8-19-2010 12:00 AM

Holomorphic k-differentials and holomorphic approximation on open Riemann surfaces

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Graduate Program in Mathematics
A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

Holomorphic k-differentials and holomorphic approximation on open Riemann surfaces

(Spine title: k-differentials and approximation on open Riemann surfaces)

(Thesis format: Monograph)

by

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Graduate Program
in
Mathematics

A thesis submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

School of Graduate and Postdoctoral Studies
The University of Western Ontario
London, Ontario, Canada

Certificate of Examination

THE UNIVERSITY OF WESTERN ONTARIO
SCHOOL OF GRADUATE AND POSTDOCTORAL STUDIES

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The thesis by

Nadya Askaripour

entitled:

Holomorphic k-differentails and holomorphic approximation on open Riemann surfaces

is accepted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

Date: ______________________
19 August 2010

Sheila Macfie
____
Chair of Examining Board

Abstract

This thesis consists of two parts: In the first part (Chapters 1 and 2) we study some spaces of holomorphic $k-$differentials on open Riemann surfaces, and make some observations about these spaces, then we obtain two main theorems about the kernel of Poincaré series map.

In the second part (Chapters 3 and 4), we study holomorphic and meromorphic approximation on closed subsets of non-compact Riemann surfaces. We add a condition to the Extension Theorem and correct its proof. The Extension Theorem was first stated and proved by G. Schmieder, but there are a few examples where the theorem as stated by Schmieder fails. That added condition is slightly affecting the definition of a class of closed sets (which is called "weakly of infinite genus") where approximation is possible.
Keywords: Riemann surface, $k-$differential, approximation

Acknowledgements

I wish to express my deep and sincere gratitude to my supervisors professor André Boivin and professor Tatyana Foth; their guidance, advices, encouragement and support from the beginning to the final stage, enabled me to develop an insight of the subject and complete my thesis.

My high appreciation to my thesis examiners, professor Paul Gauthier, professor Martin Pinsonnault, professor Serge Provost and professor Gordon Sinnamon, not only for their valuable comments, but also for their careful reading of my thesis in a short time.

My warm thanks to all the people of the Department of Mathematics at the University of Western Ontario, specially those who supported me in any respect during the years I have been a graduate student at this Department.

For their inspiring lessons, I would like to offer my regards and blessings to all the great Mathematics teachers and professors I had.

And my loving thanks to my family, specially my parents, for their invaluable support and encouragement.

To my parents

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Chapter 1

Holomorphic $k-$differentials and Poincaré series

In this chapter we review the definition of $k-$differentials and some other related concepts, which we will use in the next chapter.

1.1 $k-$differentials and related definitions

Throughout Chapters 1 and 2, we shall assume that k is a fixed positive integer, $k \geq 2$, and we shall omit k from the notation of the required function spaces, norms, inner products, Bergman kernel.

A Riemann surface R is a Hausdorff connected topological space with a collection of pairs $\{(\varphi_j, U_j) : j \in J\}$ such that:

- The sets U_j form an open cover of R, and
- φ_j is a homeomorphism of U_j onto $\varphi_j(U_j)$, where $\varphi_j(U_j)$ is an open subset of the complex plane $\mathbb{C}$,

and $\varphi_i\varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \mathbb{C}$ is a holomorphic function whenever $U_i \cap U_j \neq \emptyset$. The collection $\{(\varphi_j, U_j)\}_{j\in J}$ is referred to as a *complex structure* of R. For $p \in U_j$, $z = \varphi_j(p)$ is called a *local complex coordinate* of R.

Definition 1.1.1. *Let R be a Riemann surface and let k be a positive integer. A $k-$differential on R is a section of the holomorphic line bundle $(T^{*'}R)^{\otimes k}$, where $T^{*'}R$ is the holomorphic cotangent bundle on R.*

Another equivalent definition for a $k-$differential is as follows:

Definition 1.1.2. *A $k-$differential on R is a collection of $\mathbb{C}$-valued functions $\{\phi_\alpha(z_\alpha)\}_{\alpha\in I}$ where z_α is a local complex coordinate on an open set U_α (for an open cover $\{U_\alpha\}_{\alpha\in I}$ of R, where I is an index set), and for all $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$, over $U_\alpha \cap U_\beta$*

$$\phi_\alpha(z_\alpha) = \phi_\beta(z_\beta)\left(\frac{dz_\beta}{dz_\alpha}\right)^k. \tag{1.1}$$

A $k-$differential is usually written as $\phi(z)dz^k$, which is understood as follows: For any $\zeta \in R$, letting $\alpha \in I$ such that $\zeta \in U_\alpha$, then near ζ, $z = z_\alpha, \phi = \phi_\alpha$; this is well defined because of (1.1). Note that dz^k is a conventional notation for $dz^{\otimes k}$.

Remark 1.1.3. *Clearly it does not make sense to speak of the value of a $k-$differential φ at a point $\zeta \in R$ (since it depends on the local parameter near ζ), but it does make sense to speak of the zeroes and poles of φ (see [44, p. 18] for example).*

Definition 1.1.4. *A k-differential is called holomorphic (resp. meromorphic) if it is a holomorphic (resp. meromorphic) section of $(T^{*'}R)^{\otimes k}$, or, equivalently, if all ϕ_α, $\alpha \in I$ are holomorphic (resp. meromorphic).*

Example 1.1.5. *The k-differential $\phi \equiv 0$ exists on any Riemann surface R.*

Example 1.1.6. *Let R be a domain of the complex plane $\mathbb{C}$. The complex structure is defined by one open set, and the identity mapping. Any function $\phi(z)$ defined on R can be considered as a $k-$differential on R.*

Example 1.1.7. *Let R be the Riemann sphere. It can be covered by two open sets, $U_1 = \mathbb{C}$ and $U_2 = \mathbb{C} \cup \{\infty\} \setminus \{0\}$. As parameters we introduce z and $w = \frac{1}{z}$ respectively. Given an arbitrary function $\phi_1(z)$ which is defined in the whole plane, we can compute the function element $\phi_2(w)$ by the transformation rule (1.1), to get $\phi_2(w) = \phi_1(\frac{1}{w})(-\frac{1}{w^2})^k$.*

1.1.1 Uniformization Theorem and hyperbolic Riemann surfaces

One way to construct a Riemann surface is as the quotient of a suitable group action. For example if G is the group generated by $z \mapsto z+1$, then $G\backslash\mathbb{C}$ is the punctured plane $\mathbb{C} \setminus \{0\}$, and $G\backslash\mathbb{H}$ is the punctured disc $\{z \mid 0 < |z| < 1\}$, and in each case the quotient map is $z \mapsto exp(2\pi i z)$. These two examples give Riemann surfaces which are the quotient of the Euclidean plane $\mathbb{C}$, and the hyperbolic plane $\mathbb{H}$, respectively. For further discussion we shall need the definition of a discrete group acting properly discontinuously. Let Γ be a discrete subgroup of $SU(1,1)$, we say that Γ acts properly discontinuously on the unit disc $\Delta = \{z \in \mathbb{C} | |z| < 1\}$ if any $z \in \Delta$ has a neighborhood U_z such that $g(U_z) \cap U_z = \emptyset$ for any $g \in \Gamma$, $g \neq e$, where e is the identity element. It is a general fact that if G is any discrete group acting properly discontinuously, freely and biholomorphically on $\mathbb{C}$ or $\mathbb{H}$ then the quotient by G is a Riemann surface (see for example [9]). The Uniformization Theorem (see below) is the converse of this statement (except for the sphere). The Uniformization Theorem is a very well known theorem, which is a consequence of Koebe planarity theorem, which classifies the planar Riemann surfaces (a Riemann surface is planar if any simple closed curve contained in R divides R into two components).

Two Riemann surfaces R and R_1 are conformally equivalent if there is a conformal homeomorphism from R onto R_1. The Uniformization Theorem (see [16]) states that any Riemann surface is conformally equivalent to one of the following Riemann surfaces:

(1) The complex plane: $R = \mathbb{C}$, with the usual complex structure.

(2) The cylinder: which is the quotient of $\mathbb{C}$ by the group G, where G is the cyclic group of translations, $z \mapsto z + n$.

(3) Tori: The quotient of $\mathbb{C}$ by the group G, where, G is the group of translations,

which acts on $\mathbb{C}$ by $z \mapsto z + n\omega_1 + m\omega_2$, where n, m are integers and ω_1, ω_2 are a basis for $\mathbb{C}$ over R.

(4) The Riemann sphere: $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, which is the one point compactification of $\mathbb{C}$.

(5) Hyperbolic surfaces: Let R be the quotient of $\mathbb{H}$ factored by a discrete and torsion free group G, which is a subgroup of $PSL(2, \mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b \in \mathbb{R}, ad - bc = 1\}$. The group G is called a Fuchsian group, (see [16]).

Equivalently, a *hyperbolic Riemann surface* Σ can be represented as $\Gamma \backslash \Delta$, where Γ is a discrete subgroup of $SU(1, 1)$, acting properly discontinuously on the unit disc Δ.

Remark 1.1.8. *If R is a Riemann surface, and p_1, p_2, p_3 are three distinct points on R, then $R - \{p_1, p_2, p_3\}$ is a hyperbolic Riemann surface. This follows from the uniformization theorem.*

Remark 1.1.9. *Let $\Sigma = \Gamma \backslash \Delta$ be a hyperbolic Riemann surface, and $\pi : \Delta \to \Sigma$ be a covering map.*

The pull-back of a holomorphic (resp., meromorphic) $k-$differential on Σ to Δ under π is a Γ-invariant holomorphic (resp. meromorphic) $k-$differential on Δ. The Γ-invariance of $\varphi(z)dz^k$ is equivalent to the automorphy law

$$\varphi(gz)J(g, z)^k = \varphi(z) \tag{1.2}$$

for all $g \in \Gamma$, $z \in \Delta$, (here $J(g, z) = \frac{d(gz)}{dz}$). Conversely any $\Gamma-$invariant holomorphic (resp., meromorphic) $k-$differential on Δ defines a holomorphic (resp., meromorphic) $k-$differential on Σ. The space of holomorphic (resp., meromorphic) $k-$differential on Σ is isomorphic, as a complex vector space, to the space of holomorphic (resp., meromorphic) $\Gamma-$invariant $k-$differentials on Δ.

1.1.2 The Poincaré metric

Let U be an arbitrary open set in the extended complex plane, $\mathbb{C} \cup \{\infty\}$, whose boundary consists of more than two points. First we set

$$\omega_\Delta(z) = 1 - |z|^2, \quad z \in \Delta. \tag{1.3}$$

The *Poincaré metric* is $\mathrm{ds} = \frac{|\mathrm{dz}|}{1-|\mathrm{z}|^2}$. A simple calculation shows that for every conformal self-mapping A of Δ we have

$$\omega_\Delta(Az)|A'(z)| = \omega_\Delta(z), \quad z \in \Delta.$$

(Recall: $A(z) = \frac{az+b}{\bar{b}z+\bar{a}}$, $|a|^2 - |b|^2 = 1$.) If U is connected, the domain U has the unit disc as its universal covering space, since the boundary of U contains more than two points. Let $\pi : \Delta \to U$ be the universal holomorphic covering map. We define ω_U by

$$\omega_U(\pi(z))|\pi'(z)| = \omega_\Delta(z), \quad z \in \Delta. \tag{1.4}$$

We must verify that ω_U is well defined. If ρ is another covering map and if $\pi(z) = \rho(\zeta)$ for two points z and ζ, then there exists a conformal self-mapping A of Δ such that

$$A(\zeta) = z \quad \textit{and} \quad \pi \circ A = \rho.$$

Thus

$$\omega_U(\rho(\zeta)) = \omega_U(\zeta)|\rho'(\zeta)|^{-1} = \omega_\Delta(A\zeta)|A'(\zeta)||\pi'(A\zeta)|^{-1}|A'(\zeta)|^{-1} =$$

$$\omega_\Delta(z)|\pi'(z)|^{-1} = \omega_U(\pi(z)).$$

If $z \in U$ is arbitrary and if we choose π such that $\pi(0) = z$, then we see that $\omega_U(z) = |\pi'(0)|^{-1}$. For more information about the Poincaré metric, see [24].

1.1.3 Fundamental domain

Definition 1.1.10. *A group of holomorphic homeomorphisms of the unit disc Δ which acts properly discontinuously on Δ is called a Fuchsian group.*

Definition 1.1.11. *A fundamental domain for a Fuchsian group Γ acting on Δ is an open subset $\mathcal{D}$ of Δ such that:*

1) *Every point of Δ is $\Gamma-$equivalent to at least one point of the closure of ω.*

2) *No two points of $\mathcal{D}$ are identified by an element A of Γ.*

3) *The boundary $\partial\mathcal{D}$ of $\mathcal{D}$ in Δ can be written as a countable union of piecewise analytic arcs γ_j.*

4) *For every arc γ_j in 3), there is another arc γ_k and an element A of Γ such that $A(\gamma_j) = \gamma_k$.*

We will not need to use all of the properties of a fundamental domain. For our purpose it will suffice to know the existence of a measurable fundamental set. It is obvious that if B is a Möbius transformation and ω is a fundamental domain for Γ acting on Δ , then $B(\omega)$ is a fundamental domain for $B \circ \Gamma \circ B^{-1}$ acting on $B(\Delta)$. Here we give without proof the method for constructing the so-called Dirichlet fundamental domain. For more information on fundamental domains the reader is referred to Beardon ([5]), Ford ([12]), or Lehner ([29]).

Definition 1.1.12. *Select a point z_0 in Δ. For each A in $\Gamma - \{id\}$, let*

$$h(A) = \{z \in \Delta : d(z, z_0) < d(z, A(z_0))\}$$

where d is the Poincaré metric. The Dirichlet fundamental domain is

$$\mathcal{D}_d(z_0) = \bigcap h(A)$$

where the intersection is over all elements A in $\Gamma - \{id\}$.

1.1.4 Some function spaces of $k-$differentials

Let Σ be a hyperbolic Riemann surface, and $\pi : \Delta \to \Sigma$ be defined as in Remark 1.1.9. Let $\Lambda \subset \Sigma$ be a closed set such that $\pi^{-1}(\Sigma - \Lambda)$ is connected. Let $V = \Sigma - \Lambda$, and $U = \pi^{-1}(V)$. Clearly $V = \Gamma \backslash U$. Using the Uniformization Theorem, we conclude that U is a hyperbolic Riemann surface, and so $U = \Gamma_0 \backslash \Delta$, where Γ_0 is a discrete subgroup of Γ. Denote by $\pi_0 : \Delta \to U$ the universal covering map. Let $\mathrm{d}\mu$ be the Lebesgue measure on Δ. For a fixed positive integer $k \geq 2$, we shall define the following spaces of $k-$differentials. All of them are Banach spaces, isomorphic to function spaces on U, as explained below.

Definition 1.1.13. *Let $A^{(1)}(V)$ be the (normed) space of holomorphic $k-$differentials Φ on V such that*

$$\int_{\Gamma \backslash U} |\varphi(z)| w(z)^{k-2} \mathrm{d}\mu < \infty \tag{1.5}$$

where $z \in U$, $\varphi(z) dz^k = (\pi|_U)^* \Phi$, and the norm $\|\Phi\|_1$ is given by (1.5). The integral (1.5) can also be written as $\int_{\mathcal{F}} |\varphi(z)| w(z)^{k-2} \mathrm{d}\mu$, where $\mathcal{F} = \mathcal{D} \cap U$ and $\mathcal{D}$ is a fundamental domain for Γ.

Remark 1.1.14. *Let $A_\Gamma^{(1)}(U)$ be the space of Γ-invariant holomorphic $k-$differentials $\varphi(z) dz^k$ on U satisfying (1.5). $A^{(1)}(V)$ is isomorphic to $A_\Gamma^{(1)}(U)$ and isomorphic to the space $\mathcal{A}_\Gamma^{(1)}(U)$ of holomorphic functions $\varphi(z)$ on U satisfying (1.2) (for $g \in \Gamma$, $z \in U$) and (1.5).*

Definition 1.1.15. *Let $A^{(2)}(V)$ be the space of holomorphic $k-$differentials Φ on V such that*

$$\int_{\Gamma \backslash U} |\varphi(z)|^2 w(z)^{2k-2} \mathrm{d}\mu < \infty \tag{1.6}$$

where $z \in U$, $\varphi(z)dz^k = (\pi|_U)^*\Phi$. *The integral (1.6) can also be written as* $\int_{\mathcal{F}} |\varphi(z)|^2 w(z)^{2k-2}\mathrm{d}\mu$. $A^{(2)}(V)$ *is a normed space, with the norm*

$$\|\Phi\|_2 = (\int\limits_{\Gamma\backslash U} |\varphi|^2 w(z)^{2k-2}\mathrm{d}\mu)^{1/2}$$

Remark 1.1.16. *Let* $A^{(2)}_\Gamma(U)$ *be the space of* Γ*-invariant holomorphic* $k-$*differentials* $\varphi(z)dz^k$ *on* U *that satisfy (1.6).* $A^{(2)}(V)$ *is isomorphic to* $A^{(2)}_\Gamma(U)$ *and is isomorphic to the space* $\mathcal{A}^{(2)}_\Gamma(U)$ *of holomorphic functions* $\varphi(z)$ *on* U *satisfying (1.2) (for* $g \in \Gamma$, $z \in U$*) and (1.6).*

Definition 1.1.17. *Let* $B(V)$ *be the (normed) space of holomorphic* $k-$*differentials* Φ *on* V *such that*

$$\sup_{z\in\mathcal{F}} |\varphi(z)|w(z)^k < \infty \tag{1.7}$$

where $z \in U$, $\varphi(z)dz^k = (\pi|_U)^*\Phi$, *and the norm* $\|\Phi\|_\infty$ *is given by (1.7).*

Remark 1.1.18. *Let* $B_\Gamma(U)$ *be the space of* Γ*-invariant holomorphic* $k-$*differentials* $\varphi(z)dz^k$ *on* U *that satisfy (1.7).* $B(V)$ *is isomorphic to* $B_\Gamma(U)$ *and satisfies (1.2) (for* $g \in \Gamma$, $z \in U$*) and (1.7).*

Definition 1.1.19. *The spaces* $A^{(1)}(\Sigma)$, $A^{(2)}(\Sigma)$, $B(\Sigma)$ *are defined by setting* $\Lambda = \emptyset$ *(i.e.* $V = \Sigma$*) in the definition above.*

Definition 1.1.20. *The space* $A^{(1)}(U)$ *(respectively* $A^{(2)}(U)$ *and* $B(U)$*) is defined as the normed space of holomorphic* $k-$*differentials* $\varphi(z)dz^k$ *on* U *such that*

$$\|\varphi(z)dz^k\|_1 = \int\limits_U |\varphi(z)w(z)^{k-2}|\mathrm{d}\mu < \infty \tag{1.8}$$

(respectively

$$\|\varphi(z)dz^k\|_2 = (\int\limits_U |\varphi(z)|^2 w(z)^{2k-2}\mathrm{d}\mu)^{1/2} < \infty \tag{1.9}$$

and

$$\|\varphi(z)dz^k\|_\infty = \sup_{z\in U} |\varphi(z)|w(z)^k < \infty). \tag{1.10}$$

Remark 1.1.21. *The space $A^{(1)}(U)$ (respectively $A^{(2)}(U)$ and $B(U)$) is isomorphic to the space $\mathcal{A}^{(1)}(U)$ (respectively $\mathcal{A}^{(2)}(U)$ and $\mathcal{B}(U)$) of holomorphic functions $\varphi(z)$ on U satisfying (1.8) (respectively (1.9) and (1.10)). The isomorphism is given by the map $\varphi(z) \mapsto \varphi(z)dz^k$.*

Definition 1.1.22. *The space $A^{(1)}(\Delta)$ is defined as the normed space of holomorphic $k-$differentials $\varphi(z)dz^k$ on Δ such that*

$$\|\varphi(z)dz^k\|_1 = \int\limits_\Delta |\varphi(z)|w(z)^{k-2}\mathrm{d}\mu < \infty. \tag{1.11}$$

It is isomorphic (via $\varphi(z)dz^k \mapsto \varphi(z)$) to the space $\mathcal{A}^{(1)}(\Delta)$ of holomorphic functions $\varphi(z)$ on Δ satisfying (1.11).

Remark 1.1.23. *Spaces $A^{(1)}(\cdot)$, $A^{(2)}(\cdot)$, $B(\cdot)$, defined above, are Banach spaces. The Petersson inner product*

$$\langle \Phi, \Psi\rangle = \int\limits_{\Gamma\backslash U} \varphi(z)\overline{\psi(z)}w(z)^{2k-2}\mathrm{d}\mu,$$

where $(\pi|_U)^*\Phi$ and $(\pi|_U)^*\Psi$ are $\varphi(z)dz^k$ and $\psi(z)dz^k$, $\Phi \in A^{(1)}(V)$, $\psi \in B(V)$, establishes an antilinear isomorphism between $B(V)$ and $A^{(1)}(V)^*$.

Remark 1.1.24. *The Petersson inner product*

$$\langle \varphi(z)dz^k, \psi(z)dz^k\rangle = \int\limits_U \varphi(z)\overline{\psi(z)}w(z)^{2k-2}\mathrm{d}\mu,$$

where $\varphi(z)dz^k \in A^{(1)}(U)$ and $\psi(z)dz^k \in B(U)$ establishes an antilinear isomorphism between $B(U)$ and $A^{(1)}(U)^*$.

Remark 1.1.25. $A^{(2)}(V)$ *is a Hilbert space, with the inner product*

$$\langle \Phi, \Psi \rangle = \int\limits_{\Gamma \backslash U} \varphi(z)\overline{\psi(z)} w(z)^{2k-2} \mathrm{d}\mu \tag{1.12}$$

where $\Phi, \Psi \in A^{(2)}(V)$, $\varphi(z)dz^k = (\pi|_U)^*\Phi$, $\psi(z)dz^k = (\pi|_U)^*\Psi$.

$A^{(2)}(U)$ *is a Hilbert space, with the inner product*

$$\langle \varphi(z)dz^k, \varphi(z)dz^k \rangle = \int\limits_U \varphi(z)\overline{\psi(z)} w(z)^{2k-2} \mathrm{d}\mu.$$

Remark 1.1.26. *The space* $A^{(2)}(U)$ *admits a reproducing kernel* $K : U \times U \mapsto \mathbb{C}$. *It has, in particular, the following properties: [24, Theorem III.3.1]:*

- $K(z, \xi)$ *is holomorphic in* z *and antiholomorphic in* ξ,
- $K(z, \xi) = \overline{K(\xi, z)}$,
- *as a function of* z *(with fixed* ξ*) it belongs to* $\mathcal{A}^{(1)}(U)$ *and to* $\mathcal{A}^{(2)}(U)$,
- *for any automorphism* τ *of* U *and* $z, \xi \in U$, $K(\tau(z), \tau(\xi)) J(\tau, z)^k \overline{J(\tau, \xi)}^k = K(z, \xi)$,
- *for any function* f *in* $\mathcal{A}^{(1)}(U)$ *or in* $\mathcal{A}^{(2)}(U)$ *or in* $\mathcal{B}(U)$ *we have:*

  $$f(z) = \int\limits_U K(z, \xi) f(\xi) w(\xi)^{2k-2} \mathrm{d}\mu$$

  for any $z \in U$.

Also the operator β *defined for any* $z \in U$ *formally by*

$$(\beta f)(z) = \int\limits_U K(z, \xi) f(\xi) w(\xi)^{2k-2} \mathrm{d}\mu$$

is a bounded linear projection from $L^1(U, w(z)^{k-2}\mathrm{d}\mu)$ *(respectively* $L^2(U, w(z)^{2k-2}\mathrm{d}\mu)$*,* $L^\infty(U, sup_{z\in U}|\cdot|w(z)^k)$ *onto* $\mathcal{A}^{(1)}(U)$ *(respectively* $\mathcal{A}^{(2)}(U)$*,* $\mathcal{B}(U)$*).*
We shall denote by $L^1_\Gamma(U, w(z)^{k-2}\mathrm{d}\mu)$ *the subspace of* $L^1(U, w(z)^{k-2}\mathrm{d}\mu)$ *that consists of functions satisfying (1.2) for* $g \in \Gamma$*,* $z \in U$*.*

1.2 Poincaré series

The Poincaré series for a function $f : U \mapsto \mathbb{C}$ is formally defined as $\sum_{g\in\Gamma} f(gz)J(g,z)^k$. We shall need the following statement about the Poincaré series of $k-$differentials.

Theorem 1.2.1. *Suppose* k *is a positive integer,* $k \geq 2$*.*

(i) For any $\varphi(z)dz^k \in A^{(1)}(U)$ *the Poincaré series*

$$\theta(\varphi)(z) = \sum_{g\in\Gamma} \varphi(gz)J(g,z)^k,$$

converges absolutely and uniformly on compact sets.

(ii) (Theorem 3.3 [24]) The Poincaré series map

$$\Theta : A^{(1)}(U) \to A^{(1)}_\Gamma(U)$$

$$\varphi(z)\mathrm{d}z^k \mapsto \Theta(\varphi(z)\mathrm{d}z^k) = \theta(\varphi)(z)\mathrm{d}z^k,$$

is a surjective, bounded and linear operator with norm $\|\Theta\| \leq 1$

Proof. (i) Let $z_0 \in U$, let $r > 0$ be sufficiently small so that the closed disc $\overline{B}(z_0; r) = \{z \in \mathbb{C}||z - z_0| \leq r\}$ is in $\mathcal{F}$. Let us show that the series $\theta(\varphi)$ converges absolutely at z_0. By the Mean Value Theorem for a function f holomorphic on an open set containing $\overline{B}(z_0; r)$

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\mathrm{d}t,$$

and so

$$\int\limits_{\bar{B}(z_0;r)} f(z)dxdy = \int\limits_0^r \int\limits_0^{2\pi} f(z_0+\rho e^{it})\rho \mathrm{d}t\mathrm{d}\rho =$$

$$2\pi \int\limits_0^r f(z_0)\rho \mathrm{d}\rho = f(z_0)\pi r^2,$$

hence $f(z_0) = \frac{1}{\pi r^2}\int_{\bar{B}(z_0;r)} f(z)dxdy$. Let $m = \min_{z\in\bar{B}(z_0;r)} w(z)^{k-2}$. Note that $m > 0$. We have: for $\gamma \in \Gamma$

$$|\varphi(\gamma z_0)J(\gamma, z_0)^k| = \left|\frac{1}{\pi r^2}\int\limits_{\bar{B}(z_0;r)} \varphi(\gamma z)J(\gamma,z)^k \mathrm{d}x\mathrm{d}y\right|$$

$$\leq \frac{1}{\pi r^2}\int\limits_{\bar{B}(z_0;r)} |\varphi(\gamma z)J(\gamma,z)^k|\mathrm{d}x\mathrm{d}y$$

$$\leq \frac{1}{m\pi r^2}\int\limits_{\bar{B}(z_0;r)} |\varphi(\gamma z)J(\gamma,z)^k|w(z)^{k-2}\mathrm{d}x\mathrm{d}y,$$

and then:

$$\sum_{\gamma\in\Gamma}|\varphi(\gamma z_0)J(\gamma,z_0)^k| \leq \frac{1}{m\pi r^2}\sum_{\gamma\in\Gamma}\int\limits_{\bar{B}(z_0;r)} |\varphi(\gamma z)J(\gamma,z)^k|w(z)^{k-2}\mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{m\pi r^2}\sum_{\gamma\in\Gamma}\int\limits_{\gamma(\bar{B}(z_0;r))} |\varphi(\eta)|w(\eta)^{k-2}\mathrm{dRe}(\eta)\mathrm{dIm}(\eta)$$

$$\leq \frac{1}{m\pi r^2}\int\limits_U |\varphi(\eta)|w(\eta)^{k-2}\mathrm{dRe}(\eta)\mathrm{dIm}(\eta) < \infty.$$

Now let us choose a compact set $K \subset U$ and show that the series converges uniformly on K.

Let $r > 0$ be sufficiently small, so that the disc of radius r centered at any point of K is contained in a compact set K' and $K' \subset U$. Let q be the number of elements of Γ such that $\gamma K' \cap K' \neq \emptyset$. For $0 < \epsilon < 1$ denote by C_ϵ the closed disc centered at 0 of radius $1 - \epsilon$. There are at most finitely many $\gamma \in \Gamma$ such that $\gamma K' \cap C_\epsilon \neq \emptyset$ [43, p. 219]. Write Σ'_ϵ to denote the sum over all other $\gamma \in \Gamma$. As above, for $z \in K$, we obtain:

$$\Sigma'_\epsilon |\varphi(\gamma z) J(\gamma, z)^k| \leq \frac{q}{m_K \pi r^2} \int\limits_{U - C_\epsilon} |\varphi(z)| w(z)^{k-2} \mathrm{d}x \mathrm{d}y,$$

where $m_K = \min_{\xi \in K'} w(\xi)^{k-2}$. As $\epsilon \to 1$, the right hand side goes to zero. This proves uniform convergence.

$\square$

Chapter 2

Main results on $k-$differentials and Poincaré series

Construction of automorphic forms via Poincaré series is a classical technique, and there is a significant amount of literature on this subject. For example, it is well-known that any holomorphic k-differential (k is an integer, $k \geq 2$) on a compact Riemann surface of genus $g \geq 2$ is obtained from the Poincaré series of a polynomial in z of degree not higher than $k(2g-2)$. There are various descriptions of the kernel of the Poincaré series operator: See, in particular, [25], [26], [30], [31], [33].

The case $k = 2$ (quadratic differentials) is of special importance in Teichmüller theory and has connections to Thurston's program. The large variety of references on this subject includes, in particular, [3], [4], [15], [27], and several of McMullen's papers including [32]. This chapter contains our main results about holomorphic $k-$differentials and Poincaré series.

The results in this chapter was inspired, partially, by the questions discussed in [28], although we do not present results similar to the results of [28].

2.1 Holomorphic $k-$differentials

Let $\Theta_0 : A^{(1)}(\Delta) \longrightarrow A^{(1)}(U)$ be the Poincaré series map corresponding to π_0.

Proposition 2.1.1. *If Λ is a finite set, then $A^{(2)}(V)$ is isomorphic to $A^{(2)}(\Sigma)$, $B(V)$ is isomorphic to $B(\Sigma)$, $A^{(1)}(V)$ is isomorphic to the subspace of $A^{(1)}(\Sigma)$ that consists of meromorphic k-differentials on Σ with at most simple poles, all in Λ.*

Proof. Let Φ be a holomorphic k-differential on V. Then $(\pi|_U)^*\Phi = \phi(z)dz^k$ is a holomorphic k-differential on U. Let z_0 be a point in $\pi^{-1}(\Lambda)$. Note that $\phi(z)$ has an isolated singularity at z_0. If $\Phi \in B(V)$ then $\phi(z)$ is bounded in a small neighborhood of z_0, hence $\phi(z)$ has a removable singularity at z_0. If $\Phi \in A^{(2)}(V)$ then $\phi(z)$ has a removable singularity at z_0 (the proof is similar to the proof of 8(b) in [20, p.146]). If $\Phi \in A^{(1)}(V)$ then $\phi(z)$ has a removable singularity or a simple pole at z_0. $\square$

Proposition 2.1.2. *$A^{(1)}(U)$ and $A^{(1)}(V)$ are separable.*

Proof. The proof is analogous to the proof of Corollary 1 [16, §3-2] (see also the proof of [22, Proposition 1.3]. Since Θ and Θ_0 are continuous surjective maps, it is sufficient to show that $A^{(1)}(\Delta)$ is separable. Let f be a function in $\mathcal{A}^{(1)}(\Delta)$. For each r such that $0 < r < 1$ let f_r be the function defined by $f_r(z) = f(rz)$, $z \in \Delta$. We have:

$$\int_{\Delta} |f_r(z) - f(z)|(1 - |z|^2)^{k-2}\mathrm{d}\mu \to 0$$

as $r \to 1^-$ (to see this use that for $0 < \delta < 1$

$$\int_{\Delta} |f_r(z) - f(z)|w(z)^{k-2}\mathrm{d}\mu$$

$$\leq \int_{|z|\leq\delta} |f_r(z) - f(z)|(1 - |z|^2)^{k-2}\mathrm{d}\mu + \int_{\delta<|\mathrm{z}|<1} (|\mathrm{f_r(z)}| + |\mathrm{f(z)}|)(1 - |\mathrm{z}|^2)^{\mathrm{k}-2}\mathrm{d}\mu).$$

The function f_r can be approximated by its Taylor polynomials, and, moreover, by polynomials with rational coefficients. Thus, for any $\epsilon > 0$, there is a polynomial

$p(z)$ with rational coefficients such that

$$\int_{\Delta} |f(z) - p(z)|(1 - |z|^2)^{k-2} \mathrm{d}\mu < \epsilon.$$

$\square$

Proposition 2.1.3. *Let $\mathbb{P}$ be the set of polynomials in z. The set $\Theta_0(\{p(z)dz^k \mid p \in \mathbb{P}\})$ is dense in $A^{(1)}(U)$. The set $\Theta(\Theta_0(\{p(z)dz^k \mid p \in \mathbb{P}\}))$ is dense in $A^{(1)}(V)$. $A^{(1)}(U)$ and $A^{(1)}(V)$ are separable.*

Proof. By Theorem 1.2.1, Θ and Θ_0 are continuous surjective maps.

For any $f \in \mathcal{A}^{(1)}(\Delta)$ and for any $\epsilon > 0$ there is a polynomial $p(z)$ with rational coefficients such that

$$\int_{\Delta} |f(z) - p(z)|(1 - |z|^2)^{k-2} \mathrm{d}\mu < \epsilon$$

(the proof is analogous to the proof of [15, §3-2, Cor.1], see also the proof of Proposition 1.3 [22]). Thus $\mathcal{A}^{(1)}(\Delta)$ is separable and $\mathbb{P}$ is dense in $\mathcal{A}^{(1)}(\Delta)$. The statement follows.

$\square$

2.2 Kernel of the Poincaré series map

One of the best partial result in this direction was obtained by I. Ljan [30]. In his work Θ is viewed as an operator from $A^{(1)}(\Delta)$ onto $A^{(1)}(\Sigma)$. In the following theorem, we generalize Ljan's result to the case when Θ is an operator from $A^{(1)}(U)$ onto $A^{(1)}(V)$.

Theorem 2.2.1. *The set*

$$W = \{\beta(\chi_{g\mathcal{F}}\phi - \chi_{\gamma\mathcal{F}}\phi)(z)\mathrm{dz^k} \mid \mathrm{g}, \gamma \in \Gamma, \phi(\mathrm{z}) \in \mathrm{L}^1_\Gamma(\mathrm{U}, \mathrm{w^{k-2}d}\mu)\},$$

is dense in $\ker\Theta$.

Proof. We shall follow the idea of the proof of the main theorem in [30].

Let l be a continuous linear functional on $\ker\Theta$. It will suffice to show that if $l(f) = 0$ for all $f \in W$ then $l(f) = 0$ for all $f \in \ker\Theta$.

By the Hahn-Banach theorem, l can be extended to $\tilde{l} \in (A^{(1)}(U))^*$ such that $||\tilde{l}|| = ||l||$. By [24, Theorem 3.2.1] there exists a $\psi \in \mathcal{B}(U)$ such that

$$\tilde{l}(f) = \int\limits_U f(z)\overline{\psi(z)}w(z)^{2k-2}\mathrm{d}\mu$$

for all $f \in \mathcal{A}^{(1)}(U)$.

Now let us prove two lemmas.

Lemma 2.2.2. *If* $g, \gamma \in \Gamma$, $\phi \in L^1_\Gamma(U, w^{k-2}\mathrm{d}\mu)$, *then*

$$\beta(\chi_{g\mathcal{F}}\phi - \chi_{\gamma\mathcal{F}}\phi)(z)dz^k \in \ker\Theta.$$

Proof. $\phi \in L^1_\Gamma(U, w^{k-2}\mathrm{d}\mu)$, therefore $\beta(\chi_{g\mathcal{F}}\phi), \beta(\chi_{\gamma\mathcal{F}}\phi) \in \mathcal{A}^{(1)}(U)$, so $\beta(\chi_{g\mathcal{F}}\phi - \chi_{\gamma\mathcal{F}}\phi)(z)dz^k \in A^{(1)}(U)$. We have:

$$\beta(\chi_{g\mathcal{F}}\phi)(z) = \int\limits_{g\mathcal{F}} \phi(\xi)K(z,\xi)w(\xi)^{2k-2}\mathrm{d}\mu(\xi),$$

$$\beta(\chi_{\gamma\mathcal{F}}\phi)(z) = \int\limits_{\gamma\mathcal{F}} \phi(\eta)K(z,\eta)w(\eta)^{2k-2}\mathrm{d}\mu(\eta)$$

$$= \int\limits_{g\mathcal{F}} \phi(\gamma g^{-1}\xi)K(z,\gamma g^{-1}\xi)w(\gamma g^{-1}\xi)^{2k-2}\mathrm{d}\mu(\gamma \mathrm{g}^{-1}\xi)$$

$$= \int\limits_{g\mathcal{F}} \phi(\xi)J(\gamma g^{-1},\xi)^{-k}K(z,\gamma g^{-1}\xi)w(\xi)^{2k-2}|J(\gamma g^{-1},\xi)|^{2k}\mathrm{d}\mu(\xi)$$

$$= \int_{g\mathcal{F}} \phi(\xi) K(z, \gamma g^{-1}\xi) w(\xi)^{2k-2} \overline{J(\gamma g^{-1}, \xi)}^k \mathrm{d}\mu(\xi),$$

so that

$$\theta(\beta(\chi_{g\mathcal{F}}\phi - \chi_{\gamma\mathcal{F}}\phi))(z)$$

$$= \sum_{h\in\Gamma} \int_{g\mathcal{F}} \phi(\xi)(K(hz,\xi) - K(hz, \gamma g^{-1}\xi)\overline{J(\gamma g^{-1}, \xi)}^k) w(\xi)^{2k-2} \mathrm{d}\mu(\xi) \mathrm{J(h, z)^k},$$

which is equal to zero as

$$\sum_{h\in\Gamma} K(hz, \gamma g^{-1}\xi)\overline{J(\gamma g^{-1}, \xi)}^k J(h,z)^k$$

$$= \sum_{h\in\Gamma} K(g\gamma^{-1}hz, \xi) J(g\gamma^{-1}, hz)^k J(h,z)^k$$

$$= \sum_{h\in\Gamma} K(g\gamma^{-1}hz, \xi) J(g\gamma^{-1}h, z)^k = \sum_{\alpha\in\Gamma} K(\alpha z, \xi) J(\alpha, z)^k$$

where $\alpha = g\gamma^{-1}h$.

□

Lemma 2.2.3. *If $\psi \in \mathcal{B}(U)$ and for all $\phi \in L^1_\Gamma(U, w^{k-2}\mathrm{d}\mu)$*

$$\int_{\mathcal{F}} \phi(z)\overline{\psi(z)} w(z)^{2k-2} \mathrm{d}\mu = 0$$

then $\psi(z)$ is identically zero.

Proof. Define $\psi_1 \in L^\infty_\Gamma(U, \sup_{z\in\mathcal{F}} |\cdot| w(z)^k)$ by setting $\psi_1 = \psi$ on $\mathcal{F}$ and $\psi_1(hz) = \psi_1(z)J(h,z)^{-k}$ for all $z \in \mathcal{F}$, $h \in \Gamma$. Then ψ_1 is zero (because $L^\infty_\Gamma(U, \sup_{z\in\mathcal{F}} |\cdot| w(z)^k)$ is the dual of $L^1_\Gamma(U, w^{k-2}\mathrm{d}\mu)$ [24]). Therefore $\psi|_{\mathcal{F}} = 0$ and, since ψ is holomorphic, it must be identically zero.

□

Lemma 2.2.2 shows that $W \subset \ker\Theta$. Suppose that $\tilde{l}(f) = 0$ for all $f \in W$. We have that for all $g, \gamma \in \Gamma$, $\phi \in L^1_\Gamma(U, w^{k-2}\mathrm{d}\mu)$

$$0 = \int_U \beta(\chi_{g\mathcal{F}}\phi - \chi_{\gamma\mathcal{F}}\phi)(z)\overline{\psi(z)}w(z)^{2k-2}\mathrm{d}\mu$$

$$= \int_U\int_U (\chi_{g\mathcal{F}}\phi(\xi) - \chi_{\gamma\mathcal{F}}\phi(\xi))K(z,\xi)w(\xi)^{2k-2}\mathrm{d}\mu(\xi)\overline{\psi(\mathrm{z})}\mathrm{w}(\mathrm{z})^{2\mathrm{k}-2}\mathrm{d}\mu(\mathrm{z})$$

$$= \int_U\overline{\int_U \psi(z)K(\xi,z)w(z)^{2k-2}\mathrm{d}\mu(\mathrm{z})}(\chi_{g\mathcal{F}}\phi(\xi) - \chi_{\gamma\mathcal{F}}\phi(\xi))w(\xi)^{2k-2}\mathrm{d}\mu(\xi)$$

$$= \int_U (\chi_{g\mathcal{F}}\phi(\xi) - \chi_{\gamma\mathcal{F}}\phi(\xi))\overline{\psi(\xi)}w(\xi)^{2k-2}\mathrm{d}\mu(\xi)$$

$$= \int_{g\mathcal{F}} \phi(\xi)\overline{\psi(\xi)}w(\xi)^{2k-2}d\mu(\xi) - \int_{\gamma\mathcal{F}} \phi(\eta)\overline{\psi(\eta)}w(\eta)^{2k-2}\mathrm{d}\mu(\eta)$$

$$= \int_{\mathcal{F}} (\phi(gz)\overline{\psi(gz)}|J(g,z)|^{2k} - \phi(\gamma z)\overline{\psi(\gamma z)}|J(\gamma,z)|^{2k})w(z)^{2k-2}\mathrm{d}\mu(\mathrm{z})$$

$$= \int_{\mathcal{F}} \phi(z)(\overline{\psi(gz)J(g,z)^k} - \overline{\psi(\gamma z)J(\gamma,z)^k})w(z)^{2k-2}\mathrm{d}\mu(\mathrm{z}).$$

Hence, by Lemma 2.2.3, with $\xi = gz$,

$$\psi(\xi) = \psi(\gamma g^{-1}\xi)J(\gamma g^{-1},\xi)^k$$

for all $\xi \in U$, $\gamma, g \in \Gamma$, thus $\psi \in \mathcal{B}_\Gamma(U)$. For any $\phi(z)\mathrm{dz}^{\mathrm{k}} \in \mathrm{A}^{(1)}(\mathrm{U})$

$$\int_{\mathcal{F}} \theta\phi(z)\overline{\psi(z)}w(z)^{2k-2}\mathrm{d}\mu = \int_{\mathcal{F}}\sum_{\mathrm{h}\in\Gamma}\phi(\mathrm{hz})\mathrm{J}(\mathrm{h},\mathrm{z})^{\mathrm{k}}\overline{\psi(\mathrm{z})}\mathrm{w}(\mathrm{z})^{2\mathrm{k}-2}\mathrm{d}\mu$$

$$= \sum_{h\in\Gamma}\int_{h\mathcal{F}} \phi(\xi)\overline{\psi(\xi)}w(\xi)^{2k-2}\mathrm{d}\mu(\xi) = \int_{\mathrm{U}} \phi(\xi)\overline{\psi(\xi)}\mathrm{w}(\xi)^{2\mathrm{k}-2}\mathrm{d}\mu(\xi) = \tilde{\mathrm{l}}(\phi).$$

Thus $\tilde{l} = 0$, and so $l = 0$ on all of $\ker\Theta$, as required.

□

In the following Theorem, we obtain another description of the kernel of Poincaré series map, and as a consequence we show that there exist a subset of $Ker\Theta$ which is dense in $A^{(2)}(U)$ (if Γ is infinite). This is a new result even for the case $k = 2$.

Theorem 2.2.4. *Suppose Γ is infinite. Let P be a subset of $\mathcal{F}$ that has a limit point in $\mathcal{F}$. Let $\mathcal{P}$ be the linear span of the set*

$$\{(K(z,p) - \overline{J(g,p)}^k K(z,gp))dz^k \mid p \in P, g \in \Gamma\}.$$

Then $\mathcal{P} \subset \ker \Theta \cap A^{(2)}(U)$ and $\mathcal{P}$ is dense in $A^{(2)}(U)$.

Proof. First, we note that $K(z,p) - \overline{J(g,p)}^k K(z,gp)$, as a function of z (with fixed $g \in G$, $p \in P$), is in $\mathcal{A}^{(1)}(U)$ and $\mathcal{A}^{(2)}(U)$.

The following calculation shows that it belongs to $\ker \Theta$:

$$\sum_{\gamma \in \Gamma} (K(\gamma z, p) - \overline{J(g,p)}^k K(\gamma z, gp)) J(\gamma, z)^k$$

$$= \sum_{\gamma \in \Gamma} K(\gamma z, p) J(\gamma, z)^k - \sum_{\gamma \in \Gamma} \overline{J(g,p)}^k K(g^{-1}\gamma z, p) J(g^{-1}, \gamma z)^k \overline{J(g^{-1}, gp)}^k J(\gamma, z)^k$$

$$= \sum_{\gamma \in \Gamma} K(\gamma z, p) J(\gamma, z)^k - \sum_{\gamma \in \Gamma} K(g^{-1}\gamma z, p) J(g^{-1}\gamma, z)^k = 0.$$

Finally, let us assume that $f(z)dz^k \in A^{(2)}(U)$ and

$$\int_U f(z) \overline{(K(z,p) - \overline{J(g,p)}^k K(z,gp))} w(z)^{2k-2} \mathrm{d}\mu = 0$$

for all $p \in P$, $g \in \Gamma$. We need to show that f is identically zero. We have:

$$f(p) = \int_U f(z) K(p,z) w(z)^{2k-2} \mathrm{d}\mu =$$

$$\int_U f(z)J(g,p)^k K(gp,z))w(z)^{2k-2}\mathrm{d}\mu(z) = f(gp)J(g,p)^k$$

for all $p \in P$, $g \in \Gamma$. Therefore the holomorphic function $f(p) - f(gp)J(g,p)^k$ is zero on P; hence it is identically zero, and so $f \in \mathcal{A}_\Gamma^{(2)}(U) \cap \mathcal{A}^{(2)}(U)$ which is $\{0\}$ [24, §3.2].

$\square$

Corollary 2.2.5. *If Γ is infinite then $\ker \Theta \cap A^{(2)}(U)$ is dense in $A^{(2)}(U)$.*

Remark 2.2.6. *$\mathcal{P}$ is non-trivial. Indeed, suppose that $\mathcal{P} = \{0\}$. Then for all $p \in P$ and $g \in \Gamma$ we have:*

$$0 = K(z,p) - \overline{J(g,p)}^k K(z,gp) = K(z,p) - K(g^{-1}z,p)J(g^{-1},z)^k.$$

Hence $K(z,p)$ as a function of z (with fixed p) belongs to $\mathcal{A}_\Gamma^{(1)}(U)$. But $K(z,p) \in \mathcal{A}^{(1)}(U)$ and $\mathcal{A}_\Gamma^{(1)}(U) \cap \mathcal{A}^{(1)}(U) = \{0\}$ unless Γ is finite [24, §3.2].

Chapter 3

An Extension Theorem

3.1 Introduction

Let E be a closed subset of a non-compact Riemann surface R. In [37], I. Richards has shown that every Riemann surface can be represented (topologically) as a sphere with handles attached and points removed on the equator. For a precise statement, see [37]. From this representation, it follows easily that we can find G, $D_0, D_1, D_2, \cdots$ and $W_1, W_2, \cdots$ which are subsets of R with the following properties:

1) The domain G is an open and connected subset (that is, a domain) of R which contains E, and whose boundary consists of at most a countable number of Jordan arcs and Jordan curves.

2) Each compact subset of R meets at most a finite number of components of ∂G.

3) All the handles of G are also in E, i.e. no handle in R that starts in E but ends outside of E is contained in G.

4) The D_n's are pre-compact Jordan domains $D_0 \subset\subset D_1 \subset\subset D_2 \subset\subset \cdots$, $\bigcup_n D_n = R$ (that is, $\{D_n\}$ is an exhaustion of R) and each D_n is homeomorphic to a sphere with a finite number of circular holes, and a finite number of handles attached to them [42, p.141]. Each handle which starts in D_n also ends in D_n, and we assume that D_0 is planar. (Recall that a set M is planar if each simple closed curve that locally separates M also separates it).

5) For any D_n, $G \setminus D_n$ has only a finite number of components.

6) The W_j's are open neighborhoods around each handle in $G \cap D_n$, for each n. The closure of the $\overline{W_j}$'s are disjoint. It thus follows that $G \setminus \cup W_j$ is planar. Without loss of generality we can also assume that for $n > 0$, each $D_n \setminus \overline{D_{n-1}}$ contains just one of these neighborhoods, and so for each n, $W_n \subset\subset D_n \setminus \overline{D_{n-1}}$. Note that this implies that we are thus assuming G to be of infinite genus. For more on that, see Remark 4.2.4.

7) We also assume that the boundary of each D_n does not intersect with the"holes" of G, (by hole of G, we mean a component of $R \setminus G$ whose closure is compact in R).

3.2 Stitching up

Definition 3.2.1. *Circular domain : A domain Ω contained in the Riemann sphere $\overline{\mathbb{C}}$ is called a circular domain if its boundary consists of a finite number of disjoint non-degenerate circles.*

Let r be a planar Riemann surface which is $n-$connected, and assume $n \geq 1$. Then r can be represented by a circular domain $\Omega = \overline{\mathbb{C}} \setminus \{K_1, \cdots, K_{n+1}\}$ under a bijective meromorphic map f, where each closed disc K_i is contained in $\mathbb{C}, i = 1, \cdots, n+1$ (see [10, §15.7]).

Let $K_1 = \{z \mid |z - z_1| \leq \rho_1\}$. Set r_1 to be $r \cup K_1$, and define charts on r_1 as follows. Let $p_0 \in r_1$. If $p_0 \in r \cap r_1$, we use the charts in r containing p_0. If $p_0 \in K_1^\circ \cap r_1$, we use a small neighborhood of p_0 in $\mathbb{C}$ (small enough to be contained in K_1°) and the identity map. If $p_0 \in \partial K_1 \cap r_1$, let U be a neighborhood of p_0 in $\mathbb{C}$, and define $U' := f^{-1}((U \setminus K_1) \cap \Omega) \cup (U \cap K_1)$ to be a neighborhood of p_0 in r_1. A chart is then defined as follows:

$$F : U' \longrightarrow \mathbb{C}$$

$$p \longmapsto \begin{cases} f(p) & \text{if p} \in f^{-1}(U \setminus K_1 \cap \Omega) \\ p & \text{if } p \in U \cap K_1. \end{cases}$$

Following [41, §6.1], we now show that these charts are compatible with each other and with the original structure.

Suppose $p_0, p_1 \in \partial K_1$, let U_{p_0} and U_{p_1} be neighborhoods of p_0 and p_1 in r_1, and F_0 and F_1 be the related charts, as defined above.

Suppose $p \in U_{p_0} \cap U_{p_1}$. We then have either $p \in (U_{p_0} \cap U_{p_1}) \cap K_1$ or $p \in (U_{p_0} \cap U_{p_1}) \cap K_1^c$.

If $p \in (U_{p_0} \cap U_{p_1}) \cap K_1$, then $F_1 F_0^{-1}(p) = F_1(p) = p$, and if $p \in (U_{p_0} \cap U_{p_1}) \cap K_1^c$, then $F_1 F_0(p) = F_1(f^{-1}(p)) = f(f^{-1}(p)) = p$.
So, in either case, $F_1 {F_0}^{-1}(p) = p$, and the new charts are compatible with the original charts of r and of K_1 (as a subset of $\mathbb{C}$). Thus r_1 preserve the analytic structure of r and K_1.
The same process can be repeated for $K_2, \cdots, K_{n+1}$, to obtain a simply connected Riemann surface $\tilde{r}$. We have thus obtained the following, which is a special case of Koebe's Theorem (see for example [15, §2.2]).

Lemma 3.2.2. *(see [41, §6.1]) Let r be a planar Riemann surface which is $n-$connected, then there exists a simply connected Riemann surface $\tilde{r}$ which contains r and $\tilde{r}$ preserves the analytic structure of r.*

3.3 Constructing planar Riemann surfaces and simply connected domains

Definition 3.3.1. *(see [41]) The analytic structure of a sequence of Riemann surfaces R_n is said to encompass the analytic structure of R if, for each compact $K \subset R$, there exists $n_0 \in \mathbb{N}$ such that $K \subset R_n$, for every $n \geq n_0$. The inclusion indicates not only* *the inclusion of sets, but also that the analytic structure of K as a subset of R and K as a subset of R_n is the same. We will use the notation $R_n\}_{n\to\infty}R$.*

Note that the limit R is not unique in the above definition. For example if $R_n := \mathbb{C}\setminus[n,\infty)$, then $R_n\}_{n\to\infty}B$, for any domain $B\subset\mathbb{C}$.

The following was stated and used in [41] without proof.

Proposition 3.3.2. *Suppose $R^{n,k}\}_{k\to\infty}R^n$ and $R^n\}_{n\to\infty}R$, then there exists a sequence $k(n)$ such that $R^{n,k(n)}\}_{n\to\infty}R$.*

Proof. We can write $R=\cup_{n=0}^{\infty}D_n$, where $\{D_n\}$ is an exhaustion of R by precompact domains.

Since $R^n\}_{n\to\infty}R$ there exists $N_0\in\mathbb{N}$ such that $\overline{D_0}\subset R^n$ whenever $n\geq N_0$. Now let n be fixed with $n\geq N_0$. Because $\overline{D_0}\subset R^n$ and $R^{n,k}\}_{k\to\infty}R^n$, there exists $k_0(n)\in\mathbb{N}$ such that, $D_0\subset R^{n,k}$ whenever $k\geq k_0(n)$. It follows that for every $n\geq N_0$, $\overline{D_0}\subset R^{n,k_0(n)}$, and so we can find a sequence $R^{N_0,k_0(N_0)}, R^{N_0+1,k_0(N_0+1)},\cdots$, such that D_0 is subset of all the elements in that sequence. By choosing $k_0(n)\geq k_0(n-1)$, we can and will assume that the sequence $k_0(n)$ is non-decreasing. In the same way, it is possible to find a sequence $R^{N_1,k_1(N_1)}, R^{N_1+1,k_1(N_1+1)},\cdots$ all of its elements contains D_1, and such that $N_1\geq N_0$, $k_1(N_1)\geq k_0(N_1)$ and $k_1(n)$ is a non-decreasing sequence. For any $\overline{D_j}$ we can find a sequence $R^{N_j,k_j(N_j)}, R^{N_j+1,k_j(N_j+1)},\cdots$ such that the elements of that sequence all contains D_j, and we can suppose $N_j\geq\max(N_1,\cdots,N_{j-1})$, $k_j(N_j)\geq\max(k_0(N_j),\cdots,k_{j-1}(N_j))$ and $\{k_j(n)\}$ is non-decreasing. Let $r_j := R^{N_j+j,k_j(N_j+j)}$. For any compact set K there exists D_n such that $K\subset D_n$ and by the way r_n is defined $D_n\subset R^{N_n+n,k_n(N_n+n)}$, so $r_n\}_{n\to\infty}R$. $\square$

Lemma 3.3.3. *Let R be a non-compact Riemann surface, E be a closed subset of R, and let G and $\{D_n\}$ $(n\geq 0)$ be respectively an open domain and an exhaustion of R having the properties described above. Then:*

1) There exists a sequence of Riemann surfaces R^n, and a sequence of domains $\mathcal{G}^n \subset R^n$ such that $\mathcal{G}^n \setminus D_n$ is planar and $\mathcal{G}^n\}_{n\to\infty}G$.

2) If it is possible to find the domain G containing E in such a way that $R \setminus \overline{G}$ contains an unbounded Jordan arc, then the sequence $\mathcal{G}^n$ can be chosen such that $\mathcal{G}^n \setminus \overline{D_n}$ is simply connected for any $n \geq n_0$, for n_0 large enough.

Proof. Let $R_n := R \setminus (\bigcup_{j>n} W_j)$ and $G_n = G \setminus (\bigcup_{j>n} W_j) \subset R_n$.

Fix $n > 0$. Let $C_n := D_n \cap G_n$ and $\widehat{C_n} := D_{n+1} \cap G_n$. It follows that $\widehat{C_n} \setminus \overline{C_n}$ is planar, and consists of a finite number of components, $\mathcal{O}_1, \cdots, \mathcal{O}_N$, each component having finite connectivity (this follows from the the way G and D_n's were selected). Each $\mathcal{O}_j$ can be represented under a bijective meromorphic map θ_j by a circular domain $\overline{\mathbb{C}} \setminus (K_1^j, \cup \cdots \cup K_{m_j}^j)$ where each K_i^j is a closed disc in $\mathbb{C}$. Fix j, $1 \leq j \leq N$. Some of the holes of $\mathcal{O}_j$ may correspond precisely with a hole of G_n. Using the stitching technique of Section 3.2, we fill all those holes (and only those holes) of $\mathcal{O}_j$. Denote by $\widetilde{\mathcal{O}_j}$ the open set obtained this way. Let

$$\widetilde{d_n} := (\cup_{j=1}^N \widetilde{\mathcal{O}_j}) \cup ((\overline{D_{n+1}} \setminus \overline{D_n}) \setminus (\widehat{C_n} \setminus \overline{C_n})),$$

with the obvious identification of the corresponding boundaries of $\widetilde{\mathcal{O}_j}$ and $(\overline{D_{n+1}} \setminus D_n) \setminus (\widehat{C_n} \setminus C_n)$. Now let $R^n := \overline{D_n} \cup (\cup_{i=n}^{\infty} \widetilde{d_i})$, again with the obvious identification of the boundaries. It follows easily that $R^n\}_{n\to\infty}R$.

Let $\mathcal{G}^n := (G \cap D_n) \cup (\bigcup_{i=n}^{\infty} (\bigcup_{j=1}^{N_i} \widetilde{\mathcal{O}_j}))$, which is G_n with the holes filled outside of D_n. Obviously $\mathcal{G}^n\}_{n\to\infty}G$ and $\mathcal{G}^n \setminus D_n$ is planar for any n, and this proves the first part.

Now we prove that if $R \setminus \overline{G}$ contains an unbounded Jordan arc, then $\mathcal{G}^n \setminus D_n$ is simply connected for $n \in \mathbb{N}$ large enough: Suppose B is a bounded component of $(\mathcal{G}^n \setminus D_n)^c$. First note that $(\mathcal{G}^n \setminus D_n)^c = (\mathcal{G}^n \cap D_n^c)^c = (\mathcal{G}^n)^c \cup D_n$ and recall that the boundary of D_n does not intersect with the holes of G. Now assume $B \cap D_n = \emptyset$, and note that $B \cap G_n = \emptyset$. If B is contained in a hole of G_n, then by the way $\mathcal{G}^n$

is defined, this B must be contained in an unbounded component of G_n^c, and thus it will be unbounded, which is contrary to our hypothesis.

If $B \cap D_n \neq \emptyset$, then $D_n \subset B$ since D_n is connected. Under the extra assumption that it is possible to find a connected and unbounded Jordan arc Γ in the complement of $\overline{G}$, we have that for large enough n, say $n \geq n_0$, $D_n \cap \Gamma \neq \emptyset$ and so Γ should be contained in B, and so B is unbounded, which is again contrary to our hypothesis. Thus no such B exists and $\mathcal{G}^n \setminus D_n$ is simply connected if $n \geq n_0$.

□

Definition 3.3.4. *Ideal and real boundary points: Let A be a closed, unbounded subset of a non-compact Riemann surface R and let R' be another Riemann surface containing A such that the analytic structure of R and R' are compatible on $R \cap R'$ and such that A is bounded in R'. The non-empty set $\overline{A} \setminus A$ with respect to R' is called the ideal boundary of A with respect to R'. The boundary points of A which are in R are called real boundary points.*

Lemma 3.3.5. *As before assume R is a non-compact Riemann surface, let D be a pre-compact domain in R (bounded by a finite number of Jordan curves) and $\mathcal{G}$ be a domain in R such that $\mathcal{G} \setminus D$ ($\overline{D}$ is a compact set in R) is planar and its complement does not have any compact component, then there exists a sequence of Riemann surfaces R^k, such that:*

1) *$\mathcal{G}$ is finitely connected in R^k.*

2) *The real boundary of R^k consists of finite number of Jordan curves and Jordan arcs.*

3) *$R^k\}_{k \to \infty} R$.*

Proof. Let $\Gamma_1, \Gamma_2, \cdots$ be the unbounded components of R, let R^k be the component of $R \setminus (\cup\{\Gamma_j; k < j\})$ which contains $\mathcal{G}$. Obviously $\mathcal{G}$ is finitely connected in R^k, $\mathcal{G} \setminus \overline{D}$

does not contains any "hole", and the boundary of D_n consists of a finite number of Jordan curves, so it cannot contain more than a finite number of holes.

Only a finite number of Γ_j's intersect with K, so for k_0 large enough, K is contained in the component of $R \setminus (\cup \Gamma_j, j > k_0)$ that contains G, i.e. R^k, so $K \subset R^k$, for $k > k_0$ and $R^k\}_{k \to \infty} R$. $\square$

Lemma 3.3.6. *Suppose B is an unbounded simply connected domain with a finite number of boundary components in R, each of which is either a Jordan arc or a Jordan curve, and suppose E is a closed subset of R. Then there exists a sequence R^ν such that $R^\nu\}_{\nu \to \infty} R$ and all the points of ideal boundary of $E \cap B$ form a bounded subset of each Riemann surface R^ν.*

Proof. The domain B can be represented by a conformal map f, according to Riemann mapping theorem. Because the boundary of B is composed of Jordan arcs or curves, it follows from Carathéodory Theorem that f has a continuous extension to the ∂B (see [35, §2.1]). Note that $M := f(\partial B)$ is an open subset of the boundary of the unit disc, consisting of a finite number of arcs of the circle. The complement of M with respect to $\{|z| = 1\}$ is a finite number of points $a_1, \cdots, a_\mu$. Let $A_j^\nu := \{e^{it} | a_j - \frac{1}{\nu} \leq t \leq a_j + \frac{1}{\nu}\}$ for $\nu \geq \nu_0$. We assume ν_0 to be large enough so that for $i \neq j$, we have $A_j^{\nu_0} \cap A_i^{\nu_0} = \emptyset$.

Now, let R' be $R \setminus (\cup_{j=1}^{\mu} f^{-1}(A_j^\nu \setminus \{a_j\}))$. Take μ copies $\overline{\mathbb{C}_j}$ of $\overline{\mathbb{C}}$ and stitch up R' to circular domains $\overline{\mathbb{C}_j} \setminus \{z \in \overline{\mathbb{C}_j} \mid |z| < 1\}$ along arcs $\{e^{it} \mid a_j - \frac{1}{\nu} < t < a_j + \frac{1}{\nu}\}$, using the technique described in Section 3.1. Denote the new surface by R^ν.

From the way the conformal structure has been selected for R^ν, R and R^ν are compatible on $R \cap R^\nu$, and from the way R^ν is defined it contains all the boundary of $E \cap B$. So the ideal boundary points of $E \cap B$ in R become real boundary points in R^ν, and $E \cap B$ becomes a compact subset of R^ν. If K is compact in R then for any ν we have $K \subset R^\nu$, so $R^\nu\}_{\nu \to \infty} R$.

$\square$

3.4 Extension Theorem, statement and proof

Let E be a closed subset of a non-compact Riemann surface R, and assume that there exists a neighborhood G of E, such that G is of infinite genus and that $R \setminus G$ contains a neighborhood of a connected unbounded Jordan arc. Recall that then there always exists an exhaustion of R by Jordan domains D_n, and that there exists Jordan domains W_n, with the following properties:

a) $W_n \subset\subset D_n \setminus \overline{D_{n-1}}$ for any $n \in \mathbb{N}$ and $D_0 = \emptyset$

b) Each non-empty W_n is of finite and positive genus (we can assume of genus one).

c) $G \setminus \cup_{n\in\mathbb{N}} W_n$ is planar.

Theorem 3.4.1. *Let D_n, W_n and G satisfy the above properties, then:*

1) For all $n \in \mathbb{N}$, there exists a non-compact Riemann surface r_n on which $E_n := E \setminus \cup_{n<j} \overline{W_j}$ is relatively compact.

2) For all $n \in \mathbb{N}$, $\overline{D_n} \subset r_n$.

Proof. Suppose R is a non-compact Riemann surface, and E is a closed, unbounded subset of R. Let G be a neighborhood of E, and D_n, $n \geq 1$ an exhaustion with the properties mentioned above.

By Lemma 3.3.3, there exist sequences R_n and $\mathcal{G}_n$ such that $R_n\}_{n\to\infty} R$, $\mathcal{G}_n\}_{n\to\infty} G$ and $\mathcal{G}_n \setminus \overline{D_n}$ is simply connected.

Fix n. Then by Lemma 3.3.5 there exist sequences $R_{n,k}$ such that $R_{n,k}\}_{k\to\infty} R$ and $\mathcal{G}_n \setminus \overline{D_n}$ has a finite number of simply connected components $B_1, \cdots, B_N$ and the boundary of each B_i consists of finite number of Jordan arcs and Jordan curves. Fix n and k, By Lemma 3.3.6 we can make $E \cap B_i$ closed in a non-compact Riemann surface $R^{\nu}_{n,k}$ and $R^{\nu}_{n,k}\}_{\nu\to\infty} R$. We can repeat the same process N times, so that for any $i = 1, \cdots, N$, $E \cap B_i$ become closed in $R^{\nu}_{n,k}$. Note that $\cup_{i=1}^{N}(E \cap B_i) = E \cap \mathcal{G}_n$. Let $E_n := E \setminus \cup_{i>n} W_i$ so $E_n = E \cap \mathcal{G}_n$ and it is closed in $R^{\nu}_{n,k}$. We have:

$$R^{\nu}_{n,k}\}_{\nu\to\infty} R_{n,k}, \qquad R_{n,k}\}_{k\to\infty} R_n \quad \textit{and} \quad R_n\}_{n\to\infty} R.$$

By Proposition 3.3.2, there exist sequences $\nu(n)$ and $k(n)$ such that $R^{\nu(n)}_{n,k(n)}\}_{n\to\infty} R$. Letting $r_n := R^{\nu(n)}_{n,k(n)}$, r_n is the Riemann surface with the described property.

Note that for $r_n = R^{\nu}_{n,k(n)}$, all the discs which were attached and all the changes to R were outside of $\overline{D_n}$ so the analytic structure of R inside $\overline{D_n}$ has not been changed, and $\overline{D_n} \subset r_n$.

$\square$

Chapter 4

Holomorphic approximation on non-compact Riemann surfaces

4.1 Fusion Lemma

The Fusion Lemma of Alice Roth [8] allows for the simultaneous approximation of two meromorphic functions in the complex plane. This Lemma says the following: Let K_1 and K_2 be disjoint compact sets in $\overline{\mathbb{C}}$ and let m_1 and m_2 be meromorphic functions on $\mathbb{C}$ whose values are close to each other on a compact set K. Then one can find a third meromorphic function which is close to m_1 on $K_1 \cup K$ and close to m_2 on $K_2 \cup K$; how close depends (surprisingly) neither on the functions m_1 and m_2 nor on K, but only on K_1, K_2 and the maximum of $|m_1 - m_2|$ on K.
This phenomenon has assumed a central position in complex approximation theory (see [18]). In particular P.M. Gauthier has shown that the proof of A. Roth can also be carried over to non-compact Riemann surfaces [17, p.143]. In these earlier works, one always only found the uniform estimate $|m_j - m| < a\alpha$ on $K_j \cup K$ with $a > 2$. It can be shown that $|m_2(z) - m(z)| \to 0$ as $|z| \to \infty$, $z \in K_2$. In these earlier works, one always only found the uniform estimate $|m_j - m| < a\alpha$ on $K_j \cup K$ with $a > 2$. But in the planar case, it can be shown that $|m_2(z) - m(z)| \to 0$ as $|z| \to \infty$, $z \in K_2$. Here, without loss of generality, we assume that ∞ belongs to K_2, as in the proof of A. Roth.

In Section 4.1.2, we shall give a proof (due to G. Schmieder [41]) of a stronger version of the Fusion Lemma on Riemann surfaces, where the question of how well the function m approximate (or can approximate) the function m_1 on K_1, respectively m_2 on K_2 is considered. However, the basic idea remains the proof of A. Roth for the planar case.

A further question in this connection is whether one can omit the assumption that $K_1 \cap K_2 = \emptyset$ and instead suppose that m_1 and m_2 are close on $K_1 \cap K_2$ thus dropping K completely. This is in general not possible, in fact not even when K_1 and K_2 are rectangles as D. Gaier has shown [3]. From this fact, it is not hard to see that for numbers $a = a(K_1, K_2)$ having the above properties, it is not possible to give a common estimate valid for all disjoint K_1 and K_2.

4.1.1 Preliminaries

For a closed subset T of a Riemann surface R, we denote by $M(T)$ (respectively $H(T)$) the family of functions meromorphic (respectively holomorphic) in a neighborhood of T. Let $M_R(T)$ be the class of (continuous) functions $f : T \to \overline{\mathbb{C}}$ for which there exists a sequence $f_n \in M(R)$ such that all f_n have the same set P_f of poles on T and $|f_n - f|$ converges uniformly to zero on $T \setminus P_f$. Let $H_R(T)$ be the class of (continuous) functions $f : T \to \overline{\mathbb{C}}$ for which there exists a sequence $f_n \in H(R)$ such that that $|f_n - f|$ converges uniformly to zero on T. Now let R be a non-compact Riemann surface. According to Behnke and Stein [7], there exists a Cauchy kernel ω on R (see for example [7], [17], [38]); let one such kernel be fixed.

Lemma 4.1.1. (Pompeiu-Formula)

If $\sigma : R \longrightarrow \mathbb{C}$ is a C^1-function with compact support, then

$$\sigma(Q) = \frac{1}{2\pi i} \iint\limits_R \omega(P, Q) \wedge \overline{\partial}\sigma$$

for all $Q \in R$, where $\overline{\partial}\sigma$ is the 1-form given in the local parameters by $\frac{\partial \sigma}{\partial \overline{\zeta}}$ and $\wedge$ denotes the exterior product.

This follows from a more general theorem obtained by S. Scheinberg [39, Prop.7].

4.1.2 Formulation and proof of the Fusion Lemma

Theorem 4.1.2. (Fusion Lemma) *Let K_1 and K be compact subsets and K_2 a closed subset of an open Riemann surface R with $K_1 \cap K_2 = \emptyset$, and suppose*

$$M(K_1 \cup K_2 \cup K) \subset M_R(K_1 \cup K_2 \cup K).$$

Then, there exists a continuous function

$$C : R \to \mathbb{R}_+$$

such that for each pair $m_1, m_2 \in M(R)$ with

$$\|m_1 - m_2\|_K < \varepsilon,$$

there exists a function $m \in M(R)$ which satisfies the inequality

$$|m(p) - m_j(p)| < C(p) \cdot \varepsilon \qquad (p \in K_j \cup K, j = 1, 2)$$

The function C depends only on K_1 and K_2.

Remark 4.1.3. *If K_2 is also compact then $M(K_1 \cup K_2 \cup K) \subset M_R(K_1 \cup K_2 \cup K)$ always by [7, Thm.13]. Note that in [7], the holomorphy which is assumed in a neighborhood of $K_1 \cup K_2 \cup K$ can be circumvented. Indeed we first subtract a meromorphic function on R having the same poles and principal parts on $K_1 \cup K_2 \cup K$ and after approximation we add it again. If for a compact set K_2, we replace the function $C(p)$ by the constant $c = \|C\|_{K_1 \cup K_2 \cup K}$, then we obtain the weaker form of the Fusion Lemma (see [13], [17] and [36]).*

Remark 4.1.4. *$M(X) \subset M_{\mathbb{C}}(X)$ holds for every closed set $X \subset \mathbb{C}$. This follows from the approximation theorem of Roth [13, p.120], taking Remark 4.1.3 into consideration.*

In order to prove the Fusion Lemma (see [41]), we choose pairwise smoothly bounded neighborhoods U_1 and U_2 of K_1 and K_2 on R with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$, and we may choose U_2 so large that $R \setminus U_2$ is compact. Further, let $E = R \setminus (U_1 \cup U_2)$, and $\chi : R \to [0,1]$ be a C^1 function with $\chi|_{U_1} = 1$ and $\chi|_{U_2} = 0$ (see e.g. [34, Cor. 2.2.15]). Let ω be some Cauchy-kernel on R (see [7, Thm.12]). The function

$$B(Q) = \frac{1}{2\pi} \iint_E |\omega(P,Q) \wedge \overline{\partial}\chi(P)| \tag{4.1}$$

is well-defined and continuous on R as can be seen by passing to a local coordinate z and then changing to polar coordinate $\zeta - z = \rho e^{it}$.

Now let $q = m_1 - m_2$ be fixed. By assumption there is a neighborhood U_3 of K with $\|q\|_{\overline{U}_3} < \varepsilon$.

Now we define the function q_1 on $U_1 \cup U_2 \cup U_3$. We set $q_1 = q$ on U_3 and extend it continuously by Tietze's Theorem (see e.g. [23, p.242]) to $E \setminus U_3$ (with respect to the relative topology of E) so that $\|q_1\| < \varepsilon$ (q_1 need not be continuous on R). By passing to local coordinates we see that

$$g(Q) = -\frac{1}{2\pi i} \iint_E q_1(P)\omega(P,Q) \wedge \overline{\partial}\chi(P) \tag{4.2}$$

represents a function that is holomorphic on $R \setminus E$.

From our choice of χ it follows that the function $f = \chi q_1 + g$ is holomorphic on U_2 and meromorphic on U_1 with the same poles as q. From the Pompeiu Formula (4.1.1), we have

$$f(Q) = \frac{1}{2\pi i} \iint_E (q_1(Q) - q_1(P))\omega(P,Q) \wedge \overline{\partial}\chi(P) \tag{4.3}$$

for all $Q \in R$ with $q_1 \neq \infty$.

From the properties of the Cauchy kernel (see [7]), it follows that f is holomorphic on U_3. Thus, f is meromorphic on $U_1 \cup U_2 \cup U_3$. Now set $X = K_1 \cup K_2 \cup K$ and let $h \in H(X)$ be bounded (h may be constant). From the hypothesis that $M(X) \subset \overline{M}(X)$, it follows that there exists a function $m_3 \in M(R)$ such that

$$|m_3 - f| \leq \varepsilon |h| \qquad on \quad X. \tag{4.4}$$

Since $|q_1| < \varepsilon$ on E , we have $|g(Q)| < \varepsilon.B(Q)$, for all $Q \in R$. With $m = m_2 + m_3$ we have the following estimates:

On K_1:

$$|m - m_1| \leq |f - (m_1 - m_2)| + |m_3 - f| = |f - q| + |m_2 - f| \leq \tag{4.5}$$

$$|\chi - 1||q| + |g| + |m_3 - f| < \varepsilon \cdot B + \varepsilon \cdot |h| = (B + |h|)\varepsilon.$$

On K_2:

$$|m - m_2| \leq |m_3 - f| + |f| \leq |m_3 - f| + |\chi||q| + |g| < \tag{4.6}$$

$$\varepsilon \cdot |h| + \varepsilon \cdot B = (B + |h|)\varepsilon.$$

Taking into considration that $0 \leq \chi \leq 1$, we have on K:

$$|m - m_j| \leq |q| + |g| + |m_3 - f| < \varepsilon + \varepsilon \cdot B + \varepsilon \cdot |h| \tag{4.7}$$

for j=1,2. Now let $C : R \to \mathbb{R}_+$ be any continuous function with

$$C(P) > B(P) + |h(P)|, \quad for \;\; P \in K_1 \cup K_2, \quad and \tag{4.8}$$

$$C(P) > B(P) + 1 + |h(P)|, \quad for \;\; P \in K \setminus (K_1 \cup K_2) \tag{4.9}$$

This function satisfies the conclusion of the Fusion Lemma. $\square$

Suppose $Y(P,Q)$ is a meromorphic function on $R \times R$ and $\Omega = Y(P,Q)dP$ represents a Cauchy-kernel except on a set of the form

$$P = \bigcup_{j=1}^{M} (P_j \times R) \cup \bigcup_{i=1}^{L} (R \times Q_i)$$

for finitly many points $P_1, \cdots, P_M, Q_1, \cdots, Q_L \in R$. Then Ω is called a **pseudo-Cauchy kernel**.

Remark 4.1.5. *If Ω is a Pseudo-Cauchy Kernel with exceptional set P as above where $P_1, \cdots, P_M, Q_1, \cdots, Q_L \notin K_1 \cup K_2 \cup K$, then the above proof goes through with ω replaced by Ω. The function f can then have additional poles. However, these lie outside of $K_1 \cup K_2 \cup K$. Since Ω has an influence in the function $C(P)$, such a choice could be advantageous.*

4.1.3 The Newtonian Potential of an Annulus

It will now be shown (following [41]) that the function $C(P)$, in the Fusion Lemma can be chosen near 1 on K_1 and K_2 and can be smaller in the special case where, $R = \mathbb{C}$ and $E = \mathbb{C} \setminus (U_1 \cup U_2)$ can be chosen to be on annulus.

The Newtonian (surface) potential of the disc $|\zeta| \leq \iota$ is the function

$$F_\iota(Z) = F(\iota, z) = \int\limits_{|\zeta| \leq \iota}\int \frac{1}{|\zeta - z|} \mathrm{d}\zeta \mathrm{d}\eta$$

where $\iota = \zeta + i\eta$. If we set, as usual,

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{-1/2} \mathrm{dt}$$

$$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{1/2} \mathrm{dt},$$

then, using ([21],233,5e), we obtain by an elementary calculation

$$F(\iota, z) = \begin{cases} 4\iota E(\frac{|z|}{z}) & \text{for } |z| \leq \iota \\ 4|z|(E(\frac{\iota}{|z|}) - (1 - \frac{z^2}{|z|^2})K(\frac{\iota}{|z|})) & \text{for } |z| > \iota. \end{cases} \tag{4.10}$$

Now let $U_1 = \{|\xi| < \iota\}$ and $U_2 = \{|\xi| > \iota + \Delta\iota\}$, so that $E = \{\iota \leq |\xi| \leq \iota + \Delta\iota\}$. Let ω be the usual Cauchy kernel.

For every $\alpha > 0$, there exists a C^1-function $\sigma : \mathbb{R}_{>0} \longrightarrow [0,1]$ with $\sigma|_{[0,1]} = 1$, $\sigma|_{[\iota+\Delta\iota,\infty)=0}$ and $|\sigma\prime| \leq \frac{\iota}{\Delta\iota} + \alpha$ on $\mathbb{R}_{>0}$. Then $\chi(\zeta) = \sigma(|\zeta|)$ has the properties in the Fusion Lemma, and for $B(z)$ (see (1)) we have:

$$B(z) \leq \frac{1}{2\pi}(\frac{1}{\Delta\iota} + \alpha)(F(\iota + \Delta\iota, z) - F(\iota, z)), z \in \mathbb{C}. \tag{4.11}$$

For $B(\iota, \Delta\iota, z) = (2\pi\Delta\iota)^{-1}(F(\iota+\Delta\iota, z) - F(\iota, z))$, we have the estimate $B(\iota, \Delta\iota, z) \leq \frac{\iota}{\Delta\iota} + 1$, where for $|z| \leq \iota + \Delta\iota$ follows from (4.10), and otherwise from elementary geometry after writing F in polar coordinates.

The number α was chosen arbitrarily. Hence from the Fusion Lemma together with Remark 4.1.4, we have the following:

Corollary 4.1.6. *If K_1 is a compact subset of the disc $|\zeta| < \iota + \Delta\iota$, K_2 closed subset of $|z| > \iota + \Delta\iota$ $(\iota, \Delta\iota > 0)$, and $\delta > 0$, and K a compact set, then we have:*

If $m_1, m_2 \in M(\mathbb{C})$ with $\|m_1 - m_2\|_K < \varepsilon$. Then there exists a function $m \in M(\mathbb{C})$ such that,

$$\|m - m_j\|_{K_j} \leq (\frac{\iota}{\Delta\iota} + 1 + \delta)\varepsilon, \tag{4.12}$$

$$\|m - m_j\|_K \leq (\frac{\iota}{\Delta\iota} + 2 + \delta)\varepsilon, \quad j = 1, 2. \tag{4.13}$$

4.2 The Localization Theorem

The localization theorem has been proved by A. Roth [36, Thm.1] for planar surfaces. It allows to give a positive answer to the question: Is it possible to approximate uniformly a function f on a closed set E by global meromorphic functions when approximable on each compact subset of E.

The generalization of the theorem of A. Roth to non-compact Riemann surface was given by P.M. Gauthier [18, Thm.1], for the case when E is contained in a neighborhood which consist of disjoint components, each of finite genus. As in [41], we will now show that this localization theorem is valid for a larger class of closed sets E. Since the proof requires the Extension Theorem, we will need to impose the extra condition introduced in Chapter 3. The definition of the larger class uses Cauchy kernels which are themselves determined by the analytic structure of the Riemann surface R; to determine if a given set E belongs to this class is not as easy as in the case of Gauthier, as it is the analytic structure that manifests itself deeply through the Cauchy kernels, and not any more simply some topological conditions. Furthermore, we must take into consideration the analytic structure if we want to improve on the result of Gauthier, as many examples show ([17], [39]).

4.2.1 Special Localization on the Surfaces $r_n \cap R$

Let $E \subset R$ be a closed set which satisfies the hypothesis of the Extension Theorem in Chapter 3. Let $D_0 \subset\subset D_1 \subset\subset \ldots$ be an exhaustion of R with $\overline{D}_n \subset r_n$, as in the Extension Theorem, and let us choose the Jordan domains W_n as before. For each $W_n \neq \emptyset$, we fix a domain V_n with $W_n \subset\subset V_n$, such that the $\overline{V}_n$'s are mutually disjoint and

1) $\overline{V}_n \subset D_n \backslash D_{n-1}$ from which it follows that

2) $\overline{V}_n \subset r_n$

As before, let $E_n := E \backslash \bigcup_{n<j} W_j$. We now prove the following:

Proposition 4.2.1. *(see [41]) For a function $h \in M(\overline{D}_{n-1} \cup E_n)$, the following are equivalent:*

(i) $h \in M_R(\overline{D}_{n-1} \cup E_n)$ *with respect to the surface* $r_n \cap R$.

(ii) $h|_{K \cap (\overline{D}_{n-1} \cup E_n)} \in M_R(K \cap (\overline{D}_{n-1} \cup E_n))$ *for each compact* $K \subset r_n \cap R$

Proof. It is trivial that $(i) \mapsto (ii)$.
To prove $(ii) \mapsto (i)$, let n be fixed, $h \in M(\overline{D}_{n-1} \cup E_n)$, assume that (ii) holds and let δ be a positive number.
The argument follows closely that of A. Roth ([36]). The goal is to approach h on $\overline{D}_{n-1} \cup E_n$. Let d_j be an exhaustion of $r_n \cap R$. For the three sets bounded on r_n

$$K_1^j = \overline{d_j}, \quad K_2^j = (\overline{D}_{n-1} \cup E_n) \backslash d_j, \quad K^j = (\overline{D}_{n-1} \cup E_n) \cap \overline{d}_{j+1}$$

we choose $a_j \in \mathbb{R}_+$ replacing $C(P)$ in the Fusion Lemma (that is $a_j \geq \|C\|_{K_1^j \cup K_2^j \cup K^j}$).

Let $\delta_1, \delta_2, ...$ a sequence of positive numbers such that $\sum_1^\infty \delta_j = \delta/2$. From the hypothesis on h, there exist functions $q_j \in M(r_n \cap R)$ such that $|q_j - h| < \frac{\delta_j}{2a_j}$ on K_j. From which it follows that

$$|q_{j+1} - q_j| < \frac{\delta_j}{a_j} \quad \text{on} \quad K^j \quad \text{for all} \quad j. \tag{4.14}$$

Using the Theorem of Behnke and Stein (see [7]) we can approximate q_j uniformly on K_j by functions in $M(r_n)$; in the case where q_j has poles on K^j, we first subtract a function in $M(r_n)$ which has the same poles on K^j and we add this function after having done the approximation. We can thus suppose in what follows that $q_j \in M(r_n)$.

By the Fusion Lemma (4.1.2), there exist functions $\varphi_j \in M(r_n)$ such that

$$|\varphi_j - q_j| < \delta_j \quad \text{on} \quad \overline{d_j} \cup K^j$$

and

$$|\varphi_j - q_{j+1}| < \delta_j \quad on \quad D_{n-1} \cup E_n = K_2^j \cup K^j.$$

This implies

$$\sum_{\nu=j}^{\infty} |\varphi_\nu - q_\nu| < \sum_{\nu=j}^{\infty} \delta_\nu < \delta/2 \quad on \quad \overline{d_j}$$

Consequently, the function

$$H := q_1 + \sum_{\nu=1}^{\infty} (\varphi_\nu - q_\nu)$$

is meromorphic on $\bigcup d_j = r_n \cap R$.

On K^1, we have

$$|H - h| \leq |q_1 - h| + \sum_{\nu=1}^{\infty} |\varphi_\nu - q_\nu| \leq \frac{\delta_1}{2a_1} + \sum_{\nu=1}^{\infty} \delta_\nu < \delta.$$

and on $K^j - K^{j-1}$, $(j > 1)$, we get

$$|H-h| = |q_1 + \sum_{\nu=1}^{j-1} (\varphi_\nu - q_\nu) + h + \sum_{\nu=j}^{\infty} (\varphi_\nu - q_\nu)| = |\sum_{\nu=1}^{j-1} (\varphi_\nu - q_{\nu+1} + q_j - h + \sum_{\nu=j}^{\infty} (\varphi_\nu - q_\nu)|$$

$$\leq \sum_{\nu=1}^{j-1} \delta_\nu + \frac{\delta_j}{2a_j} + \sum_{\nu=j}^{\infty} < \delta_j/2 + \sum_{\nu=1}^{\infty} \delta_\nu < \delta$$

We thus obtain $|H - h| < \delta$ on $\overline{D}_{n-1} \cup E_n$ which is what we needed to prove. $\square$

Proposition 4.2.2. *Each function g meromorphic in a neighborhood U of $\overline{D}_{n-1} \cup E_n$ with respect to $r_n \cap R$ can be uniformly approximated on $\overline{D}_{n-1} \cup E_n$ by functions in $M(r_n \cap R)$.*

This follows from Proposition 4.2.1 and the theorem of Behnke and Stein [7, Thm. 13]. The eventual poles of g in $\overline{D}_{n-1} \cup E_n$ are treated as in the previous proof.

4.2.2 Definition of the sets "weakly of infinite genus" (wig) (=sug in German)

Let $R, E, D_n, W_n, V_n, E_n, r_n$ be defined as before. For each of the Riemann surfaces $r_n \cap R$, let w_n be a (given) Cauchy kernel (or only a pseudo Cauchy kernel (see [41, §7.2] and Section 4.1.1). For each n, let

$$K_1 = \overline{W}_n \cap E \quad , \quad K_2 = \overline{D}_{n-1} \cup (E_n \backslash V_n) \quad , \quad K = (\overline{V}_n \backslash W_n) \cap E.$$

For these sets, we now choose open neighborhoods U_1 and U_2 of K_1 and K_2 respectively with $U_1 \cap U_2 = \emptyset$, and a C^1-function with compact support
$\chi_n : r_n \cap R \longrightarrow [0,1]$ with $\chi_n \mid_{U_1} = 1$, $\chi_n \mid_{U_2} = 0$. Set $\mathcal{D}_n = (r_n \cap R) \backslash (U_1 \cup U_2)$ and

$$B_n(Q) = \frac{1}{2\pi} \int\int_{\mathcal{D}_n} |w_n(P,Q) \wedge \overline{\partial} \chi_n(P)| \quad , \quad Q \in r_n \cap R$$

where $\overline{\partial}\chi_n$ denote the 1-form which in the local coordinate ζ is given by $\frac{\partial \chi(\zeta)}{\partial \bar{\zeta}}$.

We have that $K_1 \cap K_2 \cap K = \overline{D}_{n-1} \cup E_n$. From Proposition 4.2.2, we obtain

$$M(K_1 \cup K_2 \cup K) \subset M_R(K_1 \cup K_2 \cup K).$$

We can thus apply the Fusion Lemma with K_1, K_2 and K on the Riemann surface $r_n \cap R$ with a function $C_n(P)$ satisfying

$$C_n(P) \geq B_n(P) + |\delta(P)| \quad \text{for} \quad P \in K_1 \cup K_2$$

$$C_n(P) \geq B_n(P) + 1 + |\delta(P)| \quad \text{for} \quad P \in K \setminus (K_1 \cup K_2)$$

where δ is an arbitrary function in $H(K_1 \cup K_2 \cup K)$ (see [41]). Fix such a function C_n. Moreover, let h_0 be a non-vanishing holomorphic function bounded by $\lambda_1 (> 0)$

on a neighborhood (with respect to R) of the closed planar set E_0. Now set

$$\varepsilon_0(P) := |h_0(P)| \quad \text{and} \quad e_0 := \|\varepsilon_0\|_{(\overline{V_1}\backslash W_1)\cap E}\,.$$

We define inductively the sequence of functions ε_n and a sequence of numbers e_n as follows:

$$\varepsilon_n(P) = e_{n-1}(2C_n(P)+1) \quad \text{for} \quad P \in W_n \cap E \tag{4.15}$$

$$\varepsilon_n(P) = 2e_{n-1}C_n(P) + \varepsilon_{n-1}(P) \quad \text{for} \quad P \in E_{n-1} \tag{4.16}$$

$$e_n = \|\varepsilon_n\|_{(\overline{V}_{n+1}\backslash W_{n+1})\cap E}\,. \tag{4.17}$$

We note that each ε_n is defined on E_n respectively.

Definition 4.2.3. *(see [41])*

a) A closed subset E of a non-compact Riemann surface R is said to be weakly of infinite genus-1 (wig-1) if it is of finite genus or if it is possible to carry the above construction in such a way that the following inequalities are satisfied for all $n = 0, 1, 2, \cdots$

$$(i) \quad e_n \le \frac{\lambda_{n+2}}{2\|C_{n+1}\|_{E_n\cup D_n}} \tag{4.18}$$

$$(ii) \quad e_n \le \frac{\sum_{j=1}^{n+2}\lambda_j}{\|2C_{n+1}+1\|_{W_{n+1}\cap E}} \tag{4.19}$$

where λ_n are arbitrary positive numbers satisfying $\sum \lambda_n < \infty$.

b) A closed subset $E \subset R$ is said to be weakly of infinite genus-2 (wig-2) if there exists a decomposition of E by mutually disjoint closed sets A_j such that each compact subset of R intersect with only a finite number of A_j and each set $\cup_{j=1}^{n}A_j$ is a wig-1 set $n \in \mathbb{N}$.

c) $E \subset \mathbb{R}$ is said to be weakly of infinite genus (wig) if E is a wig-1 set or a wig-2 set.

Remark 4.2.4. *Note that, this definition was given by G. Schmieder. It depends on R, E, D_n, W_n, V_n, E_n, and r_n as obtained in the Extension Theorem. The statement and the proof of this theorem needed to be fixed. This was done in Chapter 3. By doing so, a requirement was added to the set E and additional conditions were imposed on the D_n's. This in turn, has changed the class of wig sets.*

Remark 4.2.5. *The original definition of G. Schmieder for wig-1 sets, did not need a separate clause for sets of finite genus, because (based on his faulty Extension Theorem) it already contained that class. Close sets of finite genus are known to be sets of approximation (see [18]) and thus should be included in the class of wig-sets without additional hypotheses. But if we take for example $R = \mathbb{C}$ and let E be a complement of a compact set, then E is of finite genus (in fact, of genus zero) and is a set of (meromorphic) approximation. Notice that there exists no open neighborhood G for E such that $R \setminus G$ contains an unbounded and connected Jordan arc. So a definition solely based on an Extension Theorem would miss this set.*

In the case when there exist a covering of E by mutually disjoint open sets, each of finite genus, then following Gauthier, E is said to be of "essentially of finite genus", which we denote by the abbreviation efg. If E can be covered by one open set of finite genus, E will be called a set of finite genus, or in short, a fg set.

Remark 4.2.6. *Each efg set is a wig-2 set.*

Proof. If E is an efg-set, then E can be decomposed in a sequence A_j of mutually disjoint fg sets, such that each compact intersect with only a finite number of sets A_j as it follows from the definition of efg sets. (see [41, §3.2]) □

Proposition 4.2.7. *Let $E \subset R$ be a wig-1 set and $A \subset R$ be compact. Then $E \cup A$ is a wig-1 set. (see [41, §3.2])*

Proof. Let D_n, W_n, V_n, E_n, and r_n be as in Chapter 3, and assume $A \subset D_n$ for $n \geq n_0$.

For $E' = E \cup A$, choose W'_n, D'_n with the properties given in Chapter 3. This can and will be done in such a way that there is an $n_1 \in \mathbb{N}$ such that $D'_{n_1+k} = D_{n_0+k}$, $W'_{n_1+k} = W_{n_0+k}$ holds for all $k \in \mathbb{N} \cup \{0\}$. Moreover for $n \geq n_1$, we also assume that all other quantities determining e_n as well as λ_n remain the same up to indexing. Then all necessary inequalities hold for the "new" numbers e'_n, at least if $e'_0, \cdots, e'_{n-1}$ are sufficiently small. But this can be achieved by choosing suitable start function h'_0. $\square$

We will give a short heuristic interpretation of wig-sets. The reader is referred to [41, §7] for more precise statements.

If one thinks of the Riemann surface as the result of gluing handles (caps with handles) onto planar domains (suitably punctured), then we can think of $\mathcal{D}_j$ as the gluing areas (common to the domain and the lid).

Let E be a closed subset of the surface, and assume for simplicity that all $\mathcal{D}_j$ are contained in E° (in any case, the $\mathcal{D}_j$'s contained in the complement of E are irrelevant for the following discussion).

If ω_j is the usual Cauchy kernel (see [41, §7.2]), then B_j denotes the electric potential obtained by assigning the charge density $|\frac{\partial \chi_j}{\partial \bar{\zeta}}|$ to $\mathcal{D}_j$ (Newton potential); the charge density depends only on $\mathcal{D}_j$ (χ) and so C_j is an upper bound for the potential; ε_n majorize the potential of a weighted superposition of simultaneous charge on the first n sets $\mathcal{D}_j$, so ε_n becomes arbitrary small if one gets "far away" from those already charged $\mathcal{D}_j$ (i.e. when one approach the ideal boundary).

So the inequalities (i) and (ii) in the Definition 4.2.3 of wig-sets can be interpreted as follows: The numbers λ_j are given; for (4.18) to be satisfied, the larger C_{n+1} is on $E_n \cup D_n$ the smaller ε_n must be in a neighborhood of (n+1)th handle. The (n+1)th handle must therefore be at a sufficiently large distance of its predecessors for E to be a wig1-set.

4.2.3 Statement and proof of the Localization Theorem

Using the Localization Theorem, Schmieder [41] has obtained nice approximation theorems for wig-sets. Recalling that there was problem with his definition of wig-sets, we now state and prove these Theorems for the new class of wig-sets introduced in Section 4.2 the in the proofs follow the proofs found in [41].

Theorem 4.2.8. *(***Localization Theorem***) Let E be wig-set on a given Riemann surface. Given a function $f : E \to \overline{\mathbb{C}}$, the following are equivalent*

(i) $f \in M_R(E)$

(ii) $f|_{K\cap E} \in M_R(K \cap E)$ *for all compact sets* $K \subset R$.

Proof. (see [41, §3.3]) From (i) to (ii) is trivial. To prove (ii) to (i), assume first that E is a wig1-set of infinite genus (see [18] for the proof when E is of finite genus). Let $D_n, W_n, V_n, r_n, e_n, C_n$ and ϵ_n be as defined in Chapter 3. The proof will be established by constructing a sequence of functions g_n with the following properties:

1) $g_n \in M(r_n \cap R)$

2) $\|g_n - g_{n-1}\| \leq \lambda_{n+1}$

3) $|g_n - F| \leq \varepsilon_n$ pointwise on E_n, where $F = \frac{f}{\varepsilon}\sum_{j=1}^{n+1}\lambda_j$ for an arbitrary but fixed ε and λ_n are the positive numbers whose sum converges given in the definition of wig1-sets. We thus also have,

4) $\varepsilon_n \leq \sum_{j=1}^{n+1}\lambda_j$ on E_n.

As given in the definition of a wig-set, let h_0 be holomorphic on a neighborhood of E_0, without zeroes and such that $\|h_0\| < \lambda_1$. According to Proposition 4.2.2 (with $D_{-1} = \emptyset$), there exists $\tilde{g_0} \in M(r_0 \cap R)$ with the property:

$$\|\tilde{g_0} - \frac{F}{h_0}\|_{E_0} < 1. \tag{4.20}$$

Also according to Proposition 4.2.2, the function h_0 can be approximated uniformly on E_0 by functions from $M(r_0 \cap R)$. For the function $g_0 := h_o \tilde{g_0} \in M(r_0 \cap R)$, we then have,

$$|g_0 - F| < \|h_0\| = \varepsilon_0 \quad \text{pointwise on} \quad E_0. \tag{4.21}$$

Assuming now that we have already constructed $g_0, ..., g_{n-1}$, g_n can be constructed as follows:

Because of our assumption, there exists a meromorphic function h on R satisfying

$$\|h - F\|_{V_n \cap E} < e_{n-1}. \tag{4.22}$$

Because h can be approximated uniformly on $V_n \cap E$ by functions in $M(r_n \cap R)$, by the Theorem of Behnke and Stein, we can again assume that $h \in M(r_n \cap R)$. We now get:

$$\|h - g_{n-1}\|_{(\bar{V_n} \backslash W_n) \cap E} \leq \|h - F\| + \|g_{n-1} - F\| \tag{4.23}$$

$$\leq e_{n-1} + \|\varepsilon_{n-1}\|_{(\bar{V_n} \backslash W_n) \cap E} = 2e_{n-1}.$$

We now apply the Fusion Lemma with sets K_1, K_2 and K as given in the previous section; this yields a function $g_n \in M(r_n \cap R)$ with the properties,

$$|g_n - g_{n-1}| < 2\epsilon_{n-1} C_n \quad \text{pointwise on} \quad D_{n-1} \subset K_2 \tag{4.24}$$

and hence because of the assumption on the e_n, we have

$$\|g_n - g_{n-1}\| < \lambda_{n+1}. \tag{4.25}$$

On $E_{n-1} \subset K_2 \cap K$, we also have

$$|g_n - F| \leq |g_n - g_{n-1}| + |g_{n-1} - F| \leq 2e_{n-1}C_n + \epsilon_{n-1} = \epsilon_n \tag{4.26}$$

$$\leq \lambda_{n+1} + \sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{n+1} \lambda_j.$$

On $\overline{W_n} \cap E = K_1$, we have:

$$|g_n - F| \leq |g_n - h| + |h - F| \leq 2e_{n-1}C_n + e_{n-1} = e_{n-1}(2C_n + 1) = \epsilon_n \tag{4.27}$$

$$\leq \sum_{j=1}^{n+1} \lambda_j.$$

Therefore we have shown that $|g_n - F| \leq \varepsilon_n \leq \sum_{j=1}^{n+1} \lambda_j$ on all of E_n, and the induction is complete.

Because (4.25) holds, the sequence $\{g_n\}$ is uniformly convergent to a function g which is meromorphic (at least) on $\cup D_n = R$. From (3) and (4), we get $\|g - F\|_E \leq \sum_{j=1}^{\infty} \lambda_j$. Moreover for $h := \varepsilon(\sum_1^{\infty} \lambda_j)^{-1} g \in M(R)$, because $F = \frac{f}{\varepsilon} \sum_1^{\infty} \lambda_j$, we have $\|h - F\|_E < \varepsilon_0$.

Since ε was an arbitrary positive number, it follows that $f \in M_R(E)$. This proves the Localization Theorem for wig1-sets.

Now let E be a wig2-set, and $E = A_1 \cup A_2 \cup ...$ a decomposition of E as in the definition of a wig2-set. Let $G_1 \subset G_2 \subset ... \subset R$ be an exhaustion of R by relatively compact sets G_j. Because of the local finiteness of the A_j's, we may assume that G_j meets the sets $A_1, ..., A_j$ and has empty intersections with the remaining A_k's. Now let f be a function on E satisfying (ii) and let δ be a positive number.

A_1 is a wig1-set; let $f_1 \in M(R)$ such that, $\|f - f_1\|_{A_1} < \delta/2$. Assume that $f_1, ...f_{n-1} \in M(R)$ have already been constructed with the properties

$$\|f_j - f_{j-1}\|_{G_{j-1}} \leq \delta 2^{-j} \tag{4.28}$$

and

$$\|f_j - f\|_{A_1 \cup \ldots \cup A_j} \leq \delta(2^{-1} + \ldots + 2^{-j}) \quad \text{for} \quad j = 2, \ldots, n-1. \tag{4.29}$$

Thus $g_n|_{\overline{G}_{n-1} \cup A_1 \cup \ldots \cup A_n} = f_{n-1}$ and $g_n|_{A_n} = f$ defines a function g_n on the wig1-set. $X_n = G_{n-1} \cup A_1 \cup \ldots \cup A_n$ and this function satisfies (ii) with E replaced by X_n.

Moreover, there exists $f_n \in M(R)$ satisfying $\|f_n - g_n\|_{X_n} \leq \delta 2^{-n}$. This implies

$$\|f_n - f_{n-1}\|_{G_{n-1}} \leq \delta 2^{-n} \tag{4.30}$$

$$\|f_n - f\|_{A_1 \cup \ldots \cup A_{n-1}} \leq \delta(2^{-1} + \ldots + 2^{-n}) \tag{4.31}$$

and

$$\|f_n - f\|_{A_n} \leq \delta 2^{-n} \leq \delta(2^{-1} + \ldots + 2^{-n}). \tag{4.32}$$

Because (4.28) holds, the sequence of functions f_n constructed in this manner converges to a function $F \in M(R)$ that satisfies $\|F - f\|_E < \delta$, in light of (4.29). This completes the proof of the theorem for wig-sets.

□

4.3 Approximation by meromorphic function

Using the Localization Theorem, G. Schmieder [41] has obtained nice approximation results for wig-sets. Recalling that there was a problem with his definition of wig-set, we now state and prove these theorems for the class of wig-sets as defined in Section 4.2. The proofs follow verbatim the proofs found in [41].

Theorem 4.3.1. *(see [41, §4]) If E is a wig subset of a non-compact Riemann surface R, then $M(E) \subset M_R(E)$.*

Proof. This follows from the Localization Theorem in an analogous way that Proposition 4.2.2 follows from Proposition 4.2.1. □

Corollary 4.3.2. *(see [41, §4]) If $E \subset R$ is a wig-set, $f \in M(E)$ and $g \in M(R)$, then there exist a function $F \in M(R)$ such that*

$$|f(p) - F(p)| \leq |g(p)| \quad \textit{for} \quad p \in E.$$

Proof. It is suffices to approximate uniformly on E the function f/g. □

Corollary 4.3.3. *See [41, §4] Let $E \subset R$ be a wig set, $f \in M(E)$, $g \in M(E)$ non constant and bounded on E and $Z = \{p \in E | g(p) = 0\}$. Then for each $\delta > 0$, there exist a function $F \in M(R)$ such that*

(i) $\|f - F\|_E < \delta$.

(ii) $f(p) = F(p) \quad \textit{for} \quad p \in Z$.

Proof. It suffices to approximate the function $\frac{f}{g}\|g\|_E$ uniformly on E within the constant δ. □

4.4 Approximation by holomorphic functions

The first result about holomorphic approximation on unbounded subset of non-compact Riemann surface was proved by Tietz (see [45]) for disjoint union of bounded sets in $\mathbb{C}$. For closed subsets E of the complex plane, we have the well-known theorem of Arakelyan (see [13, p.129]): $A(E) \subset H_R(E)$ is equivalent to $\mathbb{C}^* \setminus E$ being connected and locally connected at the point $*$, where $\mathbb{C}^* = \mathbb{C} \cup \{*\}$ is the one-point compactification of $\mathbb{C}$. This topological condition is still necessary for $A(E) \subset H_R(E)$ if E is a closed subset of a non-compact Riemann surface R, as it was observed by Gauthier and Hengartner [19] (with the one-point compactification $R \cup \{*\}$ instead of $\mathbb{C}^*$). Following [41], we will show that for a wig-set the topological condition is also sufficient to have $A(E) \subset H_R(E)$.

4.4.1 Special approximation with fixed poles

We first observe the following.

Proposition 4.4.1. *(see [41, §5.1]) Let E be a closed subset of a non-compact Riemann surface R. The following statements are equivalent:*

1) *$(R \cup \{*\}) \setminus E$ is connected and locally connected at ∞.*

2) *There exists an exhaustion of R by Jordan domains D_n (n=-1, 0, 1, $\cdots$;$D_{-1} = \emptyset$) such that for all n=0, 1, $\cdots$ we have that $R \setminus (\overline{D}_{n-1} \cup E)$ has no bounded components.*

Proposition 4.4.2. *Let E be a closed subset with the property that $(R \cup \{*\}) \setminus E$ is connected and locally connected at point $*$. Assume morover that we are given finitely many points $P_k \in D_n \setminus (\overline{D}_{n-1} \cup E_n)$ and for each point P_k, a principal part $\mathcal{H}_k$ consisting of finitly many terms. Then there exists for each $\delta > 0$, a function $\varphi \in M(r_n)$ with the properties:*

(i) $\|\varphi\|_{\overline{D}_{n-1} \cup E_n} < \delta$.

(ii) *φ has poles exactly at the points P_k and the principal part of φ at P_k is $\mathcal{H}_k$.*

The principal parts are to be taken with respect to a fixed local parameter around the points P_k.

Proof. According to Corollary 4.1.2 there is a set $\Delta \subset r_n$ such that $r_n \setminus (\overline{D}_{n-1} \cup E_n \cup \Delta)$ has no bounded components with respect to r_n, and $\Delta \subset r_n$ contains none of the points P_k. Since $D_n \subset r_n$, the points P_k lie on r_n. Now there exists a function $\tilde{\varphi} \in M(r_n)$, whose poles are exactly at P_k with principal parts $\mathcal{H}_k$ (see [6, p.591]). According to the same Corollary, each compact set $\mathcal{K} = \overline{D_{n-1} \cup E_n \cup \Delta}$ (closure in r_n) has the property that its complement on r_n has no bounded components. Since $\tilde{\varphi} \in H(\mathcal{K})$, the theorem of Bishop (see [8] for example) implies the existence of a function $\sigma \in H(r_n)$ such that $\|\tilde{\varphi} - \sigma\|_{\mathcal{K}} < \delta$. The function $\varphi = \tilde{\varphi} - \sigma$ satisfies the conclusion of Proposition 4.4.2.

$\square$

4.4.2 Holomorphic approximation

Theorem 4.4.3. *(see [41, §5.2]) Let E be a wig-set on a non-compact Riemann surface R. The following statements are equivalent:*

(i) $(R \cup \{*\}) \setminus E$ *is connected and locally connected at* $\{*\}$.

(ii) $A(E) \subset H_R(E)$.

For the direction $(ii) \mapsto (i)$, we refer to Gauthier and Hengartner ([19]). So we only have to show the other direction.

Proof. Let $K \subset R$ be compact and $f \in A(E)$. If the complement of the set $K \cap E$ has a bounded component, then this component must be contained in E, since $R \setminus E$ has no bounded components. Extend K to the set $\hat{K}$ by taking the union of K with such components. Since $f \in A(\hat{K} \cap E)$, the approximation theorem of Bishop gives $f \in H_R(\hat{K} \cap E)$ and hence, at fortiori, $f \in M_R(K \cap E)$. Therefore f satisfies condition (ii) of the Localization Theorem. Using Proposition 4.4.2, we can choose the function g_n in the proof of the Localization Theorem to be holomorphic on $\overline{D}_{n-1} \cup E_n$. This implies that the limit g of the sequence $\{g_n\}$ is holomorphic on $\cup D_{n-1} = R$. Because the other properties of the functions g_n still holds, the claim $f \in H_R(E)$ follows.

$\square$

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