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D-dimensional Bose gases and the Lambert W function

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The applications of the Lambert W function (also known as the W function) to D-dimensional Bose gases are presented. We introduce two sets of families of logarithmic transcendental equations that occur frequently in thermodynamics and statistical mechanics and present their solution in terms of the W function. The low temperature T behavior of free ideal Bose gases is considered in three and four dimensions. It is shown that near condensation in four dimensions, the chemical potential μ and pressure P can be expressed in terms of T through the W function. The low T behavior of one- and two-dimensional ideal Bose gases in a harmonic trap is studied. In 1D, the W function is used to express the condensate temperature, T_C , in terms of the number of particles N ; in 2D, it is used to express μ in terms of T . In the low T limit of the 1D hard-core and the 3D Bose gas, T can be expressed in terms of P and μ through the W function. Our analysis allows for the possibility to consider μ , T , and P as complex variables. The importance of the underlying logarithmic structure in ideal quantum gases is seen in the polylogarithmic and W function expressions relating thermodynamic variables such as μ , T , and P . © 2010 American Institute of Physics. [doi:10.1063/1.3496906]

I. INTRODUCTION

Mathematical functions play an important role in understanding physical phenomena. Nearly all nontrivial physics problems require mathematical functions to provide numeric and/or algebraic computations. The Lambert W function is a recently defined mathematical function that is a useful tool for the physicist.

(a) Defined in computer science for the purpose of enumerating search trees,⁶ the Lambert W function has found many applications in science and engineering, usually providing solutions to transcendental equations, specifically equations involving exponentials or logarithms. Some recent applications of the Lambert W function include analytic solutions to solar cell parameters^{17,18} and the analysis of time-delay systems.^{16,43} Valluri *et al.*³⁹ show that solutions to two standard physics problems, Wien's displacement law and the fringing fields of a parallel plate capacitor, can be solved analytically using the Lambert W function. These are interesting results as they provide analytic solutions to problems previously requiring numerical solutions, thus obtaining a new elegance and convenience. Therefore, the Lambert W function has the potential to provide solutions to problems that have previously not been solved analytically, as well uncovering novel and interesting properties of previously solved problems.

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(b) In the field of statistical and plasma physics exponential and logarithmic functions are often encountered¹⁴ and the W function has already proven to be a useful tool. In this work, we will demonstrate some interesting applications of the Lambert W function to D-dimensional Bose gases. It has been the trend to compartmentalize the study of the statistical mechanics of ideal Fermi^{9,12} and Bose^{2,11,40} gases separately. This seems natural as quantum particles with integral or half integral spin obey different statistics which manifest themselves in different thermodynamic behavior. The most notable characteristics for these gases in 3D are the existence of the Fermi energy for a Fermi gas and consequently, the zero point pressure and for a Bose gas the occurrence of Bose–Einstein condensation (BEC)(Ref. 23). The basic mathematical structure underlying quantum statistics is logarithmic. When logarithms are encountered, a common trend is to “delog” by finding a way to replace it with a derivative for example. Such an elementary step will often enable one to solve a log integral but at the price of overlooking the logarithmic structure, which can be very informative. The remarkable work of the great mathematical physicist Sommerfeld on the low temperature theory of an ideal Fermi gas overlooked the rich latent structure in ideal quantum gases and cannot be easily extended nor generalized.³² Recognizing an underlying logarithmic structure in ideal quantum gases and developing a unified formulation of the grand partition function provides several special solutions such as thermodynamic equivalence in two dimensions and the μ singularity in null dimension.

(c) Mathematical functions not only assist the physicist in the analysis of physical phenomena but may also be the unifying thread which stimulates the discovery of new physics. The Riemann zeta function is an excellent example that made its advent in quantum physics in the Planck black body radiation formula and has been of fundamental importance in quantum field theory. The Riemann zeta function is also very useful in the description of BEC.²³ The renaissance of the W function and its applicability to the formalism of quantum statistics is likely to be another example of such phenomena. Moreover, the W function has an indirect connection to the polylogarithm function, which has shown the underlying logarithmic structure in the quantum statistics of ideal gases thereby providing a unification of classical and quantum statistics.^{21–23}

(d) In this work, the applications of the W function to D-dimensional Bose gases in various physical contexts will be demonstrated. In Sec. II, we provide a brief introduction to the W function and provide general solutions to two families of transcendental equations commonly occurring in thermodynamics and statistical physics. In Sec. III, a brief mention of the thermodynamic functions that occur in the analysis of our paper is given. In Sec. IV, the chemical potential and pressure of an ideal Bose gas in three and four dimensions are related to the temperature through the W function and provide new physical insight into the phenomenon of BEC. In Sec. VI, the condensate temperature of a one-dimensional ideal Bose gas in a harmonic trap and the chemical potential of the two-dimensional ideal Bose gas in a harmonic trap are derived using the W function. In Sec. V, high temperature expansions of the pressure and particle density in a one-dimensional hard-core Bose gas will be obtained by use of the W function. Also, a new relationship between the temperature, pressure, and chemical potential in terms of the W function is obtained for this system. In Sec. VII we present additional thermodynamic applications of the families of equations introduced in Sec. II. Section VIII summarizes the conclusions.

II. THE LAMBERT W FUNCTION

(e) The W function is defined as the multivalued function which solves the following equation:⁶

$$W(z) \exp(W(z)) = z \quad z \in \mathbb{C}, \quad (1)$$

or, equivalently, as the multivalued inverse of the function $f : z \rightarrow ze^z$. Equation (1) always has an infinite number of solutions, hence the multivaluedness of the W function. These solutions are indexed by the integer variable j . Thus we say that (1) is solved by the branches of the W function, W_j , for $j \in \mathbb{Z}$. Of special relevance to physics and engineering applications are the solutions of (1) when the argument is purely real. In this case there can be at most two real solutions, corresponding to the branches W_0 and W_{-1} , where W_0 is the principal branch of the W function. For real solutions to exist, we must require that $z \in (-1/e, \infty)$, in which case $W_0(z) \in [-1, \infty)$ and $W_{-1}(z) \in (-\infty, -1]$. Moreover, $W_0(z) < 0$ if $z \in (-1/e, 0)$ and $W_0(z) \geq 0$ for $z \in [0, \infty)$. The branches W_0 and W_{-1} are

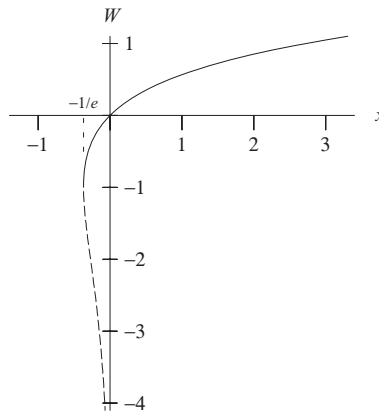


FIG. 1. The two real branches of the W function. The solid line represents W_0 and the dashed line represents W_{-1} .

monotonically increasing and monotonically decreasing, respectively. All other branches of W are always complex. The two real branches of the W function are plotted in Fig. 1.

(f) One property of the W function relevant to this work (and physics in general) is the series expansion for the principal branch about $z = 0$, given by

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n = z - z^2 + \frac{3}{2}z^3 - \dots, \quad (2)$$

where radius of convergence of the series in Eq. (2) is $1/e$. For a very good discussion on the history, properties, and applications of the W function see the work of Corless *et al.*⁶

(g) Besides the theoretical advantages of providing an adequate analytical formalism for a given problem, another advantage of solving problems in terms of the W function is the availability of libraries in computer algebra systems, which allows for a convenient way to obtain values, expansions, plots, etc. of the quantity being solved for.

(h) We can use the W function to obtain general solutions to two families of transcendental equations which appear frequently in statistical mechanics as well as applied mathematical and physics problems.^{5,13,19,21,38}

(i) Type 1:

$$x^n + B \ln x + C = 0, \quad (3)$$

where $n, B, C \in \mathbb{C}$ not depending on x and $n, B \neq 0$. Then

$$\frac{n}{B}x^n + n \ln x = -\frac{nC}{B} \Rightarrow \frac{n}{B}x^n e^{\frac{n}{B}x^n} = \frac{n}{B}e^{-\frac{nC}{B}}.$$

Applying the W function to both sides and solving for x ,

$$x = \left[\frac{B}{n} W_j \left(\frac{n}{B} \exp \left(-\frac{nC}{B} \right) \right) \right]^{1/n}. \quad (4)$$

(j) Type 2:

$$x^n \ln x + Bx^n + C = 0, \quad (5)$$

where $n, B, C \in \mathbb{C}$, not depending on x and $n, C \neq 0$. Then

$$n \ln x + nB = -Cnx^{-n} \Rightarrow -Cne^{nB} = -Cnx^{-n} e^{-Cnx^{-n}}.$$

Applying W to both sides and solving for x ,

$$x = \left[-\frac{1}{Cn} W_j(-Cne^{nB}) \right]^{-1/n}. \quad (6)$$

In both the equations W_j is a specific branch of the Lambert W function determined from the specification of the parameters in the argument.

III. THE THERMODYNAMIC FUNCTIONS

In the study of classical statistical mechanics, a system's statistical information is given by the partition function Z . If there is a high probability a system's comprising particles occupy the same single particle state, the theory is modified by replacing the partition function with the grand partition function \mathcal{Z} . From the grand partition function two important distributions in quantum statistics are derived: the distribution for number of particles over different quantum states with energy ϵ at temperature T for systems of bosons \bar{n}_{BE} and fermions \bar{n}_{FD} ,²⁹

$$\bar{n}_{BE} = -\frac{1}{\mathcal{Z}_{BE}} \frac{\partial \mathcal{Z}_{BE}}{\partial x} = \frac{1}{e^{\beta(\epsilon-\mu)} - 1}, \quad (7)$$

$$\bar{n}_{FD} = -\frac{1}{\mathcal{Z}_{FD}} \frac{\partial \mathcal{Z}_{FD}}{\partial x} = \frac{1}{e^{\beta(\epsilon-\mu)} + 1}, \quad (8)$$

where $x = \beta(\epsilon - \mu)$, ϵ is the energy, μ is the chemical potential, $\beta = 1/kT$, T is the temperature, and k is Boltzmann constant. As mentioned earlier, the study of quantum statistics is compartmentalized into the study of systems obeying one or the other of these two distribution functions. Fermi–Dirac statistics governs systems of atoms, nuclei with half-integer spins, electrons in metals, neutron stars, and white dwarfs.^{13,29} Bose–Einstein statistics deals with atoms and nuclei with integer spins and helps to understand the formation of a BEC. In this work the systems discussed will be Bose gases. We will present applications of the W function to free ideal Bose gases in two, three, and four dimensions, trapped ideal Bose gases in one and two dimensions, and to a nonideal Bose gas in one dimension. Before we discuss the ideal Bose gases in D dimensions, we summarize a few relevant formulae for thermodynamic functions such as the number of particles N , entropy S , pressure P , chemical potential μ , and energy U . They are determined from the grand partition function \mathcal{Z} ,²⁹

$$N = kT \frac{\partial(\ln \mathcal{Z})}{\partial \mu}, \quad S = -k \frac{\partial(T \ln \mathcal{Z})}{\partial T}, \quad P = kT \frac{\partial(\ln \mathcal{Z})}{\partial V}$$

$$\mu = -kT \frac{\partial(\ln \mathcal{Z})}{\partial N}, \quad U = kT^2 \frac{\partial(\ln \mathcal{Z})}{\partial T},$$

where V is the volume. The thermodynamic quantity that gives most insight into the nature of BEC is the fugacity $z = e^{\beta\mu}$. In the limit where all bosons are in the ground state, the fugacity approaches unity.^{29,30}

IV. FOUR-DIMENSIONAL IDEAL BOSE GASES

The reduced particle density for a Bose gas in D dimensions is given in terms of a polylogarithm of order related to the dimension.^{10,22,27}

$$\rho \lambda^D = Li_{m+1}(z), \quad (9)$$

where $Li_{m+1}(z)$ is the polylogarithm of order $m+1$,²⁴ $m = D/2 - 1$ and D is the dimension, $z = e^{\beta\mu}$ is the fugacity of the system, $\lambda = \sqrt{2\pi\hbar^2/kTM}$ is the de Broglie wavelength, M is the mass of the constituent particles, and the quantity $\rho \lambda^D$ is referred to as the reduced particle density. The analysis presented here relies on an expansion of the polylogarithms about $z = 1$. If $m = 0, 1, 2, (D = 2, 4, 6, \dots)$, as $z \rightarrow 1$,^{22,24}

$$Li_{m+1}(z) = \frac{(-1)^{m+1}}{\Gamma(m+1)} \left(\ln \frac{1}{z} \right)^m \left(\ln \ln \left(\frac{1}{z} \right) - \psi(m+1) + \psi(1) \right) + \sum_{r=0(r \neq m)}^{\infty} \frac{\zeta(m+1-r)(\ln z)^r}{r!}, \quad (10)$$

where $\zeta(s)$ is the Riemann zeta function, $\Gamma(s)$ is the gamma function, and $\psi(s)$ is the logarithmic derivative of the gamma function.

A. Chemical potential

In $D = 4$ dimensions, from Eq. (9), the reduced particle density is

$$\rho\lambda^4 = Li_2(z). \quad (11)$$

Equation (11) expresses the exact analytical relationship between z and the other thermodynamic quantities. To the knowledge of the authors this relationship cannot be solved exactly for z . However, if we consider the region $z \approx 1$ we can obtain an approximate solution using the W function. Using the sum from Eq. (10) and neglecting nonlinear terms, we can rewrite Eq. (11) as

$$\rho\lambda^4 = \ln\left(\frac{1}{z}\right) \left(\ln \ln\left(\frac{1}{z}\right) - \psi(2) + \psi(1) \right) + \zeta(2), \quad (12)$$

where $\psi(2) = \psi(1) + 1$ and $\psi(1) = -\gamma$, where γ is the Euler–Mascheroni constant.^{22,24} Equivalently,

$$x \ln(x) - x + (\zeta(2) - \rho\lambda^4) = 0 \quad x = -\ln z, \quad (13)$$

which we recognize as being of the form (5), with $n = 1$, $B = -1$, and $C = (\zeta(2) - \rho\lambda^4)$. Hence, the solution to (12) is

$$\begin{aligned} x = -\ln z &= \left[-\frac{1}{(1)(\zeta(2) - \rho\lambda^4)} W_j(-(\zeta(2) - \rho\lambda^4)(1) \exp((1)(-1))) \right]^{-1/1} \\ &= \left(\rho\lambda^4 - \frac{\pi^2}{6} \right) \left[W_j \left(\left[\rho\lambda^4 - \frac{\pi^2}{6} \right] e^{-1} \right) \right]^{-1}, \end{aligned}$$

where we have used $\zeta(2) = \pi^2/6$. By the definition of $W(z)$, we have $W(z)^{-1} = \exp(W(z))/z$, so we can also write the solution as

$$\mu = -kTx = -kT \exp \left(W_j \left(\frac{1}{e} \left[\rho\lambda^4 - \frac{\pi^2}{6} \right] \right) + 1 \right). \quad (14)$$

In general, (14) allows for the possibility of a complex chemical potential. By choosing $j = 0$ or $j = -1$, the chemical potential is real. To determine which of these branches to choose we consider the critical temperature, T_c , such that $\mu = 0$. In this case, the fugacity is identically unity and of BEC forms. From (14), for real branches of W , a solution exists iff

$W_j[e^{-1}(\rho\lambda^4 - (\pi^2/6))] \rightarrow -\infty$, which is only possible if $j = -1$ and $e^{-1}(\rho\lambda^4 - \frac{\pi^2}{6}) \rightarrow 0$. Recalling that $\lambda = \sqrt{2\pi\hbar^2/(MkT)}$, we have,

$$T_c = \sqrt{\frac{24\hbar^4\rho}{k^2M^2}} = \frac{2\hbar^2\sqrt{6\rho}}{kM}. \quad (15)$$

By definition, $W_{-1}(z) \in \mathbb{R} \iff z \in [-1/e, 0)$. This constraint can be expressed in terms of T_c by noting that in our case it is equivalent to $T \in [(\pi/\sqrt{6} + \pi^2)T_c, T_c) \approx [0.789T_c, T_c)$. Thus, for real values to exist in this case, T must be within about 80% and 100% the critical temperature. The analysis for temperatures to the right of this domain is not as simple. If (14) holds in this region, one of two things may occur. First, if the system is still described by the W_{-1} branch of the Lambert W function the chemical potential becomes complex. Or, if we choose to demand the chemical potential remain real, we can use the principal branch, and the chemical potential becomes discontinuous at the condensation temperature. The behavior of the chemical potential in this domain needs further study.

B. Equation of state near $z = 1$

The pressure P is a function of the chemical potential. When the volume and temperature are held fixed, according to grand ensemble theory the equation of state is given by^{29,30}

$$\frac{\rho}{kT} = \frac{\partial P}{\partial \ln z},$$

where ρ is the number density, z is the fugacity, and the pressure P is a polylogarithm whose order depends on the dimension,^{22,27,29}

$$\frac{P}{kT} = \frac{2}{\lambda^3} Li_3(z), \quad (16)$$

where V is the volume and N is the number of particles. Proceeding as in Sec. IV A, we can see from Eqs. (10) and (16) that the pressure in four dimensions can be related to the fugacity by the following expression:

$$\frac{P}{kT} = \frac{2}{\lambda^3} \left[-\frac{1}{\Gamma(3)} \left(\ln \left(\frac{1}{z} \right) \right)^2 \left(\ln \ln \left(\frac{1}{z} \right) - \psi(3) + \psi(1) \right) + \zeta(3) + \zeta(2) \ln z \right].$$

Neglecting $\ln z$ terms of order two and greater yields,

$$\frac{P}{kT} \approx \frac{2\zeta(3)}{\lambda^3} + \frac{2\zeta(2)}{\lambda^3} \ln z,$$

and using Eq. (14) we have

$$\frac{P}{kT} = \frac{2\zeta(3)}{\lambda^3} - \frac{\pi^2}{3\lambda^3} \exp \left(W_{-1} \left(\frac{1}{e} \left[\frac{\pi^2}{6} - \rho\lambda^4 \right] \right) + 1 \right). \quad (17)$$

The previous expression gives the equation of state for the four-dimensional free ideal Bose gas near $z = 1$.

V. ONE-DIMENSIONAL INTERACTING BOSE GAS

The one-dimensional hard-core Bose gas is a model for a one-dimensional system of bosons wherein the particles of the system are modeled as having a finite diameter, and it is required that they do not overlap in configuration space. The first description of gases by such a model was a classical gas, largely developed by Lewi Tonks, and is often referred to as a Tonks gas.³⁷ Recently Caillol⁴ showed the pressure and particle density of the Tonks gas are obtained in closed form through the principal branch of the W function. Here we apply similar analyses to a gas of bosons. A widely used approach to studying one-dimensional bosonic systems is the thermal Bethe ansatz method. The method was invented by Yang and Yang⁴⁴ and is called so because it starts from the Bethe ansatz wave function.^{20,28,41}

(k) In describing these systems the terminology and concepts of Fermi–Dirac statistics are used.^{25,44} Wadati⁴¹ notes that these features are overemphasized in this approach and reformulates the method using a bosonic description. Wadati and Kato⁴² further develop this work and derive the thermodynamic properties of a one-dimensional hard-core Bose gas. We will expand upon this work and show that the high temperature expansions for the pressure and particle density are obtained with the use of the W function. We also show that at low temperatures the temperature can be expressed in terms of the pressure and chemical potential through the W function.

(l) Wadati and Kato⁴² consider a one-dimensional hard-core Bose gas wherein N identical particles with mass M and diameter a move along a line. The equation of state is given as

$$\beta P = \frac{1}{1 - \left(a + \frac{\lambda}{2\sqrt{2}} \right) \rho} = \frac{\rho}{1 - a_e \rho}, \quad (18)$$

where ρ is the particle density, P is the pressure, λ is the thermal de Broglie wavelength, a is the diameter of the bosons, and a_e is the effective diameter of the bosons. This differs from the Tonks gas

by the replacement of a with $a + \lambda/(2\sqrt{2})$, thus providing a quantum correction. Here we derive the Mayer expansions using the Lambert W function. Using (18) and the thermodynamic identity,^{15,29}

$$\rho = z \frac{d}{dz} \beta P, \quad (19)$$

where $z = \exp(\beta\mu)$. From (18) and (19) we can arrive at the following transcendental equation:

$$x + \frac{1}{a_e} \ln x - \frac{\ln z}{a_e} = 0, \quad x = \beta P,$$

which is of type 1 (3), having solution

$$\beta P = \frac{1}{a_e} W_j(a_e z) = \frac{1}{a + \frac{\lambda}{2\sqrt{2}}} W_j \left[\left(a + \frac{\lambda}{2\sqrt{2}} \right) z \right],$$

or using (2),

$$\beta P = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} \left(a + \frac{\lambda}{2\sqrt{2}} \right)^{n-1} z^n,$$

or using the binomial theorem as well as (19), we express the pressure and particle density in the familiar form of the Mayer expansions,^{29,30} by writing them as

$$\beta P = \sum_{n=1}^{\infty} b_n z^n, \quad (20)$$

$$\rho = \sum_{n=1}^{\infty} n b_n z^n, \quad (21)$$

where

$$b_n = \sum_{l=0}^{n-1} \binom{n-1}{l} \left(\frac{\lambda}{2\sqrt{2}a} \right)^l \frac{(-na)^{n-1}}{n!},$$

which is similar to those obtained by Caillol for the Tonks gas. We make two observations regarding these results,

(m) We note the following two main points here:

- The high T series expansion for βP (20) of a hard-core Bose gas has been obtained and has radius of convergence $R = \frac{2\sqrt{2}}{e} \left(\frac{1}{2\sqrt{2}a + \lambda} \right)$ where a is the diameter of the bosons. This is a smaller radius of convergence from that obtained by Caillol for a classically treated system wherein the radius of convergence is $R = \frac{1}{ae}$.⁴
- The quantity βP , when real, has a branch cut along the negative real axis $-\infty < z < -R$ and a branch point at $z = -R$. As discussed by Caillol,⁴ the branch cut of βP corresponds to the distribution of the zeros of the grand partition function for this system.

VI. IDEAL BOSE GASES IN A HARMONIC POTENTIAL

The development of sophisticated techniques for confining cold atoms using magnetic traps provided the means to explore the interesting properties of quantum systems in the laboratory. Such experimental designs are usually composed of permanent magnets arranged in such a way that a minimum in the potential of the constituent particles is obtained at some known location, creating a trap.³⁵ These traps coupled with sophisticated cooling techniques provide environments in which quantum statistical effects, atomic collisions, and other low-temperature phenomena can be studied. Not long after the development of these experimental techniques the observation of BEC was achieved.^{1,3,30} To make accurate predictions concerning the behavior of quantum systems in such environments requires more complex analyses than those for the free ideal gases. As a

consequence, much recent work in mathematical physics has been devoted to deriving approximate analytic expressions for the thermodynamic properties of both Bose and Fermi gases trapped by external potentials.^{19,32,33,35} Here we consider one- and two-dimensional trapped ideal Bose gases with a finite number of particles near the BEC regime. We derive the condensate temperature for a one-dimensional ideal Bose gas in a harmonic potential in terms of the W function. For a two-dimensional trapped ideal Bose gas we express the chemical potential in terms of the temperature using the W function.

A. One-dimensional Bose–Einstein gas in a harmonic potential

In the analysis here we assume the average energy per particle is much greater than the spacing between energy levels, known as the continuum approximation.^{13,29} Formally we assume $\hbar\omega \ll kT$ where ω is the angular frequency of the oscillator, $\hbar\omega$ is the spacing between energy levels, and, by the equipartition theorem, kT is the average energy per particle. The total number of particles in the system is then given by^{8,19,29}

$$N = \sum_{j=1}^{\infty} z^j \int_0^{\infty} \rho(\epsilon) \exp(-j\beta\epsilon) d\epsilon,$$

where z is the fugacity, $\beta = 1/(kT)$, ϵ is the energy, and $\rho(\epsilon)$ is the density of states. We consider the number of particles in the excited states by replacing the lower limit of the integral with $(\hbar\omega)/2$. The density of states $\rho(\epsilon)$ is defined as the number of particle states per unit energy. For a one-dimensional harmonic oscillator the energy levels are given by $\epsilon = (n + 1/2)\hbar\omega$ and the number of individual particle states for energies up to ϵ is equal to n . The density of states is then given by $\rho(\epsilon) = \frac{dn}{d\epsilon} = \frac{1}{\hbar\omega}$. We then have

$$N = \sum_{j=1}^{\infty} z^j \int_{\frac{\hbar\omega}{2}}^{\infty} \frac{\exp(-j\beta\epsilon)}{\hbar\omega} d\epsilon = \frac{Li_1(z \exp(-\frac{\beta\hbar\omega}{2}))}{\beta\hbar\omega}$$

where $Li_1(x) = -\ln(1-x)$ is the polylogarithm of order one. Then

$$N = -\frac{1}{\beta\hbar\omega} \ln\left(1 - z \exp\left(-\frac{\beta\hbar\omega}{2}\right)\right).$$

To obtain the condensate temperature we assume $\mu \ll kT$ (i.e., $z=1$) and $\hbar\omega/2 \ll kT$, so as to neglect quadratic and higher order terms in the exponential. The above equation can then be written as

$$\beta + B \ln \beta + C = 0; \quad B = \frac{1}{N\hbar\omega}, \quad C = \frac{1}{N\hbar\omega} \ln \frac{\hbar\omega}{2}.$$

We recognize this equation as of type 1 (3) with $n = 1$ and thus having solution (4),

$$\beta = \frac{1}{N\hbar\omega} W_j \left(N\hbar\omega \exp\left(-\ln \frac{\hbar\omega}{2}\right) \right), \quad (22)$$

$$\Rightarrow T_c = \frac{N\hbar\omega}{kW_0(2N)}, \quad (23)$$

where we have chosen the principal branch to ensure that T_c is both real and positive. This result is in agreement with what is expected in the thermodynamic limit, i.e., $T_c \rightarrow 0$ as $N \rightarrow \infty$ with $N\omega$ finite.¹⁹

B. Two-dimensional Bose gas in a harmonic potential

We now consider a two-dimensional ideal Bose gas trapped in a harmonic potential. When the approximation $\hbar\omega \ll kT$ is satisfied the total number of particles N is given by^{19,21}

$$N = \left(\frac{1}{\beta\hbar\omega} \right)^2 Li_2(z),$$

where $Li_2(z)$ is the polylogarithm of order two. This expression is similar to the corresponding expression for the four-dimensional ideal gas in Eq. (11). This is expected because the two-dimensional ideal gas in a harmonic potential has the same number of degrees of freedom as the free ideal four-dimensional gas. Thus, a similar expression for the chemical potential can be derived here. Using a similar analytical technique as we did for the free four-dimensional ideal gas, we solve this equation for the chemical potential μ near $z = 1$. The solution is

$$\mu = -kT \exp \left(W_{-1} \left[\frac{1}{e} \left(N \left(\frac{\hbar\omega}{kT} \right)^2 - \frac{\pi^2}{6} \right) \right] \right),$$

where as discussed in Sec. IV A the only real solution is provided by the W_{-1} branch. The condensate temperature is then

$$T_c = \frac{\hbar\omega\sqrt{6N}}{\pi k}.$$

VII. OTHER APPLICATIONS OF TYPE 1 AND TYPE 2 EQUATION FAMILIES

A. Inversion of the temperature–chemical potential relationship

(n) In this section we apply the formulas derived in Sec. II to obtain temperature as a function of chemical potential for Maxwell–Boltzmann, Bose–Einstein, and Fermi–Dirac statistics in three dimensions. The first two cases were introduced in Ref. 38, and we show here that these solutions are special cases of the results (4) and (6). In addition we introduce a new equivalent expression for Fermi–Dirac statistics.

1. Maxwell–Boltzmann systems

(a) In Ref. 38, the authors use the W function to solve the following equation for T :

$$\mu = -kT \ln \left[\frac{V}{N} \left(\frac{2\pi mkT}{\hbar^2} \right)^{3/2} \right], \quad (24)$$

We can write this equation as

$$T \ln T + \left(\frac{2}{3} \ln \Phi \right) T + \frac{2}{3} \frac{\mu}{k}, \quad (25)$$

where $\Phi = (2\pi mk/(\hbar^2)(V/N)^{2/3}) > 0$, and we recognize this as (5) with $B = \frac{2}{3} \ln \Phi$, $n = 1$, $x = T$, and $C = \frac{2}{3} \frac{\mu}{k}$. Then using (6) the solution is

$$\begin{aligned} T &= -\frac{2}{3} \frac{\mu}{k} \left[W_j \left(-\frac{2}{3} \frac{\mu}{k} \exp \left(\frac{2}{3} \ln \Phi \right) \right) \right]^{-1}, \\ \Rightarrow T &= -\frac{2}{3} \frac{\mu}{k} \left[W_j \left(-\frac{4}{3} \frac{\mu\pi m}{\hbar^2} (V/N)^{2/3} \right) \right]^{-1}, \end{aligned} \quad (26)$$

as found in Ref. 38.

2. Bose–Einstein systems

(a) The work in Ref. 38 also presents this problem for the case of Bose–Einstein statistics. The W function is used to solve the equation,

$$\rho \approx A \frac{1}{\beta^{3/2}} e^{\beta\mu}, \quad (27)$$

where ρ is the density and $\beta = 1/kT \gg 1$. We can rewrite the equation as

$$\beta - \frac{3}{2\mu} \ln \beta - \frac{1}{\mu} \ln \left(\frac{\rho}{A} \right) = 0, \quad (28)$$

which is (3) with $x = \beta$, $n = 1$, $B = -3/(2\mu)$, and $C = -\frac{1}{\mu} \ln \left(\frac{\rho}{A} \right)$, and the solution is therefore

$$\beta = -\frac{3}{2\mu} \left[W_j \left(-\frac{2}{3} \mu \exp \left(-\frac{2}{3} \mu \frac{1}{\mu} \ln \left(\frac{\rho}{A} \right) \right) \right) \right].$$

Or

$$T(\mu) = -\frac{2}{3} \left(\frac{\mu}{k} \right) \left[W_j \left(-\frac{2}{3} \mu \left(\frac{\rho}{A} \right)^{-2/3} \right) \right]^{-1}, \quad (29)$$

the result presented in Ref. 38.

3. Fermi–Dirac systems

(a) Recently, there have been some interesting developments in calculating the chemical potential of a free Fermi gas at low temperatures based on the method of Sommerfeld.^{29,31} We employ this method in conjunction with the Lambert W function to obtain an inversion formula for temperature as a function of chemical potential.

(b) We follow Feynman's approach in Ref. 13 to find a closed form expression for the number density ρ as

$$\rho = a \int_0^\mu \epsilon^{1/2} d\epsilon - \int_0^\mu \frac{a\epsilon^{1/2}}{e^{-\beta(\epsilon-\mu)} + 1} d\epsilon + \int_\mu^\infty \frac{a\epsilon^{1/2}}{e^{\beta(\epsilon-\mu)} + 1} d\epsilon, \quad (30)$$

$$a = \frac{4\pi(2m)^{3/2}}{(2\pi\hbar)^3},$$

Using the substitutions $x = -\beta(\epsilon - \mu)$ for the second term and $x = \beta(\epsilon - \mu)$ for the third term, we can write this as

$$\rho = a \int_0^\mu \epsilon^{1/2} d\epsilon - \frac{a}{\beta} \int_0^{\beta\mu} \frac{(\mu - x/\beta)^{1/2}}{e^x + 1} dx + \frac{a}{\beta} \int_0^\infty \frac{(\mu + x/\beta)^{1/2}}{e^x + 1} dx.$$

In the method used by Feynman¹³ the approximation $\beta\mu \rightarrow \infty$ is made. Here we do not make this approximation. Evaluating these integrals we find

$$(\mu \pm x/\beta)^{1/2} \approx \mu^{1/2} \pm \frac{1}{2} \mu^{-1/2} (x/\beta) - \frac{1}{8} \mu^{-3/2} (x/\beta)^2.$$

Thus,

$$\rho = a \int_0^\mu \epsilon^{1/2} d\epsilon + \frac{a}{\beta} \mu^{1/2} \left(\int_0^\infty \frac{dx}{e^x + 1} - \int_0^{\beta\mu} \frac{dx}{e^x + 1} \right) + \frac{a}{2\beta^2} \mu^{-1/2} \left(\int_0^\infty \frac{x}{e^x + 1} dx + \int_0^{\beta\mu} \frac{x}{e^x + 1} dx \right) + \frac{a}{8\beta^3} \mu^{-3/2} \left(\int_0^{\beta\mu} \frac{x^2}{e^x + 1} dx - \int_0^\infty \frac{x^2}{e^x + 1} dx \right). \quad (31)$$

Evaluating these integrals we get

$$\begin{aligned} \rho \approx & \frac{2}{3}a\mu^{3/2} + \frac{1}{4}a\beta^{-1}\mu^{1/2}\ln(1 + e^{-\beta\mu}) - \frac{1}{4}a\beta^{-2}\mu^{-1/2}Li_2(-e^{-\beta\mu}) \\ & + \frac{1}{4}a\beta^{-3}\mu^{-3/2}Li_3(-e^{-\beta\mu}). \end{aligned} \quad (32)$$

Assuming $\beta\mu \gg 1$, which is true in most states of highly condensed matter,¹³ the last two terms can be neglected and anything but the linear term in $\ln(1 + e^{-\beta\mu})$. The resulting equation can be brought to the following form:

$$\beta + \frac{1}{\mu} \ln \beta + \frac{1}{\mu} \ln \left[\frac{4}{a\mu^{1/2}} \left(\rho - \frac{2}{3}a\mu^{3/2} \right) \right] = 0, \quad (33)$$

which is (3) with $B = \frac{1}{\mu}$, $C = \frac{1}{\mu} \ln \left[\frac{4}{a\mu^{1/2}} \left(\rho - \frac{2}{3}a\mu^{3/2} \right) \right]$, $n = 1$, and $x = \beta$, and so the solution will be

$$\begin{aligned} \beta &= \frac{1}{\mu} W_j \left(\mu \exp \left(- \ln \left[\frac{4}{a\mu^{1/2}} \left(\rho - \frac{2}{3}a\mu^{3/2} \right) \right] \right) \right), \\ \Rightarrow T &= \frac{\mu}{k} W_j \left[\frac{a\mu^{3/2}}{4 \left(\rho - \frac{2}{3}a\mu^{3/2} \right)} \right]^{-1}, \end{aligned} \quad (34)$$

where the branches of the W function are allowed to be complex.

(c) For the cases where the approximation $\rho = \frac{2}{3}a\mu_0^{3/2}$ is adequate, i.e., when $T \approx 0$,¹³ we may rewrite the formula as

$$T = \frac{\mu}{k} W_j \left[\frac{3}{8} \frac{\mu^{3/2}}{(\mu_0^{3/2} - \mu^{3/2})} \right]^{-1} \quad j \in \mathbb{Z}. \quad (35)$$

B. One-dimensional hard-core Bose gases

To analyze the low temperature behavior Wadati^{41,42} uses the following integral representation for the pressure:

$$P = \frac{1}{\pi} \int_0^\infty \frac{\epsilon^{1/2}}{e^{\beta(\epsilon - \epsilon_0)} + 1} d\epsilon, \quad (36)$$

where $\epsilon_0 = \mu - aP$ and a is the diameter of the bosons. By equivalent arguments as above, neglecting all nonlinear terms, one obtains the following approximation:

$$P \approx \frac{2}{3\pi} \epsilon_0^{3/2} + \frac{1}{4\pi\beta} \epsilon_0^{1/2} e^{-\beta\epsilon_0}$$

or equivalently,

$$\beta + \frac{1}{\epsilon_0} \ln \beta + \frac{1}{\epsilon_0} \ln D = 0, \quad D = \frac{(P - (2/3\pi)\epsilon_0^{3/2})4\pi}{\epsilon_0^{1/2}},$$

which is of the form type 1 (3) with $n = 1$, $B = 1/\epsilon_0$, $x = \beta$, and $C = (1/\epsilon_0) \ln D$, having solution

$$\beta = \frac{1}{\epsilon_0} W_j(\epsilon_0 \exp(-\ln(D))).$$

Thus,

$$T = (\epsilon_0/k) \left(W_j \left(\frac{\epsilon_0}{D} \right) \right)^{-1} = \frac{1}{k} (\mu - aP) W_j \left[\frac{(\mu - aP)^{3/2}}{4\pi \left(P - \frac{2}{3\pi} (\mu - aP)^{3/2} \right)} \right]^{-1}, \quad (37)$$

where no particular branch has been specified. Equation (37) is a new relationship between the temperature, chemical potential, and pressure of a one-dimensional hard-core Bose gas at low temperatures. Mathematically, (37) allows for the possibility of complex temperatures. Recently there have been systems in statistical mechanics where the generalization of temperature to the complex plane has been useful. Matveev and Shrock use complex temperatures to develop further information on a two-dimensional Ising model in a nonzero magnetic field.²⁶

VIII. CONCLUSIONS

We have derived new representations for the chemical potential, pressure, and temperature of Bose gases in diverse physical contexts. For the free ideal Bose gas in three and four dimensions, new relationships between the chemical potential and temperature in terms of the W function have been obtained. The expressions derived lead naturally to the concepts of complex chemical potential and temperature. The concept of a complex chemical potential has recently been used to describe decay in a BEC. Cragg and Kerman show that a complex chemical potential quantifies the decay time of an excited condensate state in a BEC.² In four dimensions an expression for small temperatures below the condensation temperature was obtained. The expression yields real values for the chemical potential for $T \in [0.789T_c, T_c)$. The analysis of the chemical potential for temperatures outside of this domain warrant further study. The expressions for the chemical potential and pressure were used to derive the equation of state in the BEC regime. These are new results and their derivation relies on the analytic properties of the W function.

The W function also proved to be a useful tool in the analysis of BEC of trapped ideal Bose gases with a finite number of particles. The condensate temperature for the one-dimensional ideal Bose gas with a finite number of particles in a harmonic trap was obtained using the W function. The condensate temperature was expressed in terms of the number of particles in the system and tends to zero in the thermodynamic limit. For the two-dimensional ideal Bose gas in a harmonic trap the chemical potential near the BEC regime was expressed in terms of the temperature and number of particles. In both the cases the condensate temperature is linearly related to the angular frequency of the harmonic oscillator.

The high temperature expansions for the pressure and particle density of a one-dimensional gas of hard-core bosons were also obtained. The result differs from the corresponding classically derived system (Tonks gas) by the radius of convergence and coefficients of the Mayer expansions. The classical case is discussed by Caillol.⁴ Both the methods used here as well as that used by Caillol rely on the properties of the W function. In the low temperature analysis a novel relationship between the temperature, chemical potential, and pressure has been derived, wherein temperature is expressed explicitly in terms of the chemical potential and pressure through the Lambert W function.

We have also presented applications of what we termed as type 1 and type 2 equations to the Bose–Einstein and Maxwell–Boltzmann systems at high temperatures. The convergence of the different statistics in the classical limit, a general characteristic of quantum statistics, is reflected in these results. These equations considered as two general families of transcendental logarithmic equations provide particular solutions when the chemical potential and temperature are also complex. The concepts of complex chemical potential and temperature are a general theme in the systems discussed here. These ideas have become important in statistical physics in recent years.^{7,26} For the four-dimensional ideal Bose gases near condensation the complex analysis of the chemical potential and pressure is incomplete. The branch cuts and singularities of these thermodynamic functions in the complex plane need further study. A similar analysis is required for the temperature dependence on chemical potential derived for the one-dimensional hard-core Bose gas at low temperatures, the ideal three-dimensional Bose gas at high temperatures, and the ideal classical gas at high temperatures.

We have given some applications of the W function that provide further evidence of the underlying logarithmic structure of quantum statistics. These are only some of the potential applications of the W function and, needless to say, the applications of this remarkable special function and its connection to the Polylogarithm will go much further beyond the field of mathematical physics. The subtle exponential and its closely connected logarithmic structure of the W function in fact appear

to be relevant to other areas of thermodynamics and statistical physics. This will be presented in future work.

When the pioneering work of Sommerfeld is viewed from today's perspectives, certain limitations are apparent. They prevent the gaining of a deeper unifying insight into the thermodynamics of Fermi–Dirac and Bose–Einstein statistics. The main problem is the absence of logarithmic structure. It is $\log \mathcal{Z}$, not \mathcal{Z} itself, which ties intimately into thermodynamics. From a deeper perspective, both the Fermi and Bose functions are inverse functions of the fugacity z , implying a logarithmic structure and the polylogarithmic formulation of $\log \mathcal{Z}$ is a manifestation of this structure. As shown in our work, the W function is sufficiently general to yield several special solutions in both the classical and nonclassical limits, from which the dualism of quantum statistics is apparent. At the classical limit, the polylogs are replaced by their fixed point indicating the absence of the logarithmic structure. In this limit both the Fermi and Bose functions are no longer an inverse function of the fugacity and this dichotomy no longer characterizes the physical state of the different systems.

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- ¹ Anderson, M. H., Ensher, J. R., Matthews, M. R., Wieman, C. E., and Cornell, E. A., "Observation of Bose–Einstein condensation in a dilute atomic vapor," *Science* **269**, 198 (1995).
- ² Bose, S. N., "Planck's gesetz und lichtquantenhypothese," *Z. Phys.* **26**, 171 (1924).
- ³ Bradley, C. C., Sackett, C. A., Tollett, J. J., and Hulet, R. G., "Evidence of Bose–Einstein condensation in an atomic gas with attractive interactions," *Phys. Rev. Lett.* **75**, 1687 (1995).
- ⁴ Caillol, J.-M. "Some applications of the Lambert W function to classical statistical mechanics," *J. Phys. A* **36**, 10431 (2003).
- ⁵ Conrath, M., Fries, N., Zhang, M., and Dreyer, M. E. "Radial capillary transport from an infinite reservoir." *Transp. Porous Media* **84**, 109 (2010).
- ⁶ Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J., and Knuth, D. E. "On the Lambert W Function." *Adv. Comput. Math.* **5**, 329 (1996).
- ⁷ Cragg, G. E. and Kerman, A. K. "Complex chemical potential: Signature of decay in a Bose–Einstein condensate." *Phys. Rev. Lett.* **94**, 190402 (2005).
- ⁸ Dalfovo, F., Giorgini, S., Pitaevskii, L. P., and Stringari, S. "Theory of Bose–Einstein condensation in trapped gases." *Rev. Mod. Phys.* **71**, 463 (1999).
- ⁹ Dirac, P. A. M. "On the theory of quantum mechanics." *Proc. R. Soc. London* **112**, 661 (1926).
- ¹⁰ Dubinov, A. E. and Dubinova, A. A. "Nonlinear isothermal waves in a degenerate electron plasma." *Plasma Phys. Rep.* **34**, 403 (2008).
- ¹¹ Einstein, A. "Quantentheorie des einatomigen idealen gases." *Sitzungsber. K. Preuss. Akad. Wiss. Phys. Math. Kl* **1**, 18 1925.
- ¹² Fermi, E. "Sulla quantizzazione del gas perfetto monoatomico." *Rend. Accad. Naz. Lincei* **3**, 145 1926.
- ¹³ Feynman, R. P. *Statistical Mechanics: A Set of Lectures*. (W. A. Benjamin, Reading, MA: 1972).
- ¹⁴ Gordienko, V. A., Dubinova, I. D., and Dubinov, A. E. "Nonlinear theory of large-amplitude stationary solitary waves in symmetric unmagnetized $e - e +$ and $C 60 - C 60 +$ plasmas." *Plasma Phys. Rep.* **32**, 910 (2006).
- ¹⁵ Hauge, E. H. and Hemmer, P. C. "On the yang-lee distribution of roots." *Physica* **29**, 1338 (1963).
- ¹⁶ Hwang, C. and Cheng, Y.-C. "A note on the use of the Lambert W function in the stability analysis of time-delay systems." *Automatica* **41**, 1979 (2005).
- ¹⁷ Jain, A. and Kapoor, A. "Exact analytical solutions of the parameters of real solar cells using Lambert W function." *Sol. Energy Mater. Sol. Cells* **81**, 269 (2004).
- ¹⁸ Jain, A., Sharma, S., and Kapoor, A. "Solar cell array parameters using Lambert W function." *Sol. Energy Mater. Sol. Cells* **90**, 25 (2006).
- ¹⁹ Ketterle, W. and van Druten, N. J. "Bose–Einstein condensation of a finite number of particles trapped in one or three dimensions." *Phys. Rev. A* **54**, 656 (1996).
- ²⁰ Komori, Y. and Wadati, M. "Bethe ansatz wavefunctions and an infinite number of conserved operators of the luttinger model." *Phys. Lett. A* **218**, 42 (1996).
- ²¹ Lee, M. H. "Chemical potential of a D-dimensional free Fermi gas at finite temperatures." *J. Math. Phys.* **30**, 1837 (1989).
- ²² Lee, M. H. "Polylogarithms and Riemann's ζ function." *Phys. Rev. E* **56**, 3909 (1997).
- ²³ Lee, M. H. "Polylogarithms and logarithmic diversion in statistical mechanics." *Acta Phys. Pol. B* **40**, 1279 (2009).
- ²⁴ Lewin, L. *Polylogarithms and Associated Function* (North Holland, Amsterdam, 1981).

- ²⁵ Lieb, E. H. and Liniger, W. "Exact analysis of an interacting bose gas. I. the general solution and the ground state." *Phys. Rev.*, **130**, 1605 (1963).
- ²⁶ Matveev, V. and Shrock, R. "Complex-temperature properties of the two-dimensional ising model for nonzero magnetic field." *Phys. Rev. E*, **53**, 254 (1996).
- ²⁷ Mladek, B. M., Kahl, G., and Neumann, M. "Thermodynamically self-consistent liquid state theories for systems with bounded potentials." *J. Chem. Phys.* **124**, 064503 (2006).
- ²⁸ Ogata, M. and Shiba, H. "Bethe-ansatz wave function, momentum distribution, and spin correlation in the one-dimensional strongly correlated hubbard model." *Phys. Rev. B* **41**, 2326 (1990).
- ²⁹ Pathria, R. K. *Statistical Mechanics*. (Butterworth-Heinemann, Oxford, UK, 1972, 1996).
- ³⁰ Sackett, C. A., Bradley, C. C., Welling, M., and Hulet, R. G., "Bose-Einstein condensation of lithium," *Braz. J. Phys.* **27**, 65 (1997).
- ³¹ Schroeder, D. V. *An Introduction to Thermal Physics* (Addison-Wesley, Reading, San Francisco, 1999).
- ³² Sommerfeld, A. "Zur elektronentheorie der metalle auf grund der fermischen statistik." *Z. Phys.* **47**, 1 (1928).
- ³³ Su, G., Chen, J., and Chen, L. "Low-temperature behavior of a weakly interacting fermi gas trapped in a power-law potential." *Phys. Lett. A* **315**, 109 (2003).
- ³⁴ Su, G., Chen, J., and Chen, L. "Formulation of the thermodynamic properties of an interacting bose gas trapped in a power-law potential." *Physica A: Statistical Mechanics and its Applications* **368**, 459 (2006).
- ³⁵ Tollett, J. J., Bradley, C. C., Sackett, C. A., and Hulet, R. G. "Permanent magnet trap for cold atoms." *Phys. Rev. A* **51**, R22 (1995).
- ³⁶ Toms, D. J. "Ideal fermi gases in harmonic oscillator potential traps." *Ann. Phys. (N.Y.)* **320**, 487 (2005).
- ³⁷ Tonks, L. "The complete equation of state of one, two and three-dimensional gases of hard elastic spheres." *Phys. Rev.*, **50**, 955 (1936).
- ³⁸ Valluri, S. R., Gil, M., Jeffrey, D. J., and Basu, S. "The Lambert W function and quantum statistics." *J. Math. Phys.* **50**, 102103 (2009).
- ³⁹ Valluri, S. R., Jeffrey, D. J., and Corless, R. M. "Some Applications of the Lambert W Function to Physics." *Can. J. Phys.* **78**, 823 (2000).
- ⁴⁰ Venkataraman, G. *Bose and his Statistics* (Sangam Books, London, UK and Hyderabad, India, 1992).
- ⁴¹ Wadati, M. "Bosonic formulation of the bethe ansatz method." *J. Phys. Soc. of Japan*, **54**, 3727 (1985).
- ⁴² Wadati, M. and Kato, G. "One-dimensional hard-core bose gas." *Chaos, Solitons and Fractals* **14**, 23 (2002).
- ⁴³ Wang, Z. H. and Hu, H. Y. "Calculation of the rightmost characteristic root of retarded time-delay systems via lambert w function." *J. Sound Vib.* **318**, 757 (2008).
- ⁴⁴ Yang, C. N. and Yang, C. P. "Thermodynamics of a One-Dimensional System of Bosons with Repulsive Delta-Function Interaction." *J. Math. Phys.* **10**, 1115 (1969).