and once with $\varepsilon = 1$. $\Box$

Proof of Theorem 8.1.10. First let $f\colon X \to Y$ be a categorical equivalence between cubical quasicategories. That $\mathsf{Ho}f$ is an equivalence of categories follows from Lemma 5.3.5 and Proposition 6.3.4 (ii). That each map $\mathrm{Map}_X(x_0, x_1) \to \mathrm{Map}_Y(fx_0, fx_1)$ is a homotopy equivalence follows from Proposition 8.1.6.

Now let $f\colon X \to Y$ be a map between cubical quasicategories inducing an equivalence on homotopy categories and homotopy equivalences on all mapping spaces. We will show that f is a categorical equivalence. By factoring an arbitrary map as a composite of a trivial cofibration with a fibration and applying the implication which we have already proven, we may assume f is a fibration. By Proposition 8.1.6, this implies that f induces fibrations on all cubical mapping spaces. Thus we wish to show that, given a fibration of cubical quasicategories $f\colon X \to Y$ such that $\mathsf{Ho}f$ is an equivalence of categories and each map $\mathrm{Map}_X(x_0, x_1) \to \mathrm{Map}_Y(fx_0, fx_1)$ is a trivial fibration, f is a trivial fibration.

We begin by showing that f has the right lifting property with respect to the map $\varnothing \to \Box^0$ – in other words, that f is surjective on vertices. To this end, let y be a vertex of Y. Then since $\mathsf{Ho}f$ is essentially surjective, there is a vertex $x\colon \Box^0 \to X$ such that $fx \cong y$ in $\mathsf{Ho}Y$. Thus we have a commuting diagram in cSet:

$$\begin{tikzcd} \square^0 \arrow[r] \arrow[d, "0"'] & X \arrow[d, "f"] \\ K \arrow[r] & Y \end{tikzcd}$$

Since f is a fibration, this diagram has a lift; the restriction of this lift to the endpoint $1 \colon \square^0 \to K$ gives a vertex $x' \colon \square^0 \to X$ with $fx' = y$.

To complete the proof, we must show that f has the right lifting property with respect to all boundary inclusions $\partial\square^{n+1} \hookrightarrow \square^{n+1}$ for $n \geq 0$. By Corollary 8.1.15, it suffices to show that f has the right lifting property with respect to all maps $\Sigma\partial\square^n \hookrightarrow \Sigma\square^n$. But by Proposition 8.1.4, this is equivalent to our assumption that f induces trivial fibrations on all mapping spaces. □

The following result shows that, in verifying the conditions of Theorem 8.1.10 for a map $f \colon X \to Y$, it suffices to show that $\mathsf{Ho}\, X$ is essentially surjective and that f induces homotopy equivalences on all mapping spaces.

Proposition 8.1.16. *Let $f \colon X \to Y$ be a map between cubical quasicategories. If f induces homotopy equivalences on all mapping spaces, then $\mathsf{Ho}\, X \to \mathsf{Ho}\, Y$ is fully faithful.*

Proof. Factoring an arbitrary map as a trivial cofibration followed by a fibration and applying Theorem 8.1.10, we see that it suffices to consider the case where f is a fibration. By Proposition 8.1.6, this implies that each map $\mathrm{Map}_X(x_0, x_1) \to \mathrm{Map}_Y(fx_0, fx_1)$ is a trivial fibration.

For $x_0, x_1 \colon \square^0 \to X$, $\mathrm{Map}_X(x_0, x_1) \to \mathrm{Map}_Y(fx_0, fx_1)$ is surjective on vertices, implying that every edge of Y from fx_0 to fx_1 is the image under f of some edge of X from x_0 to x_1. Thus $\mathsf{Ho}\, f$ is full. To see that it is faithful, let $p, q \colon \square^1 \to X$ be a pair of edges from x_0 to x_1, such that the morphisms in $\mathsf{Ho}\, Y(fx_0, fx_1)$ corresponding to fp and fq are equal. Applying Lemma 5.3.4, this implies that there is a 2-cube in Y of the form:

$$\begin{tikzcd} fx_0 \arrow[r, equal] \arrow[d, "fp"'] & fx_0 \arrow[d, "fq"] \\ fx_1 \arrow[r, equal] & fx_1 \end{tikzcd}$$

This 2-cube corresponds to an edge from fp to fq in $\mathrm{Map}_Y(fx_0, fx_1)$; thus we have a commuting diagram:

$$\begin{tikzcd} \partial\Box^1 \arrow[r, "{(p,q)}"] \arrow[d, hook] & \mathrm{Map}_X(x_0, x_1) \arrow[d] \\ \Box^1 \arrow[r] & \mathrm{Map}_Y(fx_0, fx_1) \end{tikzcd}$$

Since $\mathrm{Map}_X(x_0, x_1) \to \mathrm{Map}_Y(fx_0, fx_1)$ is a trivial fibration, this diagram has a lift, implying that $p = q$ in $\mathsf{Ho}\,X(x_0, x_1)$. Thus we see that $\mathsf{Ho}\,f$ is faithful. $\square$

The Quillen equivalences $T \dashv U$ and $Q_{W,\varepsilon} \dashv \int_{W,\varepsilon}$ relate the cubical homotopy category and mapping space constructions to their simplicial analogues, described in Section 3.2.

Lemma 8.1.17. *We have the following natural isomorphisms relating the homotopy categories of quasicategories and cubical quasicategories:*

(i) For a quasicategory X, $\mathsf{Ho}\,X \cong \mathsf{Ho}\,UX$;

(ii) For a cubical quasicategory X and $W \in \{L, R\}$, $\mathsf{Ho}\,X \cong \mathsf{Ho} \int_{W,1} X$;

(iii) For a cubical quasicategory X and $W \in \{L, R\}$, $\mathsf{Ho}\,X \cong (\mathsf{Ho} \int_{W,0} X)^{\mathrm{op}}$.

Proof. For (i), first note that X and UX have the same edges and vertices. The equivalence relations defining the morphisms of $\mathsf{Ho}X$ and $\mathsf{Ho}UX$ coincide by a simple argument involving Lemma 5.3.4 and its simplicial analogue. A similar argument proves (ii), and (iii) then follows from Lemmas 7.2.7 and 8.1.9. $\square$

Lemma 8.1.18. *For $X \in \mathsf{sSet}, x_0, x_1 \colon \Delta^0 \to X$, and $W \in \{L, R\}$, we have a natural isomorphism $U\mathrm{Map}^W_X(x_0, x_1) \cong \mathrm{Hom}_{UX}(x_0, x_1)$.*

Proof. Observe that the simplicial mapping space construction defines a functor $\mathrm{Hom} \colon \partial\Delta^1 \downarrow \mathsf{sSet} \to \mathsf{sSet}$. An argument similar to the proof of Proposition 8.1.4 shows that this functor has a left adjoint $\Sigma \colon \mathsf{sSet} \to \partial\Delta^1 \downarrow \mathsf{sSet}$, given by the following pushout diagram:

$$\begin{tikzcd} X \sqcup X \arrow[r] \arrow[d] & \partial\Delta^1 \arrow[d] \\ \Delta^1 \times X \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Thus we have the following square of adjunctions:

$$\begin{tikzcd} \partial\square^1 \downarrow \mathsf{cSet} \arrow[r, shift left] \arrow[d, shift left] & \partial\Delta^1 \downarrow \mathsf{sSet} \arrow[l, shift left] \arrow[d, shift left] \\ \mathsf{cSet} \arrow[u, shift left] \arrow[r, shift left] & \mathsf{sSet} \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd}$$

We wish to show that the square of right adjoints commutes (up to natural isomorphism); for this, it suffices to show that the square of left adjoints commutes, i.e. that $T\Sigma_W \cong \Sigma T$. To see this, we may apply T to the pushout square which defines the (left or right) suspension of a cubical set. Using Proposition 4.1.15 and the fact that T preserves pushouts, we obtain a natural isomorphism $T\Sigma_W X \cong \Sigma T X$ for $X \in \mathsf{cSet}$. $\square$

Lemma 8.1.19. *For $X \in \mathsf{cSet}$, $x_0, x_1 \colon \square^0 \to X$, we have the following natural isomorphisms:*

- $\int\limits_{L,0} \mathrm{Map}^L_X(x_0, x_1) \cong \mathrm{Hom}^R_{\int\limits_{L,0} X}(x_1, x_0)$;
- $\int\limits_{L,1} \mathrm{Map}^L_X(x_0, x_1) \cong \mathrm{Hom}^R_{\int\limits_{L,1} X}(x_0, x_1)$;
- $\int\limits_{R,0} \mathrm{Map}^R_X(x_0, x_1) \cong \mathrm{Hom}^R_{\int\limits_{R,0} X}(x_1, x_0)$;
- $\int\limits_{R,1} \mathrm{Map}^R_X(x_0, x_1) \cong \mathrm{Hom}^R_{\int\limits_{R,1} X}(x_0, x_1)$.

Proof. It suffices to prove the identity for $\int\limits_{R,1}$; the others then follow from Lemmas 7.2.7 and 8.1.5. Observe that maps $\Delta^n \to \int\limits_{R,1} \mathrm{Map}^R_X(x_0, x_1)$ correspond to maps $\Sigma^R Q^n_{R,1} \to X$ mapping the basepoints $0 \mapsto x_0, 1 \mapsto x_1$. By the universal property of the pushout, these correspond to commuting diagrams of the form:

$$\begin{tikzcd}
Q^n_{R,1} \sqcup Q^n_{R,1} \arrow[r] \arrow[d, hook] & \partial\square^1 \arrow[d] \\
Q^n_{R,1} \otimes \square^1 \arrow[r] & X
\end{tikzcd}$$

In other words, these are maps $Q^n_{R,1} \otimes \square^1 \to X$ such that for $\varepsilon \in \{0, 1\}$, the subcomplex $Q^n_{R,1} \otimes \{\varepsilon\}$ is mapped to x_ε.

On the other hand, maps $s \colon \Delta^{n+1} \to \int\limits_{R,1} X$, i.e. $Q^{n+1}_{R,1} \to X$, which map the terminal vertex to x_1 correspond to commuting diagrams of the form:

$$\begin{tikzcd}
Q^n_{R,1} \arrow[r] \arrow[d, hook, "Q^n_{R,1} \otimes \partial_{1,1}"'] & \square^0 \arrow[d, "x_1"] \\
Q^n_{R,1} \otimes \square^1 \arrow[r] & X
\end{tikzcd}$$

In other words, these are maps $Q^n_{R,1} \to X$ such that $Q^n_{R,1} \otimes \{1\}$ is mapped to x_1. By Proposition 7.2.5, the condition $s\partial_{n+1} = x_0$ corresponds to the condition that $Q^n_{R,1} \otimes \{0\}$ is mapped to x_0. $\square$

Remark 8.1.20. One may observe that applying a functor $\int\limits_{W,\varepsilon}$ to a compatible cubical mapping space always produces a simplicial right mapping space, regardless of the values of W and ε. Lemma 3.2.24 shows that the alternative definitions of $Q_{W,\varepsilon}$ discussed in Remark 7.2.6 would instead produce formulas relating cubical mapping spaces to simplicial left mapping spaces.

These results allow us to transfer Theorem 8.1.10 along the Quillen equivalence $T \dashv U$, obtaining a new proof of the analogous result for the Joyal model structure on sSet.

Proof of Theorem 3.2.26. By Proposition 2.1.37 and Theorem 7.2.1, a map of cubical quasicategories $f \colon X \to Y$ is a categorical equivalence if and only if $Uf \colon UX \to UY$ is a categorical equivalence. Similarly, by Proposition 2.1.37 and Theorem 7.2.26, each map $\mathrm{Hom}_X(x_0, x_1) \to \mathrm{Hom}_Y(fx_0, fx_1)$ is a homotopy equivalence if and only if $U\mathrm{Hom}_X(x_0, x_1) \to U\mathrm{Hom}_Y(fx_0, fx_1)$ is a homotopy equivalence. The stated result thus follows from Theorem 8.1.10, together with Lemmas 8.1.17 and 8.1.18. □

8.2 Composition in cubical quasicategories

In this section we construct a composition operation on the mapping spaces in a cubical quasicategory, which is well-defined up to homotopy.

Let $\mathcal{H}$ denote the homotopy category of the Grothendieck model structure on cSet. By [Mal09, Thm. 4.3] and [BM17, Thm. 3], binary cartesian products in cSet descend to cartesian products in $\mathcal{H}$. Our goal is to define a functor from the category cqCat of cubical quasicategories to the category $\mathsf{Cat}_{\mathcal{H}}$ of categories enriched over the cartesian monoidal structure on $\mathcal{H}$, and to show that the maps which this functor sends to equivalences of enriched categories are precisely the categorical equivalences. The analogous theory for the simplicial case is developed in [Rez20, Sec. 33.7], using inclusions of spines into simplices; as the concept of a spine is less well-suited to cubical sets, we will define and study our composition operation using the adjunction $\Sigma \dashv \mathrm{Map}$.

We begin by recalling a general model-categorical result which will be useful throughout this section.

Lemma 8.2.1. *In a model category with terminal object $*$, let $A \to B$ be a trivial cofibration between cofibrant objects, and X a fibrant object. Given any diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & * \end{tikzcd}$$

there exists a lift $B \to X$, which is unique up to homotopy. $\square$

Let $X, Y \in \mathsf{cSet}$; for ease of notation, we will denote the basepoints of ΣX by 0 and 1, and those of ΣY by 1 and 2. Given an n-cube $x \colon \square^n \to X$, we will also let x denote the corresponding n-cube $x \colon \square^{n+1} \to \Sigma X$, and similarly for $y \colon \square^n \to Y$.

We define $\Sigma X \cup_1 \Sigma Y$ by the following pushout diagram:

$$\begin{tikzcd} \square^0 \arrow[r, "1"] \arrow[d, "1"] & \Sigma Y \arrow[d] \\ \Sigma X \arrow[r] & \Sigma X \cup_1 \Sigma Y \end{tikzcd}$$

It is clear that this definition is functorial in both X and Y; equipping $\Sigma X \cup_1 \Sigma Y$ with the basepoints 0 and 2, we obtain a functor $\mathsf{cSet} \times \mathsf{cSet} \to \partial\square^1 \downarrow \mathsf{cSet}$. Our next goal will be to define a cubical set $\Sigma X \bullet \Sigma Y$, functorially in X and Y, with a natural trivial cofibration $\Sigma X \cup_1 \Sigma Y \hookrightarrow \Sigma X \bullet \Sigma Y$ as well as a natural map $\Sigma(X \times Y) \hookrightarrow \Sigma X \bullet \Sigma Y$. This object will be used in defining the composition operation.

We proceed by induction. For $m \geq -1$, we will define a cubical set $(\Sigma X \bullet \Sigma Y)^m$ admitting a trivial cofibration $(\Sigma X \bullet \Sigma Y)^{m-1} \hookrightarrow (\Sigma X \bullet \Sigma Y)^m$ for $m \geq 0$, and containing an $(n+2)$-cube $x \bullet y$ for every pair of n-cubes $x \colon \square^n \to X$, $y \colon \square^n \to Y$ with $n \leq m$, satisfying the following hypotheses:

(C1) for $1 \leq i \leq n$, $(x \bullet y)\partial_{i,\varepsilon} = (x\partial_{i,\varepsilon}) \bullet (y\partial_{i,\varepsilon})$;

(C2) $(x \bullet y)\partial_{n+1,0} = x$;

(C3) $(x \bullet y)\partial_{n+1,1} = 2$;

(C4) $(x \bullet y)\partial_{n+2,1} = y$;

(C5) for $0 \leq n \leq m-1$ and $1 \leq i \leq n+1$, $(x \bullet y)\sigma_i = (x\sigma_i) \bullet (y\sigma_i)$;

(C6) for $0 \leq n \leq m-1$ and $1 \leq i \leq n$, $(x \bullet y)\gamma_{i,\varepsilon} = (x\gamma_{i,\varepsilon}) \bullet (y\gamma_{i,\varepsilon})$.

The intuition behind this definition is that $x \bullet y$ represents a formal composition of x and y; the "composite" face $(x \bullet y)\partial_{n+2,0}$ will be identified with the cube of $\Sigma(X \times Y)$ corresponding to the pair (x, y). For instance, for $n = 0$, $x \bullet y$ is a 2-cube of the form depicted below:

$$\begin{tikzcd}
0 \arrow[r, "{(x,y)}"] \arrow[d, "x"'] & 2 \arrow[d, equal] \\
1 \arrow[r, "y"] & 2
\end{tikzcd}$$

For the base case, we set $(\Sigma X \bullet \Sigma Y)^{-1} = \Sigma X \cup_1 \Sigma Y$. Now let $m \geq 0$, and suppose we have defined $(\Sigma X \bullet \Sigma Y)^{m-1}$ satisfying the induction hypothesis. To construct $(\Sigma X \bullet \Sigma Y)^m$, we must construct a cube $x \bullet y$ satisfying (C1) through (C6) for all $x\colon \square^m \to X, y\colon \square^m \to Y$. We begin with a lemma which will be used in this construction.

Lemma 8.2.2. *For any $x\colon \square^m \to X, y\colon \square^m \to Y$, the faces specified by (C1) through (C4) define an inner open box $\widehat{\sqcap}^{m+2}_{m+2,0} \to (\Sigma X \bullet \Sigma Y)^{m-1}$.*

Proof. We must show two things: that these face assignments satisfy the cubical identity for face maps, and that the critical edge with respect to the $(m+2,0)$-face is degenerate.

For the first statement, we wish to show that the face assignments above satisfy $(x \bullet y)\partial_{j,\varepsilon'}\partial_{i,\varepsilon} = (x \bullet y)\partial_{i+1,\varepsilon}\partial_{j,\varepsilon'}$ for $j \leq i$. We proceed by case analysis, applying the induction hypothesis.

For $j \leq i \leq m-1$, we have:

$$
\begin{aligned}
(x \bullet y)\partial_{j,\varepsilon'}\partial_{i,\varepsilon} &= ((x\partial_{j,\varepsilon'}) \bullet (y\partial_{j,\varepsilon'}))\partial_{i,\varepsilon} \\
&= (x\partial_{j,\varepsilon'}\partial_{i,\varepsilon}) \bullet (y\partial_{j,\varepsilon'}\partial_{i,\varepsilon}) \\
&= (x\partial_{i+1,\varepsilon}\partial_{j,\varepsilon'}) \bullet (y\partial_{i+1,\varepsilon}\partial_{j,\varepsilon'}) \\
&= (x\partial_{i+1,\varepsilon}) \bullet (y\partial_{i+1,\varepsilon})\partial_{j,\varepsilon'} \\
&= (x \bullet y)\partial_{i+1,\varepsilon}\partial_{j,\varepsilon'}
\end{aligned}
$$

For $j \leq i = m$, $\varepsilon = 0$, we have:

$$\begin{aligned}(x \bullet y)\partial_{j,\varepsilon'}\partial_{m,0} &= ((x\partial_{j,\varepsilon'}) \bullet (y\partial_{j,\varepsilon'}))\partial_{m,0} \\ &= x\partial_{j,\varepsilon'} \\ &= (x \bullet y)\partial_{m+1,0}\partial_{j,\varepsilon'}\end{aligned}$$

For $j \leq i = m$, $\varepsilon = 1$, we have:

$$\begin{aligned}(x \bullet y)\partial_{j,\varepsilon'}\partial_{m,1} &= ((x\partial_{j,\varepsilon'}) \bullet (y\partial_{j,\varepsilon'}))\partial_{m,1} \\ &= 2 \\ &= 2\partial_{j,\varepsilon'} \\ &= (x \bullet y)\partial_{m+1,1}\partial_{j,\varepsilon'}\end{aligned}$$

For $j \leq m, i = m+1$, $\varepsilon = 1$, we have:

$$\begin{aligned}(x \bullet y)\partial_{j,\varepsilon'}\partial_{m+1,1} &= ((x\partial_{j,\varepsilon'}) \bullet (y\partial_{j,\varepsilon'}))\partial_{m,1} \\ &= y\partial_{j,\varepsilon'} \\ &= (x \bullet y)\partial_{m+2,1}\partial_{j,\varepsilon'}\end{aligned}$$

For $j = i = m+1$, $\varepsilon = 1$, $\varepsilon' = 0$, we have:

$$\begin{aligned}(x \bullet y)\partial_{m+1,0}\partial_{m+1,1} &= x\partial_{m+1,1}\\ &= 1\\ &= y\partial_{m+1,0}\\ &= (x \bullet y)\partial_{m+2,1}\partial_{m+1,0}\end{aligned}$$

Finally, for $j = i = m+1$, $\varepsilon = 1$, $\varepsilon' = 1$, we have:

$$\begin{aligned}(x \bullet y)\partial_{m+1,1}\partial_{m+1,1} &= 2\partial_{m+1,1}\\ &= 2\\ &= y\partial_{m+1,1}\\ &= (x \bullet y)\partial_{m+2,1}\partial_{m+1,1}\end{aligned}$$

Thus we see that these face assignments do satisfy the necessary cubical identity. To see that the critical edge is degenerate, observe that it is an edge of the $(m+1,1)$-face, which is degenerate at the vertex 2. $\square$

We construct $(\Sigma X \bullet \Sigma Y)^m$ from $(\Sigma X \bullet \Sigma Y)^{m-1}$ by adjoining a filler $x \bullet y$ for the inner open box described above for each pair (x, y) such that the m-cube $(x, y)\colon \square^m \to X \times Y$ is non-degenerate. In other words, if we let $(X \times Y)^{\mathrm{nd}}_m$ denote the set of non-degenerate m-cubes of $X \times Y$, then $(\Sigma X \bullet \Sigma Y)^m$ is defined by the following pushout diagram:

$$\begin{tikzcd}
\coprod\limits_{(X\times Y)_m^{\mathrm{nd}}} \widehat{\sqcap}^{m+2}_{m+2,0} \arrow[r] \arrow[d, hook] & (\Sigma X \bullet \Sigma Y)^{m-1} \arrow[d, hook] \\
\coprod\limits_{(X\times Y)_m^{\mathrm{nd}}} \widehat{\square}^{m+2}_{m+2,0} \arrow[r] & (\Sigma X \bullet \Sigma Y)^{m} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Thus the inclusion $(\Sigma X \bullet \Sigma Y)^{m-1} \hookrightarrow (\Sigma X \bullet \Sigma Y)^m$ is a trivial cofibration, as a pushout of a coproduct of trivial cofibrations.

Now we must define $x \bullet y$ for all pairs of m-cubes $x\colon \square^m \to X, y\colon \square^m \to Y$ – or in other words, for all m-cubes $(x,y)\colon \square^m \to X \times Y$. For non-degenerate (x,y), $x \bullet y$ is the filler constructed above. If $(x,y) = (x',y')\rho$ for some $n < m$, non-degenerate $(x',y')\colon \square^n \to X \times Y$, and epimorphism $\rho\colon [1]^m \to [1]^n$ in $\square$, then (C5) and (C6) require us to define $x \bullet y = (x' \bullet y')(\rho \otimes \square^2)$.

Lemma 8.2.3. $(\Sigma X \bullet \Sigma Y)^m$ *satisfies (C1) through (C6).*

Proof. For $(x,y)\colon \square^n \to X \times Y, n < m$, the identities hold by the induction hypothesis; thus we must verify that they hold for (x,y) of dimension m as well. The identities (C5) and (C6) are immediate from the definition of $x \bullet y$ (note that for $i \leq m, (\sigma_i\colon \square^{m+3} \to \square^{m+2}) = (\sigma_i\colon \square^{m+1} \to \square^m) \otimes \square^2$, and similarly for connection maps). Thus we need only verify (C1) through (C4). For non-degenerate (x,y) these hold by construction, so we only need to verify them for the case where $(x,y)\colon \square^m \to X \times Y$ is degenerate. To that end, let $(x,y) = (x',y')\rho$ for some epimorphism $\rho\colon [1]^m \to [1]^n$ in $\square$ and some non-degenerate $(x',y')\colon \square^n \to X \times Y$.

We begin with (C1). For $i \leq m, \varepsilon \in \{0,1\}$, we can compute:

$$
\begin{aligned}
(x \bullet y)\partial_{i,\varepsilon} &= (x' \bullet y')(\rho \otimes \square^2)(\partial_{i,\varepsilon} \otimes \square^2) \\
&= (x' \bullet y')(\rho\partial_{i,\varepsilon} \otimes \square^2)
\end{aligned}
$$

Considering the cubical identities, and recalling that every epimorphism in $\square$ is a composite of degeneracy and connection maps, we obtain two possible cases: either $\rho\partial_{i,\varepsilon}$ is an epimorphism, or else $n \geq 1$ and there exist $\rho' \colon [1]^{n-1} \to [1]^{m-1}$, $\partial_{j,\varepsilon} \colon [1]^{m-1} \to [1]^m$ such that $\rho\partial_{i,\varepsilon} = \partial_{j,\varepsilon}\rho'$. In the former case, we immediately obtain $(x' \bullet y')(\rho\partial_{i,\varepsilon} \otimes \square^2) = (x'\rho\partial_{i,\varepsilon}) \bullet (y'\rho\partial_{i,\varepsilon})$ by definition. In the latter case, we can apply the induction hypothesis to compute:

$$
\begin{aligned}
(x' \bullet y')(\rho\partial_{i,\varepsilon} \otimes \square^2) &= (x' \bullet y')(\partial_{j,\varepsilon}\rho' \otimes \square^2) \\
&= (x' \bullet y')(\partial_{j,\varepsilon} \otimes \square^2)(\rho' \otimes \square^2) \\
&= (x'\partial_{j,\varepsilon} \bullet y'\partial_{j,\varepsilon})(\rho' \otimes \square^2) \\
&= (x'\partial_{j,\varepsilon}\rho' \bullet y'\partial_{j,\varepsilon}\rho') \\
&= (x'\rho\partial_{i,\varepsilon}) \bullet (y'\rho\partial_{i,\varepsilon})
\end{aligned}
$$

Thus, in either case we have $(x \bullet y)\partial_{i,\varepsilon} = (x'\rho\partial_{i,\varepsilon}) \bullet (y'\rho\partial_{i,\varepsilon}) = (x\partial_{i,\varepsilon}) \bullet (y\partial_{i,\varepsilon})$. So (C1) holds.

The remaining identities concern faces of the form $(x \bullet y)(\square^m \otimes \partial_{i,\varepsilon})$ for $i \in \{1,2\}, \varepsilon \in \{0,1\}$. For every such face map, and any map $\rho \colon [1]^m \to [1]^n$ in

$\square$, we have a commuting diagram:

$$\begin{tikzcd}
\square^m \otimes \square^1 \arrow[r, "\square^m \otimes \partial_{i,\varepsilon}"] \arrow[d, "\rho \otimes \square^1"'] & \square^m \otimes \square^2 \arrow[d, "\rho \otimes \square^2"] \\
\square^n \otimes \square^1 \arrow[r, "\square^n \otimes \partial_{i,\varepsilon}"] & \square^n \otimes \square^2
\end{tikzcd}$$

Thus, for all of these maps we have:

$$\begin{aligned}
(x \bullet y)(\square^m \otimes \partial_{i,\varepsilon}) &= (x', y')(\rho \otimes \square^2)(\square^m \otimes \partial_{i,\varepsilon}) \\
&= (x', y')(\square^n \otimes \partial_{i,\varepsilon})(\rho \otimes \square^1) \\
&= (x', y')\partial_{n+1,\varepsilon}(\rho \otimes \square^1)
\end{aligned}$$

This allows us to verify the remaining identities. For (C2), we have $(x', y')\partial_{n+1,0}(\rho \otimes \square^1) = x'\rho = x$. For (C3), $(x', y')\partial_{n+1,1}$ is degenerate at the vertex 2, hence so is $(x', y')\partial_{n+1,1}(\rho \otimes \square^1)$. Finally, for (C4), we have $(x', y')\partial_{n+2,1}(\rho \otimes \square^1) = y'\rho = y$. $\square$

Thus we can define $(\Sigma X \bullet \Sigma Y)^m$ satisfying the induction hypothesis for all $m \geq -1$. We then define $\Sigma X \bullet \Sigma Y$ to be the colimit of the diagram of inclusions:

$$\Sigma X \cup_1 \Sigma Y \hookrightarrow (\Sigma X \bullet \Sigma Y)^0 \hookrightarrow (\Sigma X \bullet \Sigma Y)^1 \hookrightarrow \cdots \hookrightarrow (\Sigma X \bullet \Sigma Y)^m \hookrightarrow \cdots$$

The inclusion $\Sigma X \cup_1 \Sigma Y \hookrightarrow \Sigma X \bullet \Sigma Y$ is a trivial cofibration, as a transfinite composite of trivial cofibrations.

Given $f \colon X \to X'$, $g \colon Y \to Y'$, we obtain a map $\Sigma f \bullet \Sigma g \colon \Sigma X \bullet \Sigma Y \to \Sigma X' \bullet \Sigma Y'$ by setting $(\Sigma f \bullet \Sigma g)(x \bullet y) = fx \bullet gy$ for all $(x, y) \colon \square^n \to X \times Y$. Equipping

$\Sigma X \bullet \Sigma Y$ with the basepoints 0 and 2, we obtain a functor $\mathsf{cSet} \times \mathsf{cSet} \to \partial\square^1 \downarrow \mathsf{cSet}$. The inclusion $\Sigma X \cup_1 \Sigma Y \hookrightarrow \Sigma X \bullet \Sigma Y$ then defines a natural transformation.

We now define the canonical map $\Sigma(X \times Y) \to \Sigma X \bullet \Sigma Y$. For $x: \square^n \to X, y: \square^n \to Y$, we let (x, y) denote the $(n+1)$-cube $(x \bullet y)\partial_{n+2,0}: \square^{n+1} \to \Sigma X \bullet \Sigma Y$. Using the identities (C1) through (C6) and applying Proposition 4.1.9, we can see that the assignment $((x, y), \mathrm{id}_{[1]}) \mapsto (x, y)$ defines a map $(X \times Y) \otimes \square^1 \to \Sigma X \bullet \Sigma Y$ sending all cubes of the form $((x, y), \varepsilon)$ to ε. By the universal property of the pushout, this corresponds to a unique map of bi-pointed cubical sets $\Sigma(X \times Y) \to \Sigma X \bullet \Sigma Y$.

Lemma 8.2.4. *For all $X, Y \in \mathsf{cSet}$, the map $\Sigma(X \times Y) \to \Sigma X \bullet \Sigma Y$ constructed above is a monomorphism.*

Proof. Applying Proposition 4.1.9, we can see that for $n \geq 0$, the non-degenerate cubes of $\Sigma(X \times Y)$ correspond to non-degenerate cubes $(x, y): \square^n \to X \times Y$, and are distinct if and only if the corresponding cubes of $X \times Y$ are distinct. Thus the map $\Sigma(X \times Y) \to \Sigma X \bullet \Sigma Y$ injectively maps non-degenerate cubes to non-degenerate cubes; by Theorem 4.1.1 it is therefore a monomorphism. $\square$

This inclusion is also natural in X and Y. To summarize, for any pair of cubical set maps $X \to X', Y \to Y'$, we have a commuting diagram in $\partial\square^1 \downarrow \mathsf{cSet}$:

$$\begin{tikzcd}
\Sigma X \cup_1 \Sigma Y \arrow[r, hook, "\sim"] \arrow[d] & \Sigma X \bullet \Sigma Y \arrow[d] & \Sigma(X \times Y) \arrow[l, hook'] \arrow[d] \\
\Sigma X' \cup_1 \Sigma Y' \arrow[r, hook, "\sim"] & \Sigma X' \bullet \Sigma Y' & \Sigma(X' \times Y') \arrow[l, hook']
\end{tikzcd}$$

We are now ready to define the composition map. For a cubical quasicategory X and vertices $x_0, x_1, x_2: \square^1 \to X$, we have a map $\Sigma\mathrm{Map}_X(x_0, x_1) \cup_1 \Sigma\mathrm{Map}_X(x_1, x_2) \to X$ given by the counits $\Sigma\mathrm{Map}_X(x_0, x_1) \to X, \Sigma\mathrm{Map}_X(x_1, x_2) \to X$. Equipping X with basepoints x_0, x_2, we have the following commuting diagram in $\partial\square^1 \downarrow \mathsf{cSet}$:

$$\begin{tikzcd}
\Sigma\mathrm{Map}_X(x_0, x_1) \cup_1 \Sigma\mathrm{Map}_X(x_1, x_2) \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Sigma\mathrm{Map}_X(x_0, x_1) \bullet \Sigma\mathrm{Map}_X(x_1, x_2) \arrow[r] & \square^0
\end{tikzcd}$$

The left-hand map is a trivial cofibration, while the right-hand map is a fibration; thus the diagram admits a lift $\Sigma\mathrm{Map}_X(x_0, x_1) \bullet \Sigma\mathrm{Map}_X(x_1, x_2) \to X$. Pre-composing with the inclusion $\Sigma(\mathrm{Map}_X(x_0, x_1) \times \mathrm{Map}_X(x_1, x_2)) \hookrightarrow \Sigma\mathrm{Map}_X(x_0, x_1) \bullet \Sigma\mathrm{Map}_X(x_1, x_2)$, we obtain a map of bi-pointed cubical sets $\Sigma(\mathrm{Map}_X(x_0, x_1) \times \mathrm{Map}_X(x_1, x_2)) \to X$. We define the composition map $c \colon \mathrm{Map}_X(x_0, x_1) \times \mathrm{Map}_X(x_1, x_2) \to \mathrm{Map}_X(x_0, x_2)$ to be the adjunct of this map.

As defined above, the composition map depends on the specific choice of lift $\Sigma\mathrm{Map}_X(x_0, x_1) \bullet \Sigma\mathrm{Map}_X(x_1, x_2) \to X$. However, the following result shows that it is well-defined up to homotopy.

Proposition 8.2.5. *If $c, c' \colon \mathrm{Map}_X(x_0, x_1) \times \mathrm{Map}_X(x_1, x_2) \to \mathrm{Map}_X(x_0, x_1)$ are composition maps defined by lifts in the diagram above, then there is a homotopy $c \sim c'$ in the Grothendieck model structure on* cSet.

Proof. The objects $\Sigma\mathrm{Map}_X(x_0, x_1) \cup_1 \Sigma\mathrm{Map}_X(x_1, x_2)$ and $\Sigma\mathrm{Map}_X(x_0, x_1) \bullet \Sigma\mathrm{Map}_X(x_1, x_2)$ are cofibrant in the cubical Joyal model structure on $\partial\square^1 \downarrow \mathsf{cSet}$, since their basepoints are distinct. Thus we may apply Lemma 8.2.1 to see that the lifts defining c and c' are homotopic, hence so are their composites with the inclusion of $\Sigma(\mathrm{Map}_X(x_0, x_1) \times \mathrm{Map}_X(x_1, x_2))$ into $\Sigma\mathrm{Map}_X(x_0, x_1) \bullet \Sigma\mathrm{Map}_X(x_1, x_2)$. Since $\Sigma(\mathrm{Map}_X(x_0, x_1) \times \mathrm{Map}_X(x_1, x_2))$ is cofibrant and X is fibrant, the adjunct maps c and c' are homotopic in the Grothendieck model structure on cSet by Proposition 8.1.6. $\square$

In view of Proposition 8.2.5, we will typically refer to *the* composition map $c\colon \mathrm{Map}_X(x_0, x_1) \times \mathrm{Map}_X(x_1, x_2) \to \mathrm{Map}_X(x_0, x_2)$, distinguishing between different choices of lifts only where necessary.

Given $A, B \in \mathsf{cSet}$ equipped with maps $f\colon A \to \mathrm{Map}_X(x_0, x_1)$, $g\colon B \to \mathrm{Map}_X(x_1, x_2)$, we obtain a generalized composition map $c(f \times g)\colon A \times B \to \mathrm{Map}_X(x_0, x_2)$ by composing the map $A \times B \to \mathrm{Map}_X(x_0, x_1) \times \mathrm{Map}_X(x_1, x_2)$ with the composition map defined above. In particular, given a pair of cubes $x\colon \square^m \to \mathrm{Map}_X(x_0, x_1), x'\colon \square^n \to \mathrm{Map}_X(x_1, x_2)$, we have a map $c(x \times x')\colon \square^m \times \square^n \to \mathrm{Map}_X(x_0, x_2)$; pre-composing with the canonical inclusion $(\pi_{\square^m}, \pi_{\square^n})\colon \square^{m+n} = \square^m \otimes \square^n \to \square^m \times \square^n$, we obtain a composite cube $\square^{m+n} \to \mathrm{Map}_X(x_0, x_2)$, unique up to homotopy in $\mathrm{Map}_X(x_0, x_2)$.

Remark 8.2.6. We also have another notion of composition of cubes in mapping spaces: given $x\colon \square^n \to \mathrm{Map}_X(x_0, x_1), x'\colon \square^n \to \mathrm{Map}_X(x_1, x_2)$ we have a composite n-cube $c(x, x')\colon \square^n \to \mathrm{Map}_X(x_0, x_1) \times \mathrm{Map}_X(x_1, x_2) \to \mathrm{Map}_X(x_0, x_2)$, and again this is well-defined up to homotopy.

The following lemma provides an alternative way to construct these generalized composition maps, which will typically be more useful in practice.

Lemma 8.2.7. *Given $A, B, \in \mathsf{cSet}$, X a cubical quasicategory, vertices $x_0, x_1, x_2\colon \square^0 \to X$, and $f\colon A \to \mathrm{Map}_X(x_0, x_1), g\colon B \to \mathrm{Map}_X(x_1, x_2)$, $c(f \times g)$ is homotopic to the adjunct of the composite of $\Sigma(A \times B) \hookrightarrow \Sigma A \bullet \Sigma B$ with a lift in the diagram*

$$\begin{tikzcd} \Sigma A \cup_1 \Sigma B \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Sigma A \bullet \Sigma B \arrow[r] & \square^0 \end{tikzcd} \tag{**}$$

Proof. The diagram (**) factors as follows:

$$\begin{tikzcd} \Sigma A \cup_1 \Sigma B \arrow[r] \arrow[d, hook] & \Sigma \mathrm{Map}_X(x_0, x_1) \cup_1 \Sigma \mathrm{Map}_X(x_1, x_2) \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Sigma A \bullet \Sigma B \arrow[r] & \Sigma \mathrm{Map}_X(x_0, x_1) \bullet \Sigma \mathrm{Map}_X(x_1, x_2) \arrow[r] & \square^0 \end{tikzcd}$$

Thus, given any lift of the right-hand square, the composite with $\Sigma A \bullet \Sigma B \to \Sigma \mathrm{Map}_X(x_0, x_1) \bullet \Sigma \mathrm{Map}_X(x_1, x_2)$ defines a lift $\Sigma A \bullet \Sigma B \to X$. By Lemma 8.2.1, any lift of (**) is homotopic to this one. Thus the composites $\Sigma(A \times B) \hookrightarrow \Sigma A \bullet \Sigma B \to X$ and

$$\Sigma(A \times B) \hookrightarrow \Sigma A \bullet \Sigma B \to \Sigma \mathrm{Map}_X(x_0, x_1) \bullet \Sigma \mathrm{Map}_X(x_1, x_2) \to X$$

are homotopic. By the naturality of the inclusion $\Sigma(A \times B) \hookrightarrow \Sigma A \bullet \Sigma B$, the latter composite is equal to

$$\Sigma(A \times B) \to \Sigma(\mathrm{Map}_X(x_0, x_1) \times \mathrm{Map}_X(x_1, x_2)) \to \Sigma \mathrm{Map}_X(x_0, x_1) \bullet \Sigma \mathrm{Map}_X(x_1, x_2) \to X$$

Therefore, by Proposition 8.1.6, the adjuncts of these maps are equal; the adjunct of the latter is precisely $A \times B \to \mathrm{Map}_X(x_0, x_1) \times \mathrm{Map}_X(x_1, x_2) \to \mathrm{Map}_X(x_0, x_2)$. $\square$

Our next goal is to prove the following result:

Proposition 8.2.8. *For every cubical quasicategory X, there exists an enriched category $\mathcal{M}X \in \mathsf{Cat}_{\mathcal{H}}$ defined as follows:*

- $\mathrm{Ob}\mathcal{M}X = X_0$*;*
- *for* $x_0, x_1 \colon \square^0 \to X$, $\mathcal{M}X(x_0, x_1) = \mathrm{Map}_X(x_0, x_1)$*;*
- *for* $x_0, x_1, x_2 \colon \square^0 \to X$, *the composition map is given by the homotopy class of maps* $c \colon \mathrm{Map}_X(x_0, x_1) \times \mathrm{Map}_X(x_1, x_2) \to \mathrm{Map}_X(x_0, x_2)$*;*
- *For* $x_0 \colon \square^0 \to X$, *the identity* $\square^0 \to \mathrm{Map}_X(x_0, x_0)$ *is the vertex corresponding to the edge* $x_0\sigma_1 \colon \square^1 \to X$.

We begin by showing unitality; this will require a preliminary lemma. Observe that for any $X, Y \in \mathsf{cSet}$, there is a map of bi-pointed cubical sets $\Sigma X \cup_1 \Sigma Y \to \Sigma X$ induced by the identity on X and the map $\Sigma Y \to \Sigma X$ which is constant at the vertex 1; likewise, we have a similar map $\Sigma X \cup_1 \Sigma Y \to \Sigma Y$.

Lemma 8.2.9. *The maps $\Sigma X \cup_1 \Sigma Y \to \Sigma X, \Sigma X \cup_1 \Sigma Y \to \Sigma Y$ described above both factor through the inclusion $\Sigma X \cup_1 \Sigma Y \hookrightarrow \Sigma X \bullet \Sigma Y$. Moreover, the composite $\Sigma(X \times Y) \hookrightarrow \Sigma X \bullet \Sigma Y \to \Sigma X$ is the image under Σ of the projection map $X \times Y \to Y$, and similarly for $\Sigma(X \times Y) \hookrightarrow \Sigma X \bullet \Sigma Y \to \Sigma Y$.*

Proof. We wish to construct a map $\pi_X \colon \Sigma X \bullet \Sigma Y \to \Sigma X$ such that the following diagram commutes:

$$\begin{tikzcd}
\Sigma X \cup_1 \Sigma Y \arrow[rr, hook] \arrow[dr] & & \Sigma X \bullet \Sigma Y \arrow[dl] \\
& \Sigma X &
\end{tikzcd}$$

By the universal property of the colimit which defines $\Sigma X \bullet \Sigma Y$, to construct this map it suffices to define it on $\Sigma X \cup_1 \Sigma Y$ and then coherently extend it to the $(n+2)$-cube $x \bullet y$ for each $n \geq 0$, $(x, y)\colon \square^n \to X \times Y$. Thus we define π_X as follows:

- $\pi_X 0 = 0, \pi_X 1 = 1, \pi_X 2 = 1$;
- For $n \geq 0$ and $x\colon \square^n \to X$, $\pi_X x = x$;
- For $n \geq 0$ and $y\colon \square^n \to Y$, $\pi_X y = 1$;
- For $n \geq 0$ and $(x, y)\colon \square^n \to X \times Y$, $\pi_X(x \bullet y) = x\gamma_{n+1}$.

That this choice is coherent follows from a routine computation involving (C1) through (C6) and the cubical identities. For instance, for $i \leq n$ we have:

$$\begin{aligned}
\pi_X(x \bullet y)\partial_{i,\varepsilon} &= x\gamma_{n+1,0}\partial_{i,\varepsilon} \\
&= x\partial_{i,\varepsilon}\gamma_{n,0} \\
&= \pi_X((x\partial_{i,\varepsilon}) \bullet (y\partial_{i,\varepsilon})) \\
&= \pi_X((x \bullet y)\partial_{i,\varepsilon})
\end{aligned}$$

That the restriction of π_X to $\Sigma X \cup_1 \Sigma Y$ is as specified is immediate from the definition. To see that its composite with $\Sigma(X \times Y) \hookrightarrow \Sigma X \bullet \Sigma Y$ is as specified, we compute, for $n \geq 0, x, y\colon \square^n \to X \times Y$:

$$\begin{aligned}\pi_X(x,y) &= \pi_X((x \bullet y)\partial_{n+2,0}) \\ &= \pi_X(x \bullet y)\partial_{n+2,0} \\ &= x\gamma_{n+1,0}\partial_{n+2,0} \\ &= x\end{aligned}$$

So this composite is indeed the image under Σ of the projection $X \times Y \to X$.

To construct the map $\pi_Y \colon \Sigma X \bullet \Sigma Y \to \Sigma Y$, we set:

- $\pi_Y 0 = 0, \pi_Y 1 = 0, \pi_Y 2 = 1$;
- For $n \geq 0$ and $x \colon \Box^n \to X$, $\pi_Y x = 0$;
- For $n \geq 0$ and $y \colon \Box^n \to Y$, $\pi_Y y = y$;
- For $n \geq 0$ and $(x,y) \colon \Box^n \to X \times Y$, $\pi_Y(x \bullet y) = y\sigma_{n+2}$.

The rest of the proof is similar to the proof for π_X. $\Box$

Lemma 8.2.10. *Let X be a cubical quasicategory, and $x_0, x_1 \colon \Box^0 \to X$. The map $c(\mathrm{id} \times x_1\sigma_1)$, induced by $\mathrm{id} \colon \mathrm{Map}_X(x_0,x_1) \to \mathrm{Map}_X(x_0,x_1)$ and $x_1\sigma_1 \colon \Box^0 \to \mathrm{Map}_X(x_1,x_1)$, is homotopic to the isomorphism $\mathrm{Map}_X(x_0,x_1) \times \Box^0 \cong \mathrm{Map}_X(x_0,x_1)$. Likewise, $c(x_0\sigma_1 \times \mathrm{id})$ is homotopic to the isomorphism on $\Box^0 \times \mathrm{Map}_X(x_0,x_1) \cong \mathrm{Map}_X(x_0,x_1)$.*

Proof. We prove the result for $c(\mathrm{id} \times x_1\sigma_1)$; the proof for $c(x_0\sigma_1 \times \mathrm{id})$ is similar.

By Lemma 8.2.7, we can construct $c(\mathrm{id} \times x_1\sigma_1)$ using a lift in the following diagram:

$$\begin{tikzcd}
\Sigma \mathrm{Map}_X(x_0, x_1) \cup_1 \Sigma \square^0 \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Sigma \mathrm{Map}_X(x_0, x_1) \bullet \Sigma \square^0 \arrow[r] & \square^0
\end{tikzcd}$$

where the top map is induced by the counit $\Sigma \mathrm{Map}_X(x_0, x_1) \to X$ and the edge $x_1\sigma_1 \colon \Sigma \square^0 \cong \square^1 \to X$. We will construct such a lift explicitly, and show that the resulting generalized composition map is the identity.

The map $\Sigma \mathrm{Map}_X(x_0, x_1) \cup_1 \Sigma \square^0 \to X$ factors through $\Sigma \mathrm{Map}_X(x_0, x_1) \cup_1 \Sigma \square^0 \to \Sigma \mathrm{Map}_X(x_0, x_1)$. Therefore, by Lemma 8.2.9, the diagram above factors as:

$$\begin{tikzcd}
\Sigma \mathrm{Map}_X(x_0, x_1) \cup_1 \Sigma \square^0 \arrow[r] \arrow[d, hook] & \Sigma \mathrm{Map}_X(x_0, x_1) \arrow[r] \arrow[d, equal] & X \arrow[d] \\
\Sigma \mathrm{Map}_X(x_0, x_1) \bullet \Sigma \square^0 \arrow[r] & \Sigma \mathrm{Map}_X(x_0, x_1) \arrow[r] & \square^0
\end{tikzcd}$$

The right-hand square has a unique lift, namely the counit $\Sigma \mathrm{Map}_X(x_0, x_1) \to X$. Thus $c(\mathrm{id} \times x_1\sigma_1)$ is homotopic to the adjunct of the composite $\Sigma(\mathrm{Map}_X(x_0, x_1) \times \square^0) \hookrightarrow \Sigma \mathrm{Map}_X(x_0, x_1) \bullet \Sigma \square^0 \to \Sigma \mathrm{Map}_X(x_0, x_1) \to X$. By Lemma 8.2.9, this adjunct map is the isomorphism $\mathrm{Map}_X(x_0, x_1) \times \square^0 \cong \mathrm{Map}_X(x_0, x_1)$. $\square$

Next we show associativity; this proof will be more involved. Our strategy will be to define, for any $X, Y, Z \in \mathsf{cSet}$, an object $\Sigma X \bullet \Sigma Y \bullet \Sigma Z$ containing both $\Sigma(X \times Y) \bullet \Sigma Z$ and $\Sigma X \bullet \Sigma(Y \times Z)$ as subobjects. For a cubical quasicategory X, lifting $X \to \square^0$ against a certain map into $\Sigma \mathrm{Map}_X(x_0, x_1) \bullet \Sigma \mathrm{Map}_X(x_1, x_2) \bullet$

$\Sigma\mathrm{Map}_X(x_2, x_3)$ will allow us to show that the diagram

$$\begin{tikzcd}
\mathrm{Map}_X(x_0,x_1) \times \mathrm{Map}_X(x_1,x_2) \times \mathrm{Map}_X(x_2,x_3) \arrow[r] \arrow[d] & \mathrm{Map}_X(x_0,x_2) \times \mathrm{Map}_X(x_2,x_3) \arrow[d] \\
\mathrm{Map}_X(x_0,x_1) \times \mathrm{Map}_X(x_1,x_3) \arrow[r] & \mathrm{Map}_X(x_0,x_3)
\end{tikzcd}$$

commutes up to homotopy.

Let $X, Y, Z \in \mathsf{cSet}$; as above we denote the basepoints of ΣX by $(0,1)$ and those of Y by $(1,2)$, while those of Z will be denoted $(2,3)$. We define the object $(\Sigma X \bullet \Sigma Y) \cup_{\Sigma Y} (\Sigma Y \bullet \Sigma Z)$ by the following pushout diagram in cSet:

$$\begin{tikzcd}
\Sigma Y \arrow[r, hook] \arrow[d, hook] & \Sigma Y \bullet \Sigma Z \arrow[d] \\
\Sigma X \bullet \Sigma Y \arrow[r] & (\Sigma X \bullet \Sigma Y) \cup_{\Sigma Y} (\Sigma Y \bullet \Sigma Z) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Next we define $\Sigma Z \bullet (\Sigma Y \bullet \Sigma Z)$ by the following pushout diagram:

$$\begin{tikzcd}
\Sigma X \cup_1 \Sigma(Y \times Z) \arrow[r, hook] \arrow[d, hook] & (\Sigma X \bullet \Sigma Y) \cup_{\Sigma Y} (\Sigma Y \bullet \Sigma Z) \arrow[d, hook] \\
\Sigma X \bullet \Sigma(Y \times Z) \arrow[r, hook] & \Sigma X \bullet (\Sigma Y \bullet \Sigma Z) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Lemma 8.2.11. *The inclusion* $(\Sigma X \bullet \Sigma Y) \cup_{\Sigma Y} (\Sigma Y \bullet \Sigma Z) \hookrightarrow \Sigma X \bullet (\Sigma Y \bullet \Sigma Z)$ *is a trivial cofibration.*

Proof. The map $\Sigma X \cup_1 \Sigma(Y \times Z) \hookrightarrow \Sigma X \bullet \Sigma(Y \times Z)$ is a trivial cofibration, hence so is its pushout. □

Now we will extend $\Sigma X \bullet (\Sigma Y \bullet \Sigma Z)$ to a cubical set $\Sigma X \bullet \Sigma Y \bullet \Sigma Z$ which contains both $\Sigma(X \times Y) \bullet \Sigma Z$ and $\Sigma X \bullet \Sigma(Y \times Z)$. We will do this by successive inner open box fillings, similar to the construction of $\Sigma X \bullet \Sigma Y$.

We proceed by induction. For $m \geq -1$, we will define a cubical set $(\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^m$ admitting a trivial cofibration $\Sigma X \bullet (\Sigma Y \bullet \Sigma Z) \hookrightarrow (\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^m$, and containing an $(n+3)$-cube $x \bullet y \bullet z$ for every triple of n-cubes $x \colon \square^n \to X, y \colon \square^n \to Y, z \colon \square^n \to Z$ with $n \leq m$, satisfying the following hypotheses:

(A1) for $1 \leq i \leq n$, $(x \bullet y \bullet z)\partial_{i,\varepsilon} = (x\partial_{i,\varepsilon}) \bullet (y\partial_{i,\varepsilon}) \bullet (z\partial_{i,\varepsilon})$;

(A2) $(x \bullet y \bullet z)\partial_{n+1,0} = x \bullet y$;

(A3) $(x \bullet y \bullet z)\partial_{n+1,1} = 3$;

(A4) $(x \bullet y \bullet z)\partial_{n+2,0} = x \bullet (y, z)$;

(A5) $(x \bullet y \bullet z)\partial_{n+2,1} = x\sigma_1$;

(A6) $(x \bullet y \bullet z)\partial_{n+3,1} = y \bullet z$;

(A7) for $0 \leq n \leq m-1$ and $1 \leq i \leq n+1$, $(x \bullet y \bullet z)\sigma_i = (x\sigma_i) \bullet (y\sigma_i) \bullet (z\sigma_i)$;

(A8) for $0 \leq n \leq m-1$ and $1 \leq i \leq n$, $(x \bullet y \bullet z)\gamma_{i,\varepsilon} = (x\gamma_{i,\varepsilon}) \bullet (y\gamma_{i,\varepsilon}) \bullet (z\gamma_{i,\varepsilon})$.

For the base case, we set $(\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^{-1} = \Sigma X \bullet (\Sigma Y \bullet \Sigma Z)$. Now let $m \geq 0$, and suppose we have defined $(\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^{m-1}$ satisfying the induction hypothesis. To construct $(\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^m$, we must construct a cube $x \bullet y \bullet z$ satisfying (A1) through (A8) for all $(x, y, z) \colon \square^m \to X \times Y \times Z$.

That the face specifications (A1) through (A6) form an inner open box $\widehat{\sqcap}^{m+3}_{m+3,0} \to (\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^{m-1}$ for each (x, y, z) follows from a series of routine computations similar to the proof of Lemma 8.2.2. Thus we define $(\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^m$ by the following pushout diagram, where $(X \times Y \times Z)^{\mathrm{nd}}_m$ denotes the set of non-degenerate m-cubes of $X \times Y \times Z$:

$$\begin{tikzcd}
\coprod\limits_{(X\times Y\times Z)^{\mathrm{nd}}_m} \widehat{\sqcap}^{m+3}_{m+3,0} \arrow[r] \arrow[d, hook] & (\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^{m-1} \arrow[d, hook] \\
\coprod\limits_{(X\times Y\times Z)^{\mathrm{nd}}_m} \widehat{\square}^{m+3}_{m+3,0} \arrow[r] & (\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^{m} \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

As before, we see that $(\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^{m-1} \hookrightarrow (\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^{m}$ is a trivial cofibration, as a pushout of a coproduct of trivial cofibrations.

Now we must define $x \bullet y \bullet z$ for all m-cubes $(x, y, z) \colon \square^m \to X \times Y \times Z$. For non-degenerate (x, y, z), $x \bullet y \bullet z$ is the filler constructed above. If $(x, y, z) = (x', y', z')\rho$ for some $n < m$, non-degenerate $(x', y', z') \colon \square^n \to X \times Y \times Z$, and epimorphism $\rho \colon [1]^m \to [1]^n$ in $\square$, then (A7) and (A8) require us to define $x \bullet y \bullet z = (x' \bullet y' \bullet z')(\rho \otimes \square^3)$.

Lemma 8.2.12. $(\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^m$ *satisfies (A1) through (A8).*

Proof. Similarly to the proof of Lemma 8.2.3, (A7) and (A8) are immediate from the definition, while for non-degenerate $(x, y, z) \colon \square^m \to X \times Y \times Z$, (A1) through (A6) hold by construction. Thus it suffices to consider the case $(x, y, z) = (x', y', z')\rho$ for $(x', y', z') \colon \square^n \to X \times Y \times Z$ non-degenerate and $\rho \colon \square^m \to \square^n$ an epimorphism in $\square$. We will show the computation for (A5).

We have the following commuting diagrams:

$$\begin{tikzcd}
\square^m \otimes \square^2 \arrow[r, "\square^m \otimes \partial_{2,0}"] \arrow[d, "\rho \otimes \square^2"'] & \square^m \otimes \square^3 \arrow[d, "\rho \otimes \square^3"] \\
\square^n \otimes \square^2 \arrow[r, "\square^n \otimes \partial_{2,0}"] & \square^n \otimes \square^3
\end{tikzcd}$$

$$\begin{tikzcd}
\square^m \otimes \square^2 \arrow[r, "\square^m \otimes \sigma_2"] \arrow[d, "\rho \otimes \square^2"'] & \square^m \otimes \square^1 \arrow[d, "\rho \otimes \square^1"] \\
\square^n \otimes \square^2 \arrow[r, "\square^n \otimes \sigma_2"] & \square^{\otimes} \square^1
\end{tikzcd}$$

Thus we can compute:

$$\begin{aligned}
(x \bullet y \bullet z)\partial_{m+2,0} &= (x' \bullet y' \bullet z')(\rho \otimes \square^3)(\square^m \otimes \partial_{2,0}) \\
&= (x' \bullet y' \bullet z')(\square^n \otimes \partial_{2,0})(\rho \otimes \square^2) \\
&= z'(\square^n \otimes \sigma_2)(\rho \otimes \square^2) \\
&= z'(\rho \otimes \square^1)(\square^m \otimes \sigma_2) \\
&= z'\rho\sigma_{m+2} \\
&= z\sigma_{m+2}
\end{aligned}$$

The remaining computations are routine, and are generally similar to those used in the proof of Lemma 8.2.3. □

Thus we see that $(\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^m$ satisfies the induction hypothesis.

We define $\Sigma X \bullet \Sigma Y \bullet \Sigma Z$ to be the colimit of the following diagram of inclusions:

$$\Sigma X \bullet (\Sigma Y \bullet \Sigma Z) \hookrightarrow (\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^0 \hookrightarrow \cdots \hookrightarrow (\Sigma X \bullet \Sigma Y \bullet \Sigma Z)^m \hookrightarrow \cdots$$

Lemma 8.2.13. *The inclusion* $(\Sigma X \bullet \Sigma Y) \cup_{\Sigma Y} (\Sigma Y \bullet \Sigma Z) \hookrightarrow \Sigma X \bullet \Sigma Y \bullet \Sigma Z$ *is a trivial cofibration.*

Proof. The inclusion $(\Sigma X \bullet \Sigma Y) \cup_{\Sigma Y} (\Sigma Y \bullet \Sigma Z) \hookrightarrow \Sigma X \bullet (\Sigma Y \bullet \Sigma Z)$ is a trivial cofibration by Lemma 8.2.11. The inclusion $\Sigma X \bullet (\Sigma Y \bullet \Sigma Z) \hookrightarrow \Sigma X \bullet \Sigma Y \bullet \Sigma Z$ is a trivial cofibration, as a transfinite composite of trivial cofibrations. The result thus follows by 2-out-of-3. $\square$

Lemma 8.2.14. *For* $(x, y, z) \colon \square^n \to X \times Y \times Z$*, the* $(n+2)$*-cubes* $(x \bullet y \bullet z)\partial_{n+3,0}$ *satisfy (C1) through (C6) with respect to* $(x, y) \colon \square^n \to X \times Y$ *and* $z \colon \square^n \to Z$*.*

Proof. For (x, y, z) as above, denote $(x \bullet y \bullet z)\partial_{n+3,0}$ by $(x, y) \bullet z$. We will verify the identities by direct computation.

For (C1), (C5), and (C6), let $\rho \colon \square^m \to \square^n$ be any map in $\square$. Then we can compute:

$$
\begin{aligned}
((x, y) \bullet z)(\rho \otimes \square^2) &= (x \bullet y \bullet z)(\square^n \otimes \partial_{3,0})(\rho \otimes \square^2) \\
&= (x \bullet y \bullet z)(\rho \otimes \square^3)(\square^m \otimes \partial_{3,0}) \\
&= ((x\rho) \bullet (y\rho) \bullet (z\rho))\partial_{m+3,0} \\
&= (x\rho, y\rho) \bullet (z\rho) \\
&= ((x, y)\rho) \bullet (z\rho)
\end{aligned}
$$

For (C2), we have:

$$\begin{aligned}((x,y)\bullet z)\partial_{n+1,0} &= (x\bullet y\bullet z)\partial_{n+3,0}\partial_{n+1,0}\\ &= (x\bullet y\bullet z)\partial_{n+1,0}\partial_{n+2,0}\\ &= (x\bullet y)\partial_{n+2,0}\\ &= (x,y)\end{aligned}$$

For (C3), note that the inclusion $\Sigma(X\times Y)\bullet\Sigma Z\hookrightarrow\Sigma X\bullet\Sigma Y\bullet\Sigma Z$ identifies the vertex 2 of $\Sigma(X\times Y)\bullet\Sigma Z$ with the vertex 3 of $\Sigma X\bullet\Sigma Y\bullet\Sigma Z$. Thus the desired result follows from the computation:

$$\begin{aligned}((x,y)\bullet z)\partial_{n+1,1} &= (x\bullet y\bullet z)\partial_{n+3,0}\partial_{n+1,1}\\ &= (x\bullet y\bullet z)\partial_{n+1,1}\partial_{n+2,0}\\ &= 3\partial_{n+2,0}\\ &= 3\end{aligned}$$

For (C4), we have:

$$\begin{aligned}((x,y)\bullet z)\partial_{n+2,0} &= (x\bullet y\bullet z)\partial_{n+3,0}\partial_{n+2,1}\\ &= (x\bullet y\bullet z)\partial_{n+2,1}\partial_{n+2,0}\\ &= z\sigma_{n+2}\partial_{n+2,0}\\ &= z\end{aligned}$$

Thus we see that these cubes satisfy all the necessary identities. □

Lemma 8.2.14 shows that the inclusion $\Sigma(X \times Y) \cup_2 \Sigma Z \hookrightarrow \Sigma X \bullet \Sigma Y \bullet \Sigma Z$ extends to a map $\Sigma(X \times Y) \bullet \Sigma Z \to \Sigma X \bullet \Sigma Y \bullet \Sigma Z$; using Corollary 4.1.5 we can see that this map is a monomorphism. Thus we have a pair of commuting diagrams:

$$\begin{tikzcd} \Sigma X \cup_1 \Sigma(Y \times Z) \arrow[r, hook] \arrow[d, hook] & (\Sigma X \bullet \Sigma Y) \cup_{\Sigma Y} (\Sigma Y \bullet \Sigma Z) \arrow[d, hook] \\ \Sigma X \bullet \Sigma(Y \times Z) \arrow[r, hook] & \Sigma X \bullet \Sigma Y \bullet \Sigma Z \end{tikzcd}$$

$$\begin{tikzcd} \Sigma(X \times Y) \cup_2 \Sigma Z \arrow[r, hook] \arrow[d, hook] & (\Sigma X \bullet \Sigma Y) \cup_{\Sigma Y} (\Sigma Y \bullet \Sigma Z) \arrow[d, hook] \\ \Sigma(X \times Y) \bullet \Sigma Z \arrow[r, hook] & \Sigma X \bullet \Sigma Y \bullet \Sigma Z \end{tikzcd}$$

(The top diagram arises from the pushout construction of $\Sigma X \bullet (\Sigma Y \bullet \Sigma Z)$, while the bottom map in the bottom diagram is the map described above.)

Thus, for any $(x, y, z) \colon \Box^n \to X \times Y \times Z$, $\Sigma X \bullet \Sigma Y \bullet \Sigma Z$ contains a cube of $\Sigma X \bullet \Sigma(Y \times Z)$ which is viewed as a composite of x with (y, z), and a cube of $\Sigma(X \times Y) \bullet \Sigma Z$ which is viewed as a composite of (x, y) with z. The following result shows that these cubes are equal.

Lemma 8.2.15. *The following diagram commutes:*

$$\begin{tikzcd} \Sigma(X \times Y \times Z) \arrow[r, hook] \arrow[d, hook] & \Sigma X \bullet \Sigma(Y \times Z) \arrow[d, hook] \\ \Sigma(X \times Y) \bullet \Sigma Z \arrow[r, hook] & \Sigma X \bullet \Sigma Y \bullet \Sigma Z \end{tikzcd}$$

Proof. For $(x, y, z) \colon \Box^n \to X \times Y \times Z$, the composite $\Sigma(X \times Y \times Z) \hookrightarrow \Sigma X \bullet \Sigma(Y \times Z) \hookrightarrow \Sigma X \bullet \Sigma Y \bullet \Sigma Z$ maps the $(n+1)$-cube (x, y, z) to $(x \bullet (y, z))\partial_{n+2,0} =$

$(x \bullet y \bullet z)\partial_{n+2,0}\partial_{n+2,0}$, while $\Sigma(X \times Y \times Z) \hookrightarrow \Sigma(X \times Y) \bullet \Sigma Z \hookrightarrow \Sigma X \bullet \Sigma Y \bullet \Sigma Z$ maps it to $((x, y) \bullet z)\partial_{n+2,0} = (x \bullet y \bullet z)\partial_{n+3,0}\partial_{n+2,0}$. By the cubical identities, these cubes are equal. $\square$

We are now able to prove associativity.

Lemma 8.2.16. *Let X be a cubical quasicategory, and $x_0, x_1, x_2, x_3 \colon \square^0 \to X$. The following diagram commutes up to homotopy:*

$$\begin{tikzcd}
\mathrm{Map}_X(x_0,x_1) \times \mathrm{Map}_X(x_1,x_2) \times \mathrm{Map}_X(x_2,x_3) \arrow[r] \arrow[d] & \mathrm{Map}_X(x_0,x_2) \times \mathrm{Map}_X(x_2,x_3) \arrow[d] \\
\mathrm{Map}_X(x_0,x_1) \times \mathrm{Map}_X(x_1,x_3) \arrow[r] & \mathrm{Map}_X(x_0,x_3)
\end{tikzcd}$$

Proof. We take as given a specific pair of maps $\Sigma\mathrm{Map}_X(x_0, x_1) \bullet \Sigma\mathrm{Map}_X(x_1, x_2) \to X, \Sigma\mathrm{Map}_X(x_1, x_2) \bullet \Sigma\mathrm{Map}_X(x_2, x_3) \to X$ inducing the composition maps. For clarity and consistency with the results developed above, we will denote the basepoints of $\Sigma\mathrm{Map}_X(x_i, x_j)$ by (i, j) for $0 \leq i \leq j \leq 3$. Furthermore, for the sake of brevity, the mapping spaces $\mathrm{Map}_X(x_i, x_j)$ will be denoted $X(i, j)$.

We begin by taking a lift in the diagram

$$\begin{tikzcd}
(\Sigma X(0,1) \bullet \Sigma X(1,2)) \cup_{\Sigma X(1,2)} (\Sigma X(1,2) \bullet \Sigma X(2,3)) \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Sigma X(0,1) \bullet \Sigma X(1,2) \bullet \Sigma X(2,3) \arrow[r] & \square^0
\end{tikzcd} \qquad (\dagger)$$

By Lemma 8.2.13, the left-hand map is a trivial cofibration, so a lift does indeed exist.

By Lemma 8.2.7, to prove the stated result it suffices to construct lifts for the diagrams

$$\begin{tikzcd}
\Sigma(X(0,1) \times X(1,2)) \cup_2 \Sigma X(2,3) \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Sigma(X(0,1) \times X(1,2)) \bullet \Sigma X(2,3) \arrow[r] & \square^0
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma X(0,1) \cup_1 \Sigma(X(1,2) \times X(2,3)) \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Sigma X(0,1) \bullet \Sigma(X(1,2) \times X(2,3)) \arrow[r] & \square^0
\end{tikzcd}$$

and show that the composites of these lifts with the relevant inclusions of $\Sigma(X(0,1) \times X(1,2) \times X(2,3))$ are equal.

Both of these diagrams factor through (†), so the composite maps

$$\Sigma(X(0,1) \times X(1,2)) \bullet \Sigma X(2,3) \to \Sigma X(0,1) \bullet \Sigma X(1,2) \bullet \Sigma X(2,3) \to X$$

and

$$\Sigma X(0,1) \bullet \Sigma(X(1,2) \times X(2,3)) \to \Sigma X(0,1) \bullet \Sigma X(1,2) \bullet \Sigma X(2,3) \to X$$

define the necessary lifts. The composites of these maps with the inclusions of $\Sigma(X(0,1) \times X(1,2) \times X(2,3))$ are indeed equal by Lemma 8.2.15. $\square$

Proof of Proposition 8.2.8. The composition operation in $\mathcal{M}X$ is unital by Lemma 8.2.10, and associative by Lemma 8.2.16. $\square$

Given a cubical quasicategory X with vertices x_0, x_1, the morphisms from x_0 to x_1 in $\mathcal{M}X$, i.e. maps $\square^0 \to \mathrm{Map}_X(x_0, x_1)$ in $\mathcal{H}$, are homotopy classes of maps $\square^0 \to \mathrm{Map}_X(x_0, x_1)$ in cSet. A pair of edges f and g from x_0 to x_1 are homotopic as maps $\square^0 \to \mathrm{Map}_X(x_0, x_1)$ if and only if there is a 2-cube in X of the form

$$\begin{tikzcd} x_0 \arrow[r, equal] \arrow[d, "f"'] & x_0 \arrow[d, "g"] \\ x_1 \arrow[r, equal] & x_1 \end{tikzcd}$$

Applying Lemma 5.3.4, we thus see that the morphisms of the enriched category $\mathcal{M}X$ coincide with those of the ordinary category $\mathsf{Ho}\,X$. The following result shows that the composition operations on morphisms in $\mathcal{M}X$ and $\mathsf{Ho}\,X$ also coincide, and that $\mathcal{M}X$ can thus be viewed as an enrichment of $\mathsf{Ho}\,X$ over $\mathcal{H}$.

Proposition 8.2.17. *Let X be a cubical quasicategory, $x_0, x_1, x_2 \colon \square^0 \to X$, f an edge from x_0 to x_1, g an edge from x_1 to x_2, and h an edge from x_0 to x_2. Then $h \colon \square^0 \to \operatorname{Map}_X(x_0, x_1)$ is homotopic to $c(f \times g)$ if and only if $h = gf$ in $\mathsf{Ho}\,X$.*

Proof. The inclusion $\Sigma\,\square^0 \cup_1 \Sigma\square^0 \hookrightarrow \Sigma\,\square^0 \bullet \Sigma\square^0$ is isomorphic to the inclusion $\widehat{\sqcap}^2_{2,0} \hookrightarrow \widehat{\square}^2_{2,0}$. Therefore, by Lemma 8.2.7, we may obtain $c(f \times g)$ using a lift in the following diagram:

$$\begin{tikzcd} \widehat{\sqcap}^2_{2,0} \arrow[r, "{(f,g)}"] \arrow[d, hook] & X \arrow[d] \\ \widehat{\square}^2_{2,0} \arrow[r] & \square^0 \end{tikzcd}$$

A lift for this diagram consists of an edge h from x_0 to x_1 together with a 2-cube in X of the form:

$$\begin{tikzcd} x_0 \arrow[r, "h"] \arrow[d, "f"'] & x_2 \arrow[d, equal] \\ x_1 \arrow[r, "g"] & x_2 \end{tikzcd}$$

The inclusion $\Sigma(\square^0 \times \square^0) \hookrightarrow \Sigma\,\square^0 \bullet \Sigma\square^0$ is isomorphic to the map $\partial_{2,0} \colon \square^1 \to \widehat{\square}^2_{2,0}$. Therefore, applying Lemma 5.3.4, we see that, if $h = gf$ in $\mathsf{Ho}\,X$, then the adjunct map $h\colon \square^0 \to \mathrm{Map}_X(x_0, x_1)$ is homotopic to $c(f \times g)$.

On the other hand, suppose that for some $h\colon \square^0 \to \mathrm{Map}_X(x_0, x_1)$ we have $h \sim c(f \times g)$. By Lemma 8.2.7, we may assume that $c(f \times g)$ arises from a lift as above, so that $c(f \times g) = gf$ in $\mathsf{Ho}\, X$. By the definition of $\mathrm{Map}_X(x_0, x_2)$, $h \sim c(f \times g)$ means that there is a 2-cube in X of the form:

$$\begin{tikzcd} x_0 \arrow[r, equal] \arrow[d, "h"'] & x_0 \arrow[d, "c(f\times g)"] \\ x_2 \arrow[r, equal] & x_2 \end{tikzcd}$$

Therefore, $h = c(f \times g) = gf$ in $\mathsf{Ho}\, X$. □

Corollary 8.2.18. *An edge of a cubical quasicategory X is invertible in $\mathcal{M}X$ if and only if it is an equivalence in X.* □

Next we extend the assignment $X \mapsto \mathcal{M}X$ to a functor $\mathcal{M}\colon \mathsf{cqCat} \to \mathsf{Cat}_{\mathcal{H}}$.

Lemma 8.2.19. *Given a map of cubical quasicategories $f\colon X \to Y$, the induced maps $X_0 \to Y_0$ and $\mathrm{Map}_X(x_0, x_1) \to \mathrm{Map}_Y(fx_0, fx_1)$ for $x_0, x_1\colon \square^0 \to X$ define an enriched functor $\mathcal{M}f\colon \mathcal{M}X \to \mathcal{M}Y$.*

Proof. It is clear that $\mathcal{M}f$ preserves identities. Now we must show that it respects composition, i.e. that the following diagram commutes up to homotopy:

$$\begin{tikzcd} \mathrm{Map}_X(x_0, x_1) \times \mathrm{Map}_X(x_1, x_2) \arrow[r] \arrow[d] & \mathrm{Map}_X(x_0, x_2) \arrow[d] \\ \mathrm{Map}_Y(fx_0, fx_1) \times \mathrm{Map}_Y(fx_1, fx_2) \arrow[r] & \mathrm{Map}_Y(fx_0, fx_2) \end{tikzcd}$$

By Lemma 8.2.7, the composite $\mathrm{Map}_X(x_0, x_1) \times \mathrm{Map}_X(x_1, x_2) \to \mathrm{Map}_Y(fx_0, fx_1) \times \mathrm{Map}_Y(fx_1, fx_2) \to \mathrm{Map}_Y(fx_0, fx_2)$ can be constructed using a lift in the following diagram:

$$\begin{tikzcd}
\Sigma \mathrm{Map}_X(x_0,x_1) \cup_1 \Sigma \mathrm{Map}_X(x_1,x_2) \arrow[r] \arrow[d, hook] & Y \arrow[d] \\
\Sigma \mathrm{Map}_X(x_0,x_1) \bullet \Sigma \mathrm{Map}_X(x_1,x_2) \arrow[r] & \square^0
\end{tikzcd}$$

The adjunct of $\mathrm{Map}_X(x_0,x_1) \to \mathrm{Map}_Y(fx_0,fx_1)$ is the composite of $f\colon X \to Y$ with the counit $\Sigma \mathrm{Map}_X(x_0,x_1) \to X$. Thus this diagram factors as:

$$\begin{tikzcd}
\Sigma \mathrm{Map}_X(x_0,x_1) \cup_1 \Sigma \mathrm{Map}_X(x_1,x_2) \arrow[r] \arrow[d, hook] & X \arrow[r] \arrow[d] & Y \arrow[d] \\
\Sigma \mathrm{Map}_X(x_0,x_1) \bullet \Sigma \mathrm{Map}_X(x_1,x_2) \arrow[r] & \square^0 \arrow[r, equal] & \square^0
\end{tikzcd}$$

We can obtain a lift in the composite diagram by taking a lift in the left-hand square and composing with f. Thus we see that the composite $\mathrm{Map}_X(x_0,x_1) \times \mathrm{Map}_X(x_1,x_2) \to \mathrm{Map}_Y(fx_0,fx_1) \times \mathrm{Map}_Y(fx_1,fx_2) \to \mathrm{Map}_Y(fx_0,fx_2)$ is homotopic to the adjunct of $\Sigma(\mathrm{Map}_X(x_0,x_1) \times \mathrm{Map}_X(x_1,x_2)) \hookrightarrow \Sigma \mathrm{Map}_X(x_0,x_1) \bullet \Sigma \mathrm{Map}_X(x_1,x_2) \to X \to Y$ – but this adjunct is precisely the composite $\mathrm{Map}_X(x_0,x_1) \times \mathrm{Map}_X(x_1,x_2) \to \mathrm{Map}_X(x_0,x_2) \to \mathrm{Map}_Y(fx_0,fx_2)$. □

In view of Proposition 8.2.17, we have the following commuting diagram of functors, where $\mathsf{Cat}_{\mathcal{H}} \to \mathsf{Cat}$ is the forgetful functor taking a category enriched over $\mathcal{H}$ to the underlying ordinary category:

$$\begin{tikzcd}
\mathsf{cqCat} \arrow[rr, "\mathcal{M}"] \arrow[dr, "\mathsf{Ho}"'] & & \mathsf{Cat}_{\mathcal{H}} \arrow[dl] \\
& \mathsf{Cat} &
\end{tikzcd}$$

Theorem 8.1.10 can be interpreted as the statement that the categorical equivalences are created by the functor $\mathcal{M}$, in the following sense.

Theorem 8.2.20. *A map of cubical quasicategories $f\colon X \to Y$ is a categorical equivalence if and only if $\mathcal{M}f\colon \mathcal{M}X \to \mathcal{M}Y$ is an equivalence of enriched categories.*

Proof. By Corollary 8.2.18, $\mathsf{Ho}\, f$ is essentially surjective if and only if $\mathcal{M}f$ is essentially surjective. A map between cubical Kan complexes is an isomorphism in $\mathcal{H}$ if and only if it is a homotopy equivalence in cSet, so each map $\mathrm{Map}_X(x_0, x_1)$ is a homotopy equivalence if and only if $\mathcal{M}f$ is fully faithful. The result thus follows from Theorem 8.1.10 and Proposition 8.1.16. $\square$

Chapter 9

Cubical models of (∞, n)-categories

This chapter concerns the *comical model structures*, a family of model structures on the category of marked cubical sets which model (∞, n)-categories, analogous to the complicial model structures on marked simplicial sets. We construct these model structures via Cisinski-Olschok theory, and prove that they are Quillen-equivalent to the corresponding complicial model structures via the marked triangulation adjunction described in Section 4.3. Our overall approach is similar to that which we used for the unmarked case in Chapter 7: we extend the theory of cones developed there to the marked case, use it to develop an adjunction $Q : \mathsf{sSet}^+ \rightleftarrows \mathsf{cSet}^+ : \int$. We then show that $Q \dashv \int$ is a Quillen equivalence, and use a natural weak equivalence $TQ \Rightarrow \mathrm{id}$ to show that the same holds for $T \dashv U$. Note that this functor Q does *not* agree with the functor $Q \colon \mathsf{sSet} \to \mathsf{cSet}$ developed in Chapter 7 on underlying cubical sets; rather, it is an extension of the functor $Q_{L,0} \circ (-)^{\mathrm{op}}$ to the marked case (cf. Proposition 7.2.5 and Remark 7.2.6). The reason for this choice is that this alternate version of Q has more convenient combinatorial properties which will be of use in our proofs; see Remark 9.3.2 for instance.

We begin by constructing the comical model structures in Section 9.1. In Section 9.2, we show that marked triangulation defines a left Quillen functor between the comical and complicial model structures. In Section 9.3 we construct the marked version of Q described above, and in Section 9.4 we show that Q and T are left Quillen equivalences, following the structure of Section 7.2.

9.1 Model structure for comical sets

In this section, we introduce the notion of a comical sets and construct a model structure on cSet^+ whose fibrant-cofibrant objects are precisely the comical sets. As indicated in the introduction, our definition differs slightly from the corresponding definition given in [CKM20].

We begin by defining the (co)domains of our tentative anodyne maps.

Definition 9.1.1.

(i) For $n \geq 1$, $\varepsilon \in \{0,1\}$, and $1 \leq i \leq n$, the (i,ε)-*comical cube* in dimension n, denoted $\square^n_{i,\varepsilon}$, is the marked cubical set with underlying cubical set $\square^n$ in which a non-degenerate m-cube $\delta \colon \square^m \to \square^n$ is unmarked if and only if at least one of the following three conditions holds:

(a) the standard form of δ contains $\partial_{i,\varepsilon}$ or $\partial_{i,1-\varepsilon}$;

(b) for some $j > i$, the standard form of δ contains $\partial_{j,\varepsilon}$, as well as $\partial_{k,1-\varepsilon}$ for all $j > k > i$;

(c) for some $j < i$, the standard form of δ contains $\partial_{j,\varepsilon}$, as well as $\partial_{k,1-\varepsilon}$ for all $j < k < i$;

(ii) The (i,ε)-*comical open box* in dimension n, denoted $\sqcap^n_{i,\varepsilon}$, is the regular subcomplex of $\square^n_{i,\varepsilon}$ whose underlying cubical set is the n-dimensional (i,ε)-open box. The marked cubical set $(\square^n_{i,\varepsilon})'$ is obtained from $\square^n_{i,\varepsilon}$ by marking all $(n-1)$-cubes except for $\partial_{i,\varepsilon}$.

Definition 9.1.2.

- The (i,ε)-*comical open box inclusion* is the inclusion $\sqcap^n_{i,\varepsilon} \hookrightarrow \square^n_{i,\varepsilon}$.
- For $n \geq 2$, the *elementary* (i,ε)-*comical marking extension* is the entire map $(\square^n_{i,\varepsilon})' \to \tau_{n-2}\square^n_{i,\varepsilon}$.

Remark 9.1.3. Note that in general, this definition of comical cubes and open boxes does not coincide with that given in [CKM20], as that definition only includes conditions (b) and (c) in the cases $k = i + 1$ and $k = i - 1$, respectively. While it is not currently known whether the model structures of Theorem 9.1.7 coincide with those developed in [CKM20, Thms. 3.3 & 3.6], the comical open box inclusions and marking extensions of [CKM20] are pushouts of those defined above. As a result, we can apply closure results for anodyne maps from [CKM20] in our setting, although it will take additional work to establish such closure results in our setting.

We next define our cubical analogues of the Rezk maps; this definition is somewhat more involved than its simplicial counterpart.

Definition 9.1.4. For $x, y \in \{1, 2\}$, let $\overline{L}_{x,y}$ be given by the following pushout in cSet:

$$\begin{tikzcd} \square^1 \arrow[r, "\partial_{y,0}"] \arrow[d, "\partial_{x,1}"'] & \square^2 \arrow[d] \\ \square^2 \arrow[r] & \overline{L}_{x,y} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Let X and Y denote the 2-cubes of $\overline{L}_{x,y}$ given by the bottom and right maps in the diagram above, respectively. The marked cubical set $L_{x,y}$ has $\overline{L}_{x,y}$ as its underlying cubical set, with both non-degenerate 2-cubes X and Y marked, as well as the 1-cubes $X\partial_{x,0}, X\partial_{2-x+1,1}, Y\partial_{y,1}$, and $Y\partial_{2-y+1,0}$.

Let $L'_{x,y} = \tau_0 L_{x,y}$, i.e. the simplicial set obtained by marking the three unmarked edges of $L_{x,y}$. The (x,y)-*elementary Rezk map* is the entire map $L_{x,y} \to L'_{x,y}$.

In general, a *Rezk map* is any map of the form

$$(\partial\Box^m \hookrightarrow \Box^m)\hat{\otimes}(L_{x,y} \to L'_{x,y})\hat{\otimes}(\partial\Box^n \hookrightarrow \Box^n)$$

for $x, y \in \{1, 2\}$, $m, n \geq 0$.

As in the simplicial case, the Rezk maps capture the principle that a cube representing an invertible higher morphism should be marked. To better understand their definition, we illustrate the marked cubical set $L_{1,1}$:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\sim"'] \arrow[dr, phantom, "\sim"] & \bullet \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\sim"] & \bullet \arrow[d, "\sim"] \\
\bullet \arrow[r, "\sim"] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

We can now define comical sets, which will be the fibrant objects of our model structure.

Definition 9.1.5. A *comical set* is a marked cubical set having the right lifting property with respect to all comical open-box fillings and elementary comical marking extensions.

Definition 9.1.6. A comical set is:

- *saturated* if it has the right lifting property with respect to all Rezk maps;

- *n-trivial*, for $n \geq 0$, if it has the right lifting property with respect to all markings $\square^m \to \widetilde{\square}^m$ for $m > n$ (in other words, if all of its cubes of dimension greater than n are marked).

We are now ready to construct the desired model structures on cSet^+.

Theorem 9.1.7. *The category* cSet^+ *carries the following model structures:*

(i) A model structure for comical sets *in which*

- *cofibrations are monomorphisms;*
- *fibrant objects are comical sets;*
- *fibrations with fibrant codomain are characterized by the right lifting property with respect to comical open box inclusions and comical marking extensions.*

(ii) A model structure for saturated comical sets *in which*

- *cofibrations are monomorphisms;*
- *fibrant objects are saturated comical sets;*
- *fibrations with fibrant codomain are characterized by the right lifting property with respect to comical open box inclusions, comical marking extensions, and the Rezk maps.*

(iii) A model structure for n-trivial comical sets *for $n \geq 0$ in which*

- *cofibrations are monomorphisms;*
- *fibrant objects are n-trivial comical sets;*
- *fibrations with fibrant codomain are characterized by the right lifting property with respect to comical open box inclusions, comical marking extensions, and markings $\square^m \to \widetilde{\square}^m$ for $m > n$.*

(iv) A model structure for n-trivial saturated comical sets *in which*

- *cofibrations are monomorphisms;*
- *fibrant objects are n-trivial saturated comical sets;*
- *fibrations with fibrant codomain are characterized by the right lifting property with respect to comical open box inclusions, comical marking extensions, Rezk maps, and markings $\square^m \to \widetilde{\square}^m$ for $m > n$.*

All of these model structures are monoidal with respect to the lax Gray tensor product.

Proof. In all four cases, we will apply the Cisinski–Olschok theory (cf. Theorem 2.2.18) with the cellular model I of Lemma 4.3.14 with the cylinder functor given by $\widetilde{\square}^1 \otimes -$ and the natural transformations ∂^0, ∂^1 induced by face inclusions $\varphi\partial_{1,0}, \varphi\partial_{1,1} \colon \square_0 \to \widetilde{\square}^1$. In each case, the generating set S of anodyne maps is chosen differently.

By duality, it suffices to check that for any maps $f \in S$ and $g \in I$, the pushout product $f \hat{\otimes} g$ is again in the saturation of S. Thus we need to address the following eight cases:

$f \hat{\otimes} g$	$\partial\square^n \to \square^n$	$\square^n \to \widetilde{\square}^n$
$\sqcap^m_{i,\varepsilon} \hookrightarrow \square^n_{i,\varepsilon}$	1	2
$(\square^m_{i,\varepsilon})' \to \tau_{n-2}\square^n_{i,\varepsilon}$	3	4
$-\hat{\otimes}(L_{x,y} \to L'_{x,y})\hat{\otimes}-$	5	6
$\square^k \to \widetilde{\square}^k, k > m$	7	8

Case 5 is clear by the definition of the Rezk maps. Cases 4, 6, and 8 are pushout products of two entire maps, and hence isomorphisms by Lemma 4.3.4. Case 7 is clear since it is an entire map with no markings added in dimension m or below, and hence a pushout of markers in dimension above m.

For case 1, we consider the pushout product $(\sqcap^m_{i,\varepsilon} \hookrightarrow \square^m_{i,\varepsilon})\hat{\otimes}(\partial\square^n \to \square^n)$, which is regular by Lemma 4.3.4 and an open box inclusion on the underlying cubical sets. It therefore suffices to show that $\square^m_{i,\varepsilon} \otimes \square^n$ is a pushout of $\sqcap^{m+n}_{i,\varepsilon} \hookrightarrow \square^{m+n}_{i,\varepsilon}$. To see this, we consider the normal form of one of the faces of $\square^m_{i,\varepsilon} \otimes \square^n$, say given by $\partial_{a_1,\varepsilon_1} \ldots \partial_{a_p,\varepsilon_p} \partial_{b_1,\varepsilon'_1} \ldots \partial_{b_q,\varepsilon'_q}$, with $a_p \geq m+1$ and $b_q \leq m$. By the characterization of cubes in the geometric product, we obtain that this corresponds to a pair

$$(\partial_{b_1,\varepsilon'_1} \ldots \partial_{b_q,\varepsilon'_q}, \partial_{a_1-m,\varepsilon_1} \ldots \partial_{a_p-m,\varepsilon_p}) \in (\square^m_{i,\varepsilon})_{m-p} \times (\square^n)_{n-q}.$$

If the normal form $\partial_{a_1,\varepsilon_1} \ldots \partial_{a_p,\varepsilon_p} \partial_{b_1,\varepsilon'_1} \ldots \partial_{b_q,\varepsilon'_q}$ does not include any of the strings excluded by the definition of a comical cube, then neither does its terminal segment $\partial_{b_1,\varepsilon'_1} \ldots \partial_{b_q,\varepsilon'_q}$.

For case 2, we consider the pushout product $(\sqcap^m_{i,\varepsilon} \hookrightarrow \square^m_{i,\varepsilon})\hat{\otimes}(\square^n \to \widetilde{\square}^n)$, which is entire by Lemma 4.3.4. By definition, this is $\square^m_{i,\varepsilon} \otimes \square^n \cup \sqcap^m_{i,\varepsilon} \otimes \widetilde{\square}^n \to \square^m_{i,\varepsilon} \otimes \widetilde{\square}^n$. A face of $\square^m_{i,\varepsilon} \otimes \widetilde{\square}^n$ is marked either if its normal form does not contain any strings excluded by the definition of a comical m-cube or it does not contain any face maps with indices greater than m. Cubes satisfying the first condition are marked in the domain as well, but of the cubes satisfying the second condition, the face $\partial_{i,\varepsilon}$ is unmarked. Thus this map is a pushout of the comical marking extension $(\square^{m+n}_{i,\varepsilon})' \to \tau_{m+n-2}\square^{m+n}_{i,\varepsilon}$.

Finally, for case 3, we consider the pushout product $((\square^m_{i,\varepsilon})' \to \tau_{m-2}\square^m_{i,\varepsilon})\hat{\otimes}(\partial\square^n \to \square^n)$, which is once again entire by Lemma 4.3.4. By definition, this is the map $(\square^m_{i,\varepsilon})' \otimes \square^n \cup \tau_{m-2}\, \square^m_{i,\varepsilon} \otimes \partial\square^n \to \tau_{m-2}\, \square^m_{i,\varepsilon} \otimes \square_n$. A face is marked in the codomain if its normal form does not contain any of the strings excluded by the definition of a (i,ε)-cube or it contains at most one face map $\partial_{j,\mu}$ for $j \leq m$. The only one of these maps to be unmarked in the domain is $\partial_{i,\varepsilon}$, and hence the desired map is a pushout of a comical marking extension. $\square$

By construction, the weak equivalences of these model structures can be characterized by inducing a bijection on sets of homotopy classes of maps, where the notion of homotopy is induced by the cylinder $\widetilde{\square}^1 \otimes -$.

Corollary 9.1.8. *The weak equivalences of the model structure for (n-trivial, saturated) comical sets are maps $X \to Y$ inducing bijections $[Y, Z] \to [X, Z]$ for all (n-trivial, saturated) comical sets Z.* $\square$

In comparing the model structures of Theorem 9.1.7 with those of Example 3.3.16, we will make use of the following basic results.

Proposition 9.1.9. *For each of the model structures of Theorem 9.1.7, the self-adjunctions arising from the involutions $(-)^{\vdots}, (-)^{\mathrm{co-op}}, (-)^{\mathrm{op}} \colon \mathsf{cSet}^+ \to \mathsf{cSet}^+$ are Quillen self-equivalences.*

Proof. By Corollary 2.1.39, it suffices to show that these functors are left Quillen. For this, it suffices to show that they preserve the classes of comical open box inclusions, comical marking extensions, Rezk maps, and markers. For the elementary Rezk maps, it is easy to see that each of these involutions sends each map $L_{x,y} \to L'_{x,y}$ to some map $L_{x',y'} \to L'_{x',y'}$; the result for general Rezk maps then follows from Proposition 4.3.7. For the other three classes, it is immediate from the definitions. $\square$

9.2 Triangulation is a Quillen functor

In this section, we will show that the adjunction $T : \mathsf{cSet}^+ \rightleftarrows \mathsf{sSet}^+ : U$ is Quillen, where cSet^+ is equipped with any of the comical model structures described in the previous section, and sSet^+ is equipped with the corresponding complicial model structure.

Lemma 9.2.1. *Let $X \in \mathsf{sSet}^+$, and let $\overline{X}$ denote the precomplicial reflection of X. Let S be some set of simplices of X which are marked in $\overline{X}$, and let $X^\dagger$ denote the marked simplicial set obtained from X by marking all simplices of S. Then the entire map $X \to X^\dagger$ is a trivial cofibration.*

Proof. Any entire map is a cofibration. To see that $X \to X^\dagger$ is a weak equivalence, consider the following commuting diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \overline{X} \arrow[d, equal] \\ X^\dagger \arrow[r] & \overline{X} \end{tikzcd}$$

The map $X \to \overline{X}$ is a trivial cofibration. The square above is a pushout diagram, thus $X^\dagger \to \overline{X}$ is a trivial cofibration as well. It follows that $X \to X^\dagger$ is a weak equivalence by two-out-of-three. □

We will use $(\sqcap^n_{i,\varepsilon})^\flat$ to denote the minimal marking of the (i,ε)-open box. Note that although $(\sqcap^n_{i,\varepsilon})^\flat$ is minimally marked, its triangulation $T(\sqcap^n_{i,\varepsilon})^\flat$ is not.

Lemma 9.2.2. *Let $\phi \colon \Delta^m \to T\square^n$, and suppose that for some $i, j \in \{1, \ldots, n\}$ we have $\phi_i \leq \phi_j$. Then for any $l \leq m$ and any (composite) face map $\delta \colon \Delta^l \to \Delta^m$ we have $(\phi\delta)_i \leq (\phi\delta)_j$.*

Proof. It suffices to consider the case $l = m - 1, \delta = \partial_k$ for some $1 \leq k \leq m$. If $k < \phi_i$, then both ϕ_i and ϕ_j are lowered by 1 in computing $\phi\partial_k$, thus the inequality is preserved. Likewise, if $k \geq \phi_j$ then both ϕ_i and ϕ_j are unchanged in $\phi\partial_k$. On the other hand, if $\phi_i \leq k < \phi_j$, then ∂_k lowers ϕ_j by 1 while leaving ϕ_i unchanged. But in this case $\phi_i < \phi_j$, implying $\phi_i \leq \phi_j - 1$. □

Definition 9.2.3. Given a simplex $\phi\colon \Delta^m \to T\square^n$, a *complete substring* of ϕ is an order-preserving map $\rho\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that the composite $\phi\rho$ is equal to the inclusion $\{1, \ldots, m\} \hookrightarrow \{1, \ldots, m, \pm\infty\}$.

Intuitively, a complete substring ρ of a simplex ϕ is an increasing sequence of positions ρ_i such that $\phi_{\rho_i} = i$ for all i. For instance, the string $1\,3\,3\,2\,3$ has one complete substring, given by taking its first, fourth and fifth entries, while the string $1\,4\,2\,3\,3$ has none.

A simplex of $T\square^n$ is marked if and only if it has no complete substrings.

We will also have occasion to consider the images of simplices of triangulated cubes under cubical face maps $T\partial_{i,\varepsilon}\colon T\square^{n-1} \to T\square^n$. For $\phi\colon \Delta^m \to T\square^{n-1}$, the simplex $\partial_{i,\varepsilon}\phi$ is defined as follows:

$$(\partial_{i,\varepsilon}\phi)_j = \begin{cases} \phi_i, & j < i \\ +\infty & j = i, \varepsilon = 0 \\ -\infty & j = i, \varepsilon = 1 \\ \phi_{j-1}, & j > i \end{cases}$$

Definition 9.2.4. For $1 \leq m \leq n$, given a (composite) face map $\delta\colon \square^m \to \square^n$, the *linear simplex* of $T\square^n$ associated to δ, denoted ι_δ, is the image under δ of the m-simplex ι_m.

Lemma 9.2.5. *For $1 \leq m \leq n$, $1 \leq i \leq n, \varepsilon \in \{0, 1\}$, and a face map $\delta\colon \square^m \to \square^n$, the linear simplex associated to δ is marked in $T\square^n_{i,\varepsilon}$ if and only if δ is marked when viewed as an m-cube of $\square^n_{i,\varepsilon}$.*

Proof. This is immediate from the definition of cubical triangulation. $\square$

Example 9.2.6. We consider some examples of linear simplices to better illustrate the concept.

- For $\delta = \partial_{2,0} \colon \square^2 \to \square^3$, $\iota_\delta = 1 + 2$.
- For $\delta = \partial_{5,0}\partial_{2,1}\partial_{1,0} \colon \square^3 \to \square^6$, $\iota_\delta = + - 1\,2 + 3$.
- For any n, $\iota_{\mathrm{id}_{[1]^n}} = \iota_n$.

The following results are immediate from our earlier characterization of the actions of cubical face maps.

Lemma 9.2.7. *An m-simplex $\phi \colon \Delta^m \to T\square^n$ is linear if and only if it has a unique complete substring ρ, and for any $1 \leq i \leq n$ not in the image of ρ, $\phi_i \in \{\pm\infty\}$.* □

Lemma 9.2.8. *Let $\phi = \iota_\delta \colon \Delta^m \to T\square^n$ be the linear simplex associated to a cubical face map $\delta \colon \square^m \to \square^n$, and let ρ denote the unique complete substring of ϕ. Then $\delta = \partial_{i_1,\varepsilon_1} \ldots \partial_{i_{n-m},\varepsilon_{n-m}}$, where:*

- *the indices $i_1 > \ldots > i_{n-m}$ range over $\{1, \ldots, n\} \smallsetminus \mathrm{Im}\rho$;*
- *for $1 \leq i \leq n - m$, $\varepsilon_i = 0$ if $\phi_i = +$, while $\varepsilon_i = 1$ if $\phi_i = -$.* □

Lemma 9.2.9. *For $n \geq 1$ and $1 \leq i \leq n$, let ϕ be a linear simplex of $T\square^n$, and let ρ denote its unique complete substring. Suppose that i is in the image of ρ. If for all $\rho_{\phi_i - 1} < k < \rho_{\phi_i + 1}$ such that $k \neq i$ we have $\phi_k = -$ (resp. $\phi_k = +$), then ϕ is marked in $T\square^n_{i,0}$ (resp. $T\square^n_{i,1}$). (If $\phi_i = 1$ then we interpret ρ_0 to be 0; likewise if $\phi_i = n$ then we interpret ρ_{n+1} to be $n+1$.)*

Proof. We prove the case for $T\square^n_{i,0}$; the case for $T\square^n_{i,1}$ is similar. By Lemma 9.2.8, these are precisely the linear simplices associated to faces of $\square^n$ whose standard forms do not include $\partial_{i-1,0}, \partial_{i,0}, \partial_{i,1}, \partial_{i+1,0}$, or any string of the form $\partial_{k,0}\partial_{k-1,1} \ldots \partial_{i+1,1}$ or $\partial_{i-1,1} \ldots \partial_{k+1,1}\partial_{k,0}$. As these are precisely the marked faces of $\square^n_{i,0}$, the stated result follows from the definition of cubical triangulation. □

The non-degenerate m-simplices of $T\square^n$ are those for which the corresponding string includes all of the values $1, \ldots, m$; the interior simplices, i.e. those not contained in $T\partial\square^n$, are those for which the corresponding string does not include the values $+$ or $-$.

Definition 9.2.10. The *essential* simplices of $T\square^n$ are those which are both non-degenerate and interior. For $1 \leq m \leq n$, the set of essential m-simplices is denoted K_m.

Definition 9.2.11. Given an essential m-simplex ϕ in $T\square^n$, we define the following data:

- $P(\phi)$ is the largest value $1 \leq r \leq m$ such that for all $1 \leq i \leq r$, $\phi_j = i$ if and only if $j = i$, or 0 if no such r exists. $\Pi(\phi)$, the *preamble* of ϕ, is the initial segment of ϕ defining $P(\phi)$, i.e. the substring $1 \ldots r$, or the empty string if $P(\phi) = 0$.
- $Q(\phi) = P(\phi) + 1$. If $Q(\phi) \leq n$, then $q(\phi)$ is the value $\phi_{Q(\phi)}$; otherwise, $q(\phi) = n + 1$.

More intuitively, $P(\phi)$ is the largest r such that ϕ begins with a string of the form $\Pi(\phi) = 1 \ldots r$, none of whose entries appear in any later position of ϕ, or 0 if no such string exists. $Q(\phi)$ is the first position i such that ϕ_i is not part of such a string, either because its value $\phi_i = q(\phi)$ is greater than i itself, or because this value is repeated later on. The case $P(\phi) = n$, in which $Q(\phi) = q(\phi) = n + 1$, occurs if and only if ϕ is the n-simplex ι_n.

Example 9.2.12. We compute $\Pi(\phi), P(\phi), Q(\phi)$, and $q(\phi)$ for various essential simplices in order to better illustrate the concepts.

- For $\phi = 1\,2\,3\,5\,4, \Pi(\phi) = 1\,2\,3, P(\phi) = 3,\ Q(\phi) = 4, q(\phi) = 5$;

- For $\phi = 1\,2\,3\,4\,3$, $\Pi(\phi) = 1\,2, P(\phi) = 2$, $Q(\phi) = 3, q(\phi) = 3$;
- For $\phi = 2\,3\,1\,1\,1$, $\Pi(\phi) = \varnothing, P(\phi) = 0$, $Q(\phi) = 1, q(\phi) = 2$.

Lemma 9.2.13. *For $m \leq n$ and $\phi \in K_m$ we have $q(\phi) \geq Q(\phi)$.*

Proof. The value of ϕ at position $Q(\phi)$ cannot be less than $Q(\phi)$, as it would then be a repetition of some value in the preamble of ϕ. □

Definition 9.2.14. For $1 \leq m \leq n$, we define the following subsets of K_m:

- K^*_m, the set of *normal* essential m-simplices, consists of all simplices $\phi \in K_m$ such that the value $q(\phi)$ appears exactly once in ϕ.
- K'_m, the set of *abnormal* essential m-simplices, is $K_m \smallsetminus K^*_m$.

The following characterization of K'_m is immediate from the definition.

Lemma 9.2.15. *For $m \leq n-1$, K'_m consists of those ϕ for which the value $q(\phi)$ appears at least twice. For $m = n$, K'_m consists of the single n-simplex ι_n.* □

We next define a construction relating normal and abnormal essential simplices, which will be of significant use in proving that $T \dashv U$ is a Quillen adjunction.

Definition 9.2.16. For $1 \leq m \leq n-1$, the *normalization* of ϕ, denoted $B(\phi)$, is the $(m+1)$-simplex obtained from ϕ by raising the value of ϕ at $Q(\phi)$, and at all i such that $\phi_i > q(\phi)$, by 1.

Note that, in constructing $B(\phi)$, occurrences of the value $q(\phi)$ in positions other than $Q(\phi)$ are unchanged. For instance, $B(1\,2\,3\,2) = 1\,3\,4\,2$.

Lemma 9.2.17. *For $1 \leq m \leq n-1$, normalization defines a bijection $B\colon K'_m \to K^*_{m+1}$, with its inverse given by taking the $(q(\psi)-1)$-face of a simplex $\psi \in K^*_{m+1}$.*

Proof. Let $\phi \in K'_m$. We first show that $B(\phi)$ is essential, i.e. that it is non-degenerate and interior. It is clear from the construction of $B(\phi)$ that it does not contain any entries of the form $+$ or $-$. To see that every element of $\{1, \ldots, m+1\}$ appears in $B(\phi)$ at least once, consider the following:

- for $1 \leq i \leq q(\phi) - 1$, i appears at least once in ϕ, and these entries are unchanged in constructing $B(\phi)$;
- by Lemma 9.2.15, the value $q(\phi)$ appears at least twice in ϕ, and only one of these entries is altered in constructing $B(\phi)$;
- for $q(\phi) + 1 \leq i \leq m+1$, ϕ contains some instance of the value $i-1$ which is raised by 1 in constructing $B(\phi)$.

To see that $B(\phi)$ is in K^*_{m+1}, observe that $P(B(\phi)) = P(\phi)$, as the preamble of ϕ is unchanged in constructing $B(\phi)$; as $q(\phi) \geq Q(\phi)$, and this value is raised in constructing $B(\phi)$, the entry in position $Q(\phi)$ is not part of the preamble of $B(\phi)$). Thus $Q(B(\phi)) = Q(\phi)$, and $q(B(\phi)) = q(\phi) + 1$. Moreover, any entries having the value $q(\phi) + 1$ in ϕ are raised by 1 in constructing $B(\phi)$; thus $q(\phi) + 1$ appears exactly once in $B(\phi)$.

To see that this function is a bijection with the stated inverse, first let $\phi \in K'_m$, and consider $B(\phi)\partial_{q(B(\phi))-1} = B(\phi)\partial_{q(\phi)}$. This face is computed by lowering all entries of $B(\phi)$ greater than $q(\phi)$ by 1; as these are precisely the entries that were raised by 1 in order to obtain $B(\phi)$, this recovers the original simplex ϕ.

Now let $\psi \in K^*_{m+1}$. Since $q(\psi)$ appears exactly once in ψ by assumption, it must be greater than or equal to $P(\psi) + 2$, or else it would be part of the preamble of ψ. This implies that for some $i > Q(\psi)$ we have $\psi_i = q(\psi) - 1$. Now consider the face $\psi\partial_{q(\psi)-1}$. This face is computed by lowering every entry of ψ which is greater than or equal to $q(\psi)$ by 1. In particular, $\psi_{Q(\psi)}$ is reduced to $q(\psi) - 1$, while the preamble of ψ is unaffected. As ψ contains at least one other entry having the value $q(\psi) - 1$, which is not changed in computing this face, we see that $P(\psi\partial_{q(\psi)-1}) = P(\psi)$, $Q(\psi\partial_{q(\psi)-1}) = Q(\psi)$, and $q(\psi\partial_{q(\psi)-1}) = q(\psi) - 1$. Therefore, to compute $B(\psi\partial_{q(\psi)-1})$, we raise the entry in position $Q(\psi)$, and all entries greater than or equal to $q(\psi)$, by 1 – but these were precisely the entries of ψ that were lowered to obtain $\psi\partial_{q(\psi)-1}$. Thus $B(\psi\partial_{q(\psi)-1}) = \psi$. □

Definition 9.2.18. For $2 \leq i \leq m$, a simplex $\phi\colon \Delta^m \to T\square^n$ is *i-disordered* if it has exactly one entry with the value i, and none of its preceding entries have the value $i - 1$.

Lemma 9.2.19. *Every simplex of $T\square^n$ which is i-disordered for some i is marked.*

Proof. It is immediate from the definition that an i-disordered simplex cannot contain any complete substring. □

Lemma 9.2.20. *Let ϕ be an i-disordered m-simplex of $T\square^n$ for some $2 \leq i \leq m$, and consider a face $\phi\partial_j$. If $j \geq i+1$ then $\phi\partial_j$ is i-disordered. If $i \geq 3$ and $j \leq i-3$, then $\phi\partial_j$ is $(i-1)$-disordered.*

Proof. For the case $j \geq i+1$, the face ∂_j only lowers (or replaces with +) entries with values greater than or equal to $i+2$. Thus $\phi\partial_j$ will still have a unique entry with value i, and will no have no new entries with value $i - 1$.

Now consider the case $j \leq i-3$. In this case, ∂_j lowers all entries having the value i, $i-1$, or $i-2$. Thus $\phi\partial_j$ has a unique entry with the value $i-1$, namely that whose position coincides with that of the unique i in ϕ. Moreover, any entry having the value $i-2$ in $\phi\partial_j$ must have the value $i-1$ in ϕ; thus there is no entry preceding the unique i in $\phi\partial_j$ whose value is $i-2$. □

Lemma 9.2.21. *For $2 \leq i \leq m$, if ϕ is an i-disordered m-simplex of $T\square^n$, then ϕ is $(i-1)$-complicial.*

Proof. We must show that each simplex of the form $\phi\partial_{j_1}\ldots\partial_{j_a}\partial_{k_1}\ldots\partial_{k_b}$, where $j_1 > \ldots > j_a \geq i+1$ and $i-3 \geq k_1 > \ldots > k_b$, is marked. (Note that either or both of the strings $j_1, \ldots, j_a$ and $k_1, \ldots, k_b$ may be empty.) By repeatedly applying Lemma 9.2.20, we can see that this simplex is $(i-b)$-disordered; thus it is marked by Lemma 9.2.19. □

Corollary 9.2.22. *For $\phi \in K'_m$, the $(m+1)$-simplex $B(\phi)$ is $q(\phi)$-complicial.*

Proof. From the definition of B, we can see that $B(\phi)$ is $(q(\phi)+1)$-disordered. The statement thus follows from Lemma 9.2.21. □

Definition 9.2.23. Let ϕ be a simplex of $T\square^n$, and let ρ be a complete substring of ϕ. The *linearization* of ϕ associated to ρ is the m-simplex ϕ^ρ defined as follows:

$$\phi^\rho_i = \begin{cases} +\infty, & \phi_i = +\infty, \text{ or } \phi_i \in \{1, \ldots, m\} \text{ and } i < \rho_{\phi_i} \\ \phi_i & i = \phi_i \in \{1, \ldots, m\} \text{ and } \rho_{\phi_i} \\ -\infty, & \phi_i = -\infty, \text{ or } \phi_i \in \{1, \ldots, m\} \text{ and } i > \rho_{\phi_i} \end{cases}$$

Example 9.2.24. To illustrate the concept of a linearization, we consider the linearizations of various simplices:

- The unique linearization of $2\,1\,2\,1\,3$ is $+\,1\,2\,-\,3$.
- The linearizations of $1\,2\,-\,2$ are $1\,2\,-\,-$ and $1\,+\,-\,2$.

- The linearizations of $1\,1\,1$ are $1 - -$, $+ 1 -$, and $+ + 1$.
- The linearizations of $1\,2\,3\,2\,3$ are $1\,2\,3 - -$, $1\,2 + -3$, and $1 + + 2\,3$.
- Every linear simplex is its own unique linearization.
- A simplex of $T\square^n$ is marked if and only if it has no complete substrings, and hence no linearizations.

Lemma 9.2.25. *Cubical face maps preserve linearizations. That is, for $\phi \colon \Delta^m \to T\square^{n-1}$, for any face map $\partial_{i,\varepsilon} \colon \square^{n-1} \to \square^n$, the linearizations of $\partial_{i,\varepsilon}\phi$ are precisely the images under $\partial_{i,\varepsilon}$ of the linearizations of ϕ.*

Proof. This is immediate from the definitions of linearization and the actions of cubical face maps. $\square$

Lemma 9.2.26. *For $\phi \colon \Delta^m \to T\square^n$, the linearizations of the faces of $B(\phi)$ (other than $B(\phi)\partial_{q(\phi)} = \phi$) are as follows:*

(i) The linearizations of $B(\phi)\partial_{q(\phi)-1}$ are the linearizations of ϕ corresponding to complete substrings which include $Q(\phi)$;

(ii) The linearizations of $B(\phi)\partial_{q(\phi)+1}$ are the linearizations of ϕ corresponding to complete substrings which do not include $Q(\phi)$;

(iii) For $i < q(\phi) - 1$ or $i > q(\phi) + 1$, $B(\phi)\partial_i$ has no linearizations.

Proof. For item (i), observe that $B(\phi)\partial_{q(\phi)-1}$ is obtained from ϕ by lowering all entries of ϕ having the value $q(\phi)$, other than that in position $Q(\phi)$. Thus any linearization of $B(\phi)\partial_{q(\phi)-1}$ must include $Q(\phi)$. Moreover, it cannot include any of the entries which were changed to obtain $B(\phi)\partial_{q(\phi)-1}$, as these all appear after position $Q(\phi)$ and have value $q(\phi) - 1$. Thus we see that the complete substrings of $B(\phi)\partial_{q(\phi)-1}$ are those of ϕ which do not include any entries which are changed in $B(\phi)\partial_{q(\phi)-1}$, and these are precisely those which include $Q(\phi)$. Furthermore, note that for any such linearization ρ, those entries of ϕ which are lowered to obtain $B(\phi)\partial_{q(\phi)-1}$ are replaced by $-$ in ϕ^{ρ}, as they have value $q(\phi)$ and appear after position $Q(\phi) = \rho_{q(\phi)}$; in the corresponding linearization of $B(\phi)\partial_{q(\phi)-1}$ they will still be replaced with $-$, as they now have value $q(\phi)-1$, and appear after position $\rho_{q(\phi)-1}$.

The proof of item (ii) is similar. Observe that $B(\phi)\partial_{q(\phi)+1}$ is obtained from ϕ by raising the value in position $Q(\phi)$ to $q(\phi)+1$ and leaving all other entries unchanged. As there are no preceding entries having the value $q(\phi)$, there can be no complete substring of $B(\phi)\partial_{q(\phi)+1}$ including $Q(\phi)$. Therefore, as in the previous case, we see that the complete substrings of $B(\phi)\partial_{q(\phi)+1}$ are those of ϕ which involve only the positions whose values are unchanged in $B(\phi)\partial_{q(\phi)+1}$ – in this case, these are the positions other than $Q(\phi)$. In the linearizations of both ϕ and $B(\phi)\partial_{q(\phi)+1}$ corresponding to these complete substrings, the entry in position $Q(\phi)$ is replaced with $+$, as its value is greater than or equal to $q(\phi)$ and its position is earlier than $\rho_{q(\phi)}$.

Item (iii) is immediate from Lemma 9.2.20. □

Definition 9.2.27. For $m \geq 1$, we define a partial order on the non-degenerate m-simplices of $T\square^n$ as follows:

- if ϕ is contained in $T\partial\square^n$ or $\phi \in K_m^*$, and $\psi \in K'_m$ then $\phi < \psi$;
- for $\phi \neq \psi \in K'_m$, we have $\phi < \psi$ if either $P(\phi) < P(\psi)$ or $P(\phi) = P(\psi)$ and $q(\phi) > q(\psi)$.

The relation $<$ defined above is easily seen to be transitive and anti-symmetric; the partial order $\leq$ is defined to be its reflexive closure.

Lemma 9.2.28. *For $\phi \in K'_m$ and $0 \leq i \leq m+1$, $i \neq q(\phi)$, we have $\phi > B(\phi)\partial_i$.*

Proof. We proceed by case analysis on i.

- If $i = 0$ or $i = m+1$, then $B(\phi)\partial_i$ is part of $T\partial\Box^n$, while $\phi \in K'_m$ by assumption.

- If $1 \leq i \leq P(\phi) - 1$, then $(B(\phi)\partial_i)_i = (B(\phi)\partial_i)_{i+1} = i$, while entries before position i are the same as in ϕ. Thus $P(B(\phi)\partial_i) = i - 1 < i < P(\phi)$. (Note that this case is vacuous if $P(\phi) = 0$ or $P(\phi) = 1$.)

- If $i = P(\phi)$, then there is some $j > Q(\phi)$ such that $\phi_j = i + 1$, and this value is unchanged in $B(\phi)$, as it is less than or equal to $q(\phi)$ and not in position $Q(\phi)$. Therefore, in $B(\phi)\partial_i$ this value is lowered to i, creating a repetition. As values less than or equal to i are unchanged, we can see that $P(B(\phi)\partial_i) = i - 1 = P(\phi) - 1 < P(\phi)$.

- If $P(\phi) + 1 = Q(\phi) \leq i \leq q(\phi) - 1$, then we first note that $q(\phi) \geq P(\phi) + 2$. In computing $B(\phi)\partial_i$ from $B(\phi)$, we lower the value in position $Q(\phi)$ from $q(\phi) + 1$ to $q(\phi)$, and we also lower every occurrence of the value $q(\phi)$ to $q(\phi) - 1$. Thus $q(\phi)$ appears only once in $B(\phi)\partial_i$, in position $Q(\phi)$. Since entries less than or equal to $P(\phi)$ are unchanged, we have $P(B(\phi)\partial_i) = P(\phi)$ and $q(B(\phi)\partial_i) = q(\phi)$; thus $B(\phi)\partial_i \in K^*_m$. (Note that this case is vacuous if $q(\phi) = Q(\phi)$.)

- If $q(\phi)+1 \leq i \leq m$, then in computing $B(\phi)\partial_i$ from $B(\phi)$ we do not change any values less than or equal to $q(\phi) + 1$. Thus $P(B(\phi)\partial_i) = P(\phi)$ and $q(B(\phi)\partial_i) = q(\phi) + 1 > q(\phi)$. $\square$

Definition 9.2.29. For $n \geq 1, 1 \leq i \leq j \leq n$, we let $\omega^{n,i,j}$ denote the $(n-1)$-simplex of $T\Box^n$ defined as follows:

$$\omega_k^{n,i,j} = \begin{cases} k, & k < i \\ j, & k = i, j \leq n-1 \\ +\infty, & k = i, j = n \\ k-1, & k > i \end{cases}$$

(Note that $\omega^{n,i,n} = \iota_{\partial_{i,0}}$.) For $j \leq n-1$, we let $\Omega^{n,i,j}$ denote the n-simplex obtained from $\omega^{n,i,j}$ by raising the value of the entry in the $(j+1)$-position (i.e. the second occurrence of j in $\omega^{n,i,j}$), and all entries greater than j, by 1. We let $\Omega^{n,i,n}$ denote the n-simplex obtained from $\omega^{n,i,+}$ by changing the unique occurrence of $+$ in $\omega^{n,i,n}$ to n. More explicitly, we may define $\Omega^{n,i,j}$ for all $i \leq j \leq n$ as follows:

$$\Omega_k^{n,i,j} = \begin{cases} k, & k < i \\ j, & k = i \\ k-1, & i < k < j+1 \\ k, & i \geq j+1 \end{cases}$$

We will suppress the superscript n from the notation above where there is no risk of ambiguity, and simply write $\omega^{i,j}$ and $\Omega^{i,j}$.

Example 9.2.30. To clarify the definition of $\omega^{i,j}$ and $\Omega^{i,j}$, we state their definitions in the case $n = 5, i = 3$.

- $\omega^{3,3} = 1\,2\,3\,3\,4; \Omega^{3,3} = 1\,2\,3\,4\,5.$
- $\omega^{3,4} = 1\,2\,4\,3\,4; \Omega^{3,4} = 1\,2\,4\,3\,5.$
- $\omega^{3,5} = 1\,2 + 3\,4; \Omega^{3,5} = 1\,2\,5\,3\,4.$

In general, we always have $\Omega^{i,i} = \iota_n$.

Lemma 9.2.31. *For $n \geq 2$ and $1 \leq i \leq n-1$, we have $B(\omega^{i,j}) = \Omega^{i,j+1}$.*

Proof. From the definition of $\omega^{i,j}$ for $1 \leq j \leq n-1$, we can see that the value j appears in position i and in position $j+1$, and that all other entries appear exactly once and in order; thus $Q(\omega^{i,j}) = i$ and $q(\omega^{i,j}) = j$. To construct $B(\omega^{i,j})$, we first raise the value in position i to $j+1$, thereby obtaining $\omega^{i,j+1}$ (or $\omega^{i,j+1}$ with the $+$ in position i replaced by n, in the case $j = n$). We then raise every entry which is greater than j, aside from this first occurrence of $j+1$, by 1, thereby obtaining $\Omega^{i,j+1}$. $\square$

Lemma 9.2.32. *For $1 \leq i \leq j \leq n$ as above, $\Omega^{i,j}\partial_j = \omega^{i,j}$. Moreover, if $j \geq i+1$ then $\Omega^{i,j}\partial_{j-1} = \omega^{i,j-1}$.*

Proof. We begin by considering the first statement. For $j \leq n-1$, observe that we compute $\Omega^{i,j}\partial_j$ by lowering those entries of $\Omega^{i,j}$ which are greater than j, and these are precisely the entries of $\omega^{i,j}$ which were raised to obtain $\Omega^{i,j}$. For $j = n$, we compute $\Omega^{i,n}\partial_n$ by replacing the one occurrence of n in $\Omega^{i,n}$ by $+$, thereby obtaining $\omega^{i,n}$. The second statement is immediate from Lemmas 9.2.17 and 9.2.31. $\square$

Lemma 9.2.33. *For $1 \leq i \leq n$, an $(n-1)$-simplex $\phi \colon \Delta^{n-1} \to T\square^n$ has $\omega^{i,n}$ as a linearization if and only if $\phi = \omega^{i,j}$ for some $j \geq i$.*

Proof. For ϕ to have $\omega^{i,n}$ as a linearization, the entries of ϕ other than ϕ_i must form a complete substring ρ; in other words we must have $\phi_j = j$ for $j < i$ and $\phi_j = j-1$ for $j > i$. For ϕ_j to be replaced by $+$ rather than $-$ in the linearization associated to this substring, we must have $i < \rho_{\phi_i}$; in other words, the other occurrence of ϕ_i in ϕ must come after position i. For this to be the case, we must have $\phi_i \geq i$. We can see that these criteria are satisfied if and only if $\phi = \omega^{i,j}$ for some $j \geq i$. $\square$

Definition 9.2.34. For $n \geq 1$ and $1 \leq i \leq n$, let Ξ_i^n denote the regular subcomplex of $T\square^n$ containing all of its non-degenerate simplices except for those of the form $\omega^{i,j}$ or $\Omega^{i,j}$. Let $\partial\Xi_i^n$ denote the intersection of $T\partial\square^n$ with Ξ_i^n, i.e. the regular subcomplex of $T\square^n$ containing all non-degenerate boundary simplices except for $\omega^{i,n}$. Let $\widehat{\Xi}_i^n$ denote the regular subcomplex of $T\square_{i,0}^n$ whose underlying simplicial set is $(\Xi_i^n)^\flat$, i.e. Ξ_i^n with simplices marked whenever they are marked in $T\square_{i,0}^n$.

Proposition 9.2.35. *For $n \geq 1, 1 \leq i \leq n$, the inclusion $T(\sqcap_{i,0}^n)^\flat \hookrightarrow \Xi_i^n$ is anodyne.*

Proof. We proceed by induction on n. For the base case $n = 1$, both $T(\sqcap_{1,0}^1)^\flat$ and Ξ_i^n consist of the single vertex $-$, so the inclusion $T(\sqcap_{1,0}^1)^\flat \hookrightarrow \Xi_1^1$ is the identity.

Now let $n \geq 2$ and assume the statement holds for $n-1$. We first show that $T(\sqcap_{i,0}^n)^\flat \hookrightarrow \partial\Xi_i^n$ is anodyne. Observe that the triangulation of the missing face of the cube, $\partial_{i,0}$, consists of all simplices ϕ for which $\phi_i = +$; the non-degenerate m-simplices in the interior of this face are those in which every value between 1 and m occurs at least once, and there is no $+$ or $-$ in any position besides position i. Therefore, to construct $\partial\Xi_i^n$ from $T(\sqcap_{i,0}^n)^\flat$, we must add all such simplices except for $\omega^{i,n}$.

Identifying $\partial\Xi_{n-1}^{n-1}$ and Ξ_{n-1}^{n-1} with their images under the face map $T\partial_{i,0}\colon T\square^{n-1} \to T\square^n$, we can characterize certain subcomplexes of $T\square^n$ as follows.

- $\partial\Xi_{n-1}^{n-1}$ consists of all simplices having a $+$ in position i and a $+$ or $-$ in at least one other position, except for $\partial_{i,0}\omega^{n-1,n-1}$.
- Ξ_{n-1}^{n-1} consists of all simplices having a $+$ in position i, except for $\partial_{i,0}\omega^{n-1,n-1}$ and $\partial_{i,0}\Omega^{n-1,n-1} = \omega^{i,n}$.

- $T(\sqcap^n_{i,0})^\flat$ consists of all simplices either a $-$ in position i, or a $+$ or $-$ in some position other than i.

- $\partial \Xi^n_i$ consists of all simplices having a $+$ or $-$ in any position, other than $\partial_{i,0}\Omega^{n-1,n-1} = \omega^{i,n}$.

From this characterization, we can see that $\partial \Xi^{n-1}_{n-1}$ is the intersection of Ξ^{n-1}_{n-1} with $T(\sqcap^n_{i,0})^\flat$, while $\partial \Xi^n_i$ is their union. Thus we have the following pushout square:

$$\begin{tikzcd}
\partial \Xi^{n-1}_{n-1} \arrow[r] \arrow[d] & T(\sqcap^n_{i,0})^\flat \arrow[d] \\
\Xi^{n-1}_{n-1} \arrow[r] & \partial \Xi^n_i \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Since $\partial \Xi^{n-1}_{n-1} \hookrightarrow \Xi^{n-1}_{n-1}$ is anodyne, this implies $T(\sqcap^n_{i,0})^\flat \hookrightarrow \partial \Xi^n_i$ is anodyne as well.

Next we will show that $\partial \Xi^n_i \hookrightarrow \Xi^n_i$ is anodyne. To do this, we will add to $\partial \Xi^n_i$ every essential simplex of $T\square^n$, except for those of the form $\omega^{i,j}$ or $\Omega^{i,j}$, via a series of complicial horn fillings. We proceed by induction on dimension; for $1 \leq m \leq n-1$, let $\Xi^{n,m}_i$ denote the subcomplex of $T\square^n$ consisting of $\partial \Xi^n_i$ together with all essential simplices of dimension less than m and all normal essential simplices of dimension m. As there are no essential simplices of dimension 0 and no normal essential simplices of dimension 1, $\Xi^{n,1}_i = \partial \Xi^n_i$. We will show that for $1 \leq m \leq n-2$, the inclusion $\Xi^{n,m}_i \hookrightarrow \Xi^{n,m+1}_i$ is anodyne.

To construct $\Xi^{n,m+1}_i$ from $\Xi^{n,m}_i$, we must add all non-degenerate simplices of K'_m and K^*_{m+1} via complicial horn-filling, marking those which are marked in $T\square^n$. We proceed by induction on the partial order of Definition 9.2.27. For the base case, note that the minimal m-simplices in this partial order are those which are either on the boundary or normal, thus all minimal m-simplices are already present in $\Xi^{n,m}_i$, and this is a regular subcomplex of $T\square^n$

by definition. Now let $\phi \in K'_m$, and suppose that we have added all non-degenerate m-simplices less than ϕ, and marked those which are marked in $T\square^n$. By Lemma 9.2.28, this includes all faces of $B(\phi)$ except for $B(\phi)\partial_{q(\phi)} = \phi$ itself; by Corollary 9.2.22 these faces define a $q(\phi)$-complicial horn which we can fill to add $B(\phi)$ and ϕ. By induction, therefore, we can add ϕ and $B(\phi)$ to $\Xi_i^{n,m}$ for all $\phi \in K'_m$ via complicial horn-filling; by Lemma 9.2.17 these are all the additional simplices of $\Xi_i^{n,m+1}$. Moreover, if ϕ is marked in $T\square^n$, i.e. has no linearizations, then by Lemma 9.2.26 the same is true of all other faces of $B(\phi)$. Thus these faces are marked in $\Xi_i^{n,m}$ by the induction hypothesis; therefore, we can mark ϕ by taking a pushout of an elementary complicial marking extension. Thus we see that the inclusion $\Xi_i^{n,m} \hookrightarrow \Xi_i^{n,m+1}$ is anodyne.

Composing these anodyne maps, we see that $\partial\Xi_i^n \hookrightarrow \Xi_i^{n,n-1}$ is anodyne. To complete the proof, we must show that $\Xi_i^{n,n-1} \hookrightarrow \Xi_i^n$ is anodyne. To do this, we will add via complicial horn-filling (and mark via complicial marking extension where necessary) all remaining simplices of Ξ_i^n to $\Xi_i^{n,n-1}$ – namely, the essential simplices of dimensions $n-1$ and n, other than those of the form $\omega^{i,j}$ or $\Omega^{i,j}$.

We consider all simplices of K'_{n-1} and K^*_n not of the forms described above, once again proceeding by induction on the partial order of Definition 9.2.27 on K_{n-1}. Again, for our base case we note that all minimal non-degenerate $(n-1)$-simplices, except for $\omega^{i,n}$, are already present in $\Xi_i^{n,n-1}$, and those which are marked in $T\square^n$ are marked in $\Xi^{n,n-1}$. Now let $\phi \in K'_{n-1}$, not equal to any $\omega^{i,j}$, assume we have added all simplices less than ϕ and marked those which are marked in $T\square^n$, and consider the faces of $B(\phi)$ other than ϕ itself. By Lemma 9.2.28, all of these faces are less than ϕ. Furthermore, by Lemma 9.2.33, ϕ does not have $\omega^{i,n}$ as a linearization; therefore, none of the faces of $B(\phi)$ have $\omega^{i,n}$ as a linearization by Lemma 9.2.26. This implies that none of these faces are of the form $\omega^{i,j}$ by Lemma 9.2.33; thus all faces of

$B(\phi)$ are present except for ϕ itself. Therefore, by Corollary 9.2.22, we have a $q(\phi)$-complicial horn which we can fill to obtain $B(\phi)$ and ϕ. Moreover, as in the previous case, if ϕ is marked in $T\square^n$ then all other faces of $B(\phi)$ are marked in $T\square^n$ by Lemma 9.2.26. Thus, in this case we can mark ϕ via complicial marking extension.

By induction, then, we can add to $\Xi_i^{n,n-1}$ via complicial horn filling all essential $(n-1)$-simplices ϕ not of the form $\omega^{i,j}$, together with their associated n-simplices $B(\phi)$, and mark those which are marked in $T\square^n$ via complicial marking extension. By Lemmas 9.2.17 and 9.2.31, we see that we have therefore added all simplices of Ξ_i^n, except for the normal simplices $B(\omega^{i,j}) = \Omega^{i,j+1}$ for $i \leq j \leq n-1$, and the lone abnormal n-simplex $\iota_n = \Omega^{i,i}$. Thus $\Xi_i^{n,n-1} \hookrightarrow \Xi_i^n$ is anodyne, as a composite of complicial horn fillings and complicial marking extensions.

Thus we see that $T(\sqcap_{i,0}^n)^\flat \hookrightarrow \Xi_i^n$ is anodyne, as a composite of anodyne maps. $\square$

Next we consider how the comical marking conditions affect the simplices of $T\square^n$.

Lemma 9.2.36. *For $n \geq 1$, let X be a marked simplicial set admitting an entire map $Y \to X$, where Y is a regular subcomplex of $T\square^n$, closed under normalization. Let ϕ be a non-degenerate m-simplex of X. Then:*

- *all linearizations of ϕ are contained in X;*
- *if all linearizations of ϕ are marked in X, then ϕ is marked in the pre-complicial reflection $\overline{X}$.*

Proof. We proceed by induction on n. The case $n = 1$ is trivial, as no simplex of $T\square^1$ has a linearization other than itself.

Now let $n \geq 2$, and suppose the statement holds for $n-1$. We will prove the statement for n by induction on the partial order of Definition 9.2.27 for non-degenerate m-simplices ϕ. First we consider the case where ϕ is minimal. If ϕ is contained in $T\partial\square^n$, then $\phi = \partial_{i,\varepsilon}\psi$ for some $\psi: \Delta^m \to T\square^{n-1}$ and some cubical face map $T\partial_{i,\varepsilon}: T\square^{n-1} \to T\square^n$. By Lemma 9.2.25, the linearizations of ϕ are precisely the images under $T\partial_{i,\varepsilon}$ of the linearizations of ψ; the stated results thus follow by the induction hypothesis. On the other hand, if $\phi \in K^*_m$, then ϕ has no linearizations and is marked, so both statements are vacuously true.

Now suppose the statement has been proven for all m-simplices less than ϕ. By assumption, $B(\phi)$ and all of its faces are contained in X. As all faces of $B(\phi)$ besides ϕ are less than ϕ by Lemma 9.2.28, we can apply the induction hypothesis and Lemma 9.2.26 to see that all linearizations of ϕ are contained in X. Similarly, if all linearizations of ϕ are marked in $\overline{X}$, then the induction hypothesis and Lemma 9.2.26 show that all faces of $B(\phi)$ besides ϕ are marked in $\overline{X}$; as $B(\phi)$ is $q(\phi)$-complicial by Corollary 9.2.22, this implies that ϕ is marked in $\overline{X}$ as well. $\square$

Corollary 9.2.37. *Let ϕ be an m-simplex of $X \in \{\widehat{\Xi}^n_i, T\square^n_{i,0}\}$. Then:*

- *all linearizations of ϕ are contained in X;*
- *if all linearizations of ϕ are marked in X, then ϕ is marked in the pre-complicial reflection of X.*

Proof. By Lemma 9.2.36, it suffices to show that $\widehat{\Xi}^n_i$ and $T\square^n_{i,0}$ are closed under normalization. For $T\square^n_{i,0}$ this is trivial; for $\widehat{\Xi}^n_i$ it follows from Lemmas 9.2.17 and 9.2.31. $\square$

Lemma 9.2.38. *For $n \geq 1$ and $1 \leq i \leq j \leq n$, the n-simplex $\Omega^{i,j}$ is j-complicial in $T\square^n_{i,0}$.*

Proof. We must show that any simplex of the form $\Omega^{i,j}\partial_{k_1} \ldots \partial_{k_a}\partial_{r_1} \ldots \partial_{r_b}$, where $k_1 > \ldots > k_a \geq j+2$ and $j-2 \geq r_1 > \ldots > r_b$, is marked. (Note that either or both of the strings $k_1, \ldots, k_a$ and $r_1, \ldots, r_b$ may be empty.) Fix a particular face map δ having this form, and consider the face $\Omega^{i,j}\delta$.

We first note that when represented as a string, $\Omega^{i,j}$ contains each of the values 1 through n exactly once. In particular, $(\Omega^{i,j})_i$ is the unique entry of $\Omega^{i,j}$ having the value j, and if $j \leq n-1$ then $(\Omega^{i,j})_{j+1}$ is the unique entry having the value $j+1$. As the maps ∂_{k_t} only lower entries greater than $j+2$, this will still be true of the corresponding entries in $\Omega^{i,j}\partial_{k_1} \ldots \partial_{k_a}$.

Now let $1 \leq t \leq b$, and suppose that in $\Omega^{i,j}\partial_{k_1} \ldots \partial_{k_a}\partial_{r_1} \ldots \partial_{r_{t-1}}$, the entries in positions i and $j+1$ are the unique entries having the values $j-t+1$ and $j-t+2$, respectively. Since $r_t \leq j-t-1$, the face map ∂_{r_t} lowers these entries, along with any entries having the value $j-t$. Thus the entries in positions i and $j+1$ are the only entries of $\Omega^{i,j}\partial_{k_1} \ldots \partial_{k_a}\partial_{r_1} \ldots \partial_{r_t}$ having the values $j-(t+1)+1$ and $j-(t+1)+2$, respectively. By induction, we see that the entries in positions i and j of $\Omega^{i,j}\delta$ are the unique entries having the values $j-b$ and $j-b+1$, respectively.

Let ρ be a complete substring of $\Omega^{i,j}\delta$; for notational convenience let $j' = j-b$ and $m = n-a-b$. The discussion above shows that we must have $\rho_{j'} = i$ and $\rho_{j'+1} = j+1$. (As in the statement of Lemma 9.2.9, we interpret ρ_0 and ρ_{m+1} to be 0 and $m+1$, respectively.) Now consider k such that $\rho_{j'-1} < k < \rho_{j'+1}, k \neq i$. Observe that for such k we have $(\Omega^{i,j})_k < i \leq j = (\Omega^{i,j})_i$.

By Lemma 9.2.2, this implies $(\Omega^{i,j}\delta)_k \leq (\Omega^{i,j}\delta)_i = j'$; as $(\Omega^{i,j}\delta)_i$ is the unique entry with the value j', we in fact have $(\Omega^{i,j}\delta)_k \leq j'-1$. If $j'=1$ then this implies $(\Omega^{i,j}\delta)_k = -$. Otherwise, we see that $\rho_{(\Omega^{i,j}\delta)_k} \leq \rho_{j'-1} = w < k$. Either way, we have $(\Omega^{i,j}\delta)^\rho_k = -$. Thus $\Omega^{i,j}\delta$ is marked by Lemma 9.2.9. $\square$

Proposition 9.2.39. *For $n \geq 1$ and $1 \leq i \leq n$, the inclusion $\widehat{\Xi}^n_i \hookrightarrow T\square^n_{i,0}$ is a trivial cofibration.*

Proof. Let $(T\square^n_{i,0})^\dagger$ denote the marked simplicial set with underlying simplicial set $T\square^n$, and a simplex ϕ marked if and only if all of its linearizations are marked in $T\square^n_{i,0}$; define $(\widehat{\Xi}^n_i)^\dagger$ similarly. We have a commuting diagram:

$$\begin{tikzcd}
\widehat{\Xi}^n_i \arrow[r] \arrow[d] & (\widehat{\Xi}^n_i)^\dagger \arrow[d] \\
T\square^n_{i,0} \arrow[r] & (T\square^n_{i,0})^\dagger
\end{tikzcd}$$

The two horizontal maps are trivial cofibrations by Lemmas 9.2.1 and 9.2.36. Therefore, to prove the stated result it suffices to prove that $(\widehat{\Xi}^n_i)^\dagger \hookrightarrow (T\square^n_{i,0})^\dagger$ is anodyne.

For $i \leq j \leq n+1$, let $(\widehat{\Xi}^n_{i,j})^\dagger$ denote the regular subcomplex of $(T\square^n_{i,0})^\dagger$ consisting of $(\widehat{\Xi}^n_i)^\dagger$ together with all simplices of the form $\omega^{i,j'}$ or $\Omega^{i,j'}$ for $i \leq j' < j$. We can see that $(\widehat{\Xi}^n_{i,i})^\dagger = (\widehat{\Xi}^n_i)^\dagger$, while $(\widehat{\Xi}^n_{i,n+1})^\dagger = (T\square^n)^\dagger$; thus it suffices to show that each map $(\widehat{\Xi}^n_{i,j})^\dagger \hookrightarrow (\widehat{\Xi}^n_{i,j+1})^\dagger$ for $i \leq j \leq n$ is anodyne.

For the case $j = i$, we will show that we can add $\Omega^{i,i} = \iota_n$ and $\omega^{i,i}$ to $(\widehat{\Xi}^n_i)^\dagger$ via complicial horn-filling. Observe that for all $1 \leq k \leq n$ we have $\iota_n\partial_k = \Omega^{k,k}\partial_k = \omega^{k,k}$ by Lemma 9.2.32. From the definition of $\omega^{i,j}$ it is clear that the simplices $\omega^{k,k}$ are all distinct, thus all of these faces besides $\omega^{i,i}$ are present in $(\widehat{\Xi}^n_i)^\dagger$. Furthermore, we have $\iota_n\partial_0 = -1\,2\ldots(n-1)$, which is contained in $T\sqcap^n_{i,0} \subseteq (\widehat{\Xi}^n_i)^\dagger$. By Lemma 9.2.38, these faces define an i-complicial horn in $(\widehat{\Xi}^n_i)^\dagger$, which we can fill to obtain ι_n and its missing face $\omega^{i,i}$. Thus the inclusion $(\widehat{\Xi}^n_i)^\dagger = (\widehat{\Xi}^n_{i,i})^\dagger \hookrightarrow (\widehat{\Xi}^n_{i,i+1})^\dagger$ is anodyne.

Now consider the case $j \geq i+1$. By Lemma 9.2.32 we have $\Omega^{i,j}\partial_{j-1} = \omega^{i,j-1}$, while $\Omega^{i,j}\partial_j = \omega^{i,j}$. Furthermore, by Lemma 9.2.26, these are the only faces of $\Omega^{i,j}$ having $\omega^{i,n}$ as a linearization; therefore, by Lemma 9.2.33, no other face of $\Omega^{i,j}$ is of the form $\omega^{i,j'}$ for any j'. Thus we see that all faces of $\Omega^{i,j}$ besides $\omega^{i,j}$ are present in $(\widehat{\Xi}^n_{i,j})^\dagger$. By Lemma 9.2.38, we therefore have a j-complicial horn in $(\widehat{\Xi}^n_{i,j})^\dagger$ which we can fill to obtain $(\widehat{\Xi}^n_{i,j+1})^\dagger$. Thus the inclusion $(\widehat{\Xi}^n_{i,j})^\dagger \hookrightarrow (\widehat{\Xi}^n_{i,j+1})^\dagger$ is anodyne.

Thus we see that $(\widehat{\Xi}^n_i)^\dagger \hookrightarrow (T\square^n_{i,0})^\dagger$ is anodyne, as a composite of anodyne maps. $\square$

Corollary 9.2.40. *For $n \geq 1, l \leq i \leq n$, the inclusion $T\sqcap^n_{i,0} \hookrightarrow T\square^n_{i,0}$ is a trivial cofibration.*

Proof. The inclusion $T\sqcap^n_{i,0} \hookrightarrow \widehat{\Xi}^n_i$ is anodyne by Proposition 9.2.35, as it is a pushout of $T(\sqcap^n_{i,0})^\flat \hookrightarrow \Xi^n_i$. The inclusion $\widehat{\Xi}^n_i \hookrightarrow T\square^n_{i,0}$ is a trivial cofibration by Proposition 9.2.39. Thus $T\sqcap^n_{i,0} \hookrightarrow T\square^n_{i,0}$ is a trivia cofibrations as a composite of trivial cofibrations. $\square$

Proposition 9.2.41. *For $n \geq 2$, $1 \leq i \leq n$, the map $T(\square^n_{i,0})' \to T\tau_{n-2}\square^n_{i,0}$ is a trivial cofibration.*

Proof. As in the proof of Proposition 9.2.39, let $(T(\square^n_{i,0})')^\dagger$ denote the marked simplicial set obtained from $T(\square^n_{i,0})'$ by marking all simplices whose linearizations are all marked in $T(\square^n_{i,0})'$, and define $(T\tau_{n-2}\square^n_{i,0})^\dagger$ similarly. Note that Lemma 9.2.33 shows that the only unmarked essential simplices of $(T(\square^n_{i,0})')^\dagger$ are those of the form $\omega^{i,j}$, while all $(n-1)$-simplices of $(T\tau_{n-2}\square^n_{i,0})^\dagger$ are marked (in fact, $(T\tau_{n-2}\square^n_{i,0})^\dagger = \tau_{n-2}T\square^n_{i,0}$. Once again, we have a commuting diagram:

$$\begin{tikzcd}
T(\square^n_{i,0})' \arrow[r] \arrow[d] & (T(\square^n_{i,0})')^\dagger \arrow[d] \\
T\tau_{n-2}\square^n_{i,0} \arrow[r] & (T\tau_{n-2}\square^n_{i,0})^\dagger
\end{tikzcd}$$

Once again, Lemmas 9.2.1 and 9.2.36 show that the horizontal maps are trivial cofibrations, so it suffices to show that $(T(\sqcap^n_{i,0})')^\dagger \hookrightarrow (T\tau_{n-2}\sqcap^n_{i,0})^\dagger$ is anodyne. To this end, we will show that we may mark every simplex $\omega^{i,j}$ via complicial marking extensions, proceding by induction on j.

For the base case $j = i$, recall that $\Omega^{i,i} = \iota_n$ is i-complicial by Lemma 9.2.38. We will show that $\iota_n \partial_{i-1}$, and $\iota_n \partial_{i+1}$ in the case $i \neq n$, are both marked in $(T(\sqcap^n_{i,0})')^\dagger$. We begin with $\iota_n \partial_{i-1}$. First consider the case $i = 1$; then $\iota_n \partial_0 = -1 \ldots (n-1) = \iota_{\partial_{1,1}}$. This $(n-1)$-simplex is contained in $T\sqcap^n_{1,0}$, hence it is marked. Next consider the case $i \geq 2$; then $\iota_n \partial_i = \Omega^{i-1,i-1}\partial_{i-1} = \omega^{i-1,i-1}$. The only repeated entry of this simplex is $i-1$, which appears in positions $i-1$ and i. Thus every entry besides these two must appear in any complete substring of ρ of this simplex. Such a complete substring may have ρ_{i-1} equal to either $i-1$ or i. In the former case, the associated linearization is $\iota_{\partial_{i,1}}$, while in the latter case it is $\iota_{\partial_{i-1,0}}$. Either way, it is an $(n-1)$-simplex of $T\sqcap^n_{i,0}$, and is therefore marked in $(T(\sqcap^n_{i,0})')^\dagger$. Thus both linearizations of $\iota_n \partial_{i-1}$ are marked, hence so is $\iota_n \partial_{i-1}$ itself.

Next, assume that $i \neq n$, and consider $\Omega^{i,i}\partial_{i+1} = \iota_n \partial_{i+1}$. Similarly to the previous case, we observe that $\iota_n \partial_{i+1,i+1} = \omega^{i+1,i+1}$. If $i = n-1$ then this is $\omega^{n,n} = \iota_{\partial_{n,0}}$, hence it is marked as an $(n-1)$-simplex of $T\sqcap^n_{n-1,0}$. Otherwise, an argument similar to the above shows that it has two linearizations, namely $\iota_{\partial_{i+1,0}}$ and $\iota_{\partial_{i+2,1}}$. Again, both of these are marked as $(n-1)$-simplices of $T\sqcap^n_{n-1,0}$, hence $\iota_n \partial_{i+1}$ is marked. Thus we see that the n-simplex $\iota_n \colon \Delta^n \to T(\sqcap^n_{i,0})'$ factors through $(\Delta^n)'_i$, hence we may mark its i-face $\omega^{i,i}$ via a complicial marking extension.

Now let $i + 1 \leq j \leq n$, and assume that we have marked $\omega^{i,j-1}$. Once again, Lemma 9.2.38 shows that $\Omega^{i,j}$ is j-complicial, so we will show that the faces $\Omega^{i,j}\partial_{j-1}$ and $\Omega^{i,j}\partial_{j+1}$ (in the case $j \neq n$) are marked. For ∂_{j-1}, recall that $\Omega^{i,j}\partial_{j-1} = \omega^{i,j-1}$ by Lemma 9.2.32, thus it is marked by the induction hypothesis. For ∂_{j+1}, recall that $\Omega^{i,j} = B(\omega^{i,j-1})$ by Lemma 9.2.31. Since $q(\omega^{i,j-1}) = j-1$, this implies that $\Omega^{i,j}\partial_{j+1}$ has no linearizations by Lemma 9.2.26, and is therefore marked. Once again, therefore, we see that $\Omega^{i,j}\colon \Delta^n \to T(\square^n_{i,0})'$ factors through $(\Delta^n_j)'$, thus we may mark its j-face $\omega^{i,j}$ via complicial marking extension.

By induction, we see that we may mark all simplices $\omega^{i,j}$ via complicial marking extensions, thus the inclusion $(T(\square^n_{i,0})')^\dagger \hookrightarrow (T\tau_{n-2}\square^n_{i,0})^\dagger$ is anodyne.

$\square$

We are now able to prove the main result of this section.

Theorem 9.2.42. *The adjunction $T : \mathsf{cSet}^+ \rightleftarrows \mathsf{sSet}^+ : U$ is Quillen, where cSet^+ is equipped with any of the (saturated) (n-trivial) comical model structures, and sSet^+ is equipped with the corresponding complicial model structure.*

Proof. We must show that T sends the following maps to trivial cofibrations in sSet^+:

(i) Comical open box fillings $\sqcap^n_{i,\varepsilon} \hookrightarrow \square^n_{i,\varepsilon}$;

(ii) Comical marking extensions $(\square^n_{i,\varepsilon})' \to \tau_{n-2}\square^n_{i,\varepsilon}$;

(iii) Rezk maps, in the case where the model structures are saturated;

(iv) k-markings $\Delta^k \to \widetilde{\Delta}^k$ for $k > n$ in the case where the model structures are n-trivial for some $n \geq 0$.

By Propositions 3.3.18, 4.3.13 and 9.1.9, it suffices to show items (i) and (ii) in the case $\varepsilon = 0$. For item (i) this is Corollary 9.2.40, while for item (ii) this is Proposition 9.2.41. For item (iii), it is easy to see that T sends each elementary

Rezk map to a pushout of the simplicial elementary Rezk map; the general result then follows from Propositions 3.3.25 and 4.3.15. For item (iv), we may observe that T sends each (cubical) k-marker to a pushout of the (simplicial) k-marker, as in [CKM20, Thm. 7.2]. $\square$

9.3 The functor Q

In this section, we construct the functor $Q : \mathsf{sSet}^+ \to \mathsf{cSet}^+$. Later, we will exhibit this functor to be a homotopical inverse to the triangulation functor T, as was done in the unmarked case in Proposition 7.2.20.

We will construct this functor using the cosimpilcial object $Q_{L,0}$ of Proposition 7.2.5, pre-composed with the involution $(-)^{\mathrm{op}} : \mathsf{sSet}^+ \to \mathsf{sSet}^+$ as described in Remark 7.2.6. More explicitly, in this chapter, for $n \geq 0$ Q^n is the cubical set $C^{0,n}_{L,0}$, hereafter denoted simply by $C^{0,n}$; similarly, $C^{m,n}$ will now denote $C^{m,n}_{L,0}$. We will continue to make use of the combinatorial results and constructions of Chapter 7, but it should now be understood that they differ in the value of ε from the versions presented there.

Using Lemma 7.1.8, we can obtain the following alternative description of the objects Q^n, relating this definition of Q to that given in [KLW19].

Proposition 9.3.1. *For $n \geq 0$, Q^n is given by the pushout square*

$$\begin{tikzcd} \coprod\limits_{1 \leq i \leq n} \square^{i-1} \otimes \square^{n-i} \arrow[r] \arrow[d] & \square^n \arrow[d] \\ \coprod\limits_{1 \leq i \leq n} \square^{i-1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Q^n \end{tikzcd}$$

where the upper horizontal map restricts to $\partial_{i,0}$ on the i-th summand, and the *left vertical map is a coproduct of the projections* $\square^{i-1} \otimes \square^{n-i} \to \square^{i-1} \otimes \square^0 \cong \square^{i-1}$.

Thus each Q^n may be regarded as a quotient of $\square^n$. Then the map $Q(\partial_i) : Q^{n-1} \to Q^n$ is induced by:

- $\partial_{i+1,1} : \square^{n-1} \to \square^n$ if $i < n$; and
- $\partial_{n,0} : \square^{n-1} \to \square^n$ if $i = n$,

whereas the map $Q(\sigma_i) : Q^{n+1} \to Q^n$ is induced by:

- $\gamma_{i+1,0} : \square^{n+1} \to \square^n$ if $i < n$; and
- $\sigma_{n+1} : \square^{n+1} \to \square^n$ if $i = n$.

Remark 9.3.2. We claim that Q "preserves" normal forms. Consider a simplicial normal form $\partial_{i_m}^{n_m} \dots \partial_{i_1}^{n_1}$ with $i_m > \dots > i_1$. Then there exists a unique integer $1 \leq r \leq m + 1$ such that $i_s = n_s$ for $s \geq r$ and $i_s < n_s$ for $s < r$. By definition of Q, we have that

$$Q(\partial_{i_s}) \text{ is induced by } \begin{cases} \partial_{i_s+1,1}, & s < r, \\ \partial_{i_s,0}, & s \geq r. \end{cases}$$

Thus the only place where Q can potentially disrupt the normal form is $Q(\partial_{i_{r-1}})$ versus $Q(\partial_{i_r})$. But for Q to actually disrupt it, we must have $i_{r-1} = i_r - 1$, so

$$n_{r-1} = n_r - 1 = i_r - 1 = i_{r-1}$$

which contradicts our choice of r. This completes the proof.

Definition 9.3.3. We extend Q to a functor $Q : \Delta^+ \to \mathsf{cSet}^+$ by defining $\widetilde{Q}^n = Q(\widetilde{\Delta}^n)$ to be the marked cubical set obtained from Q^n by marking the unique non-degenerate n-cube.

Proposition 9.3.4. *The above definition indeed defines a unique functor $Q : \Delta^+ \to \mathsf{cSet}^+$, and moreover it extends to a unique-up-to-isomorphism left adjoint functor $Q : \mathsf{sSet}^+ \to \mathsf{cSet}^+$.*

Proof. Since $Q(\sigma_i) : Q^n \to Q^{n-1}$ send the unique non-degenerate n-cube to a degenerate (and so in particular marked) cube for each i, we immediately see that there is indeed such a unique functor $Q : \Delta^+ \to \mathsf{cSet}^+$. This functor induces an adjunction between $[(\Delta^+)^{\mathrm{op}}, \mathsf{Set}]$ and cSet^+ whose left adjoint we still denote by Q. Since $Q(\phi^n)$ is an epimorphism for each n, we see that the right adjoint to Q lands in the full subcategory $\mathsf{sSet}^+ \subset [(\Delta^+)^{\mathrm{op}}, \mathsf{Set}]$. Thus the restriction of this Q to sSet^+ is still a left adjoint functor; its uniqueness up to isomorphism is obvious. $\square$

Definition 9.3.5. We denote the right adjoint of Q by $\int : \mathsf{cSet}^+ \to \mathsf{sSet}^+$.

Lemma 9.3.6. *The unit* $\mathrm{id}_{\mathsf{cSet}} \Rightarrow \int Q$ *is a natural isomorphism.*

Proof. By the dual of Lemma 2.2.6, the unit of an adjunction is an isomorphism if and only if the left adjoint is fully faithful. The functor $Q : \mathsf{sSet} \to \mathsf{cSet}$ is fully faithful by Theorem 7.2.10, hence $\int QX \to X$ is an isomorphism on underlying simplicial sets, i.e. an entire map, for all X. In particular, this implies it is an isomorphism on all Δ^n; it thus suffices to show that it is also an isomorphism on the objects $\widetilde{\Delta}^n$.

For this, it suffices to show that the marked simplices of $\int \widetilde{Q}^n$ coincide with those of $\widetilde{\Delta}^n$. To see this, observe that marked m-simplices of $\int \widetilde{Q}^n$ are given by maps $\widetilde{\Delta}^m \to \int \widetilde{Q}^n$; by adjointness, these correspond to maps $\widetilde{Q}^m \to \widetilde{Q}^n$, i.e. to marked $(0, m)$-cones of $\widetilde{Q}^n$. The only such cone which is non-degenerate is given by the identity on $\widetilde{Q}^n$, which indeed corresponds to the unique non-degenerate marked simplex of $\widetilde{\Delta}^n$. $\square$

Lemma 9.3.7. *For any $X \in \mathsf{cSet}$, $Q \int X$ is the regular subcomplex of X whose n-cubes are the $(0,n)$-cones in X, and the counit is the inclusion.*

Proof. The assertion that the counit is a monomorphism and the characterization of the underlying cubical set of $Q \int X$ follow from Lemma 7.2.9. To see that the counit is regular, let x be a marked $(0,n)$-cone of X; then $x : Q^n \to X$ factors through $\widetilde{Q}^n = Q$. Thus the corresponding simplex of $\int X$, i.e. the adjunct $\overline{x} : \Delta^n \to \int X$, factors through $\int \widetilde{Q}^n \cong \widetilde{\Delta}^n$. □

Proposition 9.3.8. *There is a natural isomorphism $Q\tau_n \cong \tau_n Q$ for any $n \geq 0$.*

Proof. Since both $Q\tau_n$ and $\tau_n Q$ are cocontinuous, it suffices to check on the marked and the unmarked standard simplices. These special cases are straightforward to check. □

Now we show that Q is left Quillen with respect to the complicial model structure on sSet^+ and the comical model structure on cSet^+.

Lemma 9.3.9. *For any $1 \leq k \leq n$ and $\epsilon \in \{0,1\}$, the map $Q(\Lambda^n_k \hookrightarrow \Delta^n_k)$ is a pushout of a comical open box inclusion.*

Proof. Consider the case $k < n$ so that $Q(\partial_k) = \partial_{k+1,1}$. Then the underlying cubical map of $Q(\Lambda^n_k \hookrightarrow \Delta^n_k)$ is a pushout of the open box inclusion $\sqcap^n_{k+1,1} \hookrightarrow \square^n$ (see [DKLS20, Lemma 6.13] for details). Let $\partial_{i_m,\epsilon_m} \ldots \partial_{i_1,\epsilon_1}$ be a face of $\square^n$ in its normal form, and suppose that it does not involve $\partial_{k,1}$, $\partial_{k+1,0}$, $\partial_{k+1,1}$, or $\partial_{k+2,1}$. We wish to show that the cube in $Q^n_k = Q(\Delta^n_k)$ represented by this face map is marked.

First we treat the sub-case where $\epsilon_i = 1$ for all i. In this case we have the following commutative diagram:

$$\begin{tikzcd}
\square^{n-m} \arrow[r, "\partial_{i_1,1}"] \arrow[d] & \square^{n-m+1} \arrow[r, "\partial_{i_2,1}"] \arrow[d] & \ldots \arrow[r, "\partial_{i_m,1}"] & \square^n \arrow[d] \\
Q^{n-m} \arrow[r, "Q(\partial_{i_1-1})"] & Q^{n-m+1} \arrow[r, "Q(\partial_{i_2-1})"] & \ldots \arrow[r, "Q(\partial_{i_m-1})"] & Q^n
\end{tikzcd}$$

By our assumption, the normal form $\partial_{i_m-1} \ldots \partial_{i_1-1}$ does not involve ∂_{k-1}, ∂_k, or ∂_{k+1}, so this face is marked in Δ^n_k. It follows that the desired cube of Q^n_k is marked.

Now assume $\epsilon_r = 0$ for some $1 \leq r \leq m$. Without loss of generality we may assume that r is the smallest such integer. Write $\bar{r} = n - m + r$. Then we may further assume $(i_r, \ldots, i_m) = (\bar{r}, \ldots, n)$ for otherwise this cube is degenerate in Q^n. It follows from our choice of r that the two faces

$$\partial_{i_m,\epsilon_m} \ldots \partial_{i_1,\epsilon_1} = \partial_{n,\epsilon_m} \ldots \partial_{\bar{r}+1,\epsilon_{r+1}} \partial_{\bar{r},0} \partial_{i_{r-1},1} \ldots \partial_{i_1,1}$$

and

$$\partial_{n,0} \ldots \partial_{\bar{r}+1,0} \partial_{\bar{r},0} \partial_{i_{r-1},1} \ldots \partial_{i_1,1}$$

represent the same cube in Q^n. The latter normal form fits into the following commutative diagram:

$$\begin{tikzcd}
\square^{n-m} \arrow[r, "\partial_{i_1,1}"] \arrow[d] & \square^{n-m+1} \arrow[r, "\partial_{i_2,1}"] \arrow[d] & \ldots \arrow[r, "\partial_{i_{r-1},1}"] & \square^{\bar{r}-1} \arrow[r, "\partial_{\bar{r},0}"] \arrow[d] & \square^{\bar{r}} \arrow[r, "\partial_{\bar{r}+1,0}"] \arrow[d] & \ldots \arrow[r, "\partial_{n,0}"] & \square^n \arrow[d] \\
Q^{n-m} \arrow[r, "Q(\partial_{i_1-1})"] & Q^{n-m+1} \arrow[r, "Q(\partial_{i_2-1})"] & \ldots \arrow[r, "Q(\partial_{i_{r-1}-1})"] & Q^{\bar{r}-1} \arrow[r, "Q(\partial_{\bar{r}})"] & Q^{\bar{r}} \arrow[r, "Q(\partial_{\bar{r}+1})"] & \ldots \arrow[r, "Q(\partial_n)"] & Q^n
\end{tikzcd}$$

To prove that the desired cube is marked in Q^n_k, it suffices to show that the simplicial normal form $\partial_n \ldots \partial_{\bar{r}} \partial_{i_{r-1}-1} \ldots \partial_{i_1-1}$ does not involve ∂_{k-1}, ∂_k, or ∂_{k+1}. Note that, since $(i_r, \ldots, i_m) = (\bar{r}, \ldots, n)$ and $\partial_{i_m,\epsilon_m} \ldots \partial_{i_1,\epsilon_1}$ does not involve $\partial_{k+1,0}$ or $\partial_{k+1,1}$, we must have $\bar{r} > k + 1$. Therefore, if $Q(\partial_{k-1})$, $Q(\partial_k)$, or $Q(\partial_{k+1})$ appears in the above diagram then it must appear to the left of $Q^{\bar{r}-1}$, which is impossible since $\partial_{i_{r-1},1} \dots \partial_{i_1,1}$ does not involve $\partial_{k,1}$, $\partial_{k+1,1}$, or $\partial_{k+2,1}$. This completes the proof of the case $k < n$.

The case $k = n$ can be proven similarly. In fact, this case is easier since it is impossible for $\partial_{i_m,\epsilon_m} \dots \partial_{i_1,\epsilon_1}$ not to involve $\partial_{n,0}$ or $\partial_{n,1}$ and to have $(i_r, \dots, i_m) = (\bar{r}, \dots, n)$ at the same time, which allows us to immediately dismiss the second sub-case. □

Lemma 9.3.10. *For any $1 \le k \le n$ and $\epsilon \in \{0,1\}$, the map $Q(\Delta^n_k{}' \hookrightarrow \Delta^n_k{}'')$ is a pushout of a comical marking extension.*

Proof. Since Q is cocontinuous and commutes with trivialisations, this map fits in the dashed part of the following diagram:

$$\begin{tikzcd}
Q(\Lambda^n_k) \arrow[r] \arrow[d] & \tau_{n-2}Q(\Lambda^n_k) \arrow[d] \arrow[ddr, bend left] & \\
Q(\Delta^n_k) \arrow[r] \arrow[drr, bend right] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\
 & & \tau_{n-2}Q(\Delta^n_k)
\end{tikzcd}$$

The rest of the proof is similar to that of Lemma 9.3.9. □

Theorem 9.3.11. *The functor Q is left Quillen with respect to the (saturated) (n-trivial) complicial model structure on* sSet^+ *and the comical model structure on* cSet^+*.*

Proof. First we must show that Q preserves cofibrations. Since $\mathsf{sSet}^+ \to \mathsf{sSet}$ preserves monomorphisms and $\mathsf{cSet}^+ \to \mathsf{cSet}$ reflects them, it suffices to check that the "unmarked version" $Q : \mathsf{sSet} \to \mathsf{cSet}$ preserves them. This is easy to check (and also appears as [KLW19, Lemma 4.5]).

We have shown in Lemmas 9.3.9 and 9.3.10 Q sends the complicial horn inclusions to pushouts of comical open box inclusions, and the complicial marking extensions to pushouts of comical marking extensions. It is also easy to see from our construction of $\widetilde{Q}^n$ that Q sends each simplicial n-marker to a pushout of the cubical n-marker.

It remains to treat the saturated case. By Lemmas 3.3.20, 9.3.9 and 9.3.10, to show that Q sends all saturation maps to trivial cofibrations, it suffices to show that it sends all Rezk maps to trivial cofibrations. The object L, the domain of the elementary simplicial Rezk map, may be written as the colimit of the following diagram:

$$\begin{tikzcd}
 & \Delta^1 \arrow[dl] \arrow[dr, "\partial_1"] & & \Delta^1 \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_2"] & & \Delta^1 \arrow[dl, "\partial_1"'] \arrow[dr] & \\
\widetilde{\Delta^1} & & \widetilde{\Delta^2} & & \widetilde{\Delta^2} & & \widetilde{\Delta^1}
\end{tikzcd}$$

It follows from the above colimit description of L that we can obtain QL from two marked 2-cubes, which we call *left* and *right*, by:

- collapsing each cube to Q^2 (by collapsing $\partial_{1,0}$);
- gluing $\partial_{1,1}$ of the left cube to $\partial_{2,0}$ of the right cube; and
- marking $\partial_{2,1}$ of each cube.

Thus QA is the cubical set illustrated below on the left, while $L_{1,2}$ is illustrated on the right.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

where thick arrows indicate marked cubes, and equal signs indicate degenerate cubes. So we obtain a map $L_{1,2} \to QA$, and similarly a map $L'_{1,2} \to QB$. Since both $L_{x,y} \to L'_{x,y}$ and $QL \to QL'$ are entire, we can deduce by an easy analysis of the marked cubes that the latter is a pushout of the former. Therefore $QL \to QL'$ is a trivial cofibration in the saturated model structure.

Now fix $k \geq 0$ and consider $Q(\Delta^k \star L) \to Q(\Delta^k \star L')$. Observe that $\Delta^k \star L$ consists of two $(k+3)$-simplices in each of which a face is marked if and only if neither ∂_{k+1} nor ∂_{k+3} appears in its normal form. In other words, in each of these $(k+3)$-simplices, $\partial_0, \partial_1, \ldots, \partial_k, \partial_{k+2}$ and all their possible intersections are marked. Thus, similarly to QL, we can construct $Q(\Delta^k \star L)$ by:

- taking two $(k+3)$-cubes, called *left* and *right*;
- collapsing each of them to Q^{k+3};
- gluing $\partial_{k+2,1}$ of the left cube to $\partial_{k+3,0}$ of the right cube; and
- marking, in each of the cubes, $\partial_{1,1}, \partial_{2,1}, \ldots, \partial_{k+1,1}, \partial_{k+3,1}$ and all their possible intersections.

On the other hand, $\square^{k+1} \otimes L_{x,y}$ can be obtained by:

- taking two $(k+3)$-cubes, called *left* and *right*;
- gluing $\partial_{k+2,1}$ of the left cube to $\partial_{k+3,0}$ of the right cube; and
- marking, in each cube, any face of the form $\delta \otimes \mathrm{id}$, $\delta \otimes \partial_{1,0}$, or $\delta \otimes \partial_{2,1}$ for some face δ in $\square^{k+1}$.

Note that the last clause may be rephrased as:

- marking, in each cube, any face $\partial_{i_m,\epsilon_m} \ldots \partial_{i_1,\epsilon_1}$ (in its normal form) with
  - $i_m \leq k+1$;
  - $(i_m, \epsilon_m) = (k+2, 0)$; or

– $(i_m, \epsilon_m) = (k+3, 1)$ and $i_{m-1} \neq k+2$.

We claim that the obvious assignation defines a legitimate map $\Box^{k+1} \otimes L_{x,y} \to Q(\Delta^k \star L)$ in cSet^+. Indeed, pick a non-degenerate marked cube $\phi = \partial_{i_m,\epsilon_m} \ldots \partial_{i_1,\epsilon_1}$ in either the left or the right cube of $\Box^{k+1} \otimes L_{x,y}$.

- First consider the case where $\epsilon_r = 0$ for some r. Then we must have $r \leq k+2$. Since $\partial_{i_m,\epsilon_m} \ldots \partial_{i_1,\epsilon_1}$ cannot contain both $\partial_{k+2,\epsilon}$ and $\partial_{k+3,\epsilon'}$ at the same time, it follows that ϕ is degenerate in $Q(\Delta^k \star L)$.

- Now suppose that $\epsilon_r = 1$ for all r. Then ϕ is an intersection of some (possibly empty) combination of $\partial_{1,1}, \partial_{2,1}, \ldots, \partial_{k+1,1}, \partial_{k+3,1}$. It follows that ϕ is marked in $Q(\Delta^k \star L)$.

Similarly, we obtain a map $\Box^{k+1} \otimes L'_{x,y} \to Q(\delta^k \star L')$, and these maps fit into the following commutative square:

$$\begin{tikzcd}
\Box^{k+1} \otimes L_{x,y} \arrow[r] \arrow[d] & Q(\Delta^k \star L) \arrow[d] \\
\Box^{k+1} \otimes L'_{x,y} \arrow[r] & Q(\Delta^k \star L')
\end{tikzcd}$$

Observe that, for each vertical map (which is entire), the cubes that are marked in the codomain but not in the domain are precisely those corresponding to $\partial_{i_m,\epsilon_m} \ldots \partial_{i_1,\epsilon_1}$ with:

- either $(i_m, \epsilon_m) = (k+2, 1)$ or $(i_m, \epsilon_m) = (k+3, 0)$; and

- either $m = 1$ or $i_{m-1} \leq k+1$.

It follows that this square is a pushout, which completes the proof. $\Box$

Our next goal is to construct a natural transformation $\rho : TQ \Rightarrow \mathrm{id}$ and to exhibit it as a natural weak equivalence.

Remark 9.3.12. Throughout the remainder of this section, when we refer to a *weak equivalence* in sSet^+, we will always mean one with respect to the complicial model structure (without saturation or n-triviality), unless otherwise noted.

Note that, since T is cocontinuous, we can compute TQ^n as the following pushout:

$$\begin{tikzcd}
\coprod\limits_{1 \leq i \leq n} T(\square^{i-1} \otimes \square^{n-i}) \arrow[r] \arrow[d] & T\square^n \arrow[d] \\
\coprod\limits_{1 \leq i \leq n} T\square^{i-1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & TQ^n
\end{tikzcd}$$

It inherits a unique unmarked n-simplex from $T\square^n$, and marking this simplex yields $T\widetilde{Q}^n$.

Recall that an r-simplex in $T\square^n$ corresponds to a sequence $\phi \in \{1, \ldots, r, \pm\infty\}^n$. Since the right vertical map in the above pushout square is an epimorphism, any r-simplex in TQ^n may also be represented (not necessarily uniquely) by such ϕ. Two sequences ϕ and χ represent the same simplex if and only if, for any i with $\phi_i \neq \chi_i$, there exists $j < i$ with $\phi_j = \chi_j = +\infty$.

Definition 9.3.13. For $n \geq 0$, we define $\rho^n : TQ^n \to \Delta^n$ by sending an r-simplex represented by a sequence ϕ to $\rho^n(\phi) : [r] \to [n]$ given by

$$\rho^n(\phi)(p) = \max\Big(\{k \in [n] \mid (\forall i \leq k)\ \phi_i \leq p\} \cup \{0\}\Big).$$

The map $\widetilde{\rho}^n : T\widetilde{Q}^n \to \widetilde{\Delta^n}$ (for $n \geq 1$) has the same underlying simplicial map.

Remark 9.3.14. If we regard the vertices of $T\square^n$ as binary strings of length n, then $T\square^n \to TQ^n$ acts on those vertices by identifying any two strings that have their first 0 in the same position. Intuitively, the map $\rho^n : TQ^n \to \Delta^n$ is well defined because it essentially counts the number of 1's before the first 0. (Compare this with the corresponding unmarked map constructed in 7.2, which identifies strings having their first 1 in the same position.)

Proposition 9.3.15. *The above definitions indeed yield maps $\rho^n : TQ^n \to \Delta^n$ and $\widetilde{\rho}^n : T\widetilde{Q}^n \to \widetilde{\Delta^n}$ in* sSet^+*. Moreover these maps extend to a unique natural transformation $\rho : TQ \to \mathrm{id}$ with $\rho_{\Delta^n} = \rho^n$ and $\rho_{\widetilde{\Delta^n}} = \widetilde{\rho}^n$.*

Proof. We must show that:

(i) ρ^n at least defines a valid map between the underlying simplicial sets;

(ii) ρ^n preserves marked simplices;

(iii) ρ is natural in n; and

(iv) similarly for $\widetilde{\rho}$; and

(v) those maps indeed extends to a unique natural transformation ρ.

The proofs of (1) and (3) are analogous to those of Proposition 7.2.17 and Lemma 7.2.18, respectively. We will skip (4) since it will be almost identical to (1-3).

To prove (2), consider an r-simplex ϕ in $T\square^n$. Suppose that this simplex is:

(i) non-degenerate, or equivalently each integer in $\{1, \ldots, r\}$ appears at least once in ϕ; and

(ii) marked, or equivalently there is no sequence $1 \leq i_1 < \cdots < i_r \leq n$ such that $\phi_{i_p} = p$.

We must show that ρ^n sends the image of such ϕ in TQ^n to a marked simplex in Δ^n. We claim that there exist $1 \leq p < q \leq r$ with

$$\min\{i \mid \phi_i = p\} > \min\{i \mid \phi_i = q\}.$$

Indeed, both minima are well defined because of (i), and such p, q must exist for otherwise setting $i_p = \min\{i \mid \phi_i = p\}$ would violate (2). It follows that $\rho^n(\phi)(p) = \rho^n(\phi)(p-1)$, so the simplex $\rho^n(\phi)$ is degenerate (and hence marked) in Δ^n.

It remains to prove (5). Observe that, by construction of Q, we can regard TQ as the restriction of a cocontinuous functor $[(\Delta^+)^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{sSet}^+$ to the full subcategory sSet^+. Similarly, we may regard the identity functor on sSet^+ as the restriction of the (cocontinuous) reflection $[(\Delta^+)^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{sSet}^+$. Since $[(\Delta^+)^{\mathrm{op}}, \mathsf{Set}]$ is the free cocompletion of Δ^+, the maps ρ^n and $\widetilde{\rho}^n$ extend to a unique natural transformation between those functors $[(\Delta^+)^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{sSet}^+$. We thus obtain the desired natural transformation by restricting it to sSet^+. Its uniqueness follows from the fact that any marked simplicial set can be written as a colimit of Δ^n and $\widetilde{\Delta^n}$. □

Now we prove that ρ is a natural weak equivalence.

Lemma 9.3.16. *The component $\rho^n : TQ^n \to \Delta^n$ is a weak equivalence.*

Proof. We will exhibit Δ^n as a deformation retract of TQ^n with the retraction part given by ρ^n.

We define $\zeta^n : \Delta^n \to TQ^n$ to be the map picking out the unique unmarked n-simplex ι_n, *i.e.* the one represented by the sequence $1\,2\ldots n$. Then it is straightforward to check that $\rho^n\zeta^n = \mathrm{id}$ holds. Note that, more explicitly, ζ^n sends an r-simplex $\alpha : [r] \to [n]$ to the one represented by the sequence $\phi \in \{1, \dots, r, \pm\infty\}^n$ given by

$$\phi_i = \begin{cases} +\infty, & i > \alpha(r), \\ p, & \alpha(p-1) < i \leq \alpha(p), \\ -\infty, & i \leq \alpha(0). \end{cases}$$

We will construct a (left) homotopy between $\zeta^n\rho^n$ and the identity at TQ^n. First, we define a map $H : (\Delta^1)^n \times \Delta^1 \to (\Delta^1)^n$ in sSet so that its action on the 0-simplices (regarded as binary strings) is given by:

$$\begin{aligned} H(\epsilon_1, \epsilon_2, \dots, \epsilon_n, 0) &= \big(\min\{\epsilon_1\}, \min\{\epsilon_1, \epsilon_2\}, \dots, \min\{\epsilon_1, \dots, \epsilon_n\}\big), \\ H(\epsilon_1, \epsilon_2, \dots, \epsilon_n, 1) &= (\epsilon_1, \epsilon_2, \dots, \epsilon_n). \end{aligned}$$

In other words, $H(-,1)$ acts as the identity and $H(-,0)$ replaces all entries after the first 0 (if it exists) by 0's.

Claim. H lifts to a map $H : (\Delta^1)^{\otimes n} \times \widetilde{\Delta^1} \to (\Delta^1)^{\otimes n}$ in sSet^+.

Proof of Claim. Let us describe this map H in terms of the sequence representation of simplices. Fix an r-simplex $\phi \in \{1, \dots, r, \pm\infty\}^{n+1}$, and write $q = \phi_{n+1}$. Then H sends ϕ to $\chi \in \{1, \dots, r, \pm\infty\}^n$ given by

$$\chi_i = \begin{cases} \phi_i, & \phi_i \geq q, \\ \min\big\{\max\{\phi_1, \dots, \phi_i\}, \phi_{n+1}\big\}, & \phi_i < q. \end{cases}$$

Suppose that this χ is marked when regarded as an r-simplex in $(\Delta^1)^{\otimes n}$. Then there exist

$$1 \leq j_1 < \cdots < j_{q-1} < i_q < \cdots < i_r \leq n$$

such that

- $\phi_{i_p} = p$ for $q \leq p \leq r$; and
- $\max\{\phi_1, \ldots, \phi_{j_p}\} = p$ for $1 \leq p < q$.

(If $q = -\infty$ then this is interpreted as the existence of such $i_1, \ldots, i_r$, and if $q = +\infty$ then this is interpreted as the existence of such $j_1, \ldots, j_r$.) We can then deduce by an elementary analysis of the max function that there exist

$$1 \leq i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_{q-1} \leq j_{q-1}$$

such that $\phi_{i_p} = p$ for all $1 \leq p < q$. It follows that the projection of ϕ onto the first factor $(\Delta^1)^n$ is unmarked when regarded as an r-simplex in $(\Delta^1)^{\otimes n}$, which in turn implies that ϕ itself is unmarked as an r-simplex in $(\Delta^1)^{\otimes n} \times \widetilde{\Delta^1}$. This proves the claim. $\square$

Now it is easy to check that H restricts as

$$\begin{tikzcd}
(\Delta^1)^{\otimes(n-1)} \times \widetilde{\Delta^1} \arrow[r, "\partial_{i,0} \times \mathrm{id}"] \arrow[d, dashed, "H_i"'] & (\Delta^1)^{\otimes n} \times \widetilde{\Delta^1} \arrow[d, "H"] \\
(\Delta^1)^{\otimes(n-1)} \arrow[r, "\partial_{i,0}"'] & (\Delta^1)^{\otimes n}
\end{tikzcd}$$

for each i, and moreover each H_i descends as:

$$\begin{tikzcd}
\big((\Delta^1)^{\otimes(i-1)} \otimes (\Delta^1)^{\otimes(n-i)}\big) \times \widetilde{\Delta^1} \arrow[r] \arrow[d, "H_i"'] & (\Delta^1)^{\otimes(i-1)} \times \widetilde{\Delta^1} \arrow[d, dashed, "\overline{H_i}"] \\
(\Delta^1)^{\otimes(i-1)} \otimes (\Delta^1)^{\otimes(n-i)} \arrow[r] & (\Delta^1)^{\otimes(i-1)}
\end{tikzcd}$$

We can thus take the pushout of each row in

$$\begin{tikzcd}
(\Delta^1)^{\otimes n} \times \widetilde{\Delta^1} \arrow[d, "H"'] & \coprod_i \big((\Delta^1)^{\otimes(i-1)} \otimes (\Delta^1)^{\otimes(n-i)}\big) \times \widetilde{\Delta^1} \arrow[l] \arrow[r] \arrow[d, "\coprod_i H_i"'] & \coprod_i (\Delta^1)^{\otimes(i-1)} \times \widetilde{\Delta^1} \arrow[d, "\coprod_i \overline{H_i}"] \\
(\Delta^1)^{\otimes n} & \coprod_i (\Delta^1)^{\otimes(i-1)} \otimes (\Delta^1)^{\otimes(n-i)} \arrow[l] \arrow[r] & \coprod_i (\Delta^1)^{\otimes(i-1)}
\end{tikzcd}$$

which yields the desired left homotopy $TQ^n \times \widetilde{\Delta^1} \to TQ^n$ from $\zeta^n \rho^n$ to id. □

Lemma 9.3.17. *The component $\widetilde{\rho}^n : T\widetilde{Q}^n \to \widetilde{\Delta^n}$ is a weak equivalence.*

Proof. Fix $n \geq 1$. Then it follows from Lemma 9.3.16 that ζ^n is a trivial cofibration. Thus its pushout along the n-marker $\Delta^n \to \widetilde{\Delta^n}$ is also a trivial cofibration. But it is easy to check that this pushout is a section of $\widetilde{\rho}^n$, so the lemma follows by the 2-out-of-3 property. □

Theorem 9.3.18. *The component $\rho_X : TQX \to X$ at any $X \in \mathsf{sSet}^+$ is a weak equivalence.*

Proof. First, we prove the special case where X is n-skeletal (*i.e.* the underlying simplicial set of X is n-skeletal.) We proceed by induction on $n \geq -1$.

The base case is easy since the cocontinuity of TQ implies that $\rho_\varnothing$ is invertible. For the inductive step, fix $n \geq 0$ and assume that ρ_Y is a weak equivalence for any $(n-1)$-skeletal Y. Let X be an n-skeletal marked simlicial set and denote by X' its regular $(n-1)$-skeleton. Then we may obtain ρ_X by taking the pushout of each row in

$$\begin{tikzcd}
TQX' \arrow[d, "\rho_{X^{n-1}}"'] & \left(\coprod TQ\partial\Delta^n\right) \amalg \left(\coprod TQ\partial\Delta^n\right) \arrow[l] \arrow[r] \arrow[d, "(\coprod \rho)\amalg(\coprod \rho)"] & \left(\coprod TQ\Delta^n\right) \amalg \left(\coprod TQ\widetilde{\Delta^n}\right) \arrow[d, "(\coprod \rho)\amalg(\coprod \rho)"] \\
X' & \left(\coprod \partial\Delta^n\right) \amalg \left(\coprod \partial\Delta^n\right) \arrow[l] \arrow[r] & \left(\coprod \Delta^n\right) \amalg \left(\coprod \widetilde{\Delta^n}\right)
\end{tikzcd}$$

where, in each of the right four objects, the first (respectively second) coproduct ranges over the unmarked (resp. non-degenerate marked) n-simplices in X. The left and the middle vertical maps are weak equivalences by the inductive hypothesis. The right vertical map is also a weak equivalence by Lemmas 9.3.16 and 9.3.17 (note that the weak equivalences are closed under coproducts since they may be factorised as a trivial cofibration followed by a retraction of a trivial cofibration). Since both of the right-pointing arrows are cofibrations, it follows that the induced map ρ_X is again a weak equivalence.

Now we prove the theorem for general X. For each n, write X^n for the regular n-skeleton of X. Then $n \mapsto TQX^n$ and $n \mapsto X^n$ yield two sequences $\omega^{\mathrm{op}} \to \mathsf{sSet}^+$ of cofibrations. Since ρ provides a natural weak equivalence between these two sequences, the colimit $\rho_X : TQX \to X$ is still a weak equivalence. This completes the proof. $\square$

Corollary 9.3.19. *The functor $Q\colon \mathsf{sSet}^+ \to \mathsf{cSet}^+$ preserves and reflects weak equivalences, where sSet^+ is equipped with the model structure for (n-trivial, saturated) complicial sets, and cSet^+ is equipped with the model structure for the corresponding comical sets.*

Proof. That Q preserves weak equivalences is immediate from Theorem 9.3.11.

To see that Q reflects weak equivalences, let $X \to Y$ be a map in sSet^+, such that $QX \to QY$ is a weak equivalence. By Theorem 9.3.18, we have a commuting diagram:

$$\begin{tikzcd} TQX \arrow[r] \arrow[d] & TQY \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

in which the two vertical maps are weak equivalences. Since T preserves weak equivalences by Theorem 9.2.42, $TQX \to TQY$ is a weak equivalence as well. Thus $X \to Y$ is a weak equivalence by two-out-of-three. □

9.4 Triangulation is a Quillen equivalence

Our strategy for showing that $T \dashv U$ is a Quillen equivalence will be to first show that this is true of $Q \dashv \int$, and then apply Theorem 9.3.18. We begin by introducing the marked analogues of the objects $B^{m,n,k}$ constructed in Section 7.2.

Definition 9.4.1. For $n \geq 1$, $1 \leq i \leq n$, the *strongly* $(i,1)$-*comical cube*, denoted $\overline{\square}^n_{i,1}$, is the marked cubical set whose underlying cubical set is $\square^n$, with a non-degenerate face marked if and only if its standard form does not contain any of the maps $\partial_{i-1,1}$ (if $i > 1$), $\partial_{i,0}$, or $\partial_{i,1}$.

For $i+1 \leq j \leq n+1$, we let $\Gamma^n_{i,j}$ denote the regular subcomplex of $\overline{\square}^n_{i,1}$ consisting of all negative faces, as well as the positive faces $(k,1)$ for which $1 \leq k \leq i-1$ or $j \leq k \leq n$.

Note that in the case $j = n+1$, the only positive faces contained in $\Gamma^n_{i,j}$ are $(1,1)$ through $(i-1,1)$.

Lemma 9.4.2. *For $n \geq 1$, $1 \leq i \leq n$, let x be an n-cube in a cubical set X.*

(i) If x is strongly $(i,1)$-comical, then so is $x\sigma_j$ for $j \geq i$.

(ii) *If x is strongly $(i,1)$-comical, then $x\gamma_{j,1}$ is strongly $(i+1,1)$-comical for $j \leq i-1$.*

(iii) *If x is strongly $(i,1)$-comical, then so is $x\gamma_{j,\varepsilon}$ for $j \geq i, \varepsilon \in \{0,1\}$.*

(iv) *The cube $x\gamma_{i,1}$ is strongly $(i+1,1)$-comical.*

Proof. We prove items (i) and (iv); the proofs of items (ii) and (iii) are similar to these.

For item (i), consider a $x\sigma_i\delta$, with δ written in standard form as follows:

$$x\sigma_i\partial_{a_1,\varepsilon_1}\ldots\partial_{a_p,\varepsilon_p}\partial_{b_1,\varepsilon'_1}\ldots\partial_{b_q,\varepsilon'_q}$$

where $a_p \geq i+1$, $b_1 \leq i-1$, and if $b_1 = i-1$ then $\varepsilon_1 = 0$. First, suppose that $a_k = j$ for some $1 \leq k \leq p$; then we can rewrite this expression into standard form as:

$$x\partial_{a_1-1,\varepsilon_1}\ldots\partial_{a_{k-1}-1,\varepsilon_{k-1}}\partial_{a_{k+1},\varepsilon_{k+1}}\ldots\partial_{a_p,\varepsilon_p}\partial_{b_1,\varepsilon'_1}\ldots\partial_{b_q,\varepsilon'_q}$$

By assumption, $a_l > a_k \geq a_p \geq i+1$ for all $l < k$, so the indices $a_l - 1$ are still greater than or equal to $i+1$, while all other maps in this standard form are unchanged. Thus this face of x is marked by the assumption that x is strongly $(i,1)$-comical.

On the other hand, suppose that no a_k is equal to j; it must also be true that no b_k is equal to j as $b_k \leq i-1 < j$ for all k. Then let l be maximal such that $a_l > j$; we can rewrite this expression into standard form as:

$$x\partial_{a_1,\varepsilon_1}\ldots\partial_{a_l-1,\varepsilon_l}\partial_{a_{l+1},\varepsilon_{l+1}}\ldots\partial_{a_p,\varepsilon_p}\partial_{b_1,\varepsilon'_1}\ldots\partial_{b_q,\varepsilon'_q}\sigma_{i-p+l-q}$$

This is degenerate, hence marked.

For item (iv), consider a face of $x\gamma_{i,1}$ written in standard form as

$$x\gamma_{i,1}\partial_{a_1,\varepsilon_1}\dots\partial_{a_p,\varepsilon_p}\partial_{b_1,\varepsilon'_1}\dots\partial_{b_q,\varepsilon'_q}$$

where now $a_p \geq i+2$, $b_1 \leq i$, and if $b_1 = i$ then $\varepsilon'_1 = 0$. We can rewrite this expression as:

$$x\partial_{a_1-1,\varepsilon_1}\dots\partial_{a_p-1,\varepsilon_p}\gamma_{i,1}\partial_{b_1,\varepsilon'_1}\dots\partial_{b_q,\varepsilon'_q}$$

We consider two possible cases based on the value of b_1. If $b_1 = i$ then $\varepsilon'_1 = 0$, and we can rewrite the expression as:

$$\begin{aligned}&x\partial_{a_1-1,\varepsilon_1}\dots\partial_{a_p-1,\varepsilon_p}\partial_{b_1,\varepsilon'_1}\sigma_i\partial_{b_2,\varepsilon'_2}\dots\partial_{b_q,\varepsilon'_q}\\ =&x\partial_{a_1-1,\varepsilon_1}\dots\partial_{a_p-1,\varepsilon_p}\partial_{b_1,\varepsilon'_1}\partial_{b_2,\varepsilon'_2}\dots\partial_{b_q,\varepsilon'_q}\sigma_{i-q+1}\end{aligned}$$

On the other hand, if $b_1 < i$, then the expression becomes:

$$x\partial_{a_1-1,\varepsilon_1}\dots\partial_{a_p-1,\varepsilon_p}\partial_{b_1,\varepsilon'_1}\dots\partial_{b_q,\varepsilon'_q}\gamma_{i-q}$$

Either way this cube is degenerate, hence marked. □

Lemma 9.4.3. *For n, i as above and $i < k \leq n$, the $(k,1)$-face of $\overline{\square}^n_{i,1}$ is isomorphic to $\overline{\square}^{n-1}_{i,1}$.*

Proof. It is clear that the underlying cubical set of $\partial_{k,1}$ is $\square^{n-1}$, so it remains to be verified that the marked faces of $\partial_{k,1}$ are precisely those which are marked in $\overline{\square}^{n-1}_{i,0}$.

To see this, consider a face $\partial_{k,1}\delta$; write the standard form of δ as $\partial_{a_1,\varepsilon_1}\dots\partial_{a_p,\varepsilon_p}\dots\partial_{a_q,\varepsilon_q}$, where p is maximal such that $a_p \geq k$. Then we can rearrange $\partial_k \delta$ into standard form as:

$$\partial_{a_1+1,\varepsilon_1} \ldots \partial_{a_p+1,\varepsilon_p} \partial_{k,1} \partial_{a_{p+1},\varepsilon_{p+1}} \ldots \partial_{a_q,\varepsilon_q}$$

This cube is marked if and only if this standard form does not contain any of the maps $\partial_{i-1,1}, \partial_{i,0}$, or $\partial_{i,1}$. As $k > i$ by assumption, this holds if and only if none of these maps appear in the standard form of δ. $\square$

Lemma 9.4.4. *For n, i, j as above, the inclusion $\Gamma^n_{i,j} \hookrightarrow \overline{\square}^n_{i,1}$ is anodyne.*

Proof. We proceed by induction on n. For $n = 1$, the only case to consider is the inclusion $\Gamma^1_{1,2} \hookrightarrow \overline{\square}^n_{1,1}$, but this is isomorphic to the $(1,1)$-comical open box filling in dimension 1.

Now consider $n \geq 2$, and suppose the statement holds for $n - 1$. We first show that for $i+2 \leq j \leq n+1$, the inclusion $\Gamma^n_{i,j} \hookrightarrow \Gamma^n_{i,j-1}$ is anodyne. To see this, we consider the intersection of $\partial_{j-1,1}$ with $\Gamma^n_{i,j}$, i.e. the intersections of $\partial_{j-1,1}$ with each of the faces contained in $\Gamma^n_{i,j}$. Performing some simple calculations with the cubical identities, we see that these consist of the following faces:

- $\partial_{k,0}\partial_{j-2,1} = \partial_{j-1,1}\partial_{k,0}$ for $1 \leq k \leq j-2$;
- $\partial_{k,0}\partial_{j-1,1} = \partial_{j-1,1}\partial_{k-1,0}$ for $j-1 \leq k \leq n$;
- $\partial_{k,1}\partial_{j-2,1} = \partial_{j-1,1}\partial_{k,1}$ for $1 \leq k \leq i-1$;
- $\partial_{k,1}\partial_{j-1,1} = \partial_{j-1,1}\partial_{k-1,1}$ for $j-1 \leq k \leq n$.

By Lemma 9.4.3, this implies that $\partial_{j-1,1} \cap \Gamma^n_{i,j}$ is isomorphic to $\Gamma^{n-1}_{i,j-1}$, and that the inclusion $\Gamma^n_{i,j} \hookrightarrow \Gamma^n_{i,j-1}$ is the pushout of $\Gamma^{n-1}_{i,j-1} \hookrightarrow \overline{\square}^{n-1}_{i,0}$ along this isomorphism. Thus this inclusion is anodyne by the induction hypothesis.

To the prove the desired statement for n, it thus suffices to show that $\Gamma^n_{i,i+1} \hookrightarrow \overline{\square}^n_{i,1}$ is anodyne. But since every marked cube of $\square^n_{i,1}$ is marked in $\overline{\square}^n_{i,1}$, this map is a pushout of the $(i,1)$-comical open box inclusion. $\square$

Definition 9.4.5. For $m \geq 1, n \geq 0$, $B^{m,n}$ is the subcomplex of $C^{m,n}$ consisting of the images of the faces $\partial_{1,1}$ through $\partial_{n+1,1}$, as well as all all faces $\partial_{i,0}$, under the quotient map $\square^{m+n} \to C^{m,n}$.

The *strongly comical* (m,n)*-cone* $\overline{C}^{m,n}$ is given by the following pushout:

$$\begin{tikzcd}
\square^{m+n} \arrow[r] \arrow[d] & C^{m,n} \arrow[d] \\
\overline{\square}^{m+n}_{n+1,1} \arrow[r] & \overline{C}^{m,n}
\end{tikzcd}$$

In other words, $\overline{C}^{m,n}$ is obtained from $C^{m,n}$ by marking the images under the quotient map $\square^{m+n} \to C^{m,n}$ of the marked cubes of $\overline{\square}^{n+1,1}$. Likewise, $\overline{B}^{m,n}$ is the regular subcomplex of $\overline{C}^{m,n}$ with underlying cubical set $B^{m,n}$.

Lemma 9.4.6. *For all $m, n \geq 0$, the inclusion $\overline{B}^{m,n} \to \overline{C}^{m,n}$ is anodyne.*

Proof. We will show that the following square is a pushout:

$$\begin{tikzcd}
\Gamma^{m+n}_{n+2} \arrow[r] \arrow[d] & \overline{B}^{m,n} \arrow[d] \\
\overline{\square}^{m+n}_{m+n+1,0} \arrow[r] & \overline{C}^{m,n}
\end{tikzcd}$$

By Lemma 7.1.8, if two cubes of $\square^{m+n}$ are identified in $C^{m,n}$ then they are both contained in some face $\partial_{k,0}$, and hence in $B^{m,n}$. Thus the underlying diagram of cubical sets is a pushout, as the cubes of $C^{m,n}$ not present in $B^{m,n}$ are subject to no identifications. It thus follows that the square itself is a pushout, as the non-degenerate marked cubes of $\overline{C}^{m,n}$ are precisely the images under the quotient map $\square^{m+n} \to C^{m,n}$ of those which are marked in $\overline{\square}^{m+n}$. $\square$

We can adapt Definition 7.1.24 to the setting of comical sets. Though the identities here are essentially the same as those of Definition 7.1.24, we repeat them for ease of reference, and to highlight that all values of ε involved are different in this formulation.

Definition 9.4.7. A *coherent family of composites* θ in a comical set X is a family of functions $\theta^{m,n} \colon \mathsf{cSet}^+(C^{m,n}, X) \to \mathsf{cSet}^+(\overline{C}^{m,n+1}, X)$ for $m \geq 1, n \geq 0$, satisfying the following identities:

(Θ1) for $i \leq n$, $\theta^{m,n}(x)\partial_{i,1} = \theta^{m,n-1}(x\partial_{i,1})$;

(Θ2) $\theta^{m,n}(x)\partial_{n+1,1} = x$;

(Θ3) for $m \geq 2$ and $i \geq n+2$, $\theta^{m,n}(x)\partial_{i,0} = \theta^{m-1,n}(x\partial_{i-1,0})$;

(Θ4) for $i \geq n+2$, $\theta^{m,n}(x)\sigma_i = \theta^{m+1,n}(x\sigma_{i-1})$;

(Θ5) for $i \leq n$, $\theta^{m,n}(x)\gamma_{i,1} = \theta^{m,n+1}(x\gamma_{i,1})$;

(Θ6) for $i \geq n+2$, then $\theta^{m,n}(x)\gamma_{i,\varepsilon} = \theta^{m+1,n}(x\gamma_{i-1,\varepsilon})$;

(Θ7) $\theta^{m,n+1}(\theta^{m,n}(x)) = \theta^{m,n}(x)\gamma_{n+1,1}$

(Θ8) for $x \colon C^{m-1,n+1} \to X$, $\theta^{m,n}(x) = x\gamma_{n+1,1}$.

Note that in Section 7.2, $\theta^{0,n}$ is defined to be the degeneracy operator σ_{n+1}. This would not be appropriate in the marked setting, as $\overline{C}^{m,n}$ is only defined for $m \geq 1$. However, as the definition of $\theta^{0,n}$ is merely a notational convenience, the combinatorial proofs of Section 7.2 and Appendix A remain valid here.

Our next goal is to prove the following theorem.

Theorem 9.4.8. *Every comical set admits a coherent family of composites.*

The following lemmas will be used in defining $\theta^{m,n}$ in the inductive case.

Lemma 9.4.9. *Let $m \geq 2, n \geq 0$, and let X be a comical set equipped with functions $\theta^{m,n}$ satisfying the identities of Definition 9.4.7 for all pairs (m', n') such that $m' \leq m$, $n' \leq n$, and at least one of these two inequalities is strict. Then for any $x: C^{m,n} \to X$, the faces of $\theta^{m,n}(x)$ fixed by the identities $(\Theta 1)$ through $(\Theta 3)$ define a map $\overline{B}^{m,n+1} \to X$.*

Proof. That these faces define a map $B^{m,n+1} \to X$ is shown in the proof of Lemma 7.1.29. Thus it remains to show that this map factors through $\overline{B}^{m,n+1}$. In other words, we must show that any face of the form $\theta^{m,n}(x)\delta$ is marked if the standard form of δ is non-empty and contains only maps of the form $\partial_{k,\varepsilon}$ where either $\varepsilon = 0$ and $k \neq n+2$ or $\varepsilon = 1$ and $k \leq n$.

First suppose that the standard form of δ contains some map of the form $\partial_{k,0}$ for $k \leq n+1$. Then, since it contains no face of the form $\delta_{n+2,\varepsilon}$ by assumption, this is a degenerate face of $B^{m,n+1}$ by Lemma 7.1.10 (recalling that values of ε are reversed here compared to the unmarked setting). Thus this face is marked.

Now consider a face written in standard form as:

$$\theta^{m,n}(x)\partial_{a_1,0} \ldots \partial_{a_p,0}\partial_{b_1,1} \ldots \partial_{b_q,1}$$

where $a_p \geq n+3$ and $b_1 \leq n$. Using the identities $(\Theta 1)$ and $(\Theta 3)$, we can rewrite this as :

$$\theta^{m-p,n-q}(x\partial_{a_1-1,0} \ldots \partial_{a_p-1,0}\partial_{b_1,1} \ldots \partial_{b_q,1})$$

which is marked by assumption. (Note that $m - p \geq 1$ since there can be at most $m - 1$ face maps with index greater than or equal to $n + 3$.) $\square$

Proof of Theorem 9.4.8. As in the unmarked case, we construct the functions

$\theta^{m,n}$ by induction. For the base case $m = 1$, we set $\theta^{1,n}(x) = x\gamma_{n+1,1}$. (Once again, since every $(1,n)$-cone is a $(0,n+1)$-cone by Lemma 7.1.2, this definition is required by identity (Θ8).)

Now suppose we have defined $\theta^{m',n'}$ for all such pairs with $m' \leq m, n' \leq n$, and at least one of these inequalities strict. As in Definition 7.1.30, we will define $\theta^{m,n}$ by case analysis on $x \colon C^{m,n} \to X$, and we may reduce the number of cases to be considered using Lemma 7.1.17.

For $x \colon C^{m,n} \to X$, we define $\theta^{m,n}(x)$ as follows:

(i) If the standard form of x is $z\sigma_{a_p}$ for some $a_p \geq n+1$, then $\theta^{m,n}(x) = \theta^{m-1,n}(z)\sigma_{a_p+1}$;

(ii) If the standard form of x is $z\gamma_{b_q,1}$ for some $b_q \leq n-1$, then $\theta^{m,n}(x) = \theta^{m,n-1}(z)\gamma_{b_q,1}$;

(iii) If the standard form of x is $z\gamma_{b_q,\varepsilon}$ for some $b_q \geq n+1$, then $\theta^{m,n}(x) = \theta^{m-1,n}(z)\gamma_{b_q+1,\varepsilon}$;

(iv) If x is an $(m-1,n+1)$-cone not covered under any of cases (1) through (3), then $\theta^{m,n}(x) = x\gamma_{n+1,1}$;

(v) If $x = \theta^{m,n-1}(x')$ for some $x' \colon C^{m,n-1} \to X$ and x is not covered under any of cases (1) through (4) then $\theta^{m,n}(x) = x\gamma_{n,1}$;

(vi) If x is not convered under any of cases (1) through (5), we construct $\theta^{m,n}(x)$ by extending the map $\overline{B}^{m,n+1} \to X$ of Lemma 9.4.9 to $\overline{C}^{m,n+1}$, by taking a lift in the diagram below:

$$\begin{tikzcd}
\overline{B}^{m,n+1} \arrow[r] \arrow[d] & X \arrow[d] \\
\overline{C}^{m,n+1} \arrow[r] & \square^0
\end{tikzcd}$$

That the desired lift exists follows from Lemma 9.4.6.

That this definition satisfies identities (Θ1) through (Θ8) follows from the same combinatorial proof as in the unmarked case, given in Appendix A. Thus it remains only to be shown that $\theta^{m,n}(x)$ is strongly $(n+2,1)$-comical for all $x \colon C^{m,n} \to X$. For cases (1) through (5) this follows from Lemma 9.4.2, while for case (6) it holds by construction. $\square$

We now state some lemmas about coherent families of composites which will be of use in showing that $Q \dashv \int$ is a Quillen equivalence.

Lemma 9.4.10. *Let X be a comical set equipped with a coherent family of composites θ. For $m \geq 1, n \geq 0$ and $x \colon C^{m,n} \to X$, the $(m+n+1)$-cube $\theta^{m,n}(x)$ is $(n+1)$-comical.*

Proof. Recall that by construction, $\theta^{m,n}(x)$ is strongly $(n+2,1)$-comical, meaning that the standard form of any unmarked face must contain $\partial_{n+1,1}, \partial_{n+2,0}$, or $\partial_{n+2,1}$. Faces whose standard forms contain $\partial_{n+1,1}$ or $\partial_{n+2,1}$ are permitted to be unmarked by the definition of the $(n+2,1)$-comical $(m+n+1)$-cube, thus we may restrict our attention to those faces whose standard forms contain $\partial_{n+2,0}$.

Let δ denote a face of $\square^{m+n+1}$ whose standard form contains $\partial_{n+2,0}$ and does not contain any of the strings excluded by the definition of the $(n+2,1)$-comical $(m+n+1)$-cube, and consider the face $\theta^{m,n}(x)\delta$. Note that the standard form of δ contains no map of the form $(n+1,\varepsilon)$; therefore, if it contains any map of the form $\partial_{k,0}$ for $k \leq n$, then it is degenerate by Lemma 7.1.10, and hence marked. So assume otherwise; then we may write this face in standard form as:

$$z = \theta^{m,n}(x)\partial_{a_1,\varepsilon_1} \ldots \partial_{a_p,\varepsilon_p} \partial_{n+2,0} \partial_{b_1,1} \ldots \partial_{b_q,1}$$

where $a_p \geq n+3$. Moreover, our assumption that this standard form contains no excluded strings implies $b_1 \leq n$.

First, suppose that $\varepsilon_i = 0$ for all $1 \leq i \leq p$; then we can rewrite this expression using the identities (Θ1) and (Θ3) to obtain:

$$z = \theta^{m-p-1,n-q}(x\partial_{a_1-1,0}...\partial_{a_p-1,0}\partial_{n+1,0}\partial_{b_1,1}...\partial_{b_q,1})$$

hence this face is marked.

Now suppose that $\varepsilon_1 = 1$ for some $1 \leq i \leq p$, and consider the maximal value i such that this condition is satisfied. Then the standard form above contains the string $\partial_{a_i,1}\partial_{a_{i+1},0}...\partial_{a_p,0}\partial_{n+2,0}$. Our assumption that this standard form contains no excluded strings implies that there is a "gap" in this string, i.e. that there is some value between $n+2$ and a_i which does not appear as some a_k. In particular, this implies that there are fewer than $a_i - (n+2)$ maps in the string $\partial_{a_{i+1},0}...\partial_{a_p,0}\partial_{n+2,0}$, i.e. that $p-i+1 < a_i-(n+2)$. Applying cubical identities, we can rearrange the given standard form to move this string to the front, as follows:

$$\begin{aligned}
z &= \theta^{m,n}(x)\partial_{a_1,\varepsilon_1}\ldots\partial_{a_i,1}\partial_{a_{i+1},0}\ldots\partial_{a_p,0}\partial_{n+2,0}\partial_{b_1,1}\ldots\partial_{b_q,1} \\
&= \theta^{m,n}(x)\partial_{a_{i+1},0}\ldots\partial_{a_p,0}\partial_{n+2,0}\partial_{a_1-(p-i+1),\varepsilon_1}\ldots\partial_{a_i-(p-i+1),1}\partial_{b_1,1}\ldots\partial_{b_q,1} \\
&= \theta^{m-(p-a+1),n}(x\partial_{a_{i+1}-1,0}\ldots\partial_{a_p-1,0}\partial_{n+1,0})\partial_{a_1-(p-i+1),\varepsilon_1}\ldots\partial_{a_i-(p-i+1),1}\partial_{b_1,1}\ldots\partial_{b_q,1}
\end{aligned}$$

This expression is again in standard form, and by the inequality above, we can see that $a_i - (p-i+1) > n+2$. Thus the standard form of the face map being applied to $\theta^{m-(p-i+1),n}(x\partial_{a_{i+1}-1,0}...\partial_{a_p-1,0}\partial_{n+1,0})$ does not contain any of the maps $\partial_{n+2,0}, \partial_{n+2,1}$, or $\partial_{n+1,1}$. Since $\theta^{m-(p-i+1),n}(x\partial_{a_{i+1}-1,0} \ldots \partial_{a_p-1,0}\partial_{n+1,0})$ is strongly $(n+2,1)$-comical, this implies that z is marked. $\square$

Lemma 9.4.11. *Let X be a comical set equipped with a coherent family of composites θ. For $m \geq 1, n \geq 0$, and let $x \colon C^{m,n} \to X$, all faces of $\theta^{m,n}(x)$ other than x itself and $\theta^{m,n}(x)\partial_{n+2,1}$ are marked. Moreover, x is marked if and only if $\theta^{m,n}(x)\partial_{n+2,1}$ is marked.*

Proof. For $m = 1$ this is trivial, as $\theta^{1,n}(x)\partial_{n+2,1} = x\gamma_{n+1,1}\partial_{n+2,1} = x$, while the other two faces are degenerate.

Now consider $m \geq 2$. We begin by showing that all $(m+n)$-dimensional faces of $\theta^{m,n}(x)$, other than $\theta^{m,n}(x)\partial_{n+1,1} = x$ and $\theta^{m,n}(x)\partial_{n+2,1}$, are marked. To see this, observe that:

- for $i \leq n+1$, $\theta^{m,n}(x)\partial_{i,0}$ is degenerate by Lemma 7.1.10, hence marked;
- for $i \geq n+2$, $\theta^{m,n}(x)\partial_{i,0}$ is marked by $(\Theta 3)$;
- for $i \leq n$, $\theta^{m,n}(x)\partial_{i,1}$ is marked by $(\Theta 1)$;
- for $i \geq n+3$, $\theta^{m,n}(x)\partial_{i,1}$ is marked because $\theta^{m,n}(x)$ is strongly $(n+2,1)$-comical.

Now suppose that x is marked. Since $\theta^{m,n}(x)$ is $(n+2,1)$-comical, this implies that $\theta^{m,n}(x) \colon \square^{m+n+1} \to X$ factors through $(\square^{m+n+1}_{n+2,1})'$. Thus the following diagram admits a lift:

$$\begin{tikzcd}
(\square^{m+n+1}_{n+2,1})' \arrow[r, "\theta^{m,n}(x)"] \arrow[d] & X \arrow[d] \\
\tau_{m+n-1}\square^{m+n+1}_{n+2,1} \arrow[r] & \square^0
\end{tikzcd}$$

Thus we see that all $(m+n)$-dimensional faces of $\theta^{m,n}(x)$ are marked, including $\theta^{m,n}(x)\partial_{n+2,1}$.

In the case where $\theta^{m,n}(x)\partial_{n+2,1}$ is marked, we can show that x is marked via a similar proof using Lemma 9.4.10. □

Proposition 9.4.12. *For any comical set X, the counit $: Q \int X \hookrightarrow X$ is a comical map.*

Proof. We follow the structure of the proof of Proposition 7.2.22. By Theorem 9.4.8, we may equip X with a coherent family of composites θ. We will build X from $Q \int X$ via successive comical open box fillings and comical marking extensions, thereby showing that the inclusion of $Q \int X$ into X is anodyne.

For $m \geq 2, n \geq -1$, let $X^{m,n}$ denote the smallest regular subcomplex of X containing all (m', n')-cones of X, as well as all cones of the form $\theta^{m',n'}(x)$, for $m' < m$ or $m' = m, n' \leq n$. In particular, this means $X^{2,-1} = Q \int X$, since $(1, n)$-cubes and $(0, n+1)$-cubes coincide by Lemma 7.1.2, and all cubes in the image of $\theta^{1,n}$ are degenerate.

For $m < m'$ or $m = m', n \leq n'$, $X^{m,n}$ is a regular subcomplex of $X^{m',n'}$. Thus we obtain a sequence of regular inclusions:

$$Q \int X = X^{2,-1} \hookrightarrow X^{2,0} \hookrightarrow \ldots \hookrightarrow X^{3,-1} \hookrightarrow X^{3,0} \hookrightarrow \ldots \hookrightarrow X^{m,n} \hookrightarrow \ldots$$

Observe that the colimit of this sequence is X. Furthermore, for each m, $X^{m,-1}$ is the union of all regular subcomplexes $X^{m',n}$ for $m' < m$, i.e. the colimit of the sequence of regular inclusions $Q \int X \hookrightarrow \ldots \hookrightarrow X^{m',n} \hookrightarrow \ldots$. So to show that $Q \int X \hookrightarrow X$ is anodyne, it suffices to show that each map $X^{m,n-1} \hookrightarrow X^{m,n}$ for $n \geq 0$ is anodyne.

Fix $m \geq 2, n \geq 0$, and Let S denote the set of non-degenerate (m, n)-cones of X which are not $(m-1, n+1)$-cones, and are not in the image of $\theta^{m,n-1}$. Let S^+ denote the set of all marked cubes in S. To construct $X^{m,n}$ from $X^{m,n-1}$, we must adjoin to $X^{m,n-1}$ all (m,n)-cones of X, and images of such cones under $\theta^{m,n}$, which are not already present in $X^{m,n}$, and mark those which are marked in X. Using Lemmas 7.1.12, 7.1.17 and 7.1.33, and the identities (Θ1) to (Θ8), we can see that these are precisely the cones in S and their images under $\theta^{m,n}$.

Let $x \in S$; we will analyze the faces of $\theta^{m,n}(x)$ to determine which of them are contained in $X^{m,n-1}$. For $i \leq n$ we have $\theta^{m,n}(x)\partial_{i,1} = \theta^{m,n-1}(x\partial_{i,1})$, while for $i \geq n+2$ or $\varepsilon = 0$, $\theta^{m,n}(x)\partial_{i,\varepsilon}$ is an $(m-1, n+1)$-cone by Lemma 7.1.12. Thus we see that the only face of $\theta^{m,n}(x)$ which is not contained in $X^{m,n-1}$ is $\theta^{m,n}(x)\partial_{n+1,1} = x$. Furthermore, the faces of $\theta^{m,n}(x)$ which are contained in $X^{m,n-1}$ form an $(n+1,1)$-comical open box by Lemma 9.4.10.

We now add all cubes in S and their images under $\theta^{m,n}$ to $X^{m,n-1}$, with the cubes of S unmarked but their images under $\theta^{m,n}$ marked; this amounts to filling all of these comical open boxes. In other words, we construct the following pushout diagram:

$$\begin{tikzcd} \coprod\limits_S \sqcap^{m+n+1}_{n+1,1} \arrow[r] \arrow[d] & X^{m,n-1} \arrow[d] \\ \coprod\limits_S \square^{m+n+1}_{n+1,1} \arrow[r] & X^{m,n}_{\varnothing} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The map $X^{m,n-1} \hookrightarrow X^{m,n}_{\varnothing}$ is anodyne, as a pushout of a coproduct of anodyne maps.

To obtain $X^{m,n}$ from $X^{m,n}_{\varnothing}$, we must mark all the cubes of S^+. Let $x \in S^+$; then Lemmas 9.4.10 and 9.4.11 imply that all faces of $\theta^{m,n}(x)$ other than x are marked in X, and hence also in $X^{m,n-1}$. It follows that $\theta^{m,n}(x) \colon \square^{m+n+1} \to X^{m,n}_{\varnothing}$ factors through $(\square^{m,n}_{n+1,1})'$. We thus have the following pushout diagram:

$$\begin{tikzcd}
\coprod_{S^+} (\square^{m+n+1}_{n+1,1})' \arrow[r] \arrow[d] & X^{m,n}_{\varnothing} \arrow[d] \\
\coprod_{S^+} \tau_{m+n-1}\square^{m+n+1}_{n+1,1} \arrow[r] & X^{m,n} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Thus the map $X^{m,n}_{\varnothing} \to X^{m,n}$ is anodyne, as a pushout of a coproduct of anodyne maps. The composite map $X^{m,n-1} \hookrightarrow X^{m,n}_{\varnothing} \to X^{m,n}$ is therefore anodyne as well. $\square$

We can now prove our main results.

Theorem 9.4.13. *The adjunction $Q : \mathsf{sSet}^+ \rightleftarrows \mathsf{cSet}^+ : \int$ is a Quillen equivalence between the model structure on sSet^+ for (n-trivial, saturated) complicial sets and the corresponding model structure on cSet^+.*

Proof. By Corollary 2.1.38, it suffices to show that the left adjoint Q creates the weak equivalences of sSet^+, and that the counit $Q \int X \hookrightarrow X$ is a weak equivalence for all fibrant objects X. The former statement is Corollary 9.3.19, while the latter is immediate from Proposition 9.4.12. $\square$

Theorem 9.4.14. *The adjunction $T : \mathsf{cSet}^+ \rightleftarrows \mathsf{sSet}^+ : U$ is a Quillen equivalence between the model structure on cSet^+ for (n-trivial, saturated) comical sets and the corresponding model structure on sSet^+.*

Proof. We proceed as in the proof of Theorem 7.2.1. By Theorem 9.3.18, we have a natural weak equivalence $TQ \Rightarrow \mathrm{id}_{\mathsf{sSet}^+}$. Thus the composite of the derived functors of T and Q is naturally isomorphic to the identity on the homotopy category of sSet^+. Since the derived functor of Q is an equivalence of categories by Theorem 9.4.13, the same is therefore true of the derived functor of T. $\square$

Appendix A

Verification of identities on θ

Here we prove that the construction $\theta^{m,n}$ of Definition 7.1.30 satisfies the identities of Definition 7.1.24. Fix a cubical quasicategory X, $m \geq 2$, and $n \geq 0$, and assume that we have defined $\theta^{m',n'}$ satisfying all necessary identities for all pairs $m' \leq m, n' \leq n$ for which at least one of these inequalities is strict. Then we may define $\theta^{m,n}$ by the case analysis of Definition 7.1.30. We will show that a function $\theta^{m,n}$ defined in this way satisfies all identities of Definition 7.1.24, starting with the identities involving faces.

Proposition A.0.1. *$\theta^{m,n}$ satisfies* (Θ1) *and* (Θ2)*; that is, for $i \leq n$, $\theta^{m,n}(x)\partial_{i,0} = \theta^{m,n-1}(x\partial_{i,0})$, while $\theta^{m,n}(x)\partial_{n+1,0} = x$.*

Proof. We will prove this via a case analysis, based on the six cases of Definition 7.1.30. First, let $x = z\sigma_{a_p}$ in standard form, for $a_p \geq n + 1$. By the induction hypotheses, for $m' < m$ or $m' = m, n' < n$, $\theta^{m',n'}$ satisfies all the identities of Definition 7.1.24 (in future computations we will often use this assumption without comment). So for $i \leq n$ we have:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{i,0} &= \theta^{m-1,n}(z)\sigma_{a_p+1}\partial_{i,0} \\
&= \theta^{m-1,n}(z)\partial_{i,0}\sigma_{a_p} \\
&= \theta^{m-1,n-1}(z\partial_{i,0})\sigma_{a_p} \\
&= \theta^{m,n-1}(z\partial_{i,0}\sigma_{a_p-1}) \\
&= \theta^{m,n-1}(z\sigma_{a_p}\partial_{i,0}) \\
&= \theta^{m,n-1}(x\partial_{i,0})
\end{aligned}$$

And for $i = n+1$ we have:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{n+1,0} &= \theta^{m-1,n}(z)\sigma_{a_p+1}\partial_{n+1,0} \\
&= \theta^{m-1,n}(z)\partial_{n+1,0}\sigma_{a_p} \\
&= z\sigma_{a_p} \\
&= x
\end{aligned}$$

Now suppose that the standard form of x is $z\gamma_{b_q,0}$, where $b_q \leq n-1$. Note that we must have $b_q \geq 1$, so this case can only occur when $n \geq 2$. Now for $i \leq b_q - 1$ we have:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{i,0} &= \theta^{m,n-1}(z)\gamma_{b_q,0}\partial_{i,0} \\
&= \theta^{m,n-1}(z)\partial_{i,0}\gamma_{b_q-1,0} \\
&= \theta^{m,n-2}(z\partial_{i,0})\gamma_{b_q-1,0} \\
&= \theta^{m,n-1}(z\partial_{i,0}\gamma_{b_q-1,0}) \\
&= \theta^{m,n-1}(z\gamma_{b_q,0}\partial_{i,0}) \\
&= \theta^{m,n-1}(x\partial_{i,0})
\end{aligned}$$

For $i = b_q$ or $i = b_q + 1$ we have:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{i,0} &= \theta^{m,n-1}(z)\gamma_{b_q,0}\partial_{i,0} \\
&= \theta^{m,n-1}(z) \\
&= \theta^{m,n-1}(z\gamma_{b_q,0}\partial_{i,0}) \\
&= \theta^{m,n-1}(x\partial_{i,0})
\end{aligned}$$

For $b_q + 2 \leq i \leq n$ we have:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{i,0} &= \theta^{m,n-1}(z)\gamma_{b_q,0}\partial_{i,0} \\
&= \theta^{m,n-1}(z)\partial_{i-1,0}\gamma_{b_q,0} \\
&= \theta^{m,n-2}(z\partial_{i-1,0})\gamma_{b_q,0} \\
&= \theta^{m,n-1}(z\partial_{i-1,0}\gamma_{b_q,0}) \\
&= \theta^{m,n-1}(z\gamma_{b_q,0}\partial_{i,0}) \\
&= \theta^{m,n-1}(x\partial_{i,0})
\end{aligned}$$

And for $i = n+1$ we have:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{n+1,0} &= \theta^{m,n-1}(z)\gamma_{b_q,0}\partial_{n+1,0} \\
&= \theta^{m,n-1}(z)\partial_{n,0}\gamma_{b_q,0} \\
&= z\gamma_{b_q,0} \\
&= x
\end{aligned}$$

Next we consider the case where the standard form of x is $z\gamma_{b_q,\varepsilon}, b_q \geq n+1$. Then for $i \leq n$ we have:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{i,0} &= \theta^{m-1,n}(z)\gamma_{b_q+1,\varepsilon}\partial_{i,0} \\
&= \theta^{m-1,n}(z)\partial_{i,0}\gamma_{b_q,\varepsilon} \\
&= \theta^{m-1,n-1}(z\partial_{i,0})\gamma_{b_q,\varepsilon} \\
&= \theta^{m,n-1}(z\partial_{i,0}\gamma_{b_q-1,\varepsilon}) \\
&= \theta^{m,n-1}(z\gamma_{b_q,\varepsilon}\partial_{i,0}) \\
&= \theta^{m,n-1}(x\partial_{i,0})
\end{aligned}$$

And for $i = n + 1$ we have:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{n+1,0} &= \theta^{m-1,n}(z)\gamma_{b_q+1,\varepsilon}\partial_{n+1,0} \\
&= \theta^{m-1,n}(z)\partial_{n+1,0}\gamma_{b_q,\varepsilon} \\
&= z\gamma_{b_q,\varepsilon} \\
&= x
\end{aligned}$$

Next, we consider case (4) of Definition 7.1.30: let x be an $(m-1, n+1)$-cone not falling under any of cases (1)-(3). By Lemma 7.1.12 (i), every face $x\partial_{i,0}$ for $i \leq n$ is an $(m-1, n)$-cone, and therefore $\theta^{m,n-1}(x\partial_{i,0}) = x\partial_{i,0}\gamma_{n,0}$ by the induction hypothesis. Now, for $i \leq n$, we can compute:

$$\begin{aligned}\theta^{m,n}(x)\partial_{i,0} &= x\gamma_{n+1,0}\partial_{i,0}\\ &= x\partial_{i,0}\gamma_{n,0}\\ &= \theta^{m,n-1}(x\partial_{i,0})\end{aligned}$$

And $\theta^{m,n}(x)\partial_{n+1,0} = x\gamma_{n+1,0}\partial_{n+1,0} = x$.

Next, we consider case (5): consider an (m,n)-cone $\theta^{m,n-1}(x')$ not falling under any of cases (1) through (4). Then for $i \leq n-1$ we have:

$$\begin{aligned}\theta^{m,n}(\theta^{m,n-1}(x'))\partial_{i,0} &= \theta^{m,n-1}(x')\gamma_{n,0}\partial_{i,0}\\ &= \theta^{m,n-1}(x')\partial_{i,0}\gamma_{n-1,0}\\ &= \theta^{m,n-2}(x'\partial_{i,0})\gamma_{n-1,0}\\ &= \theta^{m,n-1}(\theta^{m,n-2}(x'\partial_{i,0}))\\ &= \theta^{m,n-1}(\theta^{m,n-1}(x')\partial_{i,0})\end{aligned}$$

For $i = n$ we have:

$$\begin{aligned}\theta^{m,n}(\theta^{m,n-1}(x))\partial_{n,0} &= \theta^{m,n-1}(x')\gamma_{n,0}\partial_{n,0}\\ &= \theta^{m,n-1}(x')\\ &= \theta^{m,n-1}(\theta^{m,n-1}(x')\partial_{n,0})\end{aligned}$$

And for $i = n+1$ we have $\theta^{m,n}(\theta^{m,n-1}(x'))\partial_{n+1,0} = \theta^{m,n-1}(x')\gamma_{n,0}\partial_{n+1,0} = \theta^{m,n-1}(x')$.

Finally, we consider case (6); in this case the identities hold by Lemma 7.1.29. □

Proposition A.0.2. *$\theta^{m,n}$ satisfies $(\Theta 3)$; that is, for all $x{:}C^{m,n} \to X^{m,n}$, $i \geq n+2$, we have $\theta^{m,n}(x)\partial_{i,1} = \theta^{m-1,n}(x\partial_{i-1,1})$.*

Proof. Throughout the proof, we fix $i \geq n+2$. First we consider case (1) of Definition 7.1.30. Suppose that the standard form of x is $z\sigma_{a_p}$, for some $a_p \geq n+1$. Here we must consider various cases based on a comparison of i with a_p. First suppose that $i \leq a_p$; note that this implies $a_p \geq n+2$. Then we have:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{i,1} &= \theta^{m-1,n}(z)\sigma_{a_p+1}\partial_{i,1} \\
&= \theta^{m-1,n}(z)\partial_{i,1}\sigma_{a_p} \\
&= \theta^{m-2,n}(z\partial_{i-1,1})\sigma_{a_p} \\
&= \theta^{m-1,n}(z\partial_{i-1,1}\sigma_{a_p-1}) \\
&= \theta^{m-1,n}(z\sigma_{a_p}\partial_{i-1,1}) \\
&= \theta^{m-1,n}(x\partial_{i-1,1})
\end{aligned}$$

To obtain the fourth equality, we have used $(\Theta 4)$ and the fact that $a_p - 1 \geq n+1$.

Next suppose that $i = a_p + 1$; then we have:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{a_p+1,1} &= \theta^{m-1,n}(z)\sigma_{a_p+1}\partial_{a_p+1,1} \\
&= \theta^{m-1,n}(z) \\
&= \theta^{m-1,n}(z\sigma_{a_p}\partial_{a_p,1}) \\
&= \theta^{m-1,n}(x\partial_{a_p,1})
\end{aligned}$$

Finally, suppose $i \geq a_p + 2$; note that this implies $a_p \geq n + 3$. Then we have:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{i,1} &= \theta^{m-1,n}(z)\sigma_{a_p+1}\partial_{i,1} \\
&= \theta^{m-1,n}(z)\partial_{i-1,1}\sigma_{a_p+1} \\
&= \theta^{m-2,n}(z\partial_{i-2,1})\sigma_{a_p+1} \\
&= \theta^{m-1,n}(z\partial_{i-2,1}\sigma_{a_p}) \\
&= \theta^{m-1,n}(z\sigma_{a_p}\partial_{i-1,1}) \\
&= \theta^{m-1,n}(x\partial_{i-1,1})
\end{aligned}$$

Next we consider case (2): suppose that $x = z\gamma_{b_q,0}$ in standard form, where $b_q \leq n - 1$. Then $i \geq b_q + 3$, and we have:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{i,1} &= \theta^{m,n-1}(z)\gamma_{b_q,0}\partial_{i,1} \\
&= \theta^{m,n-1}(z)\partial_{i-1,1}\gamma_{b_q,0} \\
&= \theta^{m-1,n-1}(z\partial_{i-2,1})\gamma_{b_q,0} \\
&= \theta^{m-1,n}(z\partial_{i-2,1}\gamma_{b_q,0}) \\
&= \theta^{m-1,n}(z\gamma_{b_q,0}\partial_{i-1,1}) \\
&= \theta^{m-1,n}(x\partial_{i-1,1})
\end{aligned}$$

Next we consider case (3): suppose that $x = z\gamma_{b_q,\varepsilon}$ in standard form, where $b_q \geq n+1$. Once again, we must perform a case analysis. First suppose that $i \leq b_q$, implying $b_q \geq n+2$. Then we can compute:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{i,1} &= \theta^{m-1,n}(z)\gamma_{b_q+1,\varepsilon}\partial_{i,1} \\
&= \theta^{m-1,n}(z)\partial_{i,1}\gamma_{b_q,\varepsilon} \\
&= \theta^{m-2,n}(z\partial_{i-1,1})\gamma_{b_q,\varepsilon} \\
&= \theta^{m-1,n}(z\partial_{i-1,1}\gamma_{b_q-1,\varepsilon}) \\
&= \theta^{m-1,n}(z\gamma_{b_q,\varepsilon}\partial_{i-1,1}) \\
&= \theta^{m-1,n}(x\partial_{i-1,1})
\end{aligned}$$

Next suppose that $i = b_q + 1$ or $b_q + 2$, and $\varepsilon = 0$. Then we have:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{i,1} &= \theta^{m-1,n}(z)\gamma_{b_q+1,0}\partial_{i,1} \\
&= \theta^{m-1,n}(z)\partial_{b_q+1,1}\sigma_{b_q+1} \\
&= \theta^{m-2,n}(z\partial_{b_q,1})\sigma_{b_q+1} \\
&= \theta^{m-1,n}(z\partial_{b_q,1}\sigma_{b_q}) \\
&= \theta^{m-1,n}(z\gamma_{b_q,0}\partial_{i-1,1}) \\
&= \theta^{m-1,n}(x\partial_{i-1,1})
\end{aligned}$$

To obtain the third equality, we used $(\Theta 3)$ for $\theta^{m-1,n}$ and the assumption that $b_q \geq n+1$. Next suppose that $i = b_q+1$ or b_q+2, and $\varepsilon = 1$. Then we have:

$$\begin{aligned}
\theta^{m,n}(x)\partial_{i,1} &= \theta^{m-1,n}(z)\gamma_{b_q+1,1}\partial_{i,1} \\
&= \theta^{m-1,n}(z) \\
&= \theta^{m-1,n}(z\gamma_{b_q,1}\partial_{i-1,1}) \\
&= \theta^{m-1,n}(x\partial_{i-1,1})
\end{aligned}$$

Finally, suppose $i \geq b_q+3$, implying $i \geq n+4$. Then we have:

$$
\begin{aligned}
\theta^{m,n}(x)\partial_{i,1} &= \theta^{m-1,n}(z)\gamma_{b_q+1,\varepsilon}\partial_{i,1} \\
&= \theta^{m-1,n}(z)\partial_{i-1,1}\gamma_{b_q+1,\varepsilon} \\
&= \theta^{m-2,n}(z\partial_{i-2,1})\gamma_{b_q+1,\varepsilon} \\
&= \theta^{m-1,n}(z\partial_{i-2,1}\gamma_{b_q,\varepsilon}) \\
&= \theta^{m-1,n}(z\gamma_{b_q,\varepsilon}\partial_{i-1,1}) \\
&= \theta^{m-1,n}(x\partial_{i-1,1})
\end{aligned}
$$

Next we consider case (4): let x be an $(m-1,n+1)$-cone not covered under any of cases (1) through (3). Then $x\partial_{i-1,1}$ is an $(m-2,n+1)$-cone by Lemma 7.1.12 (iii), so $\theta^{m-1,n}(x\partial_{i-1,1}) = x\partial_{i-1,1}\gamma_{n+1,0}$ by(Θ8) for $\theta^{m-1,n}$. Furthermore, note that by Lemma 7.1.15, $x\partial_{n+1,1} = x\partial_{m+n+1}\cdots\partial_{n+1,1}\sigma_{n+1}\cdots\sigma_{m+n}$. Using the cubical identities, we can rewrite this as $x\partial_{m+n+1}\cdots\partial_{n+1,1}\sigma_{n+1}\cdots\sigma_{n+1}$. Then for $i = n+2$, we can compute:

$$
\begin{aligned}
\theta^{m,n}(x)\partial_{n+2,1} &= x\gamma_{n+1,0}\partial_{n+2,1} \\
&= x\partial_{n+1,1}\sigma_{n+1} \\
&= x\partial_{m+n+1}\cdots\partial_{n+1,1}\sigma_{n+1}\cdots\sigma_{n+1}\sigma_{n+1} \\
&= x\partial_{m+n+1}\cdots\partial_{n+1,1}\sigma_{n+1}\cdots\sigma_{n+1}\gamma_{n+1,0} \\
&= x\partial_{n+1,1}\gamma_{n+1,0} \\
&= \theta^{m-1,n}(x\partial_{n+1,1})
\end{aligned}
$$

While for $i \geq n+3$, we have:

$$\begin{aligned}\theta^{m,n}(x)\partial_{i,1} &= x\gamma_{n+1,0}\partial_{i,1} \\ &= x\partial_{i-1,1}\gamma_{n+1,0} \\ &= \theta^{m-1,n}(x\partial_{i-1,1})\end{aligned}$$

Next we consider case (5). Let $x'\colon C^{m,n-1} \to X^{m,n}$, and consider $\theta^{m,n}(\theta^{m,n-1}(x'))$. Then we can compute:

$$\begin{aligned}\theta^{m,n}(\theta^{m,n-1}(x'))\partial_{i,1} &= \theta^{m,n-1}(x')\gamma_{n,0}\partial_{i,1} \\ &= \theta^{m,n-1}(x')\partial_{i-1,1}\gamma_{n,0} \\ &= \theta^{m-1,n-1}(x'\partial_{i-2,1})\gamma_{n,0} \\ &= \theta^{m-1,n}(\theta^{m-1,n-1}(x'\partial_{i-2,1})) \\ &= \theta^{m-1,n}(\theta^{m,n-1}(x')\partial_{i-1,1})\end{aligned}$$

Finally, in case (6), the identity holds by Lemma 7.1.29. □

Next we consider the identities involving degeneracies and connections.

Proposition A.0.3. *$\theta^{m,n}$ satisfies $(\Theta 4)$, $(\Theta 5)$, and $(\Theta 6)$. That is:*

- *If $x\sigma_i$ is an (m,n)-cone for $i \geq n+1$, then $\theta^{m,n}(x\sigma_i) = \theta^{m-1,n}(x)\sigma_{i+1}$;*
- *If $x\gamma_{i,0}$ is an (m,n)-cone for $i \leq n-1$, then $\theta^{m,n}(x\gamma_{i,0}) = \theta^{m,n-1}(x)\gamma_{i,0}$;*

- *If $x\gamma_{i,\varepsilon}$ is an (m,n)-cone for $i \geq n+1$, then $\theta^{m,n}(x\gamma_{i,\varepsilon}) = \theta^{m-1,n}(x)\gamma_{i+1,\varepsilon}$.*

Proof. For each identity, we will perform a case analysis based on the standard form of x. For $(\Theta 4)$, consider an (m,n)-cube $x\sigma_i$, where $i \geq n+1$ and the standard form of x is $y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,\varepsilon_q}\sigma_{a_1}\cdots\sigma_{a_p}$. If the string of degeneracy maps in the standard form of x is empty, or $a_p < i$, then the standard form of $x\sigma_i$ ends with σ_i, so $\theta^{m,n}(x\sigma_i) = \theta^{m-1,n}(x)\sigma_{i+1}$ by definition. So suppose that $a_p \geq i$. Then:

$$\begin{aligned}
\theta^{m,n}(x\sigma_i) &= \theta^{m,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,\varepsilon_q}\sigma_{a_1}\cdots\sigma_{a_p}\sigma_i) \\
&= \theta^{m,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,\varepsilon_q}\sigma_{a_1}\cdots\sigma_{a_{p-1}}\sigma_i\sigma_{a_p+1})
\end{aligned}$$

By assumption, all the indices $a_1, \ldots, a_{p-1}$, are less than a_p. Rearranging the expression on the right-hand side of the equation into standard form using the co-cubical identities will not increase any of these indices by more than 1, so the rightmost map in the standard form of $x\sigma_i$, i.e. the degeneracy map with the highest index, is σ_{a_p+1}. Therefore, we can compute:

$$\begin{aligned}
\theta^{m,n}(x\sigma_i) &= \theta^{m,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,\varepsilon_q}\sigma_{a_1}\cdots\sigma_{a_{p-1}}\sigma_i\sigma_{a_p+1}) \\
&= \theta^{m-1,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,\varepsilon_q}\sigma_{a_1}\cdots\sigma_{a_{p-1}}\sigma_i)\sigma_{a_p+2} \\
&= \theta^{m-2,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,\varepsilon_q}\sigma_{a_1}\cdots\sigma_{a_{p-1}})\sigma_{i+1}\sigma_{a_p+2} \\
&= \theta^{m-2,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,\varepsilon_q}\sigma_{a_1}\cdots\sigma_{a_{p-1}})\sigma_{a_p+1}\sigma_{i+1} \\
&= \theta^{m-1,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,\varepsilon_q}\sigma_{a_1}\cdots\sigma_{a_{p-1}}\sigma_{a_p})\sigma_{i+1} \\
&= \theta^{m-1,n}(x)\sigma_{i+1}
\end{aligned}$$

So $\theta^{m,n}$ satisfies (Θ4).

Next we will verify (Θ6). Consider an (m,n)-cube $x\gamma_{i,\varepsilon}$, where $i \geq n+1$ and the standard form of x is as above. If this standard form contains no degeneracy maps, and either $b_q < i$, $b_q = i$ while $\varepsilon_q \neq \varepsilon$, or x is non-degenerate, then the standard form of $x\gamma_{i,\varepsilon}$ ends with $\gamma_{i,\varepsilon}$, so the identity holds by definition. The remaining possibilities for the standard form of x can be divided into various cases. First, suppose that the string of degeneracy maps in the standard form of x is non-empty, i.e. $x = z\sigma_{a_p}$ in standard form. By Lemma 7.1.12, $x = x\gamma_{i,\varepsilon}\partial_{i,\varepsilon}$ is an $(m-1,n)$-cone, so $a_p \geq n+1$ by Lemma 7.1.17 (i). Now we must break this into further cases based on a comparison between i and a_p. If $i < a_p$ then, using the co-cubical identities, (Θ4) for $\theta^{m,n}$, and (Θ6) for $\theta^{m-1,n}$, we can compute:

$$\begin{aligned}
\theta^{m,n}(x\gamma_{i,\varepsilon}) &= \theta^{m,n}(z\sigma_{a_p}\gamma_{i,\varepsilon}) \\
&= \theta^{m,n}(z\gamma_{i,\varepsilon}\sigma_{a_p+1}) \\
&= \theta^{m-1,n}(z\gamma_{i,\varepsilon})\sigma_{a_p+2} \\
&= \theta^{m-2,n}(z)\gamma_{i+1,\varepsilon}\sigma_{a_p+2} \\
&= \theta^{m-2,n}(z)\sigma_{a_p+1}\gamma_{i+1,\varepsilon} \\
&= \theta^{m-1,n}(z\sigma_{a_p})\gamma_{i+1,\varepsilon} \\
&= \theta^{m-1,n}(x)\gamma_{i+1,\varepsilon}
\end{aligned}$$

Next we consider the case $i = a_p$:

$$\begin{aligned}
\theta^{m,n}(x\gamma_{a_p,\varepsilon}) &= \theta^{m,n}(z\sigma_{a_p}\gamma_{a_p,\varepsilon}) \\
&= \theta^{m,n}(z\sigma_{a_p}\sigma_{a_p+1}) \\
&= \theta^{m-1,n}(z\sigma_{a_p})\sigma_{a_p+2} \\
&= \theta^{m-2,n}(z)\sigma_{a_p+1}\sigma_{a_p+2} \\
&= \theta^{m-2,n}(z)\sigma_{a_p+1}\gamma_{a_p+1,\varepsilon} \\
&= \theta^{m-1,n}(z\sigma_{a_p})\gamma_{a_p+1,\varepsilon} \\
&= \theta^{m-1,n}(x)\gamma_{a_p+1,\varepsilon}
\end{aligned}$$

Now we consider the case $i > a_p$. Note that this implies $i \geq n+2$, so $i-1 \geq n+1$. Thus we can compute:

$$\begin{aligned}
\theta^{m,n}(x\gamma_{i,\varepsilon}) &= \theta^{m,n}(z\sigma_{a_p}\gamma_{i,\varepsilon}) \\
&= \theta^{m,n}(z\gamma_{i-1,\varepsilon}\sigma_{a_p}) \\
&= \theta^{m-1,n}(z\gamma_{i-1,\varepsilon})\sigma_{a_p+1} \\
&= \theta^{m-2,n}(z)\gamma_{i,\varepsilon}\sigma_{a_p+1} \\
&= \theta^{m-2,n}(z)\sigma_{a_p+1}\gamma_{i+1,\varepsilon} \\
&= \theta^{m-1,n}(z\sigma_{a_p})\gamma_{i+1,\varepsilon} \\
&= \theta^{m-1,n}(x)\gamma_{i+1,\varepsilon}
\end{aligned}$$

Next we will verify ($\Theta 6$) in the case where the standard form of x contains no degeneracy maps, and either $i < b_q$ or $i = b_q$ and $\varepsilon = \varepsilon_q$. In this case we can compute:

$$\begin{aligned}
\theta^{m,n}(x\gamma_{i,\varepsilon}) &= \theta^{m,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,\varepsilon_q}\gamma_{i,\varepsilon}) \\
&= \theta^{m,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_{q-1},\varepsilon_{q-1}}\gamma_{i,\varepsilon}\gamma_{b_q+1,\varepsilon_q})
\end{aligned}$$

Similarly to what we saw when verifying ($\Theta 4$), after we have rearranged the expression on the right-hand side of this equation into standard form, the rightmost map in the expression will still be $\gamma_{b_q+1,\varepsilon_q}$. Thus we can apply the definition of $\theta^{m,n}$ to compute:

$$\begin{aligned}
\theta^{m,n}(y\gamma_{b_1,\varepsilon}\cdots\gamma_{b_{q-1},\varepsilon_{q-1}}\gamma_{i,\varepsilon}\gamma_{b_q+1,\varepsilon_q}) &= \theta^{m-1,n}(y\gamma_{b_1\varepsilon_1}\cdots\gamma_{b_{q-1},\varepsilon_{q-1}}\gamma_{i,\varepsilon_i})\gamma_{b_q+2,\varepsilon_q} \\
&= \theta^{m-2,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_{q-1},\varepsilon_{q-1}})\gamma_{i+1,\varepsilon}\gamma_{b_q+2,\varepsilon_q} \\
&= \theta^{m-2,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_{q-1},\varepsilon_{q-1}})\gamma_{b_q+1,\varepsilon_q}\gamma_{i+1,\varepsilon} \\
&= \theta^{m-1,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,\varepsilon_q})\gamma_{i+1,\varepsilon} \\
&= \theta^{m-1,n}(x)\gamma_{i+1,\varepsilon}
\end{aligned}$$

Thus $\theta^{m,n}$ satisfies (Θ6).

Finally we will verify (Θ5). Consider an (m,n)-cube $x\gamma_{i,0}$, where $i \leq n-1$ and the standard form of x is as above. Once again, we must consider several possible cases based on the standard form of x. As with (Θ6), if the standard form of x contains no degeneracy maps, and either $b_q < i$, $b_q = i$ while $\varepsilon_q = 1$, or x is non-degenerate, then $\gamma_{i,0}$ is the rightmost map in the standard form of $x\gamma_{i,0}$, and the identity holds by definition. Once again, the remaining cases will require computation.

As above, we begin with the case where the string of degeneracy maps in the standard form of x is non-empty. By Lemma 7.1.12 (i), $x = x\gamma_{i,0}\partial_{i,0}$ is an $(m,n-1)$-cone, so $a_p \geq n$ by Lemma 7.1.17. Then, using the co-cubical identities, (Θ4) for $\theta^{m,n}$, and (Θ5) for $\theta^{m-1,n}$, we can compute:

$$
\begin{aligned}
\theta^{m,n}(x\gamma_{i,0}) &= \theta^{m,n}(z\sigma_{a_p}\gamma_{i,0}) \\
&= \theta^{m,n}(z\gamma_{i,0}\sigma_{a_p+1}) \\
&= \theta^{m-1,n}(z\gamma_{i,0})\sigma_{a_p+2} \\
&= \theta^{m-1,n-1}(z)\gamma_{i,0}\sigma_{a_p+2} \\
&= \theta^{m-1,n-1}(z)\sigma_{a_p+1}\gamma_{i,0} \\
&= \theta^{m,n-1}(z\sigma_{a_p})\gamma_{i,0} \\
&= \theta^{m,n-1}(x)\gamma_{i,0}
\end{aligned}
$$

Next we consider the cases in which the standard form of x contains no degeneracy maps; first, suppose that $b_q \geq n$. Then, using the co-cubical identities, (Θ6) for $\theta^{m,n}$, and (Θ5) for $\theta^{m-1,n}$, we can compute:

$$
\begin{aligned}
\theta^{m,n}(x\gamma_{i,0}) &= \theta^{m,n}(z\gamma_{b_q,\varepsilon_q}\gamma_{i,0}) \\
&= \theta^{m,n}(z\gamma_{i,0}\gamma_{b_q+1,\varepsilon_q}) \\
&= \theta^{m-1,n}(z\gamma_{i,0})\gamma_{b_q+2,\varepsilon_q} \\
&= \theta^{m-1,n-1}(z)\gamma_{i,0}\gamma_{b_q+2,\varepsilon_q} \\
&= \theta^{m-1,n-1}(z)\gamma_{b_q+1,\varepsilon_q}\gamma_{i,0} \\
&= \theta^{m,n-1}(z\gamma_{b_q,\varepsilon_q})\gamma_{i,0} \\
&= \theta^{m,n-1}(x)\gamma_{i,0}
\end{aligned}
$$

Next we consider the case $b_q = n-1$. Note that $x = x\gamma_{i,0}\partial_{i,0}$ is an $(m, n-1)$-

cone by Lemma 7.1.12 (i); thus $\varepsilon_q = 0$ by Lemma 7.1.17 (ii). Here we can compute:

$$\begin{aligned} x\gamma_{i,0} &= y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_{q-1},\varepsilon_{q-1}}\gamma_{n-1,0}\gamma_{i,0} \\ &= y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_{q-1},\varepsilon_{q-1}}\gamma_{i,0}\gamma_{n,0} \end{aligned}$$

As in previous cases, after rearranging this expression into standard form, the rightmost map will still be $\gamma_{n,0}$. Thus $x\gamma_{i,0}$ belongs to case (4) by Corollary 7.1.14, so:

$$\begin{aligned} \theta^{m,n}(x\gamma_{i,0}) &= x\gamma_{i,0}\gamma_{n+1,0} \\ &= x\gamma_{n,0}\gamma_{i,0} \end{aligned}$$

By Lemma 7.1.12 (i), $x = x\gamma_{i,0}\partial_{i,0}$ is an $(m, n-1)$-cone, so the fact that $b_q = n-1$ implies that x also belongs to case (4). Thus $x\gamma_{n,0} = \theta^{m,n-1}(x)$, so $(\Theta 5)$ is satisfied in this case.

Finally, we consider the case $i \leq b_q \leq n-2$. Once again, we have $\varepsilon_q = 0$ by Lemma 7.1.17 (ii). Now we can compute:

$$\begin{aligned}\theta^{m,n}(x\gamma_{i,0}) &= \theta^{m,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,0}\gamma_{i,0})\\ &= \theta^{m,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_{q-1},\varepsilon_{q-1}}\gamma_{i,0}\gamma_{b_q+1,0})\end{aligned}$$

As in previous computations, once the expression on the right-hand side of the equation has been rearranged into standard form, its rightmost map will still be $\gamma_{b_q+1,0}$. By assumption, $b_q+1 \leq n-1$, so using the co-cubical identities, the definition of $\theta^{m,n}$, and $(\Theta 5)$ for $\theta^{m,n-1}$, we can compute:

$$\begin{aligned}\theta^{m,n}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_{q-1},\varepsilon_{q-1}}\gamma_{i,0}\gamma_{b_q+1,0}) &= \theta^{m,n-1}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_{q-1},\varepsilon_{q-1}}\gamma_{i,0})\gamma_{b_q+1,0}\\ &= \theta^{m,n-2}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_{q-1},\varepsilon_{q-1}})\gamma_{i,0}\gamma_{b_q+1,0}\\ &= \theta^{m,n-2}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_{q-1},\varepsilon_{q-1}})\gamma_{b_q,0}\gamma_{i,0}\\ &= \theta^{m,n-1}(y\gamma_{b_1,\varepsilon_1}\cdots\gamma_{b_q,0})\gamma_{i,0}\\ &= \theta^{m,n-1}(x)\gamma_{i,0}\end{aligned}$$

Thus $\theta^{m,n}$ satisfies $(\Theta 5)$. □

Proposition A.0.4. *If $n \geq 1$ then $\theta^{m,n}$ satisfies* $(\Theta 7)$. *That is, for any $x\colon C^{m,n-1} \to X$, $\theta^{m,n}(\theta^{m,n-1}(x)) = \theta^{m,n-1}(x)\gamma_{n,0}$.*

Proof. We proceed by a case analysis on x, based on the cases of Definition 7.1.30. In our computations, we will freely use the identities for $\theta^{m,n}$ which we have already proven. First suppose that $x = z\sigma_{a_p}$ in standard form, for some $a_p \geq n$. Then we can compute:

$$\begin{aligned}
\theta^{m,n}(\theta^{m,n-1}(x)) &= \theta^{m,n}(\theta^{m,n-1}(z\sigma_{a_p})) \\
&= \theta^{m,n}(\theta^{m-1,n-1}(z)\sigma_{a_p+1}) \\
&= \theta^{m-1,n}(\theta^{m-1,n-1}(z))\sigma_{a_p+2} \\
&= \theta^{m-1,n-1}(z)\gamma_{n,0}\sigma_{a_p+2} \\
&= \theta^{m-1,n-1}(z)\sigma_{a_p+1}\gamma_{n,0} \\
&= \theta^{m,n-1}(z\sigma_{a_p})\gamma_{n,0} \\
&= \theta^{m,n-1}(x)\gamma_{n,0}
\end{aligned}$$

Next let the standard form of x be $z\gamma_{b_q,0}$ where $b_q \le n-2$. Then we can compute:

$$\begin{aligned}
\theta^{m,n}(\theta^{m,n-1}(x)) &= \theta^{m,n}(\theta^{m,n-1}(z\gamma_{b_q,0})) \\
&= \theta^{m,n}(\theta^{m,n-2}(z)\gamma_{b_q,0}) \\
&= \theta^{m,n-1}(\theta^{m,n-2}(z))\gamma_{b_q,0} \\
&= \theta^{m,n-2}(z)\gamma_{n-1,0}\gamma_{b_q,0} \\
&= \theta^{m,n-2}(z)\gamma_{b_q,0}\gamma_{n,0} \\
&= \theta^{m,n-1}(z\gamma_{b_q,0})\gamma_{n,0} \\
&= \theta^{m,n-1}(x)\gamma_{n,0}
\end{aligned}$$

Now let the standard form of x be $z\gamma_{b_q,\varepsilon}$ where $b_q \ge n$. Then we can compute:

$$\begin{aligned}
\theta^{m,n}(\theta^{m,n-1}(x)) &= \theta^{m,n}(\theta^{m,n-1}(z\gamma_{b_q,\varepsilon})) \\
&= \theta^{m,n}(\theta^{m-1,n-1}(z)\gamma_{b_q+1,\varepsilon}) \\
&= \theta^{m-1,n}(\theta^{m-1,n-1}(z))\gamma_{b_q+2,\varepsilon} \\
&= \theta^{m-1,n-1}(z)\gamma_{n,0}\gamma_{b_q+2,\varepsilon} \\
&= \theta^{m-1,n-1}(z)\gamma_{b_q+1,\varepsilon}\gamma_{n,0} \\
&= \theta^{m,n-1}(z\gamma_{b_q,\varepsilon})\gamma_{n,0} \\
&= \theta^{m,n-1}(x)\gamma_{n,0}
\end{aligned}$$

Next, we consider case (4): suppose that x is an $(m-1,n)$-cone not falling under any of cases (1) through (3) (when considered as an $(m,n-1)$-cone). Then $\theta^{m,n-1}(x) = x\gamma_{n,0}$. The assumption that x does not belong to any of cases (1) through (3), together with Lemma 7.1.17, implies that either it is non-degenerate, or its standard form ends with $\gamma_{n-1,0}$. Either way, the standard form of $x\gamma_{n,0}$ ends with $\gamma_{n,0}$, so it falls under case (4) by Corollary 7.1.14. Thus we can compute:

$$
\begin{aligned}
\theta^{m,n}(\theta^{m,n-1}(x)) &= \theta^{m,n}(x\gamma_{n,0}) \\
&= x\gamma_{n,0}\gamma_{n+1,0} \\
&= x\gamma_{n,0}\gamma_{n,0} \\
&= \theta^{m,n-1}(x)\gamma_{n,0}
\end{aligned}
$$

Next we consider case (5): suppose that $x = \theta^{m,n-2}(x')$ for some $x' \colon C^{m,n-2} \to X$. Then we can compute:

$$
\begin{aligned}
\theta^{m,n}(\theta^{m,n-1}(x)) &= \theta^{m,n}(\theta^{m,n-1}(\theta^{m,n-2}(x'))) \\
&= \theta^{m,n}(\theta^{m,n-2}(x')\gamma_{n-1,0}) \\
&= \theta^{m,n-1}(\theta^{m,n-2}(x'))\gamma_{n-1,0} \\
&= \theta^{m,n-2}(x')\gamma_{n-1,0}\gamma_{n-1,0} \\
&= \theta^{m,n-2}(x')\gamma_{n-1,0}\gamma_{n,0} \\
&= \theta^{m,n-1}(\theta^{m,n-2}(x'))\gamma_{n,0} \\
&= \theta^{m,n-1}(x)\gamma_{n,0}
\end{aligned}
$$

Finally, suppose x falls under case (6). Then by Lemma 7.1.33, $\theta^{m,n-1}(x)$ falls under case (5), so $\theta^{m,n}(\theta^{m,n-1}(x)) = \theta^{m,n-1}(x)\gamma_{n,0}$ by definition. □

Proposition A.0.5. $\theta^{m,n}$ *satisfies* $(\Theta 8)$. *That is, if x is an $(m-1, n+1)$-cone, then* $\theta^{m,n}(x) = x\gamma_{n+1,0}$.

Proof. As in previous proofs, we proceed via case analysis on x, based on the cases of Definition 7.1.30. First suppose that x is an $(m-1,n+1)$-cone whose standard form is $z\sigma_{a_p}$. By Lemma 7.1.17 (i), $a_p \geq n+2$. Therefore, by Lemma 7.1.12 (ii), $x\partial_{a_p,0} = z$ is an $(m-2,n+1)$-cone, so $\theta^{m-1,n}(z) = z\gamma_{n+1,0}$ by $(\Theta 8)$ for $\theta^{m-1,n}$. Thus we can compute:

$$\begin{aligned}
\theta^{m,n}(x) &= \theta^{m-1,n}(z)\sigma_{a_p+1} \\
&= z\gamma_{n+1,0}\sigma_{a_p+1} \\
&= z\sigma_{a_p}\gamma_{n+1,0} \\
&= x\gamma_{n+1,0}
\end{aligned}$$

Now let x be an $(m-1,n+1)$-cone whose standard form is $z\gamma_{b_q,0}, b_q \leq n-1$. Then by Lemma 7.1.12 (i), $x\partial_{b_q,0} = z$ is an $(m-1,n)$-cone. So by $(\Theta 8)$ for $\theta^{m,n-1}$, we have $\theta^{m,n-1}(z) = z\gamma_{n,0}$. Thus we can compute:

$$\begin{aligned}
\theta^{m,n}(x) &= \theta^{m,n-1}(z)\gamma_{b_q,0} \\
&= z\gamma_{n,0}\gamma_{b_q,0} \\
&= z\gamma_{b_q,0}\gamma_{n+1,0} \\
&= x\gamma_{n+1,0}
\end{aligned}$$

Next let x be an $(m-1,n+1)$-cone whose standard form is $z\gamma_{b_q,\varepsilon}$, where $b_q \geq n+1$. (Note that if $b_q = n+1$, then we may assume $\varepsilon = 0$ by Lemma 7.1.17

(ii).) Then by Lemma 7.1.12, $x\partial_{b_q+1,\varepsilon} = z$ is an $(m-2, n+1)$-cone, so $\theta^{m-1,n}(z) = z\gamma_{n+1,0}$ by $(\Theta 8)$ for $\theta^{m-1,n}$. Thus we can compute:

$$\begin{aligned}
\theta^{m,n}(x) &= \theta^{m-1,n}(z)\gamma_{b_q+1,\varepsilon} \\
&= z\gamma_{n+1,0}\gamma_{b_q+1,\varepsilon} \\
&= z\gamma_{b_q,\varepsilon}\gamma_{n+1,0} \\
&= x\gamma_{n+1,0}
\end{aligned}$$

Finally, case (4) consists of all $(m-1, n+1)$-cones not falling under any of the previous cases, and in this case $(\Theta 8)$ holds by definition. □

Bibliography

[AR94] Jiří Adámek and Jiří Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994, doi:10.1017/CBO9780511600579.

[Ara14] Dimitri Ara, *Higher quasi-categories vs higher Rezk spaces*, J. K-Theory **14** (2014), no. 3, 701–749, doi:10.1017/S1865243315000021, https://doi.org/10.1017/S1865243315000021.

[Bek00] Tibor Beke, *Sheafifiable homotopy model categories*, Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 3, 447–475, doi:10.1017/S0305004100004722, https://doi.org/10.1017/S0305004100004722.

[Ber07] Julia E. Bergner, *A model category structure on the category of simplicial categories*, Trans. Amer. Math. Soc. **359** (2007), no. 5, 2043–2058, doi:10.1090/S0002-9947-06-03987-0.

[Ber10] ——, *A survey of* $(\infty, 1)$*-categories*, Towards higher categories, IMA Vol. Math. Appl., vol. 152, Springer, New York, 2010, pp. 69–83, doi:10.1007/978-1-4419-1524-5_2.

[BM11] Clemens Berger and Ieke Moerdijk, *On an extension of the notion of Reedy category*, Math. Z. **269** (2011), no. 3-4, 977–1004, doi:10.1007/s00209-010-0770-x, https://doi.org/10.1007/s00209-010-0770-x.

[BM17] Ulrik Buchholtz and Edward Morehouse, *Varieties of cubical sets*, Relational and Algebraic Methods in Computer Science. RAMICS 2017. (Cham), vol. 10226, Springer, 2017, pp. 77–92.

[BR13] Julia E. Bergner and Charles Rezk, *Reedy categories and the Θ-construction*, Math. Z. **274** (2013), no. 1-2, 499–514, `doi:10.1007/s00209-012-1082-0`.

[BV73] J. Michael Boardman and Rainer M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin-New York, 1973.

[CCHM18] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg, *Cubical type theory: a constructive interpretation of the univalence axiom*, 21st International Conference on Types for Proofs and Programs, LIPIcs. Leibniz Int. Proc. Inform., vol. 69, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018, pp. Art. No. 5, 34.

[Cis06] Denis-Charles Cisinski, *Les préfaisceaux comme modèles des types d'homotopie*, Astérisque (2006), no. 308, xxiv+390.

[Cis14] ———, *Univalent universes for elegant models of homotopy types*, Preprint, 2014.

[Cis19] ———, *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2019, `http://www.mathematik.uni-regensburg.de/cisinski/CatLR.pdf`.

[CKM20] Timothy Campion, Krzysztof Kapulkin, and Yuki Maehara, *A cubical model for (∞, n)-categories*, Submitted, 2020.

[DKLS20] Brandon Doherty, Krzysztof Kapulkin, Zachery Lindsey, and Christian Sattler, *Cubical models of $(\infty, 1)$-categories*, Submitted, 2020, `arXiv:2005.04853`.

[DKM21] Brandon Doherty, Krzysztof Kapulkin, and Yuki Maehara, *Equivalence of cubical and simplicial approaches to (∞, n)-categories*, Submitted, 2021, `arXiv:2106.09428`.

[GJ99] Paul G. Goerss and John F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999, `doi:10.1007/978-3-0348-8707-6`, `https://doi.org/10.1007/978-3-0348-8707-6`.

[GM03] Marco Grandis and Luca Mauri, *Cubical sets and their site*, Theory and Applications of Categories **11** (2003), no. 8, 185–211.

[GR17a] Dennis Gaitsgory and Nick Rozenblyum, *A study in derived algebraic geometry. Vol. I. Correspondences and duality*, Mathematical Surveys and Monographs, vol. 221, American Mathematical Society, Providence, RI, 2017, `doi:10.1090/surv/221.1`, `https://doi.org/10.1090/surv/221.1`.

[GR17b] ———, *A study in derived algebraic geometry. Vol. II. Deformations, Lie theory and formal geometry*, Mathematical Surveys and Monographs, vol. 221, American Mathematical Society, Providence, RI, 2017, `doi:10.1090/surv/221.2`, `https://doi.org/10.1090/surv/221.2`.

[GZ67] Peter Gabriel and Michel Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York, 1967.

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[HKRS17] Kathryn Hess, Magdalena Kędziorek, Emily Riehl, and Brooke Shipley, *A necessary and sufficient condition for induced model structures*, J. Topol. **10** (2017), no. 2, 324–369, `doi:10.1112/topo.12011`.

[Hov99] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.

[Jar06] J. F. Jardine, *Categorical homotopy theory*, Homology Homotopy Appl. **8** (2006), no. 1, 71–144, `http://projecteuclid.org/euclid.hha/1140012467`.

[Joy02] André Joyal, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra **175** (2002), no. 1-3, 207–222, Special volume celebrating the 70th birthday of Professor Max Kelly, `doi:10.1016/S0022-4049(02)00135-4`.

[Joy07] ———, *Quasi-categories vs. simplicial categories*, Preprint, 2007, `http://www.math.uchicago.edu/~may/IMA/Incoming/Joyal/QvsDJan9(2007).pdf`.

[Joy09] ———, *The theory of quasi-categories and its applications*, Vol. II of course notes from Simplicial Methods in Higher Categories, Centra de Recerca Matemàtica, Barcelona, 2008, 2009, `http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf`.

[JT07] André Joyal and Myles Tierney, *Quasi-categories vs Segal spaces*, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 277–326, `doi:10.1090/conm/431/08278`.

[Kan55] Daniel M. Kan, *Abstract homotopy. I*, Proceedings of the National Academy of Sciences of the United States of America **41** (1955), no. 12, 1092–1096.

[Kan57] Daniel M. Kan, *On c. s. s. complexes*, Amer. J. Math. **79** (1957), 449–476.

[KLW19] Krzysztof Kapulkin, Zachery Lindsey, and Liang Ze Wong, *A co-reflection of cubical sets into simplicial sets with applications to model structures*, New York J. Math. **25** (2019), 627–641.

[Kri15] Sanjeevi Krishnan, *Cubical approximation for directed topology I*, Applied Categorical Structures **23** (2015), no. 2, 177–214.

[KV20] Krzysztof Kapulkin and Vladimir Voevodsky, *A cubical approach to straightening*, J. Topol. **13** (2020), no. 4, 1682–1700, `doi:10.1112/topo.12173`, `https://doi.org/10.1112/topo.12173`.

[Luc18] Maxime Lucas, *Cubical categories for homotopy and rewriting*, Ph.D. thesis, Univ. Paris 7, Sorbonne Paris Cité, 2018, `https://hal.archives-ouvertes.fr/tel-01668359/document`.

[Lur09a] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009, `http://www.math.harvard.edu/~lurie/papers/croppedtopoi.pdf`.

[Lur09b] ______, *On the classification of topological field theories*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 129–280.

[Mae21] Yuki Maehara, *The Gray tensor product for 2-quasi-categories*, Adv. Math. **377** (2021), 107461, 78, `doi:10.1016/j.aim.2020.107461`, `https://doi.org/10.1016/j.aim.2020.107461`.

[Mal09] Georges Maltsiniotis, *La catégorie cubique avec connexions est une catégorie test stricte*, Homology Homotopy Appl. **11** (2009), no. 2, 309–326, `http://projecteuclid.org/euclid.hha/1296138523`.

[MP89] Michael Makkai and Robert Paré, *Accessible categories: the foundations of categorical model theory*, Contemporary Mathematics, vol. 104, American Mathematical Society, Providence, RI, 1989, `doi:10.1090/conm/104`, `https://doi.org/10.1090/conm/104`.

[Ols11] Marc Olschok, *Left determined model structures for locally presentable categories*, Appl. Categ. Structures **19** (2011), no. 6, 901–938, `doi:10.1007/s10485-009-9207-2`, `https://doi.org/10.1007/s10485-009-9207-2`.

[OR20] Viktoriya Ozornova and Martina Rovelli, *Model structures for (∞, n)-categories on (pre)stratified simplicial sets and prestratified simplicial spaces*, Algebr. Geom. Topol. **20** (2020), no. 3, 1543–1600, `doi:10.2140/agt.2020.20.1543`, `https://doi.org/10.2140/agt.2020.20.1543`.

[ORV20] Viktoriya Ozornova, Martina Rovelli, and Dominic Verity, *Gray tensor product and saturated N-complicial sets*, Preprint, 2020.

[Qui67] Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967.

[Ree21] C. L. Reedy, *Homotopy theory of model categories*, Unpublished notes, accessed June 18 2021, `http://www-math.mit.edu/~psh/reedy.pdf`.

[Rez01] Charles Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007 (electronic), `doi:10.1090/S0002-9947-00-02653-2`.

[Rez20] ______, *Stuff about quasicategories*, Unpublished notes, accessed June 23 2020, `https://faculty.math.illinois.edu/~rezk/quasicats.pdf`.

[Rie14] Emily Riehl, *Categorical homotopy theory*, New Mathematical Monographs, vol. 24, Cambridge University Press, Cambridge, 2014, `doi:10.1017/CBO9781107261457`, `https://doi.org/10.1017/CBO9781107261457`.

[RS17] Emily Riehl and Michael Shulman, *A type theory for synthetic ∞-categories*, High. Struct. **1** (2017), no. 1, 147–224, `doi:10.1007/s42001-017-0005-6`, `https://doi.org/10.1007/s42001-017-0005-6`.

[RV21] Emily Riehl and Dominic Verity, *Elements of ∞-category theory*, In progress, accessed May 2021, `http://www.math.jhu.edu/~eriehl/scratch.pdf`.

[Ver08a] Dominic Verity, *Complicial sets characterising the simplicial nerves of strict ω-categories*, Mem. Amer. Math. Soc. **193** (2008), no. 905, xvi+184, `doi:10.1090/memo/0905`, `https://doi.org/10.1090/memo/0905`.

[Ver08b] ______, *Weak complicial sets. I. Basic homotopy theory*, Adv. Math. **219** (2008), no. 4, 1081–1149, `doi:10.1016/j.aim.2008.06.003`, `https://doi.org/10.1016/j.aim.2008.06.003`.