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Application of Stochastic Control to Portfolio Optimization and Energy Finance

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Abstract

In this thesis, we study two continuous-time optimal control problems. The first describes competition in the energy market and the second aims at robust portfolio decisions for commodity markets. Both problems are approached via solutions of Hamilton-Jacobi-Bellman (HJB) and HJB-Isaacs (HJBI) equations.

In the energy market problem, our target is to maximize profits from trading crude oil by determining optimal crude oil production. We determine the optimal crude oil production rate by constructing a differential game between two types of players: a single finite-reserve producer and multiple infinite-reserve producers. We extend the deterministic unbounded-production model and stochastic monopolistic game to bounded-production and construct an N-player stochastic game using analytical and numerical solutions to the corresponding HJB equation. In this way, we compute the optimal strategies of oil production for four stylized players. As an example, applying the game-theory model above, we construct a deterministic and a stochastic differential-game model between four countries, and compare the real production and the forecast production in order to test the accuracy of the model.

In the robust portfolio optimization problem, we assume the investor allocates funds among a bond, a bank account, and a commodity that either pays a mean-reverting convenience yield, or follows an exponential Ornstein-Uhlenbeck (OU) process. In our settings, the interest rate of the bond follows a Vasicek model. We optimize the expected utility of terminal wealth, solving the corresponding HJBI equation via an exponential affine ansatz, which can be used to generate an optimal portfolio strategy. As part of our study, we fit our model to prices of crude oil, gold, copper and interest rates, leading to a meaningful empirical analysis. We concluded from the suboptimal analysis that the mis-specification of parameters and incompleteness of market lead to severe wealth-equivalent losses.

Keywords: energy market, HJB, HJBI equations, differential game, robust portfolio optimization, wealth equivalent losses analysis.

Summary for Lay Audiences

This thesis addresses two problems in the theory of commodity markets, unified by the HJB stochastic optimal control methodology employed and solves associated HJB partial differential equation systems.

The first problem involves applying differential game approaches to model the production strategy of energy producers. In the game, each player determines their own production rate schedule, which affects the world energy market supply-demand relationship and hence price. We perform this in a deterministic system and in a system in which energy demand has a stochastic driver. In both cases we consider the impact of participant production and profit bounds.

The second topic treated is the robust optimization of portfolios which include commodity assets. Here an investor solves for an optimal wealth allocation in order to maximize their expected terminal wealth, in a worst-case expected return scenario. The worst-case scenario is due to ambiguity in asset return parameters. We also compute the scale of losses by, alternatively, ignoring the ambiguity but considering a case in which parameters are incorrectly estimated.

These studies will be helpful for oil or other resource producing blocs to select their production strategies and for ambiguity-averse investors to determine their optimal investment strategy.

Co-authorship Statement

I declare that this thesis incorporate the joint work with Prof. Matt Davison and Prof. Marcos Escobar-Anel of the Department of Statistical and Actuarial Sciences at Western University. Chapter 4 of this thesis has been already published. A chapter (Chapter 2) is under second-round review in a peer-reviewed journal. Details of those co-authored papers are as below:

- The research materials in Chapter 2 are based on a manuscript, co-authored with my supervisor, Matt Davison. "Deterministic Asymmetric-cost Differential Games for Energy Production with Production Bounds". This manuscript is accepted by *SN Operations Research Forum*.

- The research materials in Chapter 3 are based on a yet unsubmitted manuscript, co-authored with my supervisor, Matt Davison, and Prof. Rob Corless. "Asymmetric-cost Diferential-game Model with Stochastic Profit in Energy Market".

- The research materials in Chapter 4 are a published paper, co-authored with my supervisors, Matt Davison, Marcos Escobar-Anel, and another graduate student, Golara Zafari. "Robust Portfolio with Commodities and Stochastic Interest Rates", *Quantitative Finance*, DOI: https://doi.org/10.1080/14697688.2020.1859603.

- The research materials in Chapter 5 are based on a manuscript, co-authored with my supervisor, Marcos Escobar-Anel. "Model uncertainty on commodity portfolios, the role of Convenience Yield". This manuscript is under review in *Annals of Finance*, DOI: https://doi.org/10.1007/s10436-021-00393-5

I certify that I am the primary author of all studies presented in this thesis. I was responsible for the model development, computations of solutions to the model, numerical algorithms and completion of the studies. Prof. Matt Davison and Prof. Marcos Escobar-Anel provided valuable and useful suggestions of ideas, revisions and innovation on the model formulation and empirical analysis. I certify that this dissertation, apart from as disclosed above, is fully a product of my own works.

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Chapter 1

Background and Motivation

1.1 Introduction

Economics and finance require producers, consumers and investors to design optimal strategies aiming to maximize their wealth or minimize their cost in a certain project. Mathematical finance explores quantitative methods to actually do this. In general, producers aim to maximize their profits when selling their goods while investors hope to optimize their wealth. Control theory provides a mathematical method of optimization in economic and financial fields to address the challenges facing these producers and investors. Control theory is flexible enough to handle different settings: discontinuous time versus continuous time, single player versus multiple players, and deterministic process versus stochastic process, etc. In our studies, we consider continuous-time control problems in two areas: crude oil production and portfolio optimization.

The Bellman principle and the related Hamilton-Jacobi-Bellman (HJB) equation is one of the most popular approaches for computing the value function and the associated optimal control. Most HJB Partial Differential Equations (PDEs) are not solvable in closed form, but in many cases, for large families of underlying processes and objectives, an ansatz of the PDEs with terminal or boundary conditions is available. Moreover, one can also explore numerical solution to these PDE problems. In such case, convergence and consistency must be ensured at the expense of heavily computational algorithms due to the continuous-time nature of our setting. This thesis is not primarily about numerically solving HJB equations, instead mathematical analysis is employed to simplify and transform the PDEs to Ordinary Differential Equations (ODEs). These ODEs may require numerical solution but are simpler than the original PDE.

In the next sections we split our overview into the topics managed in the thesis: Differential games in energy market (Section 1.2 and Section 1.3) and Robust portfolio analysis (Section 1.4 and Section 1.5). In Section 1.3 and Section 1.5 we list all the contributions of our work.

1.2 Literature Review for Differential Games in Energy Markets

1.2.1 Origination of Game Theory

Many researchers were involved in presenting, motivating and formulating the mathematical settings of game theory. The single-period game among players dates back to 19th century. [Cournot (1838)] gives the very first introduction and formulation to Cournot games, where two firms maximized their profits by setting a fixed production. The Cournot game assumes the price of products follows the supply-demand relationship. It included the concept of the Nash equilibrium but did not create the generalized definition. Later, [Bertrand (1883)] and [Edgeworth (1889)] create the so-called Bertrand game based on the Cournot model, where the firms determined their own selling price of products rather than the production. In the Bertrand game, the optimal strategy for each firm sets their price equal to the cost of production hence producers cannot make extra profit.

[Von Neumann and Morgenstern (1944)] and [Nash (1951)] introduce the concept of mixed strategy and proved the existence of Nash equilibrium. Mixed strategy is an assignment of probability to a set of pure strategies. Their contribution led to a new era of modern game theory. After that, game theory has been widely applied to different fields of studies, including economics, politics, psychology, etc.

In our studies of differential game, oil producers are players in a Cournot game in which they profit from oil production net of asymmetric production costs. Producers aim to maximize their own profits, hence their optimal production strategies form a Nash equilibrium.

1.2.2 Stochastic Driver of Oil Price

Researchers and policy makers have explored oil price models through regressions, time-series analysis or stochastic differential equations (SDEs). Some models are homogeneous stochastic process which do not consider external factors. [Schwartz (1997)] presents an exponential Ornstein–Uhlenbeck (OU) process, and a three-factor model for crude oil price capable of future curve modeling using PDEs. [Morana (2001)] uses a GARCH(1,1) model to describe the volatility of crude oil price. Furthermore, [Hou and Suardi (2012)] uses different non-parametric GARCH models to describe the volatility of oil price (WTI and Brent) return. These studies of crude oil price are purely mathematical and do not consider macro-economic factors.

Oil prices are subject to supply-demand relationship, affected by the behaviors of producers. [Griffin (1985)] fits the competitive model, $\ln Q = \alpha + \gamma \ln P$, to production data of the Organization of the Petroleum Exporting Countries (OPEC), where γ is the price elasticity, $\ln Q$ is log oil production, and $\ln P$ is log price. The regression results show that elasticities γ of five OPEC countries are significantly negative. Based on this result, [Ramcharran (2002)] adds a time variable T as the trend factor into the competitive model and fits regression models to OPEC and non-OPEC countries.

[Rehrl and Friedrich (2006)] presents the Long-term Oil Price and Extraction (LOPEX) model. The model investigates oil price and production in the long-term future before 2100. They describe the production of non-OPEC countries using a model called a Hubbert curve, and obtain the production of OPEC countries via maximization of an intertemporal objective function (the total profit). [Dées et al. (2007)] subsequently analyzes models of crude oil demand, supply and price from multiple macro-economic factors, including GDP, oil stocks for OPEC and non-OPEC countries, production costs, OPEC capacity, etc. This previous study focused on the influence of OPEC behavior on the oil market. Similarly, [Hamilton (2009)] applies a regression model exploring the change of oil price using GDP, supply and demand. The article also analyzed the role of buy-sell speculation and behavior of OPEC countries.

The papers described above provide us with specific economic factors to construct an oil price model. The comparative model in [Griffin (1985)] provides the concept of linear price model. In our work, we combine the properties of stochasticity and supply-demand relationship, assuming producers control the demand factor. As [Dées et al. (2007)] and [Hamilton (2009)] indicate, long-term economic growth and inflation lead to stochastic changes in oil price. As economic growth generally leads to increased crude oil demand, we consider the stochastic demand to be a Geometric Brownian Motion (GBM).

1.2.3 Optimization and Differential-game Problems in Energy Market

A key problem of energy finance is that of production control. To make an optimal profit from the production process, optimal control methods have been applied to obtain optimal production strategies. This has been a popular topic of study for a long time. The earliest research comes from [Hotelling (1931)], where price and energy quantity follows a deterministic differential equation. Hotelling derived the dynamics of the price of exhaustible energy, demonstrating that it grows exponentially with the discount rate. [Hubbert (1956)] introduces the concept of "peak oil", in which he argued the production of exhaustible energy initially increased because of discovery, and eventually decreased due to resource exhaustion. Therefore, the shape of production curve should follow a bell shape with a peak, mathematically the derivative of a logistic curve.

[Kamien and Schwartz (1978)] investigates the optimal wealth consumption, and research & development (R&D) rate. The study analyzed the path of these two variables as a function of time. [Deshmukh and Pliska (1983)] develops a theory for optimal consumption of a exhaustible resource with two kinds of uncertainty: resource discovery and changing economic environment.

A classical problem is that of how a duopoly or oligopoly manages a renewable common resource such as fish. Fish stocks regenerate, dependent on their population, with a given growth rate and each player may harvest fish, reducing the population. The price of harvested fish satisfies the supply-demand relationship and enables players to earn profits. The so called productive asset problem is to maximize the accumulated profit of each player by deciding the production strategy for each as a control in continuous time. The resulting model is an example of a differential game. Differential game models have been widely applied to productive asset duopoly problem (see [Benchekrown (2003)]) and to the productive asset oligopoly problems (see [Benchekrown (2008)], [Benchekrown et al. (2009)] and [Colombo and Labrecciosa (2013)]).

In recent years, differential games have been widely applied to energy markets. Ronnie Sircar and his co-workers were key pioneers bringing differential game ideas to the study of energy markets. The productive asset problem can be applied to exhaustible energy production as a special case with a zero growth rate. They apply differential-game models, including the Cournot game, the Bertrand game and the mean-field game to energy production problems. For instance, [Harris et al. (2010)] uses the price model of relative prudence ρ of price P and computed a bound of $\rho = -\frac{QP''(Q)}{P'(Q)}$ in static Cournot games to ensure the existence of

Nash equilibrium. This is assuming all producers held infinite reserves. For the dynamic case, they obtained an approximate solution to dynamic games where all producers held low-cost exhaustible resources and high-cost inexhaustible resources. [Ledvina and Sircar (2011)] investigates the stochastic Bertrand oligopolistic game, where demand function was individual for each player. For a duopoly case, they obtained an approximate solution using an expansion based on the interaction parameter δ, which measures the substitutability of energy produced by different players, and a numerical solution to the HJB PDE. [Chan and Sircar (2015)] extends the stochastic Bertrand oligopolistic game into a mean-field game. They found an asymptotically approximate solution using an expansion in the interaction parameter ϵ as well.

In terms of Cournot games, [Ledvina and Sircar (2012)] finds the solution to oligopolistic games with asymmetric costs in energy market, where only one producer had an exhaustible (finite) reserve and others held renewable (infinite) reserve. They assumed a linear and deterministic price model and computed so-called "blockading points" for high-cost infinite-reserve producers. Based on the solution to asymmetric oligopolistic game, [Dasarathy and Sircar (2014)] adds the effects of R&D, varying costs, resource discovery and energy policy and computed numerical solutions to the problem.

These deterministic games can be extended by adding stochastic factors. For example, [Ludkovski and Yang (2015)] constructs a two-player differential-game model by assuming the aggregate price as a Markov switch regime M_t, where the discovery of new oil reserve is allowed with a discovery rate a_i. [Chan and Sircar (2017)] considers the mean-field games with infinite-reserve players in the continuous-time Cournot competitions via an asympotic expansion of the production cost, and concluded that producers would not decrease their production, despite falling oil price. In the meantime, [Brown et al. (2017)] analyzes the dynamic games of oil price where demand is a stochastic process. They obtained an ODE in the case that the price level Y_t followed a GBM process, and a numerical solution when Y_t followed an exponential OU process.

1.3 Introduction to Differential Game in Energy Market

Game theory uses matheamatics to study the competition between two or more players. With a long history, the first mathematical game theory problem can be traced back to the prisoner's dilemma. This two-player zero-sum game models the decisions made by two prisoners who independently choose between admitting their crimes or staying silent. Their combined selection will have different results. In order to solve this type of problem, the Nash equilibrium gives the most general mathematical formulation of this game. In the Nash equilibrium, all players aim to maximize their own profits, while assuming their opponents have already maximized their profits. In other words, each player cannot profit more by unilaterally changing their strategy. In a general mathematical setting of an N-player game, the goal for player i is to find q_i^* to achieve

$$\sup_{q_i} \pi_i(q_1^*, q_2^*, \ldots, q_i, \ldots, q_N^*). \tag{1.1}$$

where q_i is the strategy of player i, q_j^* is the optimized strategy of player j, and π_i is the profit function of player i corresponding to those strategies.

Today, there are many different types of games according to the number of players, equilibrium concepts, types of strategies, whether they are cooperative or non-cooperative, etc. [Cournot (1838)] and [Bertrand (1883)] games, which deals with production problems among multiple players, are perhaps closest to our problems. In a Cournot game, players set production, pay for the associated production costs, and take the associated market price. According to the price-demand relationship, players can maximize profits by conveniently setting their amounts of production. In contrast, players in the Bertrand game set their price, from which a quantity emerges with associated profit.

In this thesis, we study an application of differential games to the world energy market. Energy markets are impacted by diverse forces like market price, level of production, location and geography, developments in technology, geopolitics, etc. For example, in 2020, the Coronavirus pandemic spread across the whole world and dramatically cut down the demand for crude oil. Even worse, the oil price war between Saudi Arabia and Russia, as a chain reaction of Coronavirus, led to an intentional increase in oil production and a slump in oil prices. In our thesis, we focus on extraction (production) strategies of exhaustible energies among different countries as players. Exhaustible energies such as crude oil and coal are important energy sources for our daily life, because producers can cheaply extract them to produce power efficiently and smoothly at a very low cost. In particular, crude oil plays a significant role in transportations, electricity supplies, and composition of plastic products. In the world, there is a huge demand for crude oil. Countries with sufficient crude oil reserves can make considerable profits by extracting and exporting those crude oil to others. Those countries can be reasonably treated as "infinite-reserve producer". Representative countries such as Saudi Arabia, the US (shale oil) and Russia hold a large amount of oil reserves, which can be modelled as infinite. According to the U.S. Energy Information Administration*, the

*https://www.eia.gov/energyexplained/index.php?page=oil_where top three oil-producing countries each accounted for more than 12% of total production in 2018. On the other hand, countries or companies in locations with limited crude oil reserves for the foreseeable future have to treat and use those resources more efficiently/optimally. We call these countries “finite-reserve producers”.

Using a Cournot-game setting, our goal is to maximize the profit of oil-production countries by designing optimal strategies. We assume the price of crude oil follows a supply-demand relation affected by inflation. As the economy develops, the overall demand for crude oil increases and drives oil price up. The supply and demand factor can be represented by the worldwide total production and consumption, respectively.

Given the price model, we obtain the profit model for oil-production entities. Oil-producing countries must consider both their own traits and those of their opponents to make optimal profits. They should also consider the behavior of their opponents, and the balance between current and future production. Optimal strategies are functions of time. This constitutes a differential game involving those countries as players. The mathematical solution to this game yields optimal production strategies and maximum profits for those countries.

A differential-game model is a particular type of continuous-time control problem, where we need to find the possible time-varying strategies (optimal controls) for the multiple players underlying. Mathematically, the player i aims to compute the objective function in integral form,

$$J_i(t,\boldsymbol{x}) = \sup_{q_i} \mathbb{E}\left[\int_t^T \pi_i(t,\boldsymbol{X}(t),q_1^*(t),q_2^*(t),\dots,q_i(t),\dots,q_N^*(t))\,\mathrm{d}t\right]. \tag{1.2}$$

where T is either a fixed value or a stopping time, $\pi_i(\,\cdot\,)$ is the player i's profit, $\boldsymbol{X}(t)$ is the associated state variable with $\boldsymbol{X}(t)=\boldsymbol{x}$. $\boldsymbol{X}(t)$ follows a particular dynamic

$$\mathrm{d}\boldsymbol{X}(t) = \boldsymbol{a}(t,\boldsymbol{X}(t),\boldsymbol{q}(t))\,\mathrm{d}t + \boldsymbol{B}(t,\boldsymbol{X}(t),\boldsymbol{q}(t))\,\mathrm{d}\boldsymbol{W}(t), \tag{1.3}$$

where $\boldsymbol{a},\boldsymbol{B}$ are a particular vector and matrix, $\mathrm{d}\boldsymbol{W}(t)$ is a vector of Brownian motion, and $\boldsymbol{q}$ is the vector of strategies of players. An N-player differential game leads to a system of Hamilton–Jacobi–Bellman (HJB) equations. Solving the HJB equations analytically or numerically can give optimal production strategies. Depending on the type of solution, our research makes the following contributions:

- We construct asymmetric differential-game problems, obtain the feedback optimal strategy and compute the accumulated profit in both deterministic and stochastic settings as a function of reserve x and a stochastic profit level y.

- The deterministic problem is based on [Ledvina and Sircar (2012)]. We add constraints to the production and profit of the finite-reserve player, assuming the behavior of the finite-reserve player is controlled by external factors. We confirm that an upper bound on production reduces the accumulated profit of the finite-reserve producer. On the other hand, a lower bound on production to protect a minimum oil price would increase the total accumulated profit compared to the unconstrained case.

- We obtain the relationship between production and profit of each player versus the reserve x and stochastic level y. In particular, the profit of the finite-reserve player has marginal effect with an increase of y, which shows that the reserve x restricts the effect of increasing y on the profit.

- We analyze the share of world total production taking US (conventional) as an exhaustible/finite-reserve producer, as well as US (shale) + Canada, Saudi Arabia, Russia as inexhaustible/infinite-reserve players. We estimate the parametric set of differential-game models via regression of the price by the ratios production/consumption. The fitting model shows that an increase in the ratio generally lead to lower oil price.

- We forecast production using the price model above, and compare the predicted oil productions and historical real productions among selected players. The plots comparing the predicted production versus the real production shows a partially linear pattern with R^2 ranging from $0.68 - 0.82$ and F-statistic p-values < 0.001, proving that the prediction using the differential model is partially useful indication of the real production.

To sum up, Table 1.1 describes existing literatures and our studies in differential games in energy market.

1.4 Literature Review of Robust Portfolio Optimization

1.4.1 Origination of Modern Portfolio Optimization

Modern portfolio optimization could be traced back to [Markowitz (1952)], who created the mean-variance portfolio problem. The mean-variance model assumes investors maximize the

Studies	Game	Players	Descriptions
[Harris et al. (2010)]	Cournot	N	Exhaustible
[Ledvina and Sircar (2011)]	Bertrand	Duopoly	Exhaustible
[Ledvina and Sircar (2012)]	Cournot	N	One Exhaustible
[Dasarathy and Sircar (2014)]	Cournot	N	Poisson reserve, one exhaustible
[Chan and Sircar (2015)]	Bertrand	∞	Exhaustible
[Brown et al. (2017)]	Cournot	Duopoly	Stochastic demand, one exhaustible
[Chan and Sircar (2017)]	Cournot	∞	Brownian reserve, Exhaustible
Chapter 2	Cournot	N	Production bounds, one exhaustible
Chapter 3	Cournot	N	GBM demand, Exhaustible
Appendix C	Cournot	4	Application to real-world data

Table 1.1: Table of studies of differential games in energy finance

expected wealth given a particular level of risk tolerance measured in variance in a single time period. The optimal strategy achieved lies on the “efficient frontier” in the plot of variance versus portfolio return.

[Von Neumann and Morgenstern (1944)] proposes the 4-axiom expected utility theorem, in which a rational individual aims to maximize the expected value. The theorem includes a risk-aversion property if the utility function is assumed to be concave. Based on the expected utility theory, [Merton (1971)] maximizes the expectation of consumption and terminal wealth, assuming the stock follows a GBM and a risk-free bank account is available in a continuous-time scheme. Based on Merton’s results, many studies on portfolio optimization problems follow the optimal expected utility paradigm. Relevant examples with closed-form solutions are provided next. [Munk et al. (2004)] assumes asset allocations on a bond, a stock and risk-free account. The interest rate and the excess return on the stock follows the Vasicek process. [Kraft (2005)] maximizes the expected power utility using a stock and a risk-free bank account, where the stock follows a Heston’s stochastic volatility. Furthermore, [Escobar et al. (2016)] extends the problem of [Kraft (2005)] to asset-allocation problem using priciple component stochastic volatility (PCSV) model. PCSV model includes multiple correlated assets with Heston’s volatilities.

The mean-variance problem and optimization of terminal wealth are two of the main types of portfolio optimization problems developed by economists. In our thesis, we assume a risk-averse investor maximizes the expected terminal wealth using the power utility $U = x^{1-\gamma}/(1-\gamma)$. The investor can invest into three assets: a commodity, a bond and a risk-free bank account. The commodity follows either an exponential OU process, or pays a mean-reverting convenience yield, from [Schwartz (1997)].

1.4.2 Developments in Portfolio Optimization with Model Uncertainty

Model uncertainty or parameter mis-specification is known in the literature as ambiguity aversion analysis. Both types of optimization problem address parameter mis-specification. [Ellsberg (1961)] presents the earliest concept of “ambiguity”, which could not be measured in the probability distribution like “risk”. He indicated that investors were not only risk-averse, but also ambiguity-averse.

Continuous-time robust portfolio optimization considers the worst-case scenario of financial parameters, as measured in an entropy in the objective function. Since the seminal work of [Maenhout (2004)], there have been numerous studies on the topic of robust portfolio analysis, i.e. optimal portfolio allocation for risk- and ambiguity-averse investor. The author takes the setting in [Merton (1971)] with a single stock and a riskless asset and assumed ambiguity about the expected rate of return on the stock. His adaptation of the robust control framework of [Andreson et al. (2003)] permits closed-form solutions for the key objects in the analysis, namely: allocation, terminal wealth, worst-case measure and value function. Here, an investor has a favourite portfolio allocation derived from a so-called “reference model”, but the study also acknowledges that the model may be mis-specified. The investor then studies optimal allocations based on a family of alternative models and prepares for a worst-case scenario, namely the optimal alternative model, this is compatible with the axiomatic of [Gilboa and Schmeidler (1989)]. As every source of risk (probability distribution) conveys its own level of ambiguity aversion, the literature has progressively studied these sources. [Liu et al. (2005)] considers an investor who is ambiguous about the jumps of the process for the state variable and in [Liu (2011)] regime-switching expected stock return were treated. [Branger and Larsen (2013)] models the stock price by a jump-diffusion process with different levels of ambiguity aversion about the diffusion and jump parts. [Flor and Larsen (2013)] considers stochastic interest rate with an investor allocating between bonds and stocks. They assume different ambiguity aversion levels on the expected returns of short rates and stocks. [Munk and Rubtsov (2014)] studies the impact of ambiguity about expected inflation on the choice of portfolio. [Escobar et al. (2015)] and [Bergen et al. (2018)] treat the cases of ambiguity aversion on stochastic volatility, covariance and stocks for complete and incomplete markets.

Surprisingly, the topics of portfolio optimization and ambiguity aversion has been barely studied for commodity-based investors. The complexity of commodity modeling is well-known, and most literature highlights several factors that are needed to explain the term structure of future prices. For instance, [Schwartz (1997)] proposes three factors: spot price, short interest rates and convenience yield; while [Schwartz and Smith (2000)] suggests two factors, short-term and long-term dynamics. In the absence of ambiguity, the leading analysis is provided by [Mellios et al. (2016)], who established the portfolio optimization problem using a spot commodity, a future contract and a bond; the commodity price model is based on Model III of [Schwartz (1997)]. This is a complete market model, where the market price of risk is assumed a function of convenience yield and spot prices. We have found there are only two examples of robust analysis in commodities. The first is the theoretical work of [Cartea et. al (2016)], who considered ambiguity-aversion on the diffusive (short-term factor) and jump (long term) components for a model adjusted for seasonal effects and a single stochastic volatility driver. The authors do not study portfolio allocations in this context. The second is our published study described in Chapter 4, [Chen et al. (2021)]. This study involved a log Ornstein-Uhlenbeck for the spot commodity and stochastic interest rates using a combination of Models I and II of [Schwartz (1997)]; the focus is on an insurance-type investor with a Cramer-Lundberg surplus. We observed that uncertainty in the commodity asset class is more disruptive to portfolio managers than uncertainty in the equity asset class (i.e. commodity managers shall be more attentive to uncertainty than equity managers).

All these studies consistently report significant losses and under-performance for investors who acknowledge ambiguity aversion but choose to ignore such uncertainty. As revealed in the literature, some sources of ambiguity, for instance coming from changes in the distribution of interest rates, the distribution of asset returns, or the distribution of volatilities, are more harmful than others. This observation leads to the key question to be addressed in Chapter 5 of our thesis: what is the impact and the most important source of uncertainty for commodity investors?

To answer the question, we consider ambiguity as per [Ellsberg (1961)]. Due to this parametric ambiguity, the investor in our model aims to maximize the expected utility in a worst-case scenario of expected return. We take the wealth equivalent losses analysis, compare the losses in different cases of parametric mis-specifications, including min-specification of market prices of risks, volatilities, and incomplete markets.

1.5 Introduction to Robust Portfolio Optimization

Portfolio optimization aims to maximize the wealth and consumption of an investor. The investor can allocate in assets following certain stochastic processes, while consuming goods. The stochasticity of assets makes a dynamic and optimal portfolio strategy necessary. For a given objective function/utility, robust portfolio optimization maximizes the utility in the worst-case scenario. The worst-case scenario represents the worst expected return on the asset, derived from an optimal change of measure using the Girsanov theorem. In return for the decreasing expected return, we compensate the objective function by using entropy as a quadratic function, because the compensation can be interpreted as a distance to a favorable, usually a best estimate, of the targeted parameters. Mathematically, the general problem to be solved is

$$\sup_{\boldsymbol{\pi},c_t} \inf_{\boldsymbol{u}_t} \mathbb{E}\left[\underbrace{\int_t^T \boldsymbol{u}_t^T \boldsymbol{\beta}^{-1} \boldsymbol{u}_t \,\mathrm{d}s}_{\text{entropy}} + \underbrace{\int_t^T e^{-rs} U_1(c_s)\,\mathrm{d}s}_{\text{consumption}} + \underbrace{U_2(X_T)}_{\text{terminal wealth}}\right]$$

where c_t is the consumption amount, X_t is wealth at time t, $\boldsymbol{\pi}_t$ is the investment allocation, u_t is the change of parameter values using the Girsanov theorem, and $U_1(\cdot), U_2(\cdot)$ are utility functions.

In our studies, we consider two types of dynamics of commodities. The first is a generic investment company with a surplus $X(t)$ that follows a Cramér-Lundberg process and a Vasicek interest rate model correlated to a mean-reverting log-asset process representing commodity price process. This is a general setting that addresses banking and insurance companies alike. This company invests the surplus into bonds, commodities and a bank account with the goal of maximizing the expected utility of terminal wealth, but it is concerned with the uncertainty in the fixed-income and commodity markets. The second dynamics of commodity process directly comes from the three-factor model of [Schwartz (1997)], which includes mean-reverting stochastic interest rate and mean-reverting convenience yield. In this study, other alternative investment choices are completely the same as the first, including bonds and a risk-free bank account.

We are the first to consider these robust portfolio optimization problems. A closely related work is [Chiu and Wong (2013)], who studied the optimal strategy for a similar type of company with mean-reverting underlyings within expected utility theory, they provided conditions for existence of a solution but no reference to ambiguity-aversion or full estimation. We contribute to the existing literature on several levels:

- By casting our optimal asset allocations described above as an HJBI PDE, we obtain closed-form solutions for all four relevant functions: optimal allocation, worst market conditions, optimal terminal wealth and value function. Explicit conditions for existence and well-behaved solutions are also provided.

- All parameters are estimated using current data from short rates, bond prices as well as two representative commodities, West Texas Intermediate (WTI) oil and Gold. Full estimation of commodity models is rare in the literature.

- We demonstrate that ambiguity has a significant impact on optimal trading strategies and terminal wealth. In particular, plausible ambiguity levels result in decreasing kurtosis, standard deviations and excess returns on the optimal portfolio.

- We perform a wealth equivalent losses analysis thanks to quasi-closed-form solutions for relevant suboptimal optimization problems. Our analyses exhibit that investors who ignore model uncertainty incur in drastic losses. In particular, we find that ignoring commodity uncertainty is more costly than neglecting interest rate uncertainty. Commodity markets are also more sensitivity than stock markets to uncertainty.

- The importance of working on a complete market (investing in bonds) for commodity investors is confirmed, welfare-equivalent losses could easily reach 45% when working with incomplete markets.

- We also found that parameter mis-specifications, particularly incorrect larger correlation, smaller variances or differing market prices of commodity risk, could lead to drastically large wealth-equivalent losses.

- The three factors model is calibrated to empirical data via a combination of maximum likelihood estimation (MLE) and Kalman filter (KF) techniques. Moreover, we use the West Texas Intermediate (WTI) spot and future prices, and the 1-month Treasury Constant Maturity Rate. We also studied a second commodity, copper future prices, to strengthen the results (see Section E.2).

- The empirical analysis established three key insights. First, small changes in convenience yields lead to large variation on forward contract allocation; second, small variations on convenience yield covariance parameters could lead to substantial wealth equivalent losses (WELs); and finally and more importantly, uncertainty about convenience yield could be a largest contributor to the under-performance of the portfolio.

To sum up, the existing literature and our studies of robust portfolio optimization are shown in Table 1.2.

Studies	Models	Ambiguities of drifts
[Maenhout (2004)]	GBM stock	Stock
[Liu (2011)]	Region-switching return	State variable
[Branger and Larsen (2013)]	GBM stock with jumps	Jump diffusion
[Flor and Larsen (2013)]	Vasicek interest	Stock, interest
[Escobar et al. (2015)]	Stochastic volatility	Stock, volatility
[Bergen et al. (2018)]	Stochastic covariance	Stock, covariance
Chapter 4	Mean-reverting commodity	Commodity, interest
Chapter 5	Three-factor commodity	Commodity, interest, convenience yield

Table 1.2: Table of studies of differential games in robust portfolio optimization

1.6 Connections among Chapters

This thesis presents the application of stochastic control problem to energy markets and robust portfolio optimizations. Based on [Ledvina and Sircar (2012)], Chapter 2 demonstrates the effects of production bounds on the profits of players with different costs of production. Also based on [Ledvina and Sircar (2012)], Chapter 3 incorporates a stochastic factor $Y(t)$ and computes the solution to the problem by using a Puiseux series and finite difference method. Considerable numerical and mathematical challenges are shown to exist in this chapter, some of which require more exploration. Appendix C executes the application of the deterministic and stochastic models in Chapter 2 and Chapter 3 to real-world data, and compare the forecast productions to real productions by years. Those three chapters study the differential games in energy markets and are connected with one another.

Chapter 4 and Chapter 5 study the robust portfolio optimization problems for commodities following the exponential-OU process and the three-factor model in [Schwartz (1997)]. Chapter 4 combines the Vasicek interest rate model and exponential-OU process and determines the portfolio strategy over a bank account, commodities and a bond. Chapter 5 studies the effects of stochastic convenience yields and determines the optimal portfolio over a bank account, a bond and two forwards with different maturities. In addition, we also perform an empirical analysis of the two models to WTI, gold and copper data and determine the effects of parameter misspecifictions on the WELs.

All these chapters share the same mathematical techniques. We maximize the accumulated profits/terminal expected utility as objective functions, by deciding the optimal strategies, generating HJB and HJBI equations, simplifying the PDEs into ODEs and numerically solving the ODEs.

1.7 Mathematical Preliminaries

In this section, we present some general results, in order to establish notation and some key, but and well, but not widely-known mathematical techniques used in the thesis. We expect readers to have elementary knowledge of ODEs (e.g. [Coddington (2012)]), PDEs (e.g. [Strauss (2007)]), SDEs (e.g. [Øksendal and Sulem (2007)], [Øksendal (2013)]) with their numerical methods (e.g. [Corless (2013)], [Iacus (2009)]), quantitative finance (e.g. [Hull (2003)]), probability theory (e.g. [Ross (2014)]) and mathematical statistics (e.g. [Rice (2006)]). Readers not familar with those topics can refer to relevant books described above.

1.7.1 General notation

We start with abbreviations and general notations used in our thesis. Abbreviations and notations are summarized in Table 1.3, Table 1.4 and Table 1.5.

Abbreviations	Description
tr	trace of a matrix
ODE	ordinary differential equation
PDE	partial differential equation
RDE	Riccati differential equation
HJB(I)	Hamilton-Jacobi-Bellman(-Isaacs)
dim	dimension
diag	diagonal elements of a matrix
WTI	West Texas Intermediate
OU	Ornstein–Uhlenbeck
GBM	Geometric Brownian Motion
MLE	maximum likelihood estimate
WEL	wealth equivalent loss
KF	Kalman filter
csmp	crude oil consumption

Table 1.3: Abbreviation

Notation used in Chapters 2 and 3

Symbol	Description
s_i	cost of production of player i
a_n	$a_n = \frac{1+\sum_{i=1}^{n-1} s_i - ns_0}{n}$
b_n	$b_n = r\left(\frac{n+1}{n}\right)^2$
$\theta_n(x)$	$\theta_n(x) = \beta_n \mathrm{e}^{\beta_n - \frac{b_n x}{2a_n}}$
β_n	$\beta_n = -1 + \frac{\sqrt{b_n v(x_b^n)}}{a_n}$
δ_n	$\delta_n := (n+1)s_n - \left(1 + s_0 + \sum_{i=1}^{n-1} s_i\right) > 0$
$c_{k,n}$	$c_{k,n} = \frac{1+\sum_{i=0}^{n-1} s_i}{n+1} - s_k$
K	$K = \min\{k : \delta_n > 0\}$
$x, X(t)$	remaining energy reserve
v_i	accumulate profit of player i
q_i	production of player i
Q	sum of productions of opponents
r	discount rate
x_b^n	blockading point of player n
τ	stopping time
$W(\,\cdot\,)$	Lambert-W function
c	maximal production bound
$v_{c,i}$	profit of player i under limited-production constraint
$q_{c,i}$	production of player i under limited-production constraint
p	minimal profit bound
$v_{p,i}$	profit of player i under minimal-profit constraint
$q_{p,i}$	production of player i under minimal-profit constraint
$y, Y(t)$	stochastic factor
$Z(t)$	Brownian motion
μ	drift of $Y(t)$
σ	volatility of $Y(t)$
ξ	$\xi = \frac{x}{y}$
$H(\,\cdot\,)$	$v(x,y) = y^2 H\left(\frac{x}{y}\right)$

Table 1.4: Notations for Chapters 2 and 3

Notation used in Chapters 4 and 5

Symbol	Description
$X_0(t)$	Cramér-Lundberg process
c	premium
$S(t)$	commodity price
$r(t)$	interest rate
a	mean-reverting rate of commodity price
$\bar{r}$ or m	mean-level interest rate
κ or a	mean-reverting rate of interest
$P(t, r(t))$	price of the bond
$\lambda_i, \boldsymbol{\lambda}$	market price of risk
$\delta(t)$	convenience yield
α	mean level of convenience yield
κ	mean-reverting rate of convenience yield
ρ, ρ_{ij}	correlation
$W_i(t), \boldsymbol{W}(t)$	Brownian motion
$W_i^Q(t), \boldsymbol{W}^Q(t)$	Brownian motion with change of measure
$x, X(t)$	surplus/wealth process
$J(\,\cdot\,)$	value function
$\beta_i, \boldsymbol{\beta}$	ambiguity-aversion parameter
γ	level of risk aversion
$u_i, \boldsymbol{u}$	change of drift
$\pi_i, \boldsymbol{\pi}$	amount of investment
$\sigma_i, \boldsymbol{\sigma}$	volatility
$\boldsymbol{\pi}^s$	amount of investment in wealth-equivalent analysis
τ_i	time to maturity

Table 1.5: Notations for Chapters 4 and 5

1.7.2 Basic Knowledge of Stochastic Differential Equations and Optimal Control Problems

Our problems are about continuous-time optimal control problems and differential games. Therefore, we rely on mathematical techniques of HJB or HJBI equations.

Theorem 1.7.1 (Itô's Formula)**.** *Let $\boldsymbol{X}(t) \in \mathbb{R}^n$ be an n-dimensional Itô-Lévy process of the form*

$$\mathrm{d}\boldsymbol{X}(t) = \boldsymbol{\mu}(t, \boldsymbol{X}(t))\,\mathrm{d}t + \boldsymbol{\sigma}(t, \boldsymbol{X}(t))\,\mathrm{d}\boldsymbol{W}(t) + \int_{\mathbb{R}^l} \boldsymbol{\gamma}(t, \boldsymbol{X}(t), \boldsymbol{z})\boldsymbol{N}(\mathrm{d}t, \mathrm{d}\boldsymbol{z}) \tag{1.4}$$

where $\boldsymbol{\mu} \in \mathbb{R}^n, \boldsymbol{\sigma} \in \mathbb{R}^{n\times m}, \boldsymbol{\gamma} \in \mathbb{R}^{n\times l}$, $\boldsymbol{W}$ is an m-dimensional Brownian motion and $\boldsymbol{N}$ is a l-dimensional independent Poisson random measure as

$$\boldsymbol{N}^T(\mathrm{d}t, \mathrm{d}\boldsymbol{z}) = (N_1(\mathrm{d}t, \mathrm{d}z_1), \ldots, N_l(\mathrm{d}t, \mathrm{d}z_l))^T. \tag{1.5}$$

Let $f(t, \boldsymbol{X}(t)) \in C^{1,2}(\mathbb{R}, \mathbb{R}^n)$. Then

$$\begin{aligned} \mathrm{d}f(t, \boldsymbol{X}(t)) = \frac{\partial f}{\partial t}\,\mathrm{d}t &+ \frac{\partial f}{\partial \boldsymbol{x}^T}(\boldsymbol{\mu}(t, \boldsymbol{X}(t))\,\mathrm{d}t + \boldsymbol{\sigma}(t, \boldsymbol{X}(t))\,\mathrm{d}\boldsymbol{W}(t)) \\ &+ \frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\sigma}^T(t, \boldsymbol{X}(t))\frac{\partial^2 f}{\partial \boldsymbol{x}\,\partial \boldsymbol{x}^T}\boldsymbol{\sigma}(t, \boldsymbol{X}(t))\right)\mathrm{d}t \\ &+ \sum_{i=1}^{l}\int_{\mathbb{R}^l} f(t, \boldsymbol{X}(t^-) + \boldsymbol{\gamma}^{(i)}(t, \boldsymbol{X}(t), \boldsymbol{z})) - f(t, \boldsymbol{X}(t^-))\boldsymbol{N}(\mathrm{d}t, \mathrm{d}\boldsymbol{z}) \end{aligned} \tag{1.6}$$

where $\boldsymbol{\gamma}^{(k)}$ is the k-th column vector of the matrix $\boldsymbol{\gamma}$.

HJB(I) equations are the fundamental mathematical technique of this thesis. Time-invariant HJB equations without a jump are used in Chapter 2 and Chapter 3. Time-variant HJBI equations with jump diffusion are used in Chapter 4 and Chapter 5. The following theorems on HJB(I) equations are based on Section 4.2 of [Øksendal and Sulem (2007)] and Section 11.2 of [Øksendal (2013)].

Theorem 1.7.2 (Time-variant HJB equation)**.** *Assume there are N players in a differential game. The value function of player i is*

$$J_i(t, \boldsymbol{x}) = \sup_{q_i} \mathbb{E}\left[\int_t^T \pi_i(s, \boldsymbol{X}(s), q_i(s), Q^*_{-i}(s))\,\mathrm{d}s + U(\boldsymbol{X}(T))\middle|\boldsymbol{X}(0) = \boldsymbol{x}\right]. \tag{1.7}$$

where T is a fixed terminal time. q_i is the strategy of player i, and $Q_{-i} = \{q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_N\}$ is the set of undetermined strategies of the opponents.

Let $\boldsymbol{X}(t) \in \mathbb{R}^n$ be an n-dimensional Itô-Lévy process of the form

$$\begin{aligned} \mathrm{d}\boldsymbol{X}(t) =& \boldsymbol{\mu}(t, \boldsymbol{X}(t), q_i(t), Q^*_{-i}(t))\, \mathrm{d}t + \boldsymbol{\sigma}(t, \boldsymbol{X}(t), q_i(t), Q^*_{-i}(t))\, \mathrm{d}\boldsymbol{W}(t) \\ &+ \int_{\mathbb{R}^l} \boldsymbol{\gamma}(t, \boldsymbol{X}(t), q_i(t), Q^*_{-i}(t), \boldsymbol{z}) \boldsymbol{N}(\mathrm{d}t, \mathrm{d}\boldsymbol{z}) \end{aligned} \tag{1.8}$$

where the assumption of the jump diffusion process follows Theorem 1.7.1.

If T is a fixed terminal time, then the system of HJB PDEs will be

$$\begin{aligned} \frac{\partial J_i}{\partial t} &= \sup_{q_i} \Big\{ \pi_i(t, \boldsymbol{x}, q_i, Q^*_{-i}) + \frac{\partial J_i}{\partial \boldsymbol{x}^T} (\boldsymbol{\mu}(t, \boldsymbol{x}, q_i, Q^*_{-i}) \\ &\quad + \frac{1}{2} \operatorname{tr}\Big(\boldsymbol{\sigma}^T(t, \boldsymbol{x}, q_i, Q^*_{-i}) \frac{\partial^2 J_i}{\partial \boldsymbol{x}\, \partial \boldsymbol{x}^T} \boldsymbol{\sigma}(t, \boldsymbol{x}, q_i, Q^*_{-i}) \Big) \\ &\quad + \sum_{i=1}^{l} \int_{\mathbb{R}^l} J_i(t, \boldsymbol{x} + \boldsymbol{\gamma}^{(i)}(t, \boldsymbol{x}, q_i, Q^*_{-i}, \boldsymbol{z})) \boldsymbol{N}(1, \mathrm{d}\boldsymbol{z}) + J_i(t, \boldsymbol{x}) \Big\} \\ J(T, \boldsymbol{x}) &= U(x). \end{aligned} \tag{1.9}$$

Remark 1.7.1. We can think of Theorem 1.7.2 as time varying because of the explicit $\frac{\partial}{\partial t}$. But if T is not a fixed terminal time, but either infinity or a stopping time which depends only on state factor $\boldsymbol{x}$, the optimal control problem does not explicit depends on time, as the following theorem states.

Theorem 1.7.3 (Time-invariant HJB equation)**.** *Assume the value function is*

$$J_i(\boldsymbol{x}) = \sup_{q_i} \mathbb{E}\left[\int_0^\tau \mathrm{e}^{-rs} p(\boldsymbol{X}(s), q_i, Q^*_{-i})\, \mathrm{d}s + \mathrm{e}^{-r\tau} U(\boldsymbol{X}(\tau)) \middle| \boldsymbol{X}(0) = \boldsymbol{x} \right]. \tag{1.10}$$

where q_i is the strategy of player i, and $Q_{-i} = \{q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_N\}$ is the set of strategies of the opponents. Therefore, p only depends on state variable $\boldsymbol{X}(s)$ and those undetermined controls. τ is a stopping time,

$$\tau = \inf\{t \colon \boldsymbol{x} \in \mathbb{R}^n / D\} \tag{1.11}$$

where D is a dense domain of $\mathbb{R}^n$. τ can be either infinite or a finite stopping time. Let $\boldsymbol{X}(t) \in \mathbb{R}^n$ be an n-dimensional Itô-Lévy process of the form

$$\mathrm{d}\boldsymbol{X}(t) = \boldsymbol{\mu}(\boldsymbol{X}(t), q_i(t), Q^*_{-i}(t))\, \mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}(t), q_i(t), Q^*_{-i}(t))\, \mathrm{d}\boldsymbol{W}(t) \tag{1.12}$$

where $\boldsymbol{\mu}, \boldsymbol{\sigma}$ are all independent of time, $\boldsymbol{\gamma} = \mathbf{0}$ (i.e., there is no jump).

Then the HJB equations to this value function will be

$$\begin{aligned} rJ_i &= \sup_{q_i} \Big\{ p(\boldsymbol{x}, q_i, Q^*_{-i}) + \frac{\partial J_i}{\partial \boldsymbol{x}^T} (\boldsymbol{\mu}(\boldsymbol{x}, q_i, Q^*_{-i}) + \frac{1}{2} \operatorname{tr}\Big(\boldsymbol{\sigma}^T(\boldsymbol{x}, q_i, Q^*_{-i}) \frac{\partial^2 J_i}{\partial \boldsymbol{x}\, \partial \boldsymbol{x}^T} \boldsymbol{\sigma}(\boldsymbol{x}, q_i, Q^*_{-i}) \Big) \Big\} \\ J_i(\boldsymbol{x}) &= U(x) \text{ when } x \in \bar{D} \end{aligned} \tag{1.13}$$

where $\bar{D}$ is the boundary of domain D. In other words, this system of HJB PDE is independent of time.

Remark 1.7.2. If we predetermine the optimal control $q_n^*(\boldsymbol{x})$ as a function of only state variables $\boldsymbol{x}$, the objective function will, in fact, be initial-time invariant. Mathematically,

$$\begin{aligned} J(\boldsymbol{x}) =& \mathbb{E}\left[\int_0^\tau \mathrm{e}^{-rs} p(\boldsymbol{X}(s), q_i^*(\boldsymbol{X}(s)), Q_{-i}^*(\boldsymbol{X}(s)))\,\mathrm{d}s + \mathrm{e}^{-r\tau} U(\boldsymbol{X}(\tau)) \middle| \boldsymbol{X}(0) = \boldsymbol{x}\right] \\ \mathrm{e}^{rt} J(\boldsymbol{x}) =& \mathbb{E}\left[\int_t^\tau \mathrm{e}^{-rs} p(\boldsymbol{X}(s), q_i^*(\boldsymbol{X}(s)), Q_{-i}^*(\boldsymbol{X}(s)))\,\mathrm{d}s + \mathrm{e}^{-r\tau} U(\boldsymbol{X}(\tau)) \middle| \boldsymbol{X}(t) = \boldsymbol{x}\right], \end{aligned} \tag{1.14}$$

as the stopping time τ will corresponding increase t because of time-homogeneous $\boldsymbol{X}(t)$, if we take $\boldsymbol{X}(t) = \boldsymbol{x}$ as a new initial time. This property ensures the time-invariant property of the HJB equation in Theorem 1.7.3, which in turn specifies the explicit formula of $q_n^*(\boldsymbol{x})$.

Remark 1.7.3. To illustrate the time-invarant property of Theorem 1.7.3, we can recall pricing of the American put option on stock $S(t)$ with maturity T, strike K and exercise boundary $\theta(t)$ as comparison, although it is not an identical problem, where we assume

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t). \tag{1.15}$$

The pricing of American put option follows the optimal stopping problem for stopping time τ,

$$V(S,t) = \sup_{\theta(t)} \mathbb{E}[\mathrm{e}^{-r\tau}(K - S(\tau))^+ | S(t) = S] \text{ s.t. } \tau = \inf\{t : S(t) \le \theta(t)\}. \tag{1.16}$$

The corresponding PDE system is

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV \\ &\lim_{S\to\infty} V(S,t) = 0, \quad V(\theta(t),t) = K - \theta(t), \\ &\frac{\partial V}{\partial S}(\theta(t),t) = -1, \quad V(S,T) = \max(K - S, 0). \end{aligned} \tag{1.17}$$

Therefore, the control of exercise boundary depends on time t.

If $T \to \infty$, the option becomes a perpetual American put, and the time variable disappears. The system becomes

$$\begin{aligned} &\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV \\ &V(\infty) = 0, \quad V(\theta) = K - \theta. \end{aligned} \tag{1.18}$$

In this case, the exercise boundary no longer depends on time t, and the value of the perpetual put does not depend on t either.

Similarly, we also present the theorem of the HJBI equation in the form of sup inf problem.

Theorem 1.7.4 (Time-variant HJBI equation)**.** *Assume the value funtion is a* sup inf *problem,*

$$J(t,\boldsymbol{x}) = \sup_q \inf_u \mathbb{E}\left[\int_t^T \pi(s,\boldsymbol{X}(s),q(s),u(s))\,\mathrm{d}s + U(\boldsymbol{X}(T))\middle|\boldsymbol{X}(0)=\boldsymbol{x}\right]. \tag{1.19}$$

where T is a fixed terminal time. $q(s), u(s)$ are the undetermined strategies at time s.

Let $\boldsymbol{X}(t) \in \mathbb{R}^n$ be an n-dimensional Itô-Lévy process of the form

$$\begin{aligned}\mathrm{d}\boldsymbol{X}(t) =& \boldsymbol{\mu}(t,\boldsymbol{X}(t),q(t),p(t))\,\mathrm{d}t + \boldsymbol{\sigma}(t,\boldsymbol{X}(t),q(t),p(t))\,\mathrm{d}\boldsymbol{W}(t)\\ &+ \int_{\mathbb{R}^l} \boldsymbol{\gamma}(t,\boldsymbol{X}(t),q(t),p(t),\boldsymbol{z})\boldsymbol{N}(\mathrm{d}t,\mathrm{d}\boldsymbol{z})\end{aligned} \tag{1.20}$$

where the assumption of the jump diffusion process is identical to Theorem 1.7.1.

If T is a fixed terminal time, then the system of HJBI PDEs will be

$$\begin{aligned}\frac{\partial J}{\partial t} &= \sup_q \inf_u \Big\{\pi(t,\boldsymbol{x},q,u) + \frac{\partial J}{\partial \boldsymbol{x}^T}(\boldsymbol{\mu}(t,\boldsymbol{x},q,u) + \frac{1}{2}\operatorname{tr}\Big(\boldsymbol{\sigma}^T(t,\boldsymbol{x},q,u)\frac{\partial^2 J}{\partial \boldsymbol{x}\,\partial \boldsymbol{x}^T}\boldsymbol{\sigma}(t,\boldsymbol{x},q,u)\Big)\\ &\quad + \sum_{i=1}^{l}\int_{\mathbb{R}^l} J(t,\boldsymbol{x}+\boldsymbol{\gamma}^{(i)}(t,\boldsymbol{x},q,u,\boldsymbol{z}))\boldsymbol{N}(1,\mathrm{d}\boldsymbol{z}) + J(t,\boldsymbol{x})\Big\}\\ J(T,\boldsymbol{x}) &= U(x).\end{aligned} \tag{1.21}$$

Theorem 1.7.5 (Girsanov's Theorem)**.** *Set $\boldsymbol{\gamma} = \boldsymbol{0}$ in Equation (1.4). Taking the change of measure*

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \exp\Big(-\int_0^T \boldsymbol{u}^T(t)\,\mathrm{d}\boldsymbol{W}(t)\Big) \tag{1.22}$$

leads to the SDE in the measure Q, where $\boldsymbol{u} \in \mathbb{R}^m$ is an $\mathscr{F}$-adapted process.

Then

$$\mathrm{d}\boldsymbol{W}(t) = \mathrm{d}\boldsymbol{W}^Q(t) - \boldsymbol{u}(t) \tag{1.23}$$

and Equation (1.4) becomes

$$\mathrm{d}\boldsymbol{X}(t) = (\boldsymbol{\mu}(t,\boldsymbol{X}(t)) - \boldsymbol{\sigma}(t,\boldsymbol{X}(t))\boldsymbol{u}(t))\,\mathrm{d}t + \boldsymbol{\sigma}(t,\boldsymbol{X}(t))\,\mathrm{d}\boldsymbol{W}^Q(t) \tag{1.24}$$

Chapter 2

Deterministic Asymmetric-cost Differential Games for Energy Production with Production Bounds

2.1 Introduction

We study a continuous optimal control problem which models competition in the energy market. Competing agents maximize profits from crude oil by determining optimal crude oil production via solution of Hamilton-Jacobi-Bellman (HJB) equations. We design the crude oil production rate by constructing a differential game between two types of players: a single finite-reserve producer and multiple high-cost infinite-reserve producers. We extend the deterministic unbounded-production model from [Ledvina and Sircar (2012)] to a bounded-production game, in which we show that the upper (lower) bound decreases (increases) the profit of finite-reserve player and the low-cost opponents, and increases (decreases) the profit of high-cost opponents, due to the effects on the finite-reserve player's exit time and the market price.

We utilize the linear price model as profits per time unit and explain its rationale in Section 2.2 using the oil price-production data. Section 2.3 gives the complete results of the N-player asymmetric-cost static and dynamic games from [Ledvina and Sircar (2012)]. Our research makes the following contributions, as discussed in Section 2.4:

- Taking the setting of the deterministic problem based on [Ledvina and Sircar (2012)], we add production constraints to the production/price of the finite-reserve players, assuming that the behavior of the finite-reserve player is controlled by external circumstances. We confirm that an upper bound on production reduces the accumulated profit of the finite-reserve producer.

- We set a lower bound on production to protect a minimum oil price. An unexpected result shows an increase trend of the total accumulated profit with the minimal-production constraint compared to the unconstrained case.

- We analyze why the upper and lower bounds on the production of the finite-reserve producer have different directions of impacts on the opponents, depending on their costs of production. The profits of higher-cost opponents decrease with the upper bound, while the lower-cost opponents show an inverse trend. The opposite trends hold for the lower bound.

- We presents the impacts of minimum-profit and maximum-production constraints on oil prices.

2.2 Price Model

The basis of economics is the supply-demand relationship. In the website https://www.britannica.com/topic/supply-and-demand, the curve “supply and demand” indicates that the demand will decrease, as the price increases, but the supply curve shows the inverse relationship. Therefore, they form an equilibrium point called the “market equilibrium”. The curve “A shift in supply” shows that the price will increase if the supply decreases while keeping the demand level constant.

In a multi-player market, players affect the supply by changing their own production. Theoretically, each player increasing the production will push the price down, leading to decreasing production of their opponents, and vice versa. Therefore, those players form a game-theory problem in the market. Finally, each player obtains an equilibrium production point to optimize their own profit.

Figure 2.1 gives the relationship between the annual WTI oil price/GDP and world total production between 1998-2014. We can observe a clear linear relationship between them. Therefore, a linear model can be appropriately considered as a price model for crude oil, given a steady economic level.

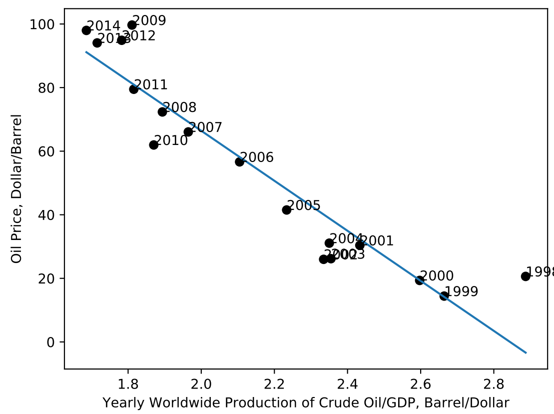


Figure 2.1: Plot of WTI price versus oil production/GDP

Typically price is a decreasing function of demand and, for a commodity that is difficult to store in large quantities like crude oil, production must nearly meet demand. As such Equation 5 of [Ledvina and Sircar (2012)] presents a non-dimensionalized linear price model, widely used by economists and statisticians because it is easy to apply linear regression between price $P(Q_{\text{total}})$ and oil production Q_{total} to fit data. Mathematically, the price model is

$$P(Q_{\text{total}}) = M - \alpha Q_{\text{total}}, \tag{2.1}$$

where Q_{total}, M and α are measured in barrel/day, dollar/barrel and dollar*day/barrel^2 respectively.

To apply the linear model, we assume Q_{total} represents production measured in barrels, and $P(Q_{\text{total}})$ is the price function. The reason for not using production directly is that the demand for oil increases with GDP. Moreover, we also assume the technology of oil production remains unchanged throughout the price model. It helps to work in nondimensional units,

$$\frac{P(Q_{\text{total}})}{M} = 1 - \frac{\alpha Q_{\text{total}}}{M}. \tag{2.2}$$

We define $P_0(Q_{\text{total}}) := \frac{P(Q_{\text{total}})}{M}$, as nondimensionalized price. Since $\frac{\alpha}{M}$ is a constant measured in 1/barrel, we can ignore the constant and control our production through $q := \frac{\alpha Q_{\text{total}}}{M}$, instead of Q_{total}. Therefore, the linear nondimensionalized price model which we will use in all subsequent sections is

$$P_0(q_{\text{total}}) = 1 - q_{\text{total}}. \tag{2.3}$$

Similarly, the cost of production S, measured in dollars per barrel, may be nondimensionalized by $s := \frac{S}{M}$, s as the corresponding nondimensional production cost.

2.3 Deterministic Game of Finite-reserve producer versus Infinite-reserve producers

We assume a differential game with N players in total, including one player with finite reserve and $N-1$ players with infinite reserves. We follow the setting of [Tsur and Zemel (2003)], [Lafforgue (2008)] and [Ledvina and Sircar (2012)], in which the finite-reserve producer has initial reserve $x(0) = x$, which depletes until exhausted thus,

$$\mathrm{d}x(t) = -q_0(t)\mathbb{1}_{\{x(t)>0\}}\,\mathrm{d}t. \tag{2.4}$$

Here $q_0(t) \geq 0$ is the rate at which the finite-reserve producer extracts the resource. It is both the rate of depletion and the rate of production of the finite-reserve producer.

We let $q_i(t)$ be the production rates of the infinite-reserve players, where $i = 1, 2, \ldots, N-1$. We need not track the reserve for infinite-reserve producers.

The solution to the differential game is based on a Nash equilibrium, in which players aim to select their optimal strategy, given that their opponents have selected their own optimal strategies. A mathematically precise definition is provided in Definition 2.3.1.

Definition 2.3.1. Assume there are N players. Player i has a control $q_i \in \mathcal{Q}_i$ and a profit function $J_i(q_i, \boldsymbol{Q}_{-i})$, where $\boldsymbol{Q}_{-i} = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_N)$ are the controls of the opponents. The Nash equilibrium is the set of controls $\boldsymbol{q} = (q_1^*, \ldots, q_n^*)$ such that

$$J_i(q_i^*, \boldsymbol{Q}_{-i}^*) \geq J_i(q_i, \boldsymbol{Q}_{-i}^*)$$

for $i = 1, \ldots, N$ and $\forall q_i \in \mathcal{Q}_i$.

The Nash equilibrium of Definition 2.3.1 means that each player will optimize their own profit function, under the assumption that the opponents have obtained their own optimal strategies. Hence, each cannot profit more by unilaterally changing their own strategy. However, the Nash equilibrium may not result in the largest-profit strategy for any given player, or even the total profit for all players, as illustrated in Section 2.4.2.

From the perspective of control q_i, Definition 2.3.2 provides different types of controls for player i.

Definition 2.3.2. A closed-loop strategy is a decision rule $q_i(x_0, x(t), t)$, which is continuous in t and uniformly Lipschitz in x for each t. This strategy depends both current state $x(t)$ and time t.

By Definition 2.3.1, the goal of the finite-reserve producer (player 0) is to find $q_0(t)$ by maximizing the total profit given $q_1^*(t), \dots, q_n^*(t)$. This is presented in Equation (2.5):

$$v(x) = \sup_{q_0(t) \geq 0} \int_0^\tau \mathrm{e}^{-rt} q_0(t) \left(1 - q_0(t) - \sum_{i=1}^{N-1} q_i^*(t) - s_0\right) \mathrm{d}t, \tag{2.5}$$

This fits in the general objective function in Equation (1.2) and price function in Equation (2.2). Similarly the infinite-reserve producer (player $n \geq 1$) wants to find $q_n(t)$ by Equation (2.6)

$$v_n(x) = \sup_{q_n(t) \geq 0} \int_0^\infty \mathrm{e}^{-rt} q_n(t) \left(1 - q_n(t) - \sum_{i=0, i \neq n}^{N-1} q_i^*(t) - s_n\right) \mathrm{d}t, \tag{2.6}$$

where $s_0, \dots, s_{N-1}$ are the cost of production, r is the interest rate, and the stopping time of the finite-reserve player is $\tau := \inf\{t \colon x(t) = 0\}$. We maintain the condition $q_n(t) \geq 0$ because oil cannot be put back into the ground. The production strategy q_i depends on the level of x, which indicates that the strategies of players should be closed-loop by Definition 2.3.2. With the production, the nondimensionalized price given value functions Equation (2.5) and Equation (2.6) is

$$P_0(q_0 + Q) = 1 - q_0 - Q \tag{2.7}$$

where $q_0, Q = \sum_{i=1}^{N-1} q_i$ denote the production of finite-reserve player, total production of infinite-reserve players, respectively. In the next sections, we will use the nondimensionalized price model in our value functions. Since this is a theoretical model, we do not consider a exact unit for time t, which is often measured in years. In the following examples, we do not specify the unit of t.

2.3.1 Monopolistic Case with One Finite-reserve Producer

In this section, we consider the case with only one finite-reserve producer ($N = 1$). In this case, our target is to find the control $q_0(t)$. This is not really a game any longer, but it is important to solve this case, of which the solution is helpful to analyze the multiple-player situation.

Solution to the HJB Equation in the Monopolistic Situation

The integrand except e^{-rt}, which can be handled with a change of variable, in time-unit profit Equation (2.5), and the dynamics Equation (2.4) of with production rate do not explicitly involve time t and hence the problem is time-invariant. The time-invariant version Theorem 1.7.3 can be used to derive the HJB equation. The HJB equation for $v(x)$ is

$$\sup_{q_0 \geq 0} q_0(1 - q_0 - s_0 - v') = rv \tag{2.8}$$

Taking the supremum in Equation (2.8) leads to $q^*(x) = \frac{1-s_0-v'(x)}{2} \geq 0$. Therefore, we must require $v'(x) \leq 1 - s_0$ for $x \geq 0$ to ensure the positivity of production rate. Inserting this q_0^* gives

$$\frac{1}{4}(1 - s_0 - v')^2 - rv(x) = 0, \tag{2.9}$$

In fact, $v'(x)$ can be recognized as a "shadow cost" which is described as the cost of producing at present and hence not in the future. The inequality shows that the shadow cost cannot be higher than $1 - s_0$, the unit profit at zero production.

To understand the novel result which follows and to build notation, it is important to quickly revisit work due to [Ledvina and Sircar (2012)].

Lemma 2.3.1. *Consider the ODE*

$$\begin{cases} (a - v')^2 = bv \\ v(x_0) = v_0, \end{cases} \tag{2.10}$$

where $v(0) \geq 0$ and $a, b > 0$. The solution to the ODE is

$$v(x) = \frac{a^2}{b}(1 + W(\theta(x - x_0)))^2 \tag{2.11}$$

where $W(\,\cdot\,)$ is the Lambert-W function, $\theta(x) = \beta \mathrm{e}^{\beta - \frac{bx}{2a}}$ and $\beta = -1 + \frac{\sqrt{bv_0}}{a}$. The Lambert-W function satisfies $z = W(z)\mathrm{e}^{W(z)}$ given $z \geq -e^{-1}$.

Proof. See Appendix A.1. □

We have the boundary condition $v(0) = 0$ because, if no resource remains, the producer cannot have any profit. So by Lemma 2.3.1, taking $a = 1 - s_0$ and $b = 4r$, the solution to Equation (2.8) is

$$v(x; r, s_0) = \frac{(1 - s_0)^2}{r} \left(\frac{1 + W\left(-\mathrm{e}^{-1-\frac{2rx}{1-s_0}}\right)}{2} \right)^2 \tag{2.12}$$

Figure 2.2 shows the plot of the profit function $v(x; r, s_0)$. As initial reserve x increases, the present value $v(x; r, s_0)$ decreases because we can obtain more profit from more crude oil resources. But the marginal profit (i.e., the increase of profit per unit of x) decreases as x increases. In the meantime, as the cost s_0 and interest rate r increases, the profit decreases. We will prove that the conclusion is financially true in the next subsection.

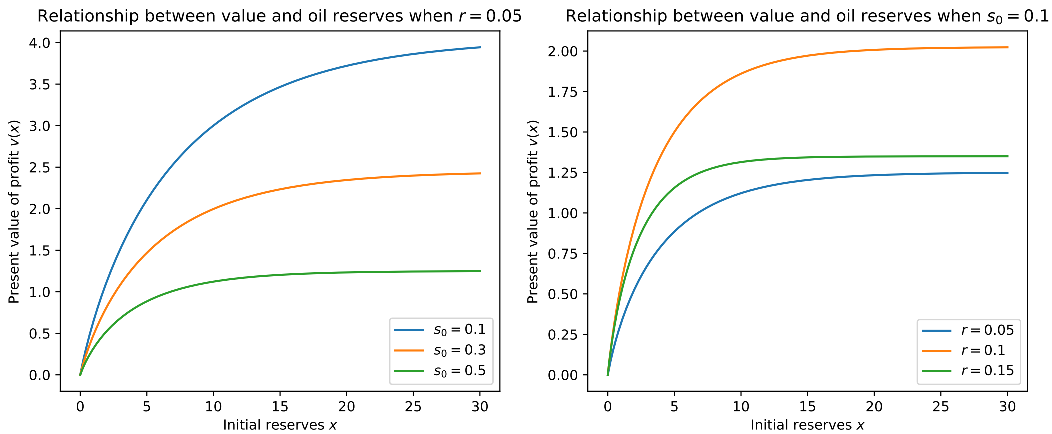


Figure 2.2: Monopolistic profit function. Left: Different s_0 given $r = 0.05$; Right: Different r given $s_0 = 0.1$.

Financial Explanation of the Solution

Since the Lambert-W function $W(\,\cdot\,)$ is an increasing function, when x increases, $v(x)$ will increase as well. By Lemma 2.3.1, the derivative of $v(x; r, s_0)$ w.r.t. x is

$$\frac{\partial v}{\partial x}(x; r, s_0) = -(1 - s_0) W\left(-\mathrm{e}^{-1-\frac{2rx}{1-s_0}}\right). \tag{2.13}$$

From the property of the Lambert-W function, $W(z) < 0$ given $-\mathrm{e}^{-1} < z < 0$. So $\frac{\partial v}{\partial x}(x; r, s_0) > 0$ for all $x > 0$. In financial words, as the finite-reserve producer possess more initial reserves, the profit will increase. This appears reasonable, as for instance, the producer with larger reserve can use the same strategy as the producer with smaller reserve. At the time when the smaller one has exhausted the entire resource, the larger one can still exploit more and profit from the resources.

Moreover, as x increases, $\frac{\partial v}{\partial x}(x; r, s_0)$ will decrease. So the rate of increasing profit is decreasing as the initial reserve increases. This phenomenon can be seen by noting that more initial reserve leads to both more future production traded at a discount rate, and more present production with reduced marginal profit. So the finite-reserve producer will have more discounted profit, given that production rate is finite. When there is infinite reserve, i.e., $x \to \infty$, $v(x; r, s_0) \to \frac{(1-s_0)^2}{4r}$.

For interest rate r, the first derivative is

$$\frac{\partial v}{\partial r}(x;r,s_0) = -\frac{(1-s_0)^2}{r^2}\left(\frac{1+W\left(-\mathrm{e}^{-1-\frac{2rx}{1-s_0}}\right)}{2}\right)^2 - \frac{1-s_0}{r}xW\left(-\mathrm{e}^{-1-\frac{2rx}{1-s_0}}\right) \tag{2.14}$$

As $r \to 0$, the profit will be

$$v(x;0,s_0) = (1-s_0)x. \tag{2.15}$$

The function in Equation (2.15) is just the product of price and reserves. This represents that any strategies are optimal whenever oil is extracted.

Intuitively, as r increases, the lifetime profit will decrease so there should be $\frac{\partial v}{\partial r}(x;r,s_0) < 0$. This is easily proven. From Equation (2.13), taking derivative w.r.t. r, we will compute

$$\frac{\partial^2 v}{\partial x\,\partial r}(x;r,s_0) = \frac{-2xW\left(-\mathrm{e}^{-1-\frac{2rx}{1-s_0}}\right)}{1+W\left(-\mathrm{e}^{-1-\frac{2rx}{1-s_0}}\right)} > 0. \tag{2.16}$$

So $\frac{\partial v}{\partial r}(x;r,s_0)$ is an increasing function w.r.t. x. Let $x \to \infty$, by $W'(0) = 1$, it can be proved that

$$\frac{\partial v}{\partial r}(x;r,s_0) < \frac{\partial v}{\partial r}(\infty;r,s_0) = -\frac{(1-s_0)^2}{4r^2} < 0. \tag{2.17}$$

for $\forall x, r \in \mathbb{R}$ and $s_0 \in [0,1]$.

Now consider the influence of s_0. Similarly, by taking derivative w.r.t. s_0,

$$\begin{aligned}\frac{\partial v}{\partial s_0}(x;r,s_0) &= -\frac{2(1-s_0)}{r}\left(\frac{1+W\left(-\mathrm{e}^{-1-\frac{2rx}{1-s_0}}\right)}{2}\right)^2 - xW\left(-\mathrm{e}^{-1-\frac{2rx}{1-s_0}}\right) \\ &= -\frac{r}{1-s_0}\frac{\partial v}{\partial r}(x;r,s_0) - \frac{1-s_0}{r}\left(\frac{1+W\left(-\mathrm{e}^{-1-\frac{2rx}{1-s_0}}\right)}{2}\right)^2\end{aligned} \tag{2.18}$$

As $s_0 = 1$, i.e., the cost is exactly equal to profit, then $v(x;r,1) = 0$. So we will make no profit in this case. From Equation (2.18), obviously, $\frac{\partial v}{\partial s_0}(x;r,s_0) < 0$ by given $\frac{\partial v}{\partial r}(x;r,s_0) < 0$. In practice, raising the production cost will decrease the profit.

Optimal Control $q_0(x)$

From the computation of the supremum in Equation (2.8), the optimal production rate for the monopolistic case is

$$q_0^*(x) = \frac{1-s_0-\frac{\partial v}{\partial x}(x;r,s_0)}{2} \tag{2.19}$$

By inserting Equation (2.13), q_0^* can be expressed as

$$q_0^*(x) = \frac{1-s_0}{2}\left(1 + W\left(-\mathrm{e}^{-1-\frac{2rx}{1-s_0}}\right)\right) \tag{2.20}$$

Since $-1 < W(z) < 0$ is a monotonically increasing function given $-\mathrm{e}^{-1} < z < 0$, $q_0^*(x) > 0$ for $x > 0$. As $x \to \infty$ (infinite reserve), the production rate will be a constant $q_0^*(\infty) = \frac{1-s_0}{2}$, which is the maximum point of the quadratic profit $q_0(1-q_0-s_0)$ in Equation (2.5). Moreover, this case is the infinite-resource case. This discussion prove that $1 - s_0 - \frac{\partial v}{\partial x}(x; r, s_0) \geq 0$ for $x \geq 0$

Given initial reserve x_0, we consider optimal control at time t. From the differential equation for $x(t)$ and the optimal extraction rate q_0^*,

$$\begin{cases} \mathrm{d}x = -q_0^*(x)\,\mathrm{d}t = -\frac{1-s_0}{2}\left(1 + W\left(-\mathrm{e}^{-1-\frac{2rx}{1-s_0}}\right)\right)\mathrm{d}t \\ x(0) = x_0. \end{cases} \tag{2.21}$$

By differentiation of Lambert-W function, as shown in Appendix A.2, computing the integral from Equation (2.21) gives

$$\begin{aligned} t = \int_0^t \mathrm{d}s &= \int_{x_0}^{x(t)} \frac{-\frac{2}{1-s_0}\,\mathrm{d}y}{1 + W\left(-\mathrm{e}^{-1-\frac{2ry}{1-s_0}}\right)} = \frac{1}{r}\int_{\frac{2rx_0}{1-s_0}}^{\frac{2rx(t)}{1-s_0}} \frac{\mathrm{e}^{-1-y}W'(-\mathrm{e}^{-1-y})\,\mathrm{d}y}{W(-\mathrm{e}^{-1-y})} \\ &= \frac{1}{r}\int_{\frac{2rx_0}{1-s_0}}^{\frac{2rx(t)}{1-s_0}} \mathrm{d}\Big(\ln\Big(-W\left(-\mathrm{e}^{-1-y}\right)\Big)\Big) = \frac{1}{r}\ln\left(\frac{W\left(-\mathrm{e}^{-1-\frac{2rx(t)}{1-s_0}}\right)}{W\left(-\mathrm{e}^{-1-\frac{2rx_0}{1-s_0}}\right)}\right). \end{aligned} \tag{2.22}$$

Taking the inverse function of Equation (2.22), we can obtain the remaining reserve at time t

$$x(t; x_0) = -\frac{1-s_0}{2r}\left(1 + rt + \ln\left(-W\left(-\mathrm{e}^{-1-\frac{2rx_0}{1-s_0}}\right)\right) + \mathrm{e}^{rt}W\left(-\mathrm{e}^{-1-\frac{2rx_0}{1-s_0}}\right)\right). \tag{2.23}$$

Inserting this returns the optimal control as a function of time t

$$q_0^*(x(t; x_0)) = \frac{1-s_0}{2}\left(1 + \mathrm{e}^{rt}W\left(-\mathrm{e}^{-1-\frac{2rx_0}{1-s_0}}\right)\right). \tag{2.24}$$

Optimal Stopping Time τ

We already define the stopping time $\tau := \inf\{t : x(t) = 0\}$. Therefore, setting initial reserve $x_0 = x$ and $x(\tau) = 0$ in Equation (2.22) returns the optimal stopping time τ as a function of initial reserve x

$$\tau(x) = -\frac{1}{r}\ln\left(-W\left(-\mathrm{e}^{-1-\frac{2rx}{1-s_0}}\right)\right). \tag{2.25}$$

Figure 2.3 plots for stopping time τ w.r.t. initial reserve x_0 and time t. From the left plot, the curve will be a nearly straight line when x is large enough, because the production rate will converge to a positive constant as $x \to \infty$. The right plot demonstrates the relationship of production rate and time t given different level of $x_0 = 10, 20, 30$, which leads to the stopping time to be around $\tau = 30, 52, 74$.

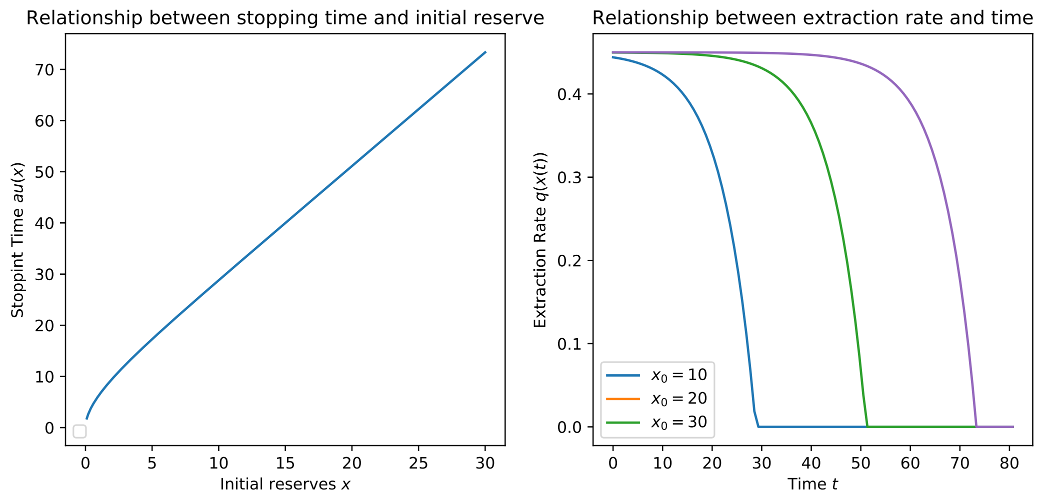


Figure 2.3: Monopoly: Left: Stopping time vs initial reserve; Right: Remaining reserve given different x_0

In fact, we can observe that $\frac{\mathrm{d}}{\mathrm{d}t}v'(x(t)) = rv'(x(t))$ because $v'(x) = -(1-s_0)W(-\mathrm{e}^{-1-\frac{2rx}{1-s_0}})$. This is the so-called Hotelling rule. By the Hotelling rule, the discounted shadow cost at time t, obtained by inserting $x = x(t)$ into $v'(x)$, will lead to

$$\mathrm{e}^{-rt}v'(x(t)) = v'(x_0). \tag{2.26}$$

Therefore, the present value of the shadow cost, in a monopoly, is constant.

2.3.2 Oligopoly with One Finite-reserve Producer versus Multiple Infinite Producers

Now turn to the asymmetric-cost oligopolistic game of [Ledvina and Sircar (2012)] in which there are $N > 1$ players. In that case, assume production from opponents with asymmetric costs $s_1 < s_2 < \cdots < s_{N-1} < 1$. Assuming s_0 as the lowest cost results in the most complicated calculations. The other cases in which s_0 is not the lowest are simpler to handle.

Since τ is the stopping time for the exhaustible energy producer, when $t > \tau$, the finite-reserve producer will exit the game. So we can divide the profit function of the infinite-reserve producers into two parts:

$$\begin{aligned} v_n(x) &= \sup_{q_n(t)\geq 0} \int_0^\infty \mathrm{e}^{-rt} q_n(t) \left(1 - q_n(t) - \sum_{i=0, i\neq n}^{N-1} q_i^*(t) - s_n\right) \mathrm{d}t \\ v_n(x) &= \sup_{q_n(t)\geq 0} \int_0^\tau \mathrm{e}^{-rt} q_n(t) \left(1 - q_n(t) - \sum_{i=0, i\neq n}^{N-1} q_i^*(t) - s_n\right) \mathrm{d}t + \int_\tau^\infty \mathrm{e}^{-rt} G_n \, \mathrm{d}t \\ &= \sup_{q_n(t)\geq 0} \int_0^\tau \mathrm{e}^{-rt} q_n(t) \left(1 - q_n(t) - \sum_{i=0, i\neq n}^{N-1} q_i^*(t) - s_n\right) \mathrm{d}t + \frac{1}{r}\mathrm{e}^{-r\tau} G_n, \end{aligned} \tag{2.27}$$

where G_n is the constant equilibrium profit of player n in a *static game* among infinite resource producers.

2.3.3 Static Game

We first discuss the simple static game with no finite-reserve player and $N-1$ infinite-reserve players. This case will occur after the finite-reserve player exits the game at $t > \tau$. This static game involves players with an invariant price function and a constant optimal strategy over time for each player. This is the *static Cournot game*. We must solve this to determine the residual values of the infinite-reserve players.

The profit of player n is

$$G_n = q_n^* \left(1 - q_n^* - \sum_{i=1; i\neq n}^{N-1} q_i^* - s_n\right) = \sup_{q_n} q_n \left(1 - q_n - \sum_{i=1; i\neq n}^{N-1} q_i^* - s_n\right). \tag{2.28}$$

By maximizing value G_n with q_n, the production rate for player n is

$$q_n^* = \max\left(\frac{1 - \sum_{i=1; i\neq n}^{N-1} q_i^* - s_n}{2}, 0\right), \tag{2.29}$$

where $n = 1, 2, \ldots, N-1$.

We present a proposition illustrating the exact number of players.

Proposition 2.3.1. *Define* $\rho_n := \frac{1+\sum_{i=1}^{n-1} s_i}{n}$ *and* $\bar{\rho} := \min\{\rho_i | i = 2, \ldots, N\}$. *In this* $N-1$-*player game, just* $n-1$ *players are active where* n *is given by*

$$n = \min\{i | \rho_i = \bar{\rho}, i = 2, \ldots, N\}. \tag{2.30}$$

Then players $1, 2, \ldots, n-1$ *are active, and players* $n, n+1, \ldots, N-1$ *are not.*

Proof. See Proposition 2.1 of [Dasarathy and Sircar (2014)] in absence of player 0. □

[Dasarathy and Sircar (2014)] makes the assumption $\rho_N > s_N$, to ensure that all infinite-reserve producers are active in the market. With the assumption, there are $N-1$ producers in the game, with production q^* solving simultaneous linear equations and

$$q_n^* = \frac{1 + \sum_{i=1}^{N-1} s_i - N s_n}{N}, \tag{2.31}$$

where we must assume $\frac{1+\sum_{i=1}^{N-1} s_i}{N} > s_n$, for all n, in order to ensure that all producers will be active.

The time-unit profit $G_n = \left(\frac{1+\sum_{i=1}^{N-1} s_i - N s_n}{N}\right)^2 = (q_n^*)^2$. Moreover, the total static production is

$$Q^* = \sum_{i=1}^{N-1} q_i^* = \frac{N - 1 - \sum_{i=1}^{N-1} s_i}{N}. \tag{2.32}$$

Example 2.3.1 (A counter-intuitive result of static game)**.** We consider an simple example of static game. If a two-player static game takes $s_1 = 0.05$, $s_2 = 0.2$, the Nash equilibrium production strategy in the static game $(q_1^*, q_2^*) = (0.3667, 0.2167)$ with computation given in Appendix A.1. We also manually increase the production of player A to be $q_1 = 0.4$ and obtain the corresponding optimal production for player B, $q_2 = 0.4$.

Performing the similar computation of profits with the setting of $q_1^* = 0.3667, q_2^* = 0.2167$, and $q_1 = 0.4, q_2 = 0.2$, we compute the profits as in Table 2.1:

		B	
	Strategy	$q_2^* = 0.2167$	$q_2 = 0.2$
A	$q_1^* = 0.3667$	$(0.1344, 0.0469)$	(0.1406,0.0467)
	$q_1 = 0.4$	$(0.1333, 0.0397)$	$(0.1400, 0.0400)$

Table 2.1: Simple game

We plot the movement from the unconstrained case to the constrained case in Figure 2.4, which shows that the constraint of $q_1 = 0.4$ will also shift the production of player 2. The orange and green contours are the profits of players 1 and 2 respectively. If the production rate of player 2 is not changed, changing player 1's production rate naturally decreases its profit, according to Definition 2.3.1 of the Nash Equilibrium. But, Figure 2.4 shows that an increase of player 1's production sacrifice player 2's production and profit. Therefore, this production constraint benefits player 1.

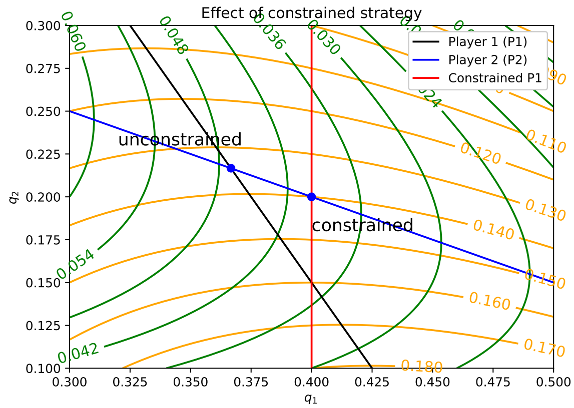


Figure 2.4: Movement from the unconstrained case to the constrained case of static game.

Therefore, although the Nash equilibrium is $(q_1^*, q_2^*) = (0.3667, 0.2167)$, setting $q_1 = 0.4$ returns a larger profit of $(0.1400, 0.0400)$ for player 1 $(0.1400 > 0.1344)$. Hence, a seemingly counter-intuitive result occurs because a constraint on the strategy of player 1 benefits that player's profit. This counter-intuitive conclusion is due to the property of the Nash equilibrium. Unlike the single-player optimization problem, the Nash equilibrium may not give the optimal strategy for any given player but an equilibrium between/among players.

2.3.4 Dynamic Game

To understand the novel result which follows, we quickly revisit and review the work of [Ledvina and Sircar (2012)]. We discuss the case of dynamic games at $0 < t < \tau$. The term "dynamic" denotes that the strategy for each player may change over time. This is because the finite-reserve player producer at each $t < \tau$, balances varying current and residual values. We need to consider the production of the finite-reserve producer. The HJB equations for players $n = 0, 1, \ldots, N-1$ are

$$\begin{aligned} \sup_{q_0 \geq 0} q_0 \left(1 - q_0 - \sum_{i=1}^{N-1} q_i^* - s_0 - v'\right) &= rv \\ \sup_{q_n \geq 0} q_n \left(1 - q_n - \sum_{i=0, i\neq n}^{N-1} q_i^* - s_0\right) &= rv + q_0^* v_n'. \end{aligned} \tag{2.33}$$

However, not all producers will produce energy at all times during the game. The level of reserve $x(t)$ affect the production of the finite-reserve player, and this production of the finite-reserve player will affect the production of these opponents. Therefore, we define the so-called *blockading points* for the infinite-reserve producers n.

Definition 2.3.3. The blockading point of player n is

$$x_b^n = \inf\{x > 0\colon q_n^*(x) = 0\}. \tag{2.34}$$

The blockading time for player n is defined to be

$$\tau_b^n = \sup\{t > 0\colon q_n^*(x(t)) = 0\}. \tag{2.35}$$

We use Figure 2.5 to explain the blockading points and blockading times. As the reserve x decreases past x_b^n, player n starts production and enters the market. Equation (2.4) demonstrates that the reserve is monotonically decreasing with regard to time t. So the number of players increases as $x(t)$ passes the blockading points x_b^n.

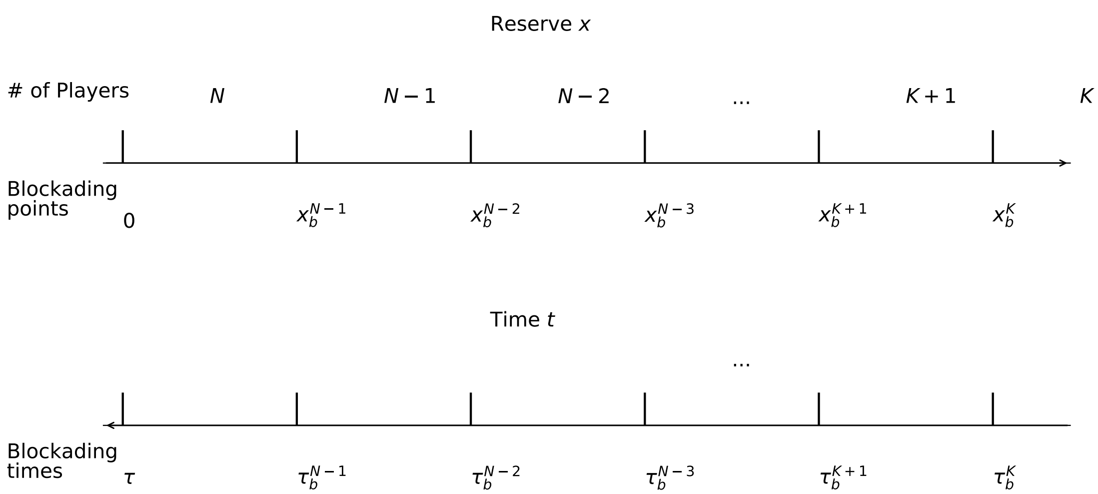


Figure 2.5: Picture explaining blockading points and blockading times

Intuitively, if all opponents have blockading points, it should be that $x_b^1 > x_b^2 > \cdots > x_b^{N-1}$ because $s_1 < s_2 < \cdots < s_{N-1}$ by assumption. In other words, producers with higher costs are more easily excluded. Then, in the interval $[x_b^n, x_b^{n-1})$, there are $n-1$ active infinite-reserve producers. Moreover, the production rate of producers would satisfy

$$\begin{aligned} q_0^* &= \frac{1 - \sum_{i=1}^{n-1} q_i^* - s_0 - v'(x)}{2} \\ q_k^* &= \frac{1 - \sum_{i=0, i\neq k}^{n-1} q_i^* - s_k}{2}, \end{aligned} \tag{2.36}$$

where $k = 1, 2, \ldots, n-1$, while $q_k^* = 0$ for $k = n, n+1, \ldots, N-1$. So, by taking $s_0 + v'(x)$ as the total "cost" for finite-reserve producer and $v'(x)$ as the "shadow cost" of depleting the reserve, we can easily obtain the production rate for all acting producers in Equation (2.37)

$$\begin{aligned} q_0^*(x) &= \frac{1 + \sum_{i=1}^{n-1} s_i - n(s_0 + v'(x))}{n+1} \\ q_k^*(x) &= \frac{1 + s_0 + v'(x) + \sum_{i=1, i\neq k}^{n-1} s_i - n s_k}{n+1}. \end{aligned} \tag{2.37}$$

However, if the costs of infinite-reserve producers are low enough, they will not have a blockading point, as every producer is profitable in all price conditions. Assume that there are $N-K$ players having blockading points $\{x_b^{N-1}, x_b^{N-2}, \ldots, x_b^K\}$. We will provide an exact formula for K later. By plugging in the q_n^*'s, the HJB equation for the finite-reserve producer will be

$$rv = \sum_{n=K}^{N} \frac{1}{(n+1)^2} \left(1 + \sum_{i=1}^{n-1} s_i - n(s_0 + v')\right)^2 \mathbb{1}_{\{x_b^n \leq x < x_b^{n-1}\}} \tag{2.38}$$

where $v(0) = 0$, $x_b^{k-1} := \infty$ and $x_b^N := 0$. Indeed, Equation (2.38) is a piecewise differential equation between two consecutive blockading points $[x_b^n, x_b^{n-1}]$, w.r.t. the reserve of the finite-reserve producers x.

Using Lemma 2.3.1 assuming $v_b^N = 0$ leads to Proposition 2.3.2.

Proposition 2.3.2. *The solution to Equation (2.38) is*

$$v(x) = \sum_{n=K}^{N} \frac{a_n^2}{b_n} (1 + W(\theta_n(x - x_b^n)))^2 \mathbb{1}_{\{x_b^n \leq x < x_b^{n-1}\}} \tag{2.39}$$

where coefficients $a_n = \frac{1+\sum_{i=1}^{n-1} s_i - n s_0}{n}$, $b_n = r\left(\frac{n+1}{n}\right)^2$, $\theta_n(x) = \beta_n e^{\beta_n - \frac{b_n x}{2a_n}}$ *and* $\beta_n = -1 + \frac{\sqrt{b_n v(x_b^n)}}{a_n}$. *The notation* $a_n, b_n, \theta_n(x), \beta_n$ *are used over this whole chapter. Moreover, since continuity of* $v(x)$ *is required at* x_b^{n-1}, *we must ensure that the terminal condition over* $[x_b^n, x_b^{n-1})$ *satisfies*

$$v(x_b^{n-1}) = \frac{a_n^2}{b_n} (1 + W(\theta_n(x_b^{n-1} - x_b^n)))^2 \tag{2.40}$$

for $n = N, N, \ldots, K+1$. *We can give an explicit solution to* $v(x)$ *recursively with initial condition* $v(x_b^N) = 0$.

Moreover, by Lemma 2.3.1, we can compute the piecewise derivative for $v(x)$ *for every interval, which is*

$$v'(x) = -\sum_{n=K}^{N} a_n W(\theta_n(x - x_b^n)) \mathbb{1}_{\{x_b^n \leq x < x_b^{n-1}\}}. \tag{2.41}$$

Proof. Over the interval $[x_b^n, x_b^{n-1})$, Equation (2.38) can be transformed into

$$\frac{r(n+1)^2}{n^2}v = \left(\frac{1+\sum_{i=1}^{n-1} s_i}{n} - s_0 - v'\right)^2. \tag{2.42}$$

Therefore, taking $a = a_n, b = b_n, v_0 = v(x_b^n)$ as parametric setting in Lemma 2.3.1 piecewisely leads to Equation (2.39). □

Blockading Points

Now we present the explicit formula for the blockading point x_b^n for $n = K, K+1, \ldots, N-1$. By the definition of blockading points in Equation (2.34) and the formula of $q_{n-1}^*(x)$ in Equation (2.37), we obtain the formula of blockading point,

$$\begin{aligned} q_{n-1}^*(x_b^{n-1}) &= \frac{1 + s_0 + v'(x_b^{n-1}) + \sum_{i=1}^{n-2} s_i - n s_{n-1}}{n} = 0 \\ v'(x_b^{n-1}) &= n s_{n-1} - \sum_{i=1}^{n-2} s_i - s_0 - 1. \end{aligned} \tag{2.43}$$

Define $\delta_n = (n+1)s_n - (1 + s_0 + \sum_{i=1}^{n-1} s_i)$. By taking $v'(x_b^{n-1})$ from Equation (2.43), the condition of existing a blockading point for n is $v'(x_b^{n-1}) = \delta_{n-1} > 0$, this is because $v(x)$ must be an increasing function. In other words, the profit function should increase with the reserve.

As [Ledvina and Sircar (2012)] indicate, intuitively, $q_k^*(x)$ is continuous w.r.t. x, because q_k^* adjusts itself with the change of reserve x. Therefore, we require the continuity of derivative $v'(x) = \sum_{n=1}^{N} a_n W(\theta_n(x - x_b^n))\mathbb{1}_{\{x_b^n \le x < x_b^{n-1}\}}$ in Equation (2.41). This gives

$$\lim_{x \to x_b^{n-1} - 0} v'(x) = v'(x_b^{n-1}) = -a_n W(\theta_n(x_b^{n-1} - x_b^n)). \tag{2.44}$$

Equating Equation (2.44) and the $v'(x)$ in Equation (2.43) gives the formula of blockading points in Proposition 2.3.3.

Proposition 2.3.3. *Let $K = \min\{n\colon \delta_n > 0\}$ and define $x_b^{K-1} := \infty$ and $x_b^N := 0$. Then the infinite-reserve producers $K, K+1, \ldots, N-1$ have blockading points, given by*

$$x_b^{N-1} = \frac{1}{\mu_N}\left(-1 + \frac{\delta_{N-1}}{a_N} - \log\left(\frac{\delta_{N-1}}{a_N}\right)\right) \tag{2.45}$$

and for $n = N-1, N-2, \ldots, K+1$,

$$x_b^{n-1} = x_b^n + \frac{1}{\mu_n}\left(\log\left(\frac{\delta_n}{\delta_{n-1}}\right) - \frac{(n+1)(s_n - s_{n-1})}{a_n}\right). \tag{2.46}$$

where we define $\mu_n = \frac{b_n}{2a_n} = \frac{r}{2a_n}\left(\frac{n+1}{n}\right)^2$.

Moreover, the value function at the blockading points $v(x_b^n), n = N-1, N-2, \ldots, K+1$ *is*

$$v(x_b^n) = \frac{1}{r}(s_n - s_0 - \delta_n)^2. \tag{2.47}$$

which ensure the continuity of $v'(x)$ *at* x_b^n*. But the second derivative* $v''(x)$ *is not necessarily continuous at* x_b^n*.*

Proof. See Appendix A.3. ☐

This proposition provides exact formulae for the blockading points. At blockading points x_b^n the second derivative $v''(x)$ is not continuous, intuitively because the production $q_k^*(x)$, which contains the item $v'(x)$ decrease with x, and suddenly stops at zero after x passes x_b^k.

Optimal Production of Each Player

From the definition of Lambert-W function, $v'(x)$ is a positive and decreasing function. So the production rate function is

$$q_0^*(x) = \sum_{n=K}^{N} \frac{na_n(1 + W(\theta_n(x - x_b^n)))}{n+1} \mathbb{1}_{\{x_b^n \le x < x_b^{n-1}\}}. \tag{2.48}$$

From the property of $W(\theta(\,\cdot\,)) \ge -1$, we can easily prove that $q_0^*(x) \ge 0$. Therefore, we confirm that the production of the finite-reserve player $q_0^*(x)$ must be positive over $x \in (0, \infty)$, and $q_0^*(x)$ always depends on x and is a closed-loop strategy. The limit

$$\lim_{x \to 0} q_0^*(x) = \frac{Na_N(1 + W(\theta_N(0)))}{N+1} = 0. \tag{2.49}$$

which that absence of reserve of the finite-reserve producer leads to zero production. Moreover, $q_0^*(x)$ is a increasing function. By taking $x \to \infty$, the supremum of production is $q_0^*(\infty) = \frac{1+\sum_{i=1}^{k-1} s_i - ks_0}{k+1}$. In this case, this game becomes a static game with N players with the finite-reverse player becoming infinite-reserve.

In fact, the condition on the cost s_0 to keep the production $q_0^* > 0$ in Equation (2.48) can be relaxed to $a_n > 0$. By assumption on ρ_n where $n = 1, 2, \ldots, N$ in Section 2.3.3, we can easily derive that $a_1 > a_2 > \cdots > a_N > 0$. Therefore, the condition for s_0 is equivalent to

$$\rho_N = \frac{1 + \sum_{i=1}^{N-1} s_i}{N} > s_0. \tag{2.50}$$

Moreover, this condition on s_0 is equivalent to the case in which the finite-reserve producer must be in the game, which becomes a N-player static game as the reserve $x \to \infty$.

Over the interval $[x_b^n, x_b^{n-1})$, we can also derive the closed-loop strategy of production of opponents by inserting this $v'(x)$ into Equation (2.37),

$$q_k^*(x) = \max\left(\frac{1 + s_0 - a_n W(\theta_n(x - x_b^n)) + \sum_{i=1, i\neq k}^{n-1} s_i - n s_k}{n+1}, 0\right). \tag{2.51}$$

where x is in the interval $[x_b^n, x_b^{n-1})$. Over the interval, the relative value of k and n decide whether player k is blockaded. In particular, when $k < n$, this production takes positive part and player k is not blockaded. Otherwise, player k is blockaded and $q_k^*(x) = 0$.

Given the production of each player, we can easily obtain the total optimal production of opponents and all players over the interval $[x_b^n, x_b^{n-1})$,

$$\begin{aligned} Q^*(x) &= \sum_{i=1}^{N-1} q_i^* = \frac{(n-1)(1 + s_0 - a_n W(\theta_n(x - x_b^n)) - 2\sum_{i=1}^{n-1} s_i)}{n+1} \\ q_0^*(x) + Q^*(x) &= \frac{n - s_0 + a_n W(\theta_n(x - x_b^n)) - \sum_{i=1}^{n-1} s_i}{n+1}. \end{aligned} \tag{2.52}$$

where we define $Q^*(x)$ as the total production of all active opponents. Since $W(\theta_n(x - x_b^n))$ increases with x, the production of opponents, $q_k^*(x), Q^*(x)$ is decreasing with x, due to increasing production $q_0^*(x)$. In contrast, the total production $q_0^*(x) + Q^*(x)$ is overall increasing. The following Example 2.3.2 gives the visual effects of the production, profit and blockading points.

Parameter	r	s_0	s_1	s_2	s_3	s_4	c	p	x_0
Values	0.05	0.05	0.3	0.32	0.5	0.52	0.1	0.05	5

Table 2.2: Parametric setting of Ex. 2 – 10

Example 2.3.2. In all examples which follow in this chapter, the set of parameters are listed in Table 2.2. Given the blockading points x_b^n in Equation (2.45) and Equation (2.46), as well as the profit function $v(x)$, the production rate of each producer $q_i^*(x)$ for $i = 0, 1, \ldots, 4$ are given in the upper-left plot. The vertical dotted lines are the two blockading points x_b^4 and x_b^3. In that plot, players $0, 1, 2$ hold their production while players $3, 4$ stop their production at $x > x_b^3 \approx 0.76$ and $x > x_b^4 \approx 0.07$ respectively. In other words, the increase of production $q_0^*(x)$ with x expels players $3, 4$ from the market.

The profit function is the upper-right plot of Figure 2.6, where we observe that $v(x)$ increases with x, including the marginal effect. The total production of opponents Q^* and all players $q_0^* + Q^*$ in Equation (2.39) are also displayed. Q^* is decreasing with x, because increases of x squeezes the market share of opponents. But the total production $q_0^* + Q^*$ increases with x overall. As a result, the lower-right plot addresses the decreasing trend of energy price. Those results naturally hold because larger x allows the finite-reserve player to account for a greater market share, and increases the accumulated production.

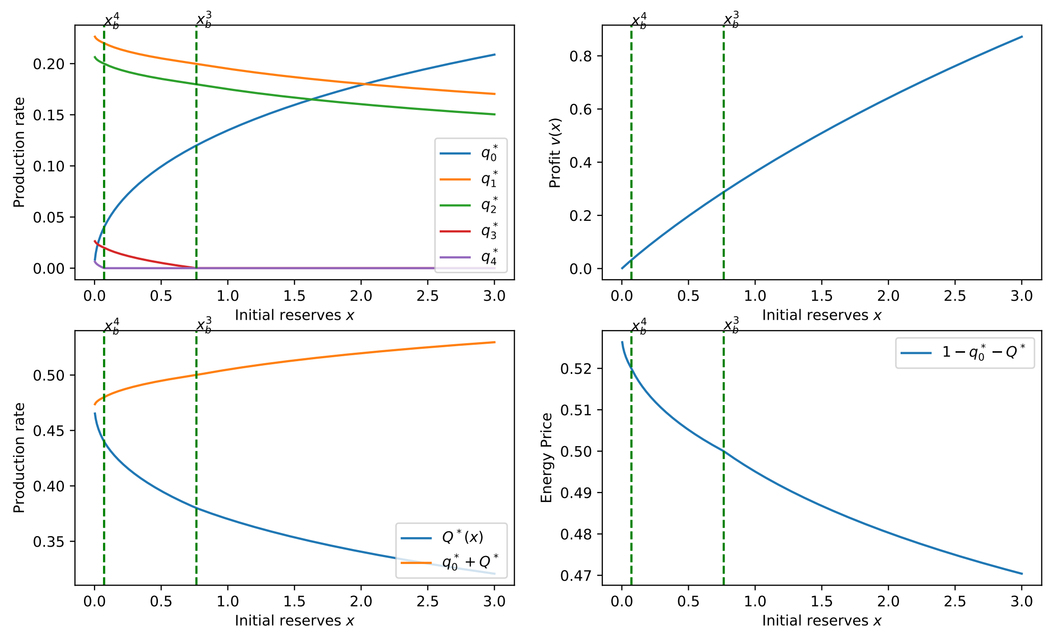


Figure 2.6: Production rate of each player (upper-left), profit (upper-right), total production (lower-left) and price (lower-right) in the game with parameters as in Table 2.2

Blockading Time and Optimal Stopping Time

Given initial reserve $x(0) = x_0$, we compute the remaining reserve $x(t)$ and production rate $q^*(x(t))$ at time t. By Equation (2.48), we obtain the ODE

$$\begin{cases} \mathrm{d}x(t) = -\sum_{n=K}^{N} \frac{n a_n\left(1+W(\theta_n(x(t)-x_b^n))\right)}{n+1}\mathbb{1}_{\{x_b^n \le x < x_b^{n-1}\}}\,\mathrm{d}t \\ x(0) = x_0. \end{cases} \tag{2.53}$$

Solving the ODE leads to the relationship between time t and reserve x as presented in Proposition 2.3.4.

Proposition 2.3.4. *Assume that $x_b^l \le x_0 < x_b^{l-1}$ and $x_b^m \le x < x_b^{m-1}$ where $m \le l$. Then* *the relationship between t and x is*

$$t(x;x_0) = \frac{2(l-l)}{rl}\ln\left(\frac{v'(x_b^{l-1})}{v'(x_0)}\right) + \sum_{n=m+1}^{l-1}\frac{2n}{r(n+1)}\ln\left(\frac{v'(x_b^{n-1})}{v'(x_b^n)}\right) + \frac{2m}{r(m+1)}\ln\left(\frac{v'(x)}{v'(x_b^m)}\right). \tag{2.54}$$

Proof. See Appendix A.4. □

Equation (2.35) defines the blockading time τ_b^l as the time at which player l starts oil production while initial reserve satisfies $x_b^m \le x < x_b^{m-1}$, over which players $m, m+1, \dots, N-1$ do not yet participate in the market. The blockading times follow from Equation (2.54),

$$\tau_b^m(x_0) = \frac{2(l-l)}{rl}\ln\left(\frac{v'(x_b^{l-1})}{v'(x_0)}\right) + \sum_{n=m+1}^{l-1}\frac{2n}{r(n+1)}\ln\left(\frac{v'(x_b^{n-1})}{v'(x_b^n)}\right) \tag{2.55}$$

where the initial number of players $l = m, m+1, \dots, N-1$. Without loss of generality, define $\tau_b^0(x_0) := 0$. If we set $m = N$, we can obtain the stopping time $\tau_N^b(x_0)$.

Moreover, we can compute the inverse function of $t(x)$, the remaining reserve at time t,

$$x(t;x_0) = x_b^m + \frac{2a_m}{b_m}\left[\beta_m + \frac{v'(x_b^{m-1})}{a_m}e^{\frac{r(m+1)}{2m}(t-\tau_b^{m-1}(x_0))} - \frac{r(m+1)}{2m}(t-\tau_b^{m-1}(x_0)) - \ln\left(\frac{-v'(x_b^{m-1})}{a_m\beta_m}\right)\right] \tag{2.56}$$

where $t \in [\tau_b^{m-1}, \tau_b^m)$ and $t - \tau_b^{m-1}$ the time remaining after the last blockading point before $x(t;x_0)$. Therefore, based on this result, the production rate at time t will be

$$\begin{aligned} q_0^*(x(t;x_0)) &= \frac{ma_m - v'(x_b^{m-1})e^{\frac{r(m+1)}{2m}(t-\tau_b^{m-1})}}{m+1} \\ q_k^*(x(t;x_0)) &= \max\left(\frac{1+s_0+v'(x_b^{m-1})e^{\frac{r(m+1)}{2m}(t-\tau_b^{m-1})}/m + \sum_{i=1,i\neq k}^{m-1}s_i - ms_k}{m+1}, 0\right). \end{aligned} \tag{2.57}$$

Example 2.3.3. The upper-left plot of Figure 2.7 gives the stopping time $\tau_N^b(x)$. The green vertical line in the right plot represents the stopping time for $\tau_b^3(5)$ and $\tau_b^4(5)$. As is shown, the stopping time $\tau(x)$ increases with the initial reserve x, containing a marginal effect. When x becomes sufficiently large, the slope becomes almost linear because $q_0^*(\infty)$ is a constant.

The production rate $q_0^*(x(t))$ given initial reserve $x_0 = 5$ is given in the upper-right plot. The stopping time is approximately $t = 34.0$. As the reserve $x(t)$ is depleted, the production rate is decreasing with higher rate. On the other hand, the opponents' total production rate $Q^*(x(t))$ shows an opposite increasing trend. Overall, the total production $q_0^*(x(t)) + Q^*(x(t))$ is decreasing. Given the connection between production and price it is not surprising that the energy price increases with t, as $x(t)$ depletes.

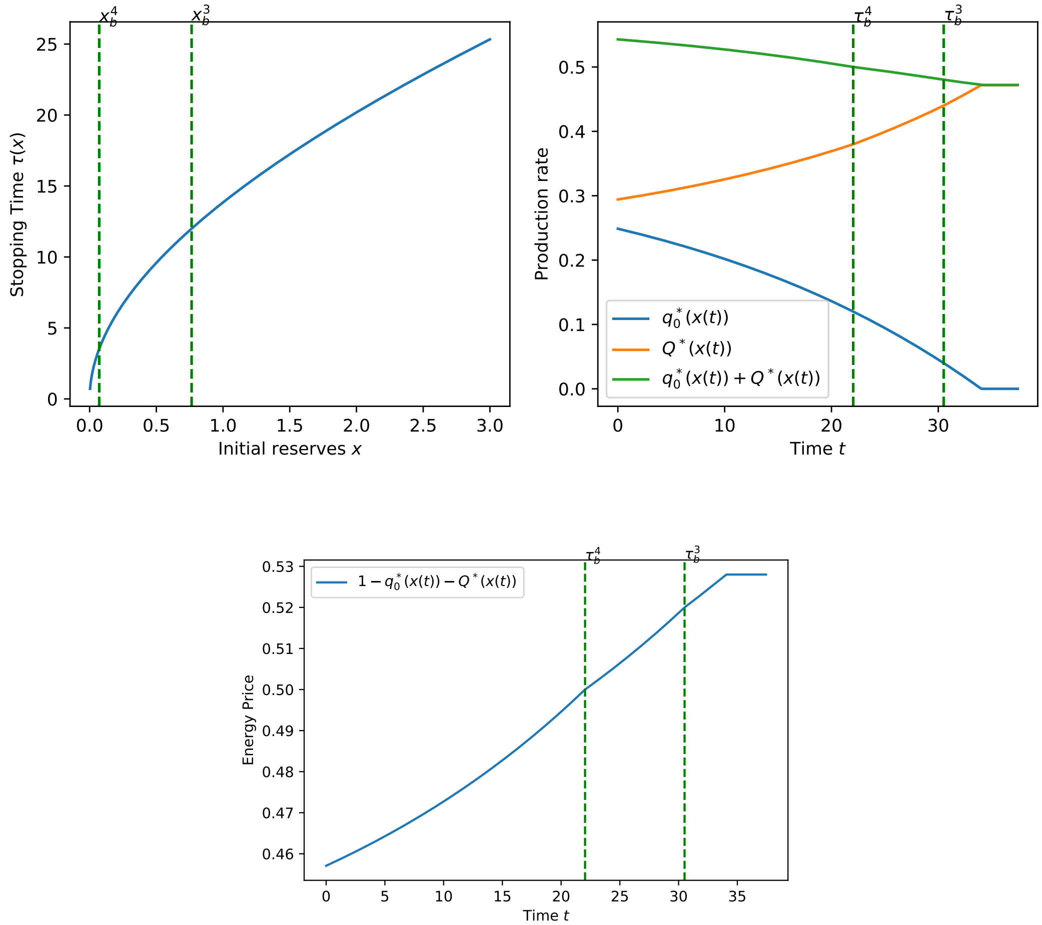


Figure 2.7: Stopping time (upper-left), production rate (upper-right) and energy price (lower) in dynamic game, parameters as in Table 2.2

This is also a game-version Hotelling's rule containing opponents. As Proposition 5.6 of [Ledvina and Sircar (2012)] indicates, taking $\frac{\mathrm{d}}{\mathrm{d}t}v'(x(t))$ leads to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}v'(x(t)) = \left(\frac{1}{2} + \frac{1}{2m}\right)rv'(x(t))$$

where $t \in [\tau_b^{m-1}, \tau_b^m)$. This result shows that the growth rate of profit, $\frac{1}{2} + \frac{1}{2m}$ depends on the number of players. More players gives an effect of low growth rate of profit. In particular in the $m = 1$ (monopoly) case, the rate equals the interest rate.

Using the result with initial reserve x_0, the present shadow cost will be

$$\mathrm{e}^{-rt}v'(x(t)) = v'(x_0)\exp\left(\frac{r}{2}\left(-\frac{(m-1)t + \tau_b^{m-1}}{m} + \sum_{n=l}^{m-1}\frac{\tau_b^n - \tau_b^{n-1}}{n}\right)\right). \qquad (2.58)$$

Profits of Opponents

We finished the discussion on the finite-reserve producer. Now consider the value function of player k, $v_k(x)$. Recall Equation (2.51) for the closed-loop production rate of player k,

$$q_k^*(x) = \begin{cases} \frac{1+s_0-a_n W(\theta_n(x-x_b^n))+\sum_{i=1,i\neq k}^{n-1} s_i - ns_k}{n+1} \text{ when } n \geq k \\ 0 \text{ Otherwise.} \end{cases} \quad (2.59)$$

Inserting this q_k^* into the HJB Equation (2.33) leads to Proposition 2.3.5:

Proposition 2.3.5. *The profit function of player k is a piecewise function over interval $[x_b^n, x_b^{n-1})$*

$$v_k(x) = \begin{cases} A_n(x)v_k(x_b^n) \text{ when } n < k \\ A_n(x)v_k(x_b^n) + \frac{c_{k,n}^2}{r}(1-A_n(x)) - \frac{4a_n c_{k,n} n}{r(n-1)(n+1)}(W(\theta_n(x-x_b^n)) - \beta_n A_n(x)) \\ \qquad -\frac{na_n^2}{r(n+1)^2}(W^2(\theta_n(x-x_b^n)) - \beta_n^2 A_n(x)) \text{ when } n \geq k, \end{cases} \quad (2.60)$$

where $A_n(x) := \left(\frac{W(\theta_n(x-x_b^n))}{\beta_n}\right)^{\frac{2n}{n+1}}$, $c_{k,n} := \frac{1+\sum_{i=0}^{n-1} s_i}{n+1} - s_k$ and the initial condition is $v_k(0) = \frac{1}{r}G_k$, given $n = N, N-1, \ldots, K$.

Proof. See Appendix A.5. □

Example 2.3.4. For each player, the plots of profits given $x \in [0, 3]$ are presented in Figure 2.8. All of the accumulated profits decrease with the initial reserve x of the finite-reserve player. But the decreasing trends of players 1 and 2 with lower costs are almost linear but not exponential and have higher profits (from 1.04 to 0.88 for player 1, and from 0.87 to 0.73 for player 2), because they are active in the game for arbitrary value of x. In comparison, players 3 and 4 with higher costs has an exponentially decreasing trend when they are expelled from the market ($x > 0.76$ for player 3, and $x > 0.07$ for player 4).

2.4 Oligopolistic Game with Constrained Production on the Finite-reserve Producer

In this section, we extend the work of [Ledvina and Sircar (2012)] to include upper and lower production constraints. [Simaan and Takayama (1978)] consider a duopoly model in which each firm has a maximum capacity. They compute some structural properties of the

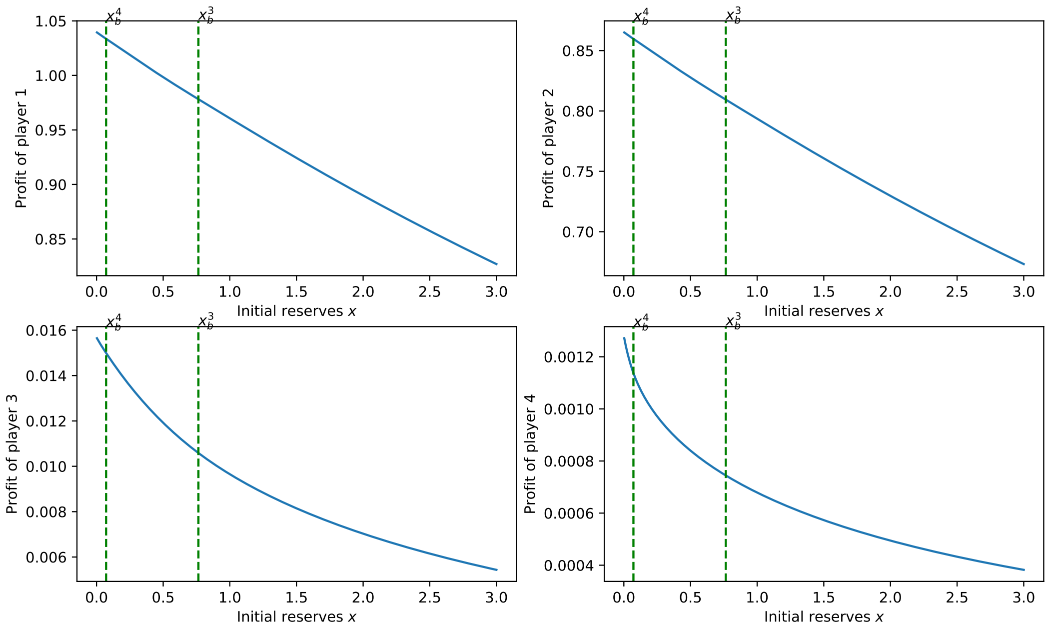


Figure 2.8: Profit function for Opponents, parameters as in Table 2.2

solution without computing it exactly. In this sense similarly, [Benchekrown (2003)], in a duopoly sharing the same renewable resource, find production restrictions result in apparently counter-intuitive behavior.

We introduce similar constant production constraints and find a similar counter-intuitive result in what follows. Additional computations from these results are also included to build intuition of how our production bounds work.

2.4.1 Limited Production for Finite-reserve producer

In this section, we consider the maximal production limit. The finite-reserve producer may encounter a case in which oil production amount is restricted. This case is possible because organizations such as OPEC may make an agreement on production reduction, or extractors have physical limitations. There may also be geological production constraints.

Assume that the maximal production for the finite-reserve producer is a constant c. In this case, the profit function in Equation (2.5) changes to

$$v_{c,0}(x) = \sup_{0 \le q_{c,0}(t) \le c} \int_0^\tau \mathrm{e}^{-rt} q_{c,0}(t) \left(1 - q_{c,0}(t) - \sum_{i=1}^{N-1} q^*_{c,i}(t) - s_0\right) \mathrm{d}t. \tag{2.61}$$

And the HJB Equation (2.33) will become

$$\sup_{0 \le q_{c,0} \le c} q_{c,0} \left(1 - q_{c,0} - \sum_{i=1}^{N-1} q^*_{c,i} - s_0 - v'_{c,0}\right) = r v_{c,0}. \tag{2.62}$$

We discuss Equation (2.62) in detail. In the last part of Section 2.3.4, we have derived the maximal production $q^*_0(\infty)$ of finite-reserve producer. So if $q^*_0(\infty) \le c$, the constraint will not bind. But for $q^*_0(\infty) > c$, the production will be limited at least some of the time. We define the constraint-touching point x_c to be the point at $q^*_0(x_c) = c$. Then the bounded closed-loop strategy of production rate will be

$$q^*_{c,0}(x) = \begin{cases} q^*_0(x) & \text{if } x \le x_c \\ c \quad \text{if } x > x_c \end{cases}. \tag{2.63}$$

Assume that x_c is on the interval $[x_b^{n_c}, x_b^{n_c-1})$, i.e., $q^*_0(x_b^{n_c}) \le c \le q^*_0(x_b^{n_c-1})$. Over this interval, players $n_c, \ldots, N-1$ are blockaded. By Equation (2.48), the expression of x_c is

$$x_c = x_b^{n_c} - \frac{2a_{n_c}}{b_{n_c}} \left[\ln\left(\frac{1}{\beta_{n_c}}\left(-1 + \frac{(n_c+1)c}{n_c a_{n_c}}\right)\right) - 1 + \frac{(n_c+1)c}{n_c a_{n_c}} - \beta_{n_c}\right] \tag{2.64}$$

The production of opponents are already computed in Equation (2.51). Therefore, with the upper bound of player 0, the closed-loop production of player k becomes

$$q^*_{c,k}(x) = \begin{cases} q^*_k(x) \text{ when } x \le x_c \\ \max\left(\frac{1 - c + \sum_{i=1, i\ne k}^{n_c-1} s_i - (n_c-1)s_k}{n_c}, 0\right) \text{ when } x > x_c \end{cases}. \tag{2.65}$$

The lower part (i.e., $x > x_c$) indicates that the number of players are n_c, where players $n_c, \ldots, N-1$ are blockaded out.

Summing production of opponents results in the HJB equation for the finite-reserve player,

$$r v_{c,0} = \begin{cases} \sum_{n=n_c}^{N} \frac{1}{(n+1)^2} \left(1 + \sum_{i=1}^{n-1} s_i - n(s_0 + v'_{c,0})\right)^2 \mathbb{1}_{\{x_b^n \le x < x_b^{n-1}\}} & \text{when } x \le x_c \\ c\left(\frac{1 - c + \sum_{i=1}^{n_c-1} s_i - n_c s_0}{n_c} - v'_{c,0}\right) & \text{when } x > x_c. \end{cases} \tag{2.66}$$

Therefore, if $x > x_c$, the HJB equation for the producer is a first-order linear ODE. To simplify, we define $a_c = \frac{1-c+\sum_{i=1}^{n_c-1} s_i - n_c s_0}{n_c}$. So the solution to Equation (2.66) is

$$v_{c,0}(x) = \begin{cases} v(x) & \text{when } x \leq x_c \\ \mathrm{e}^{-\frac{r(x-x_c)}{c}} v(x_c) + \frac{c}{r} a_c \left(1 - \mathrm{e}^{-\frac{r(x-x_c)}{c}}\right) & \text{when } x > x_c \end{cases} \tag{2.67}$$

Figure 2.9 can explain the effect of the limited-production bound. When $x \leq x_c$, the constraint is inactive and production is identical to unconstrained case. And the blockading points x_b^n for $n = N, N-1, \ldots, n_c$ are effective. But when $x > x_c$, the number of players stays at n_c, and blockading points larger than x_c no longer play a role.

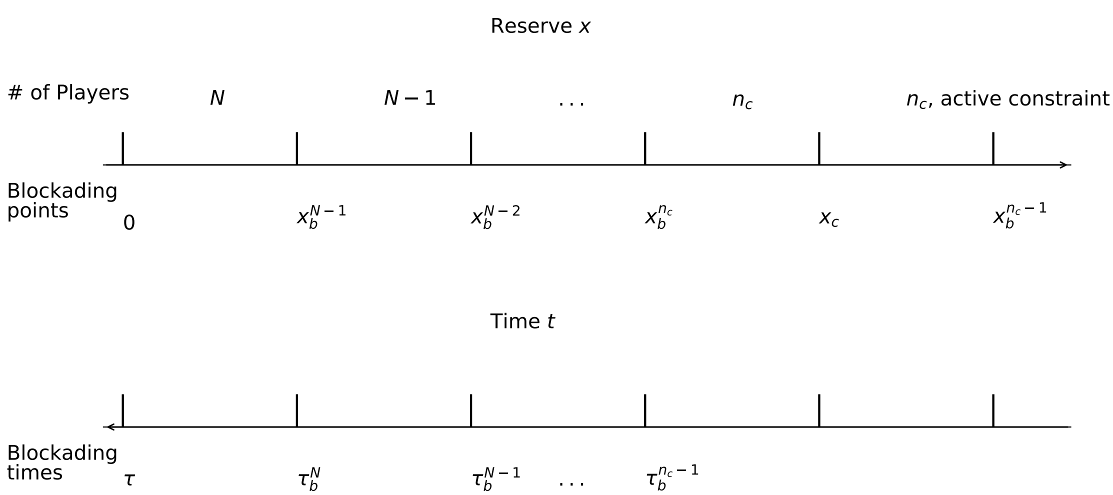


Figure 2.9: Picture explaining blockading points and times in the case of limited production

Example 2.4.1. The upper-left plot of Figure 2.10 presents production rates of the finite-reserve player $q_c^*(x)$ and the opponents $Q_c^*(x)$, where we set the limit $c = 0.15$. Corresponding to the upper limit, the red vertical line is the constraint-touching point $x_c \approx 1.3$. When $x \leq x_c$, the production is identical to the unconstrained case. In contrast, when $x > x_c$, this constraint is active hence the production q_c^* and Q_c^* are constants. Moreover, this maximal-production constraint sets a lower bound of the price at $P_0(q_c^* + Q_c^*) = 0.490$.

The lower-left plot gives the profit function $v_{c,0}(x)$, and the lower-right plot compares profits between constrained and unconstrained cases by taking the difference $v(x) - v_c(x)$. When $x > x_c$, the curve is bent upwards and indicates $v_0(x) > v_{c,0}(x)$. Therefore, the

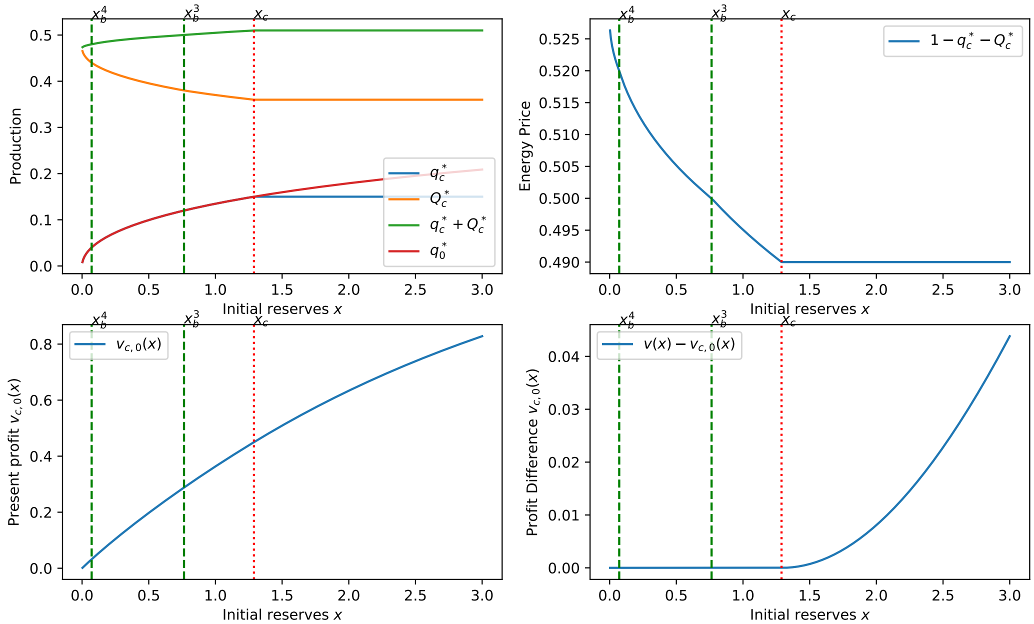


Figure 2.10: Production (upper-left), energy price (upper-right), profit (lower-left) and profit difference (lower-right) for the limited-production constraint, parameters as in Table 2.2

upper constraint has the effect of reducing profits. This result appears trivial because the production bound directly reduces the profit. However, the stopping time is extended due to the maximal-production bound. Therefore, strictly speaking, the effect of the maximal-production bound on lowering profit per time unit is larger than the effect of extending the production time.

Remaining Reserve and Stopping Time

Assume that the initial reserve is x_0. From the production rate in Equation (2.63), we can easily conclude that the remaining reserve $x_c(t)$ is

$$x_c(t;x_0) = \begin{cases} x(t;x_0) & \text{when } x_0 \leq x_c \\ x_0 - ct & \text{when } x_0 > x_c \text{ and } 0 \leq t < \frac{x_0 - x_c}{c} \\ x\left(t - \frac{x_0 - x_c}{c}; x_c\right) & \text{when } x_0 > x_c \text{ and } t \geq \frac{x_0 - x_c}{c} \end{cases} \tag{2.68}$$

where $x(t;x_0)$ is given in Equation (2.56). Moreover, we define the time $\frac{x_0 - x_c}{c}$ to be the constraint-touching time. As t passes this constraint-touching time, the constraint stops binding and production of each player is identical to that from the unconstrained case.

Inserting $x_c(t;x_0)$ into $q_c^*(x)$, we obtain the production rate at time t,

$$q_c^*(x_c(t;x_0)) = \begin{cases} q^*(t;x_0) & \text{when } x_0 \le x_c \\ c & \text{when } x_0 > x_c \text{ and } 0 \le t < \frac{x_0-x_c}{c} \\ q^*\left(t - \frac{x_0-x_c}{c}\right) & \text{when } x_0 > x_c \text{ and } t \ge \frac{x_0-x_c}{c} \end{cases}. \tag{2.69}$$

with the associated stopping time,

$$\tau_c(x_0) = \begin{cases} \tau_N^b(x_0) & \text{when } x_0 \le x_c \\ \frac{x_0-x_c}{c} + \tau_N^b(x_c) & \text{when } x_0 > x_c \end{cases} \tag{2.70}$$

where τ_N^b is given in Section 2.3.4.

Example 2.4.2. The upper-left plot of Figure 2.11 gives stopping time $\tau_N^b(x)$ versus initial reserve x. We set the limited production rate to be $c = 0.15$. The constraint-touching point x_c, as the red vertical line indicates, addresses that production rate when $x > x_c$ is a constant hence the slope is a strictly constant. Moreover, compared to the stopping time in Figure 2.7, the constraint naturally extend the stopping due to lower production when $x > x_c$.

The production rate $q_0^*(x(t))$ given initial reserve $x_0 = 5$ versus t are given in the upper-right plot. The production rate of all players are constant initially, because the constraint is active. After the time touches $\frac{x_0-x_c}{c}$ and the reserve falls below the constraint-touching point x_c, the constraint stops binding. So when $t > \frac{x_0-x_c}{c}$, the production will be complete at the same time as unconstrained case. At the stopping time $t = 40.6$, the finite-reserve player stops. For total production of opponents, Q_c^*, this constraint allows them to have the minimal production rate. But the total production of all players, $q_c^* + Q_c^*$ is constrained. As a result, as the energy price in the lower plot indicates, the constraint sets the lower price bound until $\frac{x_0-x_c}{c}$.

Comparison of Profits of Opponents

In order to compute the profits of opponents of player k, we list the ODEs for opponents depending on the constraint-touching point x_c, in a similar fashion as done in Section 2.4.1.

$$\begin{cases} rv_{c,k}(x) + q_0^*(x)v'_{c,k}(x) = (q_k^*(x))^2 & \text{when } 0 < x \le x_c \\ v_{c,k}(x) + cv'_{c,k}(x) = u_{c,k}^2 & \text{when } x > x_c. \end{cases} \tag{2.71}$$

where $u_{c,k} = \max\left(\frac{1-c+\sum_{i=1}^{n_c-1} s_i}{n_c} - s_k, 0\right)$ is a constant. When $x \le x_c$, the ODE is identical to the unconstrained case in Section 2.3.4. The constraint is active so $q_c^* = c$ when $x > x_c$. To be more specific, Proposition 2.4.1 gives the exact solution.

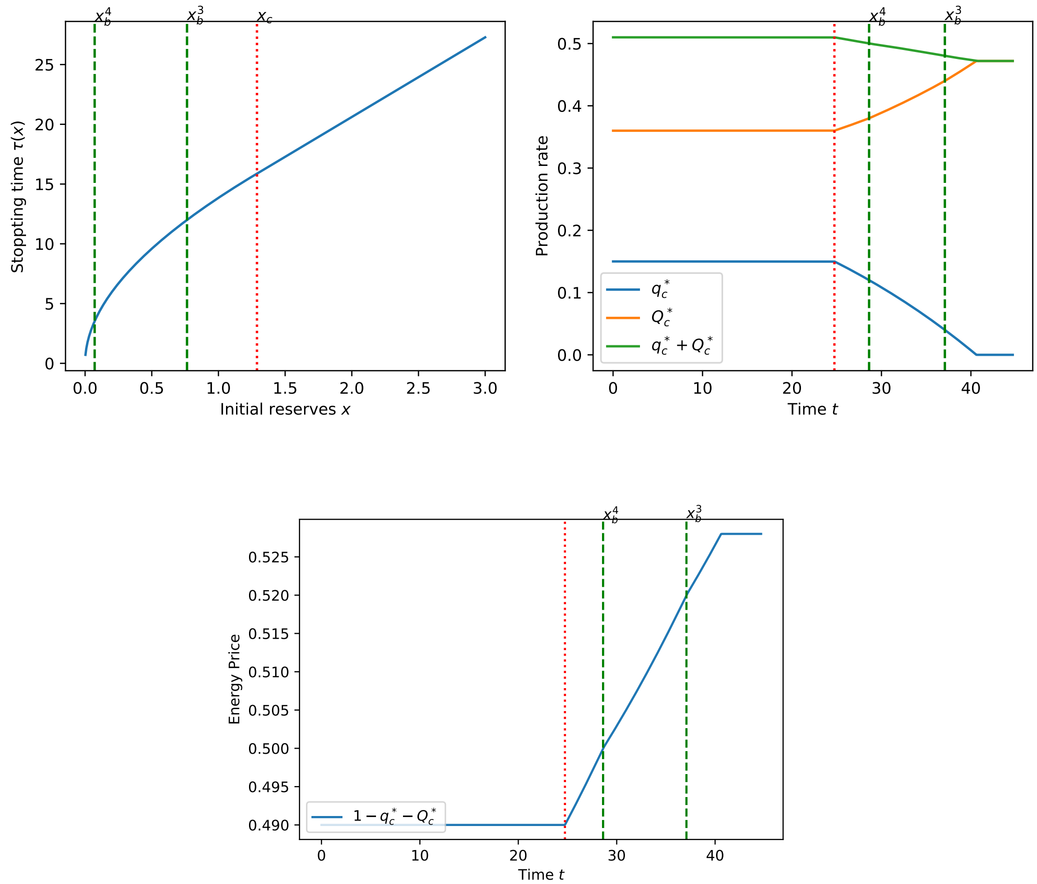


Figure 2.11: Stopping time (upper-left), production rate (upper-right) and energy price (lower) in the limited-production case

Proposition 2.4.1. *The profit function of player k with the limited-production boundary is*

$$
v_k(x) = \begin{cases} A_n(x) v_k(x_b^n) \text{ when } n < k, x \leq x_c \\ A_n(x) v_k(x_b^n) + \frac{c_{k,n}^2}{r}(1 - A_n(x)) - \frac{4 a_n c_{k,n} n}{r(n-1)(n+1)}(W(\theta_n(x - x_b^n)) - \beta_n A_n(x)) \\ \qquad -\frac{n a_n^2}{r(n+1)^2}(W^2(\theta_n(x - x_b^n)) - \beta_n^2 A_n(x)) \text{ when } n \geq k, x \leq x_c \\ \\ \exp\left(\frac{r(x - x_c)}{c}\right) v(x_c) + \frac{u_{c,k}^2}{r}\left(1 - \exp\left(\frac{r(x - x_c)}{c}\right)\right) \text{ when } x > x_c. \end{cases} \tag{2.72}
$$

where the parametric setting follows Proposition 2.3.5.

Proof. Over the interval $x \leq x_c$, the solution is identical to Proposition 2.3.5. When $x > x_c$, the ODE is a constant-coefficient first-order ODE. □

Example 2.4.3. To make a explicit comparison of profits to Figure 2.8, the difference of profits are shown in Figure 2.12. The three plots shows that the upper bound on the finite-reserve producer affects the profits of opponents in different ways. The profits of lower-cost players 1 and 2 decrease. In contrast, players 3 and 4 benefit from player 0's upper production bound. But overall, the lower plot indicates that this production bound does not benefit the whole market, as total profit $v + \sum_{k=1}^{4} v_k > v_c + \sum_{k=1}^{4} v_{c,k}$.

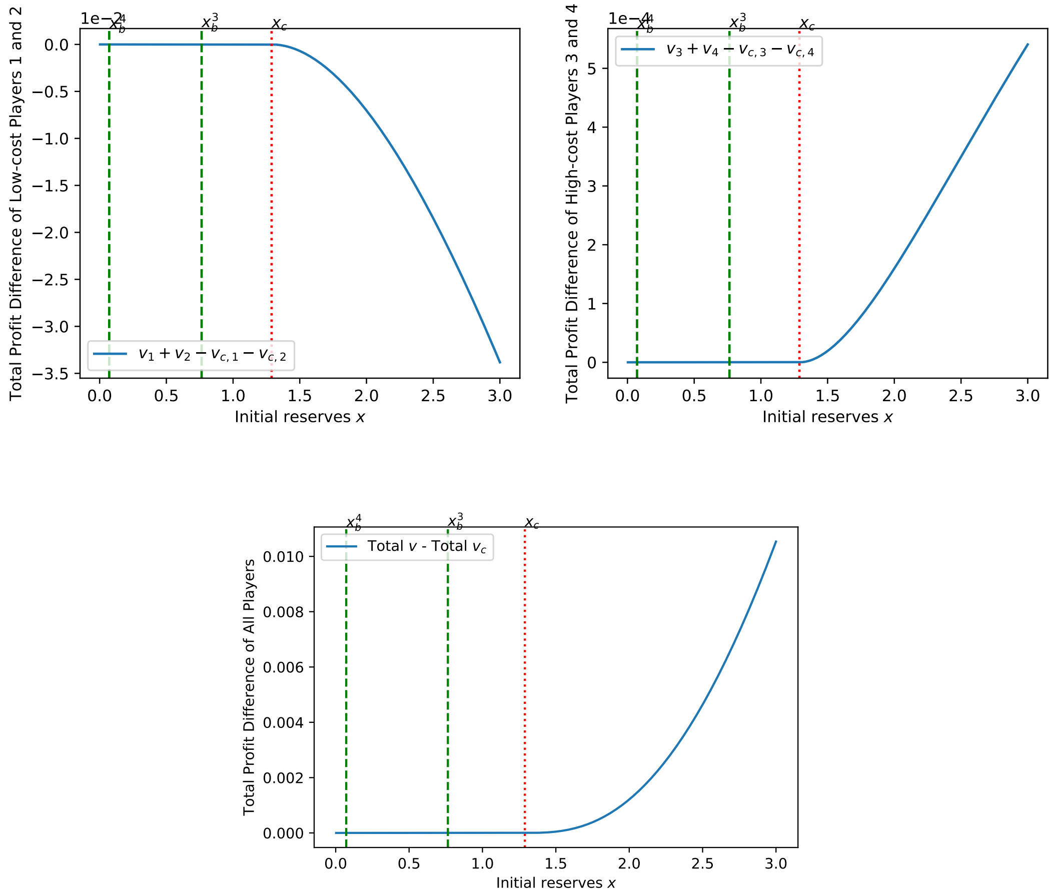


Figure 2.12: Difference of profits of low-cost opponents (upper-left), high-cost opponents (upper-right) and all players (lower) in the limited-production case

In fact, the upper bound on production of the finite-reserve player has two impacts on opponents: prolonged stopping time resulting in longer non-maximal production, and increased production with large reserve leading to more profits per time unit. Players 1 and 2 already have sufficiently large productions owing to their lower cost of production. Hence their negative effect of the prolonged stopping time overwhelms the positive effect of increased production. In contrast, this effect is opposite for higher-cost opponents 3 and 4. The high-cost opponents account for only a tiny share of the market. The limited production of q_c^* allows them to profit more because the positive effect of increased production overwhelms the negative prolonged stopping time.

2.4.2 Minimal Profit for Finite-reserve producer

In this section, we consider an entity which must maintain sufficient production to keep running. For example, an energy company may need to ensure a minimal profit per time unit for fixed consumption. A country whose economy mostly depends on oil exports, may also need to keep a level of oil production rate to maintain its economy.

Assume that p is the required minimal profit per time unit. Then by Equation (2.5), the producer must ensure that the profit satisfies:

$$q_{p,0}\left(1 - q_{p,0} - \sum_{i=1}^{N-1} q^*_{p,i} - s_0\right) \geq p. \tag{2.73}$$

Proposition 2.4.2. *The minimal profit constraint in Equation (2.73) is equivalent to setting a minimal close-loop production strategy as*

$$q^*_{p,0}(x) = \begin{cases} \frac{n_p a_{n_p} - \sqrt{n_p^2 a_{n_p}^2 - 4pn_p}}{2} & \text{when } x \leq x_p \\ q^*_0(x) \quad \text{when } x > x_p \end{cases}, \tag{2.74}$$

where we similarly define x_p to be the constraint-touching point with regard to minimal-production case. The expression of x_p is

$$x_p = x_b^{n_p} - \frac{2a_{n_p}}{b_{n_p}}\left[\ln\left(\frac{1}{\beta_{n_p}}\left(-1 + \frac{(n_p+1)q^*_{p,0}}{n_p a_{n_p}}\right)\right) - 1 + \frac{(n_p+1)q^*_{p,0}}{n_p a_{n_p}} - \beta_{n_p}\right], \tag{2.75}$$

where n_p is the number of total players when the constraint is active. In order to find n_p, the number of total players, we can test the condition $n_p = \#\{k : q^*_k > 0\} + 1$ by trying $n_p = 1, 2, \ldots, N$.

Proof. See Appendix A.6. □

Define the constant production of the finite-reserve player, $c_p = \frac{n_p a_{n_p} - \sqrt{n_p^2 a_{n_p}^2 - 4pn_p}}{2}$ when the constraint is active. The value function of the finite-reserve player becomes

$$v_{p,0}(x) = \sup_{q_0 > c_p} \int_0^\tau \mathrm{e}^{-rt} q_{p,0}(t)\left(1 - q_{p,0}(t) - \sum_{i=1}^{N-1} q^*_{p,i}(t) - s_0\right) \mathrm{d}t. \tag{2.76}$$

Given x_p and n_p in Proposition 2.4.2, we can compute closed-loop production of opponent k,

$$q^*_{p,k}(x) = \begin{cases} \max\left(\frac{1 - c_p + \sum_{i=1}^{n_p-1} s_i}{n_p} - s_k, 0\right) \text{ when } x \leq x_p \\ q^*_k(x) \text{ when } x > x_p \end{cases} \tag{2.77}$$

Therefore, summing the production of opponents from Equation (2.77) and inserting into Equation (2.62) results in the following HJB equation:

$$rv_{p,0} = \begin{cases} q^*_{p,0}\Big(\frac{1-q^*_{p,0}+\sum_{i=1}^{n_p-1} s_i - n_p s_0}{n_p} - v'_{p,0}\Big) & \text{when } 0 \le x \le x_p \\ \sum_{n=n_p}^{N} \frac{1}{(n+1)^2}\Big(1+\sum_{i=1}^{n-1} s_i - n(s_0+v'_{p,0})\Big)^2 \mathbb{1}_{\{x_b^n \le x < x_b^{n-1}\}} & \text{when } x > x_p \end{cases}. \quad (2.78)$$

Therefore, if $x \le x_p$, the HJB equation for the producer is a first-order linear ODE. For simplicity, we define $a_p = \frac{1-c_p+\sum_{i=1}^{n_p-1} s_i - n_p s_0}{n_p}$. So the solution to Equation (2.78) is

$$v_{p,0}(x) = \begin{cases} \frac{c_p}{r} a_p\Big(1-\mathrm{e}^{-\frac{rx}{c_p}}\Big) & \text{when } x \le x_p \\ \frac{a^2_{n_p}}{b_{n_p}}(1+W(\theta_{n_p}(x-x_p)))^2 & \text{when } x_p < x \le x_b^{n_p+1} \\ \sum_{n=n_p+1}^{N} \frac{a_n^2}{b_n}(1+W(\theta_n(x-x_b^n)))^2 \mathbb{1}_{\{x_b^n \le x < x_b^{n-1}\}} & \text{when } x > x_b^{n_p+1} \end{cases} \quad (2.79)$$

where $\theta_{n_p}(x) = \beta_{n_p}\mathrm{e}^{\beta_{n_p}-\frac{b_{n_p}x}{2a_{n_p}}}$ and $\beta_{n_p} = -1 + \frac{\sqrt{b_{n_p} v_{p,0}(x_p)}}{a_{n_p}}$.

Figure 2.13 can explain the effect of the limited-production bound. In contrast to the case of limited-production boundary, when $x > x_p$, the constraint is inactive and production is identical to unconstrained cases. And the blockading points x_b^n for $n = n_p, n_p+1, \ldots, K$ are effective. But when $x \le x_p$, the number of players stays at n_p, and blockading points less than x_p are not effective.

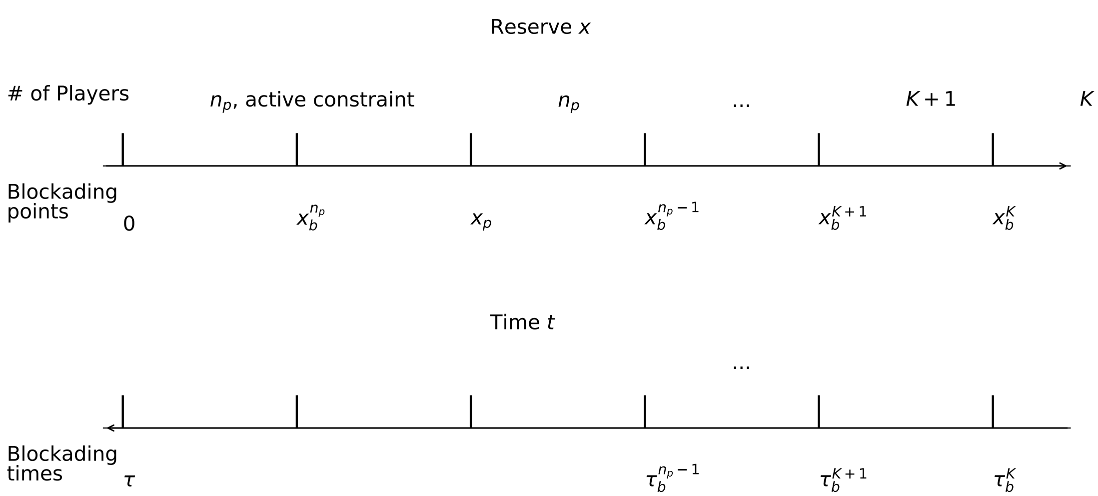


Figure 2.13: Picture showing blockading points and times in the case of minimal-profit constraint

Example 2.4.4. The upper-left plot of Figure 2.14 shows the production rates of the finite-reserve player $q_p^*(x)$ and the opponents $Q_p^*(x)$, where the minimal profit is $p = 0.05$. The red vertical dotted line is the constraint-touching point $x_p \approx 0.62$. When $x < x_p$, the constraint is active and the production rate $q_p^*(x) \approx 0.12$. When $x > x_p$, the constraint does not play a role hence $q_p^*(x)$ and $q_0^*(x)$ become equal. On the contrary, the total production of opponents has a upper bound, because of this constraint. And the total production of all players, $q_p^* + Q_p^*$, has a similar trend as q_p^*. Therefore, as the upper-right plot indicates, the energy price has an upper-bound of approximately 0.503. Moreover, there is only one green line x_b^3 in each plot of Figure 2.14, which demonstrates that the player 4 with the highest cost never participates in the game until the player 0 depletes its own reserves.

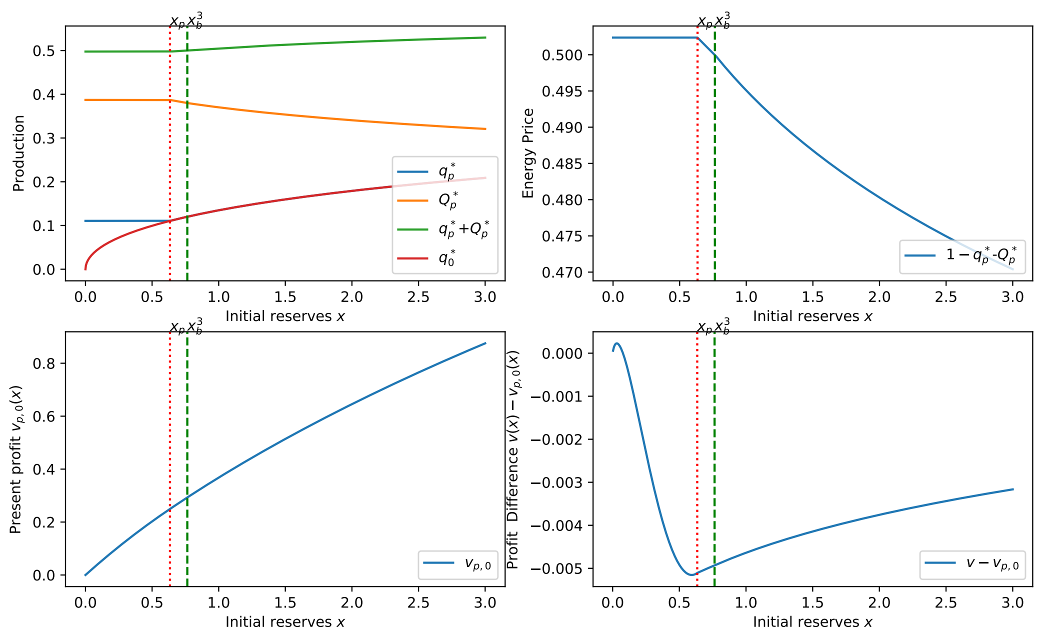


Figure 2.14: Production (upper-left), energy price (upper-right), profit (lower-left) and difference of profit (lower-right) in the minimal-profit case, parameters as in Table 2.2

The lower-left plot of Figure 2.14 gives the profit function $v_{p,0}(x)$. And the lower-right plot compares the constrained and unconstrained cases, represented by the difference $v_0(x) - v_{p,0}(x)$ are presented in the upper-right and lower plots. The difference $v(x) - v_p(x) < 0$ over the whole $[0, \infty)$ excludes the initial region, which means $v(x) < v_{p,0}(x)$. This result indicates that the profit $v_{p,0}(x)$ in the Nash equilibrium with minimal profit constraint is higher than $v(x)$ without the constraint. The result also appears counter-intuitive as discussed in Example 2.3.1.

For a simple optimization problem with a single decision maker operating outside a game setting, unconstrained control cannot perform worse than the constrained control. Otherwise, the decision maker would change the control accordingly. But a game using the Nash equilibrium does not give a strategy with the highest profit. The effect of this lower bound of production ensure the profit per time unit of the finite-reserve player, but makes an earlier stopping time. As a result, the higher profit per time unit overwhelms the harm of shorter earning time.

Remaining Reserve and Stopping Time

We can also give the remaining reserve $x(t)$ based on the result in Section 2.3.4 for this case. For initial reserve x_0, the remaining reserve is

$$x_q(t;x_0) = \begin{cases} x(t;x_0) \quad \text{when } 0 \le t < t(x_p) \text{ and } x_0 > x_p \\ x(t(x_p);x_0) - c_p(t - t(x_p)) \quad \text{when } t(x_p) \le t < t(x_p) + \frac{x_p}{c_p} \text{ and } x_0 > x_p \\ x_0 - q^*_{p,0}(t - t(x_p)) \quad \text{when } 0 \le x_0 \le x_p \end{cases} . \tag{2.80}$$

We define the time $t(x_p)$ to be the constraint-touching time of the minimal-profit bound. When $t < t(x_p)$, the bound is not active and $q^*_p(x(t)) = q^*_0(x(t))$. When $t \ge t(x_p)$, the production bound binds and $q^*_p(x(t))$ reaches the constraint.

Then, we can easily compute the rate of production at time t.

$$q^*_{p,0}(x_q(t;x_0)) = \begin{cases} q(x(t;x_0)) \quad \text{when } 0 \le t < t(x_p) \text{ and } x_0 > x_p \\ c_p \quad \text{Otherwise} \end{cases} . \tag{2.81}$$

Moreover, the stopping time $\tau_p(x_0)$ will be

$$\tau_p(x_0) = \begin{cases} \frac{x_0}{c_p} \quad \text{when } 0 \le x_0 < x_p \\ t(x_p;x_0) + \frac{x_p}{q^*_{p,0}} \quad \text{when } x_0 \ge x_p \end{cases} . \tag{2.82}$$

Example 2.4.5. The upper-left plot of Figure 2.15 gives stopping time $\tau_p(x)$. When $x < x_p$, the constraint of minimal profit is active, hence the stopping time increases linearly with a constant production rate $q^*_{p,0}$. And when $x \ge x_p$, the production rates are the same for constrained and unconstrained cases.

The upper-right plot shows the production rate $q^*_0(x(t))$ given initial reserve $x_0 = 5$. At first, the constraint makes no difference. The red vertical line $t(x_p;x_0) = 23$ is the constraint-touching time. On the approximate interval $23 < t < 29$, the production rate become a constant 0.11 to achieve a minimal profit of $p = 0.05$. When $t \approx 29$, the reserve is completely depleted hence the finite-reserve producer exits the market and the production rate falls to 0. On the contrary, the production of opponents Q_p^* jumps to 0.49.

As the result from the production, the lower plot presents the energy price as a function of t. The curve over $t < x_p$ is identical to the unconstrained case. But the energy price remain constant when the bound is active, and jump to 0.528 after the finite-reserve player depletes the reserve.

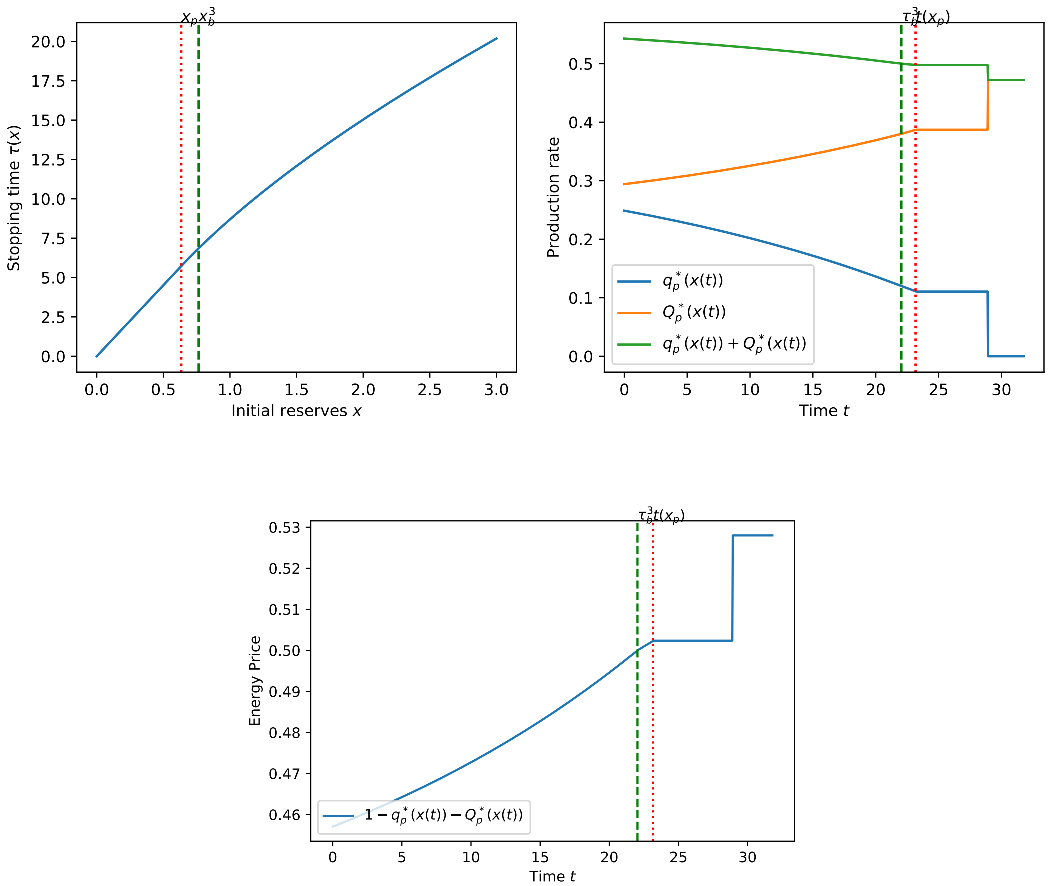


Figure 2.15: Stopping time (left) and production rate (right) in the minimal-profit case, parameters as in Table 2.2

Profits of Opponents

The computation of the profits of opponents in the minimal-profit case is similar to the limited-production case. We need to list the ODEs for opponents depending on the constraint-touching point x_p.

$$\begin{cases} v_{p,k}(x) + c_p v'_{p,k}(x) = l_{p,k}^2 \quad \text{when } 0 < x \leq x_p \\ r v_{p,k}(x) + q_0^*(x) v'_{p,k}(x) = \left(q_k^*(x)\right)^2 \quad \text{when } x > x_p \end{cases} . \tag{2.83}$$

where $l_{p,k} = \max\left(\frac{1-c_p+\sum_{i=1}^{n_p-1} s_i}{n_p} - s_k, 0\right)$ is a constant. When $x \leq x_p$, the constraint is active and $q_p^* = c_p$. When $x > x_p$, the constraint is not active hence the ODE is the same as Section 2.3.4, except for the initial conditions. Proposition 2.3.5 gives the exact solution to the ODE.

Proposition 2.4.3. *The profit function of player k with minimal-profit boundary is*

$$v_k(x) = \begin{cases} \exp\left(-\frac{rx}{c_p}\right) + \frac{l_{p,k}^2}{r}\left(1 - \exp\left(-\frac{rx}{c_p}\right)\right) \text{ when } 0 \leq x \leq x_p \\ \\ A_p(x) v_k(x_p) \text{ when } n_p < k, x_p < x \leq x_{n_p-1} \\ A_p(x) v_k(x_p) + \frac{c_{k,n_p}^2}{r}(1 - A_p(x)) - \frac{4 a_{n_p} c_{k,n_p} n_p}{r(n_p-1)(n_p+1)}\left(W(\theta_n(x - x_b^{n_p})) - W(\theta_{n_p}(x_p - x_b^n)) A_p(x)\right) \\ \qquad -\frac{n_p a_{n_p}^2}{r(n_p+1)^2}\left(W^2(\theta_{n_p}(x - x_b^{n_p})) - W^2(\theta_{n_p}(x_p - x_b^n)) A_{n_p}(x)\right) \text{ when } n_p \geq k, x_p < x \leq x_{n_p-1} \\ \\ A_n(x) v_k(x_b^n) \text{ when } n < k, x > x_{n_p-1} \\ A_n(x) v_k(x_b^n) + \frac{c_{k,n}^2}{r}(1 - A_n(x)) - \frac{4 a_n c_{k,n} n}{r(n-1)(n+1)}\left(W(\theta_n(x - x_b^n)) - \beta_n A_n(x)\right) \\ \qquad -\frac{n a_n^2}{r(n+1)^2}\left(W^2(\theta_n(x - x_b^n)) - \beta_n^2 A_n(x)\right) \text{ when } n \geq k, x > x_{n_p-1}, \end{cases} \tag{2.84}$$

where $A_p(x) = \left(\frac{W(\theta_{n_p}(x - x_b^n))}{W(\theta_{n_p}(x_p - x_b^n))}\right)^{\frac{2n_p}{n_p+1}}$*, and other parametric setting follows Proposition 2.3.5.*

Proof. When $x \leq x_p$, the ODE is a constant-coefficient first-order ODE. When $x > x_p$, the solving process is identical to the unconstrained case except that the initial condition is x_p but not the blockading points over the interval $x_p < x < x_{n_p-1}$. □

Example 2.4.6. To make an explicit comparison of profits to Figure 2.8, the profit difference of low-cost, high-cost opponents and all players are shown in Figure 2.16. The three plots show that the upper bound on the finite-reserve producer affects the profits of opponents in different ways. The profits of lower-cost players 1 and 2 decreases. In contrast, players 3 and 4 benefit from the minimal-profit production bound. But the whole market depends on the initial reserve x. Setting this minimal-profit bound, as lower plot of Figure 2.16 presents, increases (resp. decreases) the total profit when x is high (resp. low).

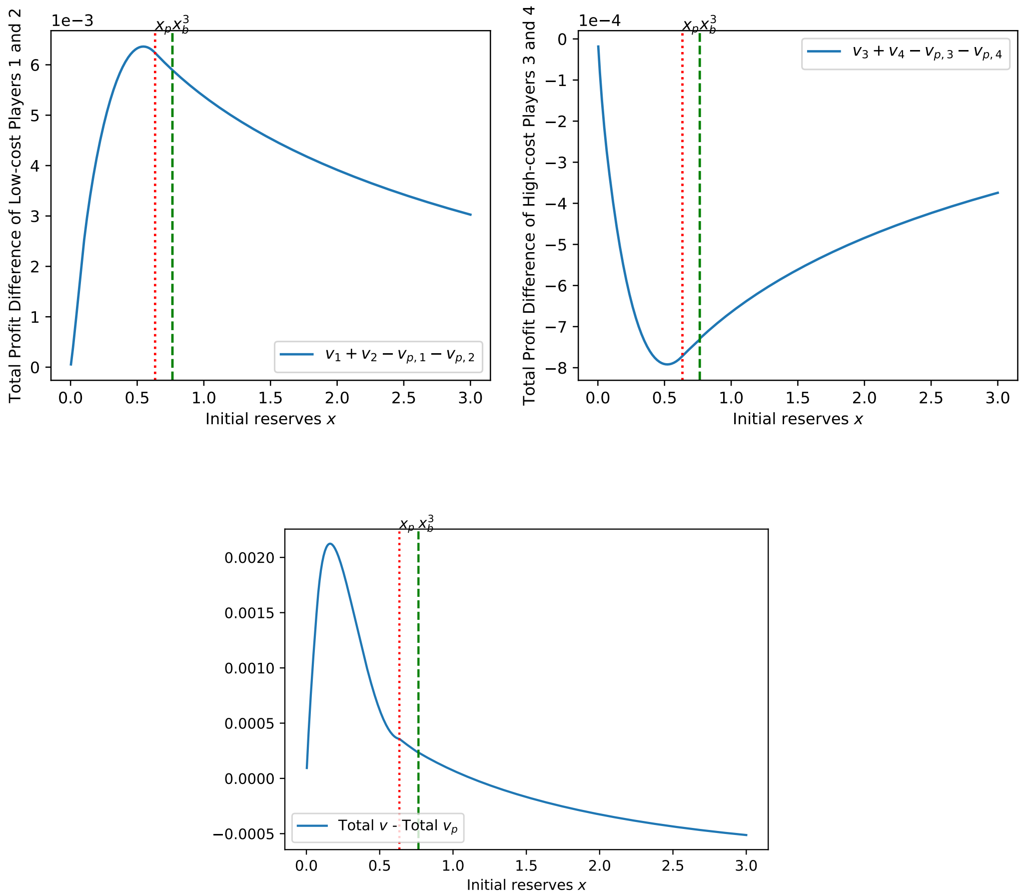


Figure 2.16: Difference of profits of high-cost opponents (upper-left), low-cost opponents (upper-right) and all players (lower) in the minimal-profit case, parameter as in Table 2.2

In contrast to Section 2.4.1, the lower bound on production of the finite-reserve player has two impacts on opponents: reduced stopping time resulting in shorter production period, and decreased production given small reserve leading to less profits per time unit. Players 1 and 2 already have sufficiently large productions owing to lower cost of production. Therefore, the decreased amount of production when the constraint is active has limited impact. In contrast, for players 3 and 4, who already account for only a tiny share of the market, the impact of their decreased production outweighs the earlier stopping time.

Similar to Example 2.3.1, we also obtain a counter-intuitive result the player 0 manually lifting the production and obtaining a higher profit. Via the analysis of the profits of opponents, the manual increase in the production has different direction impact on opponents.

But as a whole, since the lower-cost opponents take advantages over those with higher costs, who possess a much larger market share, the overall effect benefits the finite-reserve player.

2.4.3 Price comparison

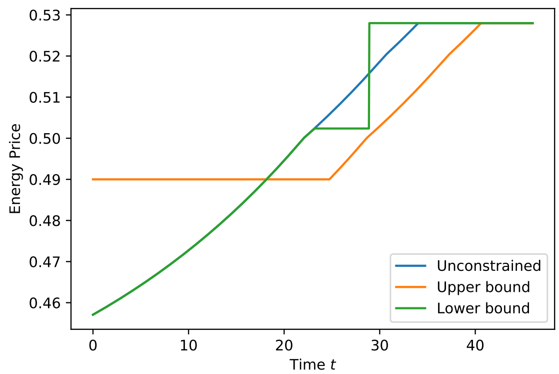


Figure 2.17: Comparison of prices over time in difference cases, parameter as in Table 2.2

The price for the unconstrained, upper-bound and lower-bound cases is available in Figure 2.17. Compared to the unconstrained case (blue curve), the upper bound (orange curve) is initially higher ($t < 18$), then presenting a parallel trend of price movement when the constraint does not bind ($t > 25$).

In contrast, the lower bound (green curve) follows identical price path when this profit constraint does not bind ($t < 23$), and presents a lower price between $23 < t < 29$. All of those price curves eventually lead to the same price of 0.53 as t passes the stopping times of $\tau = 34.0, 40.6, 28.9$, at which x depletes in the unconstrained, upper-bound and lower-bound cases, respectively.

2.5 Conclusion

In this chapter, we construct a type of differential game models between a finite-reserve player and multiple infinite-reserve players. We focus on the relationship between the production rate q_i^* and the profit $v_i(x)$ for player i w.r.t. the oil reserve x of the finite-reserve player.

Theoretically, a higher initial oil reserve x lifts the production rate of the finite-reserve player $q_0^*(x)$ and the profit $v_0(x)$ up with marginal effect because higher reserve enables the

	Stopping Time	Player 0	The High-cost	The Low-cost	All Players
Unconstrained Production	34.0	–	–	–	–
Limited Production	40.6	↓	↓	↑	↓
Minimal Profit	28.9	↑	↑	↓	Depending on x

Table 2.3: Summary of constraints on production to profits of players

finite-reserve player to produce more crude oil. But the finite-reserve player cannot increase production rate q_0^* without limit due to potential loss of profit from the effect of price model $P(Q)$ from a higher total production Q. On the other hand, the opponents have to decrease their production rate q_i^* for $i \geq 1$ because their market shares are squeezed by the finite-reserve producer.

We also manually set the cases of limited production and minimal profit for the finite-reserve producer, represented by the upper and lower production bounds. The upper and lower production bounds have opposite effects on the profits of different types of players, as Table 2.1 indicates. The upper (lower) bound increase (decrease) the profits of low-cost players and decrease (increase) the profit of the high-costs. The reason behind this result is due to the aggregated effect of profits and losses of changed production and stopping time.

Minimal-profit constraints on the finite-reserve producers help high-cost producers when they are blockaded out at high x because they are hurt by decreased market price when they are blockaded out, but are benefited by earlier exit of the finite firm. It hurts low-cost firms when they are not blockaded out. The reader may consider the impact of this game theoretic insight on the understanding of the world oil market.

Chapter 3

Asymmetric-cost Differential-game Model with Stochastic Profit in Energy Market

3.1 Introduction

In this chapter, we constructed a continuous-time, stochastic differential energy market game which includes a finite-reserve player and multiple infinite-reserve players. The study in this chapter is derived using the asymmetric-cost game from [Ledvina and Sircar (2012)] and stochastic profit factor from [Brown et al. (2017)]. Each player aims to maximize the accumulated profit affected by a stochastic factor $Y(t)$ which follows a GBM process. A Nash equilibrium setting provides a HJB representation for the problem. A similarity method transforms this PDE into a very interesting 2nd-order nonlinear ODE with internal moving boundaries and internal singularities, between which the interactions lead to numerical challenges. This system will require future study beyond the analysis in this thesis. We use the method of dominant balance, to obtain an asympotic Puiseux series solution to the ODE applicable near the initial value and extend the solution using a finite difference method. With the resulting approximate solution, we design the Nash equilibrium strategy for each player and perform the Monte-Carlo simulation of these production strategies to provide intuition.

This chapter can be read in two different ways. Those interested in the energy finance application can focus on the problem in Section 3.2 and Section 3.3, which gives the construction of the price model and derivation process of the ODE. Others, who may wish to dig right into the mathematical challenges posed by the singular ODE problem arising, can start in Section 3.4, which provides our mathematical approach to solving this problem. Both classes of reader are united in Section 3.5 where the numerical solution obtained in Section 3.3 is unpacked to give some economic insight. Finally, Section 3.6 concludes with a summary. We made the following contributions in this chapter:

- We construct a type of stochastic differential game model with a stochastic factor $Y(t)$ and obtain an interesting ODE using a similarity method.
- We used the dominant balance method to obtain a solution to the ODE on the interval $[0, \xi_0]$, and extend the solution over $[0, \infty)$ using finite difference method.
- We develop numerical techniques in order to properly treat the singularities.
- We analyze the effect of the stochastic factor $Y(t)$ and the level of reserve $X(t)$ on the profit of the finite-reserve player and production of each player.

3.2 Mathematical Assumption

3.2.1 Stochastic Profit Model

Recall the deterministic linear price model in Section 2.2, proposed by Equation (5) of [Ledvina and Sircar (2012)] and Equation (1) of [Constantinides et al. (1981)],

$$P(Q) = M - \alpha Q_{\text{total}}, \tag{3.1}$$

where M and α are measured in dollar/barrel and dollar*day/barrel^2 respectively, and Q_{total} is the actual total production measured in barrels/day. This linear model is useful to construct the relationship between production and price, given a constant demand. However, this model has its limits. For example, extremely large production leads to an improbable negative price. Therefore, we only consider the positive part of this price model.

With this price model, the economic notation of "*contribution margin*", proposed by Equation (1) of [Kim (1973)], is the profit from selling a single item (barrel), which is the price minus unit cost of production in the absence of fixed cost*

$$\Pi(Q_{\text{total}}) = P(Q_{\text{total}}) - S = M - \alpha Q_{\text{total}} - S \tag{3.2}$$

where $S < M$ is the actual cost of production measured in dollar/barrel.

In order to simplify the price model, we nondimensionalize it by dividing both sides by M and obtaining

$$P_0(q_{\text{total}}) = \frac{P(Q_{\text{total}})}{M} = 1 - q_{\text{total}} \tag{3.3}$$

where $q_{\text{total}} = \frac{\alpha}{M} Q_{\text{total}}$, as Equation (26) of [Ledvina and Sircar (2012)] indicates†. This nondimensionalized price in Equation (3.3) implies that the highest possible price, at zero market production, is 1. This is called the "choke price".

We similarly obtain a simple bilinear *nondimensionalized contribution margin* that depends on the total market production q

$$\pi(q_{\text{total}}; s) = \frac{\Pi(Q_{\text{total}})}{M} = 1 - s - q_{\text{total}}. \tag{3.4}$$

Here $s < 1$ is the nondimensionalized cost of production with $s = \frac{S}{M}$.

Following the setting of Section 3.4 of [Brown et al. (2017)]‡, we introduce a stochastic index, $Y(t)$ into the linear model. This model is fruitful and tractable, as well as plausible, as we shall see. In our setting, we take $Y(t)$ to be a driver of the economy, such as demand for energy, which affects both the price and cost of production. Increased profit, which is price minus cost, also drive up the total production Q_{total}, upon which game results depend. In fact, increased demand for an item generally leads to proportional increase demand for materials and equipment used to make this item, in turn leading to proportional increase of those costs. A related work supporting this conclusion is [Akinyemi et al. (2012)], which states that the global finance environment affects both the price and the cost of production. The figure "Cost by year for 2014 well parameters" available from [EIA (2016)], along with Figure 3.1, depicts a parallel movement of cost of oil drilling per well and WTI oil price over 2006 – 2015.

*The contribution margin is computed as the selling price per unit, minus the variable cost per unit. Also known as dollar contribution per unit, the measure indicates how a particular product contributes to the overall profit of the company. Link: https://www.investopedia.com/terms/c/contributionmargin.asp

†Equation (26) of [Ledvina and Sircar (2012)] also proposed this nondimensionized model, further simplifying it by manually setting $M = \alpha = 1$.

‡Section 3.4 of [Brown et al. (2017)] uses $P(q_{\text{total}}, Y) = Y - q_{\text{total}}$ without considering the cost of production.

Unfortunately the report only provides the plot of cost without explicit data. Therefore we can only compare and describe the plots visually. Before 2008, the cost and WTI price increased and experienced a sharp decrease between 2008 and 2009, when financial crisis happened. The cost peaked in 2012 after continuous increases and fell from 2012 – 2015. In contrast, the WTI price also experienced an increasing trend from 2009, peaked in 2013 and decreased sharply after that. Therefore, this data suggests that the oil cost and price share a parallel trend.

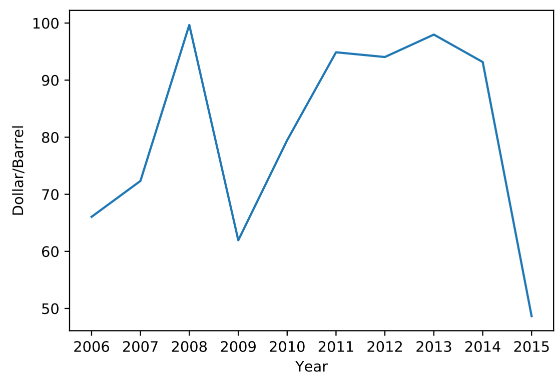


Figure 3.1: WTI oil price over 2006 – 2015.

Therefore, the nondimensionalized contribution margin is

$$\pi(q_{\text{total}}, Y(t); s) = Y(t)(1-s) - q_{\text{total}}, \tag{3.5}$$

where we assume the dynamics of $Y(t)$ is stochastic time-homogeneous (i.e., dynamics of $Y(t)$ is independent of time),

$$\mathrm{d}Y(t) = \mu(Y(t))\,\mathrm{d}t + \sigma(Y(t))\,\mathrm{d}Z(t). \tag{3.6}$$

We assume the drift $\mu(y)$ and volatility $\sigma(y)$ are both time-homogeneous. We also assume $Y(t)$ to be a positive stochastic process since the demand, in practice, should be a positive value.

The model in Equation (3.5) represents a situation in which marginal contribution of each barrel of crude oil is determined by the general economic state $Y(t)$. Increased total production brings the overall price down, so the "sales" follows $Y(t) - q_{\text{total}}$. The cost of production also scales with the general economic state $Y(t)$, for a total nondimensionalized production function.

$$Y(t)(1-s) - q = \underbrace{Y(t)}_{\text{economic index}} - \underbrace{q}_{\text{simplified production}} - \underbrace{Y(t)s}_{\text{nondimensionalized cost}}. \tag{3.7}$$

As we already indicated, the price model is a nondimensionalized model, and $Y(t)$ is an index of economic driver and does not include a unit. Instead, multiplying M on both sides can return Equation (3.5) to original contribution margin,

$$\Pi(Q_{\text{total}}) = \underbrace{MY(t)}_{\text{demand}} - \underbrace{SY(t)}_{\text{cost}} - \alpha Q_{\text{total}}. \tag{3.8}$$

where $MY(t), SY(t)$ are the practical level of demand, and real production cost, measured in dollar/barrel. Therefore, if we use the nondimensionalized model to constitute a objective function, this is equivalent to using the original model because the only difference is the division of coefficient M. The price model is

$$P(Q_{\text{total}}) = MY(t) - \alpha Q_{\text{total}}. \tag{3.9}$$

where we omit the variable cost $SY(t)$. This price model perhaps breaks the mean-reverting property of crude oil stated by [Schwartz (1997)], if $Y(t)$ is not a mean-reverting process. However, [Meade (2010)] proposes that mean-reversion model has ceased to be an suitable model for oil price since 2004.

With the nondimensionalized contribution margin $\pi(q, Y(t); s)$, the actual stochastic profit of a producer with production q is the product of contribution margin and production,

$$q\pi(q_{\text{total}}, Y(t); s) = q(Y(t)(1-s) - q_{\text{total}}). \tag{3.10}$$

In the next sections, we would use this nondimensionalized stochastic profit model as Equation (3.10) for simplicity.

3.2.2 Construction of the Game Model

In this section, we refer to the setting of [Tsur and Zemel (2003)], [Lafforgue (2008)] and [Ledvina and Sircar (2012)] to construct reserve dynamics for energy resources. Assume the finite producer, labelled by player 0, has initial reserve $X(0) = x$, and that this reserve $X(t)$ depletes as follows:

$$\mathrm{d}X(t) = -q_0(t)\mathbb{1}_{\{X(t)>0\}}\,\mathrm{d}t \tag{3.11}$$

where $q_0(t) \geq 0$ is the production rate at which the finite producer extracts the resource.

Assume there are $N-1$ opponents in the game with infinite reserve, each having an infinite-reserve labelled by player i, where $i = 1, 2, \ldots, N-1$. For those opponents, we let $q_i(t), s_i$ be the production rates and cost of production of player i. N is the total number of producers including the finite-reserve player. Infinite producers do not need differential equations to describe their reserve because their resources always remain infinite given finite production rate.

A Nash equilibrium provides the players in the game with a solution, an "optimal" strategy. In Nash equilibrium, players in the game aim to select their optimal strategy, given that their opponents have selected their own optimal strategies. Definition 3.2.1 gives a mathematically precise definition.

Definition 3.2.1. Assume there are N players in total. Player i has a control $q_i \in \mathcal{Q}_i$, where $\mathcal{Q}_i$ is a set of admissible controls, and a profit function $J_i(q_i, \boldsymbol{Q}_{-i})$, where $\boldsymbol{Q}_{-i} = (q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_N)$ are controls of opponents. The Nash equilibrium is the set of controls $\boldsymbol{q} = (q_1^*, \dots, q_n^*)$ such that

$$J_i(q_i^*, \boldsymbol{Q}_{-i}^*) \geq J_i(q_i, \boldsymbol{Q}_{-i}^*)$$

for $i = 1, \dots, N$ and $\forall q_i \in \mathcal{Q}_i$.

The Nash equilibrium in Definition 3.2.1 means that each player will optimize their own profit function, given the opponents have obtained their own optimal strategies. In other words, no player can profit more by changing their own strategy unilaterally.

Assume the discount rate agreed by each player is r, and $Y(t)$ is a positive stochastic process defined on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P}\big)$, where $\{\mathcal{F}_t\}_t$ is the filtration generated by $Y(t)$. Then for player i, the discounted profit at time t is the product of quantity and profit of each unit of energy, as Equation (3.10) indicates,

$$\mathrm{e}^{-rt} q_i(t) \pi\left(\sum_{i=0}^{N-1} q_i(t), Y(t); s_i\right) = \mathrm{e}^{-rt} q_i(t)\left(Y(t)(1-s_i) - \sum_{i=0}^{N-1} q_i(t)\right). \tag{3.12}$$

Hence integrating this leads to player i's accumulated discounted profit.

We denote $\mathbb{E}[\,\cdot\,|X(0) = x, Y(0) = y] = \mathbb{E}_{x,y}[\,\cdot\,]$. By Definition 3.2.1, with the profit at time t, the goal of the finite-reserve producer (player 0) is to find $q_0(t)$ by maximizing the expected accumulated discounted profit given $q_1^*(t), \dots, q_n^*(t)$ in Equation (3.13),

$$v(x,y) = \sup_{q_0(t)\geq 0} \mathbb{E}_{x,y}\left[\int_0^{\tau} \mathrm{e}^{-rt} q_0(t)\left(Y(t)(1-s_0) - q_0(t) - \sum_{i=1}^{N-1} q_i^*(t)\right) \mathrm{d}t\right] \tag{3.13}$$

while similarly the infinite producer (namely, player n) wants to find $q_n(t)$ by Equation (3.14)

$$v_n(x,y) = \sup_{q_n(t)\geq 0} \mathbb{E}_{x,y}\left[\int_0^{\infty} \mathrm{e}^{-rt} q_n(t)\left(Y(t)(1-s_n) - q_n(t) - \sum_{i=0, i\neq n}^{N-1} q_i^*(t)\right) \mathrm{d}t\right] \tag{3.14}$$

where $s_1 < s_2 < \cdots < s_{N-1}$ is assumed to be the unit costs of production without a loss of generality. Moreover, the boundary value is $v(0, y) = v(x, 0) = 0$. Therefore, the domain is $D = \mathbb{R}^+ \times \mathbb{R}^+$ with the boundary $\{(x, 0)\colon x > 0\} \cup \{(0, y)\colon y > 0\}$, using the setting of Theorem 1.7.3.

$X(t), Y(t)$ is time-invariant, and the integrands except for e^{-rt} in the value function in Equation (3.13) and Equation (3.14) only include undetermined controls $q_k^*(t)$ and $Y(t)$, so those integrands do not directly depend on time variable t but state variable $Y(t)$. Therefore, the time-invariant version in Theorem 1.7.3 is applied to generate the corresponding HJB equation in the following sections.

3.3 Derivation of the Differential Game

3.3.1 Infinite-reserve Producers

We first consider the differential game only involving infinite-reserve players. This is equivalent to assuming the initial reserve of the finite-reserve producer is $x = 0$ and $x = \infty$, as at $x = 0$, the finite-reserve player stops production (i.e., $q_0(0, y) = 0, \mathrm{d}X(t) = 0$) and only infinite-reserve players remain in the market. At $x = \infty$, the reserve $X(t)$ of finite-reserve producer always remain infinite, however large the production $q_0(X(t), Y(t))$. Without loss of generality, we discuss the case of $x = 0$. The movement of reserve $X(t)$ no longer plays a role in this market, as explained in the case of $X(t) = X(0) = x = 0$ in Appendix B.2.

As a result, there are only $N - 1$ active players in the game, and the only remaining variable is $Y(t)$. Using Theorem 1.7.3, the HJB equation for player k is

$$rv_k = \sup_{q_k \geq 0} q_k \left(y(1 - s_k) - q_k - \sum_{i=1, i \neq k}^{N-1} q_i^* \right) + \mu(y) \frac{\mathrm{d}v_k}{\mathrm{d}y} + \frac{1}{2}\sigma^2(y) \frac{\mathrm{d}^2 v_k}{\mathrm{d}y^2}. \tag{3.15}$$

The solution in q_k leads to a second order ODE. By taking the supremum, we can obtain the production rate for player k

$$q_k^*(y) = \frac{y(1 + \sum_{i=1, i \neq k}^{N-1} s_i - (N-1)s_k)}{N} \tag{3.16}$$

and the total production rate $Q^*(y)$

$$Q^*(y) = \sum_{i=1}^{N-1} q_i^*(y) = \frac{y(N - 1 - \sum_{i=1}^{N-1} s_i)}{N}. \tag{3.17}$$

where we make the assumption $\frac{1+\sum_{i=1,i\neq k}^{N-1} s_i-(N-1)s_k}{N} > 0$ to ensure that all players are active in the game. Obviously, the production rates only depend on, and are proportional to, the profit level y. Financially, as the profit level increases, producers will increase the production to achieve more profit.

Therefore, by inserting the total production rate $Q^*(y)$, the differential equation for player k will be

$$rv_k = y^2 w_k^2 + \mu(y)\frac{\mathrm{d}v_k}{\mathrm{d}y} + \frac{1}{2}\sigma^2(y)\frac{\mathrm{d}^2 v_k}{\mathrm{d}y^2}. \tag{3.18}$$

where $w_k := \frac{1}{N}\Big(1 + \sum_{i=1}^{N-1} s_i - N s_k\Big)$.

Now we give an example of a SDE for $Y(t)$. Although the base assumption used in the body of this chapter is to model $Y(t)$ as a Geometric Brownian Motion (GBM), Appendix B.1 provides a generalization to other price processes.

Example 3.3.1. Assume the stochastic profit dynamics follow a GBM,

$$\mathrm{d}Y(t) = \mu Y(t)\,\mathrm{d}t + \sigma Y(t)\,\mathrm{d}Z(t). \tag{3.19}$$

i.e., $\mu(y) = \mu y$ and $\sigma(y) = \sigma y$. Using Itô's lemma of $\ln Y(t)$ simply presents the expression of $Y(t)$

$$Y(t) = y\mathrm{e}^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z(t)}. \tag{3.20}$$

Then the ODE for player k will be

$$rv_k = y^2 w_k^2 + \mu y\frac{\mathrm{d}v_k}{\mathrm{d}y} + \frac{1}{2}\sigma^2 y^2\frac{\mathrm{d}^2 v_k}{\mathrm{d}y^2}. \tag{3.21}$$

The ansatz $v_k(y) = \alpha y^2$ leads to

$$r\alpha y^2 = y^2 w_k^2 + 2\mu\alpha y^2 + \sigma^2\alpha y^2. \tag{3.22}$$

Therefore, we can obtain the coefficient $\alpha = \frac{w_k^2}{r-2\mu-\sigma^2}$ and the total profit is

$$v_k(y) = \frac{w_k^2 y^2}{r - 2\mu - \sigma^2}. \tag{3.23}$$

This solution tells that the total profit is quadratic with regard to the profit level y. Moreover, in order to make $v_k(y) > 0$, the condition $r > 2\mu + \sigma^2$ must be satisfied. This condition illustrates the expected value function in Equation (3.14) at $x = 0$ should be finite, given $y > 0$. If $r \leq 2\mu + \sigma^2$, the expected accumulated profit will be infinite and $v_k(y)$ will not be a reasonable solution to the HJB equation.

In fact, Proposition 3.3.1 gives the sufficient and necessary condition for $v_k(y) = \infty$.

Proposition 3.3.1. *If $Y(t)$ follows Equation (3.19), the profit of infinite-reserve player at $x = 0$ is proportional to $Y^2(t)$ and grows with a rate of $2\mu + \sigma^2$. Therefore, $v_k(0, y) < \infty$ if and only if $r > 2\mu + \sigma^2$.*

Proof. Inserting the production in Equation (3.16) for each $k = 1, 2, \ldots, N-1$ into Equation (3.14) with $y = Y(t)$ at time t, the expected accumulated profit of player k will be the expected integral

$$\begin{aligned} v_k(0, y) &= \mathbb{E}_{0,y}\left[\int_0^\infty \mathrm{e}^{-rt} Y^2(t) w_k^2 \,\mathrm{d}t\right] \\ &= \int_0^\infty \mathrm{e}^{-rt} w_k^2 \mathbb{E}_{0,y}\left[Y^2(t)\right] \mathrm{d}t \end{aligned} \tag{3.24}$$

Therefore, the profit of the player k is $Y^2(t)w_k^2$. This profit per time unit is quadratic with the economic level $Y(t)$. Since $Y(t)$ is a GBM, the expected profit at time t can obtained by taking integral into the expectation,

$$\mathbb{E}_{0,y}[Y^2(t)] = y^2 \mathbb{E}[\mathrm{e}^{\left(2\mu - \sigma^2\right)t + 2\sigma Z(t)}] = y^2 \mathrm{e}^{(2\mu + \sigma^2)t}. \tag{3.25}$$

Therefore,

$$v_k(0, y) = w_k^2 y^2 \int_0^\infty \mathrm{e}^{-(r - 2\mu - \sigma^2)t} \,\mathrm{d}t. \tag{3.26}$$

If $r > 2\mu + \sigma^2$, the value function is finite. Otherwise, it is infinite. □

As in Assumption 5.2 of [Brown et al. (2017)]*, we proposed an assumption to ensure the value function of infinite players to be finite.

Assumption 3.3.1. *The infinite-reserve player k's value function is finite, where $k = 1, 2, \ldots, N-1$. Mathematically,*

$$v_k(0, k) = \int_0^\infty \mathrm{e}^{-rt} \mathbb{E}_{0,y}\left[q_k(t)\left(Y(t)(1 - s_k) - \sum_{i=1}^{N-1} q_i(t)\right)\right] \mathrm{d}t < \infty. \tag{3.27}$$

This assumption indicates the discount rate r larger than the expected profit growth rate $2\mu + \sigma^2$, as per Proposition 3.3.1, which corresponds to Remark 5.3 in [Brown et al. (2017)]. If we take r only as interest rate, this assumption seems unreasonable. Therefore, we cannot simply interpret r as the interest rate. Instead, the discount rate r includes not only the factor of interest rate, but also other factors representing future negative uncertainty. For example, the producers may endure potential dangers such as survival risk, destruction of their reserve,

*The assumption takes zero costs for those infinite-players. But the mathematics is identical.

and accidental collapses of production blocs. Those potential dangers also contribute to a higher discount rate.

In the next sections, $v_k(y)$ will be used as the boundary solutions at $x = \infty$ because the finite-reserve player at $x = \infty$ becomes infinite-reserve player, which leads to an N-player game with only infinite-reserve players.

3.3.2 A Finite-reserve Producer versus Infinite-reserve Producers

In this section, we maintain the GBM stochastic profit model in Equation (3.19) and assume the finite-reserve player is active in the game (i.e. $x > 0$). Therefore, the variables to be considered become (x, y).

The production rate should not take a negative value (i.e., $q_k \geq 0$). Therefore, with the change of $X(t)$ and $Y(t)$ over time will affect the production of the finite-reserve player. Also, the changed production of the finite-reserve player affects all its opponents. Therefore, producers with higher production costs are perhaps blockaded out, if the finite-reserve player has a high production rate.

We already know that $X(t)$ is a decreasing function of time. A higher reserve x seems to induce a higher production of the finite-reserve player. Later we will show our model in fact does follow this claim. As x becomes smaller, the finite-reserve player produces less, either increase the price or leave production room for competitors to produce more at the same time. As a result, more opponents will participate into this game. Similar to definition of blockading points in Equation (29) of [Ledvina and Sircar (2012)], the blockading curve is the minimum level of x at which player k does not produce. Therefore, we define the blockading curve of player k in the $x - y$ plane using the infimum of x.

In Equation (3.14), $q_k(t)$ is a undetermined production rate of player k. In the following sections we will shows that the optimal production $q_k^*(t) = q_k^*(X(t), Y(t))$, which depends on the current states $X(t), Y(t)$ only. We use the notation $q_k^*(x, y)$ to define the blockading curve.

Definition 3.3.1. The blockading curve of the player k is the set of points

$$\inf_x \{(x, y)\colon q_k^*(x, y) = 0, \forall y \in [0, \infty)\}. \tag{3.28}$$

In other words, the blockading curve should be a curve for lowest value of reserve x given any profit level of y.

Remark 3.3.1. Some low-cost players are never blockaded if the condition $q_k^*(x,y) = 0$ is never satisfied. This means that the production of player k never touches zero for any values of (x, y). In this case, we say player k does not have a blockading curve.

Assume there are currently n players in the game (i.e., $q_i^* = 0$ for $n \le i \le N-1$ so players n or greater are blockaded out). As time-variant version of Theorem 1.7.3 presents, the value function does not depend on time. The HJB equation for the finite-reserve producer is

$$\begin{aligned} rv &= \sup_{q_0 \ge 0} q_0 \left(y(1-s_0) - q_0 - \sum_{i=1}^{N-1} q_i^* - \frac{\partial v}{\partial x} \right) + \mu y \frac{\partial v}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 v}{\partial y^2}, \\ rv_k &= \sup_{q_k \ge 0} q_k \left(y(1-s_k) - q_k - \sum_{i=0, i\ne k}^{N-1} q_i^* - \frac{\partial v}{\partial x} \right) + \mu y \frac{\partial v_k}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 v_k}{\partial y^2}, \end{aligned} \tag{3.29}$$

where the sketch of derivation of the corresponding HJB equation is in Appendix B.2. This sketch explains the time-invariance of the HJB equation, because the integral of accumulated profit in Equation (3.13), and dynamics of $X(t), Y(t)$ are independent of time, except for the undetermined q_0^*. This argument is also suitable for the value function of player k in Equation (3.14) for $k = 1, 2, \ldots, N-1$.

Therefore, with the time-invariant property of the HJB Equation (3.29), the production rate can be obtained by computing the supremum.

$$\begin{aligned} q_0^*(x,y) &= \frac{y(1 + \sum_{i=1}^{n-1} s_i - n s_0) - n \frac{\partial v}{\partial x}}{n+1} \\ q_k^*(x,y) &= \frac{y(1 + s_0 + \sum_{i=1, i\ne k}^{n-1} s_i - n s_k) + \frac{\partial v}{\partial x}}{n+1}. \end{aligned} \tag{3.30}$$

where $k = 1, 2, \ldots, n-1$. Therefore, production of each player is also time-invariant. Our focus is on the profit of finite-reserve player $v(x, y)$. With production rate given as in Equation (3.30), the HJB PDE for $v(x, y)$ becomes

$$rv = \frac{n^2}{(n+1)^2} \left(y a_n - \frac{\partial v}{\partial x} \right)^2 + \mu y \frac{\partial v}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 v}{\partial y^2}. \tag{3.31}$$

where $a_n := \frac{1 + \sum_{i=1}^{n-1} s_i - n s_0}{n}$.

Using the similarity method†, we can simplify the PDE into an ODE by taking the ansatz $v(x, y) = y^2 H\left(\frac{x}{y}\right)$. The form of this ansatz, with y^2 as a multiplying factor, corresponds to the

†Example of similarity method is available in https://www.ucl.ac.uk/ũcahhwi/LTCC/sectionB-similarity.pdf. In our setting, if we introduce transformation $\bar{x} = \varepsilon^a x, \bar{y} = \varepsilon^b y, \bar{v} = \varepsilon^c v$ and insert into Equation (3.29) following the example, we would obtain $c = 2a = 2b$. This indicates $v(x, y) = y^2 H(\frac{x}{y})$ is the ansatz for the PDE.

accumulated profit $v_k(0, y)$ of the game with only infinite-reserve players in Example 3.3.1. Later we will take the corresponding result as our boundary condition at $x = \infty$. The simplified ODE is

$$(r - 2\mu - \sigma^2)H = \frac{n^2}{(n+1)^2}(a_n - H')^2 - \left(\mu + \sigma^2\right)\xi H' + \frac{1}{2}\sigma^2\xi^2 H''. \tag{3.32}$$

where we let $\xi := \frac{x}{y}$, the reserve x per unit of profit level y.

Recall Definition 3.3.1. Blockading curves describe where the infinite-reserve player k enters (resp. leaves) the game as ξ rises (resp. falls). In this sense, a blockading curve is a function $y(x)$. Intuitively, we require that $v_x > 0$, because the larger reserve of the finite-reserve producer leads to larger profit, given a constant profit level y. With this requirement, we can find the blockading curve.

To give an example, we compute the blockading curve for player $n-1$, when n players, including the finite-reserve player, are active. Among those active players, player n has the highest cost and tend to be blockaded out with increasing x. At $q^*_{n-1}(x, y) = 0$, by Equation (3.30), the partial derivative is

$$\frac{\partial v}{\partial x}(x, y) = yH'\left(\frac{x}{y}\right) = y\left[ns_{n-1} - \left(1 + s_0 + \sum_{i=1}^{n-2} s_i\right)\right] = y\delta_{n-1} \tag{3.33}$$

where we denote $\delta_n := (n+1)s_n - \left(1 + s_0 + \sum_{i=1}^{n-1} s_i\right)$. With this, we can determine the number of players with blockading curves in the following proposition.

Proposition 3.3.2. *Let $K = \min\{k : \delta_n > 0\}$. Then players $K, K+1, \dots, N-1$ have blockading curve, and players $1, 2, \dots, K-1$ do not.*

Proof. Because $v(x, y)$ should be increasing with x (i.e., $\frac{\partial v}{\partial x} > 0$), $\delta_n > 0$ should be the prerequisite of the existence of blockading curve for player n in Equation (3.33) to satisfy $q^*_{n-1}(x, y) = 0$. □

As x is sufficiently large and every player with a blockading point is expelled from the market, there are K active players in total.

The profit level y in Equation (3.33) can be cancelled on both sides. Therefore, the $x - y$ blockading curve degenerates into 1-dimensional ξ, and the blockading curves can be found by checking the relationship between $H'(\cdot)$ and δ_n. We can equivalently define the blockading point using the notation of ξ in Definition 3.3.2.

Definition 3.3.2. The blockading point of the player k using notation of ξ is

$$\xi_b^k = \inf\{\xi \colon q_k^*(y\xi, y) = 0\}. \tag{3.34}$$

Definition 3.3.2 and Equation (3.33) lead to the following proposition addressing the formula of blockading curve.

Proposition 3.3.3. *The blockading point for player n from the perspective of ξ is*

$$\xi_b^n = \inf\{\xi : H'(\xi) = \delta_n\}. \tag{3.35}$$

Transforming ξ_b^n to $x - y$ plane gives the blockading curve

$$\left\{(x, y) \colon \frac{x}{y} = \xi_b^n\right\}. \tag{3.36}$$

Moreover, for simplicity, we denote $\xi_b^N := 0$ and $\xi_b^{K-1} := \infty$ to ensure ξ to cover $\mathbb{R}^+$.

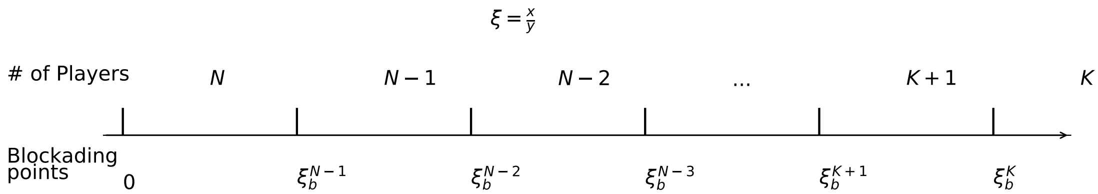
Proof. With Equation (3.33), the infimum of x s.t. $q_n^*(x, y) = 0$ is equivalent to $H'(\xi) = \delta_n$, because monotonic increasing of $v(x, y)$ w.r.t. x is equivalent to increasing of $H(\xi)$ w.r.t. ξ. □

Referring to the blockading lines, define the region

$$A_n := \left\{(x, y) \colon \xi_b^n < \frac{x}{y} < \xi_b^{n-1}\right\} = \left\{\xi \colon \xi_b^n < \xi < \xi_b^{n-1}\right\}. \tag{3.37}$$

In the region A_n, there are n active players in the game and the HJB PDE in Equation (3.31) holds. In other words, the ODE in Equation (3.32) is a piecewise ODE defined in $[\xi_b^n, \xi_b^{n-1})$. Figure 3.2 gives a representation of the blockading points of and curves described by ξ and (x, y) respectively. The upper ξ line in Figure 3.2 presents the blockading points from the perspective of ξ. The number of players is decreasing to K, as ξ goes past ξ_b^n where $n = N-1, N-2, \ldots, K$. The lower $x - y$ plane addresses the blockading curve in the plane of (x, y) with $\xi = \frac{x}{y}$. The region A_n is the region between the two straight lines $\frac{x}{y} = \xi_b^{n+1}$ and $\frac{x}{y} = \xi_b^n$.

Moreover, $\frac{\partial^2 v}{\partial x^2}(x, y) < 0$ at any value of y, or equivalently $H''(\xi) < 0$, because we assume larger reserve x has a marginal effect on the profit. $H''(\xi) < 0$ is equivalent to $H'(\xi)$ being monotonically decreasing.



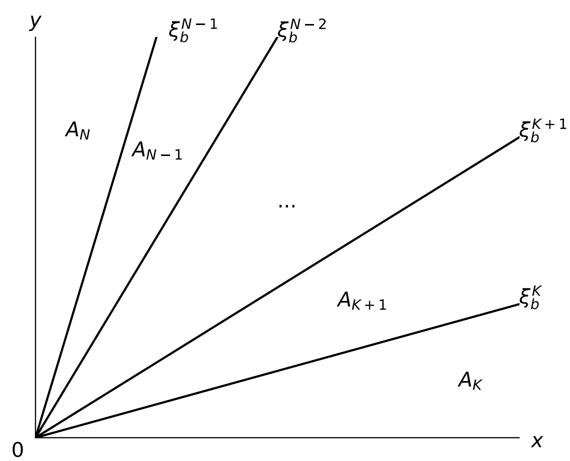


Figure 3.2: Figures explaining the blockading points and curves. Upper: from perspective of ξ. Lower: from perspective of $x-y$ plane

Therefore, in region A_n, i.e., $\xi \in [\xi_b^{n+1}, \xi_b^n)$ where $n = K, K+1, \ldots, N-1$, we can theoretically obtain the value of the blockading points ξ_b^n by setting $q_n^*(\xi_b^n) = 0$, i.e.,

$$H'(\xi_b^n) = \delta_n. \tag{3.38}$$

Since $H'(\xi)$ is strictly monotonically decreasing to zero as $\xi \to \infty$, $H'(\xi)$ is a one to one positive function. Given the prerequsite $\delta_n > 0$, the blockading point $\xi_b^n = (H')^{-1}(\delta_n)$ is well-defined. Proposition 3.3.4 proves the continuity of the piecewise ODE at blockading points.

Proposition 3.3.4. *The piecewise ODE in Equation (3.32) is continuous given the blockad-*

ing point defined in Equation (3.38). Equivalently,

$$\frac{n^2}{(n+1)^2}(a_n - \delta_{n-1})^2 = \frac{(n-1)^2}{n^2}(a_{n-1} - \delta_{n-1})^2 \tag{3.39}$$

Hence the original PDE in Equation (3.31) is also continuous.

Proof. See Appendix B.3. □

Boundary Values

In this section, we determine the boundary values for each piecewise ODE. On the interval of $[\xi_b^{n+1}, \xi_b^n)$ for $n = N, N-1, \ldots, K$, each HJB PDE is subjected to a boundary condition.

We first compute the initial boundary condition at $x = 0$. In the region A_N, where all players but the finite-reserve are active, the boundary conditions become

$$v(0, y) = 0 \tag{3.40}$$

The first boundary condition is the absence of reserve, which leads to zero profit, at any value of the profit level y.

Also, at $x = 0$, the absence of reserve leads to zero production rate. Therefore, as the first-order derivative with regard to x is given by setting the production rate $q_0^*(0, y) = 0$ in Equation (3.30)

$$\frac{\partial v}{\partial x}(0, y) = y a_N. \tag{3.41}$$

Transforming these two boundary value functions yields

$$\begin{aligned} H(0) &= \frac{1}{y^2} v(0, y) = 0 \\ H'(0) &= \frac{1}{y} \frac{\partial v}{\partial x}(0, y) = a_N. \end{aligned} \tag{3.42}$$

Then, the terminal condition at $\xi = \infty$ can be computed using Example 3.3.1, because in this case where (x, y) will finally enter the region A_{K-1}, the finite-reserve player becomes infinite-reserve, and the game includes only K infinite-reserve players with the costs $s_0, s_1, \ldots, s_{K-1}$. Players $K, K+1, \ldots, N-1$ are blockaded out. Therefore, the boundary condition becomes

$$H(\infty) = \frac{1}{y^2} v(\infty, y) = \frac{1}{r - 2\mu - \sigma^2} \left(\frac{1 + \sum_{i=0}^{K-1} s_i - K s_0}{K+1} \right)^2 \tag{3.43}$$

To ensure that the boundary value is bounded (i.e., $0 < H(\infty) < \infty$), we assume $2\mu + \sigma^2 < r$. As we indicated in Example 3.3.1, r is not an interest rate but the discount rate set by producers. μ is the drift of the economic growth and $2\mu + \sigma^2$ is the profit growth given this economic growth. Therefore, the condition is needed because financial problem occurs when cash flows, on average, grow faster than the discount rate and lead to infinite profit. Otherwise, we do not need this optimization due to this theoretic infinite profit.

3.4 Numerical Solution

To summarize the discussion of ODEs and boundary conditions from Section 3.3.2, we list the mathematical problem as a complex piecewise ODE. Given a_n, δ_n defined in Section 3.3.2,

$$(r - 2\mu - \sigma^2)H = \frac{n^2}{(n+1)^2}(a_n - H')^2 - \Big(\mu + \sigma^2\Big)\xi H' + \frac{1}{2}\sigma^2\xi^2 H'' \tag{3.44}$$

on interval $[\xi_b^{n+1}, \xi_b^n)$ where ξ_b^n is determined by $H'(\xi_b^n) := \delta_n$ for $n = N-1, N-2, \ldots, N-K$. In other words, the ODEs are different on each interval $[\xi_b^{n+1}, \xi_b^n)$ with moving boundaries. Other conditions are

$$\begin{aligned} &H(0) = 0, H'(0) = a_N \\ &H(\infty) = \frac{1}{r - 2\mu - \sigma^2}\left(\frac{1 + \sum_{i=0}^{K-1} s_i - K s_0}{K+1}\right)^2 \\ &H''(\xi) < 0 \quad \text{on } (0, \infty) \\ &H(\xi) \text{ is continuous at blockading points } \xi_b^n. \end{aligned} \tag{3.45}$$

This ODE is a non-linear piecewise second order ODE. The solution of each interval $[\xi_b^n, \xi_b^{n-1})$ will give the boundary value at ξ_b^{n-1}, because of continuity of $H(\xi)$. Therefore, tradition ansatz $a_0\xi^2 + a_1\xi + a_2$ is not suitable for this piecewise ODE, because the boundary condition will result in a "2-unknown, 3-equation problem". Instead, rearranging Equation (3.44) by letting $G = H'$ leads to the system

$$\begin{aligned} G &= H' \\ G' &= \frac{2}{\sigma^2\xi^2}\left[(r - 2\mu - \sigma^2)H - \frac{n^2}{(n+1)^2}(a_n - G)^2 + \Big(\mu + \sigma^2\Big)\xi G\right]. \end{aligned} \tag{3.46}$$

The formula of $G'(\xi)$ shows that $\lim_{\xi \to 0} H''(\xi) = \infty$ and result in a pole at $\xi = 0$. Therefore, we cannot compute the numerical solution to Equation (3.32) directly by treating it as a traditional initial value problem, or a boundary value problem. In the next sections, we introduce the asympototic expansion over the small interval on $[0, \xi_0)$.

3.4.1 Solution with Method of Dominant Balance

Before discussing the solution to the ODE, we discuss the singularity. Unfortunately, there is no common definition of singularity for a nonlinear ODE due to its complexity. To give an intuition of singularity, we present several simple examples using linear ODEs there.

Example 3.4.1. Consider the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{2t}, x(0) = 0. \tag{3.47}$$

The solution $x(t) = t^{\frac{1}{2}}$ is singular at $x = 0$ with $\lim_{t\to 0} x'(t) = \infty$. Therefore, this differential is singular at $t = 0$.

Example 3.4.2. The *Bessel differential equation*,

$$x'' + \frac{1}{t}x' + \left(1 - \frac{\alpha^2}{t^2}\right)x = 0. \tag{3.48}$$

where $\frac{1}{t}, 1 - \frac{\alpha^2}{t^2}$ have a pole of orders 1, 2 respectively. It is also singular at $t = 0$.

Therefore, we can claim that the singularity of a function $x(t)$ at $t = a$ happens if $\lim_{t\to a} x^{(n)}(t) = \infty$ for an ODE $f(t, x, x', \dots, x^n) = 0$. In other words, the singularity problem occurs when the derivatives of x, $x^k(t)$ in the ODE, perhaps has at least one point with infinite value. But this is not a formal definition of singularity, due to the complexity of nonlinear ODE.

The problem Equation (3.44) is singular at $\xi = 0$, and the most common packages for solving initial-value or boundary-value problems for ODE do not handle this case, here G' can have a point of infinite value. Specialised methods exist, see [Roswitha and Weinmüller (2001)] for instance, but in our case we can use the simple method first proposed by Euler, namely expanding in series of fractional powers in a small neighbourhood of the singular point; once an accurate solution is obtained in this small interval, standard numerical methods can then be used over the rest of the interval. See the discussion in Section 1.5 of [Hairer et al. (1993)] for a historical view of this technique. Such fractional power series are now known as Puiseux series.

We apply the method of dominant balance to obtain an approximate solution around $\xi = 0$. First, we start with $H(\xi) = a_N\xi + P_{\frac{3}{2}}\xi^{\frac{3}{2}} + O(\xi^4)$ and plug it into

$$(r - 2\mu - \sigma^2)H = \frac{N^2}{(N+1)^2}(a_N - H')^2 - \left(\mu + \sigma^2\right)\xi H' + \frac{1}{2}\sigma^2\xi^2 H'', \tag{3.49}$$

and obtain the equation

$$\left(a_N(\mu - r) + \frac{9N^2 {P_{\frac{3}{2}}}^2}{4(N+1)^2}\right)\xi + \left(-\frac{1}{8}\sigma^2 P_{\frac{3}{2}} + 1/2\, P_{\frac{3}{2}}\mu - P_{\frac{3}{2}} r\right)\xi^{\frac{3}{2}} + O(\xi^4) = 0. \tag{3.50}$$

Around $\xi = 0$, $\xi^{\frac{3}{2}}$ and $O(\xi^4)$ are the higher order terms (h.o.t.) of the equation; we hence set $a_n(\mu - r) + \frac{9n^2 {P_{3/2}}^2}{4(N+1)^2} = 0$ and compute the solution

$$P_{\frac{3}{2}} = \pm\frac{2\sqrt{a_N(r-\mu)}(N+1)}{3N} \tag{3.51}$$

where we assume $r > 2\mu + \sigma^2$. We choose the negative part of the solution because we have assumed $H''(\xi) < 0$ on $(0, \infty)$. Similarly, we can genertate a Puiseux series $P_2, P_{\frac{5}{2}}, \ldots$ by the following iterative steps:

- Insert the assumed approximate solution $H(\xi) = \sum_{i=2}^{m} P_{\frac{i}{2}}\xi^{\frac{i}{2}} + O(\xi^{\frac{m+1}{2}})$ into Equation (3.49).
- Obtain an equation $M(P_{\frac{n}{2}})\xi^{\frac{n-1}{2}} + O(\xi^n) = 0$ where $M(P_{\frac{n}{2}})$ is a certain function obtained by rearranging the ODE.
- Solve the coefficient $P_{\frac{n}{2}}$ from $M(P_{\frac{n}{2}})$.
- Repeat the whole process in the case of $n+1$ and obtain an ODE with higher order terms.

To be more specific, we insert the summation $H(\xi) = \sum_{i=2}^{\infty} P_{\frac{i}{2}}\xi^{\frac{i}{2}}$ with $P_1 = a_N$ into the ODE. Then we obtain

$$\begin{aligned}(r - 2\mu - \sigma^2)\sum_{i=2}^{\infty} P_{\frac{i}{2}}\xi^{\frac{i}{2}} = {} & \frac{N^2}{(N+1)^2}\left(\sum_{i=2}^{\infty}\frac{i}{2}P_{\frac{i}{2}}\xi^{\frac{i}{2}-1}\right)^2 - \left(\mu + \sigma^2\right)\left(\sum_{i=2}^{\infty}\frac{i}{2}P_{\frac{i}{2}}\xi^{\frac{i}{2}}\right) \\ & + \frac{1}{2}\sigma^2\left(\sum_{i=2}^{\infty}\frac{i(i-2)}{4}P_{\frac{i}{2}}\xi^{\frac{i}{2}}\right).\end{aligned} \tag{3.52}$$

Taking square and rearranging the equation yields

$$\begin{aligned}\sum_{i=2}^{\infty}\Bigg[& \left(r - \left(2 - \frac{i}{2}\right)\mu - \left(1 - \frac{i}{2} + \frac{i(i-2)}{8}\right)\sigma^2\right)P_{\frac{i}{2}} \\ & - \frac{N^2}{(N+1)^2}\left(\sum_{j=2}^{i}\frac{(i+3-j)(j+1)}{4}P_{\frac{i+3-j}{2}}P_{\frac{j+1}{2}}\right)\Bigg]\xi^{\frac{i}{2}} = 0.\end{aligned} \tag{3.53}$$

Setting the coefficients to zero from $i = 2, 3, \dots, \infty$ and solving $P_{\frac{i}{2}}$ generates the same series $\{P_{\frac{i}{2}}\}_{i \le 2}$. Moreover, the formula of coefficients shows that except the case of $i = 2$, the coefficient of $\xi^{\frac{i}{2}}$ is a linear function with regard to $P_{\frac{i+1}{2}}$ hence determining only one solution of $P_{\frac{i+1}{2}}$. Using those equations, we can imply the iteration formula for $P_{\frac{i+1}{2}}$ for $i = 3, 4, \dots,$

$$P_{\frac{i+1}{2}} = \frac{2}{3(i+1)P_{\frac{3}{2}}} \Bigg[\frac{(N+1)^2}{N^2} \left(r - \left(2 - \frac{i}{2}\right)\mu - \left(1 - \frac{i}{2} + \frac{i(i-2)}{8}\right)\sigma^2 \right) P_{\frac{i}{2}} - \sum_{j=3}^{i-1} \frac{(i+3-j)(j+1)}{4} P_{\frac{i+3-j}{2}} P_{\frac{j+1}{2}} \Bigg] \tag{3.54}$$

Given the formula for computing $P_{\frac{i}{2}}$, we observe that the Puiseux series is divergent and the solution $H(\xi)$ is actually an asymptotic solution,

$$H(\xi) \sim \sum_{i=2}^{\infty} P_{\frac{i}{2}} \xi^{\frac{i}{2}}. \tag{3.55}$$

As a result, we must truncate the series of $H(\xi)$ to a finite term of $H_m(\xi) = \sum_{i=2}^{m} P_{\frac{i}{2}} \xi^{\frac{i}{2}}$ as an approximation to the solution.

To plot the approximation effect of H_m, we assume the parametric setting of the game as shown in Table 3.1.

Parameter	r	s_0	s_1	s_2	s_3	s_4	μ	σ
Values	0.05	0.05	0.3	0.32	0.5	0.52	0.01	0.07

Table 3.1: Parametric setting

We move all terms of the ODE in Equation (3.44) to the left in the form of $D(\xi, H, H', H'') = 0$. The log-log plot of (ξ, D) in Figure 3.3 presents the absolute residual $|D(\xi, H_m, H_m', H_m'')|$ between consecutive solutions H_m of Puiseux series. The plot demonstrates that the residual decreases in the scale from 10^{-2} to 10^{-10}, as the term m increase from 2 to 9. Therefore, over the small interval around $\xi = 0$, H_m returns a good approximation.

In Appendix B.4, we presents several exact solutions when μ, r, σ^2 satisfies special relationship, by setting the terms of Puiseux series to be zero. But those exact solutions are not relevant to the discussion afterwards, because the relationship perhaps break our assumption $r > 2\mu + \sigma^2$.

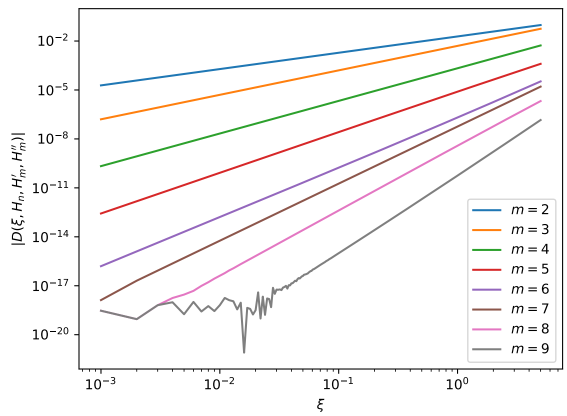


Figure 3.3: Plot of residual of the ODE with the approximation $H_m(\xi)$

3.4.2 Numerical Solution Extended to Infinity

Finite Difference Method

We constrained the interval of the approximate solution on $[0, \xi_0]$ where $H'(\xi_0) = \delta_{N-1}$. Now we extend this solution to the interval $[\xi_0, \infty)$. Define the central difference be $H_i' := \frac{H_{i+1} - H_{i-1}}{2h}$. Since the ODE is defined over $[0, \infty)$, we must assume a sufficiently large number ξ_M to replace the infinity and partition $[\xi_0, \xi_M]$ into $\{\xi_i\}_{i=0,1,\ldots,M}$. Then with the boundary condition on ξ_0, ξ_M, we can change Equation (3.44) via the finite difference method into

$$\begin{aligned} D_i(H_1, \ldots, H_{M-1}) :=& \frac{n_i^2}{(n_i+1)^2}\left(a_{n_i} - \frac{H_{i+1} - H_{i-1}}{2h}\right)^2 - \left(\mu + \sigma^2\right)\xi_i \frac{H_{i+1} - H_{i-1}}{2h} \\ &+ \frac{1}{2}\sigma^2 \xi_i^2 \frac{H_{i+1} - 2H_i + H_{i-1}}{h^2} - (r - 2\mu - \sigma^2)H_i = 0. \end{aligned} \tag{3.56}$$

where n_i is determined by checking the inequality $\delta_{n_i+1} \leq \frac{H_{i+1} - H_{i-1}}{2h} < \delta_{n_i}$ and $i = 1, 2, \ldots, M-1$. Here we use central difference for the first and second derivatives and obtain the simultaneous quadratic equations for $\{H_i\}_{i=1,2,\ldots,M-1}$ with boundary condition $H_0 = H(\xi_b^{N-1})$ and $H_M = H(\infty)$. Based on the system of quadratic eqautions in Equation (3.56), the algorithm for computing the solution is given in Table 3.2.

We take the algorithm in Table 3.2 and obtain the plot in Figure 3.4. The plot illustrates that profit is an increasing function with marginal effect, with regard to ξ, the reserve per unit profit. Moreover, the two blockading points are $\xi_{N-1} = 0.096$ and $\xi_{N-2} = 0.99$ respectively. As expected, profit of the finite-reserve player is increasing with the value of reserve x and the profit level y. The increase with x has a marginal effect as x is large enough while the increase with y does not.

Algorithm
1. Give the initial guess of solution $H^0(\xi)$, step h, sufficient large value $\xi_{\max}$ and tolerance ε. In our case, we set the initial guess be a quadratic function $H^0(\xi) = H_M + \frac{(\xi-\xi_M)^2}{(\xi_M-\xi_0)^2}(H_0 - H_M)$.
2. Partition the interval $[\xi_0, \xi_{\max}]$ into $\{\xi_i\}_{i=0,1,\ldots,M}$ and let $H^0(\xi_i) = H_i$. The guess in step 1 automatically sets the boundary conditions be $H^0(\xi_0) = H_0$ and $H^0(\xi_M) = H_M$.
3. Compute the first order derivative using central difference method $H_i' = \frac{H_{i+1}-H_{i-1}}{2h}$.
4. Determine the interval n_i for each point ξ_i by checking which of $n_i = N-2, \ldots, N-K$ satisfies the condition $\delta_{n_i+1} \leq H_i' < \delta_{n_i}$.
5. Obtain $\{H_i^{\text{new}}\}_{i=0,1,\ldots,M}$ given Equation (3.56) using a step of Newton's method and compute the error $\|H^{\text{new}} - H\|$.
6. If $\|H^{\text{new}} - H\| < \varepsilon$, stop the algorithm and make cubic intepolation; else, redo step 3 – 5.

Table 3.2: Algorithm for computing solution using finite difference method ($H_0 = H(\xi_b^{N-1})$ and $H_M = H(\infty)$)

In order to analyze the accuracy of the finite difference method, we use the backward error analysis. Given the estimated $\{(\xi_i, H_i)\}_{i=0,1,\ldots,M}$, we make a cubic interpolation of $H(\xi)$ (hence $H'(\xi), H''(\xi)$ are quadratic, linear respectively). Figure 3.5 gives $D(\xi, H, H', H'')$, the error with regard to the step $h = 10^{-1}, 10^{-2}, 10^{-3}$. As Figure 3.5 indicates, the error decreases by a scale of 10^{-2} when h increases by a scale of 10^{-1}, meaning that the central difference has an error of $O(h^2)$.

3.5 Computation of Solutions for given parameters

We have already finished the computation of the value function $H(\xi)$ in the former section. In this section, we insert this $H(\xi)$ into the production q_0^* to study the relationship of production $q_0^*(x, y)$, $Q^*(x, y)$ and x, y. Moreover, given the expressions of the dynamics $X(t), Y(t)$, we perform the Monte Carlo Simulation to make the blockading time visible.

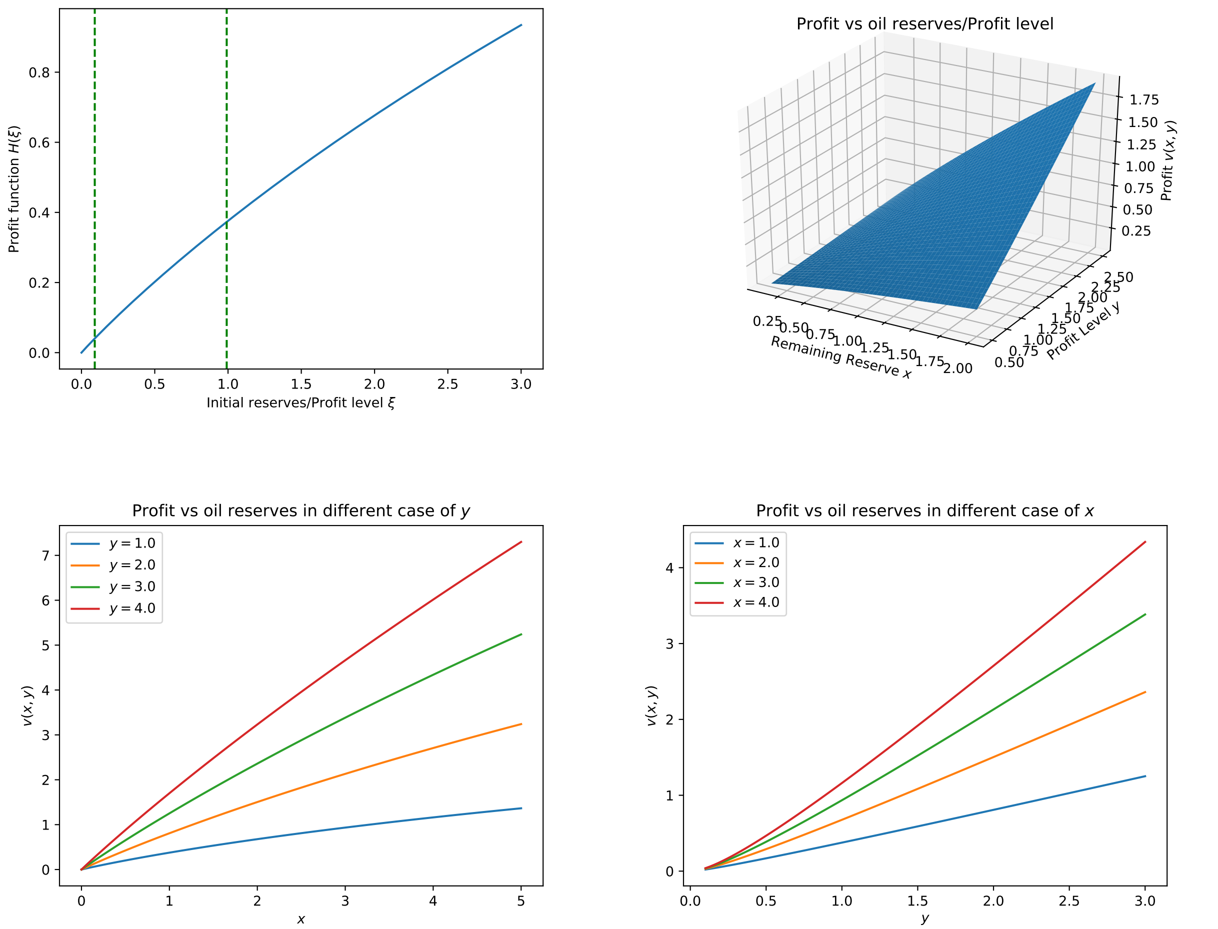


Figure 3.4: Plot of profit function
Upper-Left: $H(\xi)$ vs ξ ; Upper-Right: $H(x, y)$ vs (x, y); Lower-left: $H(x, y)$ vs x given different y; Lower-right: $H(x, y)$ vs y given different x

3.5.1 Expressions of Dynamics and Production

Dynamic of $\xi(t)$

If the solution to $H(t)$ can be solved numerically, the dynamics of the reserve $X(t)$ and the profit level $Y(t)$ can be explicitly expressed using $H(t)$. Applying Itô's lemma to the dynamics of $\xi(t) := \frac{X(t)}{Y(t)}$ leads to

$$\mathrm{d}\xi(t) = \left[-\frac{n}{n+1}(a_n - H'(\xi(t))) - (\mu - \sigma^2)\xi(t) \right] \mathrm{d}t - \sigma\xi(t)\,\mathrm{d}Z(t) \tag{3.57}$$

over the interval $\xi_b^n \leq \xi_t < \xi_b^{n-1}$. We have already ensured $\xi_b^n > \xi_b^N = 0$ hence the indicator of $X(t) > 0$ disappears. Obviously this SDE is time-homogeneous and does not depend on

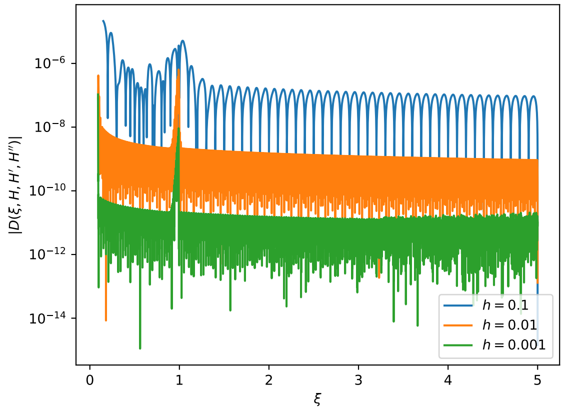


Figure 3.5: Plot of residual of the ODE w.r.t. different steps h

outer variables. Therefore, the passing time and the blockading time of player i can only depend on the property of $\xi(t)$. First we give the definition of passing time and blockading time using this $\xi(t)$.

Definition 3.5.1. The passing time for player n is the set

$$\{\xi\colon \xi(t) = \xi_b^n\}. \tag{3.58}$$

And the blockading time for player n is the supremum of the passing time,

$$\tau_b^n = \sup\{\xi\colon \xi(t) = \xi_b^n\}. \tag{3.59}$$

Those blockading points ξ_b^n's are already computed as fix values in Section 3.4.2. The property of $\xi(t)$ at these blockading points requires further investigation.

As $\xi(t)$ follows a stochastic dynamic, the random factor $\mathrm{d}Z(t)$ leads to multiple passing time. Therefore, the blockading time is the last time when $\xi(t)$ passes by the blockading point ξ_b^n. In other words, this blockading time is a random variable that we cannot find explicitly.

Formulae of Production Rates

Given Equation (3.30) and the formula of $H(\xi)$, it is easy to compute the formula of production rate

$$q_0^*(x, y) = \frac{yn}{n+1}\left(a_n - H'\left(\frac{x}{y}\right)\right) \tag{3.60}$$

when there are n active players $0, 1, \ldots, n-1$, (i.e., $\xi_b^n \leq \frac{x}{y} = \xi < \xi_b^{n-1}, n = N, N-1, \ldots, K$). As stated, over this interval $X(t) > 0$ hence the indicator ensuring the positive disappears. Therefore, the dynamics of $X(t)$ over this interval is

$$\mathrm{d}X(t) = -\frac{nY(t)}{n+1}\left(a_n - H'\left(\frac{X(t)}{Y(t)}\right)\right)\mathrm{d}t \tag{3.61}$$

For those opponents, the production rates of player $k = 1, 2, \ldots, N-1$ are

$$q_k^*(x,y) = \begin{cases} \frac{y}{n+1}\left(1 + s_0 + \sum_{i=1,i\neq k}^{n-1} s_i - ns_k + H'\left(\frac{x}{y}\right)\right) \text{ when } k < n \\ 0 \text{ Otherwise.} \end{cases} \tag{3.62}$$

Therefore, summing the equations of $q_k^*(x,y)$ above can generate the total production of all opponents

$$Q^*(x,y) = \sum_{k=1}^{n-1} q_k^*(x,y) = \frac{y}{n+1}\left((n-1)\left(1 + s_0 + H'\left(\frac{x}{y}\right)\right) - 2\sum_{k=1}^{n-1} s_k\right) \tag{3.63}$$

Inserting all production rates in Equation (3.30) into the price function Equation (3.5), the energy price $P(x,y)$ with n active players is

$$P(x,y) = y - \sum_{i=0}^{N-1} q_i^* = \frac{y}{n+1}\left(1 + \sum_{i=0}^{n-1} s_i + H'\left(\frac{x}{y}\right)\right). \tag{3.64}$$

All formulae above show that the production rates and oil price are determined by the factors $Y(t)$ and $H'(\xi(t))$. $\xi(t)$ is the ratio between the reserve and the stochastic factor. Keeping the ratio $\xi(t) = \frac{X(t)}{Y(t)}$ constant, the stochastic factor $Y(t)$ is proportional to the price $P(X(t), Y(t))$ and the production rates $q_k^*(X(t), Y(t))$. A higher $Y(t)$ leads to faster production rates and decreases the reserve $X(t)$ more rapidly.

Proposition 3.5.1. *Over the region near $\frac{x}{y} = \xi = 0$, the production of the finite-reserve player will be*

$$q_0^*(x,y) = yO\left(\left(\frac{x}{y}\right)^{\frac{1}{2}}\right). \tag{3.65}$$

In other words, given constant x, q_0^* increase with an order of $y^{\frac{1}{2}}$ as $y \to \infty$; Given constant y, q_0^* decrease with an order of $x^{\frac{1}{2}}$ as $x \to 0$.

Proof. Over the interval $0 = \xi_b^N < \xi < \xi_b^{N-1}$, given the asymptotic solution in Equation (3.55)

$$H(\xi) \sim \sum_{i=2}^{\infty} P_{\frac{i}{2}} \xi^{\frac{i}{2}} = a_N \xi + P_{\frac{3}{2}} \xi^{\frac{3}{2}} + O(\xi^2). \tag{3.66}$$

where $P_1 = a_N$. Then the production q_0^* follows

$$q_0^*(x,y) = \frac{yN}{N+1}\left(a_N - H'\left(\frac{x}{y}\right)\right) = yO\left(-P_{\frac{3}{2}}\xi^{\frac{3}{2}} + O(\xi^2)\right). \tag{3.67}$$

□

Given the formulae of $q_0^*(x,y), Q^*(x,y)$ and $P(x,y)$, the plots of production rates $q_0^*(x,y)$, $Q^*(x,y)$ are given in Figure 3.6. The upper plots presents the $q_0^*(x,y)$ with x (resp. y) given different level of y (resp. x) and the total production of all infinite-reserve players is presented in the lower plots. Proposition 3.5.1 tells us the asymptotic relationship between q_0^* and (x,y), as indicated below.

The left two plots in Figure 3.6 addresses that given different level of y a higher reserve x lead to higher production of finite-reserve player q_0^* with a marginal effect, at the cost of productions of opponents. As expected, price level y lifts productions of all players because higher profit will induce more intense production.

But the effect of y is different between the finite-reserve producer and the infinite-reserve opponents. The finite-reserve producer also has a marginal effect as profit level y goes up. This is because the production of the finite-reserve producer has a reserve limit. Even though y is high enough, the existence of the limited reserve $X(t)$ will limit the increase of production. In contrast, for the infinite-reserve player, there is no marginal effect of y.

3.5.2 Monte Carlo Simulation

With all explicit dynamics $X(t), Y(t), \xi(t)$ and formulae of $P(x,y), q_0^*(x,y), Q^*(x,y)$, we perform the Monte Carlo simulation of paths of reserves, profit level, prices and production rates, using the parametric set Table 3.1 with the initial value $X(0) = 5, Y(0) = 2$. In our simulation, we use Euler's discretization of SDE. The path of production rates, price and oil reserve are shown in Figure 3.7.

The upper-left plot is the production of the finite-reserve player 0. The plot indicates an overall decrasing trend of q_0^* with slight fluctuations. As time t goes large enough, the production decreases to almost zero, due to monotonically decrease of reserve $X(t)$ without stochasticity, as the middle-left plot indicate. In contrast, the upper-right plot shows an overall increasing trend significant fluctuations. Those fluctuation is due to the stochasticity of the profit level $Y(t)$. Similarly, the middle-right plot of the price level also indicates an overall increasing trend with fluctuations, because of apparently decreasing trend of $X(t)$

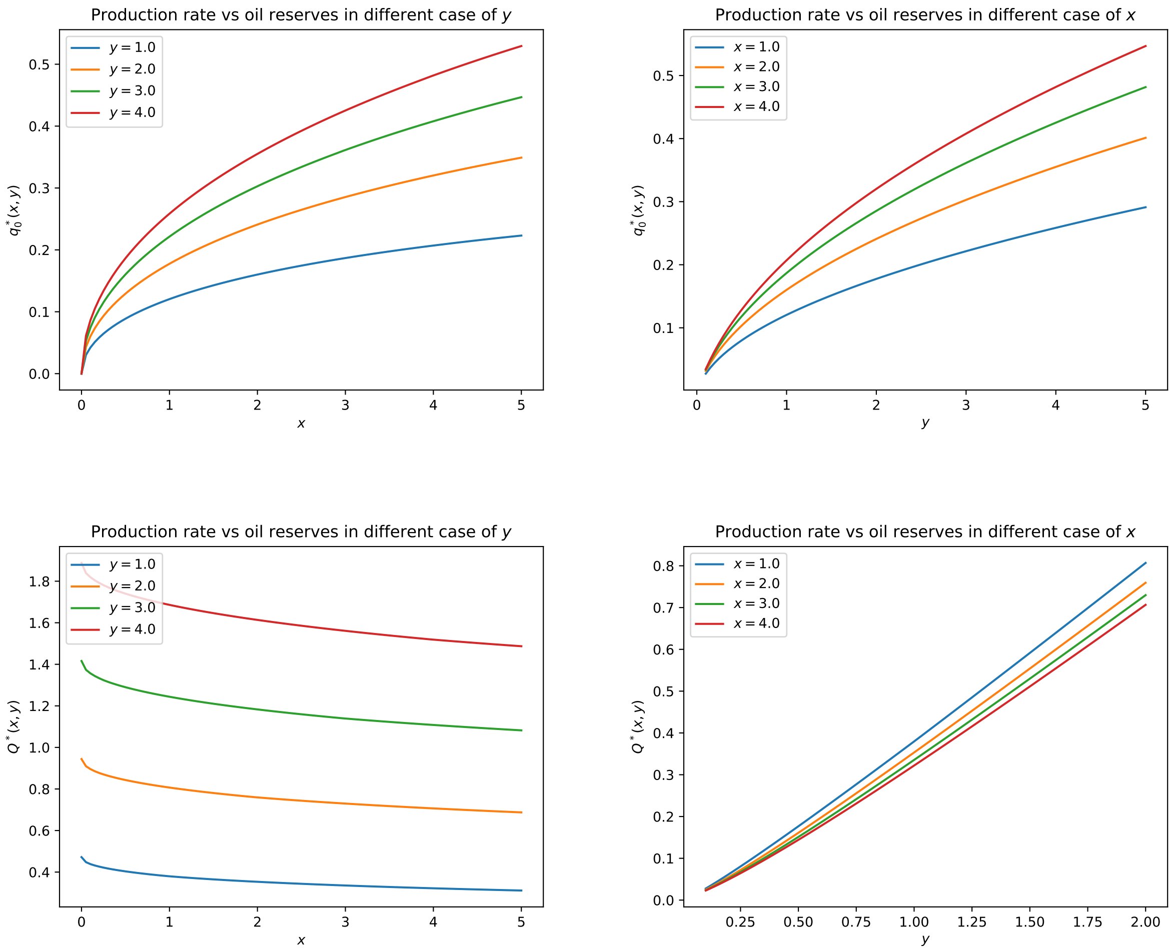


Figure 3.6: Plot of production Rate q_0^* or Q^* vs x, y

at $t < 18$ approximately. At $t > 18$, $X(t)$ does not decrease much but converges zero. As a result, the stochasticity of $Y(t)$ plays a more intense role than $X(t)$ and displays a significant fluctuation.

Moreover, the green lines are the passing times for player 4 and player 3. We can observe that the left passing time (player 4) is "heavier". This path result shows that player 4, with the highest cost of production, enter, exits and re-enter the differential game.

The bottom plot shows the relationship between $X(t)$ and $Y(t)$ in this simulation. As we indicate, $X(t)$ is monotonic decreasing without random factor. At first, the starting poing of (x, y) is on the region A_3. As $X(t)$ decreases, the point (x, y) gradually passes ξ_b^3 to the region A_4, and passes ξ_b^4. Finally, the point reach to the region A_5 where all producers start their production.

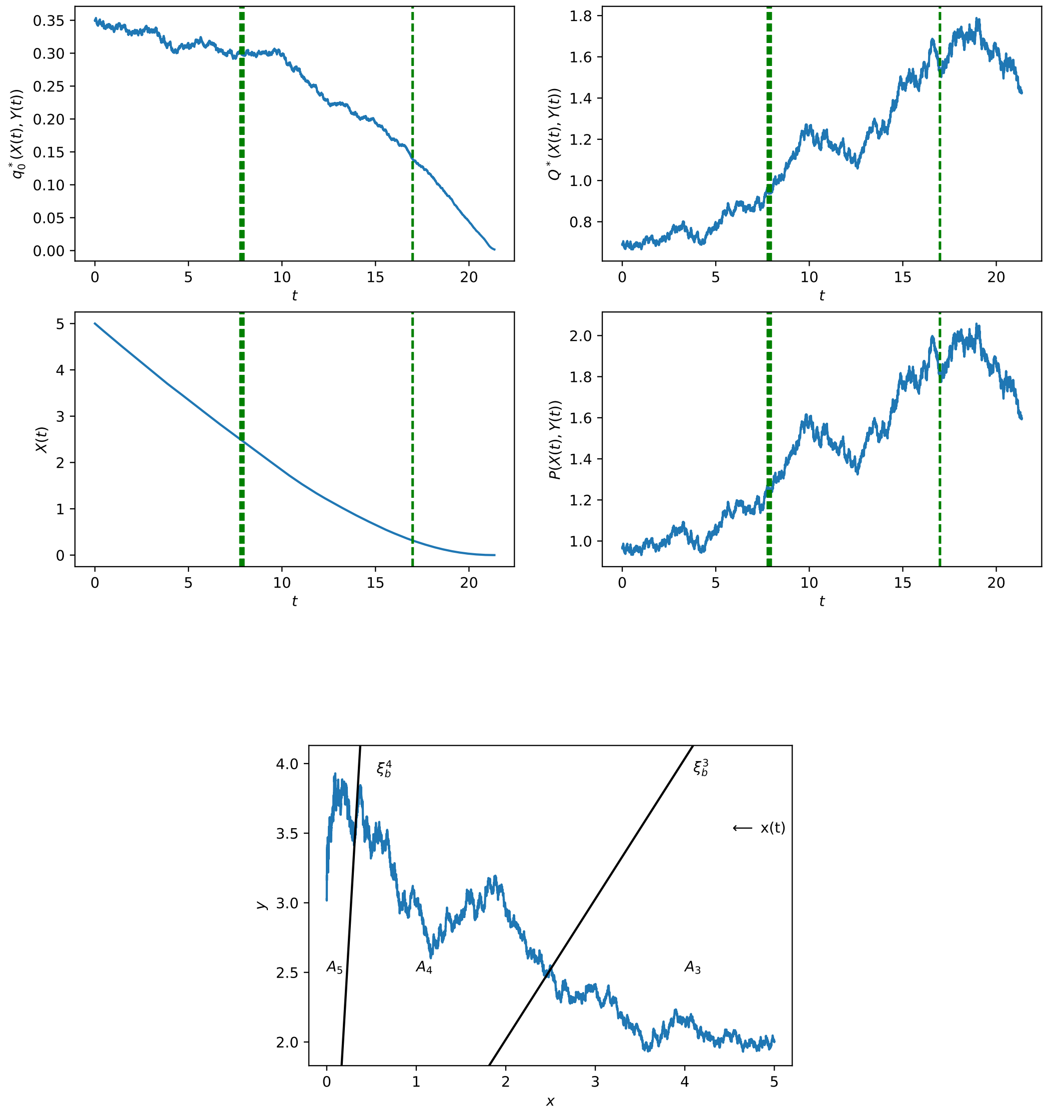


Figure 3.7: Monte Carlo Simulation

3.6 Conclusion

In this chapter, we investigate a stochastic differential game with one finite-reserve player and $N-1$ infinite-reserve players, by incoporating the stochastic factor of [Brown et al. (2017)] into the profit function in [Ledvina and Sircar (2012)]. In this stochastic model, a GBM $Y(t)$ is used to represent a general economic driver affecting both the price and the cost for each player. Then, we obtain a simplified ODE for the solution of the HJB problem by taking $\xi = \frac{x}{y}$ with a singular point at $\xi = 0$ and inserting an ansatz into the HJB PDE. This ODE allows us to construct the blockading points where the high-cost infinite-reserve player enter the market as ξ goes up.

To solve the ODE, we use the method of dominant balance to obtain the solution $H(\xi)$ on $[0, \xi_0]$. Therefore, the existence of solution at ξ_0 enables us to solve the equation on the whole domain $[0, \infty)$ via finite difference method.

We also make an analysis of the behaviour of different types of players in the market. With the solution to the HJB PDE, we study the effect of reserves $X(t)$ and profit levels $Y(t)$ on the profit and productions of each player. The increase in profit level generally increases the production of each player. Larger x presents a marginal effect on the production of the finite-reserve player, given a constant y. On the contrary, larger x decrease the production of opponents with a marginal effect. Production of each player increases with y, but only the finite-reserve player endures a marginal effect, due to the limited reserve.

Due to the stochasticity of the profit level $Y(t)$, infinite-reserve players with high costs of production may enter, leave and re-enter the market as the profit level fluctuates. But as the oil reserve decreases and the price is driven up, all players would eventually enter the market.

Chapter 4

Robust Portfolio with Commodities and Stochastic Interest Rates

Note: Brownian Motion becomes W in Chapters 5 and 6 while it was Z so far to maintain consistency with regard to published literature.

4.1 Introduction

This chapter addresses a gap in the literature concerning robust portfolio analysis for commodity markets in the presence of stochastic interest rates. For generality, we study an ambiguity-averse investor with a Cramér-Lundberg surplus to be allocated into a mean-reverting asset representing a commodity and a bond with a Vasicek interest rate model. Our framework allows for closed-form solutions for the optimal strategy, worst case measure, terminal wealth and value functions. We provide necessary conditions for a well-behaved solution. A full estimation is conducted on two commodity representatives: WTI oil prices and gold prices. We find strong evidence that optimal exposures to commodity risk and interest rate risk, as well as the performance of the portfolio, are significantly affected by the level of ambiguity aversion. Our analyses demonstrate that investors who ignore uncertainty incur in drastic equivalent welfare losses, in particular ignoring commodity uncertainty is more costly than neglecting interest rate uncertainty. In a comparison between stocks and commodities, ignoring uncertainty on the latter is also more damaging. We also confirm the importance of working on a complete market (investing in bonds) for commodity investors, otherwise welfare losses could easily reach 45%. In terms of parameter mis-specifications, we find that incorrect larger correlation, smaller variances or simply wrong market price of commodity risk, could lead to drastically large wealth-equivalent losses.

The chapter is organized as follows. Section 4.2.1 gives the formulation of surplus $X(t)$ following the Cramér-Lundberg process, Vasicek interest rate model $r(t)$ and the mean-reverting asset with correlation. Section 4.2.2 formulates the optimization problem using expected terminal surplus. The solution for the value function is found via Riccati ODEs using the exponential affine quadratic ansatz for the derived HJBI PDE, the optimal strategy $\boldsymbol{\pi}^*$ and worst change of measure are also provided explicitly. Section 4.3 provides the main elements for a full wealth-equivalent loss analysis (suboptimal analysis) $L^{\boldsymbol{\pi}^s}$ in different cases, through solving HJB PDE as well but with specially given investment (suboptimal) strategies. The empirical part, Section 4.4, describes the estimation methodology and report the numerical findings, in particular optimal strategies, probability distribution of terminal surplus and plots of equivalent losses for relevant suboptimal cases. The last section concludes. We make the following contributions:

- We construct an robust portfolio optimization for investors to maximize the expected terminal wealth and obtain a closed-form solution to the problem via an HJBI PDE. Provided the solution, the robust portfolio is obtained as a function of time, interest rate and log price.

- We compute the parameters given WTI and gold data, and analyze the effect on the optimal strategies and wealth equivalent losses due to absence of ambiguity, parametric misspecification and incomplete market.

4.2 Formulation of the Surplus Optimization Problem

We consider a company who can invest its surplus on a commodity, a bond, and a money market account. The model presented here assumes only one commodity in the portfolio but all our results can be easily extended to multiple assets following mean-reverting processes. Details on the evolution of the underlyings and the problem of interest are provided next.

4.2.1 Assumptions

A classic model for the surplus of a company is the exponential Cramér-Lundberg process which consists of a premium and decreasing jumps, this is:

$$\mathrm{d}X_0(t) = X_0(t^-)\left(c\,\mathrm{d}t - \int_{\mathbb{R}^+} yN(\mathrm{d}t, \mathrm{d}y)\right), \tag{4.1}$$

where $N(\mathrm{d}t, \mathrm{d}y)$ is a Poisson random measure with intensity, μ, $\int_{\mathbb{R}^+} \mathbb{E}[N(\mathrm{d}t, \mathrm{d}y)] = \mu\,\mathrm{d}t$, c is the premium percentage. We assume the initial surplus $X_0(0) = x$ leading to the integral form:

$$X_0(t) = x + c\int_0^t X_0(s)\,\mathrm{d}s - \sum_{i=1}^{N(t)} X_0(T_i^-)Y_i. \tag{4.2}$$

For $i = 1, 2, \ldots, N(t)$, T_i are the times when a jump occurs, and Y_i are positive random variables with first moment $\mu_1 > 0$ and second moment μ_2. In order to make sure $0 \leq Y_i \leq 1$, we refer to the setting of [Chiu and Wong (2013)], $Y_i = \mathrm{e}^{-Z_i}$ where Z_i is a non-negative random variable with well-defined MGF. In this process, the claim amounts are $X_0(T_i^-)Y_i$ which is proportional to the amount of surplus.

The company can invest this surplus into three assets: a risk-free bank account, a risky bond and a commodity. We assume the commodity follows a mean-reverting process, i.e., in real-world measure,

$$\mathrm{d}S(t) = (r(t) + \lambda_S\sigma_S - a\ln S(t))S(t)\,\mathrm{d}t + \sigma_S S(t)\left(\rho\,\mathrm{d}W_1(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_2(t)\right), \tag{4.3}$$

where $W_1(t), W_2(t)$ are independent Brownian motions, the drift is $r(t) + \lambda_S\sigma_S - a\ln S(t)$, volatility σ_S and we require $\lambda_S > 0$ to ensure that the return will be larger as the volatility increases. The interest rate $r(t)$ follows a Vasicek model, where the correlation of random part between asset and interest rate is ρ. Mathematically, under the risk-neutral measure, we have:

$$\mathrm{d}r(t) = \kappa(\bar{r} - r(t))\,\mathrm{d}t + \sigma_r\,\mathrm{d}W_1^Q(t), \tag{4.4}$$

with infinity yield $y_\infty := \bar{r} - \frac{\sigma_r^2}{2\kappa^2}$. We invest $\pi_S(t)$ into the commodity and $\pi_P(t)$ into a (rollover) bond with a fix time to maturity $T-t$. The price of the bond under the risk-neutral measure is:

$$P(t, r(t); T) := P(t, r(t)) = \exp(-I(t;T)r(t) + A(t;T)), \tag{4.5}$$

where

$$\begin{aligned} I(t;T) &= \frac{1 - \mathrm{e}^{-\kappa(T-t)}}{\kappa} \\ A(t;T) &= \left(\bar{r} - \frac{\sigma_r^2}{2\kappa^2}\right)(I(t;T) - (T-t)) - \frac{\sigma_r^2}{4\kappa}I^2(t;T). \end{aligned} \tag{4.6}$$

For simplicity, we transform the dynamics of the rollover bond into

$$\mathrm{d}P(t,r(t)) = (r(t) + \lambda_r I_\tau)P(t,r(t))\,\mathrm{d}t - I_\tau \sigma_r P(t,r(t))\,\mathrm{d}W_1(t), \tag{4.7}$$

where $I_\tau := I(t;t+\tau) = \frac{1-\mathrm{e}^{-\kappa\tau}}{\kappa}$ is a constant. λ_r is the market price of risk on the bond return. In the risk-neutral measure, the bond price can be represented via a PDE

$$P_t + P_r(\kappa(\bar{r} - r)) + \frac{\sigma_r^2}{2} P_{rr} = rP. \tag{4.8}$$

Transforming to real-world measure using Girsanov theorem, the joint dynamics for the bond and the interest rate are

$$\begin{aligned} \mathrm{d}r(t) &= [\kappa(\bar{r} - r(t)) - \lambda_r \sigma_r]\,\mathrm{d}t + \sigma_r\,\mathrm{d}W_1(t) \\ \mathrm{d}P(t,r(t)) &= (r(t) + \lambda_r I(t;T))P(t,r(t))\,\mathrm{d}t - I(t;T)\sigma_r P(t,r(t))\,\mathrm{d}W_1(t). \end{aligned} \tag{4.9}$$

We assume the jumps of the surplus, hence $N(t)$ and Y_i to be independent of $W_1(t)$ and $W_2(t)$. By letting $Z(t) = \ln S(t)$, we can simplify the representation of the asset price:

$$\mathrm{d}Z(t) = \left(r(t) + \lambda_S \sigma_S - \frac{1}{2}\sigma_S^2 - aZ(t)\right)\mathrm{d}t + \sigma_S\left(\rho\,\mathrm{d}W_1(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_2(t)\right). \tag{4.10}$$

The model in the previous equations is called the *reference model*. The surplus process becomes

$$\begin{aligned} \mathrm{d}X(t) = {} & [(c + r(t))X(t) + \pi_S(t)(\lambda_S\sigma_S - aZ(t)) + \pi_P \lambda_r I(t)]\,\mathrm{d}t \\ & + \pi_S(t)\sigma_S\left(\rho\,\mathrm{d}W_1(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_2(t)\right) - \pi_P(t)I(t)\sigma_r\,\mathrm{d}W_1(t) - X(t^-)\int_{\mathbb{R}^+} yN(\mathrm{d}t,\mathrm{d}y). \end{aligned} \tag{4.11}$$

Note X_t is not self-financing because there are cash inflows, constant premium c and cash outflows, claims Y_i. Moreover if both $c = 0$ and $\mu = 0$, the surplus $X(t)$ will be a constant hence the problem starts simply with an initial budget.

4.2.2 Optimal Portfolio Problem

We consider an ambiguous agent with constant relative risk aversion (CRRA) utility:

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \tag{4.12}$$

where the parameter controlling the level of risk aversion is γ. The CRRA utility not only gurantees a closed-form solution to the next HJBI PDE, but also ensures that a proportional change of wealth, has an idential effect on the investor regardless of initial amount. The investor wants to maximize the expected utility from the terminal wealth under worst case conditions parameterized by its level of ambiguity-aversion (ϕ), the objective function in this optimal problem is (see [Maenhout (2004)]):

$$J(t,x,r,z) = \sup_{\pi\in\Pi}\inf_{\boldsymbol{u}} \mathbb{E}^{Q^{\boldsymbol{u}}}_{x,z,r}\left[\int_t^T \frac{1}{2}\left(\frac{u_1^2(s)}{\phi_1(s)} + \frac{u_2^2(s)}{\phi_2(s)}\right)\mathrm{d}s + U(X(T))\middle|\mathcal{F}_t\right], \tag{4.13}$$

where $\mathbb{E}^{Q}_{x,z,r}[U(X(T))|\mathcal{F}_t] := \mathbb{E}^{Q}[U(X(T))|\mathcal{F}_t, X(t)=x, Z(t)=z, r(t)=r]$, Π is the set of all admissible strategies, $(\phi_1(t),\phi_2(t))$ are preference parameters for ambiguity-aversion, and $Q^{\boldsymbol{u}}$ is defined in Equation (4.14). By setting $\boldsymbol{u}=(u_1,u_2)^T$, $\mathrm{d}\boldsymbol{W}(t) = (\mathrm{d}W_1(t), \mathrm{d}W_2(t))^T$, and letting $\mathbb{Q}$ represent the set of admissible probability measures by Girsanov Theorem, we can mathematically define the Radon-Nikodym derivative as follows:

$$\begin{aligned}\mathbb{Q} = \Bigg\{Q^{\boldsymbol{u}}\colon \frac{\mathrm{d}Q^{\boldsymbol{u}}}{\mathrm{d}P}\bigg|_{\mathcal{F}_s} &= \exp\left(-\int_0^s u_1(t)\,\mathrm{d}W_1(t) - \frac{1}{2}\int_0^s u_1^2(t)\,\mathrm{d}t\right.\\ &\qquad \left.-\int_0^s u_2(t)\,\mathrm{d}W_2(t) - \frac{1}{2}\int_0^s u_2^2(t)\,\mathrm{d}t\right) \\ &= \exp\left(-\int_0^s \boldsymbol{u}^T(t)\,\mathrm{d}\boldsymbol{W}(t) - \int_0^s \boldsymbol{u}^T(t)\boldsymbol{u}(t)\,\mathrm{d}t\right)\Bigg\}.\end{aligned} \tag{4.14}$$

This means we consider changes of measure of the form:

$$\begin{cases}\mathrm{d}W_1(t) = \mathrm{d}W_1^Q(t) - u_1(t)\,\mathrm{d}t\\ \mathrm{d}W_2(t) = \mathrm{d}W_2^Q(t) - u_2(t)\,\mathrm{d}t.\end{cases} \tag{4.15}$$

Hence the dynamics under the new measures, also known as the *alternative models*, are:

$$\begin{cases}\mathrm{d}r(t) = [\kappa(\bar{r}-r(t)) - \sigma_r(\lambda_r + u_1(t))]\,\mathrm{d}t + \sigma_r\,\mathrm{d}W_1^Q(t)\\ \mathrm{d}Z(t) = \Big(r(t) + \lambda_S\sigma_S - \frac{1}{2}\sigma_S^2 - aZ(t) - \sigma_S\Big(\rho u_1(t) + \sqrt{1-\rho^2}u_2(t)\Big)\Big)\,\mathrm{d}t\\ \qquad\qquad +\sigma_S\Big(\rho\,\mathrm{d}W_1^Q(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_2^Q(t)\Big)\\ \mathrm{d}X(t) = \Big[(c+r(t))X(t^-) + \pi_S(t)\Big(\lambda_S\sigma_S - aZ(t) - \sigma_S\Big(\rho u_1(t) + \sqrt{1-\rho^2}u_2(t)\Big)\Big)\\ \qquad\qquad +(\lambda_r + u_1(t))\pi_P(t)I_\tau\sigma_r\Big]\,\mathrm{d}t + \pi_S(t)\sigma_S\Big(\rho\,\mathrm{d}W_1^Q(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_2^Q(t)\Big)\\ \qquad\qquad -\pi_P(t)I_\tau\sigma_r\,\mathrm{d}W_1^Q(t) - X(t^-)\int_{\mathbb{R}^+} yN(\mathrm{d}t,\mathrm{d}y).\end{cases} \tag{4.16}$$

Set $\boldsymbol{y}(t) = (r(t), z(t))^T$, $\mathrm{d}\boldsymbol{W}^Q(t) = (\mathrm{d}W_1^Q(t), \mathrm{d}W_2^Q(t))^T$ and $\boldsymbol{\pi} = (\pi_P, \pi_S)^T$. We can represent the dynamics compactly as:

$$\begin{cases}\mathrm{d}\boldsymbol{y}(t) = [(\boldsymbol{\theta} - \boldsymbol{A}\boldsymbol{y}(t)) - \boldsymbol{\sigma}\boldsymbol{u}]\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{W}^Q(t)\\ \mathrm{d}X(t) = [(c+r(t))X(t^-) + \boldsymbol{\pi}^T\boldsymbol{B}\boldsymbol{b}(\boldsymbol{y}(t)) - \boldsymbol{\sigma}\boldsymbol{u})]\,\mathrm{d}t + \boldsymbol{\pi}^T\boldsymbol{B}\boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{W}^Q(t) - X(t^-)\int_{\mathbb{R}^+} yN(\mathrm{d}t,\mathrm{d}y).\end{cases} \tag{4.17}$$

where vectors and matrices are

$$\boldsymbol{A} = \begin{bmatrix} \kappa & 0 \\ -1 & a \end{bmatrix}, \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_r & 0 \\ \rho\sigma_S & \sqrt{1-\rho^2}\sigma_S \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -I_\tau & 0 \\ 0 & 1 \end{bmatrix},$$
$$\boldsymbol{\theta} = \begin{bmatrix} \kappa\bar{r} - \sigma_r\lambda_r \\ \lambda_S\sigma_S - \frac{1}{2}\sigma_S^2 \end{bmatrix}, \quad \boldsymbol{b}(\boldsymbol{y}(t)) = \begin{bmatrix} -\lambda_r\sigma_r \\ \lambda_S\sigma_S - aZ(t) \end{bmatrix}. \tag{4.18}$$

Then the HJBI equation is:

$$\begin{aligned}
\sup_{\boldsymbol{\pi}\in\Pi}\inf_{\boldsymbol{u}}\Big\{&[\kappa(\bar{r}-r) - (\lambda_r + u_1)\sigma_r]J_r + \Big[\Big(r + \lambda_S\sigma_S - \frac{1}{2}\sigma_S^2 - az\Big) - \sigma_S\Big(\rho u_1(t) + \sqrt{1-\rho^2}u_2(t)\Big)\Big]J_z \\
&+ \Big[(c+r)x + \pi_S\Big(\lambda_S\sigma_S - az - \sigma_S\Big(\rho u_1(t) + \sqrt{1-\rho^2}u_2(t)\Big)\Big) + (\lambda_r + u_1)\pi_P I_\tau\sigma_r\Big]J_x \\
&+ \frac{1}{2}\sigma_r^2 J_{rr} + \frac{1}{2}\sigma_S^2 J_{zz} + (\rho\pi_S\sigma_S\sigma_r - \pi_P I_\tau\sigma_r^2)J_{rx} + \rho\sigma_S\sigma_r J_{zr} \\
&+ (\pi_S\sigma_S^2 - \pi_P I_\tau\rho\sigma_S\sigma_r)J_{zx} + \frac{1}{2}\Big((\pi_S\sigma_S\rho - \pi_P I_\tau\sigma_r)^2 + \pi_S^2\sigma_S^2(1-\rho^2)\Big)J_{xx} \\
&+ \frac{u_1^2}{2\phi_1} + \frac{u_2^2}{2\phi_2}\Big\} + \mu\mathbb{E}[J(t, x(1-Y), r, z) - J] + J_t = 0,
\end{aligned} \tag{4.19}$$

where Y is a r.v. with the same distribution as Y_i.

For analytical tractability, [Maenhout (2004)] provides suitable forms for $\phi_1(t)$ and $\phi_2(t)$ are

$$\phi_1(t) = \frac{\beta_1}{(1-\gamma)J(t,x,r,z)} > 0, \quad \phi_2(t) = \frac{\beta_2}{(1-\gamma)J(t,x,r,z)} > 0, \tag{4.20}$$

where β_1 and β_2 are the ambiguity-aversion parameters. β_1 can be interpreted as ambiguity aversion about the interest rate distribution, while β_2 is ambiguity aversion on the distribution of the commodity return. Let $\boldsymbol{\beta} = \mathrm{diag}(\beta_1, \beta_2)$ and $\Sigma = \boldsymbol{\sigma}\boldsymbol{\sigma}^T$. In matrix form, the HJBI equation becomes:

$$\begin{aligned}
\sup_{\boldsymbol{\pi}\in\Pi}\inf_{\boldsymbol{u}}\Big\{&J_t + [(\boldsymbol{\theta} - \boldsymbol{A}\boldsymbol{y}) - \boldsymbol{\sigma}\boldsymbol{u}]^T J_{\boldsymbol{y}} + [(c+r)x + \boldsymbol{\pi}^T\boldsymbol{B}(\boldsymbol{b}(\boldsymbol{y}) - \boldsymbol{\sigma}\boldsymbol{u})]J_x + \frac{1}{2}\mathrm{tr}(J_{\boldsymbol{y}\boldsymbol{y}^T}\Sigma) \\
&+ \boldsymbol{\pi}^T\boldsymbol{B}\Sigma J_{x\boldsymbol{y}} + \frac{(1-\gamma)J}{2}\boldsymbol{u}^T\boldsymbol{\beta}^{-1}\boldsymbol{u} + \frac{J_{xx}}{2}\boldsymbol{\pi}^T\boldsymbol{B}^T\Sigma\boldsymbol{B}\boldsymbol{\pi}\Big\} \\
&+ \mu\mathbb{E}[J(t, x(1-Y), r, z) - J] = 0.
\end{aligned} \tag{4.21}$$

The proposition next exhibit the implicit solution to the first order conditions.

Proposition 4.2.1. *The optimal change of measure is*

$$\boldsymbol{u}^* = \frac{\boldsymbol{\beta}\boldsymbol{\sigma}^T(J_{\boldsymbol{y}} + J_x\boldsymbol{B}\boldsymbol{\pi})}{(1-\gamma)J}. \tag{4.22}$$

The corresponding optimal investment strategy is

$$\boldsymbol{\pi}^* = -\boldsymbol{B}^{-1}\left(J_{xx}\Sigma - \frac{J_x^2}{(1-\gamma)J}\boldsymbol{\sigma\beta\sigma}^T\right)^{-1}\left(J_x\left(\boldsymbol{b}(\boldsymbol{y}) - \frac{\boldsymbol{\sigma\beta\sigma}^T J_{\boldsymbol{y}}}{(1-\gamma)J}\right) + \Sigma J_{x\boldsymbol{y}}\right). \tag{4.23}$$

The HJBI PDE becomes

$$\begin{aligned} &J_t + (\boldsymbol{\theta} - \boldsymbol{A}\boldsymbol{y})^T J_{\boldsymbol{y}} + \frac{1}{2}\operatorname{tr}(J_{\boldsymbol{y}\boldsymbol{y}^T}\Sigma) + (c + \boldsymbol{y}^T\boldsymbol{e}_1)xJ_x - \frac{J_{\boldsymbol{y}}^T\boldsymbol{\sigma\beta\sigma}^T J_{\boldsymbol{y}}}{2(1-\gamma)J} \\ &- \frac{1}{2}\left(J_x\left(\boldsymbol{\lambda} - a\boldsymbol{E}_2\boldsymbol{y} - \frac{\boldsymbol{\sigma\beta\sigma}^T J_{\boldsymbol{y}}}{(1-\gamma)J}\right) + \Sigma J_{x\boldsymbol{y}}\right)^T\left(J_{xx}\Sigma - \frac{J_x^2}{(1-\gamma)J}\boldsymbol{\sigma\beta\sigma}^T\right)^{-1} \\ &\times\left(J_x\left(\boldsymbol{\lambda} - a\boldsymbol{E}_2\boldsymbol{y} - \frac{\boldsymbol{\sigma\beta\sigma}^T J_{\boldsymbol{y}}}{(1-\gamma)J}\right) + \Sigma J_{x\boldsymbol{y}}\right) + \mu\mathbb{E}[J(t, x(1-Y), r, z) - J] = 0. \end{aligned} \tag{4.24}$$

where $\boldsymbol{I}$ *is* 2×2 *identity matrix,* $\boldsymbol{e}_1 = (1,0)^T$, $\boldsymbol{E}_2 = diag(0,1)$ *and* $\boldsymbol{\lambda} = (-\lambda_r\sigma_r, \lambda_S\sigma_S)^T$ *so* $\boldsymbol{b}(\boldsymbol{y}) = \boldsymbol{\lambda} - a\boldsymbol{E}_2\boldsymbol{y}$.

Proof. The Equation (4.19) is quadratic w.r.t. $\boldsymbol{u}$. Therefore, it is easy to obtain the optimal change of measure $\boldsymbol{u}^*$ by computing first order derivative of $\boldsymbol{u}$ and set it be $\boldsymbol{0}$. Substituting $\boldsymbol{u}^*$ into the HJB Equation (4.21), we can also compute the optimal trading strategy for $\boldsymbol{\pi}$ by setting the derivative w.r.t. $\boldsymbol{\pi}$ of RHS to be 0, because Equation (4.21) is also quadratic w.r.t. $\boldsymbol{\pi}$. We present the straightforward but laborious computational detail in Appendix C.1 using a matrix representation. Second order conditions ensure the solutions are indeed a minimum and maximum respectively. □

Before simplifying and solving the HJBI, we present Lemma 4.2.1, which gives a simple way to compute the explicit solution to a Matrix Riccati differential equation (RDE) encountered in the representation of the solution ($\boldsymbol{M}_2$ in Proposition 2.2).

Lemma 4.2.1. *Assume an* $n\times n$ *Matrix RDE* $R(t,T)$ *satisfying*

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} &= \boldsymbol{RBR} + \boldsymbol{RA} + \boldsymbol{A}^T\boldsymbol{R} + \boldsymbol{Q} \\ \boldsymbol{R}(T) &= \boldsymbol{S}, \end{aligned} \tag{4.25}$$

where $\boldsymbol{B}$, $\boldsymbol{Q}$ *and* $\boldsymbol{S}$ *are symmetric and non-negative definite. The solution to the RDE will be*

$$\boldsymbol{R} = \boldsymbol{K}_2\boldsymbol{K}_1^{-1}, \tag{4.26}$$

where $n\times n$ *matrices* $\boldsymbol{K}_1, \boldsymbol{K}_2$ *are defined by*

$$\boldsymbol{K}(t,T) = \exp\left(\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ -\boldsymbol{Q} & -\boldsymbol{A}^T \end{bmatrix}(T-t)\right)\begin{bmatrix} \boldsymbol{I}_{n\times n} \\ \boldsymbol{S} \end{bmatrix} := \begin{bmatrix} \boldsymbol{K}_1 \\ \boldsymbol{K}_2 \end{bmatrix}. \tag{4.27}$$

Proof. See [Abou Kandil et al. (2003)]. □

Now given the PDE in Proposition 4.2.1, we are ready to solve for $J(t,x,z,r)$ and compute explicitly $\boldsymbol{u}^*$ and $\boldsymbol{\pi}^*$. This is provided in the next Proposition.

Proposition 4.2.2. *The optimal value function has the representation:*

$$J(t,x,r,z) = \frac{x^{1-\gamma}}{1-\gamma}\exp\Big(M_0(t) + \boldsymbol{y}^T\boldsymbol{M}_1(t) + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{M}_2(t)\boldsymbol{y}\Big), \tag{4.28}$$

where $\boldsymbol{M}_1(t)$ is a 2×1 vector and $\boldsymbol{M}_2$ is a 2×2 symmetric matrix with terminal condition $J(T,x,r,z) = \frac{x^{1-\gamma}}{1-\gamma}$. Here the matrices $M_0(t)$, $\boldsymbol{M}_1(t)$ and $\boldsymbol{M}_2(t)$ follow Matrix Riccati ODEs:

$$\begin{cases} \boldsymbol{M}_2' + \boldsymbol{D}_0 + \boldsymbol{D}_1\boldsymbol{M}_2 + \boldsymbol{M}_2\boldsymbol{D}_1^T + \boldsymbol{M}_2\boldsymbol{D}_2\boldsymbol{M}_2 = 0 \\ \boldsymbol{M}_2(T) = \boldsymbol{0}_{2\times 2}, \end{cases} \tag{4.29}$$

$$\begin{cases} \boldsymbol{M}_1' + \boldsymbol{C}_0 + \boldsymbol{M}_2(t)\boldsymbol{C}_1 + (\boldsymbol{M}_2(t)\boldsymbol{D}_2 + \boldsymbol{D}_1)\boldsymbol{M}_1 = 0 \\ \boldsymbol{M}_1(T) = \boldsymbol{0}_{2\times 1}, \end{cases} \tag{4.30}$$

$$\begin{cases} M_0' + B_0 + \boldsymbol{M}_1^T(t)\boldsymbol{C}_1 + \frac{1}{2}\boldsymbol{M}_1^T(t)\boldsymbol{D}_2\boldsymbol{M}_1(t) = 0 \\ M_0(T) = 0, \end{cases} \tag{4.31}$$

where matrices $\boldsymbol{D}_0$, $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, vectors $\boldsymbol{C}_0$, $\boldsymbol{C}_1$ and scalar B_0 are given in Appendix C.2.

After obtaining the solution $\boldsymbol{M}_2(t)$ in Proposition 4.2.2 using Lemma 4.2.1, we can simply obtain $\boldsymbol{M}_1(t)$ and $M_0(t)$ sequentially by computing numerical solution to the corresponding ODEs. Now we are ready to compute the optimal strategies and changes of measure.

Theorem 4.2.2. *The optimal trading strategy and change of measure are*

$$\begin{aligned} \boldsymbol{\pi}^* &= x\boldsymbol{B}^{-1}\Big(\boldsymbol{\sigma}(\gamma\boldsymbol{I} + \boldsymbol{\beta})\boldsymbol{\sigma}^T\Big)^{-1}\Big[\boldsymbol{\lambda} - a\boldsymbol{E}_2\boldsymbol{y} + \boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T(\boldsymbol{M}_1(t) + \boldsymbol{M}_2(t)\boldsymbol{y})\Big] \\ \boldsymbol{u}^* &= \frac{\boldsymbol{\beta}\boldsymbol{\sigma}^T}{1-\gamma}\Big((\boldsymbol{M}_1(t) + \boldsymbol{M}_2(t)\boldsymbol{y}) + \frac{1-\gamma}{x}\boldsymbol{B}\boldsymbol{\pi}^*\Big). \end{aligned} \tag{4.32}$$

Proof. Plugging the ansatz in Equation (4.28) into $\boldsymbol{\pi}^*$ and $\boldsymbol{u}^*$ in Proposition 4.2.1 can easily derive the equations. The results with plugged $\boldsymbol{\pi}^*$ are shown in Appendix C.3. □

In fact, we can rewrite the trading strategy as follows:

$$\frac{\boldsymbol{\pi}^*}{x} = \boldsymbol{B}^{-1}\Big(\boldsymbol{\sigma}(\gamma \boldsymbol{I} + \boldsymbol{\beta})\boldsymbol{\sigma}^T\Big)^{-1}\Bigg[\boldsymbol{\lambda} + \boldsymbol{\sigma}\bigg(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\bigg)\boldsymbol{\sigma}^T \boldsymbol{M}_1(t) + \bigg(\boldsymbol{\sigma}\bigg(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\bigg)\boldsymbol{\sigma}^T \boldsymbol{M}_2(t) - a\boldsymbol{E}_2\bigg)\boldsymbol{y}\Bigg]. \tag{4.33}$$

This demonstrates that the investment fraction $\frac{\boldsymbol{\pi}^*}{x}$, and $\boldsymbol{u}^*$ are linear w.r.t. $\boldsymbol{y}$. Particularly, the part

$$\boldsymbol{B}^{-1}\Big(\boldsymbol{\sigma}(\gamma \boldsymbol{I} + \boldsymbol{\beta})\boldsymbol{\sigma}^T\Big)^{-1}\boldsymbol{\lambda}, \tag{4.34}$$

represents the standard mean-variance portfolio. Moreover, compatible with [Flor and Larsen (2013)], when $a = 0$, i.e., if there is no mean-reverting effect on the asset, the optimal investment strategy will be:

$$\frac{1}{x}\boldsymbol{\pi}^*_{a=0} = \boldsymbol{B}^{-1}\Big(\boldsymbol{\sigma}(\gamma \boldsymbol{I} + \boldsymbol{\beta})\boldsymbol{\sigma}^T\Big)^{-1}\Big(\boldsymbol{\lambda} - I_\tau \sigma_r^2[(1-\gamma) - \beta_1]\boldsymbol{e}_1\Big), \tag{4.35}$$

where the second component is from Vasicek stochastic interest rate, and the third component is from ambiguity. Therefore, the value $\frac{1}{x}(\boldsymbol{\pi}^* - \boldsymbol{\pi}^*_{a=0})$ captures the effect of the mean-reverting term.

Proposition 4.2.2 provides a candidate for the solution $J(t, x, r, z)$ to the HJBI in Equation (4.21), this solution is expressed in terms of $M_0(t)$, $\boldsymbol{M}_1(t)$ and $\boldsymbol{M}_2(t)$. In order to verify the candidate is well-defined, we must first guarantee that the change of measure determined in the worst-case scenario (the inf part) denoted $\boldsymbol{u}^*$ satisfies the Novikov's condition. In a second step we use existing results for a verification theorem on the sup part.

Theorem 4.2.3. *If Frobenius norm $\|\boldsymbol{M}_2(t)\|_F < \infty$ for $0 \le t \le T$, i.e., $\boldsymbol{M}_2$ is a well-defined matrix function, then the Novikov's condition holds for $\boldsymbol{u}^*(t)$. Moreover $\boldsymbol{\pi}^*(t)$ is the optimal strategy in the worst-case scenario $\boldsymbol{u}^*(t)$.*

Proof. See Appendix C.3. □

4.3 Wealth-equivalent Losses Analysis

In this section we quantify the wealth-equivalent utility loss the investor suffers by following meaningful suboptimal strategies. In particular, we study the effects of ignoring model uncertainty [*], the impact of ambiguity on the utility loss incurred from not investing in bonds (market incompleteness), and the impact of wrongly selecting specific parameters of the model (due to estimation error for instance). Similar to [Flor and Larsen (2013)], we measure the utility loss in terms of the percentage of wealth lost when using the suboptimal strategies.

First, we need to find a representation for the value function given a suboptimal trading strategy $\boldsymbol{\pi}^s$, this is:

$$J^{\boldsymbol{\pi}^s}(t,x,r,z) = \inf_{Q\in\mathbb{Q}} \mathbb{E}^Q_{x,r,z}\left[\int_t^T \frac{1}{2}\left(\frac{u_1^2(s)}{\phi_1(s)} + \frac{u_2^2(s)}{\phi_2(s)}\right)\mathrm{d}s + U(X(T);\boldsymbol{\pi}^s)\middle|\mathcal{F}_t\right]. \tag{4.36}$$

The wealth-equivalent loss of optimal strategy is denoted $L^{\boldsymbol{\pi}^s}$ and it is the solution to the equation:

$$J(t,x(1-L^{\boldsymbol{\pi}^s}),r,z) = J^{\boldsymbol{\pi}^s}(t,x,r,z). \tag{4.37}$$

Given the strategy $\boldsymbol{\pi}^s$, the HJBI equation is

$$\begin{aligned}\inf_{\boldsymbol{u}}\Big\{ & J_t + [(\boldsymbol{\theta}-\boldsymbol{A}\boldsymbol{y})-\boldsymbol{\sigma}\boldsymbol{u}]^T J_{\boldsymbol{y}} + [(c+\boldsymbol{y}^T\boldsymbol{e}_1)x + (\boldsymbol{\pi}^s)^T\boldsymbol{B}(\boldsymbol{\lambda}-a\boldsymbol{E}_2\boldsymbol{y}-\boldsymbol{\sigma}\boldsymbol{u})]J_x \\ & + \frac{1}{2}\operatorname{tr}(J_{\boldsymbol{y}\boldsymbol{y}^T}\Sigma) + (\boldsymbol{\pi}^s)^T\boldsymbol{B}\Sigma J_{x\boldsymbol{y}} + \frac{(1-\gamma)J}{2}\boldsymbol{u}^T\boldsymbol{\beta}^{-1}\boldsymbol{u} + \frac{J_{xx}}{2}(\boldsymbol{\pi}^s)^T\boldsymbol{B}^T\Sigma\boldsymbol{B}\boldsymbol{\pi}^s\Big\} \\ & + \mu\mathbb{E}[J(t,x(1-Y),r,z)-J] = 0.\end{aligned} \tag{4.38}$$

The proposition next provides a representation up to Riccati equations of the value function for the suboptimal problem.

Proposition 4.3.1. *Assume the suboptimal strategy is of the form $\boldsymbol{\pi}^s = x(\boldsymbol{h}(t)+\boldsymbol{H}(t)\boldsymbol{y})$, i.e., proportional to x and linear w.r.t. $\boldsymbol{y}$, then Equation (4.38) also has an exponential affine form:*

$$J^{\boldsymbol{\pi}^s}(t,x,r,z) = \frac{x^{1-\gamma}}{1-\gamma}\exp\Big(M_0^{\boldsymbol{\pi}^s}(t) + \boldsymbol{y}^T\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t) + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)\boldsymbol{y}\Big). \tag{4.39}$$

The differential equations for $M_0^{\boldsymbol{\pi}^s}(t)$, $\boldsymbol{M}_1^{\boldsymbol{\pi}^s}$ and $\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)$ are given below:

$$\begin{cases}(\boldsymbol{M}_2^{\boldsymbol{\pi}^s})' - \boldsymbol{H}^T(t)\boldsymbol{D}_0^s\boldsymbol{H}(t) - \boldsymbol{D}_3^s + \boldsymbol{D}_1^s\boldsymbol{M}_2 + \boldsymbol{M}_2(\boldsymbol{D}_1^s)^T + \boldsymbol{M}_2^{\boldsymbol{\pi}^s}\boldsymbol{D}_2^s\boldsymbol{M}_2^{\boldsymbol{\pi}^s} = 0 \\ \boldsymbol{M}_2^{\boldsymbol{\pi}^s}(T) = \boldsymbol{0}_{2\times 2}\end{cases} \tag{4.40}$$

[*] This is either investors ignoring their own level of uncertainty due to, for example, technical limitations, or ignoring market analyst recommendations about documented levels of ambiguity on the distributions of the assets at hand

$$\begin{cases}(\boldsymbol{M}_1^{\boldsymbol{\pi}^s})' + \boldsymbol{C}_0 + \boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)\boldsymbol{C}_1 + (\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)\boldsymbol{D}_2^s + \boldsymbol{D}_1^s)\boldsymbol{M}_1^{\boldsymbol{\pi}^s} + \boldsymbol{C}_0^s \\ \qquad +[(\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t))^T\boldsymbol{D}_2^s - a\boldsymbol{E}_2]\boldsymbol{B}\boldsymbol{h}(t)(1-\gamma) - (\boldsymbol{H}(t))^T\boldsymbol{D}_0^s\boldsymbol{h}(t) = 0 \\ \boldsymbol{M}_1^{\boldsymbol{\pi}^s}(T) = \boldsymbol{0}_{2\times 1}.\end{cases} \tag{4.41}$$

$$\begin{cases}(M_0^{\boldsymbol{\pi}^s})' + B_0^s + [(\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t))^T\boldsymbol{D}_2^s + \boldsymbol{\lambda}^T]\boldsymbol{B}\boldsymbol{h}(t)(1-\gamma) + \frac{1}{2}(\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t))^T\boldsymbol{D}_2\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t) \\ \qquad -\frac{1}{2}(\boldsymbol{h}(t))^T\boldsymbol{D}_0^s\boldsymbol{h}(t) = 0 \\ M_0^{\boldsymbol{\pi}^s}(T) = 0.\end{cases} \tag{4.42}$$

where matrices $\boldsymbol{D}_0^s$, $\boldsymbol{D}_1^s$, $\boldsymbol{D}_2^s$, $\boldsymbol{D}_3^s$, *vectors* $\boldsymbol{C}_0^s$ *and scalar* B_0^s *are given in Appendix C.4.*

In particular, if $\boldsymbol{\pi}^s = x\boldsymbol{h}(t)$, *the ansatz will exclude the quadratic term* $\boldsymbol{y}^T\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)\boldsymbol{y}$ *and become*

$$J^{\boldsymbol{\pi}^s}(t,x,r,z) = \frac{x^{1-\gamma}}{1-\gamma}\exp\Big(M_0^{\boldsymbol{\pi}^s}(t) + \boldsymbol{y}^T\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t)\Big). \tag{4.43}$$

Hence we only need to focus on $M_0^{\boldsymbol{\pi}^s}$ *and* $\boldsymbol{M}_1^{\boldsymbol{\pi}^s}$ *in this case. Moreover Lemma 4.2.1 can lead to the solution for* $\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)$.

Proof. See Appendix C.4. □

Proposition 4.3.1 gives us the wealth-equivalent losses in the form:

$$L^{\boldsymbol{\pi}^s} = 1-\exp\left(\frac{1}{1-\gamma}\Big((M_0^{\boldsymbol{\pi}^s}(t) - M_0(t)) + \boldsymbol{y}^T(\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t) - \boldsymbol{M}_1(t)) + \frac{1}{2}\boldsymbol{y}^T(\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t) - \boldsymbol{M}_2(t))\boldsymbol{y}\Big)\right). \tag{4.44}$$

Plugging a suboptimal parametric set denoted by hat "$(\hat{a}, \hat{\boldsymbol{\sigma}}, \dots)$" into the ODEs of $M_0(t)$, $\boldsymbol{M}_1(t)$ and $\boldsymbol{M}_2(t)$ in Equation (D.8) allow us to compute $M_0^{\boldsymbol{\pi}^s}(t)$, $\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t)$ and $\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)$ hence we can obtain the trading strategy $\boldsymbol{\pi}^s(t)$ from Theorem 4.2.2.

In other words, with $M_0^{\boldsymbol{\pi}^s}(t)$, $\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t)$ and $\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)$, the functions $\boldsymbol{h}(t)$ and $\boldsymbol{H}(t)$ of the suboptimal strategy are

$$\begin{aligned}\boldsymbol{h}(t) &= \hat{\boldsymbol{B}}^{-1}\Big(\hat{\boldsymbol{\sigma}}(\hat{\gamma}\boldsymbol{I} + \hat{\boldsymbol{\beta}})\hat{\boldsymbol{\sigma}}^T\Big)^{-1}\left(\hat{\boldsymbol{\lambda}} + \hat{\boldsymbol{\sigma}}\left(\boldsymbol{I} - \frac{\hat{\boldsymbol{\beta}}}{1-\hat{\gamma}}\right)\hat{\boldsymbol{\sigma}}^T\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t)\right) \\ \boldsymbol{H}(t) &= \hat{\boldsymbol{B}}^{-1}\Big(\hat{\boldsymbol{\sigma}}(\hat{\gamma}\boldsymbol{I} + \hat{\boldsymbol{\beta}})\hat{\boldsymbol{\sigma}}^T\Big)^{-1}\left(\hat{\boldsymbol{\sigma}}\left(\boldsymbol{I} - \frac{\hat{\boldsymbol{\beta}}}{1-\hat{\gamma}}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t) - \hat{a}\boldsymbol{E}_2\right).\end{aligned} \tag{4.45}$$

where $\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t)$, $\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)$ are solutions presented in Proposition 4.2.2 with parametric set denoted generically by $\hat{\theta}$.

We are interested in three families of suboptimal strategies, which will be studied in detail in the upcoming sections:

1. First, strategies from ambguity averse investors who choose to *ignore ambiguity* on either the distribution of the commodity $S(t)$, the interest rate $r(t)$ or both. This means three cases, $\hat{\beta}_1 = 0$, $\hat{\beta}_2 = 0$, or $\hat{\beta}_1 = \hat{\beta}_2 = 0$. This will allow us to measure the total impact of ignoring ambiguity and which source of ambiguity is more harmful. Given $\boldsymbol{\pi}^s(t)$ from Theorem 4.2.2, we can apply Proposition 4.3.1 to obtain $M_0^s(t)$, $\boldsymbol{M}_1^s(t)$ and $\boldsymbol{M}_2^s(t)$, then producing $L^{\boldsymbol{\pi}^s}$.

2. The second type of suboptimal strategies is derived by considering investor who fail to allocate to bonds as a way of hedging the interest rate risk. This is a case of incomplete markets. Here we need to find the optimal allocation on commodity given the constraint $\pi_P = 0$. This can be achieved by equivalently substituting $\boldsymbol{E}_2\boldsymbol{\pi}$ for $\boldsymbol{\pi}$ in Equation (4.21) to force $\pi_P = 0$. Using the notation derives this results leading to:

$$\begin{aligned} \boldsymbol{u}^s &= \frac{\boldsymbol{\beta}\boldsymbol{\sigma}^T(J_{\boldsymbol{y}} + J_x\boldsymbol{E}_2\boldsymbol{\pi})}{(1-\gamma)J} \\ \boldsymbol{\pi}^s &= -\boldsymbol{E}_2\boldsymbol{P}^{-1}\left(J_x\left(\boldsymbol{b} - \frac{\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T J_{\boldsymbol{y}}}{(1-\gamma)J}\right) + \Sigma J_{x\boldsymbol{y}}\right) \end{aligned} \tag{4.46}$$

   where $\boldsymbol{P}$ is the diagnoal matrix with the diagonal elements below,

$$\operatorname{diag}\left(J_{xx}\Sigma - \frac{J_x^2}{(1-\gamma)J}\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T\right) \tag{4.47}$$

   Plugging $\boldsymbol{u}^s$ and $\boldsymbol{\pi}^s$ produces a new HJBI PDE. The solution $J^{\boldsymbol{\pi}^s}(t, x, r, z)$ and loss $L^{\boldsymbol{\pi}^s}$ can be generated by taking similar steps.

3. The third and final group of suboptimal strategies arises from *wrong estimation* of the reference parameters. Some parameters in our model are either very difficult to estimate due to lack of data, like market price of rate risk $\hat{\lambda}_r$ and commodity $\hat{\lambda}_S$, or are not stable due to mis-measurement and inaccuracies in the chosen values, examples of these are volatilities $\hat{\sigma}_r$, $\hat{\sigma}_S$ and the correlation ρ. This wrong values affects the identification of the parameters in the *reference model*, therefore leading to wrongly crafted strategies.
   This direction is conceptually different to point 1 above as it escapes the worst case analyses of the ambiguity aversion framework. To see this note two aspects, first we

work in a setting of equivalent changes of measures to accommodate the worst case preferences of ambiguous investor, this does not capture concerns about covariance mis-specifications (see [Fouque et al. (2016)] for non-equivalent ambiguity framework). Second, by changing the estimates to different values, we effectively work with a different *reference model*, i.e. coming from wrong estimations, hence affecting the level of ambiguity of the *alternative models*.

4.4 Empirical Analysis

This section is divided into three part. First we provide details about the data from commodities and interest rates as well as the estimation approach. Then we report the optimal solutions, i.e. allocation, value and terminal surplus, for the representative commodity markets. The last section studies the impact of various suboptimal strategies.

4.4.1 Data and Estimation Methodology.

We target two commodities: WTI oil prices and gold prices, which can be seen as representative of this asset class. In particular, we work with weekly WTI oil prices extracted from https://fred.stlouisfed.org/series/DCOILWTICO, as well as gold prices from https://fred.stlouisfed.org/series/GOLDAMGBD228NLBM. For bond prices and interest rate calibration we use data from the Federal Reserve Banks of St.louis (FRED) †. We collect weekly US 1-month Treasury Bill rate from https://fred.stlouisfed.org/series/DGS1MO as short rates. For calibration of market price of risk of bond, we also collected the monthly 10-year govt bond yield from https://fred.stlouisfed.org/series/IRLTLT01USM156N. In all cases we choose the period from Aug 2001 to Sep 2019.

For estimation purposes, we first discretize our model, assuming a time step h with $r_i = r(t_i)$. From dynamic processes in Equation (4.9) in real-world measure, using explicit solution to OU process, we can obtain:

$$r_{i+1} = r_i \mathrm{e}^{-\kappa h} + \bar{r}^{\mathbb{P}}(1 - \mathrm{e}^{-\kappa h}) + \sigma_r^{(1)} \varepsilon_{i+1}^{(1)} \tag{4.48}$$

where for simplicity, we define the real-world interest rate be $r^{\mathbb{P}} := \bar{r} - \frac{\lambda_r \sigma_r}{\kappa}$. Both κ and $r^{\mathbb{P}}$

†see [Cooke and Gavin (2015)] and literature therein.

can be computed via a linear regression analysis. The variance would be:

$$\left(\sigma_r^{(1)}\right)^2 = \mathrm{var}\left(\sigma_r \int_0^h \mathrm{e}^{-\kappa(h-s)}\,\mathrm{d}W_1(t_i+s)\right) = \frac{1-\mathrm{e}^{-2\kappa h}}{2\kappa}\sigma_r^2. \tag{4.49}$$

Using $r(t_i)$, we can write a discretization of $Z(t)$ as follows:

$$Z_{i+1} = \mathrm{e}^{-ah}Z_i + (r_i - \bar{r}^{\mathbb{P}})\frac{\mathrm{e}^{-\kappa h} - \mathrm{e}^{-ah}}{a-\kappa} + \left(\bar{r}^{\mathbb{P}} + \lambda_S\sigma_S - \frac{1}{2}\sigma_S^2\right)\frac{1-\mathrm{e}^{-ah}}{a} + \sigma_Z^{(1)}\varepsilon_{i+1}^{(2)} \tag{4.50}$$

where the variance

$$\begin{aligned}\left(\sigma_Z^{(1)}\right)^2 =& \mathrm{var}\left(\int_0^h \sigma_r\frac{\mathrm{e}^{-\kappa(h-s)} - \mathrm{e}^{-a(h-s)}}{a-\kappa} + \sigma_S\rho\mathrm{e}^{-a(h-s)}\,\mathrm{d}W_1(t_i+s)\right.\\ &\left. + \int_0^h \sigma_S\sqrt{1-\rho^2}\mathrm{e}^{-a(h-s)}W_2(t_i+s)\right)\\ =&\frac{\sigma_r^2}{(a-\kappa)^2}\left(\frac{1-\mathrm{e}^{-2\kappa h}}{2\kappa} - \frac{2(1-\mathrm{e}^{-(\kappa+a)h})}{\kappa+a} + \frac{1-\mathrm{e}^{-2ah}}{2a}\right)\\ &+\frac{2\rho\sigma_r\sigma_S}{a-\kappa}\left(\frac{1-\mathrm{e}^{-(\kappa+a)h}}{\kappa+a} - \frac{1-\mathrm{e}^{-2ah}}{2a}\right) + \sigma_S^2\frac{1-\mathrm{e}^{-2ah}}{2a},\end{aligned} \tag{4.51}$$

and covariance between residuals is

$$\begin{aligned}&\mathrm{cov}(\sigma_r^{(1)}\varepsilon_i^{(1)}, \sigma_Z^{(1)}\varepsilon_i^{(2)})\\ =&\,\mathrm{cov}\left(\sigma_r\int_0^h \mathrm{e}^{-\kappa(h-s)}\,\mathrm{d}W_1(t_i+s), \int_0^h \sigma_r\frac{\mathrm{e}^{-\kappa(h-s)} - \mathrm{e}^{-a(h-s)}}{a-\kappa} + \sigma_S\rho\mathrm{e}^{-a(h-s)}\,\mathrm{d}W_1(t_i+s)\right.\\ &\left. + \int_0^h \sigma_S\sqrt{1-\rho^2}\mathrm{e}^{-a(h-s)}W_2(t_i+s)\right)\\ =&\frac{\sigma_r^2}{a-\kappa}\left(\frac{1-\mathrm{e}^{-2\kappa h}}{2\kappa} - \frac{1-\mathrm{e}^{-(\kappa+a)h}}{\kappa+a}\right) + \rho\sigma_r\sigma_S\frac{2(1-\mathrm{e}^{-(\kappa+a)h})}{\kappa+a}.\end{aligned} \tag{4.52}$$

The two regressions are intended to compute the coefficients $(\kappa, \sigma_r, \bar{r}^{\mathbb{P}}, a, \lambda_S, \sigma_S, \rho)$, 7 parameters in total.

Table 4.1 gives the regression result for the coefficients. Only p-values of constant of interest rate shows no significance, because of long-term quantitative easing between 2009 and 2015.

Along the lines of Appendix B in [Flor and Larsen (2013)], we compute the infinite yield as $y_\infty = \bar{r} - \frac{\sigma_r^2}{2\kappa^2}$ and the market price of risk of the bond as:

$$\lambda_r = \frac{\kappa}{\sigma_r}\left(y_\infty + \frac{\sigma_r^2}{2\kappa^2} - \bar{r}^{\mathbb{P}}\right),$$

Regression Result					
	Coef	Value	p-values	Sgn.	Overall p-values
Interest Rate	Const.	3.442×10^{-5}	0.473		0.000
	$e^{-\kappa h}$	0.9960	0.000	(***)	
	Std.err	0.1116%			
Crude Oil	Const.	0.0259	0.043	(*)	0.000
	e^{-ah}	0.9938	0.000	(***)	
	Std.err	4.1676%			
	Cor.	3.4246%			
Gold	Const.	0.0165	0.037	(*)	0.000
	e^{-ah}	0.9978	0.000	(***)	
	Std.err	1.9818%			
	Cor.	-9.4518%			

Table 4.1: Regression results.

where $y_\infty = 3.2702\%$ can be represented as the mean value of the long-term yield. From the perspective of p-value, the coefficients are all significant.

Table 4.2 summarizes all parameters from the regressions. For a comparison to existing literature, see Appendix D.5.

A visual comparison between the estimates for gold and those for oil (WTI), show important differences between these two commodities. In particular, volatility of WTI price is much larger with a higher reversion frequency, on the other hand correlation between gold and interest rate is slightly larger and negative than that of WTI and interest rates. This variety in behaviour helps cement the case for representativeness of our choices of commodity.

4.4.2 Optimal Strategy and Terminal Surplus

In this section we use the representation of the optimal terminal surplus and allocation in Proposition 4.2.2 and Theorem 4.2.2 to understand their behaviour. For this we simulate via Monte Carlo the optimal strategy $\boldsymbol{\pi}^*$ and the terminal surplus $X(T)$ given an initial value $\boldsymbol{y}$ and paths for interest rate $r(t)$ and log-price $Z(t)$. For the simulation, we use 50,000 paths and daily time step $\Delta t = 1/252$. We pick up the initial price of $S(0) = 90$ and 1200 for crude

Parameters			
Surplus	Premium percentage	c	0.1
	Claim rate	μ	5
	Claim size	e^{-Z}	$Z \sim \text{Lognormal}(-4, 1)$
	Initial surplus	$X(0)$	10
Interest Rate	Mean-reverting rate	κ	0.2062
	Risk-neutral measure interest	$\bar{r}$	3.3468%
	Real-world measure interest	$\bar{r}^{\mathbb{P}}$	0.8699%
	Market price of risk	λ_r	0.6331
	Volatility	σ_r	0.8067%
	Initial interest rate	$r(0)$	2%
Crude Oil	Market price of risk	λ_S	4.6109
	Mean-reverting rate	a	0.3229
	Volatility	σ_S	30.1460%
	Correlation	ρ	3.4246%
	Initial price (bear)	$S(0)$	110
	Initial price (bull)	$S(0)$	50
Gold	Market price of risk	λ_S	6.0297
	Mean-reverting rate	a	0.1138
	Volatility	σ_S	14.3068%
	Correlation	ρ	-9.4518%
Objective Function	Time horizon	T	10
	Relative risk aversion	γ	4
	Ambiguity parameter 1	β_1	3
	Ambiguity parameter 2	β_2	3

Table 4.2: Parameters of the problem

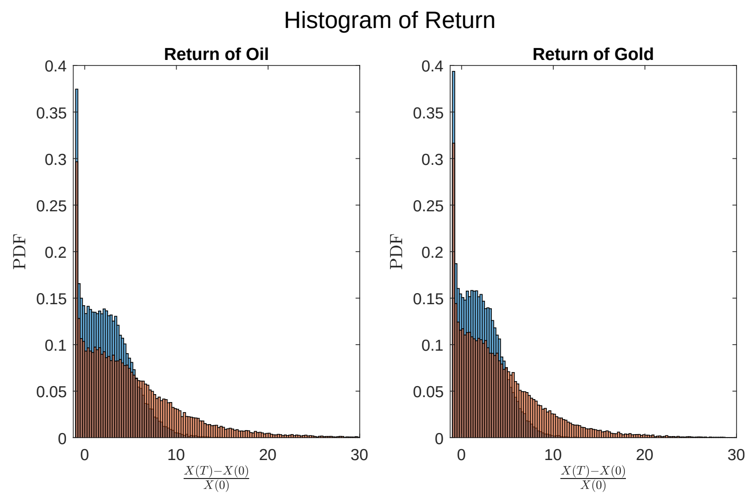


Figure 4.1: Density of $\frac{X(T)-X(0)}{X(0)}$ for $\beta_1 = \beta_2 = 3$ (blue) and $\beta_1 = \beta_2 = 0$ (orange). Oil portfolio on the left, gold portfolio on the right.

oil and gold respectively.

Figure 4.1 shows the density of the terminal return rate $\frac{X(T)-X(0)}{X(0)}$ for the ambiguous (blue) and non-ambiguous (orange) investors, as well as oil-based portfolio (left) and gold-based portfolio on the right. As expected, the absence of ambiguity-aversion ($\beta_1 = \beta_2 = 0$) leads to more extreme behaviour (higher probability on tails). This is also confirmed with the statistics for all four cases (oil, gold; absence and presence of ambiguity) reported in Table 4.3. Table 4.3 demonstrates larger moments across the board for non-ambiguous investors (twice in value as those of ambiguity-averse companies). This is also the case for allocations on the commodity, which is 1.5 times as high for non-ambiguous agents. This pictures a less risky and aggressive behaviour for ambiguity-averse investors.

Commodity	Exp. Return	Std. Deviation	Skewness	Kurtosis	$\pi_S(0)$
Oil (Ambiguity)	13.25%	26.12%	0.7756	3.4581	−14.45%
Oil (Non-ambiguity)	19.86%	56.82%	1.8810	9.2383	−21.99%
Gold (Ambiguity)	11.90%	23.03%	0.7709	3.5443	54.85%
Gold (Non-ambiguity)	17.96%	48.90%	2.0313	11.2388	87.36%

Table 4.3: Outcome of simulation results

We also perform an analysis of fractional allocation w.r.t. γ, σ_S, β_1 and β_2 in Figure 4.2 at $t = 0$. Higher risk-aversion parameter γ and volatility of commodity σ_S leads to decreasing

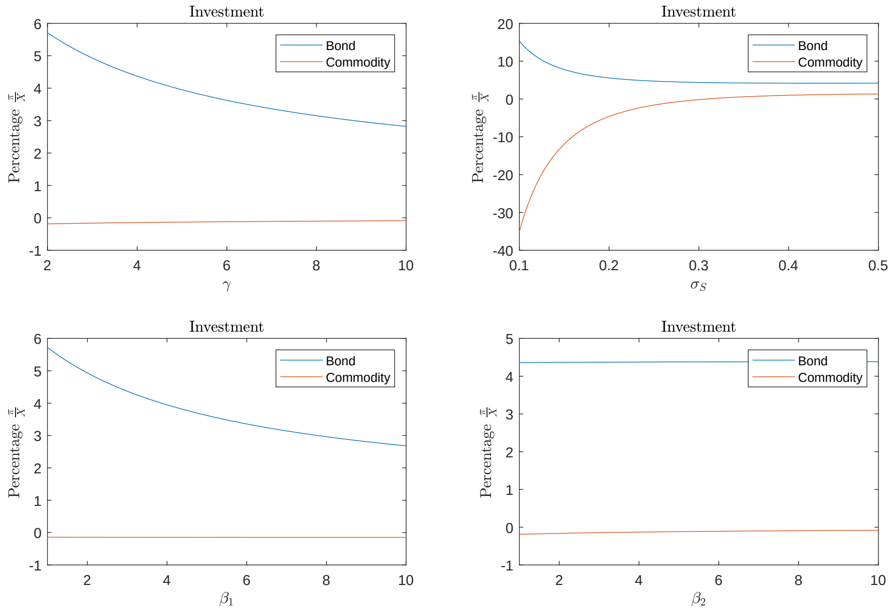


Figure 4.2: Relationship of fractional allocation at $t = 0$ with γ (upper-left), σ_S (upper-right), β_1 (lower-left), β_2 (lower-right).

long investments in the bond, and short allocations to the commodity. This is because, higher risk aversion and volatility leads to decreasing allocations in risky assets. This is identical to higher β_1 and β_2, which result in decreasing long position in the bond, and short position in the commodity, respectively.

Next we study optimal strategies for two market scenarios: first a bear market, defined as a situation where the initial price is larger than the mean reverting level, therefore the market is very likely to drop. Second a bull market case, where initial price lower than the mean reverting level, hence it is highly likely for prices to go up. Figure 4.3 and Figure 4.4 report the optimal investments in bonds, Commodity and Bank account for a path representing bear market conditions and a path representing a bull market, respectively.

In a bear market, the percentage of wealth invested in the oil portfolio drops significantly from a maximum of long 80% on the commodity to shorting 20%. The situation partially reverse in bull market conditions as per Figure 4.4. Here the investor allocates more on the commodity at the expense of less investment in bonds, this is to take advantage of the better returns in the commodity boom. The cash account does not show a clear pattern.

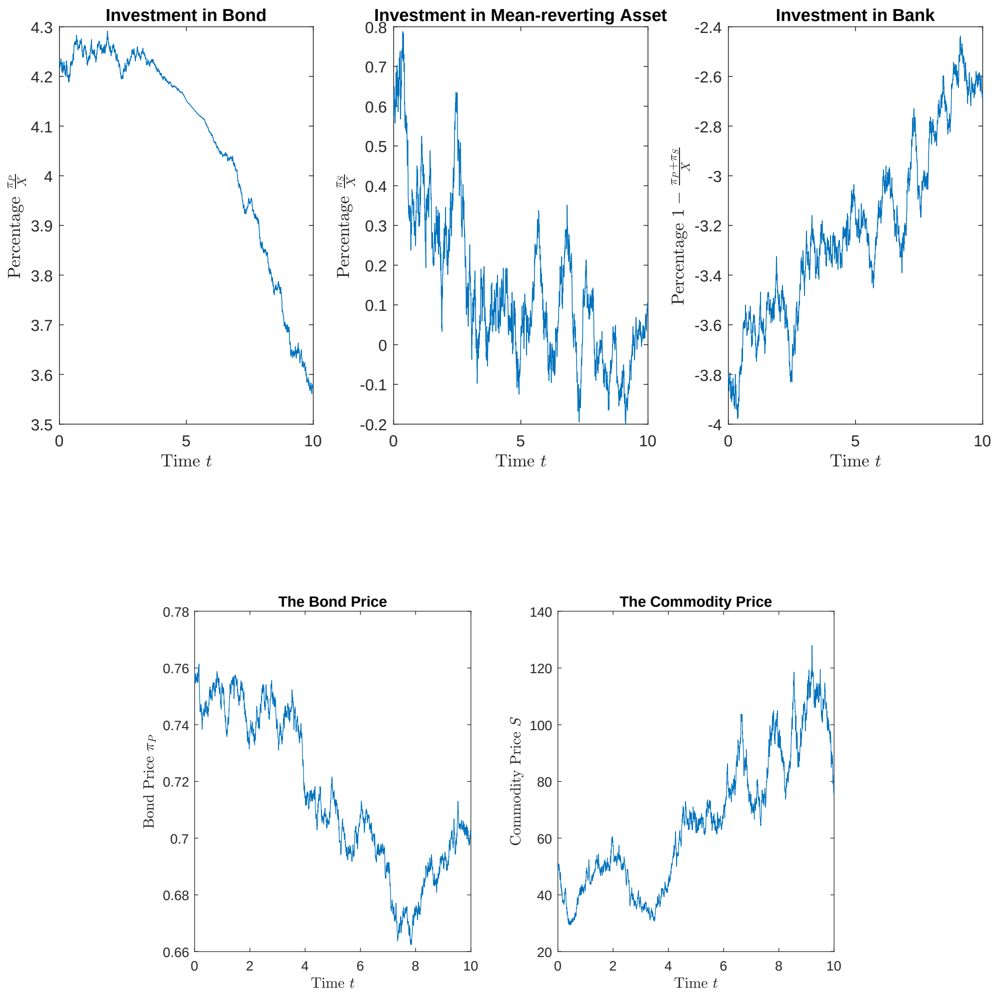


Figure 4.3: Plot of investment in a bear market, oil portfolio.

4.4.3 Equivalent Losses in Suboptimal Analysis

Here we study equivalent losses L^{π^s} from the suboptimal strategies described in Section 4.3.

Group I: Ignoring Ambiguity, Commodities versus Stocks

In this section we compare the wealth-equivalent utility loss due to ignoring ambiguity for the two commodities at hand: crude oil and gold. Figure 4.5 captures the oil portfolio while Figure 4.6 is about the gold portfolio.

The investor's ambiguity aversion level is depicted in the x-axis while the y-axis displays

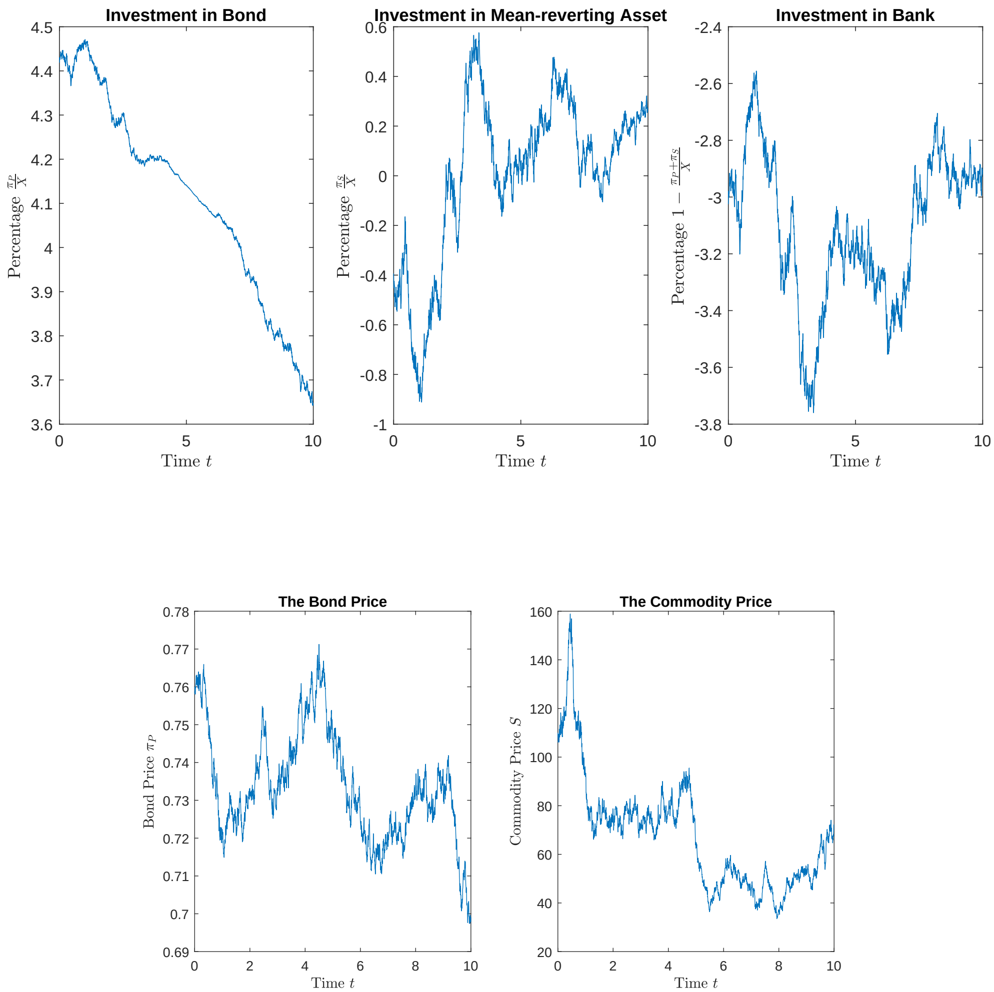


Figure 4.4: Plot of investment in a bull market, oil portfolio.

the wealth-equivalent losses from ignoring such level of ambiguity, e.g using the "suboptimal" strategy that assumes $\hat{\beta}_1 = \hat{\beta}_2 = 0$. Recall β_1 is about ambiguity-aversion on the interest rate and therefore bond market, while β_2 captures the ambiguity on the targeted commodity. It is remarkable to see that ignoring ambiguity about the commodity could be quite detrimental for an investor, the right side of Figure 4.5 and Figure 4.6 demonstrate quite large losses for even low aversion levels. The damaging effect of ignoring ambiguity depends of the mean-reverting rate a. Heavier mean-reverting effect leads to a more severe wealth-equivalent loss. As for ignoring ambiguity on bonds, the left hand side of Figure 4.5 and Figure 4.6 show a larger impact in the gold portfolio compared to the oil portfolio. This can be attributed to the larger exposure to bonds in the former portfolio compared to the later.

For the purpose of assessing the impact of ignoring ambiguity in the asset class of com-

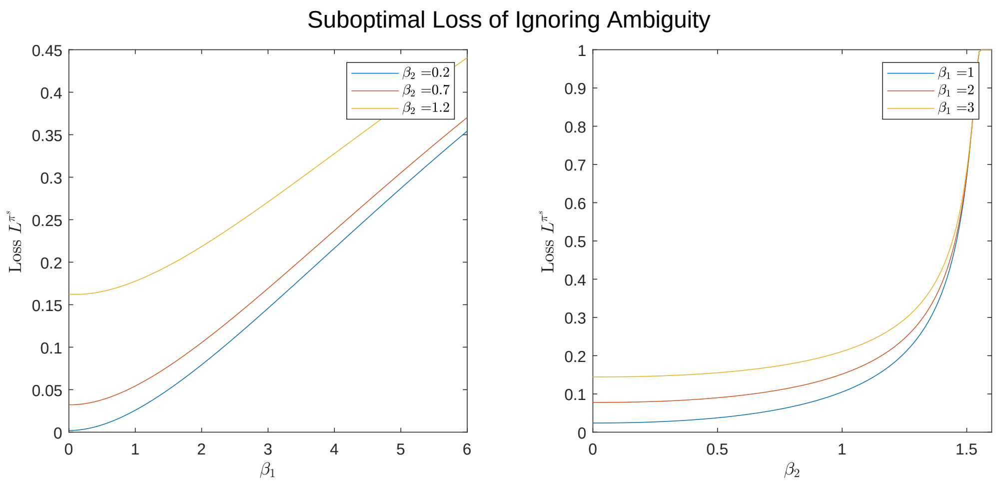


Figure 4.5: Plot of suboptimal loss given $\boldsymbol{\beta}$ using WTI crude oil price

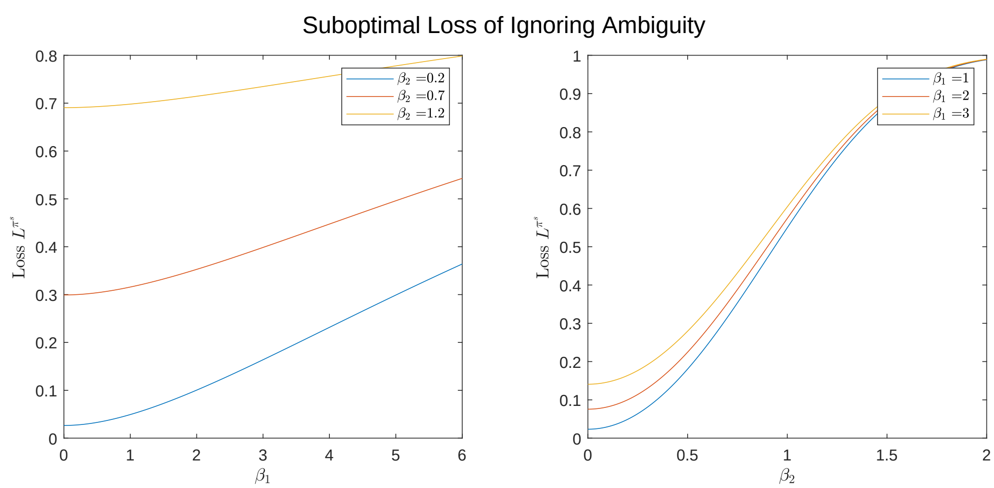


Figure 4.6: Plot of suboptimal loss given $\boldsymbol{\beta}$ using gold price

modities compared to the asset class of stocks, we also include the effect of ignoring ambiguity assuming the underlying follows a Geometric Brownian motion (GBM). This is comparable to [Flor and Larsen (2013)] where the authors study bonds and Stocks. Such setting can be accommodated here by setting the mean-reverting rate $a = 0$ and lowering the excess return hence treating the dynamics of commodities as a GBM-growth asset. Note, the resulting stocks would have acceptable volatility values (14.3% and 30.1%), as well as reasonable excess returns (7.1647% for oil and 9.3748% for gold). The suboptimal losses in this "stock market scenario" are shown in Figure 4.7 and Figure 4.8. Comparing Figure 4.5 to Figure 4.7 (oil price) and also Figure 4.6 to Figure 4.8 (gold), we can see that for the same level of ambiguity-aversion ($\beta = 1$) the wealth-equivalent utility losses are substantially larger in commodity markets compare to stock markets. The difference can be around four fold.

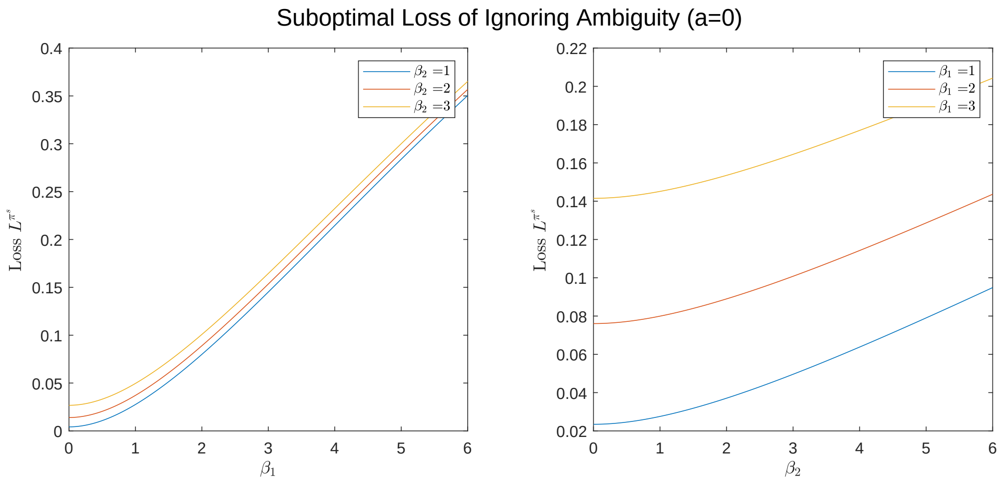


Figure 4.7: Plot of suboptimal loss given $\boldsymbol{\beta}$ and $a = 0$ using WTI crude oil price

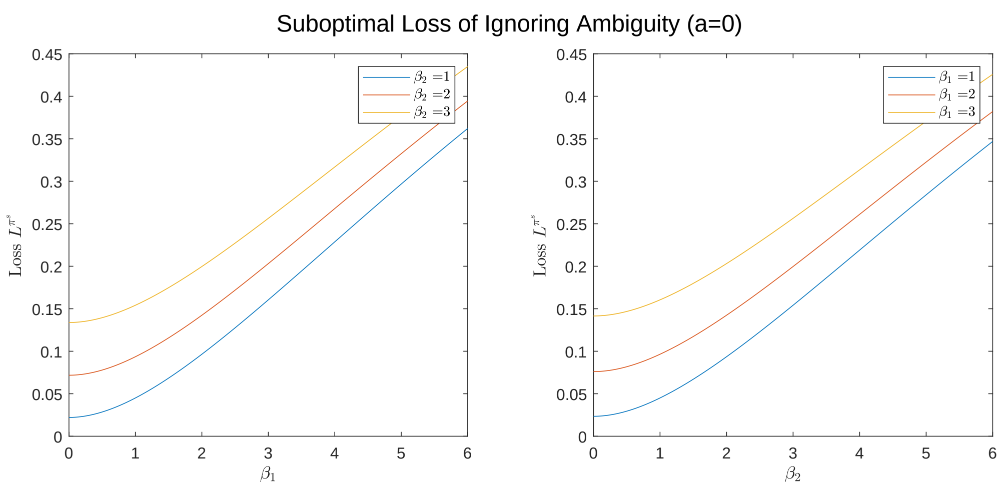


Figure 4.8: Plot of suboptimal loss given $\boldsymbol{\beta}$ and $a = 0$ using gold price

Group II: Incomplete Market: No Bond Investment

Here we explore the performance of suboptimal strategies obtained due to incomplete markets, i.e. failing to invest in bonds. Figure 4.9 and Figure 4.10 show the wealth-equivalent losses as a function of ambiguity levels: β_1 and β_2, for oil and gold portfolios respectively.

β_1 play a significant role in incomplete markets, with losses of up to 45% (3-% for gold) due to market incompleteness. On the other hand, as β_1 increases the wealth-equivalent losses decrease. This can explained as follows, ambiguity-aversion on $r(t)$ leads to higher expected short rates; in the absence of bonds, the investor can not take advantage of this improvement in performance hence suffering even worse losses.

Not surprisingly ignoring β_2 (right hand side of Figure 4.9) plays little role for crude oil, see the right plot in Figure 4.9. This is because β_2 affects only the expected return of the asset, which has very small correlation (3.42%) with interest rate. However, for larger correlations (-9.45%), the spillover is more important and as Figure 4.10 shows, one can detect significant impact of up to 25% in losses.

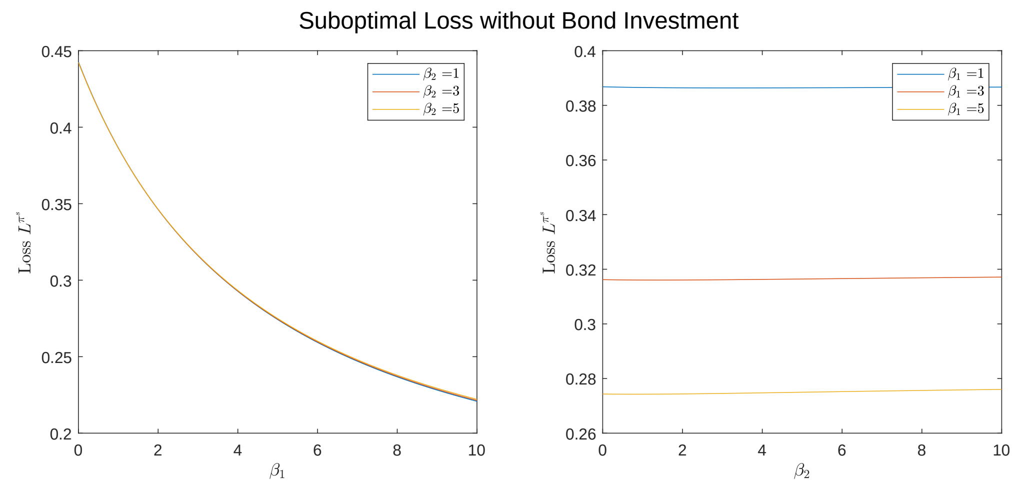


Figure 4.9: Plot of suboptimal loss without bond using crude oil price investment

Group III: Incorrect Parameters

This section focuses on wealth-equivalent losses due to mis-specification of important parameters, in particular market prices of risk, volatilities and correlations. The analysis here displays results only for oil prices, similar observations were produced with the gold portfolio.

Figure 4.11 shows the losses due to wrongly calibrated market prices of risk. We assume the correct values are those presented in Table 4.2, then the x-axis represents the chosen values: either $\hat{\lambda}_r$ (left plot) or $\hat{\lambda}_S$ (right plot). We would not be incurring in an error only if $\hat{\lambda}_r = \lambda_r$ or $\hat{\lambda}_S = \lambda_S$, in such cases we would have selected the right parameters. As one can

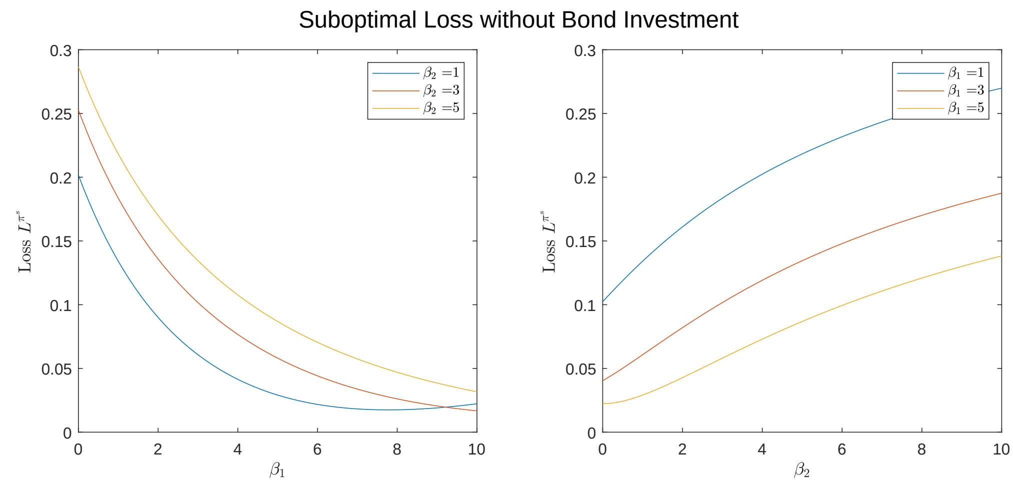


Figure 4.10: Plot of suboptimal loss without bond using gold investment

observe, the losses are far more sensitive to a wrong choice of $\hat{\lambda}_S$ than of $\hat{\lambda}_r$. This highlights the importance of a proper estimation exercise.

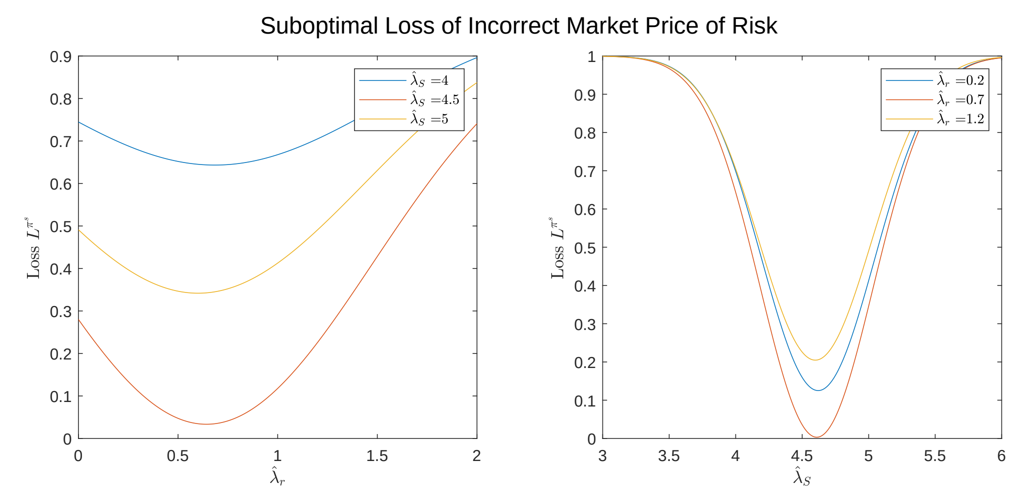


Figure 4.11: Plot of suboptimal loss given $\hat{\lambda}_r, \hat{\lambda}_S$

We also study the losses due to mis-specification of volatilities: $\hat{\sigma}_r$ and $\hat{\sigma}_S$. The patterns in losses are similar to those encountered before, emphasising the importance of estimating commodity parameters more precisely than bond's parameters. Interestingly, wrongly assuming lower values of volatilities could be far more consequential than assuming incorrect large values, which tell a story of better overestimating than underestimating the risk.

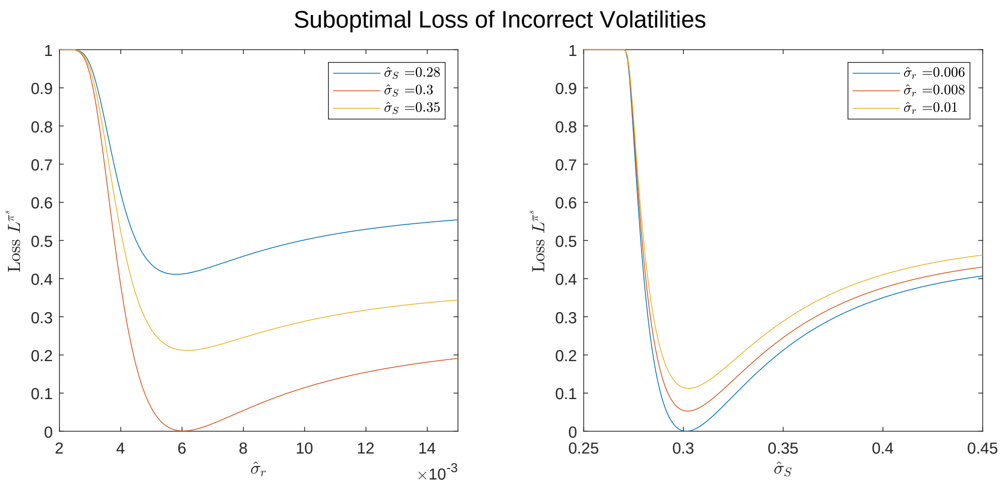


Figure 4.12: Plot of suboptimal loss given $\hat{\sigma}_r, \hat{\sigma}_S$

Lastly, we study the impact of correlation between bonds and commodities. There is a large body of literature on the absence of such dependence. Hence we assume the correct value is $\rho = 3.4246\%$ and plot the losses due to incorrectly assuming a value of $\hat{\rho}$. The further away of $\hat{\rho}$ with the true ρ, the higher the utility loss. Incorrect large positive (> 0.4) or negative (< -0.4) correlation creates heavy loss to 100%, while no obvious distinguished pattern is shown between the negative part and the positive part.

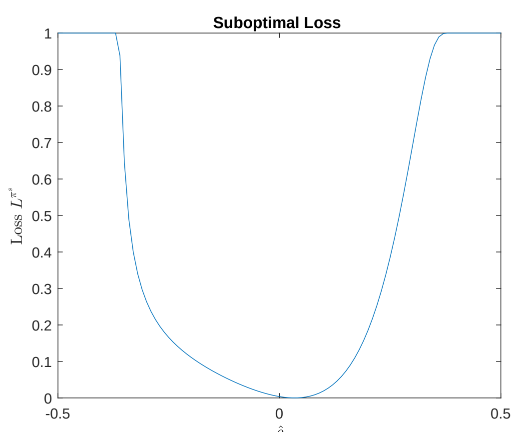


Figure 4.13: Plot of suboptimal loss given $\hat{\rho}$

4.5 Conclusion

In this chapter, we consider a robust portfolio optimization of company's surplus consisting of a bond and a mean-reverting asset representing a commodity. The surplus is assumed to be a geometric Cramér-Lundberg. We also assume a interest rate to be a Vasicek model and the commodity to be an exponential-OU process. Via a maximization of CRRA utility and applying ambiguity-aversion entropy, we generated analytical, closed-form solutions for the optimal investment, worst-case change of measure, optimal wealth and value function for the insurer (investor).

Thanks to the analytical representations we explore the behaviour of optimal solutions and the impact of various meaningful suboptimal strategies on the investor's portfolio. For these exercises we consider and estimate two assets representatives of commodity market: oil prices and gold prices. Some important findings are: ignoring ambiguity either on bonds or commodities could lead to drastic wealth-equivalent losses, harsher if ignoring ambiguity-aversion on commodities than on bonds. Even more, ignoring ambiguity-aversion on commodities is more damaging than ignoring such on the asset class of stocks (GBMs). As reported by many authors, working on an incomplete market could be detrimental for a portfolio, we demonstrate losses of up to 45% due to incompleteness in the commodity market. Lastly, mis-specifications in the parameters toward either larger correlation, smaller variances or incorrect market price of commodity risk, could lead to drastically large wealth-equivalent losses, hence an unnecessary under-performances of the investor portfolio.

Chapter 5

Model uncertainty on commodity portfolios, the role of Convenience Yield

5.1 Introduction

This chapter investigates the effect of model uncertainty on the performance of commodity-based portfolios. We consider a constant relative risk aversion (CRRA) utility maximizer investor in a complete market, with independent ambiguity-aversion levels on the three factors explaining the term structure of future prices, namely, spot prices, convenience yield and interest rates, as proposed in the seminal work of [Schwartz (1997)]. This generic investor is interested in the speculative component of the investment rather than possessing/consuming the physical commodity. We obtained closed-form solutions for optimal investments, optimal perturbations (alternative model) and value functions along in line with the robust portfolio setting of [Maenhout (2004)]. Our main focus is on the effect of convenience yield's uncertainty on the optimal analysis. We estimate the model using a combination of maximum likelihood estimation (MLE) and Kalman Filter (KF) techniques, on two commodities: West Texas Intermediate (WTI) and copper future prices. The analysis demonstrates that uncertainty on the convenience yield factor could be the largest contributor to the underperformance of a commodities portfolio, with wealth equivalent losses (WELs) in the range 33% to 88% (WTI), 7% to 31% (copper). Moreover, small variations, of up 25%, on convenience yield's covariance parameters could lead to a WEL of up to 40% (WTI, lesser volatility of convenience yield).

The chapter is structured as follows. Section 5.2 describes the mathematical model and the robust optimization problem. The analytical solutions are then detailed in Section 5.3. Thereafter, Section 5.4 first explain the estimation of parameters, before exploring optimal investment allocation with regard to covariance parameters of convenience yield. A WEL analysis is also performed for several key suboptimal cases, namely, parametric misspecification on the convenience yield, incompleteness of the market, and more importantly the impact of ignoring the various sources of uncertainty. Section 5.5 concludes the chapter. We make the following contributions:

- We construct an robust portfolio optimization for investors to maximize the expected terminal wealth and obtain a closed-form solution to the problem via an HJBI PDE. Provided the solution, the robust portfolio is obtained as a function of time.
- Using the MLE and KF techniques, we analyze the WTI and coppter future data, and successfully compute the parameters.
- The effects of absence of ambiguity, parametric misspecifiction of convenience yield and incomplete market are analyzed to present the WELs.

5.2 Mathematical Settings

Model III of [Schwartz (1997)] is presented first in this section. The model considers stochastic spot prices, stochastic interest rate and stochastic convenience yield, with the latter two following mean-reverting processes. Some results regarding the model-implied term structure of future prices, bond prices and prepaid forwards are described. The robust portfolio optimization problem is then introduced.

5.2.1 Three-Factor Model

We assume the stochastic processes describing the financial market are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with a right-continuous filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. [Schwartz (1997)] presented Model III on the risk-neutral probability measure $\mathbb{Q}$. Let $S(t)$ denote the spot price of a commodity; the model can then be described via three stochastic differential equations

(SDEs):

$$\begin{cases} \mathrm{d}S(t) = (r(t) - \delta(t))S(t)\,\mathrm{d}t + \sigma_S S(t)\,\mathrm{d}W_1^Q(t) \\ \mathrm{d}\delta(t) = \kappa(\hat{\alpha} - \delta(t))\,\mathrm{d}t + \sigma_\delta\,\mathrm{d}W_2^Q(t) \\ \mathrm{d}r(t) = a(\hat{m} - r(t))\,\mathrm{d}t - \sigma_r\,\mathrm{d}W_3^Q(t) \end{cases} \tag{5.1}$$

with correlations among all Brownian motions,

$$\mathrm{d}W_1^Q(t)\,\mathrm{d}W_2^Q(t) = \rho_{S\delta}\,\mathrm{d}t, \quad \mathrm{d}W_2^Q(t)\,\mathrm{d}W_3^Q(t) = \rho_{\delta r}\,\mathrm{d}t, \quad \mathrm{d}W_1^Q(t)\,\mathrm{d}W_3^Q(t) = \rho_{Sr}\,\mathrm{d}t. \tag{5.2}$$

Here the convenience yield $\delta(t)$ and interest rate $r(t)$ are mean-reverting processes with reverting rates of κ and a and mean values of $\hat{\alpha}$ and $\hat{m}$, respectively. Moreover, convenience yield is the benefit to the investor while holding the commodity. If convenience yield is negative, then it is equivalent to having a cost of carry. Therefore, if an investor holds the commodity directly, then the reduced amount of return on the commodity offsets the benefit; this is similar to the behavior of dividends in a stock.

The investment problem deals with the returns on the real-world probability $\mathbb{P}$. We use Girsanov's change of measure with $\mathrm{d}\boldsymbol{W}^Q(t) = \mathrm{d}\boldsymbol{W}(t) + \boldsymbol{\lambda}\,\mathrm{d}t$ where we set

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_S \\ \lambda_\delta \\ \lambda_r \end{bmatrix}, \quad \boldsymbol{W}(t) = \begin{bmatrix} W_1(t) \\ W_2(t) \\ W_3(t) \end{bmatrix} \quad \boldsymbol{W}^Q(t) = \begin{bmatrix} W_1^Q(t) \\ W_2^Q(t) \\ W_3^Q(t) \end{bmatrix}. \tag{5.3}$$

Hence, $\lambda_S, \lambda_\delta, \lambda_r$ are the market prices of risks of the commodity, convenience yield and interest rate. Under $\mathbb{P}$, Equation (5.1) becomes

$$\begin{cases} \mathrm{d}S(t) = (r(t) + \lambda_S\sigma_S - \delta(t))S(t)\,\mathrm{d}t + \sigma_S S(t)\,\mathrm{d}W_1(t) \\ \mathrm{d}\delta(t) = \kappa(\alpha - \delta(t))\,\mathrm{d}t + \sigma_\delta\,\mathrm{d}W_2(t) \\ \mathrm{d}r(t) = a(m - r(t))\,\mathrm{d}t - \sigma_r\,\mathrm{d}W_3(t) \end{cases} \tag{5.4}$$

where

$$\alpha = \hat{\alpha} + \frac{\lambda_\delta\sigma_\delta}{\kappa}, \quad m = \hat{m} - \frac{\lambda_r\sigma_r}{a}. \tag{5.5}$$

We refer to Equation (5.4) as the reference model.

5.2.2 Dynamics of Assets

Similar to [Mellios et al. (2016)], we consider three types of assets in which to invest, in addition to the cash account. These assets are a spot commodity, a prepaid forward and a bond. The dynamics of the last two assets are specified next.

Dynamics of Prepaid Forward

In the real world, commodities are usually traded in the form of futures but not immediately; these are called prepaid forwards. We first need the process of a risk-neutral future price for a maturity T, as [Schwartz (1997)] indicated:

$$F(S,\delta,r,T) = \mathbb{E}^Q[S(T)] = S\exp\left[-\frac{\delta(1-\mathrm{e}^{-\kappa T})}{\kappa} + \frac{r(1-\mathrm{e}^{-aT})}{a} + C(T)\right], \tag{5.6}$$

where $C(T)$ satisfies the formula

$$\begin{aligned} C(T) =& \frac{(\kappa\hat{\alpha} + \sigma_S\sigma_\delta\rho_{S\delta})((1-\mathrm{e}^{-\kappa T}) - \kappa T)}{\kappa^2} - \frac{\sigma_\delta^2(4(1-\mathrm{e}^{-\kappa T}) - (1-\mathrm{e}^{-2\kappa T}) - 2\kappa T)}{4\kappa^3} \\ & - \frac{(a\hat{m} + \sigma_S\sigma_r\rho_{Sr})((1-\mathrm{e}^{-aT}) - aT)}{a^2} - \frac{\sigma_r^2(4(1-\mathrm{e}^{-aT}) - (1-\mathrm{e}^{-2aT}) - 2aT)}{4a^3} \\ & + \sigma_\delta\sigma_r\rho_{\delta r}\bigg(\frac{(1-\mathrm{e}^{-\kappa T}) + (1-\mathrm{e}^{-aT}) - (1-\mathrm{e}^{-(a+\kappa)T})}{\kappa a(\kappa+a)} \\ & + \frac{a^2(1-\mathrm{e}^{-\kappa T}) + \kappa^2(1-\mathrm{e}^{-aT}) - \kappa a^2 T - a\kappa^2 T}{\kappa^2 a^2(\kappa+a)}\bigg) \end{aligned} \tag{5.7}$$

Given that the future price is the delivery price at time T, we would need to discount this future price to present value to obtain the prepaid forward, mathematically,

$$P(S,\delta,r,T) = \mathbb{E}^Q\left[\exp\left(-\int_0^T r(s)\,\mathrm{d}s\right)S(T)\middle|\mathcal{F}_t\right]. \tag{5.8}$$

where P denotes the price of a prepaid forward for the commodity. [Schwartz (1997)] presented the PDE of the prepaid forward as well as its solution:

$$\begin{aligned} &\frac{1}{2}\sigma_S^2S^2P_{SS} + \frac{1}{2}\sigma_\delta^2P_{\delta\delta} + \frac{1}{2}\sigma_r^2P_{rr} + \sigma_S\sigma_\delta\rho_{S\delta}SP_{S\delta} + \sigma_r\sigma_\delta\rho_{\delta r}P_{\delta r} + \sigma_S\sigma_r\rho_{S\delta}SP_{Sr} \\ &\qquad + (r-\delta)SP_S + \kappa(\hat{\alpha}-\delta)P_\delta + a(m^*-r)P_r - P_T = rP. \\ &P(S,\delta,r,T) = S\exp\left[-\frac{\delta(1-\mathrm{e}^{-\kappa T})}{\kappa} + D(T)\right]. \end{aligned} \tag{5.9}$$

where

$$D(T) = \frac{(\kappa\hat{\alpha} + \sigma_S\sigma_\delta\rho_{S\delta})((1-\mathrm{e}^{-\kappa T}) - \kappa T)}{\kappa^2} - \frac{\sigma_\delta^2(4(1-\mathrm{e}^{-\kappa T}) - (1-\mathrm{e}^{-2\kappa T}) - 2\kappa T)}{4\kappa^3}. \tag{5.10}$$

Therefore, by setting the maturity to be $T-t$ and denoting $P_T(t) := P(S,\delta,r,T-t)$, the dynamics of this prepaid forward is

$$\mathrm{d}P_T(t) = r(t)P_T(t)\,\mathrm{d}t + \frac{\partial P_T(t)}{\partial S}S\sigma_S\,\mathrm{d}W_1^Q(t) + \frac{\partial P_T}{\partial \delta}(t)\sigma_\delta\,\mathrm{d}W_2^Q(t). \tag{5.11}$$

Going back to the real-world probability, the dynamics would be

$$\frac{\mathrm{d}P_T(t)}{P_T(t)} = (r(t) + \lambda_S\sigma_S - A_{T-t}\lambda_\delta\sigma_\delta)\,\mathrm{d}t + \sigma_S\,\mathrm{d}W_1(t) - A_{T-t}\sigma_\delta\,\mathrm{d}W_2(t). \tag{5.12}$$

where $A_{T-t} = \frac{1-\mathrm{e}^{-\kappa(T-t)}}{\kappa} \geq 0$.

This dynamics indicates that the value of this prepaid forward $P_T(t)$ is affected by two Brownian motions: $W_1^Q(t)$ and $W_2^Q(t)$. Moreover, the excess returns (i.e. $\lambda_S\sigma_S - A_{T-t}\lambda_\delta\sigma_\delta$) on prepaid forwards are also impacted by their maturities. Larger maturities $T-t$ result in smaller returns and smaller volatilities.

Dynamics of bonds

We consider investing in a bond due to the presence of a stochastic interest rate $r(t)$. The price of the bond with a maturity of T under the risk-neutral measure is as follows:

$$P(t, r(t); T) := P(t, r(t)) = \exp(-I_{T-t}r(t) + J_{T-t}) \tag{5.13}$$

where

$$\begin{aligned} I_{T-t} &= \frac{1-\mathrm{e}^{-a(T-t)}}{a} \\ J_{T-t} &= \left(m - \frac{\sigma_r^2}{2a^2}\right)(I(t;T) - (T-t)) - \frac{\sigma_r^2}{4a}I_{T-t}^2. \end{aligned} \tag{5.14}$$

For simplicity, we can define the "infinity yield" as $y_\infty := m - \frac{\sigma_r^2}{2a^2}$.

In the risk-neutral measure, the bond price follows a PDE:

$$P_t + P_r(\kappa(\bar{r} - r)) + \frac{\sigma_r^2}{2}P_{rr} = rP. \tag{5.15}$$

Transforming to a real-world measure using Girsanov's theorem, the joint dynamics for the bond and the interest rate are

$$\begin{aligned} \mathrm{d}r(t) &= [a(m - r(t)) - \lambda_r\sigma_r]\,\mathrm{d}t - \sigma_r\,\mathrm{d}W_1(t) \\ \mathrm{d}P(t, r(t)) &= (r(t) + \lambda_r\sigma_r I_{T-t})P(t, r(t))\,\mathrm{d}t + I_{T-t}\sigma_r P(t, r(t))\,\mathrm{d}W_1(t). \end{aligned} \tag{5.16}$$

Therefore, for the return on the bond $\lambda_r\sigma_r I_{T-t}$ is the excess return.

5.2.3 Portfolio Wealth Process

Now, we consider a portfolio consisting of multiple prepaid forwards with different maturities $T_1, ..., T_n$, a bank account and the bond. Let $\pi_1, \pi_2, \ldots, \pi_n$ and π_{n+1} be the amount of wealth invested into the n prepaid forwards and the bond respectively. We write $\diamond[0,T]$ for the set of all admissible strategies; that is, $\mathcal{F}_t$-progressively measurable processes with sufficient integrability conditions for the optimization problem to be well defined.

We work with portfolio managers who prefer to invest in the commodity rather than to physically possess or consume it; they use what is called rollover investments. Here, we consider rollover investment of the prepaid forwards and the bond by setting $\tau_i = T_i - t$ and τ respectively. In this setting, the dynamics of the portfolio wealth will be

$$\mathrm{d}X(t) = \Big(r(t)X(t) + \boldsymbol{\pi}^T\boldsymbol{\sigma}\boldsymbol{\lambda}\Big)\,\mathrm{d}t + \boldsymbol{\pi}^T\boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{W}(t) \tag{5.17}$$

where

$$\boldsymbol{\pi} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_n \\ \pi_{n+1} \end{bmatrix}, \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_S & -A_{\tau_1}\sigma_\delta & 0 \\ \sigma_S & -A_{\tau_2}\sigma_\delta & 0 \\ \vdots & & \\ \sigma_S & -A_{\tau_n}\sigma_\delta & 0 \\ 0 & 0 & I_\tau\sigma_r \end{bmatrix}. \tag{5.18}$$

5.2.4 The Robust Portfolio Optimization Problem

As per the setting of [Maenhout (2004)], there is a reference model, Equation (5.4), which offers the best representation of the data to the investor. Alternative models also exist that cannot be statistically distinguished from the former model. Girsanov's theorem for correlated Brownian motions allow us to generate alternative models via a perturbation u:

$$\frac{\mathrm{d}Q^u}{\mathrm{d}P} = \exp\left(-\int_0^T \boldsymbol{u}^T(t)\,\mathrm{d}\boldsymbol{W}(t) - \int_0^T \boldsymbol{u}^T(t)\boldsymbol{\rho}^{-1}\boldsymbol{u}(t)\,\mathrm{d}t\right) \tag{5.19}$$

where

$$\boldsymbol{\rho} = \begin{bmatrix} 1 & \rho_{S\delta} & \rho_{Sr} \\ \rho_{S\delta} & 1 & \rho_{\delta r} \\ \rho_{\delta r} & \rho_{S\delta} & 1 \end{bmatrix}, \mathrm{d}\boldsymbol{W}(t) = \begin{bmatrix} \mathrm{d}W_1(t) \\ \mathrm{d}W_2(t) \\ \mathrm{d}W_3(t) \end{bmatrix}, \text{and } \boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} \tag{5.20}$$

We write $\mathcal{U}[0,T]$ for the set of all $\mathcal{F}_t$-progressively measurable processes such that the Radon–Nikodým derivative process is well defined. Using the change of measure P to Q^u, the dynamics become

$$\begin{cases} \mathrm{d}S(t) = (r(t) - \delta(t) + \lambda_S\sigma_S - \sigma_S u_1(t))S(t)\,\mathrm{d}t + \sigma_S S(t)\,\mathrm{d}W_1^{Q^u}(t) \\ \mathrm{d}\delta(t) = [\kappa(\alpha - \delta(t)) - \sigma_\delta u_2(t)]\,\mathrm{d}t + \sigma_\delta\,\mathrm{d}W_2^{Q^u}(t) \\ \mathrm{d}r(t) = [a(m - r(t)) + \sigma_r u_3(t)]\,\mathrm{d}t - \sigma_r\,\mathrm{d}W_3^{Q^u}(t) \end{cases} \tag{5.21}$$

Therefore, with the changed drift, the dynamics for the prepaid forward becomes

$$\frac{\mathrm{d}P_T(t)}{P_T(t)} = (r(t) + (\lambda_S - u_1(t))\sigma_S - A_{T-t}(\lambda_\delta - u_2(t))\sigma_\delta)\,\mathrm{d}t + \sigma_S\,\mathrm{d}W_1^{Q^u}(t) - A_{T-t}\sigma_\delta\,\mathrm{d}W_2^{Q^u}(t). \tag{5.22}$$

and the wealth process is

$$\mathrm{d}X(t) = \Big(r(t)X(t) + \boldsymbol{\pi}^T\boldsymbol{\sigma}(\boldsymbol{\lambda} - \boldsymbol{u}(t))\Big)\,\mathrm{d}t + \boldsymbol{\pi}^T\boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{W}^{\boldsymbol{Q^u}}(t). \tag{5.23}$$

Now, we construct the formal optimization problem. We consider an ambiguous investor with CRRA utility who wants to maximize the expected utility from terminal wealth X_T. As per [Andreson et al. (2003)], the value function shall include a penalty term that can be interpreted as the relative entropy for deviating from the reference model. We adopt the most analytically favorable setting of [Maenhout (2004)]:

$$J(t,x,r) = \sup_{\boldsymbol{\pi}}\inf_{\boldsymbol{u}}\mathbb{E}\left[\frac{J(1-\gamma)}{2}\int_t^T \boldsymbol{u}^T(t)\boldsymbol{\beta}_{\boldsymbol{\rho}}^{-1}\boldsymbol{u}(t)\,\mathrm{d}t + U(X_T)\right] \tag{5.24}$$

where $\boldsymbol{\rho} = \boldsymbol{B}\boldsymbol{B}^T$ is a Cholesky decomposition with

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ \rho_{S\delta} & \sqrt{1-\rho_{S\delta}^2} & 0 \\ \rho_{Sr} & \frac{\rho_{\delta r}-\rho_{S\delta}\rho_{Sr}}{\sqrt{1-\rho_{S\delta}^2}} & \frac{\sqrt{1-\rho_{\delta r}^2-\rho_{S\delta}^2-\rho_{Sr}^2+2\rho_{Sr}\rho_{\delta r}\rho_{S\delta}}}{\sqrt{1-\rho_{S\delta}^2}} \end{bmatrix}. \tag{5.25}$$

Here, $\boldsymbol{\beta} = \mathrm{diag}(\beta_1, \beta_2, \beta_3)$ is the diagonal matrix of parameters representing ambiguity-aversion levels on the underlying processes: interest rate, convenience yield and the spot commodity. For simplicity we have $\boldsymbol{\beta}_{\boldsymbol{\rho}} := \boldsymbol{B}\boldsymbol{\beta}\boldsymbol{B}^T$ and $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$.

5.3 Solution to the Robust Portfolio Problem

Given the wealth process, we derive the HJBI equation for Equation (5.24):

$$\begin{aligned}\sup_{\boldsymbol{\pi}}\inf_{\boldsymbol{u}}\Big\{&J_t + J_x(rx + \boldsymbol{\pi}^T\boldsymbol{\sigma}(\boldsymbol{\lambda}-\boldsymbol{u})) + \frac{1}{2}J_{xx}\boldsymbol{\pi}^T\boldsymbol{\sigma}\boldsymbol{\rho}\boldsymbol{\sigma}^T\boldsymbol{\pi} + [a(m-r)+\sigma_r u_r]J_r + \frac{1}{2}J_{rr}\sigma_r^2 \\ &- J_{xr}\boldsymbol{\pi}^T\boldsymbol{\sigma}\boldsymbol{\rho}_r\sigma_r + \frac{J(1-\gamma)}{2}\boldsymbol{u}^T\boldsymbol{\beta}_{\boldsymbol{\rho}}^{-1}\boldsymbol{u}\Big\} = 0,\end{aligned} \tag{5.26}$$

where we set $\boldsymbol{\rho}_r = \mathrm{d}\boldsymbol{W}(t)\,\mathrm{d}W_3(t) = [\rho_{Sr}, \rho_{\delta r}, 1]^T\,\mathrm{d}t$. Proposition 5.3.1 gives the solution to Equation (5.26), assuming an exponential affine form of the value function.

Proposition 5.3.1. *The solution to* Equation (5.26) *is* $J(t,x,r) = \frac{x^{1-\gamma}}{1-\gamma}\exp(A_0(t)+A_1(t)r)$, *where*

$$
\begin{aligned}
A_1(t) &= \frac{1-\gamma}{a}(1-\mathrm{e}^{-a(T-t)})\\
A_0(t) &= \frac{1-\gamma}{2}\boldsymbol{\lambda}^T(\gamma\boldsymbol{\rho}+\boldsymbol{\beta_\rho})^{-1}\boldsymbol{\lambda}(T-t)\\
&\quad+\Big(am+\boldsymbol{\lambda}^T(\gamma\boldsymbol{\rho}+\boldsymbol{\beta_\rho})^{-1}(-\boldsymbol{\rho}_r(1-\gamma)+\boldsymbol{\beta_\rho}\boldsymbol{e}_3)\sigma_r\Big)\frac{1-\gamma}{a}\Big[T-t-\frac{1}{a}(1-\mathrm{e}^{-a(T-t)})\Big]\\
&\quad+\frac{\sigma_r^2}{2}\Bigg(1-\frac{\boldsymbol{e}_3^T\boldsymbol{\beta_\rho}\boldsymbol{e}_3}{1-\gamma}+\frac{1}{1-\gamma}(-\boldsymbol{\rho}_r(1-\gamma)+\boldsymbol{\beta_\rho}\boldsymbol{e}_3)^T(\gamma\boldsymbol{\rho}+\boldsymbol{\beta_\rho})^{-1}\\
&\qquad\qquad\times(-\boldsymbol{\rho}_r(1-\gamma)+\boldsymbol{\beta_\rho}\boldsymbol{e}_3)\Bigg)\\
&\quad\times\frac{(1-\gamma)^2}{a^2}\Big[T-t-\frac{2}{a}(1-\mathrm{e}^{-a(T-t)})+\frac{1}{2a}(1-\mathrm{e}^{-2a(T-t)})\Big].
\end{aligned}
\tag{5.27}
$$

Proof. See Appendix E.1.1. □

We discuss the completeness of market in Proposition 5.3.2. This result demonstrates that a complete market requires at least two prepaid forwards (one could be the spot price) and a bond to control for the sources of risk in the model.

Proposition 5.3.2. *Let* $J_{\boldsymbol{\pi}|_{m,s}}$ *denote the value function when investing in* m *prepaid forwards, and* $s \in \{0,1\}$ *indicates the existence of bonds (*$s=1$ *if the bond exists,* $s=0$ *otherwise); then, we have*

- $J_{\boldsymbol{\pi}|_{n>2,s}} = J_{\boldsymbol{\pi}|_{n=2,s}} \geq J_{\boldsymbol{\pi}|_{n=1,s}}$,
- $J_{\boldsymbol{\pi}}|_{m,1} \geq J_{\boldsymbol{\pi}|_{m,0}}$

Proof. Proving that $J_{\boldsymbol{\pi}|_{n>2,s}} \geq J_{\boldsymbol{\pi}|_{n>2,s}}$ is equivalent to proving it in a semi-definite sense. For $n>2$, we can see that the supremum in Equation (E.2) can be transformed into:

$$
\begin{aligned}
\sup_{\boldsymbol{\sigma}^T\boldsymbol{\pi}}\Bigg\{&J_t+J_xrx+\boldsymbol{\pi}^T\boldsymbol{\sigma}\left(J_x\boldsymbol{\lambda}-J_{xr}\boldsymbol{\rho}_r\sigma_r+J_xJ_r\frac{\boldsymbol{\beta_\rho}\boldsymbol{e}_3\sigma_r}{J(1-\gamma)}\right)+a(m-r)J_r+\frac{1}{2}J_{rr}\sigma_r^2\\
&-\frac{J_r^2\boldsymbol{e}_3^T\boldsymbol{\beta_\rho}\boldsymbol{e}_3}{2J(1-\gamma)}\sigma_r^2+\frac{1}{2}\boldsymbol{\pi}^T\boldsymbol{\sigma}\left(J_{xx}\boldsymbol{\rho}-\frac{J_x^2}{J(1-\gamma)}\boldsymbol{\beta_\rho}\right)\boldsymbol{\sigma}^T\boldsymbol{\pi}\Bigg\}=0.
\end{aligned}
\tag{5.28}
$$

In our setting, the matrix $J_{xx}\boldsymbol{\rho} - \frac{J_x^2}{J(1-\gamma)}\boldsymbol{\beta_\rho}$ is assumed to have a full rank of 3. Therefore, to achieve the supremum, one must have span$(\boldsymbol{\sigma}^T\boldsymbol{\pi}) = \mathbb{R}^3$, which can be ensured if $n \geq 2$ with the form of $\boldsymbol{\sigma}$ in Equation (5.18).

However, if $n \leq 1$ or $s = 0$, then span$(\boldsymbol{\sigma}^T\boldsymbol{\pi})$ is a subspace of $\mathbb{R}^3$. In this case, the optimal value will not be achieved except when the optimal point $\boldsymbol{\sigma}^T\boldsymbol{\pi}^*(t)$ is in the subspace, which would not be possible. □

This means that, in our setting, the utility of investing into two prepaid forward is higher than investing in one prepaid forward. Moreover, investing in three or more prepaid forwards would not generate a higher utility. Lastly, including a bond investment also leads to a higher utility.

If we have a complete market with the least number of prepaid forwards, the matrix $\boldsymbol{\sigma}$ is an invertible 3×3 matrix. Then, the optimal investment allocation becomes

$$\begin{aligned}\boldsymbol{\pi}^*(t) &= -(\boldsymbol{\sigma}^T)^{-1}\left(J_{xx}\boldsymbol{\rho} - \frac{J_x^2}{J(1-\gamma)}\boldsymbol{\beta_\rho}\right)^{-1}\left(J_x\boldsymbol{\lambda} - J_{xr}\boldsymbol{\rho}_r\sigma_r + J_xJ_r\frac{\boldsymbol{\beta_\rho}\boldsymbol{e}_3\sigma_r}{J(1-\gamma)}\right) \\ &= x\Big[\boldsymbol{\sigma}(\gamma\boldsymbol{\rho} + \boldsymbol{\beta_\rho})\boldsymbol{\sigma}^T\Big]^{-1}\boldsymbol{\sigma}\boldsymbol{\lambda} \\ &\quad + x\Big[\boldsymbol{\sigma}(\gamma\boldsymbol{\rho} + \boldsymbol{\beta_\rho})\boldsymbol{\sigma}^T\Big]^{-1}\boldsymbol{\sigma}A_1(t)\left(-\boldsymbol{\rho}_r + \frac{\boldsymbol{\beta_\rho}\boldsymbol{e}_3}{1-\gamma}\right)\sigma_r.\end{aligned} \tag{5.29}$$

The expression of the optimal allocation has two components: a Merton's type, myopic term, which is driven by all three market prices of risk $\boldsymbol{\lambda}$ (i.e. spot price, convenience yield and interest rate) and a time-dependent term driven by the stochastic nature of the interest rates.

In addition, the expression shows that the variability of the proportion of wealth allocated to risky assets over time (i.e. $\frac{\pi}{x}$) should be small, given that σ_r, the volatility of interest rate is sufficiently small.

Theorem 5.3.1. *Minkovski's theorem holds for change of measure $\boldsymbol{u}^*(t)$, given a well defined $|A_1(t)| < \infty$ for $0 \leq t \leq T$. Hence $\boldsymbol{\pi}^*$ is the optimal strategy in the worst-case scenario provided that the change of drift term, $\boldsymbol{u}^*(t)$ is a pure function of t.*

Proof. See Appendix E.1.2. □

5.3.1 Wealth-Equivalent Loss Analysis

In this section we study WELs implied by a variety of important suboptimal allocations of the form $\boldsymbol{\pi}_s(t) = x\boldsymbol{h}(t)$, where $\boldsymbol{h}(t)$ is a deterministic function.
The objective function associated with the suboptimal strategy $\boldsymbol{\pi}_s(t)$ can be defined as follows:

$$J^{\boldsymbol{\pi}_s}(t,x,r) = \inf_{\boldsymbol{u}} \mathbb{E}\left[\frac{J(1-\gamma)}{2}\int_t^T \boldsymbol{u}^T(t)\boldsymbol{\beta}_{\boldsymbol{\rho}}^{-1}\boldsymbol{u}(t)\,\mathrm{d}t + U(X_T)\right] \tag{5.30}$$

The HJB equation for Equation (5.30) is

$$\begin{aligned}\inf_{\boldsymbol{u}}\Big\{ & J_t + J_x(rx + \boldsymbol{\pi}_s^T(t)\boldsymbol{\sigma}(\boldsymbol{\lambda}-\boldsymbol{u})) + \frac{1}{2}J_{xx}\boldsymbol{\pi}_s^T(t)\boldsymbol{\sigma}\boldsymbol{\rho}\boldsymbol{\sigma}^T\boldsymbol{\pi}_s(t) + [a(m-r)+\sigma_r u_r]J_r \\ & + \frac{1}{2}J_{rr}\sigma_r^2 - J_{xr}\boldsymbol{\pi}_s^T\boldsymbol{\sigma}\boldsymbol{\rho}_r\sigma_r + \frac{J(1-\gamma)}{2}\boldsymbol{u}^T\boldsymbol{\beta}_{\boldsymbol{\rho}}^{-1}\boldsymbol{u}\Big\} = 0.\end{aligned} \tag{5.31}$$

The solution to Equation (5.31) is given in Proposition 5.3.3.

Proposition 5.3.3. *The optimal solution to Equation* (5.31) *is* $J^{\boldsymbol{\pi}_s}(t,x,r) = \frac{x^{1-\gamma}}{1-\gamma}\exp(A_0^s(t) + rA_1^s(t))$, *where*

$$\begin{aligned} A_1^s(t) &= \frac{1-\gamma}{a}(1-\mathrm{e}^{-a(T-t)}) \\ A_0^s(t) &= \int_t^T -\frac{(1-\gamma)}{2}\boldsymbol{h}^T(s)\boldsymbol{\sigma}(\gamma\boldsymbol{\rho}+\boldsymbol{\beta}_{\boldsymbol{\rho}})\boldsymbol{\sigma}^T\boldsymbol{h}(s) + \frac{1}{2}A_1^2(s)\sigma_r^2\left(1-\frac{\boldsymbol{e}_3^T\boldsymbol{\beta}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{1-\gamma}\right) \\ &\quad + amA_1(s) + \boldsymbol{h}^T(s)\boldsymbol{\sigma}((1-\gamma)\boldsymbol{\lambda} + A_1(s)(-(1-\gamma)\boldsymbol{\rho}_r + \boldsymbol{\beta}_{\boldsymbol{\rho}}\boldsymbol{e}_3)\sigma_r)\Big]\,\mathrm{d}s \end{aligned} \tag{5.32}$$

Proof. See Appendix E.1.3. □

Now, we are ready to define the wealth-equivalent "utility" loss (WEL), which is the L solution to the equation $J^{\boldsymbol{\pi}_s}(t,x,r) = J(t, x(1-L^{\boldsymbol{\pi}_s}), r)$, this is:

$$L^{\boldsymbol{\pi}_s} = 1 - \exp\left(\frac{A_0^s(t) - A_0(t)}{1-\gamma}\right), \tag{5.33}$$

where one should note that $A_1(t) = A_1^s(t)$. We consider two general families of suboptimal strategies and therefore two sources of WEL.

Misspecification of Parameters

The most important suboptimal strategies are the result of selecting embedded models; this can hence be seen as wrong choice of parameter values. We denote the set of mis-specified parameters with a hat (e.g. $\hat{\boldsymbol{\sigma}}$). Then the suboptimal investment allocation would be:

$$\boldsymbol{h}(t) = \left[\hat{\boldsymbol{\sigma}}\left(\gamma\hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\beta}}_{\hat{\rho}}\right)\hat{\boldsymbol{\sigma}}^T\right]^{-1}\hat{\boldsymbol{\sigma}}\left(\hat{\boldsymbol{\lambda}} + A_1(t)\left(-\hat{\boldsymbol{\rho}}_r + \frac{\hat{\boldsymbol{\beta}}_{\hat{\rho}}\boldsymbol{e}_3}{1-\gamma}\right)\hat{\sigma}_r\right). \tag{5.34}$$

Plugging this investment into Equation (5.30) returns a closed-form solution for J^{π_s}, which is not listed here for the sake of simplicity. In particular, this allows us to study the impact of misspecification on the single most difficult parameter to estimate from financial time series, namely, the market price of risk (i.e. λ) for an investor who ignores ambiguity aversion on any of the factors (e.g. $\hat{\beta}_i = 0$ for some i). The next proposition provides some insight into the implied WEL.

Proposition 5.3.4. *Keeping all parameters constants, the WEL from assuming the alternative ambiguity matrix $\hat{\boldsymbol{\beta}}$ is a quadratic function of $\boldsymbol{\lambda}$.*

Proof. See Appendix E.1.4. □

We can therefore study the impact of ignoring the ambiguity-aversion on a particular factor (e.g. first factor) by setting $\hat{\boldsymbol{\beta}} = \operatorname{diag}(0, \beta_2, \beta_3)$, in terms of the factor's market price of risk (λ_1). As a result, the matrix affecting $\boldsymbol{\lambda}$ is

$$(\gamma\boldsymbol{I} + \boldsymbol{\beta})^{\frac{1}{2}}(\gamma\boldsymbol{I} + \hat{\boldsymbol{\beta}})^{-1} - (\gamma\boldsymbol{I} + \boldsymbol{\beta})^{-\frac{1}{2}} = \begin{bmatrix} \frac{\beta_1}{\gamma\sqrt{\gamma+\beta_1}} & & \\ & 0 & \\ & & 0 \end{bmatrix}. \tag{5.35}$$

Therefore, the matrix form $\boldsymbol{B}$ in Equation (5.25) gives us

$$\left((\gamma\boldsymbol{I} + \boldsymbol{\beta})^{\frac{1}{2}}(\gamma\boldsymbol{I} + \hat{\boldsymbol{\beta}})^{-1} - (\gamma\boldsymbol{I} + \boldsymbol{\beta})^{-\frac{1}{2}}\right)\boldsymbol{B}^{-1}\boldsymbol{\lambda} \tag{5.36}$$

which, together with Equation (E.17), means that WEL depends only on λ_1 in a quadratic form.

Incomplete Markets

An incomplete market means a constraint on allocation, in particular $\pi_i = 0$ for some $i \in \{1,2,3\}$ in Equation (5.26).

$$\sup_{\boldsymbol{\pi}|\pi_i=0} \inf_{\boldsymbol{u}} \Big\{ J_t + J_x(rx + \boldsymbol{\pi}^T\boldsymbol{\sigma}(\boldsymbol{\lambda} - \boldsymbol{u})) + \frac{1}{2}J_{xx}\boldsymbol{\pi}^T\boldsymbol{\sigma}\boldsymbol{\rho}\boldsymbol{\sigma}^T\boldsymbol{\pi} + [a(m-r) + \sigma_r u_3]J_r + \frac{1}{2}J_{rr}\sigma_r^2 - J_{xr}\boldsymbol{\pi}^T\boldsymbol{\sigma}\boldsymbol{\rho}_r\sigma_r + \frac{J(1-\gamma)}{2}\boldsymbol{u}^T\boldsymbol{\beta}_{\boldsymbol{\rho}}^{-1}\boldsymbol{u} \Big\} = 0. \tag{5.37}$$

In this case, the optimal solution to such constrained problem is required, which is presented in the corollary next.

First, let us use the notation "$|^{-1}_{(i,j)\in\{(i,j):\pi_i,\pi_j\neq 0\}}$" for the inverse of the matrix whose rows and columns are constrained so that the corresponding investment allocation is not 0, while keeping other elements to 0. For example,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\Bigg|^{-1}_{(i,j)\in\{1,2\}} = \begin{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix} \tag{5.38}$$

Corollary 5.3.1.1. *The optimal investment allocation using Equation (5.37) is*

$$\boldsymbol{\pi}^s(t) = x\boldsymbol{P}\boldsymbol{\sigma}\left(\boldsymbol{\lambda} + A_1(t)\left(-\boldsymbol{\rho}_r + \frac{\boldsymbol{\beta}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{1-\gamma}\right)\sigma_r\right). \tag{5.39}$$

where

$$\boldsymbol{P} = \left[\boldsymbol{\sigma}(\gamma\boldsymbol{\rho} + \boldsymbol{\beta}_{\boldsymbol{\rho}})\boldsymbol{\sigma}^T\right]\Big|^{-1}_{(i,j),\pi_i,\pi_j\neq 0}. \tag{5.40}$$

Proof. See Appendix E.1.5. □

5.4 Empirical Analysis

The data we used to estimate the parameters related to the commodity and convenience yield are spot prices from WTI quotes *. We also utilized NYMEX future prices of 1- to 4-month horizons †, which are available from the U.S. Energy Information Administration.

*https://www.eia.gov/dnav/pet/pet_pri_spt_s1_d.htm

†https://www.eia.gov/dnav/pet/pet_pri_fut_s1_d.htm

For interest rate data, we used the 1-month Treasury Constant Maturity Rate[‡] available from the Federal Reserve Bank of St. Louis. We fit the data between Aug 2001 and Jan 2020, which avoided the shock due to the Covid-19 pandemic.

Similarly to the setting of [Mellios et al. (2016)], we invested in a spot commodity ($\tau = 0$) and a prepaid forward with a maturity of one quarter ($\tau_2 = 1/4$). The parameters are estimated in Section 5.4.1 and summarized in Table 5.1. In particular, the parameters of ambiguity-aversion and risk aversion come from [Flor and Larsen (2013)], where we assumed a larger range of ambiguity aversion for convenience yield due to its unobservability (hidden Markov process) and therefore the extra difficulty for an investor to trust the reference parameters (higher uncertainty)[§]. Section 5.4.2 presents the optimal strategies and the WELs from suboptimal strategies.

5.4.1 Estimation of Parameters

Referring to [Schwartz (1997)], the parameters in a real-world measure can be estimated through discretizing Equation (5.4). We assume the time variable t is partitioned into $0 = t_0 < t_1 < \cdots < t_N = T$, where time difference $\Delta t = \frac{T}{N}$.

Define $X_i = \ln S(t_i), \delta_i = \delta(t_i)$, $r_i = r(t_i)$ and $\Delta W_j(t_i) = W_j(t_{i+1}) - W_j(t_i) \sim \mathsf{N}(0, \Delta t)$. The discretized process is as follows:

$$\begin{aligned} X_{i+1} - X_i &= (r_i + \lambda_S \sigma_S - \delta_i - \frac{1}{2}\sigma_S^2)\Delta t + \sigma_S \Delta W_1(t_i) \\ \delta_{i+1} - \delta_i &= \kappa(\alpha - \delta_i)\Delta t + \sigma_\delta \Delta W_2(t_i) \\ r_{i+1} - r_i &= a(m - r_i)\Delta t - \sigma_r \Delta W_3(t_i) \end{aligned} \tag{5.41}$$

which can help us to obtain the estimates of all above-mentioned parameters and the correlation between the Brownian motions $W_1(t), W_2(t), W_3(t)$.

For the mean-reverting level parameters $\hat{m}$ and $\hat{\alpha}$, we use the expression of log commodity future price based on Equation (5.6)

$$\ln F_{i,j} = \ln S_i - \frac{\delta_i(1 - \mathrm{e}^{-\kappa T})}{\kappa} + \frac{r_i(1 - \mathrm{e}^{-aT})}{a} + C(\tau_j) \tag{5.42}$$

where $F_{i,j} := F(S(t_i), \delta(t_i), r(t_i), T_j)$, and t_i, τ_j are current times and the time to maturities respectively. Furthermore, $\hat{m}$ and $\hat{\alpha}$ can be estimated from the formula of $C(T)$ (i.e. Equation (5.7)).

[‡]https://fred.stlouisfed.org/series/DGS1MO

[§]See [Escobar et al. (2015)] for a similar situation of unobservable processes, and a detection-error-probability analysis concluding aa feasible large range for ambiguity-aversion levels.

Given the relationship between the log future price in Equation (5.42) and the stochastic processes in Equation (5.41), we combine the MLE of parameters with a KF. We create a system of a state-space model with regard to log future prices and convenience yield, as indicated below. Given the data of future prices, spot prices and interest rate, we define the known term

$$G_{i,j} := \ln F_{i,j} - \ln S_i - \frac{r_i(1-\mathrm{e}^{-aT_j})}{a} \tag{5.43}$$

where i, j represent the time and the time to maturity, respectively. Using the notation of $G_{i,j}$, our state-space model is changed into

$$\begin{aligned} \delta_{i+1} &= \kappa\alpha\Delta t + (1-\kappa\Delta t)\delta_i + \sigma_\delta \Delta W_2(t_i) \\ \boldsymbol{G}_i &= \boldsymbol{A} + \boldsymbol{K}\delta_i + \boldsymbol{v}_i. \end{aligned} \tag{5.44}$$

where $w_i = \sigma_\delta \Delta W_2(t_i) \sim N(0, \sigma_\delta^2 \Delta t)$, $G_{i,j}$ is the given measurement, δ_i is the state model to be estimated, and

$$\boldsymbol{G}_i = \begin{bmatrix} G_{i,1} \\ G_{i,2} \\ G_{i,3} \\ G_{i,4} \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} C(\tau_1) \\ C(\tau_2) \\ C(\tau_3) \\ C(\tau_4) \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} -\frac{(1-\mathrm{e}^{-\kappa T_1})}{\kappa} \\ -\frac{(1-\mathrm{e}^{-\kappa T_2})}{\kappa} \\ -\frac{(1-\mathrm{e}^{-\kappa T_3})}{\kappa} \\ -\frac{(1-\mathrm{e}^{-\kappa T_4})}{\kappa} \end{bmatrix}, \quad \boldsymbol{v}_i = \begin{bmatrix} v_{i,1} \\ v_{i,2} \\ v_{i,3} \\ v_{i,4} \end{bmatrix} \sim N(0, R) \tag{5.45}$$

where we set $R = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$.

Due to the absence of sufficient time to maturities, we cannot directly estimate all coefficients in $C(T)$ from Equation (5.7). Rather, we approximate the term $C(T)$ via Taylor expansion on e^x,

$$\begin{aligned} \frac{(1-\mathrm{e}^{-\kappa T}) - \kappa T}{\kappa^2} &= \frac{1 - (1 - \kappa T + 1/2\kappa^2 T^2 + O(T^3)) - \kappa T}{\kappa^2} \\ &= \frac{1}{2}T^2 + O(T^3) \\ \frac{4(1-\mathrm{e}^{-\kappa T}) - (1-\mathrm{e}^{-2\kappa T}) - 2\kappa T}{4\kappa^3} &= \frac{4(\kappa T - 1/2\kappa^2 T^2 + O(T^3))}{4\kappa^3} \\ &- \frac{(2\kappa T - 1/2(2\kappa T)^2 + O(T^3)) - 2\kappa T}{4\kappa^3} = O(T^3) \\ \frac{(1-\mathrm{e}^{-\kappa T}) + (1-\mathrm{e}^{-aT}) - (1-\mathrm{e}^{-(a+\kappa)T})}{\kappa a(\kappa+a)} &+ \frac{a^2(1-\mathrm{e}^{-\kappa T}) + \kappa^2(1-\mathrm{e}^{-aT}) - \kappa a^2 T - a\kappa^2 T}{\kappa^2 a^2(\kappa+a)} \\ &= O(T^3) \end{aligned} \tag{5.46}$$

By omitting the term $O(T^3)$, the approximation of $C(T)$ is

$$C(T) \approx [(\kappa\hat{\alpha} + \sigma_S \sigma_\delta \rho_{S\delta}) - (a\hat{m} + \sigma_S \sigma_r \rho_{Sr})]\frac{T^2}{2} := bT^2, \tag{5.47}$$

and the parameters to be estimated hence become b, and $A = b[\tau_1^2, \tau_2^2, \tau_3^2, \tau_4^2]^T$.

Parameters			
Convenience Yield	Mean-reverting rate	κ	4.3799
	Risk-neutral measure	$\hat{\alpha}$	-11.94%
	Real-world measure	α	-3.76%
	Market price of risk	λ_δ	0.9344
	Volatility	σ_δ	38.33%
Interest Rate	Mean-reverting rate	a	0.2131
	Real-world measure interest	m	0.77%
	Market price of risk	λ_r	0.6731
	Volatility	σ_r	0.80%
Commodities	Market price of risk	λ_S	0.1322
	Volatility	σ_S	29.86%
	Maturity 1	τ_1	0
	Maturity 2	τ_2	1/4
Correlation	Commodity and convenience yield	$\rho_{S\delta}$	49.65%
	Commodity and interest rate	ρ_{Sr}	3.51%
	interest rate and convenience yield	$\rho_{r\delta}$	-1.04%
Objective Function	Time horizon	T	10
	Relative risk aversion	γ	4
	Ambiguity Parameter 1	β_1	3
	Ambiguity Parameter 2	β_2	3,6,9
	Ambiguity Parameter 3	β_3	3

Table 5.1: Parameters from calibration

5.4.2 Optimal Strategy and Suboptimal Analysis

In this section, we first demonstrate the impact of convenience yield parameters on the optimal strategy. Then we study WEL in three different context. First, we analyze WEL for a misspecification of covariance-related parameters in Section 5.4.2. This is a type of misspecification not accounted for by ambiguity-aversion in the setting of [Maenhout (2004)]. It addresses the impact on wealth of using a wrong value for a parameter, any strategy produced with the wrong value (e.g. bad estimate) of a parameter would be suboptimal compared to the strategy produced by the true value of the parameter. The section shed light on how serious estimation inaccuracy could be on the WEL. Section 5.4.2 addresses WEL in a second context, this is as a consequence of an investor disregarding its own true level of ambiguity-aversion. This means the investor acts as if she has perfect knowledge of the true distribution of the underlying factors (stock, interest rate and convenience yield), i.e. as if $\beta = 0$. Such behaviour would be suboptimal if there is clear evidence, e.g. due to lack of data or expertise, that the investor do not know with total confidence those distributions, i.e. $\beta > 0$. Another common reason why investors may choose the $\beta = 0$ solution, even when acknowledging aversion to ambiguity, is due to the lack of a mathematical framework and closed-form solutions to implement their intentions (e.g. solely risk-averse solutions are well known); this paper fill this gap for the first time on commodity investors. It is also important to notice that the ambiguity-averse solution ($\beta > 0$) comes from a worst-case analysis axiomatized in [Gilboa and Schmeidler (1989)]. It is common to think that investors would be better off using a non-ambiguous solution as this would avoid the apparent penalty embedded in taking a worst-case solution. To see the pitfall in the reasoning here, one can use the insight from Section 5.4.2; which clearly demonstrates that using the wrong value of the parameter (acting as if it were correct) could lead to huge WEL consequences. The robust analysis theory provides a sound and optimal approach to handle such lack of knowledge. Lastly, Section 5.4.2 focuses on a third context of WEL. Here, we study the implications of investors acting as if the market were incomplete, i.e. not hedging all sources of randomness impacting prices and therefore using the incomplete market solution. Such solution would be suboptimal in the context of solutions hedging all risk (a.k.a complete market solution). This is a very common and costly mistake, documented in many papers in the area. Given the lack of analysis on convenience yield, we focus on incompleteness due to failing to hedge convenience yield movements.

Impact of Parameters on Optimal Strategy

Figure 5.1 illustrates the relationship between the optimal allocation versus one of the four parameters associated with the convenience yield: σ_δ, $\rho_{S\delta}$, α and κ, while keeping all other parameters constant. These parameters have little impact on the size of the investment in bonds, although there is a slight decrease in the allocation to bond in terms of $\rho_{S\delta}$, α and κ. This is likely explained by the low positive correlation between interest rates and convenience yield (i.e. negative correlation between the return on the bond and convenience yield).

In contrast, all of the parameters significantly impact investments in the two prepaid forwards. The investments in the prepaid forwards looks symmetrical, which means that, on the one hand, the absolute allocation in these risky assets remain constant for changes in these parameters. On the other hand, it highlights a so-called "mis-match investment of maturity". The reason for this comes from Equation (5.12), i.e. a shorter maturity implies

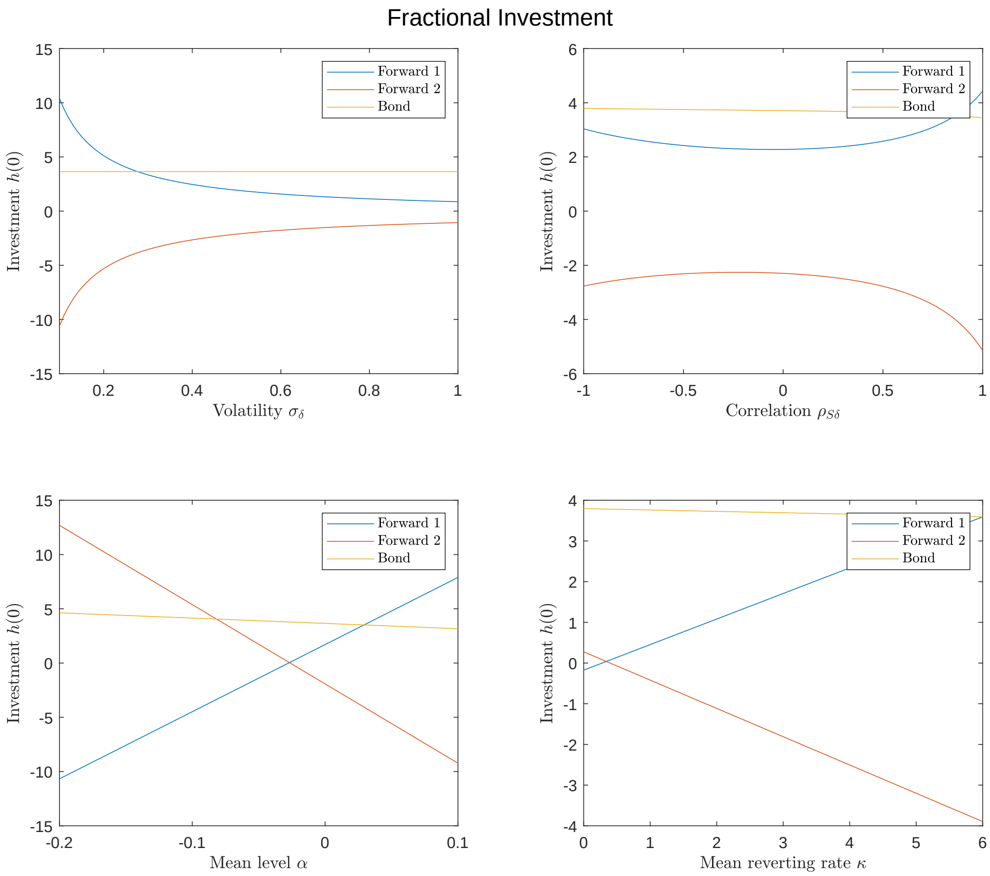


Figure 5.1: Fractional investments in assets

a larger excess return on the commodity and smaller variance (if $\rho_{S\delta} \geq 0$) hence more profitability. Moreover, a larger maturity could lead to prepaid forward negatively correlated to shorter maturities, thus explaining the shift from negative to positive positions.

A larger volatility σ_δ, and a small mean-reverting rate κ tend to decrease the scale of the mis-match investment. The decreasing trend with regard to σ_δ is due to the products becoming less appealing to risk-averse investors. Smaller mean-reverting rate κ results in a smaller market price of risk $A_{T-t}\lambda_\delta$ and volatility $A_{T-t}\sigma_\delta$, this can also be seen from Equation (5.12). As a result, the increase in overall volatility impacts more than the increase in the drift. Therefore, additional investment is allocated to the shorter-maturity Forward 1.

Parametric Misspecification of Convenience Yield

Figure 5.2 displays WELs with regard to a mis-specified volatility of the convenience yield, as well as misspecifications of the correlations between convenience yield and the other two factors (spot price and interest rate).

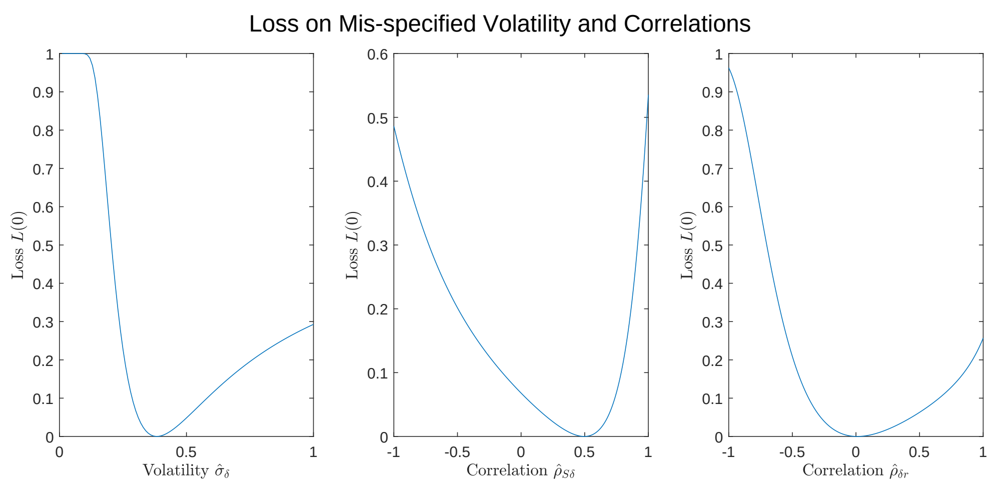


Figure 5.2: Loss versus volatility of convenience yield

All three plots exhibit relatively large losses due to discrepancies in the real value of the parameters. It must be noted that these are parameters that are not controlled by the robust approach in [Maenhout (2004)], as these are covariance related and therefore they can not be perturbed via a Girsanov change of measure.

The left plot demonstrates that working with smaller-than-true convenience yield volatilities could be more damaging than working with larger-than-true volatilities. For instance a smaller mis-specified volatility, $\hat{\sigma}_\delta$, of approximately 0.15, when the true volatility is 0.38, could result in close to 99.5% WEL. This means that a 5-cent investment by an optimal investor can produce the same utility as a 1-dollar investment for the suboptimal manager. On the other hand, using a $\hat{\sigma}_\delta$ of about only 1 leads to a 30% WEL.

For a mis-specified $\hat{\rho}_{S\delta}$, the perfect correlations (-1 and 1) lead to roughly a 50% WEL. In contrast, a mis-specified $\hat{\rho}_{\delta r}$ to -1 and 1 leads to 95% and 25% WELs respectively. This indicates that misspecifications of convenience yield correlations play a lesser role in the performance of a commodities portfolio.

Ignoring Ambiguity-Aversion

Similarly to [Branger and Larsen (2013)] and [Escobar et al. (2015)], among others, we compute the WEL when an ambiguity averse-investor decides to follow the portfolio allocation that either ignores model uncertainty or works with the wrong level.

For the left-most in Figure 5.3, we assume the actual level of ambiguity-aversion is $\beta_i = 3$, for $i = 1, 2, 3$, but the investor acts suboptimally; that is, our computation uses the strategy from Proposition 5.3.1 while forcing either $\hat{\beta}_1 = 0$ (i.e. ignoring spot price uncertainty), $\hat{\beta}_2 = 0$ (i.e. ignoring convenience yield uncertainty) or $\hat{\beta}_3 = 0$ (i.e. ignoring interest rate uncertainty). Each subfigure in Figure 5.3 hence displays the WEL for each of the three uncertainties separately, as well as a fourth curve capturing the joint WEL (i.e. the investor ignores all uncertainty $\hat{\beta}_1 = \hat{\beta}_2 = \hat{\beta}_3 = 0$). The second and third subfigures repeat the analysis assuming $\beta_2 = 6$ and $\beta_2 = 9$ respectively. This means we allow for a higher ambiguity-aversion on convenience yield because it is unobservable, and its parameters can therefore be estimated less accurately.

As can be seen, convenience yield ambiguity version plays the larger role in WEL. In particular, ignoring convenience yield uncertainty can lead to WELs ranging from 34% ($\beta_2 = 3$) to 89% ($\beta_2 = 9$). Furthermore, ignoring interest rates uncertainty leads to WELs between 17% and 52%, while spot uncertainty ranges from 0.01% to 48%. The low WELs on spot prices are due to the low value of the market price of risk on the commodity ($\lambda_S = 0.1322$), while the simultaneous increase on all WELs from Subfigure 1 Subfigure 3 is due the correlation among the factors. Particularly interesting to see is that ignoring uncertainty on all factors together leads to WEL from 43% to 91% for the WTI, highlighting the subadditivity of WEL in the context of commodity uncertainty.

Wealth-Equivalent Loss due to Incomplete Market

It is well known that incomplete markets are a major cause of poor performance in portfolios; see [Liu et al. (2005)]. In this section, we compare the WELs for three types of incomplete markets: absence of a prepaid forward of shorter maturity (e.g. spot price), the absence of longer maturity prepaid forward, and the absence of bonds. We also investigate the extreme case of a cash-only investment.

As indicated in the left of Figure 5.4, the smallest WEL comes from the absence of bonds, and it stands at 35%, which is non-negligible. Moreover, incompleteness due to not investing

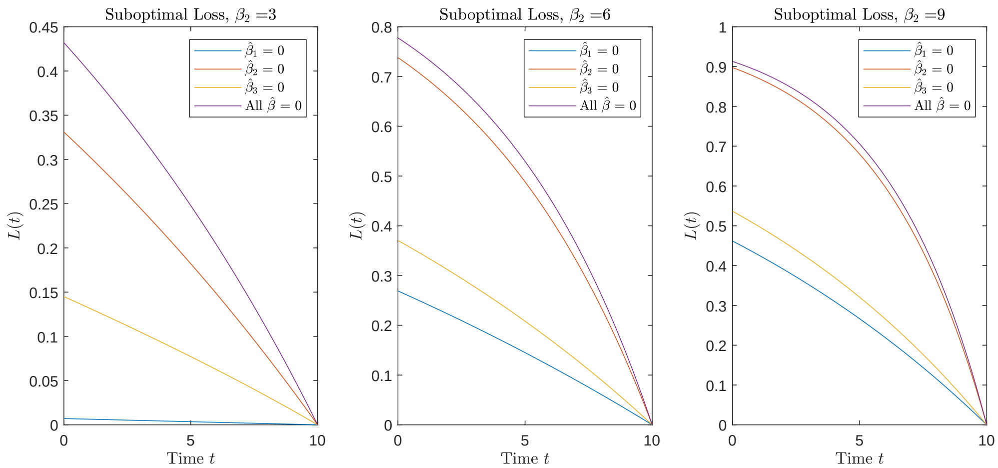


Figure 5.3: Wealth-equivalent loss for ignoring ambiguities

in the spot commodity leads to a larger WEL of approximately 49.5%, while not investing in the longer-maturity prepaid forward yields the largest WEL at 50.5%. Hence, incompleteness due to spot commodity investment is slightly less damaging than avoiding prepaid forwards.

Moreover, the absence of all assets (only investing into a bank account) leads to a WEL of about 69%, confirming again the subadditivity of the individual losses.

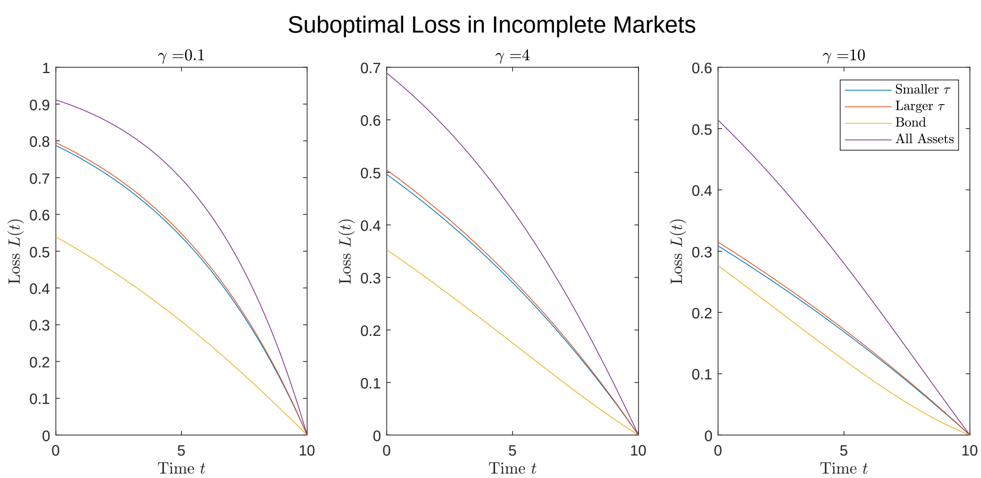


Figure 5.4: Wealth-equivalent loss comparison

5.5 Conclusion

In this thesis we studied model uncertainty in the context of portfolio analysis for the asset class of commodities. We focused on a popular model among practitioners (Model III in [Schwartz (1997)]) that highlights the three more important known factors explaining commodity term structures, namely, spot prices, convenience yields and interest rates. We are the first to compare the impact of these three potential sources of uncertainty on the performance of a portfolio.

In this study we took advantage of the affine structure of the model and the analytical robust portfolio setting of [Maenhout (2004)] to derive closed-form solutions for an ambiguity-averse utility of terminal wealth maximizer investor. We estimated the parameters of the full model using empirical data on oil future prices and short rates via a combination of MLE and KF. Our empirical analysis demonstrates that convenience yield is the most influential source of uncertainty for a commodities manager. For WTI, convenience yield is twice as critical as bond prices and substantially more important than spot prices, even under mild assumptions of ambiguity-aversion. For copper, it quickly becomes the most important factor albeit for higher ambiguity-aversion values ($\beta_2 = 7$ versus $\beta_3 = 3$). The convenience yield covariance-related parameters are also critical in terms of damages to the performance of the portfolio due to estimation imprecisions. As a byproduct, we observe that hedging the convenience yield source of risk (i.e. completing the market in terms of convenience yield) is of higher importance compared to the randomness or market-incompleteness from spot prices or interest rates.

Chapter 6

Conclusion

6.1 Summary of Contributions

In this thesis, we study applications of stochastic optimal control to two types of problems. First we anayzed differential games in energy market. We set minimal-production and minimal-profit bounds on an asymmetric-cost differential game model and compute the profit functions of each player. We also incorporate a GBM process affecting the price and production costs and successfully solve the profit function using a Puiseux series. This research makes three contributions to this topic. First, given the asymmetric-cost differential game model, we analyze the effects of bounds on the production and profit of players for different production costs. Second, we analyze the effect of a stochastic factor which affects energy prices and costs on the solution to the asymmetric-cost differential game. Third, we successfully apply the two models to real-world data and forecast production.

Next we perform a study of the construction of portfolios including commodities and fixed income instruments. We determine the robust optimal investment strategy for commodity prices following an exponential-OU model combined with stochastic interest rate. In addition, we find optimal investment for a three-factor price model. We achieve two contributions there. First, we develop the robust optimal investment into spot commodities, prepaid forwards, a bond and a risk-free bank account. Second, we analyze the effects of parametric (e.g. drifts, correlations, volatility, etc) misspecifications on the wealth-equivalent losses, with empirical analysis of WTI, gold and copper data.

Our studies provide an insight on the application of differential game models in an energy market and portfolio investments. With our differential game model, energy producers, such as oil countries and companies, may design their own production strategies and analyze the behaviours of their opponents. Commodity investors may use our results to adapt their investment strategy to better prepare for uncertainty in the distributions. Our studies are of academic value for future researchers. In this thesis, existing models are extended by adding multiple marco-economic and financial factors.

6.2 Future Works

Although we successfully make considerable contributions to differential game in energy market and robust portfolio, our models still face numerous limitations. As such, several extensions can be accomplished in the future in both theoretical and practical ways. Further investigations could, without being constrained to, include the followings:

- The single GBM stochastic factor in Chapter 3 can be developed into multiple factors affecting price and costs. This extension would enable the model to accommodate a more complicated energy market with more factors, such as extreme disasters, economic boomings, technology development, etc.

- The assumption of $r > 2\mu + \sigma^2$ in Chapter 3 directly refer to [Brown et al. (2017)]. But in reality, if r corresponds to interest rate, r cannot be larger than the economic growth. Therefore, we may set a finite time horizon over $[0, T]$.

- In the application of differential game model to real world in Appendix C, the remaining resource is not a completely observable variable, due to existence of enormous undiscovered energy source in the world. Therefore, it is meaningful to incorporate the effect of both proven reserve and undiscovered reserves.

- Chapter 4 and Chapter 5 assumed constant volatilities in the robust investment in commodities. Yet, the volatilities should be fluctuating and unobservable all the time, depending on the changing financial environments. Hence, it is worthwhile to consider stochastic volatilities in commodity price models.

- The convenience yield in Chapter 5 is a simple mean-reverting model. This model, while adding randomness, does not directly affect investor's utility of future. It would be interesting to investigate if and how a more complicated model for the convenience yield would influence investor's utility.

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Appendix A

Appendix to Chapter 2

A.1 Nash Equilibrium Computation in Example 2.3.1

$$\begin{aligned} q_1^* &= \frac{1+\sum_{i=1}^{N-1} s_i - N s_n}{N} = \frac{1+0.05+0.2}{3} - 0.05 = 0.3667 \\ q_2^* &= \frac{1+\sum_{i=1}^{N-1} s_i - N s_n}{N} = \frac{1+0.05+0.2}{3} - 0.2 = 0.2167. \end{aligned} \tag{A.1}$$

In the Nash equilibrium, the profits for the two players become

$$\begin{aligned} G_1 &= \left(\frac{1+\sum_{i=1}^{N-1} s_i - N s_n}{N}\right)^2 = 0.3667^2 = 0.1344 \\ G_2 &= \left(\frac{1+\sum_{i=1}^{N-1} s_i - N s_n}{N}\right)^2 = 0.2167^2 = 0.0469. \end{aligned} \tag{A.2}$$

If player 1 were to increase the production to $q_1 = 0.4$ slightly, then for player 2, the optimal $q_2 = \frac{1-0.4-0.2}{2} = 0.2$. Then the profit becomes

$$\begin{aligned} G_1 &= q_1(1-q_1-q_2-s_1) = 0.4(1-0.4-0.2-0.05) = 0.1400 \\ G_2 &= q_n(1-q_2-q_1-s_2) = 0.2(1-0.4-0.2-0.2) = 0.0400. \end{aligned} \tag{A.3}$$

A.2 Proof of Lemma 2.3.1

From the definition of Lambert-W function, $z = W(z)\mathrm{e}^{W(z)}$, we can compute the derivative for $W(z)$ to be

$$W'(z) = \frac{W(z)}{z(1+W(z))}. \tag{A.4}$$

Therefore, the derivative for $v(x)$ is

$$\begin{aligned} v'(x) &= \frac{2a^2}{b}\theta'(x)W'(\theta(x))(1+W(\theta(x))) \\ &= -a\theta(x)(1+W(\theta(x)))\frac{W(\theta(x))}{\theta(x)(1+W(\theta(x)))} \\ &= -aW(\theta(x)) \end{aligned} \tag{A.5}$$

Then, it is easy to see that

$$(a - v'(x))^2 = a^2(1+W(\theta(x)))^2 = bv(x). \tag{A.6}$$

So the solution satisfies the ODE. Moreover, it also satisfies the initial condition

$$v(0) = \frac{a^2}{b}(1+W(\beta e^{\beta}))^2 = \frac{a^2}{b}(1-\beta)^2 = v(0). \tag{A.7}$$

A.3 Proof of Proposition 2.3.3

By combining Equation (2.43) and Equation (2.44), and considering the first blockading point x_b^{N-1}, we can compute that

$$W(\theta_N(x_b^{N-1} - x_b^N)) = -\frac{\delta_{N-1}}{a_n}. \tag{A.8}$$

Therefore, by the definition of the Lambert-W function and since $x_b^N = 0$, the first blockading point x_b^{N-1} is

$$x_b^{N-1} = \frac{1}{\mu_N}\left(-1 + \frac{\delta_{N-1}}{a_N} - \log\left(\frac{\delta_{N-1}}{a_N}\right)\right) \tag{A.9}$$

where we define $\mu_n = \frac{b_n}{2a_n} = \frac{r}{2a_n}\left(\frac{n+1}{n}\right)^2$. Inserting Equation (A.8) into Equation (2.40) in the case of $n = N$ reveals that

$$v(x_b^{N-1}) = \frac{1}{r}(s_{N-1} - s_0 - \delta_{N-1})^2. \tag{A.10}$$

Let $K = \min\{n\colon \delta_n > 0\}$. Then the infinite-reserve producers $K, K+1, \ldots, N-1$ have blockading points. Now we find other blockading points $x_b^K, x_b^{K+1}, \ldots, x_b^{N-2}$.

By a similar calculation as for $v(x_b^{N-1})$, it can be computed that for $n \in \{K, K+1, \ldots, N-1\}$

$$v(x_b^{n-1}) = \frac{1}{r}(s_{n-1} - s_0 - \delta_{n-1})^2 = \frac{a_n^2}{b_n}(1 + W(\theta_n(x_b^{n-1} - x_b^n)))^2. \tag{A.11}$$

By taking $x_b^{n-1} - x_b^n$ out of the Lambert-W function, Equation (A.11) can indicate the relation between x_b^{n-1} and x_b^n as

$$x_b^{n-1} = x_b^n + \frac{1}{\mu_n}\left(\log\left(\frac{\delta_n}{\delta_{n-1}}\right) - \frac{(n+1)(s_n - s_{n-1})}{a_n}\right). \tag{A.12}$$

where $n = K, K+1, \dots, N-1$.

We already know that the left limit

$$\lim_{x \to x_b^n - 0} v'(x) = -a_{n+1} W(\theta_{n+1}(x_b^n - x_b^{n+1})) = \delta_n, \tag{A.13}$$

Given these $v(x_b^n) = \frac{1}{r}(s_n - s_0 - \delta_n)^2$, we can compute

$$\begin{aligned}\lim_{x \to x_b^n + 0} v'(x) &= -a_n W(\theta_n(0)) = a_n - \sqrt{b_n v(x_b^n)} = a_n - \frac{n+1}{n}(s_n - s_0 - \delta_n) \\ &= \frac{1 + \sum_{i=1}^{n-1} s_i}{n} - s_0 - \frac{n+1}{n}\left(s_n - s_0 + (n+1)s_n + \left(1 + s_0 + \sum_{i=1}^{n-1} s_i\right)\right) = \delta_n.\end{aligned} \tag{A.14}$$

Therefore, this confirms that $v'(x)$ is also continuous at x_b^n.

However, using the property to compute the second derivative of $v(x)$,

$$v''(x) = -\sum_{n=K}^{N} \frac{b_n}{2} \frac{W(\theta_n(x - x_b^n))}{1 + W(\theta_n(x - x_b^n))} \mathbb{1}_{\{x_b^n \le x < x_b^{n-1}\}}. \tag{A.15}$$

Using the left and right limits of $v'(x)$ simply leads to

$$\begin{aligned}\lim_{x \to x_b^n - 0} v''(x) &= -\frac{b_{n+1}\delta_n}{2(a_{n+1} - \delta_n)} \\ \lim_{x \to x_b^n + 0} v''(x) &= -\frac{b_n \delta_n}{2(a_n - \delta_n)}.\end{aligned} \tag{A.16}$$

This may not be equal given that different set of $\{s_i\}_{i=0,1,\dots,N-1}$ can lead to $a_n \neq a_{n+1}, b_n \neq b_{n+1}$. So the second derivative $v''(x)$ is not continuous at x_b^n.

A.4 Proof of Proposition 2.3.4

First we use the chain rule with Lemma 2.3.1 to indicate that

$$\frac{\mathrm{d}}{\mathrm{d}x} W(\theta_n(x - x_b^n)) = -\frac{b_n W(\theta_n(x - x_b^n))}{2a_n(1 + W(\theta_n(x - x_b^n)))}. \tag{A.17}$$

Over the interval $[x_b^n, x_b^{n-1})$, we transform Equation (2.4) given $q_0^*(x)$ as

$$\begin{aligned} \mathrm{d}t &= -\frac{n+1}{na_n(1+W(\theta_n(x-x_b^n)))}\,\mathrm{d}x(t) \\ &= \frac{2n}{r(n+1)W(\theta_n(x-x_b^n))}\,\mathrm{d}(W(\theta_n(x-x_b^n))) \\ &= \frac{2n}{r(n+1)}\,\mathrm{d}(\ln W(\theta_n(x-x_b^n))). \end{aligned} \tag{A.18}$$

We have demonstrated that the derivative

$$v'(x) = -\sum_{n=K}^{N} a_n W(\theta_n(x-x_b^n))\mathbb{1}_{\{x_b^n \le x < x_b^{n-1}\}} \tag{A.19}$$

which simplifies this ODE into

$$\mathrm{d}t = \frac{2n}{r(n+1)}\,\mathrm{d}(\ln v'(x)). \tag{A.20}$$

Aggregating each interval $[x_b^n, x_b^{n-1})$ from $n = l, l-1, \ldots, m$ leads to Equation (2.54).

A.5 Proof of Proposition 2.3.5

Inserting explicit formula of $q_0^*(x)$ and $q_k^*(x)$ into the HJB Equation (2.33), we can obtain for player k,

$$rv_k(x) + q_0^*(x)v_k'(x) = (q_k^*(x))^2, \tag{A.21}$$

which is a first order ODE with initial condition $v_k(0) = \frac{1}{r}G_k$ at $x = 0$. When $n < k$, the equation reduces to the homogeneous ODE

$$v_k'(x) + \frac{r(n+1)}{na_n[1+W(\theta_n(x-x_b^n))]}v_k(x) = 0. \tag{A.22}$$

The integrating factor for this ODE is $W^{-\frac{2n}{n+1}}(\theta_n(x-x_b^n))$. Then we can easily obtain the solution

$$v_k(x) = \left(\frac{W(\theta_n(x-x_b^n))}{\beta_n}\right)^{\frac{2n}{n+1}} v_k(x_b^n). \tag{A.23}$$

For $n \ge k$, player k is not blockaded and $q_k^*(x)$ takes the positive part. Using the same integrating factor, we need to find the solution to the inhomogeneous ODE

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[W^{-\frac{2n}{n+1}}(\theta_n(x-x_b^n))v_k(x)\Big] = \frac{(n+1)W^{-\frac{2n}{n+1}}(\theta_n(x-x_b^n))}{na_n[1+W(\theta_n(x-x_b^n))]}\left[c_{k,n} - \frac{a_nW(\theta_n(x-x_b^n))}{n+1}\right]^2 \tag{A.24}$$

where we define that $c_{k,n} := \frac{1+\sum_{i=0}^{n-1} s_i}{n+1} - s_k$. Taking integral on both sides, we can obtain an explicit solution

$$\begin{aligned} v_k(x) =& A_n(x)v_k(x_b^n) + \frac{c_{k,n}^2}{r}(1 - A_n(x)) - \frac{4a_n c_{k,n} n}{r(n-1)(n+1)}(W(\theta_n(x - x_b^n)) - \beta_n A_n(x)) \\ & - \frac{na_n^2}{r(n+1)^2}(W^2(\theta_n(x - x_b^n)) - \beta_n^2 A_n(x)), \end{aligned} \tag{A.25}$$

where $A_n(x) := \left(\frac{W(\theta_n(x-x_b^n))}{\beta_n}\right)^{\frac{2n}{n+1}}$.

A.6 Proof of Proposition 2.4.2

Assume n players are active in the game. First the total production of opponents is

$$Q^* = \sum_{i=1}^{N-1} q_i^* = \frac{(1-q_0)(n-1) - \sum_{i=1}^{n-1} s_i}{n} \tag{A.26}$$

given q_0 is a undetermined production, which can either be a constant or a function of x. Inserting the total production into the equality Equation (2.73) leads to

$$q_0\Big(-\frac{1}{n}q_0 + a_n\Big) \geq p, \tag{A.27}$$

which indicates the solution to this inequality,

$$\frac{na_n - \sqrt{n^2a_n^2 - 4pn}}{2} \leq q_0 \leq \frac{na_n - \sqrt{n_p^2a_n^2 - 4pn}}{2}. \tag{A.28}$$

Now we already demonstrate that $q_0^*(x) = \frac{n}{n+1}(a_n - v'(x))$ is a increasing function in Equation (2.48). Therefore, profit of the finite-reserve producer is

$$\frac{n}{(n+1)^2}\Big(na_n^2 - (n-1)a_n v'(x) - (v'(x))^2\Big). \tag{A.29}$$

over the interval $[x_b^n, x_b^{n-1})$. We have already confirmed that that $v'(x) > 0$ is a decreasing function. So this unit profit increases with x. To achieve the minimal profit, the focus should be on left interval, i.e.,

$$q_0 \geq \frac{na_n - \sqrt{n^2a_n^2 - 4pn}}{2}. \tag{A.30}$$

The remaining problem is to find the number of players the constraint-touching point x_p is achieved. To find x_p, we must testing each n so that when

$$q_0 = \frac{na_n - \sqrt{n^2a_n^2 - 4pn}}{2}, \tag{A.31}$$

the number of active players $\#\{q_k^* : q_k^* > 0\} = n-1$ from $n = N, N-1, \ldots, K$. Denote this value to be n_p. And denote the corresponding constant production as $q_{p,0}^*$.

Inserting this $q_{p,0}^*$ into formula of $q_0^*(x)$ leads to

$$x_p = x_b^{n_p} - \frac{2a_{n_p}}{b_{n_p}}\left[\ln\left(\frac{1}{\beta_{n_p}}\left(-1 + \frac{(n_p+1)q_{p,0}^*}{n_p a_{n_p}}\right)\right) - 1 + \frac{(n_p+1)q_{p,0}^*}{n_p a_{n_p}} - \beta_{n_p}\right], \tag{A.32}$$

Therefore, the production with minimal profit is

$$q_{p,0}^*(x) = \begin{cases} \frac{n_p a_{n_p} - \sqrt{n_p^2 a_{n_p}^2 - 4pn_p}}{2} & \text{when } x \leq x_p \\ q_0^*(x) \quad \text{when } x > x_p \end{cases} . \tag{A.33}$$

Appendix B

Appendix to Chapter 3

B.1 Other assumptions on the processes of $Y(t)$

Example B.1.1. Assume the dynamic of stochastic profit follows a OU process:

$$\mathrm{d}Y(t) = \kappa(\bar{y} - Y(t))\,\mathrm{d}t + \sigma\,\mathrm{d}Z(t). \tag{B.1}$$

Here a good ansatz is the quadratic model $v_k(y) = \alpha_0 + \alpha_1 y + \alpha_2 y^2$ and make the equation be

$$\alpha_2 = \frac{\omega_k}{r + 2\kappa}, \quad \alpha_1 = \frac{2\kappa\bar{y}\omega_k}{(r+2\kappa)(r+\kappa)}, \quad \alpha_0 = \frac{\omega_k}{r}\left(\frac{2\kappa^2\bar{y}^2}{(r+2\kappa)(r+\kappa)} + \frac{\sigma^2}{2(r+2\kappa)}\right) \tag{B.2}$$

Example B.1.2. Assume the dynamic of stochastic profit follows a Cinterest rate process:

$$\mathrm{d}Y(t) = \kappa(\bar{y} - Y(t))\,\mathrm{d}t + \sigma\sqrt{Y(t)}\,\mathrm{d}Z(t). \tag{B.3}$$

The ansatz here is the quadratic model $v_k(y) = \alpha_0 + \alpha_1 y + \alpha_2 y^2$ and which yields

$$\alpha_2 = \frac{\omega_k}{r + 2\kappa}, \quad \alpha_1 = \frac{\omega_k}{(r+2\kappa)(r+\kappa)}\left(2\kappa\bar{y} + \frac{\sigma^2}{2}\right), \quad \alpha_0 = \frac{\kappa\bar{y}\omega_k}{r(r+2\kappa)(r+\kappa)}\left(2\kappa\bar{y} + \frac{\sigma^2}{2}\right) \tag{B.4}$$

Example B.1.3 (Multiple Stochastic Factors)**.** This problem can be generalized into the case of multiple stochastic factor. Assume the profit of producing each unit of crude oil is

$$\pi(Q) = Y(t) - C(t) - Q \tag{B.5}$$

where $Y(t)$ and $C(t)$ can be recognized as stochastic demand and stochastic cost of production, respectively. If $Y(t)$ and $C(t)$ follow GBM, OU or Cinterest rate processes, the analytical solution to the problem can be achieved, using the objective function below,

$$v_n(y,c) = \sup_{q_n \geq 0} \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt} q_n \left(Y(t) - C(t) - q_n(t) - \sum_{i=1, i\neq n}^{N-1} q_i^*(t)\right) \mathrm{d}t\right]. \tag{B.6}$$

B.2 Sketch of Derivation of HJB Equation (3.29)

Using Bellman's principle of optimality, and denote the optimal strategy for the finite-player to be $q_0^*(t)$, the value function in Equation (3.13) can be decomposed into two components, accumulated profit over $[0,s]$, and the value function using $(X(s), Y(s))$ as a "new" initial state, given $X(0) = x > 0$.

$$\begin{aligned} v(x,y) &= \mathbb{E}_{x,y}\left[\int_0^\infty \mathrm{e}^{-rt} q_0^*(t)\left(Y(t)(1-s_0) - \sum_{i=0}^{N-1} q_i^*(t)\right) \mathrm{d}t\right] \\ &= \mathbb{E}_{x,y}\Big[\int_0^s \mathrm{e}^{-rt} q_0^*(t)\left(Y(t)(1-s_0) - \sum_{i=0}^{N-1} q_i^*(t)\right) \mathrm{d}t \\ &\quad + \mathrm{e}^{-rs}\int_s^\infty \mathrm{e}^{-r(t-s)} q_0^*(t)\left(Y(t)(1-s_0) - \sum_{i=0}^{N-1} q_i^*(t)\right) \mathrm{d}t\Big] \\ &= \mathbb{E}_{x,y}\left[\int_0^s \mathrm{e}^{-rt} q_0^*(t)\left(Y(t)(1-s_0) - \sum_{i=0}^{N-1} q_i^*(t)\right) \mathrm{d}t\right] + \mathrm{e}^{-rs}\mathbb{E}_{x,y}[v(X(s),Y(s))] \end{aligned} \tag{B.7}$$

To be specific, $q_i^*(t+s)$ for $i = 0, \dots, N-1$ is the optimal strategy for player i, taking $(X(s), Y(s))$ as our new initial state.

Using Ito's lemma results

$$\begin{aligned} \lim_{s\to 0}\Big\{\frac{1}{s}\mathbb{E}_{x,y}\left[\int_0^s \mathrm{e}^{-rt} q_0^*\left(Y(t)(1-s_0) - \sum_{i=0}^{N-1} q_i^*\right)\mathrm{d}t\right] \\ + \frac{1}{s}(\mathrm{e}^{-rs}\mathbb{E}_{x,y}[v(X(s),Y(s))] - v(x,y))\Big\} = 0 \\ q_0^*\left(y(1-s_0) - q_0^* - \sum_{i=0}^{N-1} q_i^*\right) + \lim_{s\to 0}\frac{1}{s}\Big(-v(x,y) + \mathrm{e}^{-rs}\Big(v(x,y) + \\ \mathbb{E}_{x,y}\left[\int_0^s \left(-q_0^*\frac{\partial v}{\partial x} + \mu y\frac{\partial v}{\partial y} + \frac{1}{2}\sigma^2 y^2\frac{\partial^2 v}{\partial y^2}\right)\mathrm{d}t + \int_0^s \sigma y\,\mathrm{d}Z(t)\right]\Big)\Big) = 0 \\ q_0^*\left(y(1-s_0) - q_0^* - \sum_{i=0}^{N-1} q_i^* - \frac{\partial v}{\partial x}\right) + \mu y\frac{\partial v}{\partial y} + \frac{1}{2}\sigma^2 y^2\frac{\partial^2 v}{\partial y^2} - rv(x,y) = 0 \\ \sup_{q_0\geq 0} q_0\left(y(1-s_0) - q_0 - \sum_{i=1}^{N-1} q_i^* - \frac{\partial v}{\partial x}\right) + \mu y\frac{\partial v}{\partial y} + \frac{1}{2}\sigma^2 y^2\frac{\partial^2 v}{\partial y^2} = rv \end{aligned} \tag{B.8}$$

Therefore, given x, y and other $q_i^*, i = 1, 2, \dots, N-1$ independent of time t, the optimal production q_0^* will also depend on time. This is the example for player 0. The strategy of q_i^* derived from Equation (3.14) will take a similar computation and be independent of time t.

In particular, if $X(0) = x = 0$, $X(s) = 0$ for $s > 0$, because $\mathrm{d}X(t) = q_0^*(t)\mathbb{1}_{\{X(t)>0\}}\,\mathrm{d}t = 0$. Therefore, the derivation process of the HJB equation above does not include the term of $q_0^*\frac{\partial v}{\partial x}$, where we replace $\mu(y) = \mu y, \sigma(y) = \sigma y$. Therefore, Equation (3.15) does not include any item of x.

B.3 Proof of Proposition 3.3.4

$$
\begin{aligned}
\frac{n^2}{(n+1)^2}(a_n - \delta_{n-1})^2 &= \frac{(n-1)^2}{n^2}\left(\frac{1+\sum_{i=1}^{n-1} s_i - ns_0}{n} - ns_{n-1} + \left(1 + s_0 + \sum_{i=1}^{n-2} s_i\right)\right)^2 \\
&= \frac{n^2}{(n+1)^2}\left(\frac{(1+\sum_{i=1}^{n-2} s_i)(n+1)}{n} - \frac{(n+1)(n-1)}{n}s_{n-1}\right)^2 \\
&= \frac{(n-1)^2}{n^2}\left(\frac{(1+\sum_{i=1}^{n-2} s_i)n}{n-1} - ns_{n-1}\right)^2 \\
&= \frac{(n-1)^2}{n^2}\left(\frac{1+\sum_{i=1}^{n-2} s_i}{n-1} - s_0 - ns_{n-1} + 1 + s_0 + \sum_{i=1}^{n-2} s_i\right)^2 \\
&= \frac{(n-1)^2}{n^2}(a_{n-1} - \delta_{n-1})^2.
\end{aligned} \tag{B.9}
$$

B.4 Some Special Exact Solutions Using the Puiseux Series

Using the consecutive relationship of Puiseux series in Equation (3.54), we can approximate the solution near $\xi = 0$ at $m = 8$ using Maple. The formulae of coefficients of Puiseux series of $m = 3, \ldots, 8$ are

$$
\begin{aligned}
P_{\frac{3}{2}} =& -\frac{2\sqrt{a_N(r-\mu)}(N+1)}{3N} \\
P_2 =& \frac{(N+1)^2(\sigma^2-4\mu+8r)}{48N^2} \\
P_{\frac{5}{2}} =& \frac{(N+1)^3(4\mu-8r-\sigma^2)(4r+4\mu-\sigma^2)}{2880N^3\sqrt{a_N(r-\mu)}} \\
P_3 =& -\frac{(N+1)^4(4r+4\mu-\sigma^2)(16\mu-7\sigma^2-8r)(4\mu-8r-\sigma^2)}{207360N^4a_N(r-\mu)} \\
P_{\frac{7}{2}} =& \frac{(N+1)^5(4\mu-8r-\sigma^2)(95\sigma^4-376\mu\sigma^2+188r\sigma^2+368\mu^2-368\mu r+32r^2)(4r+4\mu-\sigma^2)}{15482880N^5a_N(r-\mu)\sqrt{a_N(r-\mu)}} \\
P_4 =& -\frac{(N+1)^6(4r+4\mu-\sigma^2)(4\mu-8r-\sigma^2)}{1114767360N^6a_N^2(r-\mu)^2}\Big(-1877\sigma^6+9864\mu\sigma^4-4932r\sigma^4 \\
& -17040\mu^2\sigma^2+17040\mu r\sigma^2-2208r^2\sigma^2+9728\mu^3-14592\mu^2r+3840\mu r^2+512r^3\Big).
\end{aligned}
\tag{B.10}
$$

The coefficients of Puiseux series in Equation (B.10) also allow us to obtain several exact solutions, in the case where r, μ and σ satisfy particular equations to make the series of coefficients to be zero. For example, if we set $\sigma^2 - 4\mu + 8r = 0$, then all terms of $P_{\frac{i}{2}}$ for $i = 4, 5, \ldots$ are exactly zero. Similarly, setting $4r + 4r - \sigma^2 = 0$ also leads to $P_{\frac{i}{2}} = 0$ for $i = 5, 6, \ldots$. Therefore,

$$
H(\xi) = \begin{cases} a_N\xi - \frac{2\sqrt{a_n(r-\mu)}(N+1)}{3N}\xi^{\frac{3}{2}} \text{ when } \mu = 2r + \frac{\sigma^2}{4} \\ a_N\xi - \frac{2\sqrt{a_n(r-\mu)}(N+1)}{3N}\xi^{\frac{3}{2}} + \frac{(N+1)^2(\sigma^2-4\mu+8r)}{48N^2}\xi^2 \text{ when } \mu = \frac{\sigma^2}{4} - r. \end{cases}
\tag{B.11}
$$

where we made an unrealistic assumption that $a_N(r-\mu) < 0$, which means that the discount rate is smaller than the expected growth rate of the profit.

For another, in the case of $r = \mu$, the exact solution will be

$$
H(\xi) = a_N\xi + \frac{r(N+1)^2}{4N^2}\xi^2.
\tag{B.12}
$$

Appendix C

Application of Differential-game Model to Real-world Energy Market

C.1 Introduction

In this appendix, we apply the deterministic and stochastic differential-game model from Chapter 2 and Chapter 3 to better understand the real-world data on production, consumption, and oil reserve. We include this as an appendix because we did it as an exercise to see how empirical data might be incorporated into our model. The econometric rigor of this work is appropriate for this feasible sudy purpose but we do not feel ready for immediate publication in its current form. In the game-theory model, the finite-reserve producer is the U.S. conventional oil producer. The conventional oil reserve is assumed to be exhaustible in the foreseeable future. In fact, [Hubbert (1956)] already predicted a timeline for the exhaustion of U.S. conventional oil supplies, this is now called the "Hubbert curve". On the other hand, the other players, which include North America (U.S. shale oil and Canada heavy oil), Russia and Saudi Arabia, are infinite-reserve producers modelled to hold oil reserves which are modelled as inexhaustible. The GDP, which we relate to aggregate crude oil consumption which in turn defines production, represents the stochastic factor in our model. Using the differential-game models, we forecast productions and compare the forecast and actual productions over the period 1986 – 2016. It can be concluded that the forecast production at least visually fits the actual production. In fact, simple linear regression even show decent fits with R^2 ranging between 0.68 – 0.82.

After this introduction, Section C.2 analyzes the production data of countries accounting for the most market share, consumption and cost of production by considering the effect of specific events on the energy market. Section C.3 regresses the oil price against the GDP per unit oil consumption and analyzes the results. Section C.4 takes the consumption data as a stochastic factor to calibrate a differential game models as described in Chapter 2 and Chapter 3 and production rates. Section C.5 concludes. In this appendix, we made the following contributions:

- We describe sources of oil prices, production and consumption data. We use this to motivate a simple set of participants in our differential game, and provide numerical parameter values for the game.
- Using linear regression, we obtain the relationship between the total crude oil production and the economic growth as shown in terms of GDP and consumption. We conclude that total oil production increases with economic growth.
- We forecast the crude oil production among producers in each year using the differential game model, and make a comparison between the forecast and practical production.

C.2 Data Sources

Oil production data is summarized in Table C.1. The data in Table C.1 includes cost of production and percentage of world production between 1994-2018. In the following initial data analysis, we choose the four countries with the highest production, admittedly accounting for slightly more than 40% of the world production in aggregate and 42.95% in 2018. The U.S. data includes both conventional and shale production, so we split the data accordingly.

C.2.1 Crude Oil Production

Production data is available from several different sources. We investigated the US Energy Information Adminstration (EIA), OECD and the International Energy Agency (IEA), but in the end used EIA which provides the most comprehensive data.

Source title “international petroleum and other liquids production” from US Energy Information Administration (EIA) provides the data of production in the unit of Barrels/Day

Name	Code	Percentage (1994-2018)	Percentage (2018)	Cost of Production ($)
Brazil	BRA	2.39	-	34.99
Canada	CAN	3.66	5.15	26.64
China	CHN	5.01	4.55	-
Iran	IRN	5.24	5.13	9.08
Iraq	IRQ	3.35	5.57	10.57
Nigeria	NGA	3.06	2.40	28.99
Norway	NOR	3.30	1.83	21.31
Russia	RUS	11.72	12.99	19.21
Saudi Arabia	SAU	12.44	12.58	8.98
U.K.	GBR	2.32	1.21	44.33
U.S. (shale)	USA	11.72	13.23	23.35
U.S. (conventional)				20.99
Venezuela	VEN	3.57	1.79	27.62

Table C.1: Total production information in each country

(BPD)*. The data provides monthly world production as well as production in seven regions: Africa, Asia & Oceania, Central & South Africa, Eurasia, Europe Middle East and North America. Specifically, we consider the world production data of crude oil as the total production data Q_i.

EIA also provides monthly and annual crude oil production and consumption for certain countries†. However, only world consumption is available monthly.

Another data source is OECD (Tons/year)‡, which is only available at an annual frequency. We use the conversion of 1 Ton/year = 0.023 BPD.

At present, the top five oil producing countries are USA, Russia, Saudi Arabia, Canada and China. Figure C.1 gives the plot of production amount and percentage between 1994 and 2019. From the two plots, we can observe that Russia increased its production amount gradually, by about 3500 from 1999-2007 KBPD (Thousand Barrels/day), and maintained a relatively stable percentage (13%) after 2007. The U.S. experienced a sharp decrease until 2009, after which it gradually increased to have the highest production in the world. This growth resulted from the so-called shale oil revolution, as [Mănescu and Nuño (2015)] indi-

*https://www.eia.gov/beta/international/data/browser/

†https://www.eia.gov/totalenergy/data/browser/?tbl=T11.01B

‡https://data.oecd.org/energy/crude-oil-production.htm#indicator-chart

cates. Both Saudi Arabia and China maintained a stable production percentage (12.5% and 5% respectively), while China's production decreased very slightly after 2015. As for Canada, the production amount and percentage has been increasing over the entire period.

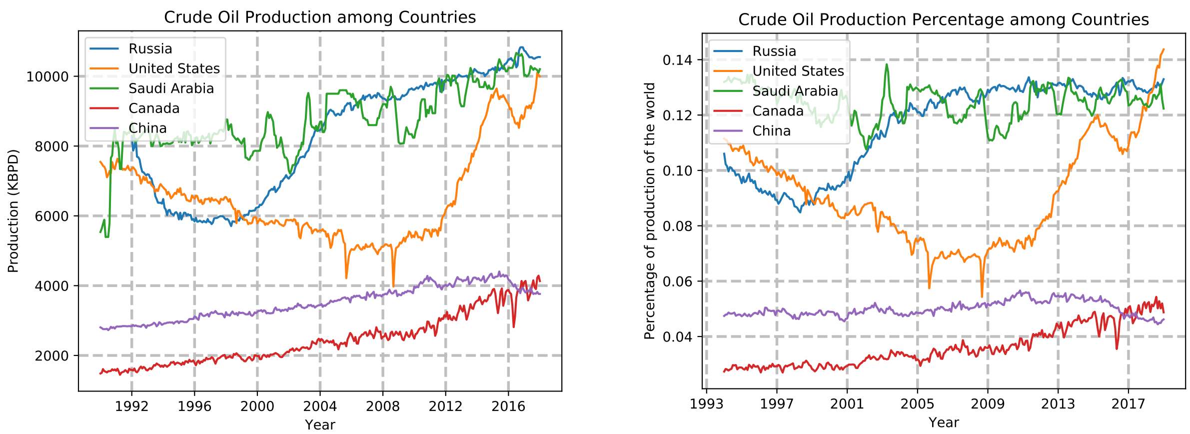

Figure C.1: Plot of production amount (left) and production percentage (right)

C.2.2 Consumption

From the same EIA source, we can also find oil consumption data among different countries and continents. Data for G7 countries is available monthly and shows seasonal periodicity which we prefer not to consider. Data for continents is available annually.

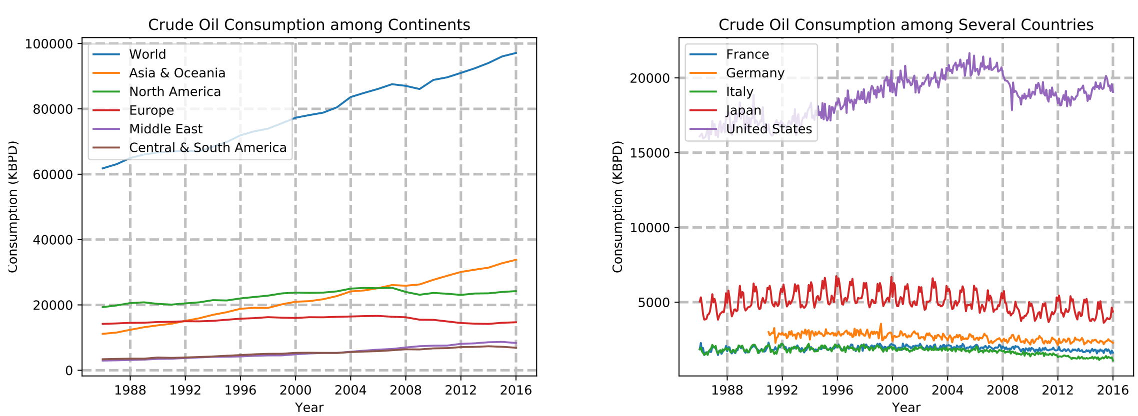

Figure C.2: Plot of oil consumption amount

Figure C.2 gives two plots of consumption data. In the left plot, annual data of continents reveal that consumption has increased during the whole period except during the financial crisis of 2007 – 2008. It is notable that the consumption in Asia & Oceania has greatly increased from about 10000 to 35000 KBPD, owing to the rapid economic growth of Asian countries. On the other hand, oil consumption for highly economically developed continents in North America and Europe has remained stable.

The right plot of Figure C.2 gives the monthly consumption of crude oil in several developed countries. This plot displays the seasonal effect of crude oil consumption. The consumption in the US increased from 16000 to 20000 KBPD approximately but experienced a heavy fall during the financial crisis. Japan, Italy and France displayed periodic monthly consumption because those countries are located in northern regions, where heating may require more energy from fuel oil.

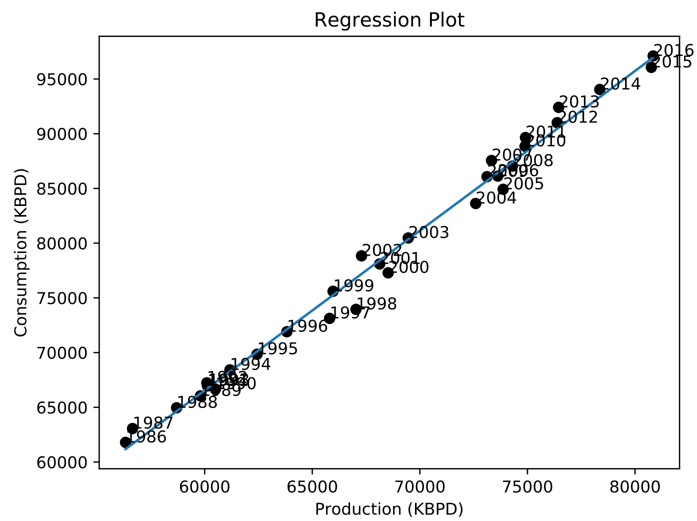

Figure C.3: Plot of oil consumption versus production

Figure C.3 shows the relationship between worldwide oil consumption and production. The linear pattern of the scatter plot is extremely obvious. A linear regression on the data confirms this, yielding the linear equation

$$\text{Consumption} = -20118.9 + 1.45 \times \text{Production}, \tag{C.1}$$

where $\mathrm{R}^2 = 0.990$ significance of coefficients (p-value < 0.001). The explanation of the plot is that, at least on an annual basis, very little crude oil is stored due to the high storage cost of crude oil. This means, we can model aggregate oil production as being driven by oil demand.

Figure C.4 also gives the relationship between productions by countries and the total consumption. There are obvious linear patterns for Saudi Arabia and Canada. Russia shows

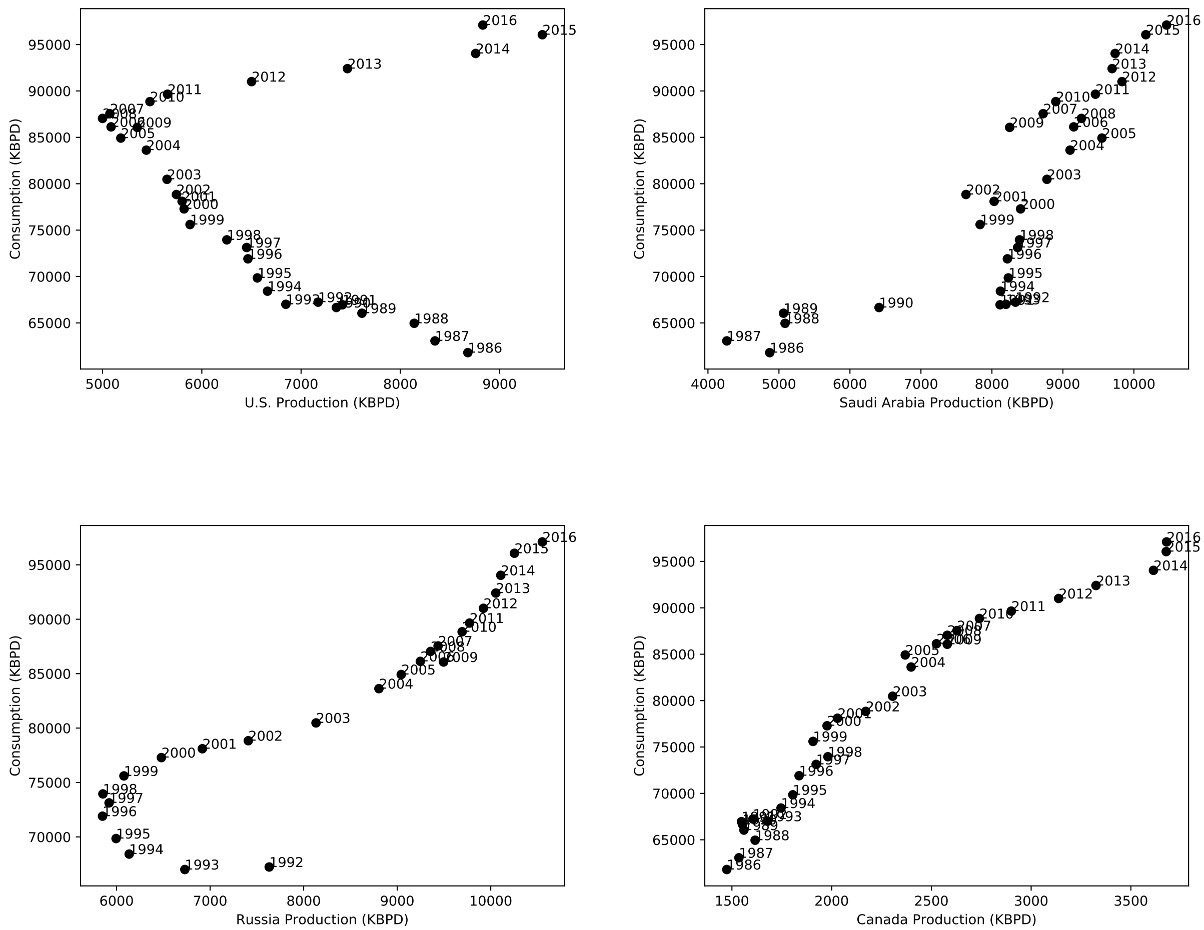

Figure C.4: Plots of oil consumption versus production by countries

a increasingly linear pattern only after 1999. There is no such an increasingly linear pattern for U.S. because the conventional oil reserves were depleted in the period 1986 – 2010.

C.2.3 Cost of Production

Recent costs of crude oil production (Dollars/barrel)[§] are available from the Wall Street Journal. The costs may be divided into several parts: gross taxes, capital spending, production costs and administrative/transportation costs. But the data is limited to just a few countries, including the U.S., Saudi Arabia, and Russia.

From Table C.1, costs of production in U.S. is divided between shale and conventional.

[§]http://graphics.wsj.com/oil-barrel-breakdown/

The production cost of shale oil ($23.35) is slightly higher than the cost of conventional oil ($20.99). Conventional oil is extracted from shallower depths to deep water, while shale oil is extracted using advanced technologies: horizontal drilling and hydraulic fracturing.

For OPEC countries, such as Iran, Iraq and Saudi Arabia, the costs of production are the lowest among almost all countries (all are around $10). The reason for these low costs is about depth of reserve and other geological properties. Oil in Russia ($19.21) is also cheaper than U.S. but more expensive than OPEC countries. The reason is that Russia has plentiful onshore oil, cheap labors and mature infrastructure.

WTI Oil Price

Historical WTI oil prices in USD/barrel ¶ are available from the St. Louis Federal Reserve Economic Database (FRED). We selected the WTI price from Jan 1994 to Jan 2019.

The crude oil price is affected by both the macro-economic environment and specific events which sharply change the supply and demand for crude oil. Wars and economic development require more crude oil while financial crisis decreases the oil demand. Moreover, development of drilling technologies may result in higher production, which increase the supply and decrease the crude oil price. On the other hand, our model considers variation in oil demand driven by GDP fluctuation.

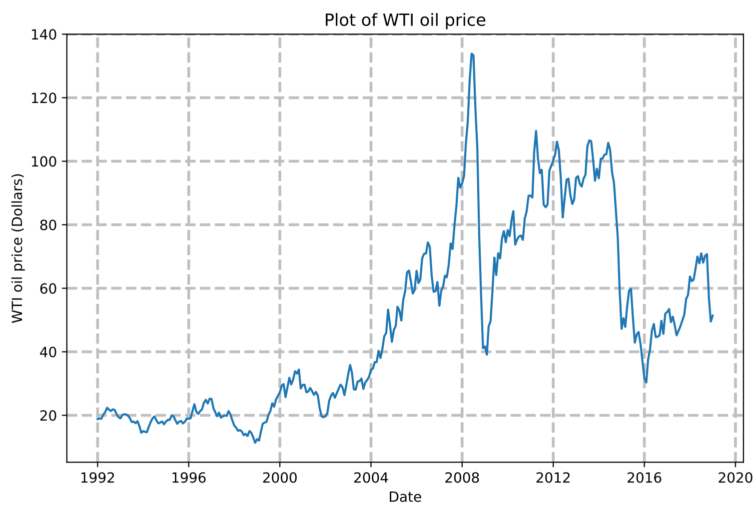

Figure C.5: Plot of oil price

¶https://fred.stlouisfed.org/series/

Figure C.5 gives the WTI oil price over the period. Starting from \$20, oil prices first vibrated around \$20, gradually went up because of the Gulf War, and peaked at about \$140 in 2009. After the recession from the 2007-2009 financial crisis, oil price went down to \$40 but shifted up again. It is worthwhile to notice that the U.S. applied "fracking" technology in 2014 and increased the production greatly. Perhaps as a result, another sharp decrease from \$105 to \$50 happened after 2014.

C.2.4 Proven Oil Reserve and Explicit Oil Production in US

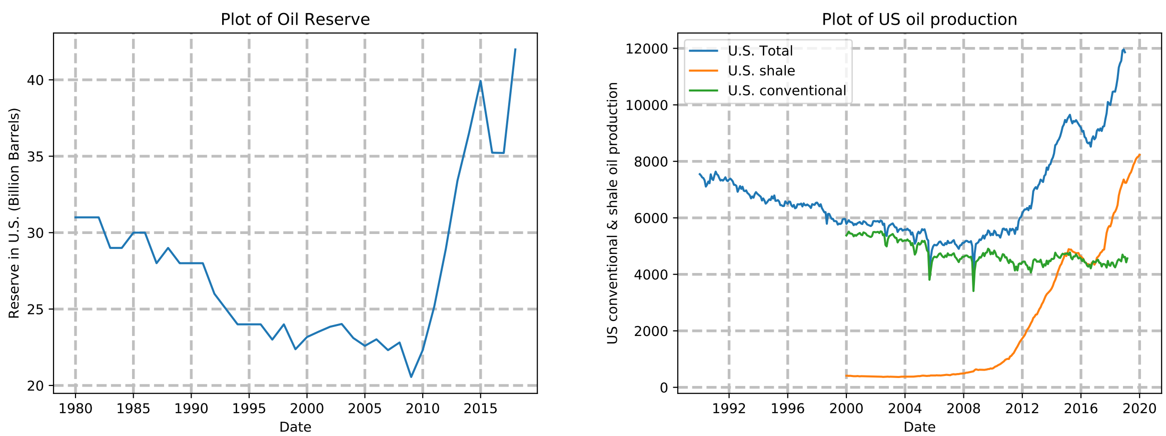

Figure C.6: Plot oil reserves (left) and daily production (right)

The left plot of Figure C.6 presents the proven oil reserve[‖] from 1980 – 2019. The proven reserve data includes shale oil and conventional oil, but not the undiscovered oil reserve. Before 2009, the amount of total oil decreased by 10 billion barrels to 21 billion barrels. This is because most of oil resources are conventional oil, as the right plot of Figure C.6 presents. Note that the data of proven oil reserve in the U.S. does not exactly represent all oil reserve in the U.S. because the undiscovered conventional oil always remains.

The right plot of Figure C.6 depicts the production of shale and conventional oil in the United States. Initially, the U.S. conventional oil production before 2009 is only 400 KBPD, accounting for only 7.5% of U.S. total production. The plot also indicates that the total production of conventional oil exceeds the decreasing amount in reserve in the meantime. Therefore, it is also reasonable to infer that conventional oil was extracted all the time, although the total conventional oil reserve is decreasing.

[‖] https://www.eia.gov/dnav/pet/hist/LeafHandler.ashx?n=PET&s=RCRR01NUS_1&f=A

After 2009, the production of shale oil increased by 8000 KBPD owing to technology breakthrough. On the other hand, the production of conventional oil is decreasing all the time. The production of shale oil exceed the conventional oil in 2016 and no longer takes the dominant portion of total U.S. production.

This information provides us with an approach to obtain a more accurate conventional oil reserve. Reserve quoted by oil companies are the quantity of oil which can be economically extended at current price conditions. Reserves, so defined, are dependent on time and price as well as on past production. In contrast, the reserve, as defined in our model, is independent of economic value. This difference in definition makes oil market modelling more challenging. We take the data in 2009 as a critical time frame, before which the shale oil does not account for oil production. Then the conventional oil reserve in the year t is

$$\text{Reserve}_t = \text{Reserve}_{2009} - \text{Total Production from 2009 to } t. \tag{C.2}$$

If $t < 2009$, the "total production" in the equation amount is negative. But even though we use the more accurate equation for oil reserve, there should be a large amount of undiscovered conventional oil reserve in the U.S. As a result, the prediction of conventional oil reserve should be underestimated.

C.2.5 Choice of Discount Rate

In Example 3.3.1, we illustrate that $r > 2\mu + \sigma^2$, which requires the discount rate of players must bigger than the growth rate of profit per time unit. Therefore, we cannot directly select r to be interest rate in our model. Moreover, the interest rate also depends on economics and hence is correlated with the oil market. Therefore, we decided to consider a alternative constant discount rate of those countries.

In this case, we decided to select $r = 0.1$ as our discount rate, which includes the interest rate and discount of future uncertainty as explained in Example 3.3.1.

C.3 Regression Model and Non-dimensionalization

In this section, we create the regression models to obtain the relationship between oil price and the economic growth. Our target in this section is to determine the price model and parameters for our differential game, and investigate the effect of each country's production. We also apply regression over production of four countries to investigate the effect of production of particular countries on the WTI price.

Theoretically, when technology does not have a dramatic development, oil prices tend to increase due to either inflation or increase of consumption. Therefore, we have two alternative choices for the growth of the oil price factor:

1. The growth of real GDP* (measured in billions of dollars). In this case we make an assumption that the profit of oil companies grows with the same speed as real GDP growth. The reasoning behind the assumption is that real GDP includes only the real economic growth without inflation. Economic growth requires the use of crude oil to provide energy for transportation or electricity.

2. The consumption of crude oil†. This choice is more reasonable because crude oil consumption directly affects oil prices and profits without considering inflation as well. But unfortunately we often lack monthly consumption data and even we have, it contains challenging seasonal effects. So we use annual data, with far fewer observations.

Since we do not consider the effect of inflation, we set the price P_i to be the WTI price discounted by CPI in year i. There are two types of regression model to be generated. The first one comes from the nondimensional price model in Chapter 2,

$$P_i = M - \alpha \frac{Q_i}{Y_i} \tag{C.3}$$

where $\frac{Q_i}{Y_i}$ is the production amount divided by the stochastic factor, K and α are coefficients. In the next section we name this model as "deterministic model".

Another model comes from the stochastic profit model in Chapter 3. In this case we made the assumption that the actual cost of production and crude oil price grows with the same rate.

$$P_i = MY_i - \alpha Q_i \tag{C.4}$$

where Y_i is the stochastic factor and Q_i is the production amount. A plausible assumption is that when the stochastic term $Y_i = 0$, there is no demand for crude oil so $Q_i = 0$ as well. Therefore, no constant term exists in this regression and the price will be $P_i = 0$. In the next section we name it as "stochastic model".

*https://fred.stlouisfed.org/series/GDPC1

†The data is available from EIA "Total oil (petroleum and other liquids) consumption"

Although the two models share the same parametric set P_i, Y_i, Q_i, their explanations are different. The model in Chapter 2 assumes $\frac{Q_i}{Y_i}$ as a whole, is the production per unit of economic level in terms of GDP or consumption. For another, the model in Chapter 3 takes Y_i as the driver of oil price and the production Q_i is the real production.

C.3.1 Growth of Real GDP

Given the data between Jan 1990 and Jan 2018, we constructed four linear regression models between monthly average WTI oil prices and monthly production/GDP w.r.t. four prime regions: World, U.S., Russia and Saudi Arabia. The coefficients for parameters are shown in Table C.2. All regression models with p-value = 0.000, which is rounded at three decimals, show excellent fits, among which the world regression has the "best" fit with $R^2 = 0.358$. However, the regression on Russian data presents the worst $R^2 = 0.087$ with an totally inverse trend with regard to production.

We also create regression models of WTI oil prices versus GDP and production. The results are shown in Table C.3. The regression of the world production has a $R^2 = 0.871$. The four regressions do not perform well with a low significance of those coefficients. Therefore, the driver of real GDP is perhaps not a good representative as driver of the market.

Regression						
Data	Param.	Coef.	Std. Error	p-value	Sgn.	R^2
World	Intercept	108.28	5.80	0.000	(***)	0.358
	WLD/GDP	-15.15	1.11	0.000	(***)	
U.S.	Intercept	60.09	2.52	0.000	(***)	0.321
	USA/GDP	-61.95	4.92	0.000	(***)	
Russia	Intercept	-12.00	7.79	0.123		0.087
	RUS/GDP	72.08	13.23	0.000	(***)	
Saudi Arabia	Intercept	78.45	4.83	0.000	(***)	0.238
	SAU/GDP	-75.08	7.34	0.000	(***)	

Table C.2: Parameters in regression using real GDP (determinstic model) corresponding to data plotted in Figure C.7

Although the real GDP is not a good indicator of the oil market, we still use the regression over the world production to test the forecast. Figure C.8 compares the predicted and real discounted prices using their ratio with the GDP-based deterministic model. If we compute

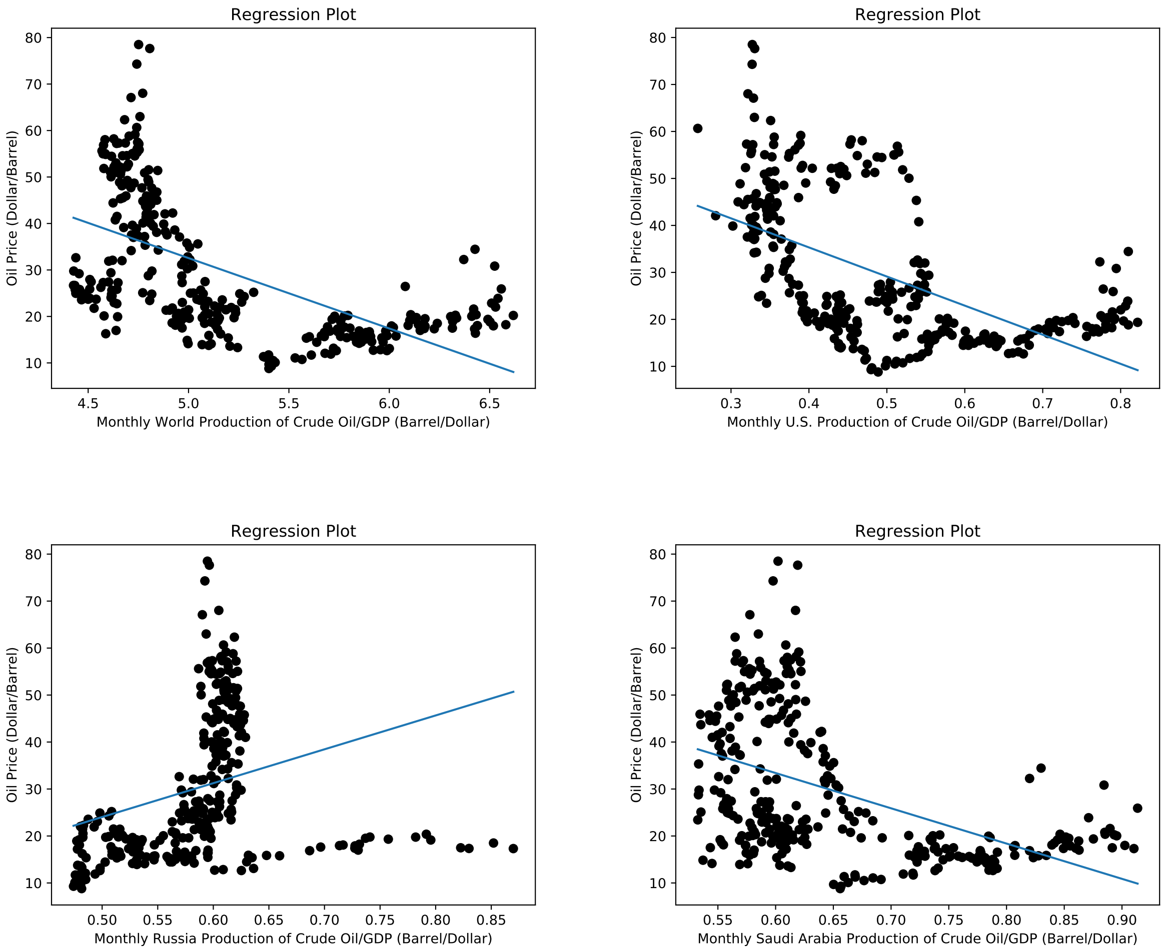

Figure C.7: Regression over world production and participants corresponding to parameters in Table C.2

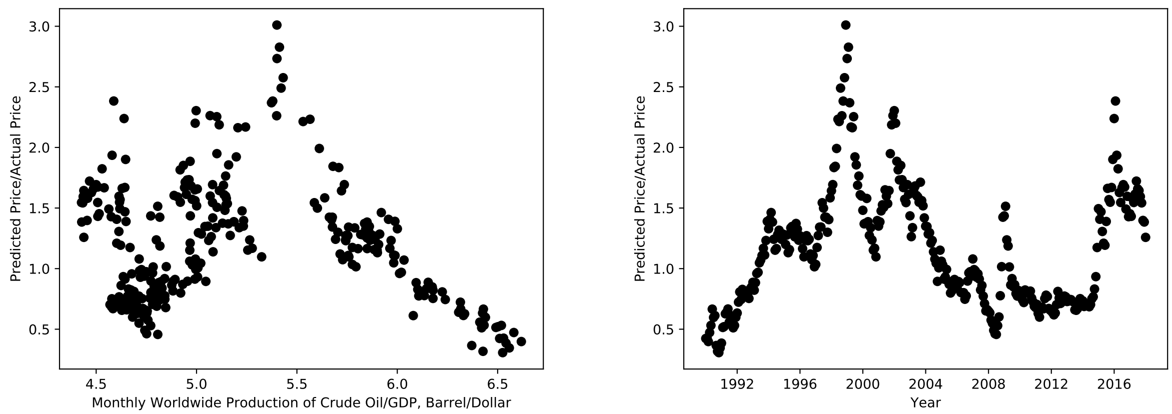

Figure C.8: Ratio of predicted and actual prices corresponding to Table C.2

the ratio of $\frac{\text{Predicted}}{\text{Actual}}$, the lowest (2.5% quantile) and highest (97.5% quantile) would be 0.46 and 2.29 respectively. The comparison is based on monthly data hence it includes numerous extreme prices due to sudden change.

Overall, the regressions using deterministic model perform well with a high significance except Russia. The negative coefficients of WLD/GDP, USA/GDP and SAU/GDP indicates a decreasing effect as production increases. Moreover, the coefficient ratios of WLD:USA:SAU (-15.15:-61.95:-75.08) indicates the effects of increased production in those countries. If booming economics drives up the production of crude oil everywhere in the same way, thus the increase of production in any particular country is assumed proportional. If we use the world GDP to fit production of a particular country, which accounts for partial world production, increase of production of this country also implicitly indicates production of other countries. Therefore, coefficients of the country are much larger than the coefficient of WLD.

The regressions using the real GDP and stochastic model do not perform very well from the perspective of parameter significance, although our target, the regression over the WLD production, gives an expected result, the higher the production, the lower the price. Moreover, the significance of the parameters in each regression are not high enough as determined by p-value. In contrast, the R^2 performs relatively better than the deterministic model, ranging from 0.87 to 0.88. In the regressions over those particular countries, the coefficients of the deterministic model also indicates a similar conclusion as the deterministic model, increase of production of each country is also proportional (WLD:USA:ASU = -0.36:-2.52:-1.86) given the driver of the economics.

Regression (Scale: $\times 10^{-3}$)						
Data	Param.	Coef.	Std. Error	p-value	Sgn.	R^2
World	GDP	3.90	1.51	0.016	(*)	0.871
	WLD	−0.36	0.30	0.236		
U.S.	GDP	3.27	0.61	0.000	(***)	0.882
	USA	−2.52	1.26	0.055	(·)	
Russia	GDP	−1.99	1.92	0.309		0.881
	RUS	6.84	3.22	0.044	(*)	
Saudi Arabia	GDP	3.24	1.27	0.017	(*)	0.868
	SAU	−1.86	2.02	0.365		

Table C.3: Parameters in regression using real GDP (stochastic model). Multiple regression results does not directly correspond to any figure plotted.

C.3.2 Growth of Consumption

In this section, we use the growth of the consumption as the driver of the oil market. Table C.4 and Table C.5 give the regression results of deterministic and stochastic models respectively. The data are annual from 1986-2016. Therefore, the number of data points is much fewer than for the monthly dataset in Section C.3.1. In those tables, the term "csmp" represents the crude oil consumption.

The regression of our deterministic model in Table C.4 shows that the world regression gives a good fit with $R^2 = 0.453$ and p-values < 0.001. Like Table C.2, the regression with regard to Russia performs the worst with a relatively insignificant evidence, the p-value $= 0.081$ and a relatively low $R^2 = 0.127$. All those regression models present a better significance level and accuracy of the deterministic price model than Table C.2. Another improvement is that all of those coefficients present a negative trend of the price over productions, which fits the intuition of supply-demand relationship. Moreover, the ratio of WLD/csmp:USA/csmp:RUS/csmp:SAU/csmp = -53.5:-234.68:-544.66:-577.48 indicates the proportional decreasing price given the drive of economic level. The reason behind this is identical to our discussion of GDP regression. Increase of production in a particular country includes increase of production in other countries.

For the regression in Table C.5, all models shows evident significance with a high R^2 ranging from 0.866 to 0.922. The p-value does not presents much better performance compared to GDP regression in Table C.3. But the factor of fewer available observations must be considered so those low p-values are more acceptable than those in Table C.3. Moreover, the negative coefficients of WLD, USA and SAU also present a negative trend, with the ratio of WLD:USA:SAU = -2.94:-3.01:-1.24. The coefficient of WLD is not much closer to zero because the coefficient of "csmp" w.r.t. world regression is much larger than others $(2.86 > 0.57, 0.45)$, compared to Table C.4.

In contrast to Figure C.8, Figure C.10 shows the ratio of predicted and actual discounted price using consumption-based deterministic model. The corresponding 2.5% (resp. 97.5%) quantile is 0.34 (resp. 1.33). Compared to the result $(0.46, 2.29)$, this prediction addresses an overall lower ratio range. Ratios of 97.5% over 2.5% quantiles are 4.98 for GDP and 3.92 for consumption respectively. Therefore, from the perspective of forecasting prices, consumption performs better.

Overall, compared to Table C.2 and Table C.3, regressions using consumption data provide results with stronger significance and higher accuracy in terms of acceptable p-values; this is due to fewer observations, and higher R^2. Therefore, in the next section, we will use parameters extracted from consumption regressions.

Regression						
Data	Param.	Coef.	Std. Error	p-value	Sgn.	R^2
World	Intercept	54.97	6.45	0.000	(***)	0.453
	WLD/csmp	−53.50	10.92	0.000	(***)	
U.S.	Intercept	38.36	8.942	0.000	(***)	0.336
	USA/csmp	−234.68	61.32	0.001	(***)	
Russia	Intercept	134.92	38.58	0.002	(**)	0.127
	RUS/csmp	−544.66	297.94	0.081	(·)	
Saudi Arabia	Intercept	141.64	16.17	0.000	(***)	0.501
	SAU/csmp	−577.48	107.01	0.000	(***)	

Table C.4: Parameters in regression using consumption (deterministic model) corresponding to data plotted in Figure C.9

C.4 Differential-game Model

In this section, we apply our models to the data using the estimated parameters. According to the significance levels, p-values and values of R^2 in Table C.2 – Table C.5, we select the consumption model with the best performance as the stochastic factor in our differential

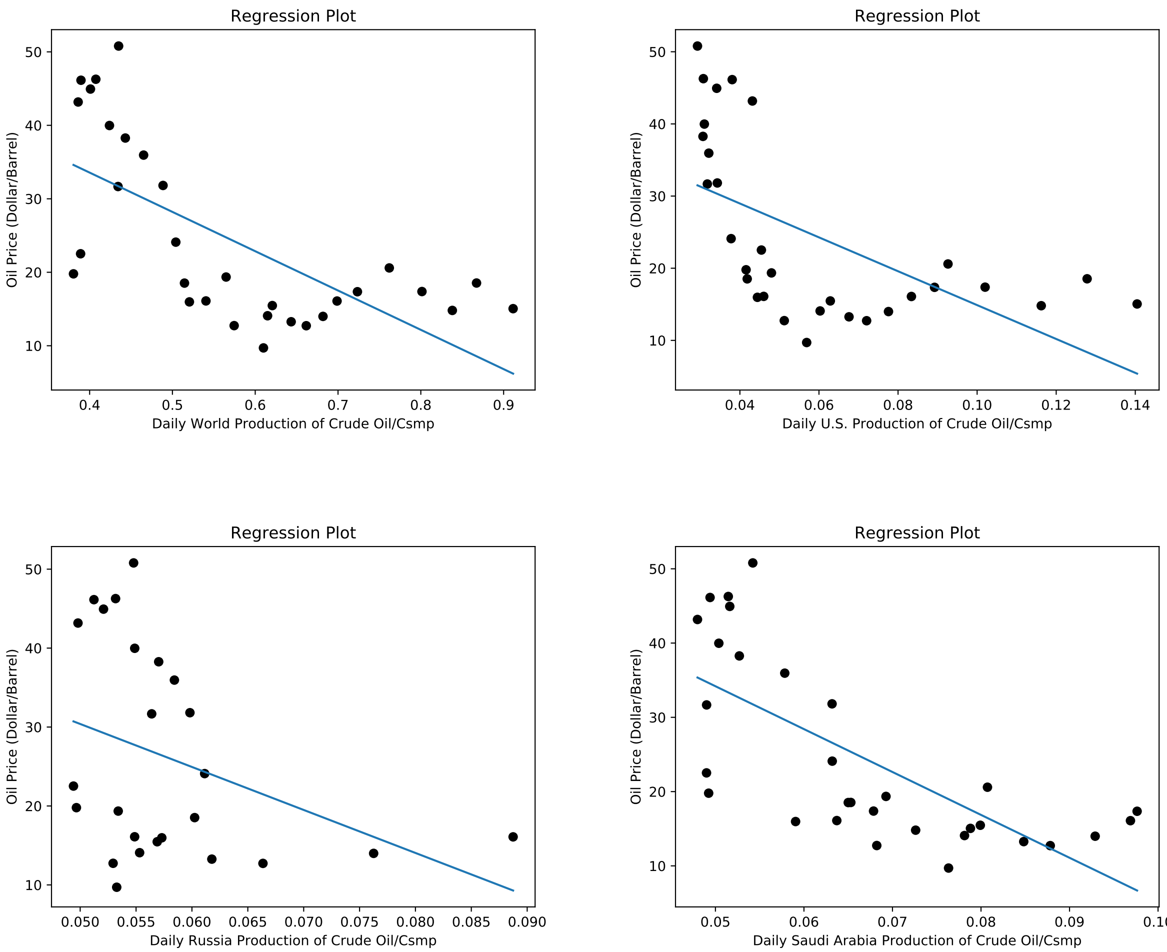

Figure C.9: Regression over world production and participants corresponding to parameters in Table C.4

game model. In our case, the coefficients for the "world" regression are the corresponding nondimensional parameter.

C.4.1 Deterministic Oil Price Model

In the deterministic model, we assume there is no stochastic term in real GDP and consumption. This assumption seems to be unreasonable over a long period so this model can only be applied over a short time interval where the stochastic term does not change much.

Table C.6 give the non-dimensionalized costs. The non-dimensionalized costs are the ratios $\frac{\text{Cost}}{K}$ where K is the constant coefficient in the regressions (deterministic).

Regression (Scale: $\times 10^{-3}$)						
Data	Param.	Coef.	Std. Error	p-value	Sgn.	R^2
World	Csmp	2.86	0.059	0.000	(***)	0.917
	WLD	−2.94	0.069	0.000	(***)	
U.S.	Csmp	0.57	0.069	0.000	(***)	0.897
	USA	−3.01	1.098	0.005	(**)	
Russia	Csmp	−0.51	0.193	0.014	(*)	0.922
	RUS	8.33	1.90	0.456		
Saudi Arabia	Csmp	0.45	0.022	0.050	(*)	0.866
	SAU	−1.24	2.09	0.557		

Table C.5: Parameters in regression using consumption (stochastic model). Multiple regression results does not directly correspond to any figure plotted.

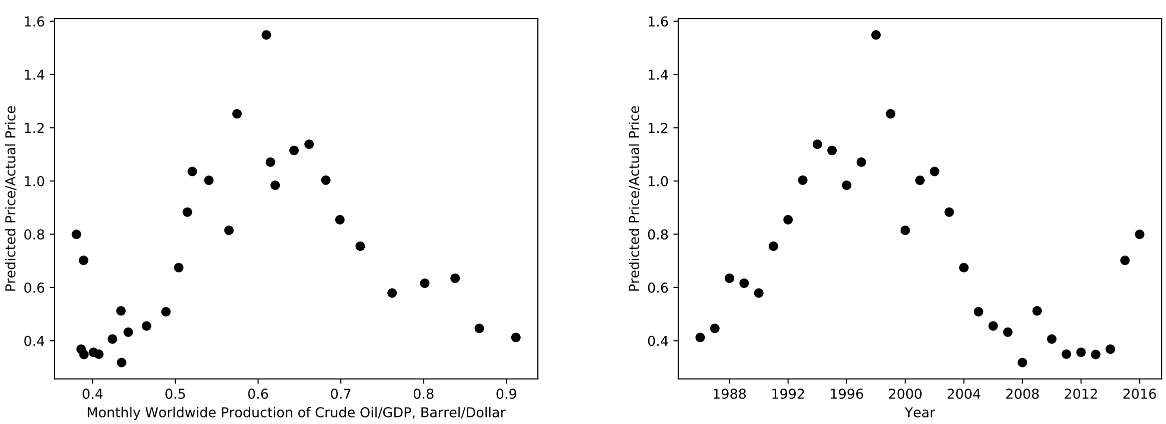

Figure C.10: Ratio of predicted and actual prices corresponding to Table C.4

	Regression	USA(shale)	USA(conventional)	SAU	RUS
Modified Cost	Real GDP	0.1156	0.1039	0.0446	0.0951
	Consumption	0.1952	0.1745	0.0747	0.1597

Table C.6: Nondimensionalized Cost of Crude Oil Production

C.4.2 Stochastic Profit Model

In this section, we work on the data using model in Chapter 3. The non-dimensional costs are the ratios between the cost and the predicted discounted 2016 price, $\frac{\text{Cost}}{Y_{2016}}$, when the costs were generated.

	Regression	USA(shale)	USA(conventional)	SAU	RUS
Modified Cost	Real GDP	0.1822	0.1638	0.0701	0.1499
	Consumption	0.0840	0.0755	0.0323	0.0691

Table C.7: Nondimensionalized Cost of Crude Oil Production (stochastic)

Figure C.11 and Figure C.12 plot the comparisons between the predicted production using formulae of stochastic and deterministic cases and the actual production in Chapter 2 and Chapter 3. From the form of q_i^*, we know that the production will be affected by $Y(t)$ and $X(t)$. q_i^* is proportional to $Y(t)$ for constant $\frac{X(t)}{Y(t)}$.

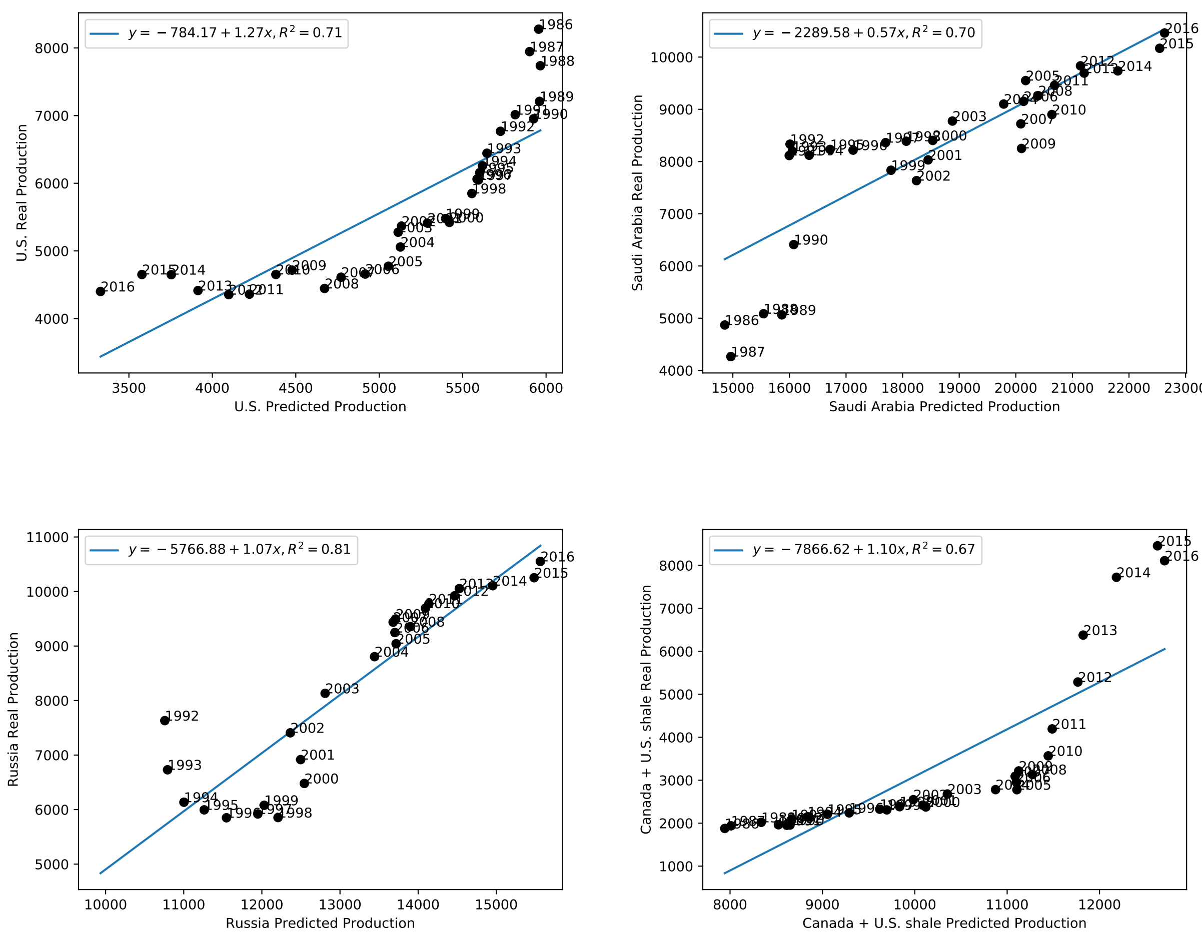

Figure C.11: Predicted versus real production (deterministic model)

The deterministic and stochastic plots comparing the forecast production and historical real production in Figure C.11 and Figure C.12 indicate a new linearly increasing trend between forecast and real production. The upper-left plots, “U.S. predicted production versus

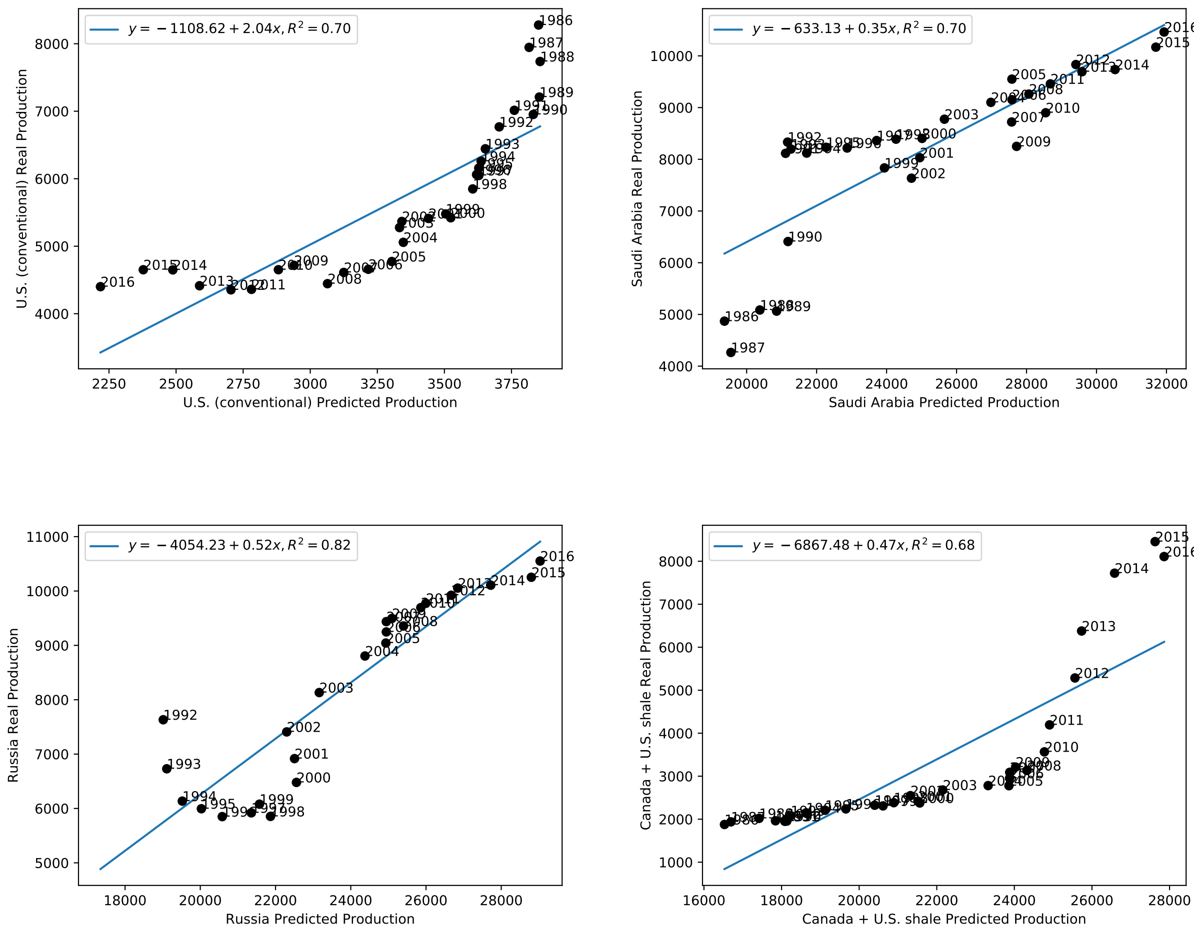

Figure C.12: Predicted versus real production (stochastic model)

real production" in Figure C.11 and Figure C.12, shows a bent shape among data points. The bent shape comes from underestimation of the conventional oil reserve, as Section C.2.4 indicates. The unavoidable underestimation of U.S. conventional oil reserve leads to reduction of the forecast. However, all prediction plots present high linear fit R^2 ranging from 0.67 to 0.83, which illustrates that the prediction gives a relative accurate result.

Moreover, although we assume only four primary players in the oil production role in the differential-game model, this is not a practical assumption, because in real world, almost all countries plays a greater or lesser role in oil production. As a result, the scales of x-axis and y-axis in each plot are different, as the Figure C.11 and Figure C.12 indicate.

C.5 Conclusion

This study applies the differential-game models from Chapter 2 and Chapter 5 to a dataset of oil production, oil reserves and consumptions from EIA. First we apply the linear regression between WTI oil price and productions among countries, which shows significant linear relationship (p-value < 0.001 and $R^2 = 0.990$). We conclude that oil demand proportionally increases oil supply in each year studied.

To study the world oil demand and individual countries, the data also shows an overall increasing trend except for U.S. production. We also compare the predicted production using the differential-game model versus the real production in the past. The pattern between the predicted and the real shows a increasing, albeit nonlinear trend, which fits our expectation. But the exact value of predicted production and the real production are different in scale and present partially linear shape, one reason may be that we only consider four representative countries as players. In reality, there are more players in the world. For example, Iran and Iraq account for great market share with low costs of production. If we take Saudi Arabia as the infinite-reserve player, those two countries are included in the predicted production because they share the similar condition. There are several other probable reasons: energy markets in those participating countries do not follow the worldwide energy market trend; the assumption of finite reserve of conventional oil is inappropriate, etc.

To sum up, this appendix created a model simple enough to be tractable but complex enough to capture real world behaviour, and illustrated present challenges in adapting the model to real data. Despite the challenges of our model, it did still have some explanatory power and some ability to assist in anomaly detection. For example, Russia's production between 1993 – 2002 did not present a reasonable trend due to instability in the country. We believe that the partial success of this exercise shows that future investigation of differential game in world oil market may bear fruitful results.

Appendix D

Appendix to Chapter 4

D.1 Proof of Proposition 4.2.1

The first derivative of the HBJI w.r.t. $\boldsymbol{u}$ leads to:

$$\begin{aligned} 0 &= (-J_{\boldsymbol{y}}^T\boldsymbol{\sigma} - J_x\boldsymbol{\pi}^T\boldsymbol{B}\boldsymbol{\sigma}) + (1-\gamma)J(\boldsymbol{u}^*)^T\boldsymbol{\beta}^{-1} \\ \boldsymbol{u}^* &= \frac{\boldsymbol{\beta}\boldsymbol{\sigma}^T(J_{\boldsymbol{y}} + J_x\boldsymbol{B}\boldsymbol{\pi})}{(1-\gamma)J}. \end{aligned} \tag{D.1}$$

More explicitly,

$$\begin{aligned} u_1^* &= \frac{\beta_1(\sigma_S\rho J_z + \sigma_r J_r + (\pi_S\sigma_S\rho - \pi_P I_\tau\sigma_r)J_x)}{(1-\gamma)J} \\ u_2^* &= \frac{\beta_2\sqrt{1-\rho^2}(\sigma_S J_z + \pi_S\sigma_S J_x)}{(1-\gamma)J}. \end{aligned} \tag{D.2}$$

Plugging $\boldsymbol{u}$ into Equation (Equation (4.21)), produces:

$$\begin{aligned} \sup_{\boldsymbol{\pi}\in\Pi}\Bigg\{ & J_t + \left[(\boldsymbol{\theta} - \boldsymbol{A}\boldsymbol{y}) - \frac{\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T(J_{\boldsymbol{y}} + J_x\boldsymbol{B}\boldsymbol{\pi})}{(1-\gamma)J}\right]^T J_{\boldsymbol{y}} + \frac{1}{2}\operatorname{tr}(J_{\boldsymbol{y}\boldsymbol{y}^T}\Sigma) + \boldsymbol{\pi}^T\boldsymbol{B}\Sigma J_{x\boldsymbol{y}} \\ & + \left[(c+r)x + \boldsymbol{\pi}^T\boldsymbol{B}\left(\boldsymbol{b} - \frac{\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T(J_{\boldsymbol{y}} + J_x\boldsymbol{B}\boldsymbol{\pi})}{(1-\gamma)J}\right)\right]J_x + \frac{J_{xx}}{2}\boldsymbol{\pi}^T\boldsymbol{B}^T\Sigma\boldsymbol{B}\boldsymbol{\pi} \\ & + \frac{(J_{\boldsymbol{y}} + J_x\boldsymbol{B}\boldsymbol{\pi})^T\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T(J_{\boldsymbol{y}} + J_x\boldsymbol{B}\boldsymbol{\pi})}{2(1-\gamma)J}\Bigg\} + \mu E[J(t, x(1-Y), z, v) - J] = 0. \end{aligned} \tag{D.3}$$

This is also quadratic w.r.t. $\boldsymbol{\pi}$. Taking derivative for $\boldsymbol{\pi}$ again, we can obtain

$$\begin{aligned} 0 &= \boldsymbol{B}\left(J_x\left(\boldsymbol{b} - \frac{\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T J_{\boldsymbol{y}}}{(1-\gamma)J}\right) + \Sigma J_{x\boldsymbol{y}}\right) + \boldsymbol{B}\left(J_{xx}\Sigma - \frac{J_x^2}{(1-\gamma)J}\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T\right)\boldsymbol{B}\boldsymbol{\pi}^* \\ \boldsymbol{\pi}^* &= -\boldsymbol{B}^{-1}\left(J_{xx}\Sigma - \frac{J_x^2}{(1-\gamma)J}\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T\right)^{-1}\left(J_x\left(\boldsymbol{b} - \frac{\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T J_{\boldsymbol{y}}}{(1-\gamma)J}\right) + \Sigma J_{x\boldsymbol{y}}\right) \end{aligned} \tag{D.4}$$

By plugging u_1^* and u_2^* into from Equation (Equation (D.2)), the explicit formulation is

$$\boldsymbol{\pi}^* = \begin{bmatrix} \pi_P^* \\ \pi_S^* \end{bmatrix} = \boldsymbol{P}^{-1}\boldsymbol{q} \tag{D.5}$$

where the matrix $\boldsymbol{P}$ and vector $\boldsymbol{q}$ are

$$\begin{aligned} \boldsymbol{P} &= \begin{bmatrix} I_\tau^2\sigma_r^2\Big(J_{xx} - \frac{\beta_1}{(1-\gamma)J}J_x^2\Big) & -\rho\sigma_S\sigma_r I_\tau\Big(J_{xx} - \frac{\beta_1}{(1-\gamma)J}J_x^2\Big) \\ -\rho\sigma_S\sigma_r I_\tau\Big(J_{xx} - \frac{\beta_1}{(1-\gamma)J}J_x^2\Big) & \sigma_S^2\Big(J_{xx} - \frac{\beta_1\rho^2+\beta_2(1-\rho^2)}{(1-\gamma)J}J_x^2\Big) \end{bmatrix} \\ \boldsymbol{q} &= \begin{bmatrix} I_\tau\Big(\rho\sigma_S\sigma_r\Big(J_{xz} - \frac{\beta_1}{(1-\gamma)J}J_xJ_z\Big) + \sigma_r^2\Big(J_{xr} - \frac{\beta_1}{(1-\gamma)J}J_xJ_r\Big) - \lambda_r\sigma_rJ_x\Big) \\ \sigma_S^2\Big(\frac{\beta_1\rho^2+\beta_2(1-\rho^2)}{(1-\gamma)J}J_xJ_z - J_{xz}\Big) + \rho\sigma_S\sigma_r\Big(\frac{\beta_1}{(1-\gamma)J}J_xJ_r - J_{xr}\Big) - (\lambda_S\sigma_S - az)J_x \end{bmatrix} \end{aligned} \tag{D.6}$$

Plugging $\boldsymbol{\pi}^*$ into the HJBI PDE, we can obtain the final PDE in Equation (Equation (4.24)).

D.2 Proof of Proposition 4.2.2

Inserting the ansatz in Equation (Equation (4.28)) into the PDE in Equation (Equation (4.24)) and grouping conveniently, we obtain a system of ODE

$$\begin{aligned} &M_0' + \boldsymbol{y}^T\boldsymbol{M}_1' + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{M}_2'\boldsymbol{y} + \boldsymbol{\theta}^T\boldsymbol{M}_1 + \boldsymbol{y}^T(\boldsymbol{M}_2^T\boldsymbol{\theta} - \boldsymbol{A}^T\boldsymbol{M}_1) - \frac{1}{2}\boldsymbol{y}^T(\boldsymbol{A}^T\boldsymbol{M}_2 + \boldsymbol{M}_2^T\boldsymbol{A})\boldsymbol{y} \\ &\quad + \frac{1}{2}\operatorname{tr}(\boldsymbol{M}_2\Sigma) + c(1-\gamma) + \boldsymbol{y}^T\boldsymbol{e}_1(1-\gamma) + \frac{1}{2}\boldsymbol{M}_1^T\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1 \\ &\quad + \boldsymbol{y}^T\boldsymbol{M}_2^T\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1 + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{y} \\ &\quad + \frac{1}{2}\left(\boldsymbol{\lambda} + \boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1\right)^T\left(\boldsymbol{\sigma}\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\right)^{-1}\left(\boldsymbol{\lambda} + \boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1\right) \\ &\quad + \boldsymbol{y}^T\left(\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_2 - a\boldsymbol{E}_2\right)^T\left(\boldsymbol{\sigma}\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\right)^{-1}\left(\boldsymbol{\lambda} + \boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1\right) \\ &\quad + \frac{1}{2}\boldsymbol{y}^T\left(\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_2 - a\boldsymbol{E}_2\right)^T\left(\boldsymbol{\sigma}\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\right)^{-1}\left(\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_2 - a\boldsymbol{E}_2\right)\boldsymbol{y} \\ &\quad + \mu\mathbb{E}[(1-Y)^{1-\gamma} - 1] = 0. \end{aligned} \tag{D.7}$$

Arranging constant parts, first order part $\boldsymbol{y}^T(\,\cdot\,)$ and second order part $\frac{1}{2}\boldsymbol{y}^T(\,\cdot\,)\boldsymbol{y}$ can result in the Riccati ODE:

$$\begin{cases} M_0' + c(1-\gamma) + \mu\mathbb{E}[(1-Y)^{1-\gamma} - 1] + \frac{1}{2}\operatorname{tr}(\boldsymbol{M}_2\Sigma) + \frac{1}{2}\boldsymbol{\lambda}^T\Big(\boldsymbol{\sigma}\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\Big)^{-1}\boldsymbol{\lambda} \\ \qquad +\boldsymbol{M}_1^T\Big[\boldsymbol{\theta} + \boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)^{-1}\boldsymbol{\sigma}^{-1}\boldsymbol{\lambda}\Big] \\ \qquad +\frac{1}{2}\boldsymbol{M}_1^T\boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\Big(\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)^{-1} + \Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)^{-1}\Big)\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\boldsymbol{M}_1 = 0 \\ M_0(T) = 0, \end{cases}$$

$$\begin{cases} \boldsymbol{M}_1' - a\boldsymbol{E}_2\Big(\boldsymbol{\sigma}\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\Big)^{-1}\boldsymbol{\lambda} - \Big(\boldsymbol{A}^T + a\boldsymbol{E}_2(\boldsymbol{\sigma}^T)^{-1}\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)^{-1}(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma})\boldsymbol{\sigma}^T\Big)\boldsymbol{M}_1 \\ \qquad +\boldsymbol{M}_2^T\Big(\boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)^{-1}\boldsymbol{\sigma}^{-1}\boldsymbol{\lambda} + \boldsymbol{\theta}\Big) + (1-\gamma)\boldsymbol{e}_1 \\ \qquad +\boldsymbol{M}_2^T\boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\Big(\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)^{-1} + \Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)^{-1}\Big)\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\boldsymbol{M}_1 = 0 \\ \boldsymbol{M}_1(T) = \boldsymbol{0}_{2\times 1} \end{cases} \tag{D.8}$$

$$\begin{cases} \boldsymbol{M}_2' + a^2\boldsymbol{E}_2\Big(\boldsymbol{\sigma}\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\Big)^{-1}\boldsymbol{E}_2 - \Big(\boldsymbol{A}^T + a\boldsymbol{E}_2(\boldsymbol{\sigma}^T)^{-1}\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)^{-1}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\Big)\boldsymbol{M}_2 \\ \qquad -\boldsymbol{M}_2^T\Big(\boldsymbol{A}^T + a\boldsymbol{E}_2(\boldsymbol{\sigma}^T)^{-1}\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)^{-1}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\Big)^T \\ \qquad +\boldsymbol{M}_2^T\boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\Big(\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)^{-1} + \Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)^{-1}\Big)\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\boldsymbol{M}_2 = 0 \\ \boldsymbol{M}_2(T) = \boldsymbol{0}_{2\times 2} \end{cases}$$

For simplicity, we set the notation

$$\begin{aligned} \boldsymbol{D}_0 &:= a^2\boldsymbol{E}_2\Big(\boldsymbol{\sigma}(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma})\boldsymbol{\sigma}^T\Big)^{-1}\boldsymbol{E}_2 \\ \boldsymbol{D}_1 &:= -\Big(\boldsymbol{A}^T + a\boldsymbol{E}_2(\boldsymbol{\sigma}^T)^{-1}\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)^{-1}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\Big) \\ \boldsymbol{D}_2 &:= \boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\Big(\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)^{-1} + \Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)^{-1}\Big)\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T \\ \boldsymbol{C}_0 &:= (1-\gamma)\boldsymbol{e}_1 - a\boldsymbol{E}_2\Big(\boldsymbol{\sigma}\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\Big)^{-1}\boldsymbol{\lambda} \\ \boldsymbol{C}_1 &:= \Big(\boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)^{-1}\boldsymbol{\sigma}^{-1}\boldsymbol{\lambda} + \boldsymbol{\theta}\Big) \\ B_0 &= c(1-\gamma) + \mu\mathbb{E}[(1-Y)^{1-\gamma} - 1] + \frac{1}{2}\operatorname{tr}(\boldsymbol{M}_2(t)\Sigma) + \frac{1}{2}\boldsymbol{\lambda}^T\Big(\boldsymbol{\sigma}\Big(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\Big)^{-1}\boldsymbol{\lambda}. \end{aligned} \tag{D.9}$$

to transform the Riccati ODEs in Equation (D.8) into the three ODEs in Proposition 4.2.2.

D.3 Proof of Theorem 4.2.3

We first rewrite the optimal change of measure $\boldsymbol{u}^*(t)$ after inserting explicit $\boldsymbol{\pi}^*(t)$

$$\boldsymbol{u}^* = \boldsymbol{A}_1(t) + \boldsymbol{A}_2(t)\boldsymbol{y}(t) \tag{D.10}$$

where

$$\begin{aligned}\boldsymbol{A}_1(t) &= \frac{\boldsymbol{\beta}\boldsymbol{\sigma}^T}{1-\gamma}\left(\boldsymbol{M}_1(t) + (1-\gamma)\Big(\boldsymbol{\sigma}(\gamma\boldsymbol{I}+\boldsymbol{\beta})\boldsymbol{\sigma}^T\Big)^{-1}\left(\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1(t) + \boldsymbol{\lambda}\right)\right.\\ \boldsymbol{A}_2(t) &= \frac{\boldsymbol{\beta}\boldsymbol{\sigma}^T}{1-\gamma}\left[\boldsymbol{M}_2(t) + (1-\gamma)\Big(\boldsymbol{\sigma}(\gamma\boldsymbol{I}+\boldsymbol{\beta})\boldsymbol{\sigma}^T\Big)^{-1}\left(\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_2(t) - a\boldsymbol{E}_2\right)\right].\end{aligned} \tag{D.11}$$

Now we return to the lemma. In order to prove Novikov's condition, we rewrite

$$\begin{aligned}&\mathbb{E}^P\left[\exp\left(\frac{1}{2}\int_0^T \|\boldsymbol{u}^*(t)\|^2\,\mathrm{d}t\right)\right]\\ =&\mathbb{E}^P\left[\exp\left(\frac{1}{2}\int_0^T \boldsymbol{A}_1^T(t)\boldsymbol{A}_1(t) + 2\boldsymbol{A}_1^T(t)\boldsymbol{A}_2(t)\boldsymbol{y} + \boldsymbol{y}^T(t)\boldsymbol{A}_2^T(t)\boldsymbol{A}_2(t)\boldsymbol{y}(t)\,\mathrm{d}t\right)\right]\end{aligned} \tag{D.12}$$

Therefore, we need to prove the boundness of $\boldsymbol{A}_1(t), \boldsymbol{A}_2(t)$ and $\boldsymbol{y}(t)$. By ODE of $\boldsymbol{M}_1$ in Theorem 4.2.2, the Lipschitz condition of existence and uniqueness of $\boldsymbol{M}_1$ holds for $0 \le t \le T$ if $\boldsymbol{M}_2$ holds. Therefore, $\|\boldsymbol{M}_1(t)\| < \infty$ is a well-defined vector function.

Also by the form of $\boldsymbol{A}_1(t)$ and $\boldsymbol{A}_2(t)$. Obviously $\|\boldsymbol{A}_1(t)\| < \infty$ and $\|\boldsymbol{A}_2(t)\|_F < \infty$ by Minkovski's inequality and Schwartz's inequality.

Now we plug $\boldsymbol{u}^*(t)$ to obtain the dynamic of $\boldsymbol{y}(t)$ in worst-case scenario

$$\mathrm{d}\boldsymbol{y}(t) = [\boldsymbol{\theta} - \boldsymbol{A}_1(t) - (\boldsymbol{A} + \boldsymbol{A}_2(t))\boldsymbol{y}(t)]\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{W}^Q(t). \tag{D.13}$$

The Lipschitz condition for $\boldsymbol{y}(t)$ to be well-defined also holds. Therefore,

$$\mathbb{E}\left[\int_0^T \|y\|^2\,\mathrm{d}t\right] < \infty. \tag{D.14}$$

Using Minkovski's inequality and Schwartz's inequality again for Equation (Equation (D.12)), we successfully prove

$$\mathbb{E}^P\left[\exp\left(\frac{1}{2}\int_0^T \|\boldsymbol{u}^*(t)\|^2\,\mathrm{d}t\right)\right] < \infty. \tag{D.15}$$

With the explicit formula for $\boldsymbol{y}(t)$ in the worst-case scenario, the sup-inf problem degenerates into an HJB problem studied in [Chiu and Wong (2013)]. The proof of the second part follows along the lines of their Section 3.2. The only difference is that in their thesis the mean-reverting matrix A is constant, while our mean-reverting matrix is $\boldsymbol{A}+\boldsymbol{A}_2(t)$ in the worst-case scenario.

D.4 Proof of Proposition 4.3.1

Taking derivative on the left side of Equation Equation (4.38), we can obtain

$$\begin{aligned} 0 &= -\boldsymbol{\sigma}^T(J_{\boldsymbol{y}} + J_x \boldsymbol{B}\boldsymbol{\pi}^s) + (1-\gamma)\boldsymbol{\beta}^{-1}\boldsymbol{u}^* \\ \boldsymbol{u}^* &= \frac{\boldsymbol{\beta}\boldsymbol{\sigma}^T(J_{\boldsymbol{y}} + J_x \boldsymbol{B}\boldsymbol{\pi}^s)}{(1-\gamma)J}. \end{aligned} \tag{D.16}$$

Plugging $\boldsymbol{u}^*$ back into the HJB in Equation (Equation (4.38)), we obtain

$$\begin{aligned} &J_t + (\boldsymbol{\theta} - \boldsymbol{A}\boldsymbol{y})^T J_{\boldsymbol{y}} + (c+r)xJ_x - \frac{J_{\boldsymbol{y}}^T \boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T J_{\boldsymbol{y}}}{2(1-\gamma)J} + \frac{1}{2}\operatorname{tr}(J_{\boldsymbol{y}\boldsymbol{y}^T}\Sigma) \\ &\quad + (\boldsymbol{\pi}^s)^T \boldsymbol{B}\left((\boldsymbol{\lambda} - a\boldsymbol{E}_2\boldsymbol{y})J_x + \Sigma J_{x\boldsymbol{y}} - \frac{J_x \boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T J_{\boldsymbol{y}}}{(1-\gamma)J}\right) \\ &\quad + \frac{1}{2}(\boldsymbol{\pi}^s)^T \boldsymbol{B}\boldsymbol{\sigma}\left(J_{xx}\boldsymbol{I} - \frac{J_x^2}{(1-\gamma)J}\boldsymbol{\beta}\right)\boldsymbol{\sigma}^T \boldsymbol{B}\boldsymbol{\pi}^s + \mu E[J(t, x(1-Y), z, v) - J] = 0. \end{aligned} \tag{D.17}$$

Using the ansatz $J(t,x,r,z) = \frac{x^{1-\gamma}}{1-\gamma}\exp\Big(M_0^{\pi^s}(t) + \boldsymbol{M}_1^{\pi^s}(t)\boldsymbol{y} + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{M}_2^{\pi^s}(t)\boldsymbol{y}\Big)$ and the form $\boldsymbol{\pi}^s = x(\boldsymbol{h} + \boldsymbol{H}\boldsymbol{y})$ where $\boldsymbol{h} := \boldsymbol{h}(t)$ and $\boldsymbol{H} := \boldsymbol{H}(t)$, we obtain

$$\begin{aligned} &(M_0^{\pi^s})' + \boldsymbol{y}^T(\boldsymbol{M}_1^{\pi^s})' + \frac{1}{2}\boldsymbol{y}^T(\boldsymbol{M}_2^{\pi^s})'\boldsymbol{y} + (\boldsymbol{\theta} - \boldsymbol{A}\boldsymbol{y})^T(\boldsymbol{M}_1^{\pi^s} + \boldsymbol{M}_2^{\pi^s}\boldsymbol{y}) + (c + \boldsymbol{y}^T\boldsymbol{e}_1)(1-\gamma) \\ &\quad - \frac{(\boldsymbol{M}_1^{\pi^s} + \boldsymbol{M}_2^{\pi^s}\boldsymbol{y})^T\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T(\boldsymbol{M}_1^{\pi^s} + \boldsymbol{M}_2^{\pi^s}\boldsymbol{y})}{2(1-\gamma)} + \frac{1}{2}(\boldsymbol{M}_1^{\pi^s} + \boldsymbol{M}_2^{\pi^s}\boldsymbol{y})^T\Sigma(\boldsymbol{M}_1^{\pi^s} + \boldsymbol{M}_2^{\pi^s}\boldsymbol{y}) \\ &\quad + (\boldsymbol{h} + \boldsymbol{H}\boldsymbol{y})^T\boldsymbol{B}[((\boldsymbol{\lambda} - a\boldsymbol{E}_2\boldsymbol{y}) + \Sigma(\boldsymbol{M}_1^{\pi^s} + \boldsymbol{M}_2^{\pi^s}\boldsymbol{y}))(1-\gamma) - \boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T(\boldsymbol{M}_1^{\pi^s} + \boldsymbol{M}_2^{\pi^s}\boldsymbol{y})] \\ &\quad - \frac{1}{2}(\boldsymbol{h} + \boldsymbol{H}\boldsymbol{y})^T\boldsymbol{B}\boldsymbol{\sigma}[(1-\gamma)(\gamma\boldsymbol{I} + \boldsymbol{\beta})]^{-1}\boldsymbol{\sigma}^T\boldsymbol{B}(\boldsymbol{h} + \boldsymbol{H}\boldsymbol{y}) + \frac{1}{2}\operatorname{tr}(\boldsymbol{M}_2^{\pi^s}\Sigma) \\ &\quad + \mu\mathbb{E}[(1-Y)^{1-\gamma} - 1] = 0 \end{aligned} \tag{D.18}$$

Then we can arrange Equation (D.18) to obtain the differential equation for $M_0^{\pi^s}$, $\boldsymbol{M}_1^{\pi^s}$ and $\boldsymbol{M}_2^{\pi^s}$ in Equation (D.19) below. For simplicity, we just represent them as M_0, $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$.

$$
\begin{cases}
M_0' + c(1-\gamma) + \mu\mathbb{E}[(1-Y)^{1-\gamma} - 1] + \frac{1}{2}\operatorname{tr}(\boldsymbol{M}_2\Sigma) + \frac{1}{2}\boldsymbol{M}_1^T\boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\boldsymbol{M}_1 \\
\qquad + \boldsymbol{h}^T\boldsymbol{B}\Big[\boldsymbol{\lambda} + \boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\boldsymbol{M}_1\Big](1-\gamma) + \boldsymbol{\theta}^T\boldsymbol{M}_1 \\
\qquad - \frac{1}{2}\boldsymbol{h}^T\boldsymbol{B}\boldsymbol{\sigma}[(1-\gamma)(\gamma\boldsymbol{I} + \boldsymbol{\beta})]^{-1}\boldsymbol{\sigma}^T\boldsymbol{B}\boldsymbol{h} = 0 \\
M_0(T) = 0,
\end{cases}
$$

$$
\begin{cases}
\boldsymbol{M}_1' + \Big[(1-\gamma)\boldsymbol{H}^T\boldsymbol{B}\boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T - \boldsymbol{A}^T\Big]\boldsymbol{M}_1 + \boldsymbol{y}^T\boldsymbol{M}_2\boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\boldsymbol{M}_1 \\
\qquad + (1-\gamma)\boldsymbol{e}_1 + \boldsymbol{M}_2^T\boldsymbol{\theta} + \Big[\boldsymbol{M}_2^T\boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T - a\boldsymbol{E}_2\Big]\boldsymbol{B}\boldsymbol{h}(1-\gamma) \\
\qquad + \boldsymbol{H}^T\boldsymbol{B}\boldsymbol{\lambda}(1-\gamma) - \boldsymbol{H}^T\boldsymbol{B}\boldsymbol{\sigma}[(1-\gamma)(\gamma\boldsymbol{I} + \boldsymbol{\beta})]^{-1}\boldsymbol{\sigma}^T\boldsymbol{B}\boldsymbol{h} = 0 \\
\boldsymbol{M}_1(T) = \boldsymbol{0}_{2\times 1}
\end{cases}
\tag{D.19}
$$

$$
\begin{cases}
\boldsymbol{M}_2' - \boldsymbol{H}^T\boldsymbol{B}\boldsymbol{\sigma}[(1-\gamma)(\gamma\boldsymbol{I} + \boldsymbol{\beta})]^{-1}\boldsymbol{\sigma}^T\boldsymbol{B}\boldsymbol{H} - a(1-\gamma)(\boldsymbol{H}^T\boldsymbol{B}\boldsymbol{E}_2 + \boldsymbol{E}_2\boldsymbol{B}\boldsymbol{H}) \\
\qquad + \boldsymbol{M}_2^T\boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\boldsymbol{M}_2 + \Big[(1-\gamma)\boldsymbol{H}^T\boldsymbol{B}\boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T - \boldsymbol{A}^T\Big]\boldsymbol{M}_2 \\
\qquad + \boldsymbol{M}_2^T\Big[(1-\gamma)\boldsymbol{H}^T\boldsymbol{B}\boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T - \boldsymbol{A}^T\Big]^T = 0 \\
\boldsymbol{M}_2(T) = \boldsymbol{0}_{2\times 2}
\end{cases}
$$

For simplicity, if we let

$$
\begin{aligned}
\boldsymbol{D}_0^s &= \boldsymbol{B}\boldsymbol{\sigma}(1-\gamma)(\gamma\boldsymbol{I} + \boldsymbol{\beta})\boldsymbol{\sigma}^T\boldsymbol{B} \\
\boldsymbol{D}_1^s(t) &= (1-\gamma)\boldsymbol{H}^T(t)\boldsymbol{B}\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T - \boldsymbol{A}^T \\
\boldsymbol{D}_2^s &= \boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T \\
\boldsymbol{D}_3^s(t) &= a(1-\gamma)(\boldsymbol{H}^T(t)\boldsymbol{B}\boldsymbol{E}_2 + \boldsymbol{E}_2\boldsymbol{B}\boldsymbol{H}(t)) \\
\boldsymbol{C}_0^s(t) &= (1-\gamma)\boldsymbol{e}_1 + \boldsymbol{M}_2^T\boldsymbol{\theta} + \boldsymbol{H}^T(t)\boldsymbol{B}\boldsymbol{\lambda}(1-\gamma) \\
\boldsymbol{B}_0^s(t) &= c(1-\gamma) + \mu\mathbb{E}[(1-Y)^{1-\gamma} - 1] + \frac{1}{2}\operatorname{tr}(\boldsymbol{M}_2(t)\Sigma) + \boldsymbol{\theta}^T\boldsymbol{M}_1,
\end{aligned}
\tag{D.20}
$$

we can obtain the ODEs in Proposition 4.3.1.

Moreover, if we set an strategy with $H(t) = 0$, we can easily derive the differential for $\boldsymbol{M}_2$ in Equation (D.19) to obtain:

$$
\begin{cases}
\boldsymbol{M}_2' + \boldsymbol{M}_2^T\boldsymbol{\sigma}\Big(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\Big)\boldsymbol{\sigma}^T\boldsymbol{M}_2 = 0 \\
\boldsymbol{M}_2(T) = \boldsymbol{0}_{2\times 2}
\end{cases}
\tag{D.21}
$$

Which leads to $\boldsymbol{M}_2(t) = \boldsymbol{0}_{2\times 2}$. Then the quadratic term disappears in the exponentially affine form.

D.5 Our Estimates and those in [Schwartz (1997)]

In order to compare our model with Schwartz's one-factor model of oil, we make a match between the two parameter sets. We extract the stochastic process of oil price using parametric set of 1/2/90 to 2/17/95 in Table IV from [Schwartz (1997)]. Plugging the parameters and assuming a constant interest rate of 6%, we obtain

$$\mathrm{d}X(t) = (0.06 + 6.301 \times 0.326 - 0.694X(t))\,\mathrm{d}t + 0.326\,\mathrm{d}W(t). \tag{D.22}$$

Table D.1 shows all coefficients of the WTI oil price as well as those from Schwartz's thesis in our notation. We observe that the estimated parameters can match parameters from Schwartz's thesis quite closely.

Parameters			
Asset	Market price of risk	λ_S	6.5253
	Mean-reverting rate	a	0.6539
	Volatility	σ_S	31.281%
	Correlation	ρ	0.5317%
Schwartz	Market price of risk	λ_S	6.301
	Mean-reverting rate	a	0.694
	Volatility	σ_S	32.6%

Table D.1: Parameters, comparison to [Schwartz (1997)]

Appendix E

Appendix to Chapter 5

E.1 Proofs

E.1.1 Proof of Proposition 5.3.1

Taking the infimum w.r.t $\boldsymbol{u}$, the change of measure $\boldsymbol{u}$ is

$$\boldsymbol{u}^* = \frac{1}{J(1-\gamma)}\boldsymbol{\beta}_{\boldsymbol{\rho}}(J_x\boldsymbol{\sigma}^T\boldsymbol{\pi} - J_r\sigma_r\boldsymbol{e}_3) \tag{E.1}$$

Plugging the result lead to the PDE

$$\begin{aligned}\sup_{\boldsymbol{\pi}}\Bigg\{&J_t + J_x rx + \boldsymbol{\pi}^T\boldsymbol{\sigma}\left(J_x\boldsymbol{\lambda} - J_{xr}\boldsymbol{\rho}_r\sigma_r + J_xJ_r\frac{\boldsymbol{\beta}_{\boldsymbol{\rho}}\boldsymbol{e}_3\sigma_r}{J(1-\gamma)}\right) + a(m-r)J_r + \frac{1}{2}J_{rr}\sigma_r^2\\ &-\frac{J_r^2\boldsymbol{e}_3^T\boldsymbol{\beta}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{2J(1-\gamma)}\sigma_r^2 + \frac{1}{2}\boldsymbol{\pi}^T\boldsymbol{\sigma}\left(J_{xx}\boldsymbol{\rho} - \frac{J_x^2}{J(1-\gamma)}\boldsymbol{\beta}_{\boldsymbol{\rho}}\right)\boldsymbol{\sigma}^T\boldsymbol{\pi}\Bigg\} = 0.\end{aligned} \tag{E.2}$$

The expression of $\boldsymbol{\sigma}$ indicates a rank of 3. So $\boldsymbol{\sigma}^T\boldsymbol{\pi}$ can be projected into the full 3-dimension space. Taking supremum w.r.t $\boldsymbol{\sigma}^T\boldsymbol{\pi}$ with a full rank, we obtain

$$(\boldsymbol{\sigma}^T\boldsymbol{\pi})^* = -\left(J_{xx}\boldsymbol{\rho} - \frac{J_x^2}{J(1-\gamma)}\boldsymbol{\beta}_{\boldsymbol{\rho}}\right)^{-1}\left(J_x\boldsymbol{\lambda} - J_{xr}\boldsymbol{\rho}_r\sigma_r + J_xJ_r\frac{\boldsymbol{\beta}_{\boldsymbol{\rho}}\boldsymbol{e}_3\sigma_r}{J(1-\gamma)}\right) \tag{E.3}$$

and the final PDE,

$$
\begin{aligned}
&J_t + J_x r x + a(m-r)J_r + \frac{\sigma_r^2}{2}\left(J_{rr} - \frac{J_r^2 \boldsymbol{e}_3^T \boldsymbol{\beta}_{\boldsymbol{\rho}} \boldsymbol{e}_3}{J(1-\gamma)}\right) \\
&- \frac{1}{2}\left(J_x \boldsymbol{\lambda} - J_{xr}\boldsymbol{\rho}_r \sigma_r + J_x J_r \frac{\boldsymbol{\beta}_{\boldsymbol{\rho}} \boldsymbol{e}_3 \sigma_r}{J(1-\gamma)}\right)^T \left(J_{xx}\boldsymbol{\rho} - \frac{J_x^2}{J(1-\gamma)}\boldsymbol{\beta}_{\boldsymbol{\rho}}\right)^{-1} \\
&\times \left(J_x \boldsymbol{\lambda} - J_{xr}\boldsymbol{\rho}_r \sigma_r + J_x J_r \frac{\boldsymbol{\beta}_{\boldsymbol{\rho}} \boldsymbol{e}_3 \sigma_r}{J(1-\gamma)}\right) = 0.
\end{aligned}
\tag{E.4}
$$

Substituting the ansatz $J(t,x,r) = \frac{x^{1-\gamma}}{1-\gamma}\exp(A_0(t) + A_1(t)r)$ gives the PDE

$$
\begin{aligned}
&A_0' + amA_1 + \frac{\sigma_r^2}{2}\left(1 - \frac{\boldsymbol{e}_3^T \boldsymbol{\beta}_{\boldsymbol{\rho}} \boldsymbol{e}_3}{1-\gamma}\right)A_1^2 + \frac{1-\gamma}{2}\left(\boldsymbol{\lambda} + \left(-\boldsymbol{\rho}_r + \frac{\boldsymbol{\beta}_{\boldsymbol{\rho}} \boldsymbol{e}_3}{1-\gamma}\right)\sigma_r A_1\right)^T (\gamma\boldsymbol{\rho} + \boldsymbol{\beta}_{\boldsymbol{\rho}})^{-1} \\
&\times \left(\boldsymbol{\lambda} + \left(-\boldsymbol{\rho}_r + \frac{\boldsymbol{\beta}_{\boldsymbol{\rho}} \boldsymbol{e}_3}{1-\gamma}\right)\sigma_r A_1\right) + r(A_1' + 1 - \gamma - aA_1) = 0.
\end{aligned}
\tag{E.5}
$$

Utilizing the separation of variables w.r.t. r returns the ODEs with terminal conditions,

$$
\begin{aligned}
&\begin{cases}
A_0' + amA_1 + \frac{\sigma_r^2}{2}\left(1 - \frac{\boldsymbol{e}_3^T \boldsymbol{\beta}_{\boldsymbol{\rho}} \boldsymbol{e}_3}{J(1-\gamma)}\right)A_1^2 + \frac{1-\gamma}{2}\left(\boldsymbol{\lambda} + \left(-\boldsymbol{\rho}_r + \frac{\boldsymbol{\beta}_{\boldsymbol{\rho}} \boldsymbol{e}_3}{1-\gamma}\right)\sigma_r A_1\right)^T (\gamma\boldsymbol{\rho} + \boldsymbol{\beta}_{\boldsymbol{\rho}})^{-1} \\
\qquad\qquad \times\left(\boldsymbol{\lambda} + \left(-\boldsymbol{\rho}_r + \frac{\boldsymbol{\beta}_{\boldsymbol{\rho}} \boldsymbol{e}_3}{1-\gamma}\right)\sigma_r A_1\right) = 0 \\
A_0(T) = 0
\end{cases} \\
&\begin{cases}
A_1' + 1 - \gamma - aA_1 = 0 \\
A_1(T) = 0.
\end{cases}
\end{aligned}
\tag{E.6}
$$

Solving the simultaneous ODEs returns the solution in Proposition 5.3.1.

E.1.2 Proof of Theorem 5.3.1

First, we present the change of measure $\boldsymbol{u}^*$ in the worst-case scenario as a pure function of t,

$$
\begin{aligned}
\boldsymbol{u}^*(t) &= \frac{1}{J(1-\gamma)}\boldsymbol{\beta}_{\boldsymbol{\rho}}(J_x \boldsymbol{\sigma}^T \boldsymbol{\pi} - J_r \sigma_r \boldsymbol{e}_3) \\
&= \frac{\boldsymbol{\beta}_{\boldsymbol{\rho}}}{1-\gamma}\left((1-\gamma)(\gamma\boldsymbol{\rho} + \boldsymbol{\beta}_{\boldsymbol{\rho}})^{-1}\left(\boldsymbol{\lambda} + A_1(t)\left(-\boldsymbol{\rho}_r + \frac{\boldsymbol{\beta}_{\boldsymbol{\rho}} \boldsymbol{e}_3}{1-\gamma}\right)\sigma_r\right) - A_1(t)\sigma_r \boldsymbol{e}_3\right)
\end{aligned}
\tag{E.7}
$$

Therefore, given a well-defined set of parameters and the function $|A_1(t)| < \infty$ for $0 \le t \le T$, the Novikov's condition for Girsanov's theorem holds, i.e.,

$$
\mathbb{E}^P\left[\exp\left(\frac{1}{2}\int_0^T \|\boldsymbol{u}^*(t)\|\right)\right] < \infty. \tag{E.8}
$$

Hence the change of measure in the worst-case scenario is well-defined. Following Corollary 1.2 in [Kraft (2012)] provides the proof that $\boldsymbol{\pi}^*$ is in fact the optimal strategy.

E.1.3 Proof of Proposition 5.3.3

Taking infimum of Equation (5.31) w.r.t. $\boldsymbol{u}$, we obtain

$$\boldsymbol{u}^* = \frac{1}{J(1-\gamma)}\boldsymbol{\beta_\rho}(J_x\boldsymbol{\sigma}^T\boldsymbol{\pi}_s(t) - J_r\sigma_r\boldsymbol{e}_3) \tag{E.9}$$

The HJB equation is changed into

$$\begin{aligned} &J_t + J_x r x + \boldsymbol{\pi}_s^T(t)\boldsymbol{\sigma}\left(J_x\boldsymbol{\lambda} - J_{xr}\boldsymbol{\rho}_r\sigma_r + J_xJ_r\frac{\boldsymbol{\beta_\rho}\boldsymbol{e}_3\sigma_r}{J(1-\gamma)}\right) + a(m-r)J_r + \frac{1}{2}J_{rr}\sigma_r^2 \\ &- \frac{J_r^2\boldsymbol{e}_3^T\boldsymbol{\beta_\rho}\boldsymbol{e}_3}{2J(1-\gamma)}\sigma_r^2 + \frac{1}{2}\boldsymbol{\pi}_s^T(t)\boldsymbol{\sigma}\left(J_{xx}\boldsymbol{\rho} - \frac{J_x^2}{J(1-\gamma)}\boldsymbol{\beta_\rho}\right)\boldsymbol{\sigma}^T\boldsymbol{\pi}_s(t) = 0. \end{aligned} \tag{E.10}$$

To simplify the notation, we omit the superscript s of A_0^s and A_1^s. Plugging the ansatz $J(t,x,r) = \frac{x^{1-\gamma}}{1-\gamma}\exp(A_0(t) + A_1(t)r)$ and the assumption $\boldsymbol{\pi}_s(t) = x\boldsymbol{h}(t)$ leads to:

$$\begin{aligned} &A_0' + A_1'r + r(1-\gamma) + (1-\gamma)\boldsymbol{h}^T(t)\boldsymbol{\sigma\lambda} - \frac{\gamma(1-\gamma)}{2}\boldsymbol{h}^T(t)\boldsymbol{\sigma\rho\sigma}^T\boldsymbol{h}(t) + a(m-r)A_1 \\ &- \frac{1}{2(1-\gamma)}\Big((1-\gamma)^2\boldsymbol{h}^T(t)\boldsymbol{\sigma\beta_\rho\sigma}^T\boldsymbol{h}(t) - 2(1-\gamma)\boldsymbol{h}^T(t)\boldsymbol{\sigma\beta_\rho}\boldsymbol{e}_3\sigma_rA_1 + \boldsymbol{e}_3^T\boldsymbol{\beta_\rho}\boldsymbol{e}_3\sigma_r^2A_1^2\Big) \\ &+ \frac{1}{2}A_1^2\sigma_r^2 - (1-\gamma)A_1\boldsymbol{h}^T(t)\boldsymbol{\sigma\rho}_r\sigma_r = 0 \end{aligned} \tag{E.11}$$

Via separation of variables w.r.t. r, the ODE for the HJB equation is:

$$\begin{cases} A_1' + 1 - \gamma - aA_1 = 0 \\ A_1(T) = 0 \end{cases}$$
$$\begin{cases} A_0 + \Big[(1-\gamma)\boldsymbol{h}^T(s)\boldsymbol{\sigma\lambda} - \frac{(1-\gamma)}{2}\boldsymbol{h}^T(s)\boldsymbol{\sigma}(\gamma\boldsymbol{\rho} + \boldsymbol{\beta_\rho})\boldsymbol{\sigma}^T\boldsymbol{h}(s) + amA_1(s) \\ +\frac{1}{2}A_1^2(s)\sigma_r^2\left(1 - \frac{\boldsymbol{e}_3^T\boldsymbol{\beta_\rho}\boldsymbol{e}_3}{1-\gamma}\right) + A_1(s)\boldsymbol{h}^T(s)\boldsymbol{\sigma}(-(1-\gamma)\boldsymbol{\rho}_r + \boldsymbol{\beta_\rho}\boldsymbol{e}_3)\sigma_r\Big] = 0 \\ A_0(T) = 0. \end{cases} \tag{E.12}$$

which returns to the solution in Proposition 5.3.3.

E.1.4 Proof of Proposition 5.3.4

Taking $\hat{\boldsymbol{\beta}}$ gives us the suboptimal allocation

$$\boldsymbol{h}(t) = \left[\boldsymbol{\sigma}\left(\gamma\boldsymbol{\rho} + \hat{\boldsymbol{\beta}}_{\boldsymbol{\rho}}\right)\boldsymbol{\sigma}^T\right]^{-1}\boldsymbol{\sigma}\left(\boldsymbol{\lambda} + A_1(t)\left(-\boldsymbol{\rho}_r + \frac{\hat{\boldsymbol{\beta}}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{1-\gamma}\right)\sigma_r\right). \tag{E.13}$$

With the notation $\boldsymbol{\rho} = \boldsymbol{B}\boldsymbol{B}^T$ and $\boldsymbol{\beta}_{\boldsymbol{\rho}} = \boldsymbol{B}\boldsymbol{\beta}\boldsymbol{B}^T$, plugging the investment allocation into Equation (5.32) gives us

$$\begin{aligned} A_0^s(t) = \int_t^T &-\frac{1-\gamma}{2}\left(\boldsymbol{\lambda} + A_1(t)\left(-\boldsymbol{\rho}_r + \frac{\hat{\boldsymbol{\beta}}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{1-\gamma}\right)\sigma_r\right)^T (\boldsymbol{B}^T)^{-1}(\gamma\boldsymbol{I} + \hat{\boldsymbol{\beta}})^{-1} \\ &\times (\gamma\boldsymbol{I} + \boldsymbol{\beta})(\gamma\boldsymbol{I} + \hat{\boldsymbol{\beta}})^{-1}\boldsymbol{B}^{-1}\left(\boldsymbol{\lambda} + A_1(t)\left(-\boldsymbol{\rho}_r + \frac{\hat{\boldsymbol{\beta}}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{1-\gamma}\right)\sigma_r\right) \\ &+ \frac{1}{2}A_1^2(s)\sigma_r^2\left(1 - \frac{\boldsymbol{e}_3^T\boldsymbol{\beta}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{1-\gamma}\right) + amA_1(s) \\ &+ (1-\gamma)\left(\boldsymbol{\lambda} + A_1(t)\left(-\boldsymbol{\rho}_r + \frac{\hat{\boldsymbol{\beta}}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{1-\gamma}\right)\sigma_r\right)^T (\boldsymbol{B}^T)^{-1}(\gamma\boldsymbol{I} + \hat{\boldsymbol{\beta}})^{-1} \\ &\times \boldsymbol{B}^{-1}\left(\boldsymbol{\lambda} + A_1(s)\left(-\boldsymbol{\rho}_r + \frac{\boldsymbol{\beta}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{1-\gamma}\right)\sigma_r\right) \mathrm{d}s \end{aligned} \tag{E.14}$$

In order to compute the WEL in Equation (5.33), we first compute the difference

$$\begin{aligned} A_0^s(t) - A_0(t) = \int_t^T &-\frac{1-\gamma}{2}\left(\boldsymbol{\lambda} + A_1(s)\left(-\boldsymbol{\rho}_r + \frac{\hat{\boldsymbol{\beta}}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{1-\gamma}\right)\sigma_r\right)^T (\boldsymbol{B}^T)^{-1}(\gamma\boldsymbol{I} + \hat{\boldsymbol{\beta}})^{-1} \\ &\times (\gamma\boldsymbol{I} + \boldsymbol{\beta})(\gamma\boldsymbol{I} + \hat{\boldsymbol{\beta}})^{-1}\boldsymbol{B}^{-1}\left(\boldsymbol{\lambda} + A_1(s)\left(-\boldsymbol{\rho}_r + \frac{\hat{\boldsymbol{\beta}}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{1-\gamma}\right)\sigma_r\right) \\ &+ (1-\gamma)\left(\boldsymbol{\lambda} + A_1(s)\left(-\boldsymbol{\rho}_r + \frac{\hat{\boldsymbol{\beta}}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{1-\gamma}\right)\sigma_r\right)^T (\boldsymbol{B}^T)^{-1}(\gamma\boldsymbol{I} + \hat{\boldsymbol{\beta}})^{-1} \\ &\times \boldsymbol{B}^{-1}\left(\boldsymbol{\lambda} + A_1(s)\left(-\boldsymbol{\rho}_r + \frac{\boldsymbol{\beta}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{1-\gamma}\right)\sigma_r\right) \\ &- \frac{1-\gamma}{2}\left(\boldsymbol{\lambda} + A_1(s)\left(-\boldsymbol{\rho}_r + \frac{\boldsymbol{\beta}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{1-\gamma}\right)\sigma_r\right)^T (\boldsymbol{B}^{-1})^T(\gamma\boldsymbol{I} + \boldsymbol{\beta})^{-1}\boldsymbol{B}^{-1} \\ &\times \left(\boldsymbol{\lambda} + A_1(s)\left(-\boldsymbol{\rho}_r + \frac{\boldsymbol{\beta}_{\boldsymbol{\rho}}\boldsymbol{e}_3}{1-\gamma}\right)\sigma_r\right) \mathrm{d}s. \end{aligned} \tag{E.15}$$

We can easily obtain the quadratic form

$$A_0^s(t) - A_0(t) = \int_t^T -\frac{1-\gamma}{2}\Bigg((\gamma\boldsymbol{I}+\boldsymbol{\beta})^{\frac{1}{2}}(\gamma\boldsymbol{I}+\hat{\boldsymbol{\beta}})^{-1}\boldsymbol{B}^{-1}\Bigg(\boldsymbol{\lambda}+A_1(s)\Bigg(-\boldsymbol{\rho}_r+\frac{\hat{\boldsymbol{\beta}}_\rho\boldsymbol{e}_3}{1-\gamma}\Bigg)\sigma_r\Bigg)$$
$$-(\gamma\boldsymbol{I}+\boldsymbol{\beta})^{-\frac{1}{2}}\boldsymbol{B}^{-1}\Bigg(\boldsymbol{\lambda}+A_1(s)\Bigg(-\boldsymbol{\rho}_r+\frac{\boldsymbol{\beta}_\rho\boldsymbol{e}_3}{1-\gamma}\Bigg)\sigma_r\Bigg)\Bigg)^T$$
$$\Bigg((\gamma\boldsymbol{I}+\boldsymbol{\beta})^{\frac{1}{2}}(\gamma\boldsymbol{I}+\hat{\boldsymbol{\beta}})^{-1}\boldsymbol{B}^{-1}\Bigg(\boldsymbol{\lambda}+A_1(s)\Bigg(-\boldsymbol{\rho}_r+\frac{\hat{\boldsymbol{\beta}}_\rho\boldsymbol{e}_3}{1-\gamma}\Bigg)\sigma_r\Bigg)$$
$$-(\gamma\boldsymbol{I}+\boldsymbol{\beta})^{-\frac{1}{2}}\boldsymbol{B}^{-1}\Bigg(\boldsymbol{\lambda}+A_1(s)\Bigg(-\boldsymbol{\rho}_r+\frac{\boldsymbol{\beta}_\rho\boldsymbol{e}_3}{1-\gamma}\Bigg)\sigma_r\Bigg)\Bigg)\,\mathrm{d}s. \tag{E.16}$$

Rewriting it:

$$A_0^s(t) - A_0(t) = \int_t^T -\frac{1-\gamma}{2}\Bigg(\Big((\gamma\boldsymbol{I}+\boldsymbol{\beta})^{\frac{1}{2}}(\gamma\boldsymbol{I}+\hat{\boldsymbol{\beta}})^{-1}-(\gamma\boldsymbol{I}+\boldsymbol{\beta})^{-\frac{1}{2}}\Big)\boldsymbol{B}^{-1}\Bigg(\boldsymbol{\lambda}+A_1(s)\Bigg(-\boldsymbol{\rho}_r+\frac{\hat{\boldsymbol{\beta}}_\rho\boldsymbol{e}_3}{1-\gamma}\Bigg)\sigma_r\Bigg)\Bigg)^T$$
$$\Bigg(\Big((\gamma\boldsymbol{I}+\boldsymbol{\beta})^{\frac{1}{2}}(\gamma\boldsymbol{I}+\hat{\boldsymbol{\beta}})^{-1}-(\gamma\boldsymbol{I}+\boldsymbol{\beta})^{-\frac{1}{2}}\Big)\boldsymbol{B}^{-1}\Bigg(\boldsymbol{\lambda}+A_1(s)\Bigg(-\boldsymbol{\rho}_r+\frac{\hat{\boldsymbol{\beta}}_\rho\boldsymbol{e}_3}{1-\gamma}\Bigg)\sigma_r\Bigg)\Bigg)\,\mathrm{d}s. \tag{E.17}$$

Therefore, the WEL is an quadratic function of $\boldsymbol{\lambda}$.

E.1.5 Proof of Corollary 5.3.1.1

To simplify and generalize the problem, we consider the quadratic maximization problem of

$$\max_{\boldsymbol{x}|\boldsymbol{x}_2=0} \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}^T\boldsymbol{x}+c \tag{E.18}$$

where we set

$$\boldsymbol{x}=\begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{bmatrix},\quad \boldsymbol{b}=\begin{bmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\end{bmatrix},\quad \boldsymbol{A}=\begin{bmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{12}^T & \boldsymbol{A}_{22}\end{bmatrix} \tag{E.19}$$

Then the quadratic maximizer point would be:

$$\boldsymbol{x}^* = -\boldsymbol{A}_1^{-1}\boldsymbol{b}_1 = -\begin{bmatrix}\boldsymbol{A}_{11}^{-1} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\boldsymbol{b} \tag{E.20}$$

E.2 Estimation and Results with Copper Future Data

The copper future price data is available from Wiki Continuous Futures* using the code "CME_HG1.". To fit the data structure, we chose weekly future prices with maturity of 1 – 9 months over the period of Aug 2001 – Jan 2020 (the future prices in Nov 2011 are missing so we ignored this period). However, the spot copper price is not available via this website; hence, for simplicity, and adapting the formulas accordingly, we took the 1-month future as the spot price, and n-month future to be the $(n-1)$-month future price. Applying MLE and KF with the available dataset leads to the estimations provided in Table E.1 below.

Parameters			
Convenience Yield	Mean-reverting rate	κ	1.7874
	Risk-neutral measure	$\hat{\alpha}$	-2.33%
	Real-world measure	α	1.60%
	Market price of risk	λ_δ	0.5122
	Volatility	σ_δ	13.70%
Interest Rate	Mean-reverting rate	a	0.2798
	Real-world measure interest	m	0.90%
	Market price of risk	λ_r	0.6803
	Volatility	σ_r	0.98%
Commodities	Market price of risk	λ_S	0.4270
	Volatility	σ_S	27.47%
	Maturity 1	τ_1	0
	Maturity 2	τ_2	1/12
Correlation	Commodity & convenience yield	$\rho_{S\delta}$	31.04%
	Commodity & interest rate	ρ_{Sr}	6.76%
	interest rate & convenience yield	$\rho_{r\delta}$	12.58%

Table E.1: Parameters from Calibration

We created for copper data figures similar to the WTI analyses in Figure 5.3 and Figure 5.4. In Figure E.2, the plot depicts the WEL due to ignoring ambiguities. The absence of ambiguities on the spot commodity, convenience yield, and interest rate lead to WEL of 7%, 6% and 19.5% respectively. For a larger ambiguity aversion level on convenience yield (i.e. $\beta_2 = 9$, the last subfigure), the WEL from convenience yield goes up to 31%, larger than the 27% from ignoring ambiguity on bonds or the 16% from ignoring ambiguity on spot prices.

*https://www.quandl.com/data/CHRIS-Wiki-Continuous-Futures

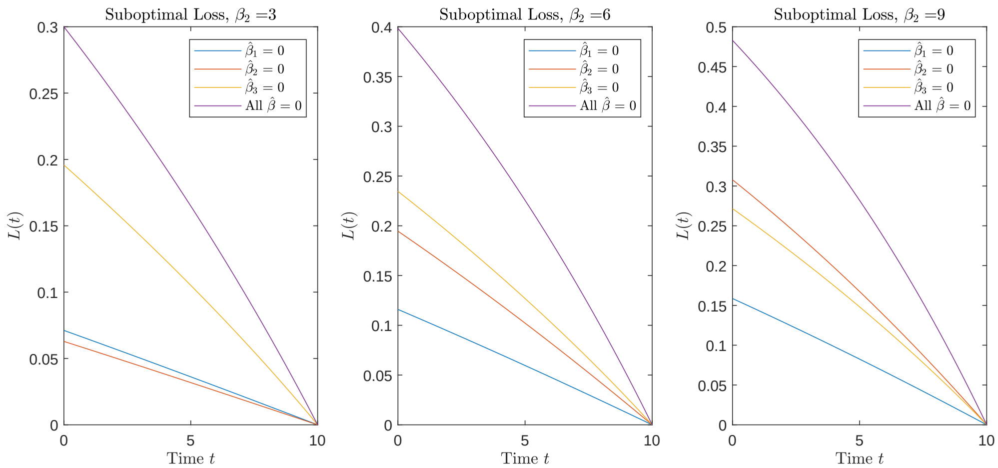

Figure E.1: Wealth-equivalent loss for ignoring ambiguities

As for incompleteness of the market, the plot in Figure E.2 illustrates that not investing in bond accounts for 44% of WELs, which is a larger than WELs due to incompleteness of the spot (15%) and the longer-maturity prepaid forward (18%). Moreover, the absence of all assets leads to WEL of 55%, also satisfying the subadditivity of individual losses.

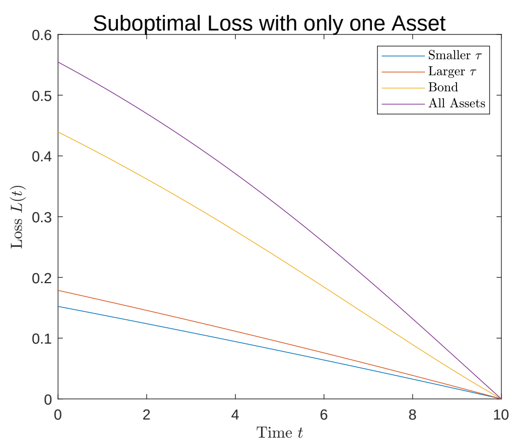

Figure E.2: Wealth-equivalent loss for Incomplete Markets

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Chen, J., Davison, M. (2021). Deterministic Asymmetric-cost Differential Games for Energy Production with Production Bounds. Accepted by *SN Operational Research Forum.*

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