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Essays on Private Information and Monetary Policy

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Abstract

In our economy, many interactions between individuals involve one party possessing more or better information than the other party (i.e., private information). For example, in asset markets, sellers often have better knowledge about the quality of the assets than buyers. Private information also exists when governments collect taxes, because taxpayers typically have more information about their income compared to the government. In my thesis, I explore the implications of private information in three novel contexts.

In Chapter 2, I study the implications of tax evasion for the design of central bank digital currency, which is an emerging payment instrument. I build a general equilibrium framework to explicitly allow tax evasion by agents and tax audits by a government. I find that introducing a deposit-like CBDC can increase welfare and reduce tax evasion. Furthermore, a deposit-like CBDC needs not increase the funding costs of private banks or decrease bank lending and investment. However, paying a high interest rate on CBDC will decrease the central bank's net interest revenue, which may jeopardize central bank independence.

Chapter 3 examines how multi-dimensional private information affects asset market equilibrium. I find that when asset quality is the only source of private information, sellers with high-quality assets signal their quality to buyers through partial retention of assets if and only if their liquidity holdings are large. However, when sellers' valuations of liquid assets are also private information, some sellers with high-quality assets signal their quality even if their liquidity holdings are small. I extend the model to study of the implications for discount window lending and government asset purchases.

Chapter 4 contains a study of illiquidity and partial retention of assets as signals of asset quality in markets with private information. I find that both signals are used in equilibrium. However, sellers with high-quality assets prefer illiquidity over partial retention in the sense that among these sellers, those with higher-quality assets sell marginally fewer assets but with significantly lower probability. In comparison, sellers with low-quality assets prefer partial retention over illiquidity in the sense that among these sellers, those with higher-quality assets sell significantly fewer assets but with only marginally lower probability.

Keywords: Private Information, Monetary Policy, Tax Compliance, Central Bank Digital Currency, Financial Markets, Competitive Search, Signaling

Summary for Lay Audience

People face information disadvantages and advantages in everyday life. For example, when buying a used car, one recognizes that there is typically hidden information (i.e., private information) about the quality of the car. In contrast, when filing tax returns, taxpayers usually have more information about their income compared to the government. There is a large literature that studies private information in various contexts, and my thesis contributes to this literature by investigating how private information affects monetary policy and asset pricing.

Chapter 2 of my thesis studies the implications of tax evasion for the design of central bank digital currency, which is a digital form of central bank money and an emerging payment instrument. I find that introducing a deposit-like CBDC can increase welfare and reduce tax evasion. Furthermore, competition from a deposit-like CBDC needs not increase the deposit rate and reduce bank lending. However, paying a high interest rate on CBDC will increase the expense of the central bank and may jeopardize its independence.

Chapter 3 examines how asset markets are affected by two types of private information that may coexist in markets. The first type of private information considered is asset quality, while the second type is how much sellers value liquid assets. I find that asset prices and the quantities of assets for sale depend not only on how much liquid assets sellers need but also on what private information is present in the market. Building on the findings, I study the monetary and fiscal policies during financial crises.

Chapter 4 studies two types of signals of asset quality that are often observed empirically in asset markets, i.e., signaling via illiquidity (that is sellers with high-quality assets sell with a lower probability) and signaling via partial retention (that is sellers with high-quality assets sell a smaller quantity). I find that, for sellers with high-quality assets, illiquidity is preferred over partial retention, while the opposite is true for sellers with low-quality assets. Building on these results, I study aggregate liquidity and quality shocks that often happen during financial crises.

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Chapter 1

Introduction

In our economy, many interactions between individuals involve one party possessing more or better information than the other party. The most cited example is used car markets, where sellers have better knowledge of the quality of the cars than buyers. Private information, however, does not only exist in asset markets. For example, when reporting income for tax purposes, taxpayers also have better information regarding their income compared to the government. The literature on private information spurred by the seminal work of Akerlof (1970) has demonstrated that compared to environments with complete information, the existence of private information often leads to drastically changes in economic outcomes and policy prescriptions.

My thesis contains three chapters that explore private information in three different contexts. In Chapter 2, I study the implications of tax evasion for the design of central bank digital currency, an emerging payment instrument. In Chapter 3, I study multi-dimensional private information in asset market equilibrium and policy implications for discount window lending and government asset purchases. In the final chapter, I study how private information interacts with search friction in asset markets, and how such interactions may help explain some puzzling phenomena observed in asset markets.

Central bank digital currency is a digital form of central bank money and is typically referred to as CBDC. A survey by the Bank for International Settlements finds that currently, more than 80% of central banks are actively researching CBDC (Boar et al., 2020). Although there is not yet a consensus on how CBDC should be designed, many central bankers argue that the potential use of CBDC in illicit activities should be taken into account. One of the frequently mentioned illicit activities is tax evasion, which is an important problem in many countries. A report by the Internal Revenue Service estimates that between 2011 and 2013, the loss in tax revenue in the US due to tax evasion was 14.2% of total Federal tax revenue and 2.4% of US GDP (Internal Revenue Service, 2019). The same report also finds that the tax evasion is closely related to the use of cash as a payment instrument. Specifically, nearly half of the loss in tax revenue came from individual businesses that were cash-intensive. To study the implications of tax evasion for the design of central bank digital currency, I build a general equilibrium framework to explicitly allow tax evasion by agents and tax audits by a government. I find that as long as CBDC offers less anonymity than cash, introducing CBDC will decrease tax evasion. However, if CBDC is "*cash-like*" in the sense that it still offers relatively high level of anonymity but low interest rate, then introducing CBDC will decrease the output from not only agents who evade taxes but also agents who report their income truthfully. If CBDC is instead "*deposit-like*" in the sense that it offers low anonymity but high interest rate, then introducing CBDC will increase output and aggregate welfare. Furthermore, introducing deposit-like CBDC needs not increase the funding costs of private banks or decrease bank lending and investment. However, paying a high interest rate on CBDC will decrease the central bank's net interest revenue, which may jeopardize the central bank's independence.

Chapter 3 examines how the liquid assets sellers possess before they trade in asset markets affect the equilibrium when sellers also have private information about asset quality and how much they value liquidity.[1] To see why sellers' liquidity is important for asset market outcomes, consider a firm trying to issue equity to fund a new project. It may choose to offer investors fewer shares at a (potentially) higher price if it already has some liquid assets on hand. Conversely, consider a hedge fund facing sudden redemptions during a financial crisis. It may deplete its liquidity reserves and be forced to sell large quantities of assets at a loss. Most papers that study private information in asset markets, especially those that study two-dimensional private information, shut down the liquidity channel completely by assuming either indivisible assets or linear preferences. The main result of this chapter is that depending on sellers' liquidity holdings, they may or may not reveal the quality of their assets to buyers through prices of the assets and the quantities for sale. I find that when asset quality is the only source of private information, sellers with high-quality assets signal their quality to buyers through partial retention of assets if and only if their liquidity holdings are large. However, when sellers' valuations of liquid assets are also private information, some sellers with high-quality assets signal their quality even if their liquidity holdings are small. The model is then extended to study the implications for discount window lending and government asset purchases. I find that it is possible to have only sellers with high-quality assets borrow from the discount window even though the discount window does not attempt to screen the borrowers. I also find

[1] A version of this chapter has been published in the Journal of Economic Theory (Wang, 2020).

that after a negative quality shock causes the asset market to freeze, the government can unfreeze the market by purchasing bad assets. However, it is not optimal for the government to purchase all bad assets from the market.

Chapter 4 studies how search frictions interact with private information about asset quality. This chapter is motivated by the observations that signaling via illiquidity (i.e. sellers with high-quality assets sell with a lower probability) and signaling via partial retention (i.e. sellers with high-quality assets sell a smaller quantity) are both observed in asset markets. However, the differences between these two signals are not well-understood because it is difficult to induce sellers to use both signals in a theoretical framework. In this chapter, I find that both signals are used in equilibrium. However, among sellers with relatively high quality assets, those with higher-quality assets sell marginally fewer assets but with significantly lower probability. In comparison, among sellers with relatively low quality assets, those with higher-quality assets sell significantly fewer assets but with only marginally lower probability. Building on these results, I study aggregate liquidity and quality shocks. For sellers with high-quality assets, the shocks generate larger changes in trading probability than in trading volume, while the opposite happens to sellers with low-quality assets.

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Chapter 2

Tax Compliance and Central Bank Digital Currency

2.1 Introduction

Central bank digital currency (CBDC), which is a digital form of central bank money, has attracted worldwide attention in recent years. A survey by the Bank for International Settlements finds that currently, more than 80% of central banks are actively researching CBDC (Boar et al., 2020). Although there is not yet a consensus on how CBDC should be designed, many central bankers argue that the potential use of CBDC in illicit activities should be taken into account.[1] One of the frequently mentioned illicit activities is tax evasion, which is an important problem in many countries. A report by the Internal Revenue Service estimates that between 2011 and 2013, the loss in tax revenue in the US due to tax evasion was 14.2% of total Federal tax revenue and 2.4% of US GDP (Internal Revenue Service, 2019).[2] The same report also finds that the tax evasion is closely related to the use of cash as a payment instrument. Specifically, nearly half of the loss in tax revenue came from individual businesses that were cash-intensive. The goal of this chapter is therefore to study how CBDC should be designed when tax evasion is a concern in the economy. I ask how the introduction of CBDC affects the choice of payment methods, economic output, and aggregate welfare.

[1] See for example Lagarde (2018), Powell (2020), Bank of England (2020), and Bank of Canada (2020). See also Bank of Canada et al. (2020) for an overview of central banks' motivations for introducing CBDC.

[2] Rogoff (2017) argues that tax evasion in Europe is likely to be more severe compared to the US due to Europe's larger informal economy. Canada Revenue Agency (2017, 2019) estimates that in 2014, tax evasion in corporate income tax, Goods and Services Tax, and personal income tax amounts to $26B or 9.6% of the total federal tax revenue. See Rogoff (2017) for discussions about other countries.

To answer these questions, I develop a general equilibrium framework based on the model of Lagos and Wright (2005) to allow tax evasion by agents and tax audits by a government. A key feature of the framework is that the problems of tax evasion and tax audits are studied jointly with the problem of payment choice. Specifically, in the economy, there are buyers and sellers who trade a consumption good, and sellers can choose the payment instrument(s) they accept. After each period, sellers are required to file income reports to the government and pay an income tax. The government may collect the tax based on reported income, or it can conduct costly audits on sellers. After an audit, the probability of the government observing a seller's income depends on the payment instrument(s) in which the income is received. If sellers are found evading taxes, the government can punish them by confiscating their income.

As a benchmark, I study a scenario where only cash and bank deposits are available as payment instruments. In equilibrium, some sellers choose to evade taxes, while other sellers choose to report their income truthfully. The benefit of evading taxes is that sellers can potentially have a higher income, while the cost is the risk of being audited and punished by the government. I find that the risk of punishment acts as a proportional tax on sellers who evade taxes and creates distortions in sellers' production decisions. As a result, sellers who evade taxes produce less and receive smaller payments compared to sellers who report their income truthfully. I also find that as long as it is sufficiently easier to hide cash income compared to deposit income, sellers who evade taxes accept cash despite its lower return.

The main results of this chapter concern the optimal design of CBDC when tax evasion creates inefficiency and cash facilitates tax evasion. I define central bank digital currency (CBDC) to be another payment instrument (in addition to cash) that is issued by the central bank. It is stored in accounts managed by the central bank, and it can be held by all agents. CBDC can be different from cash and bank deposits depending on two design choices of the central bank: the first is the interest rate on CBDC, and the second is how much anonymity CBDC offers.[3] In the context of tax evasion, the degree of anonymity of CBDC will determine the probability of the income received in CBDC being observed by the government after an audit.

I find that as long as CBDC offers less anonymity than cash, then introducing CBDC will decrease tax evasion. However, if CBDC is "*cash-like*" in the sense that it still offers relatively high level of anonymity but low interest rate, introducing CBDC will decrease

[3]The anonymity of CBDC can indeed be a choice of the central bank. For example, Darbha and Arora (2020) show that combinations of cryptographic techniques and operational arrangements can be used to achieve fine-grained privacy designs. ESCB (2019) demonstrates that through "anonymity vouchers", CBDC can offer anonymity while also complying with anti-money laundering and anti-terrorism laws.

the output from not only sellers who evade taxes but also sellers who report their income truthfully. Specifically, in equilibrium, although CBDC offers less anonymity, sellers who evade taxes substitute cash with CBDC because the latter pays a positive interest rate. However, because the visibility of sellers' income increases, the government has more incentive to conduct audits, which decreases tax evasion. For sellers who evade taxes, higher audit probability increases the risk of being punished by the government, so they produce less for buyers and receive smaller payments.

Perhaps more interestingly, the output from sellers who report their income truthfully will also decrease. In equilibrium, because the interest rate on cash-like CBDC is low, sellers who report truthfully prefer to accept bank deposits. The decrease in tax evasion prompts more sellers to switch to bank deposits. As a result, the demand for bank deposits increases, which drives down the deposit rate. This lowers the income of sellers who report truthfully and reduces their output. In addition, a higher interest rate on CBDC will further reduce tax evasion, because it increases the income of sellers who evade taxes, which gives the government more incentive to audit. However, in equilibrium, the higher audit probability will offset the increase in the return on CBDC and causes the output from sellers who evade taxes to decrease. A decrease in tax evasion will also worsen the shortage of bank deposits and further lower the deposit rate. As a result, the output from sellers who report truthfully will decrease.

It is, however, possible to design CBDC in a way that decreases tax evasion while also avoiding a shortage of bank deposits. Specifically, the central bank sets the interest rate on CBDC to be sufficiently high so that sellers who report their income truthfully are willing to accept CBDC. Meanwhile, CBDC offers the least amount of anonymity such that sellers who evade taxes are also willing to accept CBDC. In equilibrium, the deposit rate will be equal to the interest rate on CBDC. As a result, sellers who report their income truthfully are indifferent between accepting deposits and CBDC, while sellers who evade taxes prefer CBDC to cash. I refer to this type of CBDC as "*deposit-like*". Similar to cash-like CBDC, introducing deposit-like CBDC will decrease tax evasion and increase the demand for bank deposits. However, because the interest rate on CBDC is equal to the deposit rate, some sellers who report their income truthfully are willing to substitute bank deposits with CBDC. This avoids a shortage of bank deposits and therefore facilitates the transactions between buyers and sellers. Consequently, introducing CBDC increases the output and aggregate welfare by reducing the distortions created by tax evasion and the real resources devoted to tax audits.

It has been argued that CBDC may compete with bank deposits and increase the funding costs of private banks.[4] Indeed, Keister and Sanches (2019) show that while CBDC tends to promote efficiency in exchange, it also crowds out bank deposits and decreases investment in the economy. However, I find that this is not the case no matter CBDC is cash-like or deposit-like. When CBDC is cash-like, introducing CBDC will lower the deposit rate and decrease the funding costs of private banks. When CBDC is deposit-like, it indeed competes with bank deposits. However, introducing CBDC also increases the demand for payment instruments by increasing the output of the economy. I show that it is possible for private banks' funding costs to remain unchanged after the introduction of CBDC.

Although aggregate welfare is higher with deposit-like CBDC, the central bank has to pay higher interest, which may decrease the central bank's net interest revenue. Specifically, the central bank in the model, similar to its counterparts in reality (e.g., the Federal Reserve), earns revenue through the interest payments on its assets (i.e., government bonds). The central bank's expenses include the interest payments on its liabilities (CBDC) and the costs of operating the central bank (e.g., personnel costs). To ensure a central bank's independence of the fiscal authority, it may be necessary for the central bank to cover the expenses with its revenue rather than rely on transfers from the fiscal authority. For example, when discussing its independence, the Federal Reserve emphasizes that it "does not receive funding through the congressional budgetary process".

I find that if CBDC is cash-like, introducing CBDC decreases the central bank's net interest revenue. This is because, firstly, the demand for central bank money is lower due to fewer sellers evading taxes. Secondly, the decrease in tax evasion increases the demand for bank deposits, which are partially backed by government bonds in equilibrium. Higher demand for government bonds then decreases the bond rate and reduces interest payments the central bank receives. If CBDC is deposit-like, the reduction in the central bank's net interest revenue will be even larger because the interest rate on CBDC is higher. Hence, no matter CBDC is cash-like or deposit-like, introducing CBDC will have a negative impact on the central bank's net interest revenue. While introducing deposit-like CBDC will increase output and aggregate welfare, it may jeopardize the central bank's independence.

2.1.1 Related Literature

This chapter is related to the vast theoretical literature on tax compliance.[5] Some of the earliest work includes Allingham and Sandmo (1972) and Reinganum and Wilde (1985,

[4]See for example Garcia et al. (2020), Bank of England (2020), and Bank of Canada (2020).
[5]For reviews of this literature, see Andreoni et al. (1998), Slemrod (2007), and Alm (2019).

1986). Allingham and Sandmo (1972) assume the audit probability is exogenous and study how tax evasion responds to tax rates. Reinganum and Wilde (1985, 1986) assume the income distribution is exogenous and study the strategic interactions between taxpayers and the tax authority. Building on Reinganum and Wilde (1986), Erard and Feinstein (1994) assume that a portion of agents are honest and always report truthfully. More recently, Bassetto and Phelan (2008) also assume an exogenous income distribution and study optimal taxation using a mechanism design approach. They find that there exists an equilibrium where households under-report their incomes because other households are expected to do so as well. Compared to the theoretical literature on tax compliance, the main contribution of this chapter is to incorporate the choice of payment methods and the audit game into a general equilibrium framework. Such a setup allows me to study how the intention to evade taxation affects the choice of payment methods, and how the characteristics of payment instruments affect tax evasion.

Papers that also study tax evasion and informal economy in the Lagos and Wright (2005) framework include Gomis-Porqueras et al. (2014), Aruoba (2018), Aït Lahcen (2020), Bajaj and Damodaran (2020), and Kwon et al. (2020).[6] Gomis-Porqueras et al. (2014) assume that cash transactions are not observable to the government but credit transactions are, and that the government cannot audit agents. They find a negative relationship between tax evasion and inflation. Aruoba (2018) studies a Ramsey optimal taxation problem with a focus on tax enforcement capabilities. The government can choose to audit agents but does so randomly because agents do not report their income. Similar to Gomis-Porqueras et al. (2014), Aït Lahcen (2020) does not allow the government to audit agents and obtains a similar negative relationship between tax evasion and inflation. Bajaj and Damodaran (2020) assume that the fiscal authority can observe all transactions in cash, and that it only chooses the effort spent in collecting taxes. They find that the effective tax rate is low because cash payments tend to be small, which reduces the fiscal authority's incentive to collect taxes. Lastly, Kwon et al. (2020) also assume the government cannot observe transactions in cash but can observe perfectly transactions in CBDC and deposits. They find that the distortion from tax evasion can be corrected by implementing high inflation on cash and using the seigniorage income to finance a high interest rate on CBDC.

This chapter is also related to the emerging literature on CBDC. Using a dynamic general equilibrium model, Barrdear and Kumhof (2016) find that introducing CBDC can stimulate macroeconomic activity as well as bank lending. By focusing on the digital nature of CBDC, Davoodalhosseini (2018) shows that the central bank can in princi-

[6]For more work on informal economy, see Canzoneri and Rogers (1990), Nicolini (1998), Cavalcanti and Villamil (2003), and Yesin (2004, 2006).

ple cross-subsidize different types of agents and improve welfare, which is not possible with cash. Keister and Sanches (2019) find that CBDC can help alleviate frictions that prevent the efficient level of investment, but it also competes with bank deposits and increases the funding cost of financial institutions. Brunnermeier and Niepelt (2019) derives conditions under which the issuance of CBDC does not alter equilibrium allocations. This suggests that CBDC does not have to reduce credit or crowd out investment. Chiu et al. (2019) and Andolfatto (2020) drop the assumption of competitive banking markets common in the literature. Chiu et al. (2019) find that CBDC can promote the competition in the deposit market and increase bank lending. Andolfatto (2020) shows that although the introduction of CBDC increases the deposit rate and reduces bank revenue, it also increases deposit demand and promotes saving. Williamson (2019a) considers an environment where privacy is demanded in some transactions and banks are subject to limited commitment. CBDC may be designed to offer privacy like cash while being more efficient than bank deposits because the central bank is immune from limited commitment. Fernández-Villaverde et al. (2020), Keister and Monnet (2020), and Williamson (2020) study CBDC and financial stability. Fernández-Villaverde et al. (2020) find that CBDC may attract deposits away from the commercial banking sector because it is more stable during bank runs. Keister and Monnet (2020) show that CBDC is beneficial because real-time information on transactions is available to the central bank and regulators. Such information mitigates the moral hazard problem and improves financial stability. Williamson (2020) studies flight to safety when CBDC is designed to be a safe asset. CBDC is found to reduce the damages resulted from a banking panic as it is less disruptive of retail payments.[7]

The rest of the chapter is organized as follows. Section 2.2 describes the environment. Section 2.3 solves a benchmark model. Section 2.4 introduces CBDC. Section 2.5 discuss aggregate welfare and central bank net revenue. Section 2.6 concludes the chapter.

2.2 Model Environment

The model builds on the Lagos and Wright (2005) framework. Time is discrete and continues forever. Each period is divided into two subperiods: the decentralized market (DM) and the centralized market (CM). There is measure one of infinitely-lived buyers and measure $\alpha > 1$ of infinitely-lived sellers. In the DM, buyers consume a DM good that can only be produced by sellers. In the CM, sellers consume a CM good that can

[7] For more discussions on the benefits and costs of CBDC, see Bordo and Levin (2017), Berentsen and Schar (2018), Ricks et al. (2018), and Kahn et al. (2020).

be only produced by buyers. The CM good also serves as the numéraire. A buyer's instantaneous utility is given by

$$u(g_t) - l_t, \tag{2.1}$$

where g_t is the consumption of DM good, and l_t is the labor supplied in the CM. One unit of labor can be turned into one unit of CM good. I assume $u'(g) > 0$, $u''(g) < 0$, $u(0) = 0$, $u'(\infty) = 0$, $u'(0) = \infty$, and $-gu''(g)/u'(g) < 1$. A seller's instantaneous utility is given by

$$-h_t + x_t, \tag{2.2}$$

where h_t is the labor supplied in the DM, and x_t is the consumption of CM good. I assume one unit of labor in the DM can be turned into one unit of DM good. All agents discount future utility using $\beta \in (0, 1)$. Neither good can be carried across periods.

In the DM, buyers and sellers trade the DM good, and the terms of trade are determined via price posting. Specifically, the market in the DM consists of different submarkets. Each submarket is identified by its terms of trade, (q, p), where q represents the quantity of DM good a seller in this submarket offers to produce, and $p \equiv (m, c, d)$ represents the payment that the seller expects to receive, with m, c, and d denoting payments in cash, central bank digital currency (CBDC), and bank deposits, respectively. The terms of trade for the DM of period t are posted in the CM of period $t-1$, and sellers can commit to the terms of trade they post. Buyers observe all terms of trade before they decide which market they will visit and how much cash, CBDC, and bank deposits they will carry. Let $n(q, p)$ denote the buyer-to-seller ratio in each submarket. I assume a buyer meets a seller with probability $\min\{1, 1/n(q, p)\}$ and a seller meets a buyer with probability $\min\{1, n(q, p)\}$. Because by assumption there are more sellers than buyers, some sellers may not meet a buyer in the DM. This means that although all agents are ex ante homogeneous, sellers may be ex post heterogeneous in their income.

Next, at the beginning of the CM, sellers are required to report their income to a fiscal authority and pay an income tax. After receiving the income reports, the fiscal authority can choose to either collect the tax based on reported income or audit sellers. Each audit costs the fiscal authority C units of CM good. I assume that after an audit, the fiscal authority observes sellers' income received in the form of bank deposits with probability one. However, the income received in cash is only observed with exogenous probability $\rho^m \in (0, 1)$. As for CBDC, I consider various regimes where income in CBDC is more or less likely to be observed compared to other payment methods (see Section 2.4). If a seller is found evading taxes, the fiscal authority may confiscate his or her income. After the tax is collected and all audits are finished, a perfectly competitive market opens for agents to trade the CM good. Sellers may use their post-tax and post-audit income to purchase the CM good for consumption, while buyers may sell the CM they produce to acquire cash, CBDC, and bank deposits for the next DM.

In the CM, there is also measure one of bankers and entrepreneurs who are active. Bankers create deposits that are used by buyers to purchase the DM good. They derive linear utility from the CM good like sellers, and they can produce the CM good with the same linear technology that buyers use. I assume bankers have limited commitment and can choose to default on their deposit liabilities Therefore, bank deposits must be backed by other assets such as cash, CBDC, government bonds, and loans to entrepreneurs. I assume entrepreneurs are born in each CM with a one-period project that takes the CM good as input (denoted by k) and yields the CM good in the CM of the next period as output (denoted by $f(k)$), where $f'(k) > 0$, $f''(k) < 0$, $f'(0) = \infty$, and $f'(\infty) = 0$. Entrepreneurs derive linear utility from consuming the CM good in the second CM of their lives before they die and are replaced with a new set of entrepreneurs. I assume entrepreneurs are born without any funds and they cannot work in the CM. Therefore, they must borrow from bankers and use the output of their projects as collateral. I assume both the loan market and the deposit market are perfectly competitive.

The government in the model consists of the fiscal authority and the central bank. In addition to taxing sellers, the fiscal authority issues one-period nominal government bonds that are traded in a competitive bond market in the CM. Following Andolfatto and Williamson (2015), Williamson (2016), Williamson (2019a), and Williamson (2019b), I assume the fiscal authority determines the supply of government bonds, while the central bank determines the supply of cash and CBDC through open market purchases and sales of government bonds. Let price of the CM good be p_t. The central bank's objective is to adjust the supply of cash and CBDC to achieve a certain inflation target $\mu = (p_{t+1} - p_t)/p_t$. Now, let M_t and C_t denote the total supply of cash and CBDC in period t. Let B_t^c denote the government bonds held by the central bank. Let the nominal bond rate and the nominal interest rate on CBDC be R^b and R^c, respectively. The central bank's budget constraint is

$$\frac{M_{t+1} - M_t}{p_t} + \frac{C_{t+1} - C_t}{p_t} + \frac{(1 + R_t^b)B_t^c}{p_t} = \frac{R^c C_t}{p_t} + \frac{B_{t+1}^c}{p_t} + \mathcal{E}_t + \mathcal{T}_t^c. \tag{2.3}$$

The left-hand side of (2.3) represents the per-period income of the central bank, which consists of the revenue from issuing new cash and CBDC, and the revenue from redeeming the government bonds purchased in the last period. The expenses of the central bank consist of the interest payment on CBDC, the purchase of government bonds, and the cost of operating the central bank $\mathcal{E}_t$. After paying the expenses, the central bank transfers the rest of its income to the fiscal authority ($\mathcal{T}_t^c$).

Finally, denote the total supply of government bonds in a period as B_t. The budget constraint of the fiscal authority is

$$\frac{B_{t+1}}{p_t} + \bar{\tau}_t + \mathcal{T}_t^c = T_t + \frac{(1+R_t^b)B_t}{p_t}. \tag{2.4}$$

The left-hand side of (2.4) denotes the per-period income of the fiscal authority, which consists of the income from issuing new government bonds, net tax revenue, and the transfer from the central bank. The right-hand side denotes the per-period expenses of the fiscal authority, which consists of a non-negative lump-sum transfer to buyers in the CM (T_t) and the redemption value of the government bonds issued in the previous period. Throughout this chapter, I assume that the fiscal authority chooses the total amount of government debt ($\mathcal{D} = \frac{(1+R_t^b)B_t}{p_t}$) and the tax income schedule, and lets T_t adjust passively so that (2.4) holds. In Appendix A.3, I consider an alternative setup where there is no lump-sum transfer (i.e., $T_t = 0$), and the government balances the budget by adjusting $\mathcal{D}$ instead.

2.3 A Benchmark Model: No CBDC

As a benchmark, I consider a scenario where the only government-issued money is cash. I restrict my attention to stationary equilibria where all real variables remain constant.

2.3.1 Tax Compliance and Sellers' Problem

First, let y denote the reported income of a seller, and let $\tau(y) : \mathbb{R}_+ \to \mathbb{R}_+$ denote the tax schedule. In the first part of this section, I solve sellers' and the fiscal authority's problems with a general tax schedule, which is only required to satisfy $\tau(y) \leq y$. In the second part, I solve for the optimal tax schedule that maximizes the total surplus in the DM subject to the fiscal authority raising a given amount of net tax revenue.

Let R^d denote the nominal interest rate on bank deposits. Suppose a buyer decides that he or she will visit market (q, p) in the next DM, where $p = (m, d)$ represents the (real) amount of payment in cash and deposits. Accumulating (m, d) requires the following amount of labor in the CM.

$$(1+\mu)m + \frac{(1+\mu)d}{1+R^d}. \tag{2.5}$$

Recall that a buyer meets a seller with probability $\min\{1, 1/n(q,p)\}$. Then, the expected surplus of a buyer choosing to visit (q,p) is given by

$$-(1+\mu)m - \frac{(1+\mu)d}{1+R^d} + \beta \min\{1, 1/n(q,p)\}[u(q) - (m+d)] + \beta(m+d). \tag{2.6}$$

Sellers maximize their utility by choosing q, m, d, n, and y subject to (q,p) providing expected surplus equal to $\mathcal{S}$ to buyers, where $\mathcal{S}$ is expected surplus a buyer can obtain from his or her best alternative. Let $\mathbb{E}_y(\eta)$ denote a seller's expectation about the audit probability η conditional y. A seller's problem is given by

$$\begin{aligned}
\max_{q,m,d,y,n} \; & \{[(1-\mathbb{E}_y(\eta))(m+d-\tau(y)) + \mathbb{E}_y(\eta)(1-\rho^m)(m+d-\tau(y))]\mathbb{1}(d \le y < d+m) \\
& + [(1-\mathbb{E}_y(\eta))(m+d-\tau(y)) + \mathbb{E}_y(\eta)(1-\rho^m)m]\mathbb{1}(y<d) \\
& + [m+d-\tau(y)]\mathbb{1}(y = m+d) - q\} \min\{1,n\} \qquad (2.7) \\
\text{s.t.} \; & -(1+\mu)m - \frac{(1+\mu)d}{1+R^d} + \beta \min\{1, 1/n\}[u(q)-(m+d)] + \beta(m+d) = \mathcal{S}. \qquad (2.8)
\end{aligned}$$

First, note that $\mathbb{1}(.)$ is an indicator function that takes the value of one if the statement in the bracket is true and zero otherwise. Second, recall that if a seller is audited, the fiscal authority can observe directly income in bank deposits, but it can only observe income in cash with probability ρ^m. If $y < d$, the seller always loses his or her income in bank deposits if he or she is audited, but he or she only loses his or her cash income if it is observed. If $d \le y < d+m$, a seller loses his or her entire income when he or she is audited *and* his or her cash income is observed, but does not lose any income if his or her cash income is not observed.

Next, after receiving the income reports, the fiscal authority chooses η to maximize its net tax revenue given $\tau(y)$ and its beliefs about m and d conditional on y.

$$\begin{aligned}
\max_{\eta} \; & \eta \left[\mathbb{E}_y\{(d+\rho^m m - \tau(y))\mathbb{1}(y<d) + \rho^m(d+m-\tau(y))\mathbb{1}(d \le y < d+m)\} - C\right] \\
& + \tau(y), \qquad (2.9)
\end{aligned}$$

where the expectation is taken over m and d. Lastly, recall that there is measure one of buyers and measure α of sellers. Let Φ denote the set of posted terms of trade. Let

$F(q,p)$ denote the distribution of (q,p). Then, $n(q,p)$ must satisfy

$$\int_{(q,p)\in\Phi} n(q,p)\,\mathrm{d}F(q,p) = \frac{1}{\alpha}. \tag{2.10}$$

Now, I solve for the equilibrium for a given $\tau(y)$. The equilibrium concept is Perfect Bayesian Equilibrium (PBE), which requires that
(1) the choices of (q,m,d,y,n) and η are sequentially rational given the fiscal authority's beliefs about m and d conditional on y; and
(2) the fiscal authority's beliefs are derived from Bayes' rule whenever possible.
For any y that is not reported in equilibrium, conditions (1) and (2) put no restrictions on the fiscal authority's beliefs. Therefore, there may exist many equilibria supported by various off-equilibrium path beliefs, and some form of equilibrium refinement is necessary. However, standard refinement methods such as the intuitive criterion may not apply in this environment, because sellers' true income is determined by sellers' own choices (i.e., (q,m,d,n)).

To refine the equilibrium, I require that for any y, a seller's choices of (q,m,d,n) and the fiscal authority's choice of η constitute a Nash equilibrium. Specifically, given $(\mathcal{S},y,\eta)$, sellers solve

$$\begin{aligned}\max_{q,m,d,n}\ &\min\{1,n\}\{[(1-\eta)(m+d-\tau(y))+\eta(1-\rho^m)(m+d-\tau(y))]\mathbb{1}(d\le y<d+m)\\ &+[(1-\eta)(m+d-\tau(y))+\eta(1-\rho^m)m]\mathbb{1}(y<d)\\ &+[m+d-\tau(y)]\mathbb{1}(y=m+d)-q\} \end{aligned}\tag{2.11}$$

$$\text{s.t.}\ -(1+\mu)m-\frac{(1+\mu)d}{1+R^d}+\beta\min\{1,1/n\}[u(q)-(m+d)]+\beta(m+d)=\mathcal{S}. \tag{2.12}$$

The fiscal authority's choice of η conditional y and the seller's strategy (q,m,d,n) is optimal.

$$\eta\begin{cases}=0,\ \text{if } y\ge m+d \text{ or } d+\rho^m m-\tau(y)<C;\\ \in[0,1],\ \text{if } y<d \text{ and } d+\rho^m m-\tau(y)=C,\ \text{or } d\le y<m+d \text{ and } \rho^m[d+m-\tau(y)]=C;\\ =1,\ \text{if } y<d \text{ and } d+\rho^m m-\tau(y)>C,\ \text{or } d\le y<m+d \text{ and } \rho^m[d+m-\tau(y)]>C.\end{cases}\tag{2.13}$$

Note that for any y that is on the equilibrium path, (q,m,d,n) must solve problem (2.11) and η must satisfy expression (2.13). The proposed refinement simply requires the same to be true for any off-equilibrium choice of y. This refinement is in the same vein of the Reordering Invariance (RI) equilibrium proposed by In and Wright (2018). See Appendix

A.1 for more discussion on equilibrium refinement.

In the next proposition, I describe all possible equilibria of the game between sellers and the fiscal authority given $\tau(y)$, R^d, μ, and ρ^m.

Proposition 2.1 *Assume $R^d > 0$ and $C/\rho^m < \tilde{q}$ where $\tilde{q}$ solves $u'(q) = \frac{1+\mu}{\beta}$. Then,*
(1) *Sellers who fail to meet buyers in the DM report $y = 0$ and produce $q = 0$.*
(2) *Sellers who meet buyers in the DM randomize over strategies that include:*
(a) $q > 0$, $m > 0$, $d > 0$, *and* $y = d$; (b) $q > 0$, $m = 0$, $d > 0$, *and* $0 < y < d$;
(c) $q > 0$, $m = 0$, $d > 0$, *and* $y = d$; (d) $q > 0$, $m = 0$, $d > 0$, *and* $y = 0$;
(e) $q > 0$, $m > 0$, $d = 0$, *and* $y = 0$.
In particular, in any equilibrium, either (d) *or* (e) *is played with probability strictly between zero and one.*
Proof: see Appendix A.2.

Note that if $R^d = 0$, there is no benefit from using bank deposits. To make the problem interesting, I assume $R^d > 0$. If a seller fails to meet a buyer in the DM, it is his or her dominant strategy to report $y = 0$ and produce $q = 0$ since sellers do not derive utility from the DM good.

Now, suppose a seller meets a buyer in the DM. Consider strategies (a)-(e). Strategy (a) says that the seller under-reports his or her income, and he or she accepts both cash and bank deposits as payment. The amount of income the seller reports, y, is equal to the amount he or she accepts in bank deposits. Accepting both cash and deposits allows the seller to benefit from both the higher return of deposits and the feature of cash that allows it to be hidden with probability $1 - \rho^m$. To see why $y = d$, note that for all $d \leq y$, the larger d is, the more the seller benefits from the higher return of deposits. If the seller is audited, then as long as the fiscal authority fails to observe his or her cash income, he or she does not lose any income (bar the tax due). However, if $d > y$ and the seller is audited, he or she will lose all income in bank deposits with certainty. If the following relationship holds in equilibrium

$$R^d < \frac{(1-\rho^m)\eta}{1-\eta}, \tag{2.14}$$

then the tax-evasion benefit of cash outweighs the higher return of deposits. Hence, the seller chooses to receive the rest of the payment in cash. Note that (2.14) implies that for cash to be used in equilibrium, the return on deposits must not be too high.

If R^d is instead high, i.e., $R^d > \frac{(1-\rho^m)\eta}{1-\eta}$, then it is beneficial to accept only bank deposits (strategy (b)). In this case, even though the seller loses all of his or her income if he or she is audited, the higher return on deposits outweighs the tax-evasion benefit of cash. Depending on $\tau(y)$, it may be that reporting truthfully (i.e., strategy (c)) offers higher surplus compared to the under-reporting strategies of (a) and (b). In this case, cash loses its tax-evasion benefit. Then, as long as $R^d > 0$, the seller accepts only bank deposits.

Lastly, consider strategies (d) and (e). The proposition says that either one of (d) and (e) must be played with probability strictly between 0 and 1. To see why, first note that because there are more sellers than buyers, a positive measure of sellers do not meet buyers in the DM and will report $y = 0$. If the fiscal authority does not audit sellers who report $y = 0$, then all sellers including those who have met buyers will report $y = 0$. However, if the fiscal authority audits sellers who report $y = 0$ with probability one, all sellers will report truthfully, and then the fiscal authority will not have the incentive to audit sellers. Hence, in equilibrium, it must be that the fiscal authority audits sellers who report $y = 0$ with probability strictly between 0 and 1, and some (but not all) sellers who have met buyers report $y = 0$.

Now, I am ready to solve for the optimal tax schedule that maximizes the total surplus in the DM subject to raising a given amount of net tax revenue.

Proposition 2.2 *The optimal tax schedule is* $\tau^*(y) = \min\{y, \tilde{\tau}\}$.
Proof: see Appendix A.2.

The tax schedule $\tau^*(y)$ is optimal for two reasons. First, it does not distort sellers' decisions in the DM as long as sellers report their income truthfully. Second, as shown in the proof of Proposition 2.2, it eliminates strategies (a) and (b) from the equilibrium. That is, under $\tau^*(y)$, sellers who meet buyers in the DM either report $y = 0$ or report truthfully. This way, $\tau^*(y)$ lowers the audit costs and increases the fiscal authority's net tax revenue.

The equilibrium of the audit game under $\tau^*(y)$ is given by the following.

(1) With probability $1-\gamma$, a matched seller chooses $q = q^h$, $m = 0$, and $d = y = d^h$, where q^h solves

$$u'(q) = \frac{1+\mu}{\beta(1+R^d)}, \tag{2.15}$$

and d^h is given by

$$d^h = \frac{1+R^d}{1+\mu}[\beta u(q^h) - \mathcal{S}]. \tag{2.16}$$

(2) With probability γ, a matched seller chooses $y = 0$. Given R^b and $\tilde{\tau}$, there exists $\rho^{m\prime}$ such that if $\rho^m < \rho^{m\prime}$, the seller accepts only cash and is audited with probability η^0. The seller chooses $q = q^0$ and $m = m^0$, where q^0 solves

$$u'(q) = \frac{1+\mu}{\beta(1-\rho^m\eta^0)}, \tag{2.17}$$

and m^0 is given by

$$m^0 = \frac{1}{1+\mu}[\beta u(q^0) - \mathcal{S}]. \tag{2.18}$$

(3) $\mathcal{S}$ is such that $d^h - q^h - \tilde{\tau} = 0$. η^0 is such that $(1-\rho^m\eta^0)m^0 - q^0 = 0$. The fiscal authority does not audit sellers who report $y = d^h$.
(4) γ solves

$$\frac{\gamma\rho^m m^0}{\alpha - 1 + \gamma} = C. \tag{2.19}$$

It should be noted that if $\rho^m \geq \rho^{m\prime}$, then sellers who report $y = 0$ may also use bank deposits, and hence the demand for cash is zero. In what follows, I assume that ρ^m is sufficiently small so that sellers who evade taxes use cash. Note also that the term $\frac{\gamma}{\alpha-1+\gamma}$ in (2.19) represents the probability that a seller who reports $y = 0$ is evading taxes. Hence, (2.19) guarantees that the fiscal authority is indifferent between auditing and not auditing a seller who reports $y = 0$.

It is worth noting that because (by assumption) there are more sellers than buyers in the economy and that matching is efficient, sellers' surplus is driven to zero in equilibrium. To see why, first note that in equilibrium there must not exist a submarket where sellers are strictly better off while buyers are at least as well off. Second, because there are more sellers than buyers, it must be that $n(q, p) < 1$ for some (q, p). If sellers' surplus is positive, then consider a submarket with (q, p, n') where $n' > n(q, p)$. In this submarket, buyers are as well off but sellers are strictly better off, which is a contradiction. If matching is not efficient, then both $\mathcal{S}$ and seller's surplus may be strictly positive. In such case, the determination of $\mathcal{S}$ is more complex, but the main findings in this chapter remain unchanged.

The next proposition shows the effects of $\tilde{\tau}$, R^d, and ρ^m on equilibrium outcomes.

Proposition 2.3 (1) *An increase in $\tilde{\tau}$ leads to a decrease in q^0, no change in q^h, and an increase in γ.* (2) *An increase in R^d leads to an increase in q^h, an increase in q^0, and a decrease in γ.* (3) *An increase in ρ^m leads to a decrease in γ but no changes in q^0 and*

q^h.
Proof: see Appendix A.2.

An increase in $\tilde{\tau}$ makes tax evasion more attractive. Therefore, the audit probability on sellers who report $y = 0$ must increase. This leads to a decrease in production by these sellers (i.e., q^0) and a decrease in payment to these sellers (i.e., m^0). Then, there must be more sellers who meet buyers in the DM reporting $y = 0$ so that (2.19) holds. An increase in R^d has the opposite effect: it makes reporting truthfully more attractive. Therefore, the audit probability on sellers who report $y = 0$ must decrease. This then leads to an increase in q^0 and a decrease in γ.

Notice that even though the tax schedule $\tau^*(y)$ is optimal, it mitigates but does not eliminate distortions in the economy. Specifically, for sellers who report $y = 0$ but have met buyers in the DM, the risk of being audited by the fiscal authority acts effectively as a proportional tax. As a result, these sellers produce less in equilibrium. This means that when tax evasion is possible, distortions caused by taxes cannot be eliminated solely through the optimal design of the tax schedule.

Lastly, result (3) shows that a change in ρ^m does not have any effect on q^0 and q^h. This is because any change in ρ^m is offset by a change in η^0 in the opposite direction so that $1 - \rho^m \eta^0$, the probability of successful tax evasion, is kept constant. In fact, q^0 only depends on $\mathcal{S}$ in equilibrium. To see this, note that the surplus of sellers who report $y = 0$ satisfies

$$(1 - \rho^m \eta^0) m^0 - q^0 = 0, \tag{2.20}$$

where

$$m^0 = \frac{1}{1+\mu}[\beta u(q^0) - \mathcal{S}], \tag{2.21}$$

$$u'(q^0) = \frac{1+\mu}{\beta(1 - \rho^m \eta^0)}, \tag{2.22}$$

$$\mathcal{S} = \beta u(q^h) - \frac{(1+\mu)(q^h + \tilde{\tau})}{1 + R^d}. \tag{2.23}$$

Hence, we have

$$\mathcal{S} = \beta u(q^0) - \frac{1+\mu}{1 - \rho^m \eta^0} q^0 = \beta u(q^0) - \beta q^0 u'(q^0). \tag{2.24}$$

That is, as long as $\mathcal{S}$ does not change, q^0 will not change. And $\mathcal{S}$ will not change as long as μ, R^d, and $\tilde{\tau}$ do not change. This means that even if the increase in ρ^m is sufficiently large so that sellers who report $y = 0$ switch to accepting bank deposits, it will have no effect on q^0 and q^h. However, it is easy to show that γ will decrease and net tax revenue will increase.

2.3.2 Bankers' Problem in the CM

Recall that the markets for bank deposits and bank loans are perfectly competitive, so bankers take the nominal deposit rate R^d and the nominal loan rate R^k as given.

I assume that if a banker chooses to default, he or she can abscond with a fraction θ of the collateral. Bankers' problem is given by

$$\max_{d^B, m^B, b^B, k^B} \{\beta(k^B + m^B + b^B - d^B) - e\} \tag{2.25}$$

$$\text{s.t.} \quad \frac{(1+\mu)k^B}{1+R^k} + (1+\mu)m^B + \frac{(1+\mu)b^B}{1+R^b} = \frac{(1+\mu)d^B}{1+R^d} + e, \tag{2.26}$$

$$(k^B + m^B + b^B)(1-\theta) \geq d^B, \tag{2.27}$$

$$k^B, m^B, b^B, d^B \geq 0. \tag{2.28}$$

The amount of bank deposits (in real term) created by a banker is d^B, and e is the amount of CM good produced by the banker using the same technology as buyers. This means that, to satisfy the incentive constraint (2.27), bankers supply its own capital (i.e., "sweat equity"). Finally, a banker's holdings of cash, government bonds, and loans are m^B, b^B, and k^B, respectively.

Next, given R^d, the demand for deposits by a buyer who decides to visit (q^h, d^h) is

$$d^h = q^h + \tilde{\tau}, \tag{2.29}$$

where

$$u'(q^h) = \frac{1+\mu}{\beta(1+R^d)}. \tag{2.30}$$

Since a fraction $1-\gamma$ of sellers accept deposits in the DM, to clear the deposit market, R^d must be such that $d^B = (1-\gamma)d^h$.

Given R^k, the demand for loans by entrepreneurs is given by

$$f'(k) = \frac{1+R^k}{1+\mu}. \tag{2.31}$$

Recall that there is measure one of entrepreneurs and bankers. Hence, R^k must be such that $\frac{(1+\mu)k^B}{1+R^k} = k$.

Finally, depending on the return on loans, government bonds, and cash, the bankers may hold one or more types of assets as collateral. In what follows, I focus on equilibria where (2.27) binds. If bankers hold both government bonds and loans, then R^k and R^b must satisfy

$$\frac{1+\mu}{1+R^b} = \frac{1+\mu}{1+R^k} = \beta\theta + \frac{(1-\theta)(1+\mu)}{1+R^d}. \tag{2.32}$$

2.3.3 Equilibrium

An equilibrium is a vector $\{q^h, q^0, k, \gamma, \eta^0, R^d, R^b, R^k\}$ that solves the audit game between sellers and the fiscal authority and bankers' problems in the CM, and satisfies the budget constraints (2.3) and (2.4) given fiscal and monetary policies $(\tilde{\tau}, \mathcal{D}, \mu)$.

Proposition 2.4 *There exists μ' such that if (1) $\beta - 1 < \mu < \mu'$, (2) $\frac{C}{\rho^m} \leq \tilde{\tau}$, and (3) $kf'(k)$ is increasing in k, then there exists a unique equilibrium.*
Proof: see Appendix A.2.

To solve the equilibrium, one may first derive the total demand for cash from (2.19).

$$\bar{m} = \frac{(\alpha - 1 + \gamma)C}{\rho^m}. \tag{2.33}$$

This shows that the demand for cash only depends on other parameter values through its dependence on γ, the proportion of sellers who evade taxes. Next, the total demand for government bonds by bankers is given by

$$\bar{b} = \frac{1-\gamma}{1-\theta}(q^h + \tilde{\tau}) - k^B. \tag{2.34}$$

In the proof, I show that if $\frac{C}{\rho^m} \leq \tilde{\tau}$, then $\bar{m} + \bar{b}$ is strictly increasing in R^b. Hence, there exists a unique R^b such that $\bar{m} + \bar{b} = \mathcal{D}$. For an equilibrium to exist, it is also necessary that μ is not too large. This is because if μ is large, the cost of holding cash may be so high that even if the fiscal authority conducts no audits, sellers accepting cash will not be able to compete with sellers who accept bank deposits. In such case, the demand for cash is zero.

Next, I consider the effects of increasing the inflation target μ on the equilibrium. In the model, this is achieved through open market purchases of government bonds.

Proposition 2.5 *Suppose the assumptions in Proposition* 2.4 *hold. Then, an increase in* μ *leads to increases in both* q^h *and* q^0*. In addition,* γ *and* $\frac{1+R^b}{1+\mu}$ *increase, while* η^0 *and* k *decreases.*
Proof: see Appendix A.2.

Proposition 2.5 says that when μ is small, increasing μ will increase q^h, q^0, and the proportion of sellers who evade taxes. This is because higher inflation reduces the income of sellers who evade taxes. Tax evaders' lower cash income means that even if they are found evading taxes, the punishment the fiscal authority can impose is small. This decreases the fiscal authority's incentive to conduct audits. The drop in the level of tax enforcement (i.e., η^0) leads to a larger share of sellers evading taxes. It can be shown that as long as $\frac{C}{\rho^m} \leq \tilde{\tau}$, the drop in enforcement always offsets the increase in inflation. As a result, sellers who accept cash and evade taxes can produce more for buyers. More sellers accepting cash also means a lower demand for bank deposits. Consequently, the demand for government bonds decreases, and the real bond rate increases. This allows sellers who accept bank deposits and report truthfully to produce more for buyers. However, a higher government bond rate crowds out bank lending to entrepreneurs, so k decreases.

The general equilibrium effects of inflation on real allocations in Proposition 2.5 emerge in this environment for two reasons. First, the value of cash as a payment instrument is determined not only by inflation but also by the fiscal authority's audit strategy. A change in inflation leads not only to a change in the cost of carrying cash but also to a change in the audit probability. Second, the total supply of government liabilities, $\mathcal{D} = \bar{m} + \bar{b}$, is too low to support efficient consumption in the DM. This means that a decrease in demand for bank deposits will lead to an increase in government bond rate and the deposit rate, and hence a higher q^h. In comparison, if $\mathcal{D}$ is sufficiently large so that $R^d = R^b = (1+\mu)/\beta - 1$, constraint (2.27) will not bind and bankers will be indifferent between holding or not holding one extra unit of government bonds. In this case, an increase in inflation will increase γ and q^0, but will have no effect on q^h.

2.4 Central Bank Digital Currency: Equilibrium

I define central bank digital currency (CBDC) to be another type of payment instrument (in addition to cash) that is issued by the central bank. It is stored in accounts managed by the central bank, and it can be held by all agents. Throughout this section, I assume that the central bank does not withdraw cash from circulation. However, I will show that depending on the characteristics of CBDC, cash may not be used in equilibrium because CBDC may be a superior payment instrument.

It is also important to be clear about how CBDC is introduced into the economy. Recall that in the benchmark model, the central bank can increase or decrease the supply of cash through open market purchases or sales of government bonds. The central bank's balance sheet is

Assets of the central bank	**Liabilities of the central bank**
Government bonds	Cash

Table 2.1: Central Bank Balance Sheet: Before the Introduction of CBDC

Similar to cash, the central bank introduces CBDC by using it to purchase government bonds from the bond market. The central bank's balance sheet after the introduction of CBDC is

Assets of the central bank	**Liabilities of the central bank**
Government bonds	Cash
	Central bank digital currency

Table 2.2: Central Bank Balance Sheet: After the Introduction of CBDC

Now, I discuss the characteristics of CBDC. I assume that CBDC can be (potentially) different from cash and bank deposits in two dimensions: the first is the interest rate on CBDC (R^c), and the second is how much anonymity CBDC offers. In the context of tax evasion, the degree of anonymity of CBDC will affect the probability of the income received in CBDC being observed by the fiscal authority after an audit (ρ^c). I assume that both R^c and ρ^c are the choices of the central bank.[8] Specifically, I consider the scenarios where $(R^c, \rho^c) \in \mathbb{R}_+ \times [\rho^m, 1]$. Recall that ρ^m is the probability of income received in cash being observed by the fiscal authority after an audit. Since CBDC competes with cash, if $\rho^c \geq \rho^m$, then it must be that $R^c \geq 0$, because otherwise sellers will strictly prefer cash over CBDC.[9]

In what follows, I first consider a scenario where both the interest rate on CBDC and the degree of anonymity of CBDC are too low so that CBDC is not used in equilibrium. I then show what other types of equilibria may exist depending on R^c and ρ^c.

[8]Darbha and Arora (2020) show that combinations of several cryptographic techniques and operational arrangements can be used to achieve fine-grained privacy designs. ESCB (2019) demonstrates that through "anonymity vouchers", CBDC can offer anonymity while also complying with anti-money laundering and anti-terrorism laws.

[9]If $\rho^c < \rho^m$, then even if the interest rate on CBDC is negative, it may still be accepted by sellers. I discuss this case in Appendix A.4. I also consider a scenario where CBDC offers *less* anonymity than bank deposits. Specifically, I assume that the fiscal authority can costlessly observe any income sellers receive in CBDC, but audits are necessary for the fiscal authority to observe cash and deposit income.

2.4.1 Type-1 Equilibrium: CBDC is not Used

Denote deposit rate in the benchmark as R^{bench}. Denote the audit probability in the benchmark as η^{bench}. Recall that in the benchmark model, sellers who choose to evade taxes accept only cash, and they solve the following problem

$$\max_{q,m}\{(1-\rho^m\eta^{bench})m-q\} \text{ s.t. } -(1+\mu)m+\beta u(q)=\mathcal{S}, \tag{2.35}$$

where $\mathcal{S}$ is highest surplus buyers can obtain elsewhere in the market, and sellers take it as given. If a seller chooses to accept CBDC, then he or she solves

$$\max_{q,c}\{(1-\rho^c\eta^{bench})c-q\} \text{ s.t. } -\frac{1+\mu}{1+R^c}c+\beta u(q)=\mathcal{S}, \tag{2.36}$$

where c represents the payment in CBDC. Then, it is easy to see that as long as

$$\frac{1+\mu}{(1-\rho^c\eta^{bench})(1+R^c)} > \frac{1+\mu}{1-\rho^m\eta^{bench}}, \tag{2.37}$$

sellers who evade taxes will strictly prefer cash over CBDC. The left-hand side of (2.37) represents the marginal cost of accepting CBDC, while the right-hand side represents the marginal cost of accepting cash. While CBDC may offer a positive interest rate (i.e., $R^c>0$), it may also offer less anonymity compared to cash (i.e., $\rho^c>\rho^m$). If R^c is too low and ρ^c is too high, then after factoring the risk of being audited and punished by the fiscal authority, the cost of accepting cash is strictly lower compared to the cost of accepting CBDC.

Next, consider sellers who report their income truthfully. In the benchmark model, these sellers accept bank deposits, and they solve

$$\max_{q,d}\{d-\tilde{\tau}-q\} \text{ s.t. } -\frac{1+\mu}{1+R^{bench}}d+\beta u(q)=\mathcal{S}. \tag{2.38}$$

If a seller chooses to accept CBDC, then he or she solves

$$\max_{q,c}\{c-\tilde{\tau}-q\} \text{ s.t. } -\frac{1+\mu}{1+R^c}c+\beta u(q)=\mathcal{S}. \tag{2.39}$$

Then as long as $R^c<R^{bench}$, sellers who report their income truthfully will have no incentive to accept CBDC. If neither sellers who evade taxes nor sellers who report truthfully accept CBDC, then the equilibrium will be identical to the benchmark equilibrium in Section 2.3.

Now, suppose that R^c and ρ^c are such that

$$\frac{1+\mu}{(1-\rho^c\eta^{bench})(1+R^c)} < \frac{1+\mu}{1-\rho^m\eta^{bench}}. \tag{2.40}$$

This means that for sellers who evade taxes, the marginal cost of accepting CBDC is lower compared to the marginal cost of accept cash. In this case, depending on R^c and ρ^c, sellers who report their income truthfully may accept only bank deposits (a type-2 equilibrium; see Section 2.4.2), or some of them may switch to accepting CBDC (a type-3 equilibrium; see Section 2.4.3).

Finally, suppose that

$$\frac{1+\mu}{(1-\rho^c\eta^{bench})(1+R^c)} > \frac{1+\mu}{1-\rho^m\eta^{bench}} \tag{2.41}$$

but $R^c > R^{bench}$. In this case, sellers who report their income truthfully have the incentive to accept CBDC. However, for sellers who evade taxes, the marginal cost of accepting cash is still lower compared to that of accepting CBDC (a type-4 equilibrium; see Section 2.4.4).

2.4.2 Type-2 Equilibrium: CBDC Replaces Cash

Assume that

$$\frac{1+\mu}{(1-\rho^c\eta^{bench})(1+R^c)} < \frac{1+\mu}{1-\rho^m\eta^{bench}}. \tag{2.42}$$

Then, sellers who evade taxes have the incentive to switch to accepting CBDC. Now, suppose that R^c is low so that sellers who report truthfully prefer to accept bank deposits. The equilibrium is given by the following.

(1) With probability $1-\gamma$, a matched seller produces $q = q^h$, demands deposit payment $d = d^h$, and reports $y = d$, where q^h solves

$$u'(q) = \frac{1+\mu}{\beta(1+R^d)}, \tag{2.43}$$

and d^h is given by

$$d^h = \frac{1+R^d}{1+\mu}[\beta u(q^h) - \mathcal{S}]. \tag{2.44}$$

(2) With probability γ, a matched seller produces $q = q^0$, demands CBDC payment

$c = c^0$, and reports $y = 0$, where q^0 solves

$$u'(q) = \frac{1+\mu}{\beta(1-\rho^c\eta)(1+R^c)}, \tag{2.45}$$

and c^0 is given by

$$c^0 = \frac{1+R^c}{1+\mu}[\beta u(q^0) - \mathcal{S}]. \tag{2.46}$$

(3) $\mathcal{S}$ is such that $d^h - q^h - \tilde{\tau} = 0$. η^0 is such that $(1-\rho^c\eta)c^0 - q^0 = 0$.
(4) γ solves

$$\frac{\gamma\rho^c c^0}{\alpha - 1 + \gamma} = C. \tag{2.47}$$

A banker's problem is similar to the benchmark model, but now the bankers can hold CBDC. However, note that similar to the benchmark, in equilibrium,

$$\frac{1+\mu}{1+R^b} = \frac{1+\mu}{1+R^k} = \beta\theta + \frac{(1-\theta)(1+\mu)}{1+R^d}, \tag{2.48}$$

where R^b and R^k are government bond rate and nominal loan rate, respectively. Since $R^b > R^d$ and $R^d > R^c$, the interest rate on CBDC is lower than the interest rate on government bonds. This means that bankers will not hold CBDC. Finally, let $\bar{c}$ denote the the demand for CBDC. Let $\bar{b}$ denote the demand for government bonds from bankers. Then

$$\bar{c} = \frac{(\alpha - 1 + \gamma)C}{\rho^c}, \tag{2.49}$$

$$\bar{b} = \frac{(1-\gamma)(q^h + \tilde{\tau})}{1-\theta} - f'(k)k, \tag{2.50}$$

where $\frac{(1-\gamma)(q^h+\tilde{\tau})}{(1-\theta)}$ is the total demand for bank deposits, k is the investment in each entrepreneur's project, and $f'(k)k$ is the total value of loans. In equilibrium, R^d is such that $\mathcal{D} = \bar{c} + \bar{b}$, where $\mathcal{D}$ is the total supply of government bonds.

Now I derive the effects of R^c and ρ^c on the equilibrium.

Proposition 2.6 *Increasing R^c or ρ^c while holding μ constant lowers q^h and q^0. In addition, γ, R^d, and R^b will decrease, while k will increase.*
Proof: see Appendix A.2.

First, a higher return on CBDC leads to larger payments to sellers who evade taxes, which gives the fiscal authority more incentive to audit sellers. Such an increase in tax enforcement leads to fewer sellers evading the tax and lower output from those who choose to evade. Because more sellers are accepting deposits, the demand for deposits increases. This drives down the deposit rate and the government bond rate. Consequently, the output produced by sellers who report truthfully decreases as well. A lower government bond rate also means that the loan rate will decrease and bank lending will increase.

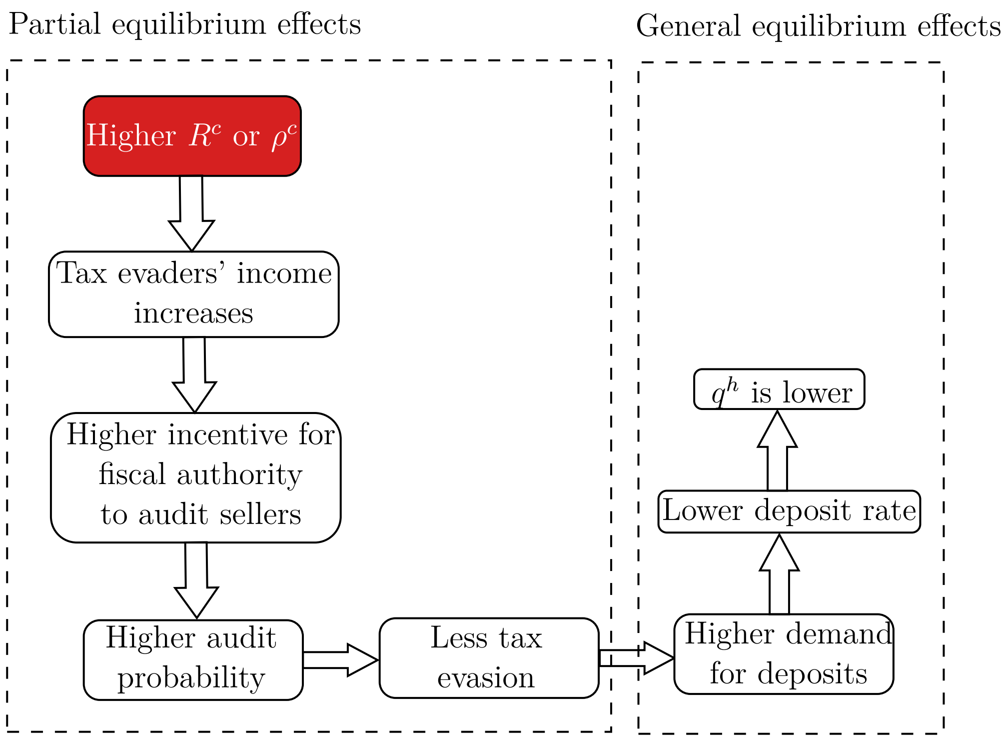


Figure 2.1: Effects of Increasing R^c in a Type-2 Equilibrium

Second, a higher ρ^c increases the visibility of sellers' income, which also makes the fiscal authority's more willing to audit sellers. This decreases tax evasion and increases the demand for deposits. Similar to increasing R^c, the deposit rate will be lower, and the income and output of sellers who report truthfully will decrease.

Figure 2.1 illustrates the mechanism. Note that an increase in R^c or ρ^c does not directly affect q^h, since sellers who report truthfully do not use CBDC, and they are not audited. In a partial equilibrium where the deposit rate is taken as given, increasing R^c or ρ^c will only affect the level of tax evasion (γ) and the output from sellers who evade taxes (q^0). However, in a general equilibrium, the decrease in tax evasion raises the demand for bank deposits, which then affects the output from sellers who report truthfully through its effect on the deposit rate.

Because R^d is decreasing in R^c and ρ^c, for sufficiently large R^c and ρ^c, we will have

$R^c = R^d$. In such case, sellers who report truthfully will be willing to switch to accepting CBDC as well. I discuss this case in the next sub-section.

2.4.3 Type-3 Equilibrium: CBDC Replaces Cash and (some) Bank Deposits

Before I discuss type-3 equilibria, it should be made clear that while CBDC may completely replace cash (a type-2 equilibrium), it will not completely replace bank deposits. The reason is that if bank deposits are not used in equilibrium, bankers will have to fund the loans to entrepreneurs through working in the CM. In such case, the real loan rate $\frac{1+R^k}{1+\mu}$ will be equal to $\frac{1}{\beta}$, and the real deposit rate will be equal to $\frac{1}{\beta}$ as well. This means that unless the real interest rate on CBDC is greater or equal to $\frac{1}{\beta}$, sellers will prefer to use bank deposits. Note that the demand for CBDC will be infinite if the real interest rate on CBDC is greater than $\frac{1}{\beta}$. Hence, in any equilibrium, sellers who report their income truthfully will either strictly prefer bank deposits, or be indifferent between accepting CBDC and accepting bank deposits.

Now, I discuss a type-3 equilibrium where some sellers who report truthfully accept CBDC, while other sellers who report truthfully accept bank deposits. In such case, the interest rate on CBDC must be equal to the deposit rate. Let $R \equiv R^c = R^d$. The equilibrium is given by the following.

(1) With probability $1-\gamma$, a matched seller reports truthfully and produces $q = q^h$, where q^h solves

$$u'(q) = \frac{1+\mu}{\beta(1+R)}. \tag{2.51}$$

Among sellers who report truthfully, a fraction ϵ accept bank deposits, while the rest accept CBDC. Let a^h denote the payment received in bank deposits or CBDC by these sellers. Then a^h is given by

$$a^h = \frac{1+R}{1+\mu}[\beta u(q^h) - \mathcal{S}]. \tag{2.52}$$

(2) With probability γ, a matched seller produces $y = 0$, reports an income equal to zero, and demands CBDC payment $c = c^0$, where q^0 solves

$$u'(q) = \frac{1+\mu}{\beta(1-\rho^c\eta^0)(1+R)}, \tag{2.53}$$

and c^0 is given by

$$c^0 = \frac{1+R}{1+\mu}[\beta u(q^0) - \mathcal{S}]. \tag{2.54}$$

(3) $\mathcal{S}$ is such that $a^h - q^h - \tilde{\tau} = 0$, and η^0 is such that $(1 - \rho^c \eta^0)c^0 - q^0 = 0$.
(4) γ solves

$$\frac{\gamma \rho^c c^0}{\alpha - 1 + \gamma} = C. \tag{2.55}$$

(5) The demand for government bonds from bankers is given by

$$\bar{b} = \frac{\epsilon(1-\gamma)(q^h + \tilde{\tau})}{1-\theta} - f'(k)k, \tag{2.56}$$

where $\frac{\epsilon(1-\gamma)(q^h+\tilde{\tau})}{1-\theta}$ is the total demand for bank deposits, k is the investment in each entrepreneur's project, and $f'(k)k$ is the total value of loans.
(6) The total demand for CBDC is given by

$$\bar{c} = (1-\gamma)(1-\epsilon)(q^h + \tilde{\tau}) + \frac{(\alpha - 1 + \gamma)C}{\rho^c}, \tag{2.57}$$

where the first term represents the demand from sellers who report truthfully, and the second term represents the demand from sellers who evade taxes.
(7) The fraction of sellers who report truthfully and accept deposits, ϵ, is such that $\bar{b} + \bar{c} = \mathcal{D}$.

Finally, a banker's problem is similar to the benchmark model. In equilibrium,

$$\frac{1+\mu}{1+R^b} = \beta\theta + \frac{(1-\theta)(1+\mu)}{1+R^d}. \tag{2.58}$$

Hence, $R^b > R^d = R^c$. This means that the interest rate on CBDC is lower than the interest rate on government bonds, so bankers will not hold CBDC.

Now, I discuss the effects of R^c and ρ^c on the equilibrium.

Proposition 2.7 (1) *Increasing R^c while holding μ and ρ^c constant increases q^h and q^0. In addition, γ, ϵ, and k will decrease.* (2) *Increasing ρ^c while holding μ and R^c constant has no effect on q^h, q^0, and k. However, γ, η^0, and ϵ will decrease.*
Proof: see Appendix A.2.

First, higher R^c increases the income of sellers who accept CBDC and makes them willing to produce more for buyers. However, because sellers who evade taxes faces audits from the fiscal authority (which effectively serves as a proportion tax), they benefit less from the increase in R^c compared to sellers who report truthfully. This means that more sellers will choose to report truthfully. The decrease in tax evasion is sufficiently large to offset the increase in payment received by sellers who evade taxes. As a result, audit probability decreases. Second, a higher ρ^c has no direct effect on sellers who report their income truthfully because the interest rate on CBDC and bank deposits are unchanged. However, because ρ^c is larger, the visibility of sellers' income increases. This decreases sellers' incentive to evade taxes and reduces tax evasions. Since fewer sellers are evading taxes, the audit probability decreases, which compensates for the increase in ρ^c. As a result, q^0 is also unchanged.

To understand why ϵ decreases with R^c and ρ^c, first recall that CBDC is created by the central bank through purchasing government bonds with CBDC, so one unit of government bonds can be used to create one unit of CBDC. However, because bankers have limited commitment and can abscond with a fraction θ of assets, they can only create $1-\theta$ units of bank deposits with one unit of government bonds. This means that the central bank is more efficient at creating payment assets compared to bankers.[10]

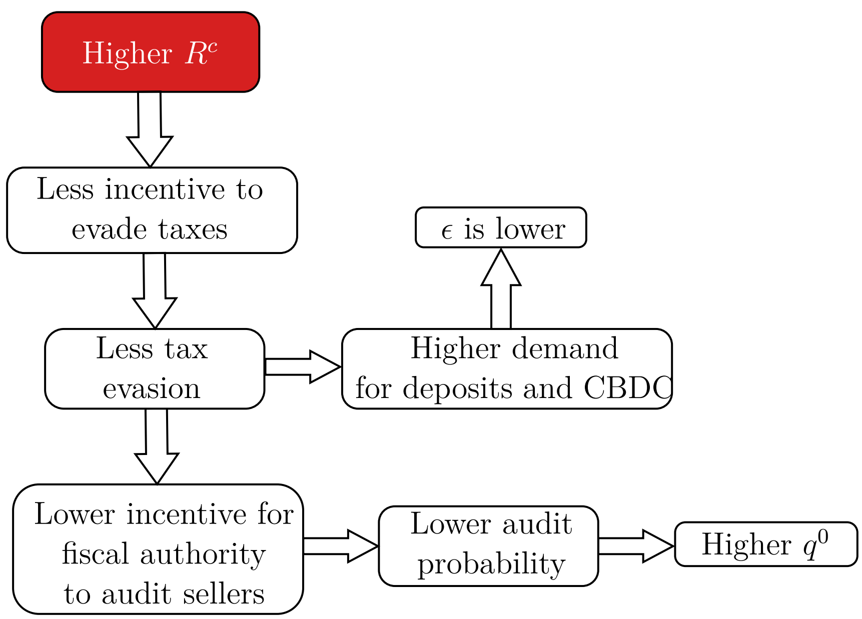


Figure 2.2: Effects of Increasing R^c in a Type-3 Equilibrium

In equilibrium, an increase in R^c or ρ^c will lead to an increase in the demand for bank deposits and CBDC. The central bank then increases the supply of CBDC by purchasing more government bonds from the bond market. Since the total supply of

[10] I discuss this property in more details in Section 2.5.1.

government bonds is fixed, bankers decrease their holdings of government bonds and reduce the supply of deposits. In equilibrium, the total supply of CBDC *and* bank deposits increases, and a larger share of sellers who report truthfully accept CBDC. The mechanisms are summarized in Figure 2.2.

It should be noted that in both type-2 and type-3 equilibria, a higher R^c decreases tax evasion. However, in a type-3 equilibrium, a higher R^c increases rather than decreases R^d, q^h, and q^0. To see why, first note that in a type-2 equilibrium, CBDC is not accepted by sellers who report their income truthfully. A higher R^c leads to more sellers accepting deposits, which decreases the deposit rate and lowers the income and output of sellers who report truthfully. In a type-3 equilibrium, an increase in R^c also increases the demand for bank deposits, but because the interest rate on CBDC is high, some sellers who report truthfully substitute CBDC for bank deposits. Since the central bank is more efficient at creating payment assets, the total supply of CBDC *and* bank deposits increases. As a result, the deposit rate and the government bond rate increase rather than decrease, and all sellers produce more in equilibrium.

In Figure 2.3, I fix ρ^c and show how R^d varies with R^c for a given ρ^c. When R^c is low, CBDC is not used in equilibrium, and R^d is equal to the deposit rate in the benchmark. When R^d is large enough for sellers who evade taxes to be willing to accept CBDC, the economy is in a type-2 equilibrium. In such case, R^d is decreasing in R^c. When $R^d = R^c$, the economy transitions into a type-3 equilibrium, and R^d is decreasing in R^c. Notice that $R^c < R^{bench}$ when the equilibrium switches from type-2 to type-3. This means that even if the interest rate on CBDC is lower than the deposit rate in the benchmark, sellers who report their income truthfully may still accept CBDC after its introduction.

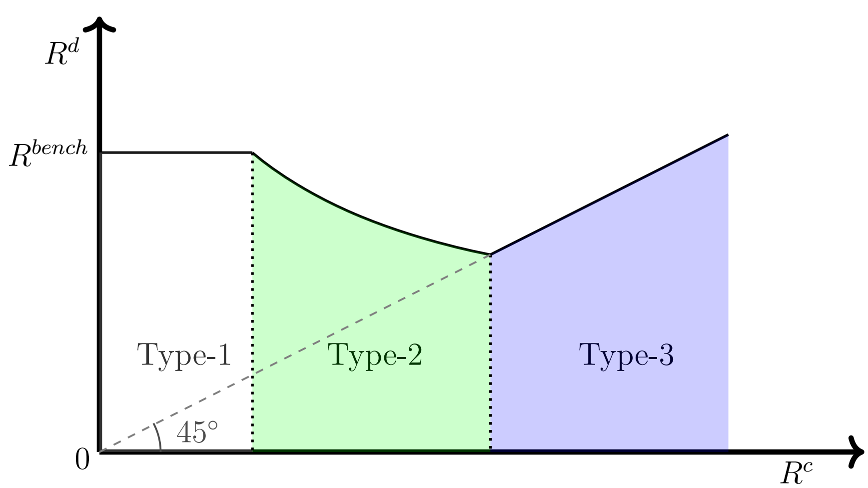


Figure 2.3: Relationship Between R^d and R^c

Finally, suppose $R^c < R^{bench}$ and consider an increase in ρ^c. If R^c and ρ^c satisfy

$$\frac{1+\mu}{(1-\rho^c\eta^{bench})(1+R^c)} > \frac{1+\mu}{1-\rho^m\eta^{bench}}, \tag{2.59}$$

then sellers who evade taxes will switch back to accepting cash. Since $R^c < R^{bench}$, sellers who report their income truthfully will not accept CBDC either. That is, the equilibrium transitions from type-3 to type-1. Now, suppose $R^c > R^{bench}$ and consider again an increase in ρ^c. If (2.59) holds, then sellers who report truthfully will accept CBDC but sellers who evade taxes will not. I discuss this type of equilibria in the next sub-section.

2.4.4 Type-4 Equilibrium: CBDC Replaces (some) Bank Deposits

Assume that $R^c > R^{bench}$, and that R^c and ρ^c satisfy

$$\frac{1+\mu}{(1-\rho^c\eta^{bench})(1+R^c)} > \frac{1+\mu}{1-\rho^m\eta^{bench}}. \tag{2.60}$$

Then, sellers who evade taxes prefer to accept cash, while sellers who report their income truthfully have the incentive to accept CBDC. Recall that in any equilibrium, sellers who report their income truthfully must either strictly prefer bank deposits or be indifferent between accepting CBDC and accepting bank deposits (see Section 2.4.3). This means that R^d must increase in equilibrium so that $R^d = R^c$. Now, define $R \equiv R^d = R^c$. The equilibrium is given by the following.

(1) With probability $1-\gamma$, a matched seller reports truthfully and produces $q = q^h$, where q^h solves

$$u'(q) = \frac{1+\mu}{\beta(1+R)}. \tag{2.61}$$

Among sellers who report truthfully, a fraction ϵ accept bank deposits, while the rest accept CBDC. Let a^h denote the payment received in bank deposits or CBDC. Then a^h is given by

$$a^h = \frac{1+R}{1+\mu}[\beta u(q^h) - \mathcal{S}]. \tag{2.62}$$

(2) With probability γ, a matched seller produces $y = 0$, reports an income equal

to zero, and demands CBDC payment $m = m^0$, where q^0 solves

$$u'(q) = \frac{1+\mu}{\beta(1-\rho^m\eta^0)}, \tag{2.63}$$

and m^0 is given by

$$m^0 = \frac{1}{1+\mu}[\beta u(q^0) - \mathcal{S}]. \tag{2.64}$$

(3) $\mathcal{S}$ is such that $a^h - q^h - \tilde{\tau} = 0$, and η^0 is such that $(1-\rho^m\eta^0)m^0 - q^0 = 0$.
(4) γ solves

$$\frac{\gamma\rho^m m^0}{\alpha - 1 + \gamma} = C. \tag{2.65}$$

(5) The demand for government bonds from bankers is given by

$$\bar{b} = \frac{\epsilon(1-\gamma)(q^h+\tilde{\tau})}{1-\theta} - f'(k)k, \tag{2.66}$$

where $\frac{\epsilon(1-\gamma)(q^h+\tilde{\tau})}{1-\theta}$ is the total demand for bank deposits, k is the investment in each entrepreneur's project, and $f'(k)k$ is the total value of loans.
(6) The total demand for CBDC is given by

$$\bar{c} = (1-\gamma)(1-\epsilon)(q^h+\tilde{\tau}). \tag{2.67}$$

(7) The total demand for cash is given by

$$\bar{m} = \frac{(\alpha-1+\gamma)C}{\rho^m}. \tag{2.68}$$

(8) The fraction of sellers who report truthfully and accept deposits, ϵ, is such that $\bar{m} + \bar{b} + \bar{c} = \mathcal{D}$.

In equilibrium, R^b and R^d satisfy

$$\frac{1+\mu}{1+R^b} = \beta\theta + \frac{(1-\theta)(1+\mu)}{1+R^d}. \tag{2.69}$$

Hence, $R^b > R^d = R^c$. This means that the interest rate on CBDC is lower than the interest rate on government bonds, so bankers will not hold CBDC.

In a type-4 equilibrium, CBDC is not accepted by sellers who evade taxes. This means that how much anonymity CBDC offers will not have any impact on the equilibrium. Now, consider the effects of R^c on equilibrium outcomes.

Proposition 2.8 *Increasing R^c while holding μ constant increases q^h and q^0. In addition, γ, ϵ, and k will decrease.*
Proof: see Appendix A.2.

Increasing R^c has similar effects in type-3 and type-4 equilibria. Specifically, a higher R^c increases the income and output of sellers who report truthfully and lowers the incentive for sellers to evade taxes. The decrease in tax evasion lowers the fiscal authority's incentive to audit, which allows sellers who evade taxes to also produce more for buyers. The decrease in tax evasion also increases the demand for bank deposits and CBDC. To increase the supply of CBDC, the central bank purchases more government bonds from the bond market. Recall that because the central bank is immune from the limited commitment issue, for the same quantity of government bonds the central bank can create more payment assets compared to bankers. Since the total supply of government bonds is fixed, the competition between the central bank and bankers drives out deposits. As a result, ϵ decreases in equilibrium.

I summarize the four equilibrium types discussed in the following table.

Equilibrium Types	**Payment instrument(s) accepted by honest sellers**	**Payment instrument(s) accepted by dishonest sellers**
Type-1 (benchmark)	Bank deposits	Cash
Type-2	Bank deposits	CBDC
Type-3	CBDC and bank deposits	CBDC
Type-4	CBDC and bank deposits	Cash

Table 2.3: Equilibrium Types

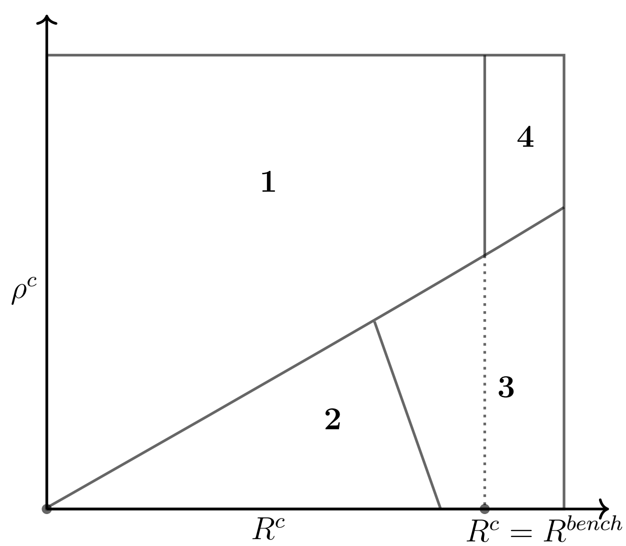


Figure 2.4: Equilibrium Types

In Figure 2.4, I show how equilibrium types depend on R^c (the horizontal axis) and ρ^c (the vertical axis). The origin represents $(R^c, \rho^c) = (0, \rho^m)$. In general, CBDC replaces only cash if it offers high anonymity but low interest rate (a type-2 equilibrium). CBDC replaces (some) bank deposits if it offers low anonymity but high interest rate (a type-4 equilibrium). If CBDC offers a combination of high interest rate and *higher* anonymity than deposits, then it replaces cash and (some) bank deposits (a type-3 equilibrium).

2.5 Aggregate Welfare and Central Bank Net Revenue

In the first part of this section, I discuss the welfare effects of introducing a central bank digital currency (CBDC). Aggregate welfare is measured using an equal-weighted sum of all agents' utility. Specifically, let x^b denote the net consumption of bankers in the DM (recall that bankers may both produce and consume in the CM). Let x^e denote the consumption of entrepreneurs born in the previous period. Recall that l is the labor supplied by buyers in the CM, and x is the consumption of sellers in the CM. Furthermore, q^h and q^0 are the output from sellers who report their income truthfully and the output from sellers who evade taxes, respectively. Finally, γ is the proportion of sellers who evade taxes. Then, aggregate welfare is given by

$$\mathcal{W} = (1-\gamma)[u(q^h) - q^h] + \gamma[u(q^0) - q^0] + x^b + x^e + x - l. \tag{2.70}$$

In the second part of this section, I discuss the effect of introducing CBDC on the central bank's net revenue. The central bank in the model, similar to its counterparts in reality (e.g., the Federal Reserve), earns revenue through the interest payments on its assets (government bonds). The interest expenses of the central bank are the interest payments on its liabilities (CBDC and/or cash). Since the central bank does not pay interest on cash, the interest expense of the central bank is zero in the benchmark. In practice, the expenses of central banks may also include costs of producing and replacing physical cash, and the costs of operating the central bank (e.g., personnel costs).[11] To ensure a central bank's independence of the fiscal authority, it may be necessary for the central bank to cover its own expenses rather than rely on transfers from the fiscal authority. For example, when discussing its independence, the Federal Reserve emphasizes that it

[11] The interest expense of the Federal Reserve in 2019 is \$41B, and the operating expense is \$7B. Together, they account for 85% of the Federal Reserve's total expenses. In comparison, the interest expense and the operating expense of the Bank of Canada in 2019 are \$406M and \$519M, respectively, and they account for 94% of the Bank of Canada's total expenses.

"does not receive funding through the congressional budgetary process".[12] It is therefore of interest to understand how introducing CBDC, especially interest-bearing CBDC, affects the central bank's net revenue.

2.5.1 Aggregate Welfare

Recall that η^0 is the audit probability and C is the cost per audit. In equilibrium, the fiscal authority only audits sellers who report zero income, which include measure $\alpha - 1$ of sellers who did not meet buyers in the DM, and measure γ of sellers who choose to evade taxes. Hence, the total audit costs are equal to $(\alpha - 1 + \gamma)\eta^0 C$. Recall also that k is the amount of CM good invested in each entrepreneur's project. Then, $f(k) - k$ represents the net output from entrepreneurs' projects. Since the CM good cannot be carried into the next period, the resource constraint in the CM implies that net CM consumption $x^b + x^e + x - l$ must satisfy

$$x^b + x^e + x - l = f(k) - k - (\alpha - 1 + \gamma)\eta^0 C. \tag{2.71}$$

Aggregate welfare can then be divided into three components.

$$\mathcal{W} = \underbrace{(1-\gamma)[u(q^h) - q^h] + \gamma[u(q^0) - q^0]}_{\text{Total surplus in the DM}} + \underbrace{f(k) - k}_{\text{Net output from entrepreneurs}} - \underbrace{(\alpha - 1 + \gamma)\eta^0 C}_{\text{Total audit costs}}. \tag{2.72}$$

In what follows, to study the effect of introducing CBDC on aggregate welfare, I compare the three components with their counterparts in the benchmark equilibrium.

I. Total surplus in the DM

Recall that R^c denotes the interest rate on CBDC, and ρ^c denotes the probability of the income received in CBDC being observed by the fiscal authority after an audit. Depending on the central bank's choices of R^c and ρ^c, there are four types of equilibria (see Table 2.3 and Figure 2.4). Let R^{bench} denote the deposit rate in the benchmark equilibrium.

First, consider the effect of introducing CBDC on γ.

[12] See `https://www.federalreserve.gov/faqs/about_12799.htm`. Currently, both the Federal Reserve and the Bank of Canada earn more than enough to cover their expenses, and they transfer the remaining revenue to the fiscal authorities at the end of each year. In 2019, the Federal Reserve and the Bank of Canada transferred \$54.8B and \$1.2B to the federal governments of the US and Canada, respectively.

Proposition 2.9 *Compared to the benchmark, γ is smaller in type-2, type-3, and type-4 equilibria. Given ρ^c, γ is smaller in type-3 equilibria than in type-2 equilibria. Given R^c, γ is also smaller in type-3 equilibria than in type-4 equilibria.*
Proof: see Appendix A.2.

In a type-2 equilibrium, CBDC is only accepted by sellers who evade taxes. Recall that ρ^m is the probability of cash being observed by the fiscal authority after an audit. Because $\rho^c > \rho^m$, introducing CBDC increases the visibility of sellers' income. This gives the fiscal authority more incentive to audit sellers, which decreases tax evasion. In a type-4 equilibrium, CBDC is only accepted by sellers who report their income truthfully, so the anonymity of CBDC is irrelevant to tax evasion. However, in this case, CBDC competes with bank deposits and increases the deposit rate, which attracts more sellers to report their income truthfully. Finally, in a type-3 equilibrium, CBDC is accepted by both sellers who report their income truthfully and sellers who evade taxes. Therefore, CBDC not only competes with bank deposits but also makes tax audits more effective. The result is that the reduction in tax evasion is larger compared to any type-4 equilibria given R^c and any type-2 equilibria given ρ^c.

Next, consider the effects of introducing CBDC on q^0 and q^h.

Proposition 2.10 *Compared to the benchmark,* (1) *q^0 and q^h are smaller in type-2 equilibria, and larger in type-4 equilibria; and* (2) *q^0 and q^h are smaller in type-3 equilibria if $R^c < R^{bench}$ but larger if $R^c > R^{bench}$.*
Proof: see Appendix A.2.

In equilibrium, q^0 and q^h only depend on the deposit rate in the economy. To see this, note that q^h solves

$$u'(q^h) = \frac{1+\mu}{\beta(1+R^d)}. \tag{2.73}$$

Recall that sellers who evade taxes and sellers who report truthfully offer the same surplus $\mathcal{S}$ to buyers. By following (2.20)-(2.24), it is easy to derive that

$$\mathcal{S} = \beta u(q^h) - \beta u'(q^h)(q^h + \tilde{\tau}) = \beta u(q^0) - \beta u'(q^0)q^0. \tag{2.74}$$

In type-3 and type-4 equilibria, because sellers who report truthfully are indifferent between CBDC and bank deposits, $R^c = R^d$. Hence, q^0 and q^h are larger than their counterparts in the benchmark equilibrium if and only if $R^c \geq R^{bench}$. In type-2 equilibria, the decrease in tax evasion causes a shortage of bank deposits, which lowers the deposit rate. Hence, in any type-2 equilibrium, $R^d < R^{bench}$, and q^0 and q^h are smaller compared to the benchmark.

The intuition for the relationship between q^0 and q^h (i.e., equation (2.74)) is as follows. First, q^h does not depend on the fiscal authority's audit strategy because it is the output from sellers who report truthfully. Second, if q^h changes, the fiscal authority must change the audit probability so that sellers who evade taxes produce exactly the q^0 that satisfies (2.74). This ensures that sellers who evade taxes have no disadvantage nor advantage when competing with sellers who report truthfully, because in equilibium sellers must be indifferent between evading taxes and not evading taxes.

Finally, consider how introducing CBDC affects the total surplus in the DM.

Proposition 2.11 *For any ρ^c, there exists $R^{c\prime} < R^{bench}$ such that for any $R^c \geq R^{c\prime}$, the total surplus in DM is higher in type-3 equilibria than in the benchmark equilibrium. There also exists $R^{c\dagger}$ such that for any $R^c \geq R^{c\dagger}$, the total surplus in DM is higher in type-3 equilibria than in any type-2 equilibria. Finally, given R^c, the total surplus in DM is higher in type-3 equilibria than in any type-4 equilibria.*

Proof: see Appendix A.2.

Proposition 2.11 follows from Proposition 2.9 and 2.10. In particular, it says that even if $R^{c\prime} \leq R^c < R^{bench}$, as long as R^c and ρ^c are such that the equilibrium is type-3, introducing CBDC will increase DM surplus compared to the benchmark. In such case, although q^0 and q^h are lower than their counterparts in the benchmark equilibrium, γ is also smaller. Recall that $q^0 < q^h$ because the risk of being audited and punished by the fiscal authority acts as a proportional tax on sellers who evade taxes. Hence, the total surplus in the DM increases even though the deposit rate is lower than the benchmark.

Note that increasing R^c in a type-3 equilibrium will increase q^0 and q^h and decrease γ (see Proposition 2.7). Hence, the total surplus in the DM will be higher. However, it will also increase the funding costs of bankers, which will in turn raise the loan rate and lower the investment in entrepreneurs' projects. I discuss this effect in Part II.

II. Net output from entrepreneurs' projects

The investment in an entrepreneur's project, k, is given by

$$f'(k) = \frac{1 + R^k}{1 + \mu}, \tag{2.75}$$

where R^k is the nominal loan rate. From bankers' problem in Section 2.3.2, we know that the relationship between R^k and the deposit rate R^d is given by

$$\frac{1+\mu}{1+R^k} = \beta\theta + \frac{(1-\theta)(1+\mu)}{1+R^d}. \tag{2.76}$$

In a type-2 equilibrium, the deposit rate is lower than the benchmark because the decrease in tax evasion causes a shortage in bank deposits. Hence, k is larger than the benchmark. In a type-3 equilibrium, the deposit rate is equal to the interest rate on CBDC, so the deposit rate is larger than the benchmark if $R^c < R^{bench}$ and smaller if $R^c > R^{bench}$. In a type-4 equilibrium, CBDC competes with bank deposits and increases the deposit rate. Hence, k is smaller than the benchmark.

However, a higher k does not necessarily mean higher net output from entrepreneurs' projects ($f(k) - k$). When the total supply of assets in the economy (government bonds plus loans to entrepreneurs) is low, the deposit rate carries a large liquidity premium (i.e., R^d is low). This means the loan rate is low and investment k is inefficiently high. Specifically, if $R^{bench} < \mu$, then an increase in k compared to the benchmark will decrease the net output from entrepreneurs' projects. The next proposition follows directly from the above arguments.

Proposition 2.12 *Assume $R^{bench} < \mu$. Compared to the benchmark, $f(k) - k$ is lower in type-2 equilibria. In addition, there exists $R^{c\prime\prime}$ such that if $R^{bench} < R^c \leq R^{c\prime\prime}$, $f(k)-k$ is higher in both type-3 and type-4 equilibria. If $R^{bench} > R^c$, $f(k) - k$ is lower in type-3 equilibria.*

If $R^{bench} > \mu$ and $R^c > R^{bench}$, then compared to benchmark, $f(k) - k$ is lower in both type-3 and type-4 equilibria. Recall that under the same conditions, the total surplus in the DM surplus is higher compared to the benchmark. Hence, a trade-off exists in such cases: introducing CBDC promotes the trade in the DM, but it also increases the funding costs of bankers and decreases the output from entrepreneurs.

III. Total audit costs

The number of audits that the fiscal authority conducts depends on the measure of sellers who report zero income ($\alpha - 1 + \gamma$) and the audit probability (η^0). Proposition 2.9 shows that introducing CBDC decreases tax evasion and reduces the measure of sellers who report zero income. However, the audit probability may be higher or lower compared to the benchmark depending on R^c and ρ^c.

Proposition 2.13 *Compared to the benchmark, η^0 is lower in type-4 equilibria but higher in type-2 and type-3 equilibria.*

Proof: see Appendix A.2.

Hence, compared to the benchmark, the total audit costs are lower in type-4 equilibria. For type-2 and type-3 equilibria, the total audit costs may be higher or lower depending on R^c and ρ^c.

Because all three components of aggregate welfare can in principle move in different directions, in order to derive clear welfare implications, I restrict my attention to some subsets of CBDC configurations. First, assume $R^c > R^{bench}$. Recall that in this case, only type-3 and type-4 equilibria exist (see Figure 2.4). Now, define $\rho^c(R^c)$ to be the highest ρ^c such that given (R^c, ρ^c), the equilibrium is type-3. Specifically, $\rho^c(R^c)$ solves

$$\frac{1+\mu}{(1-\rho^c\eta^0)(1+R^c)} = \frac{1+\mu}{1-\rho^m\eta^0}, \tag{2.77}$$

where η^0 is given by the solution to a type-3 equilibrium in Section 2.4.3.

Proposition 2.14 *For any $R^c > R^{bench}$, $(R^c, \rho^c(R^c))$ offers the highest aggregate welfare.*
Proof: see Appendix A.2.

First, in type-3 and type-4 equilibria, $R^c = R^d$. From Part I and II, we know that q^0, q^h, and k only depend on R^d. This means that for any given $R^c > R^{bench}$, aggregate welfare varies only because of γ and η^0. By following Proposition 2.7, it is easy to plot the relationship between γ and ρ^c and the relationship between η^0 and ρ^c.

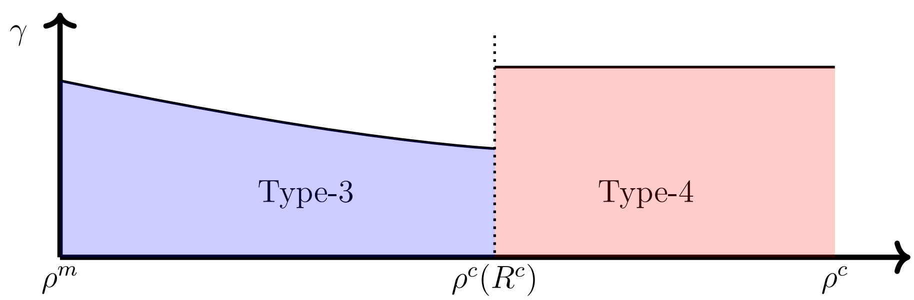


Figure 2.5: Relationship Between γ and ρ^c when $R^c > R^{bench}$

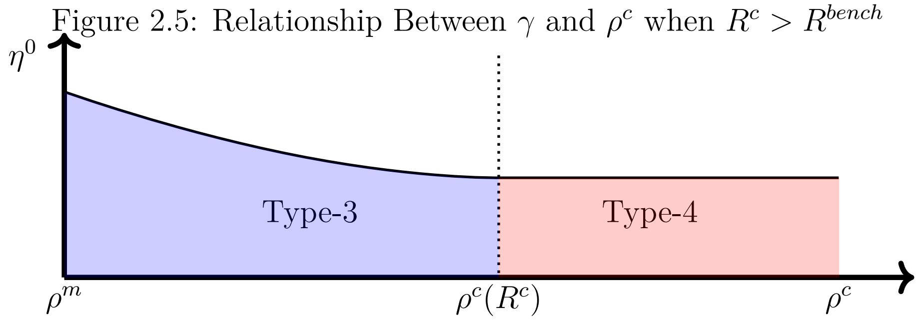


Figure 2.6: Relationship Between η^0 and ρ^c when $R^c > R^{bench}$

That is, for any given $R^c > R^{bench}$, γ and η^0 reach minima when $\rho^c = \rho^c(R^c)$. Hence, for any given $R^c > R^{bench}$, aggregate welfare reaches a unique maximum when $\rho^c = \rho^c(R^c)$. Note that in type-4 equilibria, equilibrium outcomes do not depend on ρ^c since CBDC is only accepted by sellers who report truthfully.

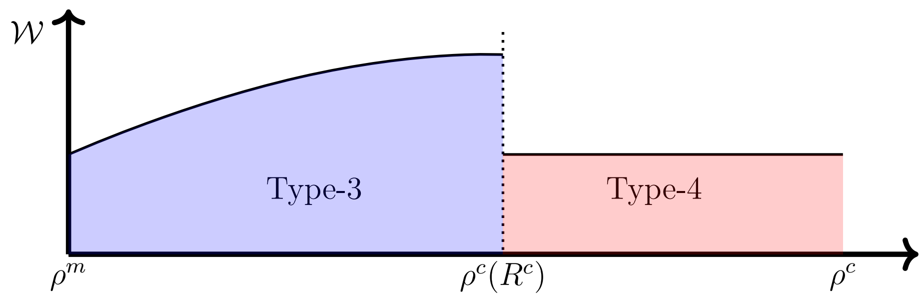


Figure 2.7: Relationship Between $\mathcal{W}$ and ρ^c when $R^c > R^{bench}$

When $R^c \leq R^{bench}$, how introducing CBDC affects aggregate welfare is in general ambiguous. The reason is that changes in net output from entrepreneurs are often in the opposite directions of changes in DM surplus. Nevertheless, if the question one wishes to answer is whether introducing CBDC *can* increase aggregate welfare, then we may restrict our attention to R^c's that are close to R^{bench}. In such cases, k is similar to its counterpart in the benchmark equilibrium. I refer to this type of CBDC as "**deposit-like**" CBDC. Formally, a CBDC is deposit-like if
(1) (R^c, ρ^c) is in the neighborhood of $B \equiv (R^{bench}, \rho^c(R^{bench}))$, and
(2) the equilibrium given (R^c, ρ^c) is type-3.

I consider "**cash-like**" CBDC. Formally, a CBDC is **cash-like** if
(1) (R^c, ρ^c) is in the neighborhood of $A \equiv (0, \rho^m)$, and
(2) the equilibrium given (R^c, ρ^c) is type-2.
The following figure shows the locations of A and B in the R^c-ρ^c space.

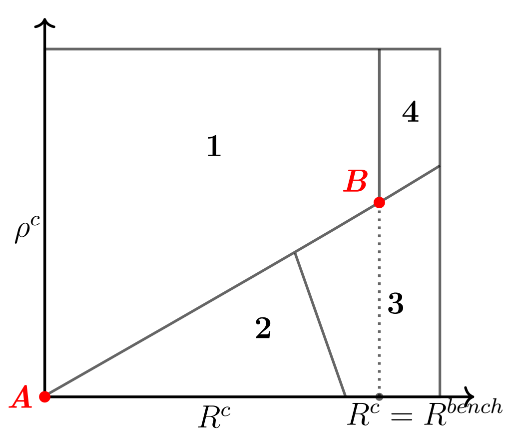


Figure 2.8: Locations of A and B

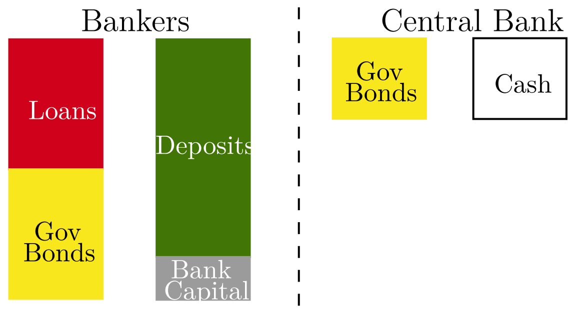
Proposition 2.15 *Introducing deposit-like CBDC increases aggregate welfare. Furthermore, aggregate welfare is higher with deposit-like CBDC than with cash-like CBDC.*
Proof: see Appendix A.2.

By following Proposition 2.11, it is easy to show that compared to the benchmark equilibrium and any type-2 equilibria with cash-like CBDC, the total surplus in the DM is higher. Furthermore, the total audit costs is also lower with deposit-like CBDC because the reduction in tax evasion is larger in type-3 equilibria than in type-2 equilibria.

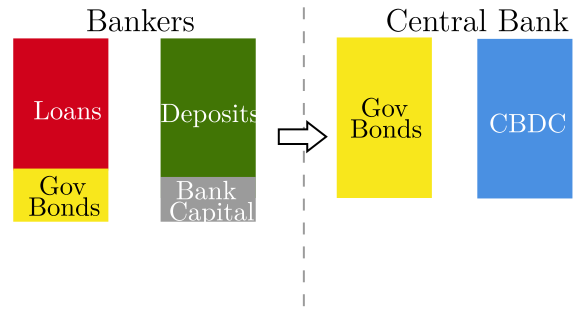
I conclude this subsection by providing some intuition for why deposit-like CBDC provides higher aggregate welfare compared to cash-like CBDC. First, because tax evasion is distortionary, ceteris paribus, a decrease in tax evasion always increases aggregate welfare. However, if CBDC is cash-like, sellers who report their income truthfully accept only bank deposits, so a decrease in tax evasion will increase the demand for deposits. The resulted shortage of bank deposits will lower the deposit rate and hinder the transactions between buyers and sellers. To solve this problem, the design of CBDC must achieve two objectives. First, introducing CBDC must increase the supply of payment assets in the economy. Second, the interest rate on CBDC must be sufficiently high for sellers who report truthfully to accept CBDC. Note that achieving only one of the two objectives is not sufficient to avoid the shortage of bank deposits: the shortage will remain if either CBDC is not accepted by sellers who report truthfully, or introducing CBDC does not increase the total supply of payment assets.

Achieving the second objective requires the interest rate on CBDC to be sufficiently high, but how does introducing CBDC increase the supply of payment assets? Recall that bankers have limited commitment so they can abscond with a portion θ of their assets. By assumption, the central bank is immune from the limited commitment problem. This means that bankers must hold more assets than their liabilities so that they do not have the incentive to default. The difference between bankers' assets and liabilities is bank capital, which bankers accumulate through working in the CM. In comparison, the central bank holds the same quantity of assets (government bonds) and liabilities (cash and/or CBDC). See Figure 2.9(a) for an illustration. Recall that the central bank introduces CBDC by purchasing government bonds from the bond market with CBDC (and it introduces cash in the same way). Since there are fewer government bonds available, bankers issue fewer deposits. See Figure 2.9(b) for an illustration.

Next, note that with each unit of government bonds, the central bank can create one unit of CBDC, while bankers can only create $1-\theta$ units of deposits. As a result, after the introduction of CBDC, the total supply of bank deposits and deposit-like CBDC is larger compared to the supply of deposits in the benchmark. In other words, the introduction of CBDC increases the supply of payment assets. See Figure 2.10 for an illustration.



(a) Before Deposit-like CBDC



(b) After Deposit-like CBDC

Figure 2.9: The Balance Sheet Effects of Introducing Deposit-like CBDC

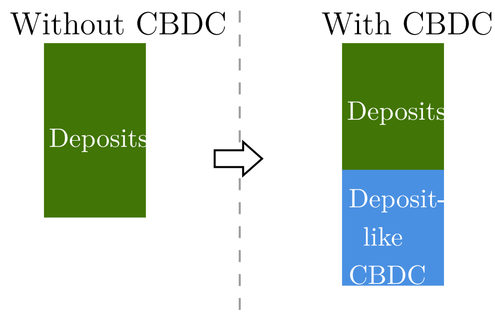


Figure 2.10: Supply of Bank Deposits and Deposit-like CBDC

Regardless of whether CBDC is cash-like or deposit-like, introducing CBDC will decrease tax evasion and increase the demand for payment assets. However, with deposit-like CBDC, the increase in the demand for payment assets is satisfied by the increase in the supply of payment assets (see Figure 2.12). This effect is what allows the total output and surplus in the DM to increase after the introduction of CBDC. Without this effect, the increase in demand for deposits will drive down the deposit rate, which will then decrease the income and output of sellers who accept bank deposits (see Figure 2.11).

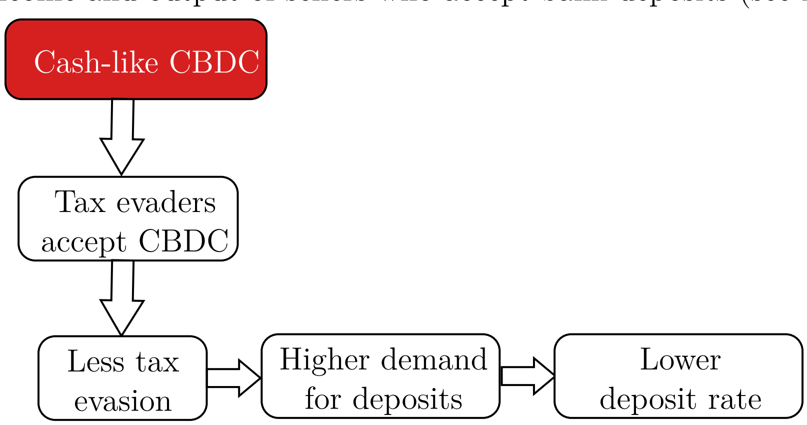


Figure 2.11: Effects of Introducing Cash-like CBDC

It should be noted that the above results do not imply that the central bank should replace private banking because bankers also make loans to entrepreneurs. In practice, central banks may lack the expertise to make such loans. It should also be noted that if $R^c = R^{bench}$, the supply of loans is not affected by the introduction of CBDC. Although bank deposits are crowded out by CBDC, they are (a portion of) the deposits that are backed by government bonds (see Figure 2.9b).

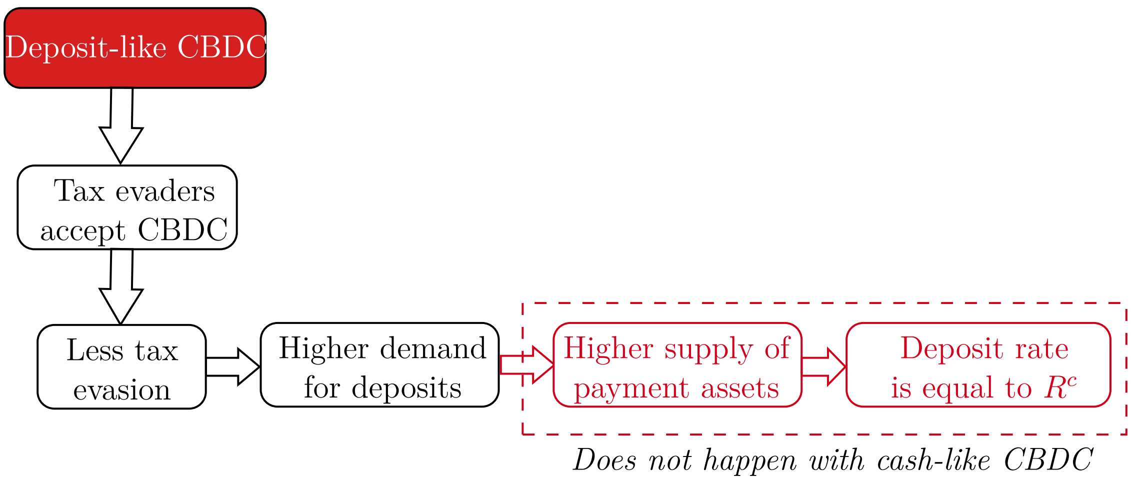


Figure 2.12: Effects of Introducing Deposit-like CBDC

2.5.2 Central Bank Net Revenue

Although aggregate welfare is higher with deposit-like CBDC, the central bank also has to pay higher interest. The goal of this section is to understand the impact of an interest-bearing CBDC on the central bank's net revenue. The central bank's budget constraint is given by the following.

$$\frac{M_{t+1} - M_t}{p_t} + \frac{C_{t+1} - C_t}{p_t} + \frac{(1 + R^b)B^c_t}{p_t} = \frac{R^c C_t}{p_t} + \frac{B^c_{t+1}}{p_t} + \mathcal{E}_t + \mathcal{T}^c_t. \tag{2.78}$$

Recall that $\mathcal{E}_t$ is the central bank's operating expenses. In practice, the operating expenses of a central bank can be large. For example, the operating expenses of the Bank of Canada is equal to 28% of its net interest income. Since the central bank uses the cash and CBDC it creates to purchase government bonds, we have $M_t + C_t = B^c_t$ for all t. Now, define $\tilde{c} = \frac{C_t}{p_t}$ and $\tilde{m} = \frac{M_t}{p_t}$. The central bank's net revenue can be written as

$$\Pi = (R^b - R^c)\tilde{c} + R^b \tilde{m} - \mathcal{E}. \tag{2.79}$$

Note that $\tilde{m} = 0$ in type-2 and type-3 equilibria, but $\tilde{m} > 0$ in type-4 equilibria. Note also that Π is equal to the transfer to the fiscal authority, $\mathcal{T}^c$. In principle, Π can be negative, in which case the central bank receives transfers from the fiscal authority.

The effect of introducing CBDC on central bank net revenue depends on the interest rate spread $R^b - R^c$ and the demand for central bank liabilities, $\tilde{c}$ and $\tilde{m}$. In a type-2 equilibrium, R^b is smaller compared to the benchmark (see Proposition 2.10). The demand for CBDC is also lower compared to the demand for cash in the benchmark, because CBDC is accepted by sellers who evade taxes, and there is less tax evasion. Hence, the central bank's net revenue is lower. In a type-3 equilibrium, the interest rate spread $R^b - R^c$ is smaller than in a type-2 equilibrium because R^c is higher. However, the demand for CBDC is also higher because CBDC is accepted by both sellers who evade taxes and (some) sellers who report truthfully.

Now, denote the deposit rate and government bond rate in the benchmark equilibrium as R^{bench} and $R^{b\prime}$, respectively. They satisfy the following relationship

$$\frac{1+\mu}{1+R^b} = \beta\theta + \frac{(1-\theta)(1+\mu)}{1+R^{bench}}. \tag{2.80}$$

Let the level of tax evasion in the benchmark equilibrium be γ^{bench}. Recall that $\mathcal{D} = \frac{(1+R^b)B_t}{p_t}$ is the total supply of government debt. I show that if the following assumption about the benchmark equilibrium holds, the central bank's net revenue is lower with deposit-like CBDC than with cash-like CBDC.

Assumption 2.5.1 $\frac{(R^b - R^{bench})\mathcal{D}}{1+R^d} < \frac{R^b(\alpha - 1 + \gamma^{bench})C}{\rho^m}$.

The assumption is satisfied as long as θ is sufficiently small. Recall that the central bank creates CBDC by purchasing government bonds with CBDC. In the extreme case where the central purchases all government bonds from the market, the supply of CBDC is equal to the total supply of government bonds, $\mathcal{D}/(1 + R^b)$. Then, the left-hand side of the inequality represents the highest possible central bank net revenue when $R^c = R^{bench}$. The right-hand side represents the central bank's net revenue in the benchmark. Intuitively, Assumption 2.5.1 ensures that in a type-3 equilibrium, the increase in demand for CBDC does not offset the decrease in the interest rate spread $R^b - R^c$.

Proposition 2.16 *Suppose Assumption* 2.5.1 *holds. Then, introducing either deposit-like CBDC or cash-like CBDC will decrease central bank net revenue. Furthermore, central bank net revenue will be lower with deposit-like CBDC than with cash-like CBDC.*
Proof: see Appendix A.2.

If the decrease in net revenue is significant, the central bank may not be able to cover its own expense and may require transfers from the fiscal authority. Such transfers may be feasible because the reduction in tax evasion increases the fiscal authority's tax revenue. However, relying on the fiscal authority's transfers may undermine the independence of the central bank.

2.6 Conclusion

The goal of this chapter is to study the implications of tax evasion for the optimal design of central bank digital currencies (CBDC). To accomplish this goal, I incorporate an audit game between taxpayers and the fiscal authority into a general equilibrium model. As a benchmark, I consider a scenario where only cash and bank deposits are available as payment. Bank deposits have a higher return than cash, but it is easier to conceal cash from the fiscal authority. I find that under the optimal tax schedule, an increase in inflation prompts agents to substitute away from cash, but this also lowers the fiscal authority's incentive to audit agents. In equilibrium, the decrease in tax enforcement leads to more agents evading taxes and hence a higher demand for cash.

When CBDC is introduced as a new payment instrument, the effect on tax evasion depends crucially on the degree of anonymity of CBDC, which determines the probability of the income received in CBDC being observed by the government. I find that as long as CBDC offers less anonymity than cash, introducing CBDC will decrease tax evasion. However, if CBDC is cash-like in the sense that it still offers relatively high anonymity but low interest rate, then introducing CBDC will decrease the output from not only agents who evade taxes but also agents who report their income truthfully. If CBDC is instead deposit-like in the sense that it offers low anonymity but high interest rate, then introducing CBDC will increase output and aggregate welfare. Furthermore, introducing deposit-like CBDC needs not increase the funding costs of private banks or decrease bank lending and investment. However, paying the high interest rate on CBDC will decrease the central bank's net interest revenue, which may jeopardize the central bank's independence.

To emphasize the mechanisms through which tax evasion affects payment choice and CBDC design, I abstract from several important issues that could be addressed in future research. First, in this paper, all agents are assumed to be ex ante homogeneous. The only uncertainty in the economy comes from search friction, and it only affects agents who are risk-neutral. These simplifying assumptions remove the income redistribution concerns when I design the tax schedule. Income redistribution is certainly an important issue, and CBDC may be a useful tool in this regard as well (Davoodalhosseini, 2018). Second, in this paper, there are no concerns of privacy other than tax evaders trying to conceal their income from the government. In practice, there are many reasons why individuals may want to hide their transactions from other individuals and the government (Kahn et al., 2020). The origins of such privacy concerns will likely determine the type(s) of transactions where CBDC will be accepted. This in turn can have important implications for tax compliance and the central bank's revenue. Finally, it has been argued that CBDC may help promote financial inclusion (Lagarde, 2018). Specifically, individuals may prefer cash not because they want to evade taxes, but because the transaction costs are lower compared to bank deposits and bank credit (Aït Lahcen and Gomis-Porqueras, 2019). If CBDC can be easy and cheap to use, it may promote both transaction efficiency and tax compliance.

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Chapter 3

Liquidity and Private Information in Asset Markets

3.1 Introduction

In asset markets, liquid assets that sellers possess before trading play an important role in determining market outcomes. For example, a firm issuing equity to fund a new project may choose to offer investors fewer shares at a (potentially) higher price if it already has some liquid assets on hand. Conversely, a hedge fund facing sudden redemptions during a financial crisis may deplete its liquidity reserves and be forced to sell large quantities of assets at a loss. However, it is not well understood how the liquidity possessed by sellers interacts with information friction in equilibrium. Specifically, how does the seller's liquidity affect market equilibrium when asset quality is the seller's private information? How will the equilibrium change if sellers' valuations of liquid assets are also private information? What implications can be derived concerning government interventions during crises, such as central bank lending and government asset purchases?

To answer these questions, I build a model where agents can choose to hold a portfolio of fiat money and other assets, which I assume to be one-period lived Lucas trees. After choosing their portfolios, some agents receive an opportunity to consume a good that can only be purchased with money. These agents have the incentive to sell their assets for money. Because agents' preferences over the consumption good are assumed to be concave, there exists an optimal amount of after-trading money holdings. This means that the more money asset sellers have before trading, the fewer assets they need to sell in the asset market. Asset quality can be either low or high, and this information is privately held by asset owners. Asset sellers' demand for money may also be private information: specifically, some sellers may have a greater demand for the consumption good and hence a greater need for money, but this cannot be directly observed by asset buyers.

There are two main findings. First, as a benchmark, I assume that all sellers have the same demand for the consumption good, and asset quality is the only source of private information. I find that the equilibrium depends on sellers' money holdings. When sellers' money holdings are large, sellers with high-quality assets lower the quantity they sell in order to signal asset quality. Although these sellers obtain less money, they sell at a higher price. When sellers' money holdings are small, the opportunity cost of signaling by lowering sale quantity is large. Sellers with high-quality assets choose to pool with other sellers and make the same offer to buyers. Although the asset price is lower for sellers with high-quality assets, they sell more assets and obtain more money.

Second, I assume that some sellers have a greater demand for money and it is their private information. I interpret this scenario as a flight-to-safety episode when some sellers become distressed and are pressured to liquidate their assets. I find that the equilibrium can be separating, pooling, or "semi-pooling" depending on sellers' money holdings and the number of distressed sellers. A semi-pooling equilibrium emerges when many sellers are distressed. In such an equilibrium, distressed sellers and undistressed sellers with low-quality assets pool and make the same offer to buyers. Undistressed sellers with high-quality assets separate from other sellers by charging a higher price and selling a smaller quantity. In contrast to the benchmark, under certain conditions, the semi-pooling equilibrium is the unique equilibrium even if sellers' money holdings are small. This finding shows that the effect of sellers' money holdings on the equilibrium depends crucially on the private information present in the asset market.

In related papers, Chang (2018), Guerrieri and Shimer (2018), and Williams (2016) also study two-dimensional private information in asset markets. Chang (2018) assumes indivisible assets and constructs a semi-pooling equilibrium to study the fire sale phenomenon, while Williams (2016) solves on a fully separating equilibrium with divisible assets. Guerrieri and Shimer (2018) focus on the welfare implications of multiple equilibria. In these papers, sellers' preferences over liquid assets are assumed to be heterogeneous but linear. As a result, if there are no market frictions, sellers will sell their entire stock of assets regardless of the liquidity they already have in their possession. In this chapter, the marginal value of liquidity decreases to zero as a seller obtains more liquidity. This setup allows me to show how sellers' liquidity holdings before trading and the private information in asset markets together determine the equilibrium.

To explore policy implications, I extend the model to include discount window lending and government asset purchases. In the case of discount window lending, I assume sellers can use their assets as collateral to borrow from the discount window before trading in the asset market. I show that there exists an equilibrium where only sellers with high-quality assets borrow from the discount window. This result is in contrast with the existing theoretical literature on discount window lending, which finds that the discount window lends mostly, if not entirely, to borrowers with low-quality assets.[1] In another related paper, Madison (2017) also allows sellers to choose between collateralized loans and asset sales, and finds that they are equivalent in separating equilibria. I find that as long as the discount rate is not too high, discount window loans benefit sellers with high-quality assets regardless of whether the equilibrium is separating or pooling.

The model in this chapter is based on Geromichalos and Herrenbrueck (2016), who extend the Lagos and Wright (2005) framework by introducing an over-the-counter asset market to allow agents to rebalance their portfolios.[2] The closest work that studies private information in asset markets is Madison (2019), who also builds on Geromichalos and Herrenbrueck (2016). The main difference is that Madison (2019) focuses on the effects of the distribution of asset quality on the market equilibrium, while this chapter focuses on the effects of sellers' holdings of liquid assets. In another closely related paper, Rocheteau (2011) shows that as long as the supply of bonds (akin to fiat money in this chapter) is not too small, the market equilibrium is separating. This echoes the finding in this chapter that the equilibrium in the asset market is separating when sellers' money holdings are large. However, Rocheteau (2011) only provides sufficient conditions for separating equilibria to exist, while this chapter derives sufficient and necessary conditions for the equilibrium to be separating or pooling. Papers that also study the endogenous selection between pooling and separating equilibria include Rocheteau (2008), Madison (2017), and Bajaj (2018). Similar to Madison (2019), they focus solely on the effects of the distribution of asset quality.

Finally, the chapter belongs to the vast literature on private information and signaling spurred by Akerlof (1970) and Leland and Pyle (1977). Some recent work that also studies private information in asset markets includes Eisfeldt (2004), Kurlat (2013), Guerrieri and Shimer (2014), Chiu and Koeppl (2016), and Choi (2018). Eisfeldt (2004), Kurlat (2013), and Chiu and Koeppl (2016) restrict their attention to pooling equilibria by

[1] See for example Ennis and Weinberg (2013), Ennis (2019), Gauthier et al. (2015), Li et al. (2016) and Gorton and Ordoñez (2016).

[2] See also Geromichalos et al. (2016), Herrenbrueck and Geromichalos (2017), Geromichalos and Herrenbrueck (2017), and Herrenbrueck (2019). Other papers that explore the idea of rebalancing asset portfolios before consumption include Kocherlakota (2003), Boel and Camera (2006), Berentsen et al. (2007), Berentsen and Waller (2011), Li and Li (2013), and Jacquet (2018). None of these papers study the private information problem.

requiring agents to trade at one price or not trade at all. Guerrieri and Shimer (2014) focus on separating equilibria and show that asset markets shut down when there are worthless assets in the market. I show that this needs not be the case if sellers are allowed to pool. Lastly, Choi (2018) assumes that asset buyers do not inherit sellers' private information and have to learn it themselves. Instantaneous learning is found to be welfare-maximizing, and public disclosure of private information is not always welfare-improving.[3]

The chapter is organized as follows. Section 3.2 describes the model environment. Section 3.3 solves the equilibrium. Section 3.4 studies policy implications. Section 3.5 concludes the chapter.

3.2 Model Environment

Time is discrete and continues forever. Each period is divided into three subperiods: the asset market (AM), the goods market (GM), and the centralized market (CM). There is measure one of consumers and producers. In the GM, there is a GM good that is consumed by consumers, but it can only be produced by producers. In the CM, there is a CM good that can be produced and consumed by all agents. The CM good also serves as the numeraire. A consumer's instantaneous utility is given by

$$\eta_t u(g_t) + c_t, \tag{3.1}$$

where g_t and c_t are the consumption of the GM good and the CM good, respectively. I assume η_t is stochastic. Specifically,

$$\eta_t = \begin{cases} 0, \text{ with probability } 1-\alpha; \\ 1, \text{ with probability } \alpha(1-\pi); \\ \eta^d \geq 1, \text{ with probability } \alpha\pi. \end{cases} \tag{3.2}$$

If $\eta_t = 0$, a consumer does not derive utility from the DM good and is referred to as a "non-shopper". If $\eta_t > 0$, a consumer derives utility from the DM good and is referred to as a "shopper". In particular, if $\eta^d > 1$, then some shoppers have a greater demand for

[3]Other work that also studies private information in asset markets includes Williamson and Wright (1994), Li et al. (2012), Golosov et al. (2014), Camargo and Lester (2014), Chari et al. (2014), Carapella and Williamson (2015), Lauermann and Wolinsky (2016), and Ozdenoren et al. (2019). A recent paper by Cai and Dong (2020) studies the effects of adverse selection on the secular migration of asset trading from centralized markets to decentralized markets.

the DM good than other shoppers. The realization of η_t is *i.i.d.* across consumers and time. I assume that $u'(.) > 0$, $u''(.) < 0$, $u'(0) > 1$, and $u'(g^*) = 1$ for some $g^* < \infty$. I also assume $g_t \geq 0$, but c_t can be negative, in which case it is interpreted as production in the CM: one unit of labor in the CM can be turned into one unit of CM good. Next, the instantaneous utility of a producer is given by

$$-h_t + X_t, \tag{3.3}$$

where h_t is the amount of labor supplied in the GM, and X_t is the consumption of the CM good. I assume one unit of labor in the GM can be turned into one unit of the GM good. Neither goods can be carried across periods. All agents discount future utility using $\beta \in (0, 1)$.

There are two types of assets in the economy: (fiat) money and perfectly divisible real assets. Money is issued by a government, and the real assets are endowed to consumers in each CM. In the CM of the next period, each unit of assets produces a dividend of δ units of CM good before depreciating by 100%. I assume δ is stochastic. I also follow Plantin (2009)'s approach and assume that by holding the real assets, consumers learn their quality.[4] Specifically, at the beginning of the AM, with probability Δ, a consumer learns that her real assets have low quality, and each unit of the assets will produce $\delta_l > 0$ units of the CM good. With probability $1 - \Delta$, a consumer learns that her real assets have high quality, and each unit will produce $\delta_h > \delta_l$ units of the CM good. δ is a consumer's private information and cannot be observed by other agents. The realization of δ is independent across consumers and is independent of the realization of η_t.

I assume agents are anonymous in all three subperiods. Therefore, a medium of exchange is necessary for the trade between shoppers and producers in the GM. I assume producers only accept money for payments. Hence, shoppers may want to sell their real assets to non-shoppers for money. In the AM, after consumers learn η_t and the quality of their assets, a market opens for asset trade. A seller is randomly matched with a buyer. Once matched, the seller makes take-it-or-leave-it offer to the buyer. The offer consists of a unit price, ψ, and a quantity for sale, s. If an offer is accepted, the seller receives ψs units of money, and the buyer receives s units of assets. To simplify the analysis, I assume an asset seller is matched with a buyer with probability one.[5]

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[4] Plantin (2009) shows that this assumption is of particular relevance to assets like collateralized debt obligations and privately placed debt, which are securities sold to selected investors and are bundled with future access to privileged information about the assets. Investors may not be able to fully diversify their portfolios because these assets are not sold publicly, and the access to privileged information creates private information. Similar assumptions can be found in Rocheteau (2011) and Madison (2017, 2019).

[5] This assumption requires α, the proportion of shoppers, to be less than 1/2.

In the GM, a shopper is matched with a producer with probability one, and shoppers make take-it-or-leave-it offers to producers. Next, in the CM, agents produce, trade, and consume the CM good. The real assets are also traded in the CM. Lastly, let M_t denote the supply of money.

$$M_{t+1} = (1+\mu)M_t, \tag{3.4}$$

where μ is the money growth rate. I assume $\mu > \beta - 1$. Money is injected to (or withdrawn from) the economy by the government via a lump-sum transfer (or tax) in each CM.

3.3 Equilibrium

In this section, I first study the case where $\eta^d = 1$ and all shoppers have the same demand for the GM good. Next, I solve the case where $\eta^d > 1$ and some shoppers have a greater demand for the GM good. I consider in particular the possibility that η_t is shoppers' private information and focus on a two-dimensional private information problem in the AM. Throughout this section, I restrict my attention to stationary equilibria where all real variables stay constant.

3.3.1 Equilibrium with One-dimensional Private Information

It is convenient to start with the goods market (GM). Since shoppers make take-it-or-leave-it offers to producers, the price of the GM good (in terms of the CM good) is one. Denote the amount of real balances carried by a shopper as $\tilde{z}$. As is standard in the Lagos and Wright (2005) models, shoppers' value function in the CM is linear in $\tilde{z}$. Hence, the cost of purchasing g units of GM good is g. The shoppers solve the following problem

$$\max_{g} \quad u(g) - g \quad \text{s.t. } g \leq \tilde{z}, \tag{3.5}$$

where g represents the GM good consumed. Let $g(\tilde{z})$ be the solution to the above problem. It is easy to see that $g(\tilde{z}) = \min\{\tilde{z}, g^*\}$ where $u'(g^*) = 1$. That is, shoppers either spend all money if it is not enough to purchase the efficient amount of consumption (g^*), or consume g^* if $\tilde{z} \geq g^*$.

Next, let z_b denote the real balances a non-shopper has at the beginning of the AM. Let z_s and a denote the real balances and the real assets held by a shopper at the beginning of the AM, respectively. If $z_s \geq g^*$, shoppers have enough real balances to consume efficiently in the GM. If $z_s < g^*$, shoppers may want to sell their real assets for money. Non-shoppers are willing to purchase assets because they have no use for money. In what follows, I assume $z_s < g^*$.

Now, consider a seller with asset quality δ_j offering (ψ_j, s_j) where $j \in \{l, h\}$ to her matched buyer. Recall that ψ is the unit price of the real assets (in real term), and s is the quantity for sale. If the offer is accepted by the buyer, the seller's surplus is given by

$$u(z_s + \psi_j s_j) - u(z_s) - \delta_j s_j \quad \text{s.t. } s_j \leq a \text{ and } \psi_j s_j \leq z_b. \tag{3.6}$$

In words, (3.6) says that the seller obtains $\psi_j s_j$ units of money, which allows her to consume $\psi_j s_j$ units more GM goods in the following GM.[6] In exchange, she gives up s_j units of assets, which will produce $\delta_j s_j$ units of dividends in the CM. The offer (ψ_j, s_j) must be feasible given z_b and a. Next, the surplus of a buyer receiving an offer (ψ, s) is given by

$$\left[s \sum_{j=l,h} \gamma(j|\psi, s)\delta_j - \psi s \right] \mathbb{1}(\psi, s). \tag{3.7}$$

In words, (3.7) says that if the buyer accepts the offer, she receives s units of assets, which she expects to produce $s \sum_{j=l,h} \gamma(j|\psi, s)\delta_j$ units of dividends in the CM. In exchange, she hands over ψs units of money to the seller. $\gamma(j|\psi, s)$ is the buyer's belief about the probability of the quality being δ_j given the offer (ψ, s). $\mathbb{1}(\psi, s)$ is the buyer's acceptance rule. $\mathbb{1}(\psi, s) = 1$ if the buyer accepts the offer (ψ, s), and $\mathbb{1}(\psi, s) = 0$ if she does not. Now, I define the equilibrium in the AM.

Definition 3.1 *The equilibrium in the AM is a set of offers* $\{(\psi_l, s_l), (\psi_h, s_h)\}$*, an acceptance rule* $\mathbb{1}(\psi, s)$*, and a belief function* $\gamma(j|\psi, s)$ *such that*
(1) *Conditional on* $\gamma(j|\psi, s)$*, the strategies* (ψ_l, s_l)*,* (ψ_h, s_h)*, and* $\mathbb{1}(\psi, s)$ *are sequentially rational;*
(2) $\gamma(j|\psi, s)$ *is derived from Bayes' rule whenever it is possible; and*
(3) *The equilibrium is undefeated.*

Conditions (1) and (2) define a Perfect Bayesian Equilibrium (PBE), but they put no restrictions on beliefs about out-of-equilibrium actions. As a result, conditions (1) and (2) are satisfied by a continuum of PBE. To obtain meaningful results, in condition (3), I apply the undefeated equilibrium refinement by Mailath et al. (1993). A formal definition of the undefeated equilibrium can be found in Appendix B.1. Informally, a

[6] I show in Appendix B.2 that in equilibrium, $z_s + \psi_j s_j \leq g^*$ for all $z_s < g^*$.

PBE ("PBE-1") is defeated by another PBE ("PBE-2") if (1) there exists an offer that is not made by any sellers in PBE-1 but is made by a set of sellers (denoted as K) in PBE-2; and (2) all sellers in K must be weakly better off in PBE-2 while some sellers in K are strictly better off. In Appendix B.2, I show that to find undefeated equilibria, one needs only check if high-quality sellers have the incentive to deviate unilaterally ($K = \{h\}$) or if both types of seller have the incentive to deviate ($K = \{l, h\}$). This is because low-quality sellers do not benefit from deviating unilaterally. The undefeated equilibrium refinement is therefore equivalent to selecting the PBE that maximizes the surplus of high-quality sellers.

In similar models of asset markets, the Intuitive Criterion (Cho and Kreps, 1987) is sometimes used as a refinement method (see Rocheteau (2011) and Madison (2019)). However, the undefeated equilibrium refinement has several advantages in this environment. First, both refinement methods select the least inefficient separating equilibria among all separating equilibria (Riley, 1979). However, the undefeated equilibrium refinement also allows pooling equilibria if they are Pareto-superior. For example, if $\delta_l = 0$, the Intuitive Criterion predicts a complete shutdown of trading. However, both types of sellers may prefer a pooling equilibrium if the proportion of low-quality assets (Δ) is small. Second, as mentioned above, the undefeated equilibrium refinement is equivalent to selecting the PBE that maximizes the surplus of high-quality sellers. This greatly simplifies the implementation of the refinement.

The equilibrium is solved in Appendix B.2. The next proposition characterizes the equilibrium.

Proposition 3.1 *There exists a cutoff value $z_s' \in [0, g^*)$ such that*
(1) *For all $z_s < z_s'$, the equilibrium is pooling and $(\psi_l, s_l) = (\psi_h, s_h) = (\psi_p, s_p)$;*
(2) *For all $z_s \geq z_s'$, the equilibrium is separating and $(\psi_l, s_l) \neq (\psi_h, s_h)$;*
(3) *$\psi_l < \psi_p < \psi_h$, $s_l > s_p > s_h$ and $\psi_l s_l \geq \psi_p s_p > \psi_h s_h$. And z_s' is decreasing in z_b and a.*
Proof: see Appendix B.2.

The proposition says that there exists a threshold z_s' such that, if z_s is smaller than z_s', low-quality sellers and high-quality sellers offer the same price and sell the same quantity. If z_s is larger than z_s', high-quality sellers signal their quality to buyers by charging a higher price and selling a smaller quantity. The equilibrium depends on sellers' liquidity needs because high-quality sellers face different costs in pooling and separating equilibria. In a pooling equilibrium, the sale price is discounted for high-quality sellers because buyers are only willing to pay for the average quality in the market. In a separating equilibrium, the sale quantity is low for high-quality sellers in order to prevent low-quality sellers from mimicking high-quality sellers' offers. When sellers' liquidity needs are large (i.e., z_s is small), high-quality sellers prioritize obtaining more liquidity and accept the price discount. When sellers' liquidity needs are small (i.e., z_s is large), high-quality sellers can afford to lower the quantity they sell in order to charge a higher price.

Another finding is that the threshold z_s' depends on z_b and a. The reason is that the asset trade in the AM may be constrained by either buyer's real balances or seller's real assets. Then, a larger z_b or a can increase the trading volume for both types of assets in a separating equilibrium. This is because to signal quality, high-quality sellers only have to sell less relative to the quantities low-quality sellers sell. Hence, high-quality sellers can sell more if low-quality sellers sell more. In other words, for high-quality sellers, the signaling cost is decreasing in z_b and a.

Next, I turn to consumers' choices of z and a in the CM to bring to the next period. I restrict my attention to symmetric solutions where all consumers choose to carry the same z and a. Let $V(\hat{z}, \hat{a})$ be the expected utility for a consumer choosing portfolio $(\hat{z}, \hat{a})$.

$$V(\hat{z}, \hat{a}) = \alpha[\Delta(u(g_l) - g_l) + (1 - \Delta)(u(g_h) - g_h)] + \hat{z} + \bar{\delta}\hat{a} \tag{3.8}$$

First, $(g_l, g_h) = (\hat{z} + \psi_l s_l, \hat{z} + \psi_h s_h)$ represents the GM consumption of shoppers with low-quality and high-quality assets, respectively. In words, (3.8) says that with probability α, a consumer is a shopper, and her surplus from consuming in the GM is equal to $\Delta(u(g_l) - g_l) + (1 - \Delta)(u(g_h) - g_h)$. If the consumer is a non-shopper, the expected value of the portfolio $(\hat{z}, \hat{a})$ is $\hat{z} + \bar{\delta}\hat{a}$.

Now, denote the price of money and the real assets as ϕ_t and κ_t, respectively. In a stationary equilibrium, $\kappa_t = \kappa$, and the (gross) inflation rate $\frac{\phi_t}{\phi_{t+1}}$ is equal to $1 + \mu$ where μ is the money growth rate. Consumers in CM solve the following problem

$$\max_{\hat{z}, \hat{a}} \quad -(\mu + 1)\hat{z} - \kappa\hat{a} + \beta V(\hat{z}, \hat{a}). \tag{3.9}$$

In Appendix B.2 (Proposition B.5), I show that there exist $\mu_1 < \mu_2$ such that the AM equilibrium is separating if $\mu \leq \mu_1$ and pooling if $\mu \geq \mu_2$. If $\mu_1 < \mu < \mu_2$, then under certain conditions, no symmetric solutions exist. The reason is that when $\mu_1 < \mu < \mu_2$, asset trade in the AM is constrained by buyer's money, and the optimal choices of $\hat{z}$ and $\hat{a}$ depend on other consumers' choices of $\hat{z}$. Conditional on other consumers expecting the AM equilibrium to be pooling (or separating), a consumer may deviate to choose a different portfolio and make a separating (or pooling) offer in the AM. When such deviations are profitable, there are no symmetric solutions.

3.3.2 Equilibrium with Two-dimensional Private Information

In this section, I assume $\eta^d > 1$ so some asset sellers have a greater demand for the GM good. I interpret $\eta_t = \eta^d$ as an unforeseen liquidity shock that happens with probability π and increases a seller's demand for money. The liquidity shock is akin to a flight-to-safety episode that often happens during runs on financial institutions or crises in repo markets (Shleifer and Vishny, 2011; Martin et al., 2014). In this context, I refer to sellers with $\eta_t = \eta^d$ as distressed sellers and sellers with $\eta_t = 1$ as undistressed sellers. I denote π as the market distress level.

The sellers' and buyers' problems are similar to the benchmark case in Section 3.3.1. There are now four types of sellers: undistressed (and distressed) low-quality sellers and undistressed (and distressed) high-quality sellers. Denote the offer of a seller as (ψ_j^i, s_j^i) where the subscript $j \in \{l, h\}$ denotes the quality of her assets and the superscript $i \in \{u, d\}$ denotes the seller's distress status. Let $\eta^u = 1$. If the offer is accepted by a buyer, the seller's surplus is

$$\eta^i u(z_s + \psi_j^i s_j^i) - u(z_s) - \delta_j s_j^i \quad \text{s.t. } s_j^i \leq a \text{ and } \psi_j^i s_j^i \leq z_b. \tag{3.10}$$

The surplus of a buyer receiving an offer (ψ, s) is given by

$$\left[s \sum_{j=l,h} \gamma(i, j|\psi, s)\delta_j - \psi s \right] \mathbb{1}(\psi, s), \tag{3.11}$$

Similar to the benchmark case, $\gamma(i, j|\psi, s)$ represents the buyer's belief about the probability of the seller being type (i, j) given the offer (ψ, s). And $\mathbb{1}(\psi, s)$ represents the buyer's acceptance rule. $\mathbb{1}(\psi, s) = 1$ if the buyer accepts the offer (ψ, s), and $\mathbb{1}(\psi, s) = 0$ if she does not. Now, I define the equilibrium in the AM.

Definition 3.2 *The equilibrium in the AM is a set of offers* $\{(\psi_l^u, s_l^u), (\psi_h^u, s_h^u), (\psi_l^d, s_l^d), (\psi_h^d, s_h^d)\}$, *an acceptance rule* $\mathbb{1}(\psi, s)$, *and a belief function* $\gamma(i, j|\psi, s)$ *such that*
(1) *Conditional on* $\gamma(i, j|\psi, s)$, *the strategies* (ψ_l^u, s_l^u), (ψ_h^u, s_h^u), (ψ_l^d, s_l^d), (ψ_h^d, s_h^d), *and* $\mathbb{1}(\psi, s)$ *are sequentially rational;*
(2) $\gamma(i, j|\psi, s)$ *is derived from Bayes' rule whenever it is possible; and*
(3) *The equilibrium is undefeated.*

The liquidity shock brings two changes to the AM equilibrium. First, since there are now four types of sellers, the equilibrium is not restricted to be either pooling or separating. In particular, there may exist equilibria where some sellers pool while others separate. Second, the AM equilibrium depends crucially on whether a seller's distress status can be observed by other agents in the AM or not, and in what follows, I discuss these two cases separately.

I. Distress Status Revealed

First, suppose that sellers' distress statuses are revealed to all agents. Recall that Proposition 3.1 shows that if $z_s < z_s'$, sellers would pool in the AM without the liquidity shock, and if $z_s \geq z_s'$, sellers would separate. The next proposition shows how the liquidity shock affects the AM equilibrium.

Proposition 3.2 *Assume η^d is large and $\delta_h/\bar{\delta} < 2 - \delta_l/\delta_h$.*
(1) *Suppose $z_s < z_s'$. All sellers pool in equilibrium.*
(2) *Suppose $z_s \geq z_s'$. Distressed sellers pool while undistressed sellers separate in equilibrium.*
Proof: see Appendix B.3.

First, I explain the assumptions in Proposition 3.2. If η^d is large, a sufficient condition for distressed sellers to pool is $\delta_h/\bar{\delta} < 2 - \delta_l/\delta_h$. To see why, first note that when η^d is large, compared to a separating equilibrium, the extra amount of liquidity distressed high-quality sellers can obtain from pooling is small. This is because, first, asset trade in a pooling equilibrium can be constrained by sellers' assets or buyers' money. Second, in a separating equilibrium, the increase in liquidity needs makes it less attractive for distressed low-quality sellers to mimic high-quality sellers' offer. This allows distressed high-quality sellers to sell more in a separating equilibrium. Now, if $\delta_h/\bar{\delta}$ is large, so is the price discount in a pooling equilibrium. In such case, the extra amount of liquidity sellers obtain from pooling may not be large enough to justify the price discount. As a result, distressed sellers may separate after a large liquidity shock even if their money holdings are small.

Next, in case (1), if sellers pool without the liquidity shock (i.e., $z_s < z_s'$), they continue to pool after receiving the shock. In addition, although distressed sellers demand more liquidity than undistressed sellers, they make the same pooling offer because asset trade is constrained either by buyers' money or by sellers' assets. In fact, when the AM equilibrium is pooling, asset trade is constrained with or without the liquidity shock. If asset trade was not constrained, consumers could increase their expected utility by carrying less money from the CM. Hence, the liquidity shock does not affect the AM equilibrium if $z_s < z_s'$.

Now, suppose sellers separate without the liquidity shock (i.e., $z_s \geq z_s'$). Then distressed sellers choose to pool, while undistressed sellers remain separated in equilibrium.

In this case, distressed sellers do not need to prevent undistressed low-quality sellers from mimicking their offers. This is because if undistressed low-quality sellers chose to deviate, they would be identified by buyers as low-quality sellers, since undistressed high-quality sellers do not have the incentive to deviate. Therefore, when sellers' distress statuses are revealed, the equilibrium price and sale quantity for undistressed sellers are unaffected by either the distressed level of other sellers (η^d) or the distress level of the market (π). However, this is not the case if sellers' distress statuses are not revealed.

II. Distress Status not Revealed

Suppose buyers cannot tell if a seller is distressed or not. Now there exist two dimensions of private information: sellers' distress statuses and the quality of their assets. In Section 3.3.1, I show that a PBE is undefeated if there does not exist another PBE where high-quality sellers are strictly better off. This result can be generalized to the two-dimensional case. Specifically, a PBE is undefeated if there does not exist another PBE that satisfies one of the following three conditions.

(1) All high-quality sellers are weakly better off while some high-quality sellers (distressed and/or undistressed) are strictly better off.

(2) Distressed and undistressed high-quality sellers do not pool. Undistressed high-quality sellers are strictly better off while distressed high-quality sellers are strictly worse off.

(3) Distressed and undistressed high-quality sellers do not pool. Distressed high-quality sellers are strictly better off while undistressed high-quality sellers are strictly worse off.

Similar to the one-dimensional case, a PBE is defeated if all high-quality sellers are better off in an alternative PBE. However, unlike the one-dimensional case, a PBE can be defeated by another PBE where some high-quality sellers are strictly worse off. For example, suppose that for some PBE, condition (2) is satisfied. Then, undistressed high-quality sellers have the incentive to deviate even though distressed high-quality sellers are strictly worse off in the alternative PBE. Now, suppose instead that in some PBE, distressed high-quality sellers are strictly better off while undistressed high-quality sellers are strictly worse off, and all high-quality sellers pool (and potentially they pool with other low-quality sellers as well). This means that distressed high-quality sellers can only profit from a deviation if undistressed high-quality sellers also deviate. Because the latter do not have incentive to deviate, the alternative PBE does not defeat the original PBE.

Now, I describe the AM equilibrium. First, I define a "semi-pooling" equilibrium.

Definition 3.3 *A semi-pooling equilibrium is such that distressed sellers pool with undistressed low-quality sellers while undistressed high-quality sellers separate from other sellers.*

The next proposition shows that the equilibrium now depends on market distress level, π, which is equal to the proportion of distressed sellers. In addition, unlike when the distress statuses are revealed, undistressed sellers may also be affected by the liquidity shock in equilibrium.

Proposition 3.3 *Assume η^d is large and $\delta_h/\bar{\delta} < 2 - \delta_l/\delta_h < \bar{\delta}/\delta_l$.*
(1) *Suppose $z_s < z'_s$. If π is small, all sellers pool in equilibrium. If π is large and a is small, the equilibrium is semi-pooling.*
(2) *Suppose $z_s \geq z'_s$. If π is small and z_b and a are large, distressed sellers pool while undistressed sellers separate in equilibrium. If π is large and a is small, the equilibrium is semi-pooling.*
Proof: see Appendix B.3.

When market distress level is low, the equilibrium is identical to when sellers' distress statuses are revealed. Specifically, if the amount of money sellers have is small ($z_s < z'_s$), all sellers pool, and the liquidity shock has no effect on the equilibrium. If $z_s \geq z'_s$ and z_b and a are large, distressed sellers pool while undistressed sellers separate. In the latter case, if either z_b or a is small, undistressed low-quality sellers may have the incentive to mimic distressed sellers' pooling offer. Because sellers' distress statuses are not revealed, buyers cannot detect such deviation, and distressed sellers may be forced to pool with undistressed low-quality sellers. I discuss this case in the proof of Proposition 3.3.

When market distress level is high, there exists a semi-pooling equilibrium where distressed sellers pool with undistressed low-quality sellers. To see why such an equilibrium exists, first suppose that $z_s < z'_s$. Without the liquidity shock, high-quality sellers prefer to pool. However, when undistressed low-quality sellers pool with distressed sellers, the former are better off than they are in a separating equilibrium. This lowers undistressed high-quality sellers' signaling cost should they choose to separate from other sellers. Now, in a semi-pooling equilibrium, undistressed low-quality sellers' surplus increases with π. This means undistressed high-quality sellers will prefer to separate from other sellers if π is large. Second, suppose that $z_s \geq z'_s$. In this case, distressed sellers pool with undistressed low-quality sellers because when distress statuses are not revealed, the former cannot prevent the latter from mimicking the pooling offer. Lastly, if both π and a are large, there may be no undefeated equilibria. I discuss this case in the proof of Proposition 3.3.

III. Welfare

Part I and II show that the main implication of two dimensions of private information is the existence of a semi-pooling equilibrium. In this part, I compare sellers' surplus in the semi-pooling equilibrium when sellers' distress statuses are not revealed to their surplus when the distressed statuses are revealed. Recall that buyers' expected surplus is always zero because sellers make take-it-or-leave-it offers. Sellers' surplus is given by (3.10).

Proposition 3.4 *Assume that η^d is large, $\delta_h/\bar{\delta} < 2 - \delta_l/\delta_h < \bar{\delta}/\delta_l$, π is large, and a is small.*
(1) *If $z_s < z_s'$, in the semi-pooling equilibrium, distressed sellers and undistressed low-quality sellers are worse off while undistressed high-quality sellers better off.*
(2) *If $z_s \geq z_s'$, in the semi-pooling equilibrium, distressed sellers are worse off while undistressed sellers are better off.*
Proof: see Appendix B.3.

When $z_s < z_s'$, all sellers pool if the distress statuses are revealed. Distressed sellers are worse off in the semi-pooling equilibrium because the average asset quality of the pool of sellers, which is equal to $[\Delta\delta_l + (1-\Delta)\pi\delta_h]/[\Delta + (1-\Delta)\pi]$, is lower than the average asset quality of the market (i.e., the average quality when all sellers pool). This also means that undistressed low-quality sellers are worse off. Undistressed high-quality sellers are better off because the pooling of the distressed sellers and undistressed low-quality sellers lowers the signaling cost.

When $z_s \geq z_s'$ and the distress statuses are revealed, distressed sellers pool while undistressed sellers separate. Distressed sellers are worse off in the semi-pooling equilibrium because they have to pool with undistressed low-quality sellers, who are simultaneously better off because they now pool with high-quality sellers. Similar to the case where $z_s < z_s'$, undistressed high-quality sellers are better off because the signaling cost is lower in the semi-pooling equilibrium.

3.4 Policy Implications

In this section, I extend the model to study two policy experiments: discount window lending and government asset purchases.

3.4.1 The Discount Window and Discount Window Stigma

The discount window is a facility where the Federal Reserve makes (collateralized) loans to depository institutions. While there is a large literature that studies discount window lending,[7] much less attention has been paid to environments with private information. This has inspired a developing literature that studies discount window lending when the quality of the collateral is unobservable (Ennis and Weinberg, 2013; Gauthier et al., 2015; Gorton and Ordoñez, 2016; Li et al., 2016; Ennis, 2019). The literature finds that due to adverse selection, discount window borrowers are mostly, if not entirely, borrowers with low-quality assets. By removing these borrowers from the market, the discount window reduces information friction, but incurs a net loss. In this section, I show that under certain conditions, fully collateralized discount window loans can attract exclusively borrowers with high-quality assets, and the discount window can make a positive profit.

I assume there exists a discount window that opens at the beginning of the AM, after agents observe η_t and asset quality but before they trade assets. The (gross) lending rate of discount window loans is $R^D \geq 1$, and repayments are due in the CM of the same period. The loans must be fully collateralized so that borrowers with low-quality assets do not have the incentive to default. That is, the highest amount an agent with a units of real assets can borrow is $\delta_l a / R^D$. The assets pledged as collateral are held by the discount window until loan repayments are received. I also assume that borrowing from the discount window is *not* observable to other agents in the market. In practise, the information on discount window loans is disclosed with a two-year lag.

It is clear that asset buyers do not have the incentive to borrow from the discount window. Now, I set up sellers' problems. For simplicity, I assume $\eta^d = 1$ and asset trade is not constrained by z_b or a. Let m^D denote the amount of discount window loan (in real term), and let $(\psi_j^D s_j^D)$, $j \in \{l, h\}$, denote a seller's offer in the asset market. A seller solves

$$\max_{m^D, \psi_j^D, s_j^D} u(z_s + m^D + \psi_j^D s_j^D) - R^D m^D - \delta_j s_j^D \tag{3.12}$$

subject to buyers' beliefs about asset quality. Buyers' problem is the same as (3.7) and the equilibrium definition is similar to Definition 3.1. The next proposition shows that when sellers' money holdings are large (i.e.„ $z_s \geq z_s'$), the discount window can attract exclusively sellers with high-quality assets. I discuss the case where sellers' money holdings are small at the end of this section.

[7] See Ennis (2016) for a review of the discount window literature.

Proposition 3.5 *Suppose $z_s \geq z_s'$. There exists an $R^{D\prime} > 1$ such that*
(1) *If $R^D = 1$, low-quality sellers and high-quality sellers only borrow from the discount window.*
(2) *If $R^D \in (1, R^{D\prime})$, high-quality sellers borrow from the discount window and sell in the asset market, while low-quality sellers only sell in the asset market.*
(3) *If $R^D \geq R^{D\prime}$, low-quality sellers and high quality only sell in the asset market.*
Proof: see Appendix B.4.

I focus my attention on case (2), where the cost of discount window loans is strictly positive but not too large. In this case, only high-quality sellers borrow from the discount window. However, high-quality sellers do not borrow all the liquidity they need, as a portion of their real assets is sold later in the asset market. The higher R^D is, the less they borrow from the discount window and the more they sell in the asset market. Low-quality sellers do not mimic this strategy because, firstly, collateralized loans allow sellers to obtain liquidity while keeping their assets. Low-quality sellers do not benefit from keeping their assets because the market price is either equal to their assets' value (in a separating equilibrium) or above their assets' value (in a pooling equilibrium). However, for high-quality sellers, a combination of borrowing and selling allows them to retain the dividends of some of their assets and to charge a high price on the assets they sell. In equilibrium, the discount window increases high-quality sellers' utility while earning a net profit.[8]

These findings have novel implications on discount window stigma, which refers to depository institutions' reluctance to borrow from the discount window during financial crises.[9] Existing theoretical literature shows that for borrowers with low-quality assets, the discount window offers more favorable terms compared to the market.[10] Consequently, it is predicted that the discount window attracts mostly, if not entirely, borrowers with low-quality assets. Borrowing from the discount window therefore becomes a signal of low quality.

However, this mechanism is inconsistent with the Federal Reserve's discount window policies, because all discount window loans must be collateralized and the lending Reserve

[8]In a related paper, Ozdenoren et al. (2020) show that safe assets (akin to money in this chapter) can help reduce information friction when they are pooled with information-sensitive assets. In contrast, in this chapter, discount window loans reduce information friction by lowering high-quality sellers' needs to sell information-sensitive assets.

[9]For empirical evidence, see Furfine (2003), Armantier et al. (2015), Anbil (2018) and Vossmeyer (2019).

[10]In Ennis and Weinberg (2013) and Li et al. (2016), the discount rate is lower than the market rate for borrowers with low-quality assets. In Ennis (2019), the market rate contains a risk premium, but the discount rate does not. In Gorton and Ordoñez (2016), the discount window may accept bad assets as collateral.

Banks apply haircuts based on asset types. Also, there has been no report of default in discount window loans.[11] In this chapter, discount window loans are fully collateralized, and borrowers do not default. Discount window stigma does not appear in equilibrium: rather than avoiding discount window loans, high-quality sellers use them to lower the signaling cost.

Lastly, if sellers' money holdings and asset holdings are small, discount window lending may hurt aggregate welfare, which is measured by the total consumption in the GM.[12] In this case, sellers may pool in the asset market after borrowing from the discount window because their liquidity needs are still high. Since the discount window treats all assets as low-quality assets, compared to the pooling equilibrium in the benchmark model, sellers obtain less liquidity when they borrow from the discount window. Despite obtaining less liquidity, high-quality sellers are better off because they can keep the dividends of some of their assets. However, low-quality sellers are strictly worse off compared to when there is no discount window and the equilibrium is pooling. Because sellers bring less liquidity to the GM, the total consumption is lower.

3.4.2 Market Freeze and Government Asset Purchase

In 2007, the collapse of the housing market surprised investors and led to widespread market dysfunction (Gorton, 2008; Shleifer and Vishny, 2011). There was virtually no trading in the market of mortgages-backed securities and other asset-backed securities (Benmelech et al., 2019). Such market freezes happen frequently during financial crises (Benmelech and Bergman, 2018). The goal of this section is to study an unforeseen quality shock that makes the quality of low-quality assets, δ_l, zero. First, I show that depending on sellers' liquidity needs, this shock may or may not cause a market freeze. Second, because $\delta_l = 0$, collateralized lending is not feasible. I study how government asset purchases may unfreeze the market and improve welfare.

I assume the unforeseen quality shock happens at the beginning of the AM. It is common knowledge that the shock has happened, but asset quality is still private information. For simplicity, I assume $\eta^d = 1$ and asset trade is not constrained by z_b or a. If the AM equilibrium is separating, the trading volume of all assets (i.e., asset price times sale quantity) approaches zero as δ_l approaches zero. This is because the opportunity cost of selling assets is zero for low-quality sellers, who will then mimic any offers from high-quality sellers. However, this does not mean the market will freeze because sellers

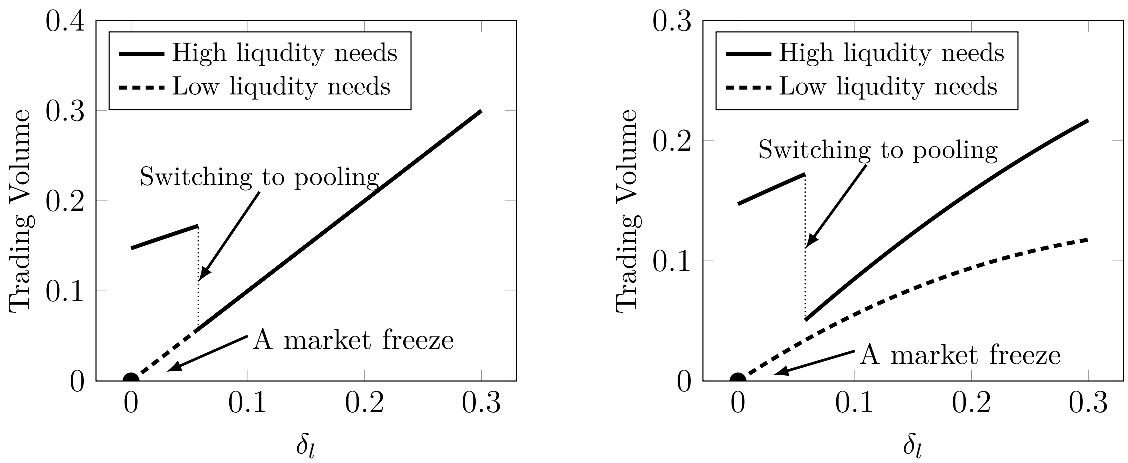
[11] Data on discount window loans: `https://www.federalreserve.gov/regreform/discount-window.htm`.

[12] A formal analysis can be found in the proof of Proposition 3.5 in Appendix B.4.

can choose to pool. The following proposition shows that as long as sellers' liquidity needs are sufficiently high, high-quality sellers prefer pooling to not selling at all.

Proposition 3.6 *There exists a* $z_s^{\diamond}$ *such that*
(1) *For all* $z_s \leq z_s^{\diamond}$*, there exists a* $\delta_l^{\diamond}$ *such that if* $\delta_l \in [0, \delta_l^{\diamond})$*, the equilibrium is pooling;*
(2) *For all* $z_s > z_s^{\diamond}$*, there exists a* $\delta_l^{\diamond\diamond}$ *such that if* $\delta_l \in (0, \delta_l^{\diamond\diamond})$*, the equilibrium is separating. If* $\delta_l = 0$*, a market freeze happens.*
Proof: see Appendix B.4.

The proposition shows that when sellers' liquidity needs are low, a market freeze happens when $\delta_l = 0$. This is because for high-quality sellers, the price discount in a pooling equilibrium is high relative to the benefit of obtaining extra liquidity. Figure 3.1 provides an illustration of the results.



(a) Low-quality Assets (b) High-quality Assets

Figure 3.1: Market Freezes

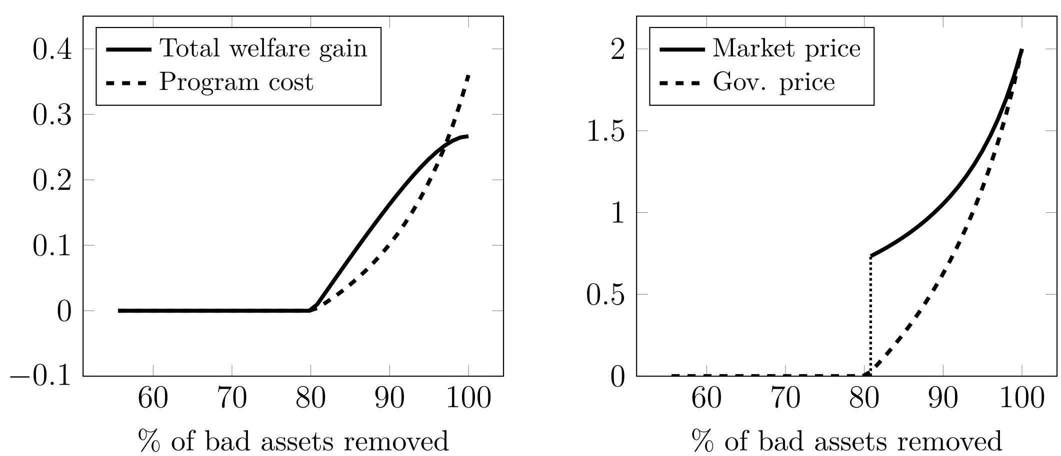
The findings in Proposition 3.6 indicate that if a government can remove (purchase) the bad assets from the market and improve the market price, it may be able to unfreeze the market. Now, suppose a market freeze happens and the government implements an asset purchase program. The program is financed by a lumpsum tax in the CM. The government makes two announcements: (1) the price at which it will purchase the real assets; and (2) the total quantity of real assets it will purchase. All sellers have an equal chance to contact the government and the contact is frictionless. If a seller chooses to sell to the government, she is required to sell all of her real assets so that sellers with bad assets are removed from the market. This increases the pooling price and may unfreeze the market. Sellers who choose not to sell to the government proceed to trade in the asset market.

In equilibrium, only bad assets are sold to the government. This is consistent with the findings in Philippon and Skreta (2012) and Tirole (2012). Hence, the government can unfreeze the market if it purchases enough bad assets. The government asset purchase has the following two properties.

Proposition 3.7 (1) *It is not optimal to purchase all bad assets from the market.*
(2) *The government can purchase assets at a price below the market price.*
Proof: see Appendix B.4.

I explain the first result with an example in the panel (a) of Figure 3.2. First, if not enough bad assets are removed from the market, the market stays frozen so the curves are flat. Once the government unfreezes the market, the cost of the program increases rapidly because removing bad assets increases the market price, and hence the government has to pay more to compete with the market. However, the marginal benefit of the program (the slope of the solid line) decreases as the trading in the asset market improves. As a result, there exists an optimal purchase quantity, and it is less than the quantity of bad assets in the economy.

Second, the government price is below the market price because sellers with bad assets sell more to the government than in the market. Due to the price discount in the pooling equilibrium, high-quality sellers in general do not sell all of their assets. However, sellers with bad assets sell all of their assets to the government. In equilibrium, these sellers are indifferent between selling less in the market for a higher price and selling more to the government for a lower price.



(a) Welfare Gain and Program Cost (b) Asset Prices

Figure 3.2: Welfare and Asset Prices

3.5 Conclusion

This chapter examines how asset sellers' private information and the liquid assets they possess before trading together affect asset market outcomes. In equilibrium, sellers' liquidity interacts with private information in the asset market and generates various types of equilibria. Specifically, when asset quality is the source of private information, the equilibrium is separating if sellers' liquidity holdings are large, and pooling if sellers' liquidity holdings are small. This is no longer true when another dimension of private information, i.e., sellers' valuations of liquid assets, is introduced to the environment. The equilibrium can be semi-pooling and feature sellers with high-quality assets separating from other sellers even when sellers liquidity holdings are small.

I also extend the model to study government interventions in the forms of discount window lending and asset purchases. I find that it is possible that only sellers with high-quality assets borrow from the discount window even though the discount window does not attempt to screen the borrowers. This shows that discount window stigma may not be the result of private information about quality in the asset market. I also find that after a negative quality shock causes the asset market to freeze, a government can unfreeze the market by purchasing bad assets. However, it is not optimal for the government purchase all bad assets from the market.

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Chapter 4

Signaling in Competitive Search Equilibrium

4.1 Introduction

Consider an asset seller with private knowledge that her assets have high quality. How does the seller signal this information to buyers? A large literature spurred by Leland and Pyle (1977) has studied partial retention of assets as a signal. Specifically, compared to sellers with low-quality assets, sellers with high-quality assets may sell a smaller quantity because their assets have a higher continuation value. Alternatively, sellers with high-quality assets may charge a higher price if it means to sell with a lower probability (e.g., illiquidity) (Guerrieri and Shimer, 2014). Illiquidity can also be a credible signal of quality because sellers with high-quality assets benefit more from keeping their assets compared to sellers with low-quality assets. Although both types of signals have been studied in the literature, how they differ from each other is not well understood. To asset sellers, what are the relative costs of signaling via illiquidity and partial retention? How does the use of illiquidity and partial retention as signals depend on the quality of assets? How is signaling affected by shocks to seller's liquidity demand or asset quality?

To answer these questions, I build a model to study both signals in a general framework. To allow illiquidity as a signal, I follow Guerrieri and Shimer (2014) to assume agents trade assets via competitive search. To allow partial retention as a signal, I deviate from Guerrieri and Shimer (2014) by assuming that assets are perfectly divisible. To sell assets, sellers post a price and a quantity for sale. The probability of a seller meeting a buyer (and vice versa) is determined by the buyer-to-seller ratio associated with each price and quantity. After paying an entry cost, buyers observe all offers and the associated buyer-to-seller ratios before deciding where to buy.

There are two types of perfectly divisible assets: fruit and trees. Fruit can be consumed at anytime, but trees cannot be consumed directly. Instead, trees will bear fruit sometime in the future. Fruit is meant to represent liquid assets that can be used to purchase consumption goods or fund investment, while trees cannot be used for those purposes. All agents are endowed with some fruit and trees. I assume only the owners of the trees can observe the quality, which is measured by the amount of fruit each unit of trees will eventually produce. There are two types of agents, early consumers and late consumers. Early consumers may want to sell their trees to late consumers for fruit so that they can consume immediately.

There are two main findings. First, all assets sellers (i.e., early consumers) except those with the lowest-quality trees use both types of signals in equilibrium. Specifically, I find that both trading probability and trading volume (which is the amount of fruit exchanged in trade) decrease with the quality of seller's trees. In a related paper, Williams (2016) also studies both signals in a similar framework. However, he shows that sellers do not use partial retention as a signal unless other information frictions are introduced. This is because in Williams (2016), sellers' preferences over fruit are assumed to be linear. If a seller is able to signal through illiquidity the quality of her trees, it is optimal to sell all of her trees. In this paper, asset sellers' preferences over fruit are strictly concave. Compared to using only illiquidity as a signal, sellers benefit from lowering trading volume in exchange for higher trading probability. In other words, there exists a trade-off between illiquidity and partial retention that is absent in models with linear preferences.

Second, I find that how the two signals are used depends on the quality of sellers' trees. Specifically, among sellers with high quality trees, those with relatively higher-quality trees sell marginally fewer trees but with significantly lower probability. In comparison, among sellers with low quality trees, those with relatively higher-quality trees sell significantly fewer trees but with only marginally lower probability. This is because for sellers with low-quality trees, trading volume is high and the marginal value of fruit is low, so they prefer lowering trading volume to lowering trading probability. For sellers with high-quality trees, trading volume is low so the marginal value of fruit is high. trading probability is then a less costly signal.

Building on these results, I study how the use of signals react to shocks to the economy. First, an aggregate liquidity shock that increases all sellers' demand for fruit leads to higher trading probability and trading volume for all sellers. However, for sellers with high-quality trees, trading probability increases more than trading volume. This is because for sellers with high-quality trees, a small increase in trading volume has a large effect on a seller's utility since the marginal utility of fruit is large. Hence, for sellers with high-quality trees, a large increase in trading volume would lead to sellers with lower-quality trees mimicking their offers. I also find that a quality shock that lowers the quality of low-quality trees has similar effects. The shock lowers trading probability and trading volume for all sellers. However, for sellers with high-quality trees, the decrease in trading probability is larger than the decrease in trading volume.

The relationship between illiquidity and partial retention as signals is also explored in Williams (2016). After introducing sellers' preferences as a second dimension of private information, Williams (2016) shows that there exists a fully separating equilibrium where both types of signals are used. Sellers with the same continuation value of assets charge the same price. Among such sellers, those with higher-quality assets sell less but with a higher probability. This also implies that buyers purchase assets with different quality at the same price in equilibrium. In this chapter, sellers use both types of signals even though there is only one dimension of private information. The trade-off emerges as the relative costs of illiquidity and partial retention changes with trading volume. Furthermore, in an extension I show that the main findings of this chapter holds when there is a fixed supply of buyers as opposed to free entry of buyers.

This chapter is related to literature that studies asset markets with private information and search friction. Building on Guerrieri et al. (2010), Chang (2018) studies a two-dimensional private information problem similar to Williams (2016). However, Chang (2018) assumes the asset is indivisible and focuses instead on semi-pooling equilibria that can be used to explain the fire sale phenomena. The two-dimensional private information problem is also studied by Guerrieri and Shimer (2018). Their focus is the welfare implications of multiple equilibria. Both Chang (2018) and Guerrieri and Shimer (2018) assume linear preferences. Therefore, the trade-off between illiquidity and partial retention does not emerge in their models.

This chapter belongs to the vast literature on private information spurred by Akerlof (1970) and Leland and Pyle (1977). Some recent work that also studies private information in asset markets includes Eisfeldt (2004), Kurlat (2013), Chiu and Koeppl (2016), and Choi (2018). Eisfeldt (2004), Chiu and Koeppl (2016), and Kurlat (2013) restrict their attention to pooling equilibria by requiring agents to trade at one price or not trade at all. Choi (2018) assumes that asset buyers do not inherit sellers' private information after trade and have to learn it, and finds that public disclosure of private information is not always welfare-improving. The notion that trade in asset markets is motivated by agents' different needs for liquidity is based on Geromichalos and Herrenbrueck (2016a), who extend the Lagos and Wright (2005) framework by allowing agents to rebalance their portfolios before consumption.[1]

In what follows, Section 4.2 describes the physical environment. Section 4.3 solves the equilibrium. Section 4.4 analyzes liquidity and quality shocks. Section 4.5 extend the model to allow a fixed supply of buyers. Section 4.6 concludes the chapter.

4.2 Model Environment

There are three periods, $t = 0, 1, 2$, and two types of agents, early consumers and late consumers. In period 0, a unit measure of early consumers are born and endowed with a units of trees and b units of fruit. Both fruit and trees are perfectly divisible. Fruit may be consumed in period 1 or period 2, but trees cannot be directly consumed. Instead, trees will bear fruit in period 2. An early consumer's utility is given by

$$u(c_1) + c_2^e, \tag{4.1}$$

where c_1 is the consumption in period 1 and c_2^e is the consumption in period 2. I assume $u'(.) > 0$, $u''(.) < 0$, $u'(b) > 1$, and $u'(c^*) = 1$ for some $c^* < \infty$. There is also a large measure of late consumers who only consumes in period 2. Like early consumers, they are also endowed with fruit and trees, and their preferences are linear in their period-2 consumption.

I assume early consumers differ in the quality of the trees they hold. Let δ denote the quality of an early consumer's trees. It means each unit of the early consumer's trees will produce δ units of fruit in period 2. A consumer's trees belong to one of J different quality types, i.e., $\delta \in \mathcal{J} = \{\delta_j\}_{j=1}^J$ where $\delta_1 < \delta_2 < ... < \delta_J$. The distribution of δ is

$$\Pr(\delta = \delta_j) = \Delta_j, \text{ for all } \delta_j \in \mathcal{J}. \tag{4.2}$$

I assume only owners' of the trees observe the quality. However, whether an agent is an early consumer or a late consumer is public information.[2]

In period 0, an asset market opens for agents to trade fruit and trees. An agent who wants to sell trees posts an offer (z, s) where z is the amount of fruit asked by the agent,

[1] See also Geromichalos and Herrenbrueck (2016b), Herrenbrueck and Geromichalos (2017), Geromichalos and Herrenbrueck (2017), and Herrenbrueck (2019). None of these papers study the private information problem.

[2] Wang (2019) studies the case where agents' preferences are also private information. In such case, sellers may have different trading motives, and agents' welfare depends on whether trading motives are revealed.

and s is the quantity of trees offered by the agent. An agent who wants to buy trees must pay an entry cost k. Following the competitive search literature, I refer to (z, s) as a "location". Denote the buyer-seller ratio in each location as $\theta(z, s)$. I assume a buyer meets a seller with probability $q(\theta)$, while a seller meets a buyer with probability $p(\theta) = \theta q(\theta)$. I assume $q'(\theta) < 0$, $p'(\theta) > 0$ and $p''(\theta) < 0$. Buyers observe all (z, s) pairs and the associated buyer-seller ratios before they decide which location to visit. After the asset market, early consumers decide how much to consume in period 1 and 2. For simplicity, I assume that $a \geq (c^* - b)/\delta_1$ where c^* solves $u'(c^*) = 1$, and that late consumers have deep pockets. This to guarantee that trade in the asset market is not constrained by sellers' or buyers' assets, but they are not crucial to the results of this chapter. Lastly, in period 2, trees bear fruit and the agents consume fruit.

4.3 Equilibrium

4.3.1 Agents' Problems

To characterize the equilibrium, it is convenient to start with early consumers' problem after asset trade. Let $b + \tilde{z}$ be the amount of fruit an early consumer has after selling their assets. $\tilde{z}$ is the amount of fruit obtained from the asset market. Let $\tilde{a}$ denote the trees the early consumer has. Then the early consumer solves

$$\max_{c_1, c_2} u(c_1) + c_2^e \tag{4.3}$$

$$\text{s.t. } c_1 \leq b + \tilde{z}, \tag{4.4}$$

$$c_2^e \leq b + \tilde{z} - c_1 + \delta\tilde{a}. \tag{4.5}$$

Let $c_1(b+\tilde{z})$ denote the optimal c_1. Then it is easy to see that $c_1(b+\tilde{z}) = \min\{b+\tilde{z}, c^*\}$ where $u'(c^*) = 1$. Recall that I assume $u'(b) > 1$. That is, the fruit endowment is not large enough for early consumers to consume c^*. Since all agents' preferences in fruit are linear in period 2 but fruit has a higher marginal value to early consumers in period 1, they will want to sell trees in exchange for fruit. Now recall that to sell trees, sellers has to post an offer (z, s). If buyers accept this offer, she must give the seller z units of fruit. In return, she receives s units of trees. I assume and later verify that in equilibrium, $z \leq c^* - b$ so $c_1(b + z) = b + z$. The expected utility of a seller who has quality δ trees and posts (z, s) is given by

$$p(\theta(z, s))[u(b + z) - \delta s] + (1 - p(\theta(z, s)))u(b) + \delta a, \tag{4.6}$$

where $\theta(z,s)$ is the market tightness at location (z,s). In words, (4.6) says that with probability $p(\theta(z,s))$, a seller meets a buyer and obtains z units of fruit, which allows her to consume in total $b+z$ units of fruit in period 1. In exchange, the seller transfers s units of trees to the buyer. With probability $1-p(\theta(z,s))$, the seller does not meet a buyer and consumes b in period 1. Next, the expected utility of a buyer who chooses (z,s) to visit is

$$q(\theta(z,s))\left[s\sum_{j'=1}^{J}\gamma(z,s;\delta_{j'})\delta_{j'}-z\right], \tag{4.7}$$

where $\gamma(z,s;\delta_{j'})$ is the buyer's belief about the probability of tree quality in location (z,s) being $\delta_{j'}$. Note that $\sum_{j'=1}^{J}\gamma(z,s;\delta_{j'})=1$. In words, (4.7) says that if a buyer meets a seller at location (z,s), the buyer expects each unit of trees to produce on average $\sum_{j'=1}^{J}\gamma(z,s;\delta_{j'})\delta_{j'}$ units of fruit in period 2. In exchange for s units of trees, the buyer gives the seller z units of fruit.

Sellers and buyers can choose not to participate in the asset market. Denote not participating as $\emptyset$. I define the competitive equilibrium in the asset market.

Definition 4.1 *A competitive equilibrium is a set Ψ of seller offers, a vector $\{v^*_{s,j}\}_{j=1}^{J}$, a market tightness functions $\theta:\Psi\to[0,\infty]$, and a belief function $\gamma:\Psi\times\mathcal{J}\to[0,1]$ that satisfy*

(1) *Given $\gamma(z,s;\delta_j)$, $v^*_{s,j}$ is given by*

$$v^*_{s,j}=\max_{(z,s)\in\Psi\cup\emptyset}p(\theta(z,s))[u(b+z)-u(b)-\delta_j s], \tag{4.8}$$

and the free entry condition holds:

$$\max_{(z,s)\in\Psi\cup\emptyset}q(\theta(z,s))\left[s\sum_{j'=1}^{J}\gamma(z,s;\delta_{j'})\delta_{j'}-z\right]=k; \tag{4.9}$$

(2) *$\gamma(z,s;\delta_j)$ is given by Bayes' Rule whenever possible;*

(3) *For all (z,s), $(z,s)\in\Psi$ only if it solves the maximization problem in (4.9); and*

(4) *There do not exist $S\subset\mathcal{J}$ and $(z',s',\theta')\in\mathbb{R}^3_+$ such that for any $\gamma'(z',s';\delta_j)$,*

$$p(\theta')[u(b+z')-u(b)-\delta_j s']>v^*_{s,j} \text{ for all } \delta_j\in S, \tag{4.10}$$

$$p(\theta')[u(b+z')-u(b)-\delta_{j'} s']\le v^*_{s,j'} \text{ for all } \delta_{j'}\in \mathcal{J}\backslash S, \tag{4.11}$$

$$q(\theta')s'\sum_{j'=1}^{J}\gamma'(z',s';\delta_{j'})\delta_{j'}-z'\ge k \text{ for all } \gamma'(z',s';\delta_j) \text{ with support } S. \tag{4.12}$$

Conditions (1)-(3) define a Perfect Bayesian Equilibrium (PBE), but they put no restrictions on beliefs about out-of-equilibrium actions. As a result, there exist a plethora of PBE that satisfy conditions (1)-(3). To obtain meaningful results, condition (4) requires the equilibrium to satisfy the Intuitive Criterion (Cho and Kreps, 1987). It says that for any PBE to be an equilibrium, there must not exist an off-equilibrium offer (z', s') such that (i) it only makes sellers in set S strictly better off, and (ii) it is accepted by buyers as long as buyers' beliefs about δ conditional on (z', s') put no weights on sellers in $\mathcal{J}\backslash S$. Condition (i) means that sellers in $\mathcal{J}\backslash S$ do not have the incentive to deviate. Condition (ii) means that buyers will accept this off-equilibrium offer as long as their beliefs are consistent with sellers' strategies. If such offer exists, sellers in S will always deviate and the original equilibrium will cease to exist.

The equilibrium is solved in Appendix C.1. Now let (z_j, s_j) denote the offer posted by sellers with quality δ_j trees. Let $\theta_j = \theta(z_j, s_j)$.

Proposition 4.1 *Assume $k < u(c^*) - u(b) - (c^* - b)$. Then there exists a unique competitive equilibrium. In addition, for all $1 \leq j' < j'' \leq J$, $\theta_{j'} > \theta_{j''}$, $s_{j'} > s_{j''}$ and $z_{j'} > z_{j''}$.*
Proof: see Appendix C.1.

There are two important characteristics of the equilibrium. First, it is fully separating in the sense that sellers with different quality trees sell with different probabilities and receive different amounts of money. Second, in equilibrium, both illiquidity and partial retention are used by sellers. Sellers with higher-quality trees sell with lower probability, and they also sell smaller quantities. This is in stark contrast with the findings in Williams (2016), who shows that when there is only one dimension of private information (i.e., asset quality), sellers do not use partial retention as a signal. Williams (2016) assumes sellers' preferences over fruit are linear, while in this chapter sellers' preferences over fruit in period 1 are strictly concave. To see why sellers use both signals, suppose for a moment that sellers do not use partial retention. Then, for all sellers, $c_1 = c^*$ and the marginal utility of fruit would be equal to one. Since the opportunity cost of consuming in period 1 is also one, lowering trading volume only has second-order effects on early consumers' utility. However, a lower trading volume allows high-quality sellers to increase trading probability, which has a first order effect on their utility. Hence, there exists a trade-off between illiquidity and partial retention.

4.3.2 Illiquidity vs. Partial Retention

To demonstrate the trade-off between illiquidity and partial retention as signals of asset quality, I calculate the elasticities of trading probability $(p(\theta_j))$ and trading volume (z_j)

with respect to tree quality (δ_j). From Proposition 4.1, we know that $\mathrm{d}p(\theta_j)/\,\mathrm{d}\delta_j < 0$ and $\mathrm{d}z_j/\,\mathrm{d}\delta_j < 0$. Now define

$$\epsilon_{p(\theta_j)} = \frac{\mathrm{d}p(\theta_j)/p(\theta_j)}{\mathrm{d}\delta_j/\delta_j} \text{ and } \epsilon_{z_j} = \frac{\mathrm{d}z_j/z_j}{\mathrm{d}\delta_j/\delta_j}. \tag{4.13}$$

Define

$$D(j) = \frac{\epsilon_{p(\theta_j)}}{\epsilon_{z_j}}. \tag{4.14}$$

And define

$$U(z) = u(b+z) - u(b) - zu'(b+z). \tag{4.15}$$

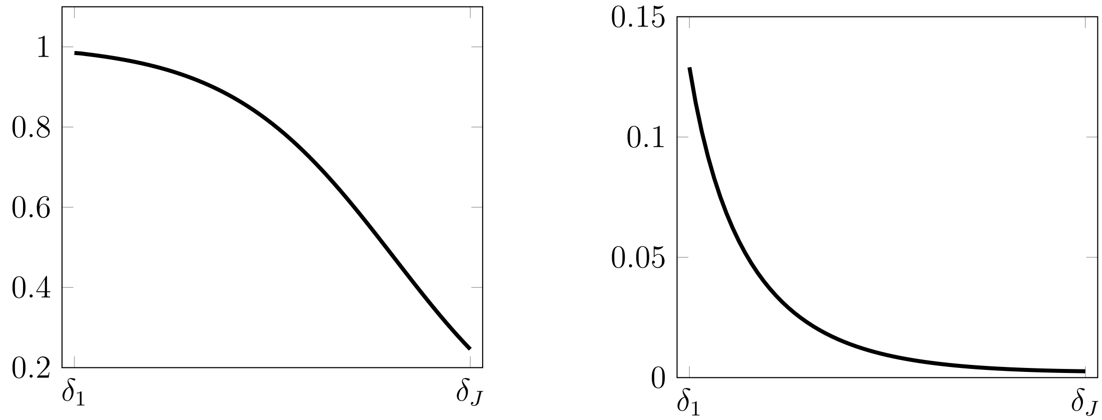
The next proposition shows how the trade-off changes with the quality of sellers' trees.

Proposition 4.2 *Assume $\frac{p'(\theta)}{p(\theta)p''(\theta)}$ is increasing in θ and $\frac{-zU'(z)}{U(z)}$ is increasing in z. Then $D(j') < D(j'')$ for all $1 < j' < j'' \leq J$.*
Proof: see Appendix C.1.

The assumption on $p(\theta)$ is satisfied by the Cobb-Douglas matching function (i.e., $p(\theta) = \theta(1/\theta)^\epsilon$ where $\epsilon \in (0,1)$). If θ is not too large, the assumption is also satisfied by the urn-ball matching function (i.e., $p(\theta) = \theta(1-e^{-1/\theta})$) and the telephone-line matching function (i.e., $p(\theta) = \theta[\alpha/(\alpha+\theta^\gamma)]^{1/\gamma}$ where $\alpha \in (0,1]$ and $\gamma \geq 1$). The assumption on $U(z)$ is satisfied by CRRA utility function (i.e., $u(c) = (c^{1-\sigma}-1)/(1-\sigma)$) for all $\sigma > 0$. It should be noted that these two assumptions are sufficient but not necessary for the proposition to hold.

Now, to understand Proposition 4.2, consider sellers with δ_j and δ_{j+1} trees as well as sellers with $\delta_{j'}$ and $\delta_{j'+1}$ trees, where $j < j'$. From Proposition 4.1, we know that to prevent δ_j sellers from deviating, it must be that $p(\theta_{j+1}) < p(\theta_j)$ and $z_j < z_{j+1}$. Similarly, to prevent $\delta_{j'}$ sellers from deviating, it must be that $p(\theta_{j'+1}) < p(\theta_{j'})$ and $z_{j'+1} < z_{j'}$. Proposition 4.2 says that compared to δ_{j+1} sellers, $\delta_{j'+1}$ sellers reduce trading probability more than they reduce trading volume. In other words, sellers with higher-quality trees prefer illiquidity to partial retention. The intuition is that when the tree quality is relatively low, trading volume is high so the marginal value of fruit is low. To signal their quality, sellers prefer lowering trading volume to lowering trading probability. When the tree quality is relatively high, trading volume is low so the marginal value of fruit is high. Lowering trading volume becomes too costly compared to lowering trading probability.

In Figure 4.1, I provide a numerical example using the urn-ball matching function.[3]



(a) Trading probability ($p(\theta)$)

(b) Trading volume (z)

Figure 4.1: Equilibrium Trading Probability and Trading Volume

In panel (a) and (b), trading probability ($p(\theta)$) and trading volume (z) are plotted against δ, respectively. Panel (a) shows that $p(\theta)$ decreases slowly at first but accelerates when δ is larger. Panel (b) shows that z decreases quickly at first but decelerates when δ is larger. This means that compared to sellers with low-quality trees, sellers with high-quality trees are more willing to sacrifice trading probability than trading volume.

4.4 Liquidity and Quality Shocks

In this section, first I study a liquidity shocks that increases early consumers' demand for fruit. Second, I discuss a shock to the quality distribution that disproportionately affects low-quality trees. Such a scenario is similar to the subprime mortgages crisis in 2007-08 when the subprime mortgages have a much higher delinquency rate.

4.4.1 Liquidity Shock

Assume that before the asset market opens, an aggregate liquidity shock happens and it increases early consumers' marginal utility for Period 1 consumption. Specifically, let the utility of early consumers in Period 1 be given by

$$u(.) = \eta \hat{u}(.),$$

where $\hat{u}'(.) > 0$, $\hat{u}''(.) < 0$, $\hat{u}'(b) > 1$, and $\hat{u}'(c^*) = 1$ for some $c^* < \infty$. Note that

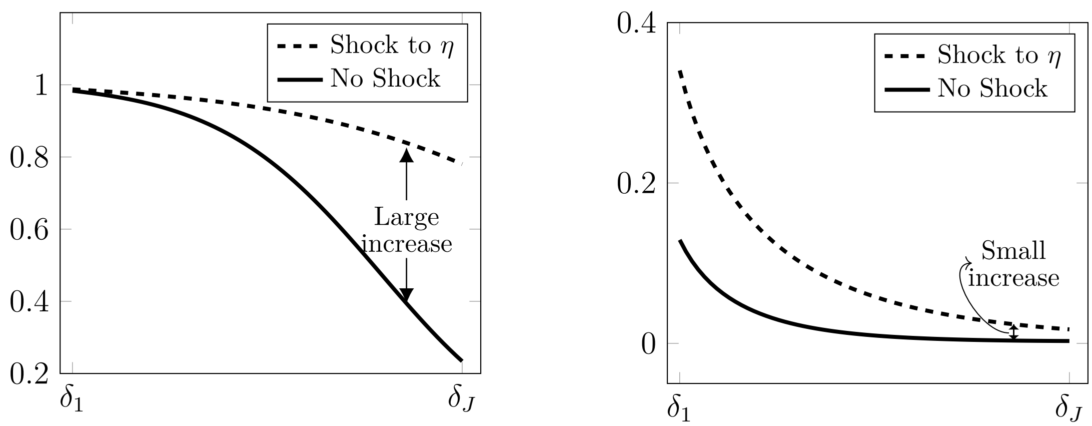
[3] The utility function used is $\sqrt{c}$.

$\eta = 1$ in the benchmark model, and I assume $\eta > 1$ when the liquidity shock happens. A straightforward interpretation of such liquidity shock is a surprise consumption need. Alternatively, the shock can be interpreted as a flight-to-safety episode that often happens during runs on financial institutions or crises in repo markets (Shleifer and Vishny, 2011; Martin et al., 2014).[4]

The next proposition shows how the equilibrium responds to the shock.

Proposition 4.3 *After the liquidity shock, $p(\theta_j)$ and z_j increase for all j.*
Proof: see Appendix C.2.

The liquidity shock increases trading probability and trading volume for all quality types. In particular, for low-quality sellers, the increase in the marginal utility of fruit increases the quantity they sell. This makes mimicking high-quality sellers' offers less attractive because high-quality sellers sell less than low-quality sellers. As a result, high-quality sellers are also able to sell more without making low-quality sellers deviate.



(a) Trading probability ($p(\theta)$)

(b) Trading volume (z)

Figure 4.2: Liquidity Shock and Equilibrium

Another interesting result is that as j increases, the increase in $p(\theta_j)$ becomes larger and larger compared to the increase in z_j, as shown in Figure 4.2. This is the result of the trade-off between illiquidity and partial retention (see Proposition 4.2). When δ_j is large, z_j is small and the marginal utility of fruit is large. As a result, a small increase in z has a large effect on a seller's utility. This means z_j cannot increase much without triggering the deviation from sellers with lower-quality trees. On the other hand, the marginal effect of increasing $p(\theta_j)$ is much smaller. Therefore, for sellers with high-quality trees, after a

[4] Wang (2020) studies an idiosyncratic liquidity shock and shows that the equilibrium depends crucially on whether or not asset buyers can observe which sellers have received the shock.

liquidity shock, the increase in trading probability is larger compared to the increase in trading volume.

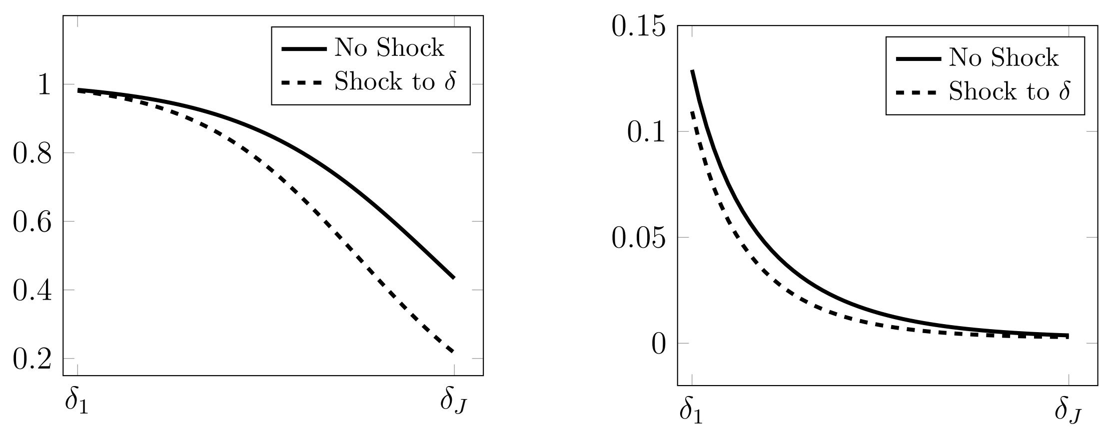
4.4.2 Quality Shock

Now assume that before the asset market opens, a quality shock happens and it decreases the quality of sellers' trees. Specifically, let $\{\delta_j^\dagger\}_{j=1}^J$ denote the tree quality after the shock. I assume $\delta_j^\dagger = h(\delta_j)\delta_j$ where $h(\delta_j) \in (0,1]$ and $h(\delta_j) \leq h(\delta_{j'})$ for all $j < j'$. This means that the shock disproportionately affects low-quality trees.

The next proposition shows how the equilibrium responds to the shock.

Proposition 4.4 *Equilibrium trading probability $p(\theta_j)$ and equilibrium trading volume z_j decrease for all $j > 1$ after the quality shock.*
Proof: see Appendix C.2.

The quality shock decreases trading probability and trading volume for all quality types even if it only affects the trees with the lowest quality (i.e., if $h(\delta_1) < 1$ while $h(\delta_j) = 1$ for all $j > 1$). The reason is that to prevent sellers with δ_1 trees from deviating, all sellers must lower trading probability and trading volume. Hence, sellers with high-quality trees will be affected by a quality shock even if it does not directly affect high-quality trees.



(a) Trading probability ($p(\theta)$)

(b) Trading volume (z)

Figure 4.3: Quality Shock and Equilibrium

Figure 4.3 shows that sellers with high-quality trees face larger decreases in trading probability than sellers with low-quality trees, even though the shock affects low-quality trees more than high-quality trees.

Similar to the equilibrium under the liquidity shock, for sellers with high-quality trees, the decreases in trading volume are small compared to the decreases in trading probability. This is because sellers with high-quality trees substitute trading probability for trading volume, since it is expensive to decrease trading volume when it is already small. It can therefore be concluded that after a liquidity shock or a quality shock, for sellers with high-quality trees, the changes in trading probability are always larger than the changes in trading volume.

4.5 Extension: Fixed Supply of Buyers

In this section, I assume there is a fixed supply of late consumers as a robustness check of the results in this chapter. This modification offers two new insights. First, buyer's surplus is no longer pinned down by the entry cost. Instead, it is endogenously determined by the number of buyer relative to sellers in the asset market. Second, the fruit available for asset trade is limited to the fruit held by the fixed supply of buyers. Recall that fruit is meant to represent liquid assets. Then, buyers having limited amount fruit in possession is akin to the situation where there is insufficient liquidity in asset markets. In what follows, I solve the equilibrium in the new environment.

Let the total measure of early and late consumers be normalized to one. Let α denote the measure of early consumers so $1-\alpha$ is the measure of late consumers. Also, let b^e and b^l denote the amount of fruit held by early and late consumers before trading. The problems of the late and early consumers are identical to those in Section 4.3 except now the trade may be constrained by b^l.

Definition 4.2 *A competitive equilibrium is a set Ψ of seller offers, a vector $\{v^*_{s,j}\}_{j=1}^J$, a scalar v^*_b, a market tightness functions $\theta : \Psi \to [0,\infty]$, a belief function $\gamma : \Psi \times \mathcal{J} \to [0,1]$, and an accumulative price distribution function $F : \Psi \to [0,1]$ that satisfy*
(1) *Given $\gamma(z,s;\delta_j)$, $v^*_{s,j}$ is given by*

$$v^*_{s,j} = \max_{(z,s)\in\Psi\cup\emptyset} p(\theta(z,s))[u(b^e+z) - u(b^e) - \delta s], \tag{4.16}$$

and v^*_b is given by

$$v^*_b = \max_{(z,s)\in\Psi\cup\emptyset} q(\theta(z,s)) \left[s \sum_{j'=1}^{J} \gamma(z,s;\delta_{j'})\delta_{j'} - z \right]; \tag{4.17}$$

(2) *$\gamma(z,s;\delta_j)$ is given by Bayes' Rule whenever possible;*
(3) *For all (z,s), $(z,s)\in\Psi$ only if it solves the maximization problem in (4.17), and it is feasible: $z \le b^l$;*

(4) *For all $j \in \mathcal{J}$ and for all $(z,s) \in \Psi$,*

$$\Delta_j = \int_{\Psi} \gamma(z,s;\delta_j) dF(z,s) \text{ and } \frac{\alpha}{1-\alpha} = \frac{1}{\int_{\Psi} \theta(z,s) dF(z,s)};$$

and

(5) *There do not exist $S \subset \mathcal{J}$ and $(z', s', \theta') \in \mathbb{R}^3_+$ such that $z' \leq b^l$ and*

$$p(\theta')[u(b^e + z') - u(b^e) - \delta_j s'] > v^*_{s,j} \text{ for all } j \in S, \tag{4.18}$$

$$p(\theta')[u(b^e + z') - u(b^e) - \delta_{j'} s'] \leq v^*_{s,j'} \text{ for all } j' \in \mathcal{J} \backslash S, \tag{4.19}$$

$$q(\theta')s' \sum_{j' \in S'} \gamma'(z', s'; \delta_{j'})\delta_{j'} - z' \geq v^*_b \text{ for all } \gamma'(z', s'; \delta_j) \text{ with support } S. \tag{4.20}$$

Conditions (1)-(3) are the same as conditions (1)-(3) in Definition 4.1 except that the entry cost k is replaced with the endogenous buyer surplus v^*_b. Condition (4) is new and it requires that the measure of buyers in each location adds up to the total measure of buyers in the market. Lastly, condition (5) is the same as the condition (4) in Definition 4.1. It requires the equilibrium to satisfy the intuitive criterion.

The equilibrium is solved in two steps. The first step takes v_b, the surplus of buyers, as given. It solves the partial equilibrium by requiring sellers to offer at least v_b to buyers. This step is identical to how the equilibrium is solved in Section 4.3. The second step is to find the equilibrium v^*_b that satisfies condition (4). The next proposition describes the equilibrium.

Proposition 4.5 *For all $\alpha \in (0,1)$, there exists a unique competitive equilibrium. In addition, for all $1 < j' < j'' \leq J$, $\theta_{j'} > \theta_{j''}$, $s_{j'} > s_{j''}$ and $z_{j'} > z_{j''}$.*
Proof: see Appendix C.3.

The equilibrium is very similar to the equilibrium found in Proposition 4.5: it is fully separating, and sellers use both illiquidity and partial retention as signals. The main finding of this chapter, i.e., the trade-off between illiquidity and partial retention, also holds in this equilibrium. Similar to Section 4.3.2, I compare the elasticities of θ_j and z_j with respect to δ_j. Recall that $D(j) = \frac{\mathrm{d}p(\theta_j)}{\mathrm{d}\delta_j} \Big/ \frac{\mathrm{d}z_j}{\mathrm{d}\delta_j}$ and $U(z) = u(b^e+z) - u(b^e) - zu'(b^e+z)$.

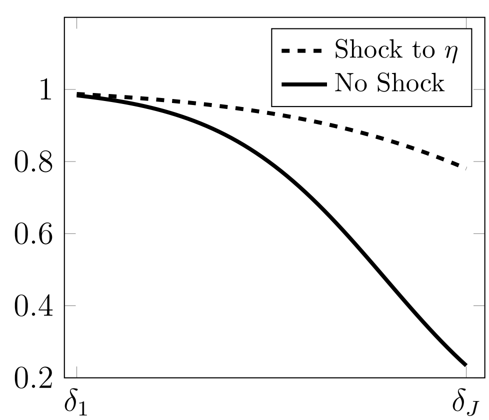
Proposition 4.6 *Assume $\frac{p'(\theta)}{p(\theta)p''(\theta)}$ is increasing in θ and $\frac{-zU'(z)}{U(z)}$ is increasing in z. Then $D(j') < D(j'')$ for all $1 < j' < j'' \leq J$.*
Proof: see Appendix C.3.

Now, I show how buyers' surplus in equilibrium, v^*_b, depends on the share of sellers, seller's demand for fruit, and the amount of fruit each buyer has. To discuss sellers' demand for fruit, I reuse the utility function introduced in Section 4.4.1: $u(.) = \eta\hat{u}(.)$.

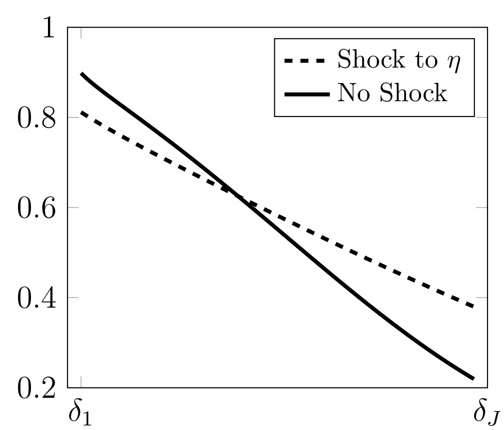
Proposition 4.7 *v_b^* is increasing in α, η, and b^l.*
Proof: see Appendix C.3.

First, when the share of sellers or seller's demand for fruit increases, sellers in each location have the incentive to attract buyers by increasing the value they offer to buyers. Hence, in equilibrium, v_b^* is higher. Second, if b^l is larger, the total surplus of match between buyers and sellers are larger. Since buyers are in fixed supply, sellers have the incentive to attract buyers to their locations, and hence v_b^* is higher.

Since v_b^* is determined endogenously as opposed to being pinned down by the entry cost, the equilibrium also responds to shocks differently. For example, suppose a liquidity shock happens and it increases η. Not only will trading probability $p(\theta)$ and trading volume z react to the shock, but v_b^* will also change. Hence, there exists a general equilibrium effect that is not present in the model with free entry. Figure 4.4 provides an example of how the responses differ in two environments. I reproduce panel (a) of Figure 4.2 in the panel (a) of Figure 4.4 for comparison. In panel (b) where the supply of buyers is fixed, trading probability is lower for low-quality sellers but higher for high-quality sellers after the liquidity shock. In comparison, when there is free entry of buyers, trading probability is higher for all sellers following the shock.



(a) Free Entry



(b) Fixed Supply of Buyers

Figure 4.4: Liquidity Shock and Trading probability ($p(\theta)$)

4.6 Conclusion

I develop a simple competitive search model to study illiquidity and partial retention of assets as signals of asset quality in asset markets with private information. I show that both signals are used by sellers in equilibrium. However, for sellers with high-quality assets, illiquidity is preferred over partial retention in the sense that among these sellers, those with higher-quality assets sell with significantly lower probability but only marginally less. I show the opposite is true for sellers with low-quality assets. Building on these results, I study aggregate liquidity shocks that increase asset sellers' need for liquidity, and quality shocks that lower the quality of low-quality assets. The model predicts that after the shocks, for sellers with high-quality assets, the changes in trading probability are larger than the changes in trading volume, while the opposite happens to sellers with low-quality assets. Lastly, as a robustness check, I show that the main results in this chapter hold if instead of free entry of buyers, a fixed supply of buyers is assumed. I provide an example of how the general equilibrium effects in this modified setup affects equilibrium responses to shocks.

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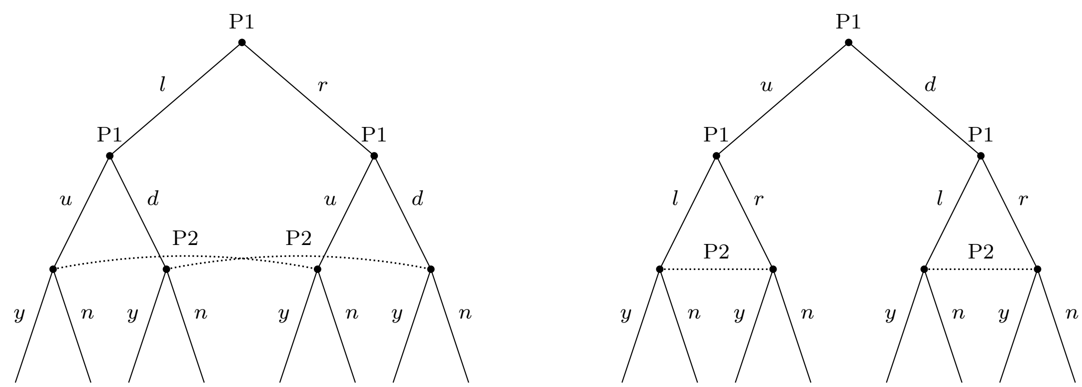
Appendix A

Appendices to Chapter 2

A.1 Discussion: Endogenous Signaling Games

Consider an "endogenous signaling" game in Figure A.1a where player 1 (P1) makes an unobservable move (l or r) and an observable move (u or d) before player 2 (P2) moves (y or n).



(a) Original Game (b) Reordered Game

Figure A.1: Reordering Invariance Equilibrium

In this game, PBE does not put any restrictions on player 2's beliefs about player 1's unobservable move on off-equilibrium paths. In and Wright (2018) note that the order of player 1's two moves does not matter for equilibrium outcomes, because the player does not gain any payoff-relevant information between moves. However, if player 1 makes the observable move first, his subsequent unobservable move and player 2's move constitutes a proper subgame of the reordered game (Figure A.1b). By solving the Nash equilibrium of the subgame, one can confine player 2's beliefs about player 1's unobservable move in a logically consistent way. The Reordering Invariance (RI) equilibrium then selects the equilibrium outcomes that are also equilibrium outcomes of the reordered game. Reordering Invariance equilibrium has been used in Li et al. (2012), Gomis-Porqueras et al. (2017), Kang (2017), and Berentsen et al. (2017) to study the problem of counterfeit assets.

In this paper, the game between sellers and the fiscal authority is similar to the endogenous signaling game. A seller makes an unobservable action (posting (q, p)) and an observable action (reporting y) before the fiscal authority decides η. For any given y, the game between sellers and the fiscal authority, which is described by (2.11) and (2.13), is similar to the subgame in the reordered endogenous signaling game. Similar to the Reordering Invariance (RI) equilibrium, the proposed refinement requires the same to be true for any off-equilibrium choice of y.

A.2 Proofs

Proof of Proposition 2.1: This proof has two parts. In part I, I solve for the Nash equilibrium for any given $\tau(y)$ and y. In part II, I solve for the optimal choice of y. Throughout this proof, I assume $R^d > 0$. The case with $R^d = 0$ is straightforward to solve using this proof.

Part I. In this part, I first solve the Nash equilibrium for any given $\tau(y)$ and y. It is easy to see that (2.12) must bind because if it does not, a seller can be better off by choosing a smaller q. Also, it must be that $n \leq 1$ because otherwise a seller can lower n and make (2.12) slack while not affecting his surplus.

Next, there are two possible cases. First, suppose that $R^d < (1-\rho^m)\eta/(1-\eta)$. This means that the marginal cost of using cash is lower than bank deposits. However, since $R^d > 0$, the seller benefits from using some bank deposits. Specifically, the seller will choose $d = y$. The seller solves

$$\max_{q,m}\{(1-\rho^m\eta)(y+m-\tau(y))-q\} \text{ s.t. } -(1+\mu)m-\frac{1+\mu}{1+R^d}y+\beta u(q)=\mathcal{S}. \tag{A.1}$$

The first order condition is

$$u'(q)=\frac{1+\mu}{\beta(1-\rho^m\eta)}. \tag{A.2}$$

In a Nash equilibrium, it must be that $\rho^m(d+m-\tau(y)) = C$ and $\eta \in (0,1)$. Then, η must be such that

$$\frac{1}{1+\mu}\left[\beta u(q)-\mathcal{S}\right]=\tau(y)-\frac{R^d y}{1+R^d}+\frac{C}{\rho^m}. \tag{A.3}$$

Hence, q is increasing in y and therefore η is decreasing in y. Then, there exists $y^\dagger$ such that for all $y \geq y^\dagger$, if the seller continues using both bank deposits and cash, then $R^d \geq (1-\rho^m)\eta/(1-\eta)$. In such case, the marginal cost of using cash becomes higher than bank deposits. This means the seller has the incentive to deviate and use bank deposits. Suppose the seller chooses to use only bank deposits. The seller solves

$$\max_{q,d}\{(1-\eta)(d-\tau(y))-q\} \text{ s.t. } -\frac{(1+\mu)d}{1+R^d}+\beta u(q)=\mathcal{S}. \tag{A.4}$$

The first order condition is

$$u'(q)=\frac{1+\mu}{\beta(1-\eta)(1+R^d)}. \tag{A.5}$$

In a Nash equilibrium, it must be that $d-\tau(y) = C$ and $\eta \in (0,1)$. That is, η solves

$$d=\frac{1+R^d}{1+\mu}[\beta u(q)-\mathcal{S}]=\tau(y)+C. \tag{A.6}$$

It is easy to see that q is increasing in y and therefore η is decreasing in y. This means that there exists $y^\ddagger$ such that for all $y < y^\ddagger$, if the seller accepts only bank deposits, then $R^d < (1-\rho^m)\eta/(1-\eta)$. This means that the seller has the incentive to deviate and use both bank deposits and cash. Note that $y^\dagger < y^\ddagger$. To see this, suppose the seller uses bank deposits and $y = y^\ddagger$. If the seller switches to a mixture of bank deposits and cash, then q will not change because $R^d = (1-\rho^m)\eta/(1-\eta)$. However, since $\rho^m(y+m-\tau(y)) < d-\tau(y)$ because $R^d > 0$ and $\rho^m < 1$, the fiscal authority has to lower η so that $\rho^m(y+m-\tau(y)) = C$. Then it must be that $R^d > (1-\rho^m)\eta/(1-\eta)$, i.e., $y > y^\dagger$. This means that for any $y^\dagger < y < y^\ddagger$, there does not exist an equilibrium where the seller plays a pure strategy. Instead, for all $y^\dagger < y < y^\ddagger$, $\eta = \eta' = R^d/(1-\rho^m+R^d)$. For any $y^\dagger < y < y^\ddagger$, let $(q^\dagger, m^\dagger)$ solve

$$u'(q^\dagger) = \frac{1+\mu}{\beta(1-\rho^m\eta')}, \tag{A.7}$$

$$m^\dagger = \frac{1}{1+\mu}\left[\beta u(q^\dagger) - \mathcal{S} - \frac{1+\mu}{1+R^d}y\right]. \tag{A.8}$$

Let $d^\ddagger$ solve

$$d^\ddagger = \frac{1+R^d}{1+\mu}[\beta u(q^\dagger) - \mathcal{S}]. \tag{A.9}$$

It is clear that $\rho^m(y+m^\dagger-\tau(y)) < C$ and $d^\ddagger - \tau(y) > C$ for any $y^\dagger < y < y^\ddagger$. Also, both $\rho^m(y+m^\dagger-\tau(y))$ and $d^\ddagger - \tau(y)$ are decreasing in y. For any $y^\dagger < y < y^\ddagger$, there exists $\epsilon(y) \in (0,1)$ such that

$$\epsilon(y)\rho^m(y+m^\dagger-\tau(y)) + (1-\epsilon(y))(d^\ddagger-\tau(y)) = C. \tag{A.10}$$

And $\epsilon(y)$ is decreasing in y. Then, for all $y^\dagger < y < y^\ddagger$, the seller chooses $(q^\dagger, m^\dagger)$ with probability $\epsilon(y)$ and $(q^\ddagger, d^\ddagger)$ with probability $1-\epsilon(y)$.

To summarize, for any $\tau(y)$ and y, the Nash equilibrium is given by the following. Define $\eta' = R^d/(1-\rho^m+R^d)$. Let $q^\dagger$ solve

$$u'(q^\dagger) = \frac{1+\mu}{\beta(1-\rho^m\eta')}. \tag{A.11}$$

Define $y^\dagger$ to be such that

$$\frac{1}{1+\mu}\left[\beta u(q) - \mathcal{S}\right] = \tau(y) - \frac{R^d y}{1+R^d} + \frac{C}{\rho^m}. \tag{A.12}$$

Define $y^\ddagger$ to be such that

$$\frac{1+R^d}{1+\mu}[\beta u(q) - \mathcal{S}] = \tau(y) + C. \tag{A.13}$$

Then for all $y \le y^\dagger$, $d = y$. m and q solve

$$u'(q) = \frac{1+\mu}{\beta(1-\rho^m\eta)}, \tag{A.14}$$

$$m = \frac{1}{1+\mu}\left[\beta u(q) - \mathcal{S} - \frac{1+\mu}{1+R^d}y\right] = \tau(y) - y + \frac{C}{\rho^m}. \tag{A.15}$$

For all $y \ge y^\ddagger$, $m = 0$. d and q solve

$$u'(q) = \frac{1+\mu}{\beta(1-\eta)(1+R^d)}, \tag{A.16}$$

$$d = \frac{1+R^d}{1+\mu}[\beta u(q) - \mathcal{S}] = \tau(y) + C. \tag{A.17}$$

For all $y^\dagger < y < y^\ddagger$, the seller chooses $(q^\dagger, m^\dagger)$ with probability $\epsilon(y)$ and $(q^\dagger, d^\ddagger)$ with probability $1-\epsilon(y)$, where $(q^\dagger, m^\dagger)$ solve (A.7) and (A.8) and $d^\ddagger$ solves (A.9). $\epsilon(y)$ is given by (A.10).

Part II. In this part, I solve for the optimal y. First, suppose that the seller plays a pure strategy and accepts both cash and bank deposits. Note that in such case, we have $\rho^m(d+m-\tau(y)) = C$. Hence, the seller's surplus is given by

$$\frac{(1-\rho^m\eta)C}{\rho^m} - q = \frac{(1+\mu)C}{\beta\rho^m u'(q)} - q, \tag{A.18}$$

where the equality uses the first order condition. Then given q, the choice of y solves

$$\frac{1}{1+\mu}\left[\beta u(q) - \mathcal{S}\right] = \tau(y) - \frac{R^d y}{1+R^d} + \frac{C}{\rho^m}. \tag{A.19}$$

Take the derivative of $\frac{(1+\mu)C}{\beta\rho^m u'(q)} - q$ with respect to q and we have

$$-\frac{(1+\mu)Cu''(q)}{\beta\rho^m[u'(q)]^2} - 1. \tag{A.20}$$

Second, suppose that the seller plays a pure strategy and accepts only bank deposits. Note that in such case, we have $d - \tau(y) = C$. The seller's surplus is then

$$(1-\eta)C - q = \frac{(1+\mu)C}{\beta(1+R^d)u'(q)} - q.$$

Take the derivative of the right-hand side with respect to q to get

$$-\frac{(1+\mu)Cu''(q)}{\beta(1+R^d)[u'(q)]^2} - 1. \tag{A.21}$$

Third, suppose the seller plays a mixed strategy. In such case, his surplus is equal to when $y = y^\dagger$ and the seller accepts both cash and deposits, or when $y = y^\ddagger$ and the seller accepts only deposits.

Fourth, suppose the seller reports truthfully. Since $R^d > 0$, the seller will only accept deposits. The seller's problem becomes

$$\max_{q,d}\{d - \tau(d) - q\} \text{ s.t. } -\frac{1+\mu}{1+R^d}d + \beta u(q) = \mathcal{S}. \tag{A.22}$$

Then, to determine the equilibrium, we can compare the seller's surplus when he or she misreports his income and when he or she reports truthfully.

Lastly, in equilibrium it must be that $n(q,p) < 1$ for some (q,p) since there are more sellers than buyers. This means that (1) $\mathcal{S}$ must be such that sellers' surplus in equilibrium is zero; and (2) some sellers do not meet buyers in the DM. For these sellers, the dominant strategy is to produce $q = 0$ and report $y = 0$. If the fiscal authority does not audit sellers who report zero income, then all sellers will report zero income. Hence, these sellers must be audited with a positive probability so that sellers who meet buyers are indifferent between reporting an income of zero or a positive income. Consider one of such sellers. If the seller chooses to accept bank deposits, he or she solves

$$\max_{q,d}\{(1-\eta)d - q\} \text{ s.t. } -\frac{(1+\mu)d}{1+R^d} + \beta u(q) = \mathcal{S}. \tag{A.23}$$

For this to be part of an equilibrium, it must be that $R^d > (1-\rho^m)\eta/(1-\eta)$. If the seller chooses to accept cash, he or she solves

$$\max_{q,m}\{(1-\rho^m\eta)m - q\} \text{ s.t. } -(1+\mu)m + \beta u(q) = \mathcal{S}. \tag{A.24}$$

For this to be part of an equilibrium, it must be that $R^d \leq (1-\rho^m)\eta/(1-\eta)$. Now let γ denote the share of sellers who post either (q,d) or (q,m) that solves the above problems. In equilibrium, if the seller chooses to accept bank deposits, it must be that

$$\frac{\gamma d}{\alpha - 1 + \gamma} = C. \tag{A.25}$$

If the seller chooses to accept cash, it must be that

$$\frac{\gamma \rho^m m}{\alpha - 1 + \gamma} = C. \tag{A.26}$$

□

Proof of Proposition 2.2: I prove the optimality of $\tau^*(y)$ by showing that for any $\tau(y)$, there exists a $\tilde{\tau}$ such that all agents are (weakly) better off while the fiscal authority receives (weakly) higher net tax revenue.

First, for any $\tau(y)$, let Y denote the set of income reported in equilibrium. Let y° be given by

$$y^\circ \in \arg\max_{y \in Y} \tau(y). \tag{A.27}$$

There are two possible cases: (1) strategies (a) and (b) (see Proposition 2.1) are not played in the equilibrium; and (2) either strategy (a) and/or (b) are played in the equilibrium.

First, in case (1), let $\tilde{\tau} = \tau(y^\circ)$. Then, as long as no sellers who meet buyers report $y \in (0, \tilde{\tau})$, agents under $\tau^*(y)$ are (weakly) better-off compared to $\tau(y)$ and the fiscal authority receives (weakly) higher net tax revenue. Now, suppose sellers under $\tau^*(y)$ have the incentive to report some $y^\ominus \in (0, \tilde{\tau})$. Assume sellers under $\tau(y)$ also report $y^\ominus$. If $\tau(y^\ominus) = y^\ominus$, then the seller obtain the same surplus by reporting $y^\ominus$ under $\tau(y)$, which means the seller must have the incentive to report $y^\ominus$ under $\tau(y)$ as well, a contradiction. If $\tau(y^\ominus) < y^\ominus$, consider strategy (a). It is easy to show that η must be larger so that $\rho^m(d + m - \tau(y)) = C$ holds. This means q must be smaller. In other words, the seller obtains a higher surplus reporting $y^\ominus$ under $\tau(y)$ compared to reporting $y^\ominus$ under $\tau^*(y)$. If the seller plays strategy (b), then again the seller obtain the same surplus when reporting $y^\ominus$ under $\tau(y)$ or $\tau^*(y)$. In both cases, the seller has the incentive to report $y^\ominus$ under $\tau(y)$ as well, a contradiction.

Second, in case (2), let $\tilde{\tau} = \tau(y^\circ) - \upsilon$ where $\upsilon > 0$ is a small constant. Then, as long as no sellers who meet buyers report $y \in (0, \tilde{\tau})$, agents under $\tau^*(y)$ are strictly better-off compared to $\tau(y)$ and the fiscal authority receives a strictly higher net tax revenue. To see the latter point, note that if all sellers who meet buyers report truthfully, the fiscal authority saves the audit costs. Then as long as υ is small, the fiscal authority receives strictly higher net tax revenue. Now, suppose sellers under $\tau^*(y)$ have the incentive to report some $y^\ominus \in (0, \tilde{\tau})$. Similar to case (1), sellers under $\tau(y)$ can make the same report and receive at least the same surplus. Since sellers who report truthfully are better-off under $\tau^*(y)$, this means sellers who reports $y^\ominus$ under $\tau(y)$ are better-off than they are under $\tau(y)$, a contradiction. □

Proof of Proposition 2.3: First, it is easy to see that an increase in $\tilde{\tau}$ leads to a decrease in $\mathcal{S}$. Now consider sellers who meet buyers in the DM but report $y = 0$. Since it must be that $(1 - \rho^m \eta^0) m^0 - q^0 = 0$ and m^0 is given by

$$m^0 = \frac{1}{1 + \mu} [\beta u(q^0) - \mathcal{S}], \tag{A.28}$$

then η^0 must decrease. We also have

$$\mathcal{S} = \beta u(q^0) - \frac{1 + \mu}{1 - \rho^m \eta^0} q^0 = \beta u(q^0) - \beta q^0 u'(q^0) \tag{A.29}$$

and

$$\frac{\mathrm{d}q^0}{\mathrm{d}\mathcal{S}} = -\frac{1}{\beta q^0 u''(q^0)}. \tag{A.30}$$

Hence,

$$\frac{\mathrm{d}m^0}{\mathrm{d}\mathcal{S}} = \frac{1}{1+\mu}\left[-\frac{u'(q^0)}{q^0 u''(q^0)} - 1\right] > 0. \tag{A.31}$$

Since

$$\frac{\gamma\rho^m m^0}{\alpha - 1 + \gamma} = C, \tag{A.32}$$

then γ must increase.

Second, suppose R^b increases. It is easy to see that it leads to increases in q^h and $\mathcal{S}$. Then from the last case, we know that η^0 must decrease and q^0 must increase. In addition, m^0 will increase, and hence γ must decrease. Lastly, suppose ρ^m increases. Then η^0 must decrease to keep $(1-\rho^m\eta^0)$ constant. There are otherwise no changes in the equilibrium. □

Proof of Proposition 2.4: First, given R^d and μ, q^h solves

$$u'(q) = \frac{1+\mu}{\beta(1+R^d)}. \tag{A.33}$$

Next, given $\tilde{\tau}$, η^0 is such that

$$\frac{1-\rho^m\eta^0}{1+\mu}\left[\beta u(q^0) - \beta u(q^h) + \frac{(1+\mu)(q^h+\tilde{\tau})}{1+R^d}\right] - q^0 = 0, \tag{A.34}$$

$$u'(q^0) = \frac{1+\mu}{\beta(1-\rho^m\eta^0)}. \tag{A.35}$$

γ is such that

$$\frac{\gamma\rho^m m^0}{\alpha - 1 + \gamma} = C. \tag{A.36}$$

The demand for cash is given by

$$\bar{m} = \gamma m^0 = \frac{(\alpha - 1 + \gamma)C}{\rho^m}. \tag{A.37}$$

Note that $\frac{1+\mu}{1+R^b} = \beta\theta + \frac{(1-\theta)(1+\mu)}{1+R^d}$. Then, R^b is such that

$$f'(k) = \frac{1+R^b}{1+\mu}, \tag{A.38}$$

$$(1-\theta)(k^B + \bar{b}) = (1-\gamma)(q^h + \tilde{\tau}). \tag{A.39}$$

Note that (A.39) defines $\bar{b}$ as a function of R^b. Then,

$$\bar{m} + \bar{b} = \frac{q^h + \tilde{\tau}}{1-\theta} - \frac{\gamma}{1-\theta}\left(q^h + \tilde{\tau} - \frac{C(1-\theta)}{\rho^m}\right) - f'(k)k + \frac{(\alpha-1)C}{\rho^m}. \tag{A.40}$$

Assume $\tilde{\tau} - \frac{C(1-\theta)}{\rho^m} \geq 0$. Then $\bar{m} + \bar{b}$ is decreasing in γ. Note that γ is decreasing in m^0, which is increasing in $\mathcal{S}$. q^h and $\mathcal{S}$ are increasing in R^b. Hence, γ is decreasing in R^b. Since $kf'(k)$ is increasing in k, k is decreasing in R^b. Hence, $\bar{m} + \bar{b}$ is increasing in R^b, and there exists a unique equilibrium. □

Proof of Proposition 2.5: Consider an increase in μ and assume that $\frac{1+R^d}{1+\mu}$ is unchanged.

Then $\mathcal{S}$ is unchanged. This means that q^0 is unchanged as well, Since

$$u'(q^0) = \frac{1+\mu}{\beta(1-\rho^m\eta^0)} \tag{A.41}$$

and μ is higher, $1-\rho^m\eta^0$ must increase. Because $m^0 = q^0/(1-\rho^m\eta^0)$, it is lower. Since

$$\frac{\gamma\rho^m m^0}{\alpha-1+\gamma} = C, \tag{A.42}$$

it means γ is larger. Now, recall that

$$\bar{m}+\bar{b} = \frac{q^h+\tilde{\tau}}{1-\theta} - \frac{\gamma}{1-\theta}\left(q^h+\tilde{\tau}-\frac{C(1-\theta)}{\rho^m}\right) - k + \frac{(\alpha-1)C}{\rho^m} \tag{A.43}$$

is decreasing in γ so long as $\tilde{\tau}-\frac{C(1-\theta)}{\rho^m} \geq 0$. This means that if μ is higher and $\frac{1+R^d}{1+\mu}$ is unchanged, $\bar{m}+\bar{b} < \mathcal{D}$. Because $\bar{m}+\bar{b}$ is increasing in R^b, in equilibrium it must be that $\frac{1+R^d}{1+\mu}$ is larger. Then, $\mathcal{S}$ and q^h must be larger in equilibrium, and so is q^0, which means that η^0 must be smaller. Now, if γ is unchanged, then $\bar{m}$ is unchanged. However, because q^h is larger and $\frac{1+R^d}{1+\mu}k$ is smaller, (A.39) suggests that $\bar{b}$ is larger. Hence, $\bar{m}+\bar{b} > \mathcal{D}$, so R^b needs to be smaller. Since γ is decreasing in R^b, γ must be larger, and $\bar{m}$ must be larger and $\bar{b}$ must be smaller.

Lastly, the above results hold as long as there exist $\eta^0 \in (0,1)$ and $\gamma \in (0,1)$ that are part of an equilibrium. To see what this means, note that as μ increases, $\frac{1+R^d}{1+\mu}$ increases and η^0 decreases. Now let $\frac{1+R^d}{1+\mu} = \frac{1}{\beta}$. Then even if $\eta^0 = 0$, it is not possible for (A.34) to hold. Hence, there exists μ' such that if $\mu \geq \mu'$, the equilibrium described above does not exist. □

Proof of Proposition 2.6: First, given R^d and μ, q^h solves

$$u'(q) = \frac{1+\mu}{\beta(1+R^d)}. \tag{A.44}$$

Next, given $\tilde{\tau}$, η^0 is such that

$$\beta u(q^h) - \beta u'(q^h)(q^h+\tilde{\tau}) = \beta u(q^0) - \beta u'(q^0)q^0, \tag{A.45}$$

$$u'(q^0) = \frac{1+\mu}{\beta(1-\rho^c\eta^0)(1+R^c)}. \tag{A.46}$$

Let $c^0 = q^0/(1-\rho^c\eta^0)$. Then, γ is such that

$$\frac{\gamma\rho^c c^0}{\alpha-1+\gamma} = C. \tag{A.47}$$

The demand for CBDC is given by

$$\bar{c} = \gamma c^0 = \frac{(\alpha-1+\gamma)C}{\rho^c}. \tag{A.48}$$

Note that $\frac{1+\mu}{1+R^b} = \beta\theta + \frac{(1-\theta)(1+\mu)}{1+R^d}$. Then, R^b is such that

$$f'(k) = \frac{1+R^b}{1+\mu}, \tag{A.49}$$

$$(1-\theta)(k^B+\bar{b}) = (1-\gamma)(q^h+\tilde{\tau}). \tag{A.50}$$

Then,

$$\bar{c}+\bar{b} = \frac{q^h+\tilde{\tau}}{1-\theta} - \frac{\gamma}{1-\theta}\left(q^h+\tilde{\tau}-\frac{C(1-\theta)}{\rho^c}\right) - f'(k)k + \frac{(\alpha-1)C}{\rho^c}. \tag{A.51}$$

Now, consider an increase in R^c. If R^d is unchanged, then q^h and q^0 will be unchanged. But then $1-\rho^c\eta^0$ will decrease, which means c^0 will increase and γ will decrease. Hence, R^d must decrease, which means q^h and q^0 will decrease. Finally, consider an increase in ρ^c. If R^d is unchanged, then q^h, q^0, and c^0 will be unchanged, but γ will decrease. Hence, R^d must decrease, which means q^h and q^0 will decrease. □

Proof of Proposition 2.7: First, given R and μ, q^h solves

$$u'(q) = \frac{1+\mu}{\beta(1+R)}. \tag{A.52}$$

Then $d^h = q^h + \tilde{\tau}$. Next, given $\tilde{\tau}$, η^0 is such that

$$\beta u(q^h) - \beta u'(q^h)(q^h + \tilde{\tau}) = \beta u(q^0) - \beta u'(q^0)q^0, \tag{A.53}$$

$$u'(q^0) = \frac{1+\mu}{\beta(1-\rho^c\eta^0)(1+R)}. \tag{A.54}$$

Let $c^0 = q^0/(1-\rho^c\eta^0)$. Then, γ is such that

$$\frac{\gamma\rho^c c^0}{\alpha - 1 + \gamma} = C, \tag{A.55}$$

The demand for government bonds is given by

$$f'(k) = \frac{1+R^b}{1+\mu}, \tag{A.56}$$

$$(1-\theta)(k^B + \bar{b}) = \epsilon(1-\gamma)(q^h + \tilde{\tau}). \tag{A.57}$$

where R^b is given by $\frac{1+\mu}{1+R^b} = \beta\theta + \frac{(1-\theta)(1+\mu)}{1+R}$. Then, we have

$$\bar{c} + \bar{b} = \frac{(\alpha-1+\gamma)C}{\rho^c} + (1-\epsilon)(1-\gamma)(q^h+\tilde{\tau}) + \frac{\epsilon(1-\gamma)(q^h+\tilde{\tau})}{1-\theta} - f'(k)k. \tag{A.58}$$

Now, consider an increase in R. Then q^h and q^0 will increase. In addition, c^0 will increase because $c^0 = q^0/(1-\rho^c\eta^0) = \beta(1+R)q^0u'(q^0)/(1+\mu)$. This means γ and ϵ will decrease. Finally, consider an increase in ρ^c. If R^c is unchanged, then q^h, q^0, and c^0 will be unchanged. However, this means that γ will decrease. □

Proof of Proposition 2.8: First, given R^c and μ, q^h solves

$$u'(q) = \frac{1+\mu}{\beta(1+R^c)}. \tag{A.59}$$

Next, given $\tilde{\tau}$, η^0 is such that

$$\beta u(q^h) - \beta u'(q^h)(q^h + \tilde{\tau}) = \beta u(q^0) - \beta u'(q^0)q^0, \tag{A.60}$$

$$u'(q^0) = \frac{1+\mu}{\beta(1-\eta^m\eta^0)}. \tag{A.61}$$

Let $m^0 = q^0/(1-\rho^m\eta^0)$. And γ is such that

$$\frac{\gamma\rho^m m^0}{\alpha - 1 + \gamma} = C. \tag{A.62}$$

The demand for government bonds is given by

$$f'(k) = \frac{1+R^b}{1+\mu}, \tag{A.63}$$

$$(1-\theta)(k^B + \bar{b}) = \epsilon(1-\gamma)(q^h + \tilde{\tau}). \tag{A.64}$$

Then,

$$\bar{m}+\bar{c}+\bar{b}=\frac{(\alpha-1+\gamma)C}{\rho^m}+(1-\epsilon)(1-\gamma)(q^h+\tilde{\tau})+\frac{\epsilon(1-\gamma)(q^h+\tilde{\tau})}{1-\theta}-f'(k)k. \tag{A.65}$$

Now, consider an increase in R^c while μ is held constant. Then q^h and q^0 will increase. In addition, m^0 will increase because $m^0 = q^0/(1-\rho^m\eta^0) = \beta q^0 u'(q^0)/(1+\mu)$. Hence, γ will decrease. Since the demand for government liabilities increases, ϵ must decrease. □

Proof of Proposition 2.9: Let γ^{bench} and η^{bench} denote the share of sellers who evade taxes and the audit probability in the benchmark equilibrium, respectively. First, in a type-2 equilibria, assume

$$\frac{1+\mu}{(1-\rho^c\eta^{bench})(1+R^c)}=\frac{1+\mu}{1-\rho^m\eta^{bench}}. \tag{A.66}$$

Then q^0 and q^h will be unchanged. However, $c^0 = q^0/(1-\rho^c\eta^{bench}) > m^0 = q^0/(1-\rho^m\eta^{bench})$ because either $R^c > 0$ or $\rho^c > \rho^m$. This means that

$$\frac{\gamma^{bench}\rho^c c^0}{\alpha-1+\gamma}>C. \tag{A.67}$$

Hence, R^d must decrease and $\gamma < \gamma^{bench}$. Now, from Proposition 2.6 we know that the γ is decreasing in R^c and ρ^c. This means that γ in a type-3 equilibrium must be smaller than in a type-2 equilibrium.

Finally, given R^c, q^0 are the same in both type-3 equilibria and type-4 equilibria. However, $m^0 = q^0/(1-\rho^m\eta^0)$ in type-4 equilibria but $c^0 = q^0/(1-\rho^c\eta^0)$. Since $\rho^c \geq \rho^m$, $m^0 \geq c^0$. In a type-4 equilibrium, γ is given by

$$\frac{\gamma\rho^m m^0}{\alpha-1+\gamma}=C. \tag{A.68}$$

In a type-3 equilibrium, γ is given by

$$\frac{\gamma\rho^c c^0}{\alpha-1+\gamma}=C. \tag{A.69}$$

Hence, γ is smaller in a type-3 equilibrium than in a type-3 equilibrium. □

Proof of Proposition 2.10: Note that q^h is given by

$$u'(q^h)=\frac{1+\mu}{\beta(1+R^d)}. \tag{A.70}$$

And q^0 is given by

$$\mathcal{S}=\beta u(q^h)-\beta u'(q^h)(q^h+\tilde{\tau})=\beta u(q^0)-\beta u'(q^0)q^0. \tag{A.71}$$

Hence, q^h and q^0 only depend on R^d. From the proof of Proposition 2.9, we know that R^d is lower in a type-2 equilibrium than in the benchmark equilibrium. In a type-3 equilibrium or a type-4 equilibrium, $R^d = R^c$. Hence, compared to the benchmark, q^0 and q^h are smaller in type-2 equilibria, and larger in type-4 equilibria. Also, q^0 and q^h are smaller in type-3 equilibria if $R^c < R^{bench}$ but larger if $R^c > R^{bench}$. □

Proof of Proposition 2.11: Since given R^c, γ is smaller in type-3 equilibria than in type-4 equilibria while q^0 and q^h are equal in the two types of equilibria, it is clear that the total surplus in DM is the higher in type-3 equilibria than type-4 equilibria given R^c. Next, because when $R^c = R^{bench}$, q^0 and q^h are equal in type-3 equilibria and the benchmark equilibrium but γ is smaller in type-3 equilibria, the total surplus in DM is the higher in type-3 equilibria. This means that there exists $R^{c\prime} < R^{bench}$ such that for any $R^c \geq R^{c\prime}$, the total surplus in DM is higher in type-3 equilibria than in the benchmark equilibrium. Finally, recall that q^0 and q^h are lower in type-2 equilibria than in the benchmark equilibrium, while γ is lower in type-3 equilibria than in type-2 equilibria given ρ^c. This means that there exists $R^{c\dagger}$ such that for any $R^c \geq R^{c\dagger}$, the total surplus in DM is higher in type-3 equilibria than in type-2 equilibria. □

Proof of Proposition 2.13: First, in type-2 equilibria, suppose that for some $\rho^c \geq \rho^m$, R^c is such that

$$\frac{1+\mu}{(1-\rho^c\eta^{bench})(1+R^c)} = \frac{1+\mu}{1-\rho^m\eta^{bench}}. \tag{A.72}$$

Because $c^0 = q^0/(1-\rho^c\eta^{bench}) > m^0 = q^0/(1-\rho^m\eta^{bench})$, it must be that

$$\frac{\gamma^{bench}\rho^c c^0}{\alpha - 1 + \gamma} > C. \tag{A.73}$$

Hence, R^d must decrease and $\eta^0 > \gamma^{bench}$. Next, from the proof of Proposition 2.6, it is easy to see that η^0 will further increase as R^c increases. Hence, η^0 is larger in type-2 equilibria than in the benchmark equilibrium.

Second, on the boundary of type-1 and type-3 equilibria, it must be that

$$\frac{1+\mu}{(1-\rho^c\eta^{bench})(1+R^c)} = \frac{1+\mu}{1-\rho^m\eta^{bench}}. \tag{A.74}$$

From Proposition 2.7, we know that η^0 will further increase as ρ^c decreases. Hence, η^0 is larger in type-3 equilibria than in the benchmark equilibrium.

Finally, in a type-4 equilibrium, q^0 is larger than its counterpart in the benchmark. Since

$$u'(q^0) = \frac{1+\mu}{\beta(1-\rho^m\eta^0)}, \tag{A.75}$$

it must be that η^0 is larger in type-4 equilibria than in the benchmark equilibrium. □

Proof of Proposition 2.14: I only need to show that for any given R^c, when $\rho^c = \rho^c(R^c)$, γ is larger in a type-4 equilibrium than in a type-3 equilibrium, while η^0 is equal in type-3 and type-4 equilibria. Note that on the boundary of type-3 and type-4 equilibria, it must be that

$$\frac{1+\mu}{(1-\rho^c\eta^0)(1+R^c)} = \frac{1+\mu}{1-\rho^m\eta^0}. \tag{A.76}$$

Hence, η^0 is equal in type-3 and type-4 equilibria. Next, note that $c^0 = q^0/(1-\rho^c\eta^{bench}) > m^0 = q^0/(1-\rho^m\eta^{bench})$. In addition, in a type-4 equilibriumγ is given by

$$\frac{\gamma\rho^m m^0}{\alpha - 1 + \gamma} = C. \tag{A.77}$$

In a type-3 equilibrium, γ is given by

$$\frac{\gamma\rho^c c^0}{\alpha - 1 + \gamma} = C. \tag{A.78}$$

Hence, γ is larger in a type-4 equilibrium than in a type-3 equilibrium. □

Proof of Proposition 2.15: Following Proposition 2.9-2.13, if $(R^c, \rho^c) = (R^{bench}, \rho^c(R^{bench}))$, then η^0, q^0, q^h, and k in a type-3 equilibrium are all identical to their counterparts in the benchmark equilibrium. However, γ is strictly lower. This means that the DM surplus is strictly higher, while the audit costs are strictly lower. Hence, the aggregate welfare is strictly higher in a type-3 equilibrium than in the benchmark equilibrium. In other words, if CBDC is deposit-like, then introducing CBDC increases aggregate welfare. Next, consider cash-like CBDC. We know that as long as (R^c, ρ^c) are close to point A, then the change in γ and k will be small compared to the benchmark. However, η^0 is strictly higher, while q^0 and q^h are strictly lower. Hence, aggregate welfare is higher with deposit-like CBDC than with cash-like CBDC. □

Proof of Proposition 2.16: If Assumption 2.5.1 holds, because the demand for CBDC is strictly lower than $\mathcal{D}/(1+R^d)$, compared to the benchmark, the central bank's net revenue is lower after introducing deposit-like CBDC. Next, consider cash-like CBDC. As long as (R^c, ρ^c) are close to point A, the central bank's revenue will be close to (but strictly lower than) its revenue in the benchmark. □

A.3 Extension: Zero Lump-sum Transfer

In this appendix, I consider an alternative setup where the fiscal authority sets $T = 0$. It balances the budget through issuing government bonds. I restrict my attention to the benchmark case.

The equilibrium characterization is similar to the original model. The main difference is how the nominal government bond rate, R^b, is determined in equilibrium. In the original model, the demand for cash, $\bar{m}$, and the demand for government bonds by bankers, $\bar{b}$, must satisfy

$$\bar{m} + \bar{b} = \mathcal{D}, \tag{A.79}$$

where $\mathcal{D}$ is determined by the fiscal authority. It can be shown that $\bar{m}$ is given by

$$\bar{m} = \frac{(\alpha - 1 + \gamma)C}{\rho}, \tag{A.80}$$

where γ is the proportion of sellers who evade taxes. $\bar{b}$ is given by

$$\bar{b} = \frac{1-\gamma}{1-\theta}(q^h + \tilde{\tau}) - \frac{1+R^d}{1+\mu}k. \tag{A.81}$$

where q^h is the amount of DM good produced by sellers who report their income truthfully, and k is loans to entrepreneurs. It can be shown that γ and k is decreasing in R^b, while q^h is increasing in R^b. If $\frac{C}{\rho} \leq \tilde{\tau}$, then $\bar{m} + \bar{b}$ is strictly increasing in R^b. Hence, there exists a unique R^b such that $\bar{m} + \bar{b} = \mathcal{D}$.

Now, suppose that $\tilde{\tau}$ is fixed, while $\bar{b}$ is adjusted to balance the budget. Define $\mathcal{G}(\bar{b})$ to be

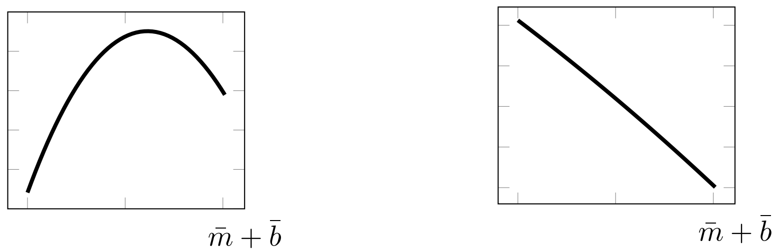
$$\mathcal{G}(\bar{b}) = \mu\bar{m} - \left(1 - \frac{1+\mu}{1+R^b}\right)\bar{b} + (1-\gamma)\tilde{\tau}. \tag{A.82}$$

Then $\mathcal{G}(\bar{b})$ represents the fiscal authority's net revenue. One can conclude from the benchmark model that $\bar{b}$ is increasing in R^b. Hence, $\bar{m}$ is decreasing in $\bar{b}$. Because γ is decreasing in R^b, the net tax revenue is increasing in $\bar{b}$. Depending on parameter values, $\mathcal{G}(\bar{b})$ may not be monotonic in $\bar{b}$, as demonstrated by the numerical examples in Figure A.2.[1]

First, in panel (a) of Figure A.2, $\tilde{\tau}$ is relatively large. Net government revenue first increases with $\bar{b}$, because the increase in net tax revenue offsets the increase in the cost of servicing government bonds (the second term in (A.82)). However, as $\bar{b}$ continues to increase, the cost of servicing government bonds becomes the dominant force and net government revenue decreases. In panel (b) of Figure A.2, $\tilde{\tau}$ is low. In this case, the increase in the cost of servicing government bonds is always larger than the increase in net tax revenue. Hence, the net government revenue is always decreasing in $\bar{b}$. In both cases, seigniorage income (the first term in (A.82)) also decreases with $\bar{b}$. However, as long as the seigniorage income is small, which happens when γ is small, its

[1] In both panels, $u(c) = 2x^{0.5}$, $f(k) = k^{0.3}$, $\mu = 1\%$, $\beta = 0.99$, $\rho = 0.5$, $C = 0.1$, and $\alpha = 1.5$. In panel (a), $\tilde{\tau} = 0.45$. In panel (b), $\tilde{\tau} = 0.2$.

impact on net government revenue is insignificant.



(a) High $\tilde{\tau}$ (b) Low $\tilde{\tau}$

Figure A.2: Net Government Revenue and the Supply of Government Liability ($\bar{m} + \bar{b}$)

If net government revenue is a non-monotonic function of government liabilities, then there may exist multiple equilibria as shown in Figure A.3. Specifically, there may exist an equilibrium where $\bar{b}$ and $\bar{b} + \bar{m}$ are small and R^b is low, and another equilibrium where $\bar{b}$ and $\bar{b} + \bar{m}$ are large and R^b is high. Multiple equilibria exist because a higher supply of government bonds drives up R^b. For bankers, a higher bond return decreases the cost of holding government bonds as collateral, which translates into a higher deposit rate. Then, fewer sellers choose to under-report their income, and the net tax revenue increases. The fiscal authority can then use the additional tax revenue to pay for the increase in the cost of servicing government bonds, which is why the fiscal authority can maintain a net revenue of zero. Note that buyers are better off in the high R^b equilibrium.

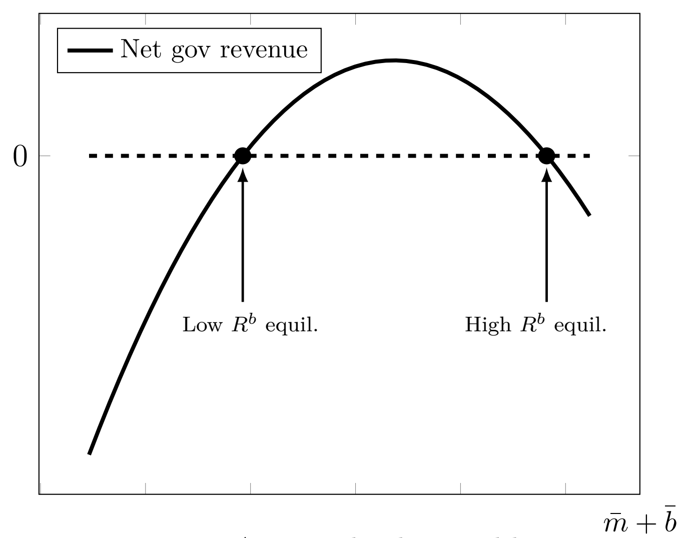


Figure A.3: Multiple Equilibria

A.4 Extension: Other Equilibria with CBDC

A.4.1 CBDC offers less Anonymity than Bank Deposits

I assume that the fiscal authority can costlessly observe any income sellers receive in CBDC. However, to observe income in bank deposits, the fiscal authority has to audit sellers. In equilibrium, CBDC is accepted by sellers who report truthfully, while bank deposits compete with cash in transactions involving tax evasion.

The strategies of sellers who evade taxes can be divided into the following three cases:
(1) Only cash is accepted by sellers who evade taxes: for this to be part of an equilibrium, it must be that $1 + R^d < (1 - \rho\eta^0)/(1 - \eta^0)$.
(2) Only bank deposits are accepted by sellers who evade taxes: for this to be part of an equilibrium, it must be that $1 + R^d > (1 - \rho\eta^0)/(1 - \eta^0)$.
(3) Sellers who evade taxes are indifferent between cash and bank deposits: for this to be part of an equilibrium, it must be that $1 + R^d = (1 - \rho\eta^0)/(1 - \eta^0)$.

In what follows, I focus my attention on case (3). Let d^0 and m^0 represent the deposit and cash payments to sellers. Let the proportion of sellers who use bank deposits be γ^d and let the proportion of sellers who use cash be γ^m. In equilibrium, the fiscal authority must be indifferent between auditing sellers or not. Hence, γ^d and γ^m satisfy

$$\frac{\gamma^d d^0}{\alpha - 1 + \gamma^d + \gamma^m} + \frac{\gamma^m \rho m^0}{\alpha - 1 + \gamma^d + \gamma^m} = C. \tag{A.83}$$

The next proposition shows the existence and uniqueness of the equilibrium.

Proposition A.1 *There exist $\mathcal{D}^\diamond$ and $\mathcal{D}^{\diamond\diamond}$ such that if $\mathcal{D} \in [\mathcal{D}^\diamond, \mathcal{D}^{\diamond\diamond}]$ and C is small, there exists a unique equilibrium where both cash and bank deposits are used by sellers who evade taxes. In addition, increasing R^c while holding μ constant leads to increases in q^h, q^0, and decreases in R^b and η^0.*

Proof: First, given R^c and μ, q^h solves

$$u'(q) = \frac{1 + \mu}{\beta(1 + R^c)}. \tag{A.84}$$

Next, given $\tilde{\tau}$, η^0 is such that

$$\frac{1 - \rho\eta^0}{1 + \mu}\left[\beta u(q^0) - \beta u(q^h) + \frac{(1 + \mu)(q^h + \tilde{\tau})}{1 + R^d}\right] - q^0 = 0, \tag{A.85}$$

$$u'(q^0) = \frac{1 + \mu}{\beta(1 - \rho\eta^0)}. \tag{A.86}$$

And R^d is given by $1 + R^d = (1 - \rho\eta^0)/(1 - \eta^0)$. It is easy to see that the following conditions hold.

$$\frac{(1 - \eta^0)(1 + R^d)}{1 + \mu}\left[\beta u(q^0) - \beta u(q^h) + \frac{(1 + \mu)(q^h + \tilde{\tau})}{1 + R^c}\right] - q^0 = 0, \tag{A.87}$$

$$u'(q^0) = \frac{1 + \mu}{\beta(1 - \eta^0)(1 + R^d)}. \tag{A.88}$$

γ^d and γ^m are such that

$$\frac{\gamma^d d^0}{\alpha - 1 + \gamma^d + \gamma^m} + \frac{\gamma^m \rho m^0}{\alpha - 1 + \gamma^d + \gamma^m} = C, \tag{A.89}$$

where $d^0 = q^0/(1 - \eta^0)$ and $m^0 = q^0/(1 - \rho\eta^0)$. The demand for government liabilities is given by

$$\tilde{\mathcal{D}} = (1 - \gamma^d - \gamma^m)(q^h + \tilde{\tau}) + \frac{\gamma^d d^0}{1 - \theta} + \gamma^m m^0. \tag{A.90}$$

where $\tilde{\mathcal{D}} = \mathcal{D} + k$. Then γ^d and γ^m solve (A.89) and (A.90). We have

$$\gamma^d = \frac{(\rho m^0 - C)(\tilde{\mathcal{D}} - q^h - \tilde{\tau}) - (m^0 - q^h - \tilde{\tau})(\alpha - 1)C}{(d^0/(1-\theta) - q^h - \tilde{\tau})(\rho m^0 - C) - (d^0 - C)(m^0 - q^h - \tilde{\tau})}, \tag{A.91}$$

$$\gamma^m = \frac{(d^0/(1-\theta) - q^h - \tilde{\tau})(\tilde{\mathcal{D}} - q^h - \tilde{\tau}) - (d^0 - C)(\alpha - 1)C}{(d^0/(1-\theta) - q^h - \tilde{\tau})(\rho m^0 - C) - (d^0 - C)(m^0 - q^h - \tilde{\tau})} \tag{A.92}$$

Then it is easy to see that there exist $\mathcal{D}^\diamond$ and $\mathcal{D}^{\diamond\diamond}$ such that if $\mathcal{D} \in [\mathcal{D}^\diamond, \mathcal{D}^{\diamond\diamond}]$, γ^d and γ^m will take values between 0 and 1. □

An equilibrium where sellers who evade taxes play a mixed strategy exists provided the total supply of government debt, $\mathcal{D}$, is neither too large nor too small. This is because if $\mathcal{D}$ is too large, R^b may be so high that sellers who evade taxes only accept bank deposits. Then, the equilibrium is the same as when CBDC replaces cash. If $\mathcal{D}$ is too small, government liabilities may not be able to support DM consumption, which is determined solely by R^c, ρ, and μ but not $\mathcal{D}$.

Now, suppose the central bank increases R^c while holding μ constant. An increase in R^c attracts sellers to accept CBDC and report their income truthfully. This lowers the fiscal authority's incentive to audit sellers, which leads to an increase in $\gamma^c + \gamma^m$. A lower η^0 first increases q^0 and d^0, but the subsequent increase in the demand of government bonds drives down R^b. Nevertheless, the decrease in audit probability outweighs the decrease in R^b. In equilibrium, q^0 increases.

A.4.2 CBDC offers More Anonymity than Cash

I assume that if an agent is audited, the fiscal authority observes his or her income in CBDC with probability $\rho^c < \rho^m$. Now, suppose a seller choose to evade taxes. If the seller chooses to accept cash, he or she solves

$$\max_{q,m} \{(1 - \rho\eta^0)m - q\} \text{ s.t. } -(1+\mu)m + \beta u(q) = \mathcal{S}. \tag{A.93}$$

For this to be part of an equilibrium, it must be that $1 + R^d \leq (1 - \rho\eta^0)/(1 - \eta^0)$ and $1 + R^c \leq (1 - \rho\eta^0)/(1 - \rho^c\eta^0)$. The first condition guarantees that the seller does not have the incentive to use bank deposits, and the second condition guarantees that the seller does not have the incentive to use CBDC. If the seller chooses to accept CBDC, he or she solves

$$\max_{q,d} \{(1 - \rho^c\eta^0)c - q\} \text{ s.t. } -\frac{(1+\mu)c}{1+R^c} + \beta u(q) = \mathcal{S}. \tag{A.94}$$

For this to be part of an equilibrium, it must be that $(1 + R^d)/(1 + R^c) \leq (1 - \rho\eta^0)/(1 - \eta^0)$ and $1 + R^c \geq (1 - \rho\eta^0)/(1 - \rho^c\eta^0)$. The first condition guarantees that the seller does not have the incentive to use bank deposits, and the second condition guarantees that the seller does not have the incentive to use cash.

There are three possible equilibria:
(1) Only cash is accepted by sellers who evade taxes: for this to be part of an equilibrium, it must be that $1 + R^c < (1 - \rho\eta^0)/(1 - \rho^c\eta^0)$.
(2) Only CBDC is accepted by sellers who evade taxes: for this to be part of an equilibrium, it must be that $1 + R^c > (1 - \rho\eta^0)/(1 - \rho^c\eta^0)$.
(3) Sellers who evade taxes are indifferent between cash and CBDC: for this to be part of an equilibrium, it must be that $1 + R^c = (1 - \rho\eta^0)/(1 - \rho^c\eta^0)$.

In what follows, I focus on case (3). Let c^0 and m^0 represent the CBDC and cash payments to sellers. Let the proportion of sellers who use CBDC be γ^c and let the proportion of sellers who use cash be γ^m. In equilibrium, it must be that the fiscal authority is indifferent between auditing sellers or not. Hence, γ^c and γ^m must satisfy

$$\frac{\gamma^c \rho^c c^0}{\alpha - 1 + \gamma^c + \gamma^m} + \frac{\gamma^m \rho m^0}{\alpha - 1 + \gamma^c + \gamma^m} = C, \tag{A.95}$$

where $\frac{\gamma^c}{\alpha-1+\gamma^c+\gamma^m}$ $(\frac{\gamma^m}{\alpha-1+\gamma^c+\gamma^m})$ is the probability that a seller who reports an income of zero received CBDC (cash). To solve for the equilibrium, note that given R^c, ρ, and ρ^c, the condition $1 + R^c = (1 - \rho\eta^0)/(1 - \rho^c\eta^0)$ determines η^0. This means that q^0 and $\mathcal{S}$ are determined by μ:

$$u'(q^0) = \frac{1+\mu}{\beta(1-\rho\eta^0)}, \tag{A.96}$$

$$\mathcal{S} = \beta u(q^0) - \frac{(1+\mu)q^0}{1-\rho\eta^0}. \tag{A.97}$$

Then, because sellers who report truthfully use bank deposits, $\mathcal{S}$ and μ determine R^d:

$$u'(q^h) = \frac{1+\mu}{\beta(1+R^d)}, \tag{A.98}$$

$$\mathcal{S} = \beta u(q^h) - \frac{(1+\mu)q^h}{1+R^d}. \tag{A.99}$$

This means that the terms of trade in the DM are determined by only four parameters: R^c, ρ, ρ^c, and μ. Because $1 + R^c = (1 - \rho\eta^0)/(1 - \rho^c\eta^0)$, sellers who evade taxes produce the same amount of DM good regardless of the payment method they use. Lastly, γ^c and γ^m are such that (A.95) holds, and the total demand for government liabilities (cash, CBDC, and government bonds) is equal to the supply of government liabilities prescribed by the fiscal authority, $\mathcal{D}$. The next proposition shows the effects of R^c on the equilibrium.

Proposition A.2 *Suppose the central bank increases R^c while holding μ constant. If $\rho^c < \rho^m$, then η^0 and k will decrease, while q^h, q^0, and R^b will increase.*

Proof: First, if $\rho^c < \rho$, then η^0 is decreasing in R^c. This means q^0 and $\mathcal{S}$ must increase because

$$u'(q^0) = \frac{1+\mu}{\beta(1-\rho\eta^0)}, \tag{A.100}$$

$$\mathcal{S} = \beta u(q^0) - \frac{(1+\mu)q^0}{1-\rho\eta^0}. \tag{A.101}$$

Since

$$u'(q^h) = \frac{1+\mu}{\beta(1+R^d)}, \tag{A.102}$$

$$\mathcal{S} = \beta u(q^h) - \frac{(1+\mu)q^0}{1+R^b}, \tag{A.103}$$

q^h and R^b will increase as well. Next, note that because

$$c^0 = \frac{q^0}{1-\rho^c\eta^0} = \frac{\beta(1+R^c)q^0 u'(q^0)}{1+\mu}, \tag{A.104}$$

$$m^0 = \frac{q^0}{1-\rho\eta^0} = \frac{\beta q^0 u'(q^0)}{1+\mu}, \tag{A.105}$$

and q^0 increases, then m^0 will increase but c^0 will decrease. □

If $\rho^c < \rho$, the effect of an increase in R^c on the equilibrium is the opposite because $(1 - \rho\eta^0)/(1 - \rho^c\eta^0)$ is decreasing in η^0. Intuitively, because $\rho^c < \rho$, the payment in CBDC is smaller than the payment in cash (in real terms). An increase in R^c attracts sellers who evade taxes to accept CBDC and decreases the left-hand side of (A.95). Hence, η^0 must decrease. As a result, more sellers evade taxes, and $\gamma^c + \gamma^m$ increases. Note that in this case, R^c must be strictly negative for sellers who evade taxes to accept cash.

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Appendix B

Appendices to Chapter 3

B.1 Definition of Undefeated Equilibria

In this appendix I provide the definition of Undefeated Equilibria when there are two or more types of sellers. Let $j = 1, 2, ..., J$ denote the types of sellers. Let Δ_j denote the proportion of type j sellers.

Definition B.1 (Mailath et al., 1993) *A Perfect Bayesian Equilibrium characterized by* $\{(\psi_j^*, s_j^*)\}_{j=1}^J$ *and* $\gamma^*(j|\psi, s)$ *is defeated by an alternative Perfect Bayesian Equilibrium characterized by* $\{(\psi_j', s_j')\}_{j=1}^J$ *and* $\gamma'(j|\psi, s)$ *if*
(a) $\exists(\psi'', s'') \in \{(\psi_j', s_j')\}_{j=1}^J$ *such that* $(\psi'', s'') \notin \{(\psi_j^*, s_j^*)\}_{j=1}^J$, *and* $K = \{j|(\psi_j', s_j') = (\psi'', s'')\} \neq \emptyset$;
(b) $\forall j \in K$, *their utility is (weakly) higher in the alternative equilibrium;* $\exists j \in K$ *such that her utility is strictly higher in the alternative equilibrium; and*
(c) $\exists j' \in K$ *such that* $\gamma^*(j'|\psi'', s'') \neq \Delta_{j'}\pi(j')/(\sum_{j=1}^J \Delta_j \pi(j))$ *for any* $\pi : \{l, h\} \to [0, 1]$ *satisfying*
i. $\pi(j) = 1$ *if* $j \in K$ *and her utility is strictly higher in the alternative equilibrium; and*
ii. $\pi(j) = 0$ *if* $j \notin K$.

B.2 Proofs for Section 3.3.1

In this appendix I solve agents' AM problem in Section 3.3.1 and prove Proposition 3.1. I also solve agents' problem in the CM (see Proposition B.5). Before I solve the AM problem, it is convenient to define a belief system that will be useful later. First, define v_j^o, $j \in \{l, h\}$, to be the lowest possible surplus any asset seller can obtain from the AM. It is given by

$$v_j^o = \max_{\psi, s} \quad u(z + \psi s) - u(z) - \delta_j s \tag{B.1}$$

$$\text{s.t. } s[\delta_l - \psi] \geq 0, \tag{B.2}$$

$$\psi s \leq z \text{ and } s \leq a. \tag{B.3}$$

Next, let $v_j(\psi, s) = u(z + \psi s) - u(z) - \delta_j s$ for some (ψ, s). Now, consider the following belief system, $\gamma^*(j|\psi, s)$. For $(\psi, s) \in \{(\psi_l^\dagger, s_l^\dagger), (\psi_h^\dagger, s_h^\dagger)\}$ where $\{(\psi_l^\dagger, s_l^\dagger), (\psi_h^\dagger, s_h^\dagger)\}$ is some set of seller offers, let $\gamma^*(j|\psi,s)$ be given by

$$\gamma^*(\delta_h|\psi,s) = \frac{\mathbb{1}(v_h(\psi,s) > v_h^o)(1-\Delta)}{\mathbb{1}(v_l(\psi,s) > v_l^o)\Delta + \mathbb{1}(v_h(\psi,s) > v_h^o)(1-\Delta)} \tag{B.4}$$

and

$$\gamma^*(\delta_l|\psi,s) = \frac{\mathbb{1}(v_l(\psi,s) \geq v_l^o)\Delta}{\mathbb{1}(v_l(\psi,s) \geq v_l^o)\Delta + \mathbb{1}(v_h(\psi,s) > v_h^o)(1-\Delta)}, \tag{B.5}$$

where $\mathbb{1}(.)$ is an indication function that equals one if the statement in the bracket is true. For all $(\psi,s) \notin \{(\psi_l^\dagger, s_l^\dagger), (\psi_h^\dagger, s_h^\dagger)\}$, $\gamma(\delta_l|\psi,s) = 1$.

To solve the equilibrium, I consider two possibilities: pooling (i.e., $(\psi_l, s_l) = (\psi_h, s_h)$) and separating (i.e., $(\psi_l, s_l) \neq (\psi_h, s_h)$). In each case, I prove the proposed equilibrium satisfies Definition 3.1. Lastly, I compare these two cases and determine which equilibrium satisfies condition (3) of Definition 3.1.

Proposition B.1 *Define $\bar{\delta} = (1-\Delta)\delta_h + \Delta\delta_l$. The unique pooling equilibrium that satisfies Definition* 3.1 *is characterized by $\gamma^*(j|\psi,s)$ and $(\psi_l^\dagger, s_l^\dagger) = (\psi_h^\dagger, s_h^\dagger) = (\psi_p, s_p)$ where $\psi_p = \bar{\delta}$, $s_p = m_p/\psi_p$, and m_p is given by* (B.11).

Proof: Consider the following problem

$$\max_{\psi,s} u(z_s + \psi s) - u(z_s) - \delta_h s \tag{B.6}$$

$$\text{s.t.} u(z_s + \psi s) - u(z_s) - \delta_l s \geq v^*, \tag{B.7}$$

$$\bar{\delta} - \psi = 0, \tag{B.8}$$

$$\psi s \leq z_b \text{ and } s \leq a \tag{B.9}$$

where $v^* = v_l^o$. To solve the problem, I find it is easier to define $m = \psi s$. Let m_l^* solves $u'(z_s + m_l^*) = 1$. We have

$$v^* = u(z_s + m) - u(z_s) - m \text{ where } m = \min\{m_l^*, z_b, \delta_l a\}. \tag{B.10}$$

Next, let m_h^* solves $u'(z_s + m_h^*) = \delta_h/\bar{\delta}$ and let m_h solves $u(z_s + m_h) - u(z_s) - \delta_l/\bar{\delta} m_h = v^*$. Then

$$m_p = \min\{\max\{m_h^*, m_h\}, z_b, \bar{\delta}a\}. \tag{B.11}$$

Now, I claim that $(\psi_p, s_p) = (\bar{\delta}, m_p/\bar{\delta})$ and $\gamma^*(j|\psi,s)$ constitute a pooling equilibrium that satisfies Definition 3.1. It is easy to see that conditions (1) and (2) are satisfied. Now, consider condition (3) (see Definition B.1). First, there is obviously no other pooling equilibrium that offers higher utility to high-quality sellers. Second, condition (c) of Definition B.1 is violated if low-quality sellers want to deviate since $\gamma^*(\delta_l|\psi,s) = 1$ for all $(\psi,s) \neq (\psi_p, s_p)$. Hence the proposed equilibrium is undefeated.

To show it is unique, note that high-quality sellers will be strictly worse off if they deviate from (ψ_p, s_p) because it maximizes their surplus given the constraints. Then, such equilibria are defeated by any equilibrium where high-quality sellers are strictly better off while low-quality sellers are at least as well off. In other words, the only undefeated pooling equilibrium is the one that maximizes high-quality sellers' surplus. □

Now, I turn to separating equilibria.

Proposition B.2 *The unique separating equilibrium that satisfies Definition* 3.1 *is characterized by $\gamma^*(j|\psi,s)$ and $\psi_l^\dagger = \delta_l$, $\psi_h^\dagger = \delta_h$, $s_l^\dagger = m_l/\psi_l$, and $s_h^\dagger = m_h/\psi_h$, where m_l and m_h are given by* (B.17) *and* (B.18).

Proof: Consider the following problem

$$\max_{\psi,s} u(z_s+\psi s)-u(z_s)-\delta_h s \tag{B.12}$$

$$\text{s.t. } s(\delta_h-\psi)=0, \tag{B.13}$$

$$v^* \geq u(z_s+\psi s)-u(z_s)-\delta_l s. \tag{B.14}$$

where $v^* = v_l^o$. One can rewrite the first problem as

$$\max_m \left[u(z_s+m)-u(z_s)-m\right] \tag{B.15}$$

$$\text{s.t. } v^* \geq \left[u(z_s+m)-u(z_s)-m\frac{\delta_l}{\delta_h}\right] \tag{B.16}$$

where $v^* = u(z_s+m_l)-u(z_s)-m_l$, and

$$m_l = \min\{m^*, z_b, \delta_l a\} \tag{B.17}$$

and $u'(m^*)=1$. Now let the m_h be given by

$$v^* = u(z_s+m_h)-u(z_s)-m_h\frac{\delta_l}{\delta_h} \tag{B.18}$$

Then m_h is the solution to the above problem.

Now I show that the proposed solution is an equilibrium that satisfies Definition 3.1. It is easy to see that conditions (1) and (2) are satisfied. Now consider condition (3). Among all the separating equilibrium, this equilibrium maximizes high-quality sellers' utility. Hence, sellers have no incentive to deviate. Any other separating equilibrium is therefore defeated by the proposed equilibrium. □

The last step is to choose between the separating equilibrium and the pooling equilibrium.

Proposition B.3 *Whichever equilibrium that offers higher utility to high-quality sellers is undefeated.*

Proof: Firstly, if the pooling equilibrium offers high-quality sellers strictly higher utility, high-quality sellers have the incentive to deviate. In addition, the low-quality sellers will be strictly better off in the pooling equilibrium as well. To see this, note that

$$\begin{aligned}
&u(z_s+m_p)-u(z_s)-m_p\frac{\delta_h}{\bar{\delta}} > u(z_s+m_h)-u(z_s)-m_h \\
\Rightarrow &u(z_s+m_p)-u(z_s)-m_p\frac{\delta_l}{\bar{\delta}}+\left(\frac{\delta_l-\delta_h}{\bar{\delta}}\right)m_p > u(z_s+m_h)-u(z_s)-m_h\frac{\delta_l}{\delta_h}+\left(\frac{\delta_l-\delta_h}{\delta_h}\right)m_h \\
\Rightarrow &u(z_s+m_p)-u(z_s)-m_p\frac{\delta_l}{\bar{\delta}} > u(z_s+m_h)-u(z_s)-m_h\frac{\delta_l}{\delta_h} = v^*
\end{aligned}$$

where the last inequality is because $m_p > m_h$ and $\left(\frac{\delta_l-\delta_h}{\bar{\delta}}\right)m_p < \left(\frac{\delta_l-\delta_h}{\delta_h}\right)m_h < 0$. Then it is easy to check that the separating equilibrium is defeated by the pooling equilibrium.

Next, if the separating equilibrium offers high-quality sellers strictly higher utility, high-quality sellers have the incentive to deviate. Then it is easy to check that the pooling equilibrium is defeated by the separating equilibrium. □

Before I discuss how the equilibrium depends on z_s, I use the following lemma to show how z_s affects constraint (B.7). Define $L_1 = \min\{z_b, \bar{\delta}a\}$ and $L_2 = \min\{z_b, \delta_l a\}$.

Proposition B.4 *Given L_1 and L_2, there exists $0 \leq z_s'' < \hat{g}$ such that for all $z_s < z_s''$, constraint* (B.7) *does not bind; for all $z_s \geq z_s''$, constraint* (B.7) *binds.*

Proof: Define $\hat{g}$ to be such that $u'(\hat{g}) = \frac{\delta_h}{\bar{\delta}}$. First, suppose that $L_1 < \hat{g}$. Consider the case where $z_s + L_1 < \hat{g}$. In this case, $m_p = L_1$ and (B.7) does not bind. Now consider the case where $z_s + L_1 \geq \hat{g}$. If (B.7) does not bind, a low-quality seller's surplus in a pooling equilibrium is given by

$$v_l^p(z_s) = u(\hat{g}) - \frac{\delta_l}{\bar{\delta}}\hat{g} + \frac{\delta_l}{\bar{\delta}}z_s \tag{B.19}$$

while their utility under complete information is

$$v_l^c(z_s) = \begin{cases} u(z_s + L_2) - L_2, \text{ if } z_s + L_2 \leq g^*, \\ u(g^*) - g^* + z_s, \text{ if otherwise.} \end{cases} \tag{B.20}$$

Then we have $\frac{\mathrm{d}v_l^p(z_s)}{dz_s} = \frac{\delta_l}{\bar{\delta}} < 1$ and $\frac{\mathrm{d}v_l^c(z_s)}{dz_s} \geq 1$. Note also that $v_l^p(\hat{g} - L_1) > v_l^c(\hat{g} - L_1)$ and (B.7) binds when $z_s = \hat{g}$. Hence, there exists $z_s'' \in (\hat{g} - L_1, \hat{g})$ such that $v_l^p(z_s'') = v_l^c(z_s'')$. For all $z_s < z_s''$, constraint (B.7) does not bind. For all $z_s \geq z_s''$, constraint (B.7) binds.

Next, assume $L_1 \geq \hat{g}$. It is easy to see that this case is identical to the above case where $z_s + L_1 \geq \hat{g}$. That is, there exists $z_s'' < \hat{g}$ such that for all $z_s < z_s''$, constraint (B.7) does not bind and that for all $z_s \geq z_s''$, constraint (B.7) binds. □

It should be noted that, since $\hat{g} < g^*$ for all $\delta_l < \delta_h$ and $\Delta > 0$, the proof of Proposition B.4 suggests that constraint (B.7) must bind for some $z_s < g^*$. Now I am ready to prove Proposition 3.1.

Proof of Proposition 3.1: The proof has two parts. The first part shows that there exists a cutoff value z_s' such that sellers pool if $z_s < z_s'$ and separate if $z_s \geq z_s'$. Part II shows z_s' is decreasing in z_b and a. Throughout this proof, the quantities m_l, m_h and m_p are given by Proposition B.1 and Proposition B.2. Note that they are functions of z_s, z_b and a. I omit this dependence so long as there is no confusion.

Part I. The cutoff value, z_s', may depend on z_b and a because trade in the AM can be constrained by either z_b or a. First, assume $\min\{z_b, \bar{\delta}a\} < \hat{g}$. I show that if the incentive constraint in the pooling equilibrium, (B.7), binds, then the separating equilibrium is undefeated. Assume (B.7) binds, we have

$$u(z_s + m_p) - u(z_s) - \frac{\delta_l}{\bar{\delta}}m_p = u(z_s + m_h) - u(z_s) - \frac{\delta_l}{\delta_h}m_h. \tag{B.21}$$

Note that $m_p > m_h$. Hence,

$$u(z_s + m_p) - u(z_s) - \frac{\delta_h}{\bar{\delta}}m_p < u(z_s + m_h) - u(z_s) - m_h. \tag{B.22}$$

because $\frac{\delta_l}{\bar{\delta}}m_p > \frac{\delta_l}{\delta_h}m_h$. That is, high-quality sellers strictly prefer the separating equilibrium.

In what follows, I assume (B.7) does not bind. i.e.„ $z_s \leq z_s''$ where z_s'' is defined by Proposition B.4. Define $v_h^{pool}(z_s) = u(z_s + m_p) - \frac{\delta_h}{\bar{\delta}}m_p$ and $v_h^{sep}(z_s) = u(z_s + m_h) - m_h$. Now, from previous propositions, we know that

$$\frac{\mathrm{d}v_h^{pool}(z_s)}{\mathrm{d}z_s} = \begin{cases} u'(z_s + m_p) > \frac{\delta_h}{\bar{\delta}}, \text{ if } z_s + m_p < \hat{g}; \\ \frac{\delta_h}{\bar{\delta}}, \text{ if otherwise.} \end{cases} \tag{B.23}$$

And

$$\frac{\mathrm{d}v_h^{sep}(z_s)}{\mathrm{d}z_s} = u'(z_s + m_h) + [u'(z_s + m_h) - 1]\frac{u'(z_s + m_l) - u'(z_s + m_h)}{u'(z_s + m_h) - \frac{\delta_l}{\delta_h}} > 0. \tag{B.24}$$

Note that $z_s + m_l$ and $z_s + m_h$ both are increasing in z_s. Note also that $\frac{\delta_h}{2\delta_h - \delta_l} > \frac{\delta_l}{\delta_h}$. Hence,

$\frac{\mathrm{d}v_h^{sep}(z_s)}{\mathrm{d}z_s}$ decreases with z_s. i.e.„ v_h^{sep} is strictly increasing and strictly concave.

Suppose that $z_s + m_p < \hat{g}$, then $m_l \leq m_p$ and $u'(z_s + m_l) \geq u'(z_s + m_p)$.[1] We have

$$\frac{\mathrm{d}v_h^{sep}(z_s)}{\mathrm{d}z_s} - \frac{\mathrm{d}v_h^{pool}(z_s)}{\mathrm{d}z_s} \geq \frac{1 - \frac{\delta_l}{\delta_h}}{u'(z_s + m_h) - \frac{\delta_l}{\delta_h}}[u'(z_s + m_h) - u'(z_s + m_l)] > 0. \tag{B.25}$$

That is, $v_h^{sep}(z_s)$ is steeper than $v_h^{pool}(z_s)$ at least when $z_s + m_p < \hat{g}$.

Now define z_s' be such that $v_h^{pool}(z_s') = v_h^{sep}(z_s')$. If such z_s' does not exist, then the equilibrium is always separating, as it is not possible that the equilibrium is always pooling because Proposition B.4 shows constraint (B.7) must bind at some point. If it does exist, there are two different cases to discuss.

Case I: $z_s' + m_p(z_s', z_b, a) < \hat{g}$. First, because $v_h^{sep}(z_s)$ is steeper, we have $v_h^{pool}(z_s) > v_h^{sep}(z_s)$ for all $z_s < z_s'$ and $v_h^{pool}(z_s) < v_h^{sep}(z_s)$ for all $z_s' < z_s \leq \hat{g} - m_p(z_s', z_b, a)$. Second, for $z_s > \hat{g} - m_p(z_s', z_b, a)$, we have $\frac{\mathrm{d}v_h^{pool}(z_s)}{\mathrm{d}z_s} = \frac{\delta_h}{\bar{\delta}}$. Recall that v_h^{sep} is strictly concave and $v_h^{pool}(z_s'') < v_h^{sep}(z_s'')$. Then we must have $v_h^{pool}(z_s) < v_h^{sep}(z_s)$ for all $\hat{g} - m_p(z_s', z_b, a) < z_s \leq z_s''$. If not, because of the strict concavity of v_h^{sep} and the linearity of $v_h^{pool}(z_s)$ when $z_s > \hat{g} - m_p(z_s', z_b, a)$, there must exist $z_s^\dagger \in (\hat{g} - m_p(z_s', z_b, a), z_s'')$ such that $v_h^{pool}(z_s) > v_h^{sep}(z_s)$ for all $z_s \in (z_s^\dagger, z_s'']$, which contradicts with the earlier result that $v_h^{pool}(z_s'') < v_h^{sep}(z_s'')$.

Case II: $z_s' + m_p(z_s', z_b, a) \geq \hat{g}$. First, note that if $z_s' + m_p(z_s', z_b, a) \geq \hat{g}$, then it must be that $v_h^{pool}(\hat{g} - m_p(z_s', z_b, a)) \geq v_h^{sep}(\hat{g} - m_p(z_s', z_b, a))$, because as shown in case I, if $v_h^{pool}(\hat{g} - m_p(z_s', z_b, a)) < v_h^{sep}(\hat{g} - m_p(z_s', z_b, a))$, then $v_h^{pool}(z_s) < v_h^{sep}(z_s)$ for all $z_s \geq \hat{g} - m_p(z_s', z_b, a)$. Second, because (1) $v_h^{pool}(z_s'') < v_h^{sep}(z_s'')$, (2) v_h^{sep} is strictly concave, and (3) $v_h^{pool}(z_s)$ is linear, there must exist a unique z_s' such that $v_h^{pool}(z_s) > v_h^{sep}(z_s)$ for all $z_s < z_s'$ and $v_h^{pool}(z_s) \leq v_h^{sep}(z_s)$ for all $z_s \geq z_s'$. Otherwise, there exists $z_s^{\dagger\dagger} \in (\hat{g} - m_p(z_s', z_b, a), z_s'')$ such that $v_h^{pool}(z_s) > v_h^{sep}(z_s)$ for all $z_s \in (z_s^{\dagger\dagger}, z_s'']$, which again contradicts with the earlier result that $v_h^{pool}(z_s'') < v_h^{sep}(z_s'')$.

Lastly, suppose $\min\{z_b, \bar{\delta}a\} \geq \hat{g}$. This case is similar to case II. Again, if z_s' does not exist, the equilibrium is always separating.

Part II. I now discuss the effects of z_b and a. First, suppose the trade in both separating and pooling equilibrium is constrained by z_b. We have

$$u(z_s + z_b) - z_b = u(z_s + m_h) - \frac{\delta_l}{\delta_h} m_h \tag{B.26}$$

Suppose $\frac{m_h}{z_b} \geq \left(\frac{\delta_h}{\bar{\delta}} - 1\right) \Big/ \frac{\delta_h - \delta_l}{\delta_h}$, then

$$v_h^{pool} = u(z_s + z_b) - z_b - z_b\left(\frac{\delta_h}{\bar{\delta}} - 1\right) \geq u(z_s + m_h) - \frac{\delta_l}{\delta_h} m_h - m_h\left(\frac{\delta_h - \delta_l}{\delta_h}\right) = v_h^{sep}. \tag{B.27}$$

That is, to obtain the effect of z_b on v_h^{pool} and v_h^{sep}, one needs only consider its effect on $\frac{m_h}{z_b}$. We have

$$\frac{\mathrm{d}\left(\frac{m_h}{z_b}\right)}{dz_b} = \frac{\frac{\mathrm{d}m_h}{\mathrm{d}z_b} z_b - m_h}{z_b^2} = \frac{z_b[u'(z_s + z_b) - 1] - m_h[u'(z_s + m_h) - \delta_l/\delta_h]}{z_b^2}[u'(z_s + m_h) - \delta_l/\delta_h]^{-1}. \tag{B.28}$$

Now consider the function $f(\delta_l/\delta_h) = m_h[u'(z_s + m_h) - \delta_l/\delta_h]$. It is easy to see that $f(1) =$

[1] If the trade is constrained by a, then $m_l < m_p$, otherwise $m_l = m_p$.

$z_b[u'(z_s+z_b)-1]$. Now, note that $\frac{\mathrm{d}m_h}{d(\delta_l/\delta_h)} = \frac{m_h}{u'(z_s+m_h)-\delta_l/\delta_h}$. Hence,

$$f'(\delta_l/\delta_h) = m_h + m_h\left[\frac{m_h u''(z_s+m_h)}{u'(z_s+m_h)-\delta_l/\delta_h} - 1\right] = \frac{m_h u''(z_s+m_h)}{u'(z_s+m_h)-\delta_l/\delta_h} < 0. \tag{B.29}$$

That is, $\frac{\mathrm{d}\frac{m_h}{z_b}}{dz_b} < 0$ for all $\delta_l/\delta_h < 1$. Now, for some z_s, let $z_b(z_s)$ be such that $v_h^{sep}(z_s, z_b(z_s)) = v_h^{pool}(z_s, z_b(z_s))$. First, suppose $z_s + z_b(z_s) < \hat{g}$. It is easy to see that, if $z_b(z_s)$ exists, $v_h^{pool}(z_s, z_b(z_s)) > v_h^{sep}(z_s, z_b(z_s))$ for all $z_b < z_b(z_s)$ and $v_h^{pool}(z_s, z_b(z_s)) \leq v_h^{sep}(z_s, z_b(z_s))$ for all $z_b(z_s) \leq z_b < \hat{g} - z_s$.

Next, suppose the trade in the pooling equilibrium is not constrained by z_b. We have $\frac{\mathrm{d}v_h^{pool}}{\mathrm{d}z_b} = 0$ and

$$\frac{\mathrm{d}v_h^{sep}}{\mathrm{d}z_b}\begin{cases} > 0, & \text{if } z_s + z_b < g^*; \\ = 0, & \text{otherwise.}\end{cases} \tag{B.30}$$

Again, if $z_b(z_s)$ exists, it is easy to see that $v_h^{pool}(z_s, z_b(z_s)) > v_h^{sep}(z_s, z_b(z_s))$ for all $z_b < z_b(z_s)$ and $v_h^{pool}(z_b) \leq v_h^{sep}(z_b)$ for all $z_b \geq z_b(z_s)$. In conclusion, in all cases, an increase in z_b will increase $v_h^{sep}(z_s, z_b(z_s))$ relative to $v_h^{pool}(z_s, z_b(z_s))$, which means z_s' must be decreasing in z_b.

Now suppose the trade is constrained by a. Suppose the trade in both separating and pooling equilibrium is constrained by a. Define $\delta_l a = \hat{z}$ and let $\hat{\delta}(\hat{z})$ be such that

$$u(z_s' + \bar{\delta}a) - \delta_h a = u(z_s' + \bar{\delta}/\delta_l \hat{z}) - \frac{\delta_h}{\delta_l}\hat{z} = u(z_s' + \hat{z}) - \frac{\hat{\delta}(\hat{z})}{\delta_l}\hat{z} \tag{B.31}$$

Following the arguments in the last case, the equilibrium is pooling if $\frac{m_h}{\hat{z}} \geq \left(\frac{\hat{\delta}(\hat{z})}{\delta_l} - 1\right) \Big/ \frac{\delta_h - \delta_l}{\delta_h}$ and separating if otherwise, where m_h is given by

$$u(z_s + \hat{z}) - \hat{z} = u(z_s + m_h) - \frac{\delta_l}{\delta_h} m_h. \tag{B.32}$$

We already know that $\frac{\mathrm{d}\frac{m_h}{\hat{z}}}{\mathrm{d}\hat{z}} < 0$ for all $\delta_l/\delta_h < 1$. Then $g(\hat{z}) = \frac{m_h}{\hat{z}} - \left(\frac{\hat{\delta}(\hat{z})}{\delta_l} - 1\right) \Big/ \frac{\delta_h - \delta_l}{\delta_h}$ is decreasing in $\hat{z}$ if

$$\frac{\mathrm{d}\frac{m_h}{\hat{z}}}{\mathrm{d}\hat{z}} - \hat{\delta}'(\hat{z})\frac{\delta_h}{\delta_l(\delta_h - \delta_l)} < 0, \tag{B.33}$$

which holds if $\hat{\delta}'(\hat{z}) > 0$. Then if there exists a $\hat{z}'$ such that $g(\hat{z}) = 0$, it must be that the equilibrium is pooling if $\hat{z} < \hat{z}'$ and separating if $\hat{z} \geq \hat{z}'$. Now I show $\hat{\delta}'(\hat{z}) > 0$. Note that

$$\begin{aligned}\hat{\delta}'(\hat{z}) &= \frac{\delta_l u'(z_s' + \hat{z}) - \bar{\delta}u'(z_s' + \bar{\delta}/\delta_l \hat{z}) + \delta_h - \hat{\delta}(\hat{z})}{a} \\ &= \frac{\delta_l a u'(z_s' + \delta_l a) - u(z_s' + \delta_l a) - [\bar{\delta}a u'(z_s' + \bar{\delta}a) - u(z_s' + \bar{\delta}a)]}{a^2}.\end{aligned} \tag{B.34}$$

where the second equality is because by definition of $\hat{\delta}$, we have $u(z_s' + \bar{\delta}a) - \delta_h a = u(z_s' + \delta_l a) - \hat{\delta}a$. Now consider the function $h(x) = xu'(z_s' + x) - u(z_s' + x)$. It is easy to show that $h'(x) < 0$. Hence, $\hat{\delta}'(\hat{z}) > 0$.

Lastly, suppose the trade in the pooling equilibrium is not constrained by a. Then an increase in a has no effect on v_h^{pool} but it increases v_h^{sep}. Then it is easy to see that z_s' must be decreasing in a. □

In the last part of this appendix, I solve the agents' problem in the CM. The following proposition describes how the solution depends on μ.

Proposition B.5 *Given μ and A, there exist $\mu_1 < \mu_2$ such that*
(1) *For all $\mu \geq \mu_2$ and $\mu \leq \mu_1$, there exists a unique solution to problem* (3.9)*;*
(2) *For all $\mu \geq \mu_2$, the AM equilibrium is pooling; for all $\mu \leq \mu_1$, the AM equilibrium is separating.*
(3) *For all $\mu_1 < \mu < \mu_2$, under certain conditions, symmetric solutions do not exist.*

Proof: A symmetric solution to consumers' problem (3.9) means that all consumers will choose the same $\hat{z}$ and $\hat{a}$, but a consumer still takes other consumers' choices as given when choosing $\hat{z}$ and $\hat{a}$. When $z_s = z_b$ in the AM, one can show, by following the proof of Proposition 3.1, that given $\hat{a}$, there exists a z' such that the AM equilibrium is pooling for all $z_s = z_b < z'$, and separating for all $z_s = z_b \geq z'$. For the ease of exposition, I denote the trading volume in an AM pooling equilibrium $\psi_p s_p$ as m_p, and the trading volume in an AM separating equilibrium $\psi_l s_l$ and $\psi_h s_h$ as m_l and m_h. (ψ_p, s_p), (ψ_l, s_l) and (ψ_h, s_h) are given by Proposition 3.1. Note that m_l, m_h and m_p are functions of $(\hat{z}, \hat{a})$.

First, I take $\hat{a}$ as given. One can rewrite the objective function (3.9) as

$$f_i(\hat{z}, \hat{a}, \mu) = \beta S^i(\hat{z}, \hat{a}) - (\mu + 1 - \beta)\hat{z} - \kappa\hat{a} + \beta\bar{\delta}\hat{a} \tag{B.35}$$

where $i = \{s, p\}$ denotes separating or pooling equilibria and $S^i(\hat{z}, \hat{a})$ denotes consumers' surplus. In a separating equilibrium, $S^s(\hat{z}, \hat{a})$ is given by

$$S^s(\hat{z}, \hat{a}) = \alpha\{\Delta[u(\hat{z} + m_l) - (\hat{z} + m_l)] + (1 - \Delta)[u(\hat{z} + m_h) - (\hat{z} + m_h)]\}. \tag{B.36}$$

First note the following useful results

$$\begin{cases} \frac{\mathrm{d}m_l}{d\hat{z}} = -1, \text{ if } u'(\hat{z} + m_l) = 1; \\ \frac{\mathrm{d}m_l}{d\hat{z}} = 0, \text{ if } u'(\hat{z} + m_l) > 1; \\ \frac{\mathrm{d}m_h}{d\hat{z}} = \frac{u'(\hat{z}+m_l) - u'(\hat{z}+m_h)}{u'(\hat{z}+m_h) - \delta_l/\delta_h}. \end{cases} \tag{B.37}$$

Then

$$\frac{\partial S^s(\hat{z}, \hat{a})}{\partial \hat{z}} = \alpha\left\{\Delta[u'(\hat{z} + m_l) - 1] + (1 - \Delta)[u'(\hat{z} + m_h) - 1]\left[\frac{u'(\hat{z} + m_l) - u'(\hat{z} + m_h)}{u'(\hat{z} + m_h) - \delta_l/\delta_h} + 1\right]\right\}.$$

Hence, the FOC is

$$(\mu + 1)/\beta - 1 = \alpha\left\{\Delta[u'(\hat{z} + m_l) - 1] + (1 - \Delta)[u'(\hat{z} + m_l) - \delta_l/\delta_h]\left[\frac{u'(\hat{z} + m_h) - 1}{u'(\hat{z} + m_h) - \delta_l/\delta_h}\right]\right\}. \tag{B.38}$$

If a separating equilibrium is undefeated, there exists a unique solution to the above equation. To see this, $\frac{u'(\hat{z}+m_h)-1}{u'(\hat{z}+m_h)-\delta_l/\delta_h} < 1$ and hence is decreasing in $\hat{z} + m_h$. And $\frac{\mathrm{d}(\hat{z}+m_h)}{\mathrm{d}\hat{z}} = \frac{u'(\hat{z}+m_l)-u'(\hat{z}+m_h)}{u'(\hat{z}+m_h)-\delta_l/\delta_h} + 1 > 0$ means that $\hat{z} + m_h$ is decreasing in $\hat{z}$. That is, the RHS of (B.38) is strictly decreasing in $\hat{z}$.

Now I turn to the pooling equilibrium. $S^p(\hat{z})$ is given by

$$S^p(\hat{z}, \hat{a}) = \alpha[u(\hat{z} + m_p) - (\hat{z} + m_p)]. \tag{B.39}$$

Recall that if the pooling equilibrium is undefeated, (B.7) does not bind. Hence

$$\begin{cases} \frac{\mathrm{d}m_p}{d\hat{z}} = -1, \text{ if } u'(\hat{z} + m_p) = \delta_h/\bar{\delta}; \\ \frac{\mathrm{d}m_p}{d\hat{z}} = 0, \text{ if } u'(\hat{z} + m_p) > \delta_h/\bar{\delta}. \end{cases} \tag{B.40}$$

And

$$\frac{\partial S^p(\hat{z},\hat{a})}{\partial \hat{z}} = \begin{cases} \alpha[u'(\hat{z}+m_p)-1], \text{ if } u'(\hat{z}+m_p) \geq \delta_h/\bar{\delta}; \\ 0, \text{ if } u'(\hat{z}+m_p) < \delta_h/\bar{\delta}. \end{cases}$$

Hence, in this case, $\hat{z}$ solves

$$\begin{cases} (\mu+1)/\beta - 1 = \alpha[u'(\hat{z}+m_p)-1], \text{ if } \mu \geq \beta\{\alpha[\delta_h/\bar{\delta}-1]+1\}-1; \\ u'(\hat{z}+m_p) = \delta_h/\bar{\delta}, \text{ if otherwise.} \end{cases} \tag{B.41}$$

In the latter case, if consumers carry more than $\hat{z}$, they will just sell less assets in the AM should they become shoppers. Since it is costly to carry money, consumers do not carry more than $\hat{z}$.

Now, Let $z^s(\mu)$ and $z^p(\mu)$ denote the solutions in the separating and pooling cases, respectively. Let $\mu^\ddagger = \beta\{\alpha[\delta_h/\bar{\delta}-1]+1\}-1$. Note that the second term in the bracket of (B.38), $[u'(\hat{z}+m_l)-\delta_l/\delta_h]\left[\frac{u'(\hat{z}+m_h)-1}{u'(\hat{z}+m_h)-\delta_l/\delta_h}\right]$, is strictly larger than the first term, $u'(\hat{z}+m_l)-1$, because $m_l < m_h$ so

$$\frac{u'(\hat{z}+m_l)-1}{u'(\hat{z}+m_l)-\delta_l/\delta_h} < \frac{u'(\hat{z}+m_h)-1}{u'(\hat{z}+m_h)-\delta_l/\delta_h}. \tag{B.42}$$

Suppose $\mu \geq \mu^\ddagger$. Then $m_l \leq m_p$. If $z^p(\mu) = z^s(\mu)$, then $\frac{\partial S^s(\hat{z},\hat{a})}{\partial \hat{z}} > \frac{\partial S^p(\hat{z},\hat{a})}{\partial \hat{z}}$. Hence, it must be $z^p(\mu) < z^s(\mu)$. Since $z^p(\mu) = z^p(\mu^\ddagger)$ for all $\mu < \mu^\ddagger$, we have $z^p(\mu) < z^s(\mu)$ for all μ.

Now, let $\hat{a} = A$ and define $\kappa^p(\mu)$ and $\kappa^s(\mu)$ to be the equilibrium asset prices in the two scenarios. If the trade in the AM is not constrained by assets, then $\kappa^p(\mu) = \kappa^s(\mu) = \beta\bar{\delta}$. If the trade in the AM is constrained by assets, then

$$\kappa^p(\mu) = \beta\bar{\delta}\left\{1+\alpha[1+\alpha[u'(z^p(\mu)+m_p)-1]]\right\}, \tag{B.43}$$

$$\kappa^s(\mu) = \beta\left\{\bar{\delta}+\alpha\delta_l\left[\Delta[u'(z^s(\mu)+m_l)-1]+(1-\Delta)[u'(z^s(\mu)+m_h)-1]\frac{u'(z^s(\mu)+m_l)-1}{u'(z^s(\mu)+m_h)-\delta_l/\delta_h}\right]\right\}. \tag{B.44}$$

We need to see if the above solutions, $(z^s(\mu),\kappa^s(\mu))$ and $(z^p(\mu),\kappa^p(\mu))$, are consistent with Proposition 3.1. We have three possible cases: **Case I:** $z^p(\mu) < z^s(\mu) < z'$. **Case II:** $z' < z^p(\mu) < z^s(\mu)$. **Case III:** $z^p(\mu) < z' < z^s(\mu)$. For $z^p(\mu)$ or $z^s(\mu)$ or both to be equilibrium solutions, one needs to first check **(1)** conditional on other consumers choosing $z^p(\mu)$ (or $z^s(\mu)$) and given $\kappa = \kappa^p(\mu)$ (or $\kappa = \kappa^s(\mu)$), whether a consumer have the incentive to deviate, choose a different $\hat{z}$ and $\hat{a}$, and make a separating (or pooling) offer in the AM; and **(2)** whether such deviation is consistent with Proposition 3.1. If the answers to both questions are yes, then $z^p(\mu)$ (or $z^s(\mu)$) is not a solution. I show next that if μ is sufficiently large or sufficiently small, there exists a unique solution.

First, define μ' to be such that $u'(z^p(\mu')+m_p) = \delta_h/\bar{\delta}$. Define μ'' to be such that $z^s(\mu'') = z'$. Let $\mu_2 = \max\{\mu',\mu''\}$. If $\mu > \mu_2$, we have case I, so $(z^s(\mu),\kappa^s(\mu))$ is not a solution. First, suppose A is large so the trade in the AM is not constrained by assets. Consider a consumer who deviates by holding more money and making a separating offer in the AM. Because in this case $m_l = m_p$, the consumer's expected surplus in a separating AM equilibrium is lower. Next, suppose the trade in the AM is constrained by assets. Consider a joint deviation where a consumer choose $\hat{z}' \neq z^p(\mu)$ and $\hat{a}' \neq A$ given $\kappa^p(\mu)$ and μ, and make a separating offer in the AM.[2] Note that the

[2] I thank one of the anonymous referees for pointing out the possibility of joint deviations.

marginal value of $\hat{a}$ in a separating equilibrium is

$$\frac{\partial S^s(\hat{z}', \hat{a})}{\partial \hat{a}} = \beta \left\{ \bar{\delta} + \alpha \delta_l \left[\Delta[u'(\hat{z} + m_l) - 1] + (1 - \Delta)[u'(\hat{z} + m_h) - 1] \frac{u'(\hat{z} + m_l) - 1}{u'(\hat{z} + m_h) - \delta_l / \delta_h} \right] \right\}. \tag{B.45}$$

The optimal $\hat{z}$ and $\hat{a}$ must satisfy $\frac{\partial S^s(\hat{z}', \hat{a})}{\partial \hat{z}} = \frac{\partial S^p(\hat{z}, \hat{a})}{\partial \hat{z}} = (\mu + 1)/\beta - 1$. However, this means $\frac{\partial S^s(\hat{z}', \hat{a})}{\partial \hat{a}} < \kappa^p$. In other words, $\kappa^p(\mu)$ is too high and the consumer is better off holding only money. Hence, $\hat{a}' = 0$, which obviously does not offer higher utility compared to the pooling equilibrium.

Next, I show that the separating equilibrium is the unique equilibrium for small μ. First, given $\hat{a}$, for all $\mu < \mu'$, consumers' value in a pooling solution is

$$f_p(\mu) = \beta \alpha [u(\hat{g}) - \hat{g}] - (\mu + 1 - \beta)(\hat{g} - m_p) - \kappa \hat{a} + \beta \bar{\delta} \hat{a}, \tag{B.46}$$

which does not approach the first best (i.e.„ $g = g^*$) as $\mu \to \beta - 1$. Hence, there must exist $\mu^\dagger$ such that for all $\mu < \mu^\dagger$, consumers prefer choosing a larger $\hat{z}$ and separating in the AM. Now consider if a consumer would deviate from separating to pooling. The highest this consumer can get from pooling is

$$f_p(\mu) = \beta \alpha [u(\hat{g}) - \hat{g}] - \kappa \hat{a} + \beta \bar{\delta} \hat{a}, \tag{B.47}$$

in which case the consumer does not carry any money because other consumers carry enough to support $\hat{g}$. But this is still not the first best, so as long as μ is sufficiently small, consumers do not deviate. Lastly, consider a joint deviation where a consumer chooses $\hat{z}'' \neq z^s(\mu)$ and $\hat{a}'' \neq A$ given $\kappa^s(\mu)$ and μ, and makes a pooling offer in the AM. Because $\kappa^s > \beta \bar{\delta}$, the consumer will carry just enough assets to consume $\hat{g}$. That is, $\bar{\delta} \hat{a} = \hat{g}$. However, as long as μ is sufficiently small, such deviation is not profitable.

Lastly, when $\mu_1 \leq \mu \leq \mu_2$, a symmetric solution may not exist. For example, suppose that A is large so the trade is not constrained by assets. A consumer may deviate by choosing a different $\hat{z}$. First, assume all other consumers choose $z^p(\mu)$ and $z^p(\mu) < z'$. Suppose there exists $z_1^d \geq z'$ such that

$$\beta \hat{S}^s(z_1^d, A) - (\mu + 1 - \beta) z_1^d - \kappa A + \beta \bar{\delta} A > \beta S^p(z^p(\mu), A) - (\mu + 1 - \beta) z^p(\mu) - \kappa A + \beta \bar{\delta} A \tag{B.48}$$

where

$$\begin{aligned} \hat{S}^s(z_1^d, A) = \alpha \{ &\Delta[u(z_1^d + m_l(z^p(\mu))) - (z_1^d + m_l(z^p(\mu)))] \\ &+ (1 - \Delta)[u(z_1^d + m_h(z^p(\mu))) - (z_1^d + m_h(z^p(\mu)))] \}. \end{aligned}$$

I use $m_l(z^p(\mu))$ and $m_h(z^p(\mu))$ to emphasize the dependence of m_l and m_h on $z^p(\mu)$. If the inequality holds, a deviation to (z_1^d, A) is profitable. Next, assume all the other consumers choose $z^s(\mu)$ and $z^s(\mu) \geq z'$. Suppose there exists $z_2^d < z'$ such that

$$\beta \hat{S}^p(z_2^d, A) - (\mu + 1 - \beta) z_2^d - \kappa A + \beta \bar{\delta} A > \beta S^s(z^s(\mu), A) - (\mu + 1 - \beta) z^s(\mu) - \kappa A + \beta \bar{\delta} A \tag{B.49}$$

where

$$\hat{S}^p(z_2^d, A) = \alpha [u(z_2^d + m_p(z^s(\mu))) - (z_2^d + m_p(z^s(\mu)))].$$

Then a deviation to (z_2^d, A) is profitable. If z_1^d and z_2^d exist simultaneously, there does not exist a symmetric solution. $\square$

B.3 Proofs for Section 3.3.2

Proof of Proposition 3.2: There are in principle fifteen types of equilibria (compared to only two types in the benchmark model). The list of possible types of perfect Bayesian equilibria is given by the following.
(1) $(\psi_l^u, s_l^u) \neq (\psi_h^u, s_h^u) \neq (\psi_l^d, s_l^d) \neq (\psi_h^d, s_h^d)$; (2) $(\psi_l^u, s_l^u) = (\psi_l^d, s_l^d) \neq (\psi_h^u, s_h^u) \neq (\psi_h^d, s_h^d)$;
(3) $(\psi_l^u, s_l^u) \neq (\psi_l^d, s_l^d) \neq (\psi_h^u, s_h^u) = (\psi_h^d, s_h^d)$; (4) $(\psi_l^u, s_l^u) = (\psi_l^d, s_l^d) \neq (\psi_h^u, s_h^u) = (\psi_h^d, s_h^d)$;
(5) $(\psi_l^u, s_l^u) = (\psi_h^u, s_h^u) \neq (\psi_l^d, s_l^d) \neq (\psi_h^d, s_h^d)$; (6) $(\psi_l^u, s_l^u) \neq (\psi_h^u, s_h^u) \neq (\psi_l^d, s_l^d) = (\psi_h^d, s_h^d)$;
(7) $(\psi_l^u, s_l^u) = (\psi_h^u, s_h^u) \neq (\psi_l^d, s_l^d) = (\psi_h^d, s_h^d)$; (8) $(\psi_l^u, s_l^u) = (\psi_h^d, s_h^d) \neq (\psi_h^u, s_h^u) \neq (\psi_l^d, s_l^d)$;
(9) $(\psi_l^u, s_l^u) \neq (\psi_h^d, s_h^d) \neq (\psi_h^u, s_h^u) = (\psi_l^d, s_l^d)$; (10) $(\psi_l^u, s_l^u) = (\psi_h^d, s_h^d) \neq (\psi_l^d, s_l^d) = (\psi_h^u, s_h^u)$;
(11) $(\psi_l^u, s_l^u) = (\psi_h^d, s_h^d) = (\psi_h^u, s_h^u) \neq (\psi_l^d, s_l^d)$; (12) $(\psi_h^d, s_h^d) = (\psi_h^u, s_h^u) = (\psi_l^d, s_l^d) \neq (\psi_l^u, s_l^u)$;
(13) $(\psi_l^u, s_l^u) = (\psi_h^u, s_h^u) = (\psi_l^d, s_l^d) \neq (\psi_h^d, s_h^d)$; (14) $(\psi_h^d, s_h^d) = (\psi_l^u, s_l^u) = (\psi_l^d, s_l^d) \neq (\psi_h^u, s_h^u)$;
(15) $(\psi_h^d, s_h^d) = (\psi_l^u, s_l^u) = (\psi_l^d, s_l^d) = (\psi_h^u, s_h^u)$.
Some cases can be ruled out. Specifically, when sellers' distress statuses are revealed, undistressed low-quality (or high-quality) sellers cannot pool with other sellers without also pooling with undistressed high-quality (or low-quality) sellers. It is the same for distressed sellers. To see why, suppose, for example, that undistressed low-quality sellers are in a pool of sellers without undistressed high-quality sellers. Then, buyers will recognize undistressed low-quality sellers and only accept a low price $\psi = \delta_l$ from them. This eliminates cases (8)-(14).

Now I solve the equilibrium. In cases (1)-(4), low-quality sellers and high-quality sellers make different offers. Hence, these cases can all be solved by following the proof of Proposition B.2. Specifically, for (ψ_l^u, s_l^u) and (ψ_h^u, s_h^u), the solution is the same as those proposed in Proposition B.2. However, (ψ_h^d, s_h^d) must be such that undistressed sellers do not have the incentive to mimic this offer. Case (5) can be solved by solving a pooling problem for undistressed sellers and a separating problem from distressed sellers. However, the offers must be such that distressed high-quality sellers and undistressed high-quality sellers do not have the incentive to mimic each other's offers. Next, for case (6), one can solve (ψ_l^u, s_l^u) and (ψ_h^u, s_h^u) by following the proof of Proposition B.2 and (ψ_l^d, s_l^d) and (ψ_h^d, s_h^d) by following the proof of Proposition B.1. Similar to (5), undistressed high-quality sellers must not have the incentive to make the pooling offer and distressed high-quality sellers must not have the incentive to make the separating offer. Case (7) is similar to (5) and (6). Lastly, in case (15), an equilibrium is such that low-quality sellers' participation constraints are satisfied, and distressed and undistressed high-quality sellers cannot both be made better-off.

Next, I apply the undefeated equilibrium to select the equilibrium. First, suppose $z_s \geq z_s'$ so without the liquidity shock sellers separate in the equilibrium. When η^d is small, all sellers may separate in equilibrium. Now, suppose η^d is large. Case (6) is the unique undefeated equilibria if distressed sellers prefer to pool while undistressed sellers prefer to separate. The latter is true by the assumption that $z_s \geq z_s'$. Now, consider the former. Suppose that η^d is sufficiently large so that $\eta^d u'(z_s + \min\{\delta_l a, z_b\}) > 1$ and $\eta^d u'(z_s + \min\{\bar{\delta} a, z_b\}) > \frac{\delta_h}{\bar{\delta}}$. Define

$$v_h^{pool}(\eta^d) = \eta^d u(z_s + m_p) - \frac{\delta_h}{\bar{\delta}} m_p \tag{B.50}$$

where $m_p = \min\{\bar{\delta} a, z_b\}$. Define

$$v_h^{sep}(\eta^d) = \eta^d u(z_s + m_h^d) - m_h^d; \tag{B.51}$$

where m_h^d is given by

$$\eta^d u(z_s + m_l) - m_l = \eta^d u(z_s + m_h^d) - \frac{\delta_l}{\delta_h} m_h^d, \tag{B.52}$$

and $m_l = \min\{\delta_l a, z_b\}$. One can rewrite $v_h^{sep}(\eta^d)$ as $v_h^{sep}(\eta^d) = \eta^d u(z_s+m_l) - m_l - (1-\delta_l/\delta_h)m_h^d$. Then

$$\frac{\mathrm{d}v_h^{pool}(\eta^d)}{\mathrm{d}\eta^d} = u(z_s + m_p) > u(z_s + m_l) - (1 - \delta_l/\delta_h)\frac{\mathrm{d}m_h^d}{\mathrm{d}\eta^d} = \frac{\mathrm{d}v_h^{sep}(\eta^d)}{\mathrm{d}\eta^d} \tag{B.53}$$

because $m_p \geq m_l$ and $\frac{\mathrm{d}m_h^d}{\mathrm{d}\eta^d} > 0$. First, suppose $m_p = \bar{\delta}a$ and $m_l = \delta_l a$. Then, for sufficiently large η^d,

$$\begin{aligned} v_h^{pool}(\eta^d) =& \eta^d u(z_s + m_p) - m_p - (\delta_h/\bar{\delta} - 1)m_p \\ >& \eta^d u(z_s + m_l) - m_l - (1 - \delta_l/\delta_h)m_h^d = v^{sep}(\eta^d). \end{aligned} \tag{B.54}$$

Second, suppose that $m_p = m_l = z_b$ and $\delta_h/\bar{\delta} < 2 - \delta_l/\delta_h$. Because $\lim_{\eta^d\to\infty} m_h^d = m_l = m_p$, for sufficiently large η^d, $(\delta_h/\bar{\delta} - 1)m_p < (1 - \delta_l/\delta_h)m_h$, and hence $v_h^{pool}(\eta^d) > v^{sep}(\eta^d)$.

Lastly, suppose $z_s < z_s'$ so without the liquidity shock sellers pool in the equilibrium. From the proof of Proposition B.5, we know that in this case, asset trade is constrained either by sellers' assets or buyers' money. Then, case (15) can be solved by following the proof of Proposition B.1. Note that because asset trade is constrained, maximizing undistressed high-quality sellers' surplus will also maximize distressed high-quality sellers' surplus. Similar to the case when $z_s \geq z_s'$, distressed sellers prefer to pool if η^d is large and $\delta_h/\bar{\delta} < 2 - \delta_l/\delta_h$. Since without the liquidity shock sellers pool in equilibrium, all sellers are better off in case (15) than in cases (1)-(7). Hence, the only undefeated case is (15). ☐

Proof of Proposition 3.3: I start by ruling out equilibria that cannot exist. First, suppose distressed low-quality sellers and undistressed low-quality sellers do not pool (i.e., $(\psi_l^u, s_l^u) \neq (\psi_l^d, s_l^d)$). We have

$$u(z_s + \psi_l^u s_l^u) - \delta_l s_l^u \geq u(z_s + \psi_l^d s_l^d) - \delta_l s_l^d, \tag{B.55}$$

$$\eta^d u(z_s + \psi_l^u s_l^u) - \delta_l s_l^u \leq \eta^d u(z_s + \psi_l^d s_l^d) - \delta_l s_l^d, \tag{B.56}$$

where at least one inequality is strict. For both inequalities to hold at the same time, it must be that $\psi_l^u s_l^u < \psi_l^d s_l^d$. Note that since

$$u(z_s + \psi_l^d s_l^d) - u(z_s + \psi_l^u s_l^u) \leq \delta_l(s_l^d - s_l^u) < \delta_h(s_l^d - s_l^u), \tag{B.57}$$

undistressed high-quality sellers do not pool with distressed low-quality sellers either. Similarly, if distressed high-quality sellers and undistressed high-quality sellers do not pool (i.e., $(\psi_h^u, s_h^u) \neq (\psi_h^d, s_h^d)$), then distressed low-quality sellers do not pool with undistressed high-quality sellers either. This eliminates cases (9), (10), (12) and (13).

Now I solve the equilibrium. First, cases (1)-(4) are similar to when distressed status is revealed. Second, consider cases (5)-(8). Unlike when distressed status is revealed, undistressed or distressed low quality sellers can deviate unilaterally without being detected by buyers. Hence, it is not enough that high-quality sellers do not have the incentive to deviate – low-quality sellers must not have the incentive to deviate either. Third, consider cases (11), (14), and (15). In (11) and (15), an equilibrium is such that low-quality sellers' participation constraints are satisfied, and distressed and undistressed high-quality sellers cannot both be made better off. (14) can be solved by maximizing distressed high-quality sellers' utility subject to the participation constraints of low-quality sellers. The pooling offer in (11) must also prevent distressed low-quality sellers from deviating, while undistressed high-quality sellers in (14) must prevent other sellers that pool from deviating.

Next, I apply the undefeated equilibrium to select the equilibrium.

Part I. Suppose that $z_s < z_s'$. As shown in the proof of Proposition 3.2, distressed sellers prefer to pool if η^d is large and $\delta_h/\bar{\delta} < 2 - \delta_l/\delta_h$. Since undistressed sellers also prefer to pool, cases (1)-(7) are defeated by (15). Hence, we are only left with cases (8), (11), (14), and (15). Consider cases (8) and (14). Let $\hat{\delta}_{(8)} = [\Delta(1-\pi)\delta_l + (1-\Delta)\pi\delta_h]/[\Delta(1-\pi) + (1-\Delta)\pi]$ and $\hat{\delta}_{(14)} = [\Delta\delta_l + (1-\Delta)\pi\delta_h]/[\Delta + (1-\Delta)\pi]$ be the average quality of the asset pool in (8) and (14). Let $m_p = \psi_p s_p$ (in case (8) and (14)), $m_h = \psi_h^u s_h^u$, and $m_l = \psi_l^d s_l^d$ (in case (8) only). We have

$$\eta^d u(z_s + m_h) - m_h \leq \eta^d u(z_s + m_p) - \frac{\delta_h}{\hat{\delta}_{(8)}} m_p \quad \text{(replace } \hat{\delta}_{(8)} \text{ with } \hat{\delta}_{(14)} \text{ in (14))}, \tag{B.58}$$

$$u(z_s + m_h) - \frac{\delta_l}{\delta_h} m_h \leq u(z_s + m_p) - \frac{\delta_l}{\hat{\delta}_{(8)}} m_p \quad \text{(replace } \hat{\delta}_{(8)} \text{ with } \hat{\delta}_{(14)} \text{ in (14))}, \tag{B.59}$$

$$\eta^d u(z_s + m_p) - \frac{\delta_l}{\hat{\delta}_{(8)}} m_p \leq \eta^d u(z_s + m_l) - m_l, \tag{B.60}$$

$$\eta^d u(z_s + m_h) - \frac{\delta_l}{\delta_h} m_h \leq \eta^d u(z_s + m_l) - m_l. \tag{B.61}$$

(B.58) and (B.59) ensure that pooling sellers do not have the incentive to deviate, and (B.60) and (B.61) ensure that in case (8), distressed low-quality sellers do not have the incentive to deviate. It is easy to show that as long as $\eta^d \delta_l/\delta_h > 1$, (B.59) binding implies (B.58). (B.58) and (B.60) together imply (B.61). This means that in cases (8) and (14), undistressed high-quality sellers only have to prevent undistressed low-quality sellers from deviating. In other words, (B.59) always binds.

Now, suppose that (i) π is large, (ii) a is small (so asset trade is constrained by sellers' assets), and (iii) $2 - \delta_l/\delta_h < \bar{\delta}/\delta_l$. First, undistressed low-quality sellers are better off in (14) than in (8). To see this, first note that m_p in (8) never exceeds m_l. If a is small and π is large, $m_l = \delta_l a$ and in (14) $m_p = \hat{\delta}_1 a$. Then

$$\begin{aligned} \delta_l(1 - \delta_l/\hat{\delta}_{(8)}) < \hat{\delta}_{(14)}(1 - \delta_l/\hat{\delta}_{(14)}) &\Rightarrow \delta_l(u'(z_s) - \delta_l/\hat{\delta}_{(8)}) < \hat{\delta}_{(14)}(u'(z_s) - \delta_l/\hat{\delta}_{(14)}) \\ \Rightarrow u(z_s + m_l) - \frac{\delta_l}{\hat{\delta}_{(8)}} m_l < u(z_s + m_p) - \frac{\delta_l}{\hat{\delta}_{(14)}} m_p &\text{ when } a \text{ is small.} \end{aligned} \tag{B.62}$$

If $2 - \delta_l/\delta_h < \bar{\delta}/\delta_l$, then $\delta_l(1 - \delta_l/\hat{\delta}_{(8)}) < \hat{\delta}_{(14)}(1 - \delta_l/\hat{\delta}_{(14)})$ when π is large. This means that for undistressed low-quality sellers, the larger trading volume in (14) provides a higher surplus than the higher price in (8). Because (B.59) binds, undistressed high-quality sellers are better off in (14) as well. Since distressed sellers prefer to pool, they are better off in (14) too. This means (8) is defeated by (14). Next, undistressed low-quality sellers and distressed sellers are better off in (14) than in (11) because one, the larger trading volume in (14) provides a higher surplus than the higher price in (11), and two, distressed sellers prefer to pool. When π is large, undistressed high-quality sellers are better off in (8) than in (11). To see this, note that distressed and undistressed low-quality sellers' surplus in (8) and (11) are similar because if π is large, pooling sellers' average quality is similar with or without undistressed high-quality sellers. Then undistressed high-quality sellers can benefit from making a separating offer that satisfies (B.59). This means undistressed high-quality sellers are better off in (14) than in (11) as well. Lastly, when π is large, undistressed high-quality sellers are better off in (14) than in (15) because distressed and undistressed low-quality sellers' surplus in (14) and (15) is similar, and undistressed high-quality sellers can benefit from making a separating offer. In conclusion, only (14) is undefeated.

Next, suppose π is small so that $u(z_s + m_p) - \frac{\delta_l}{\hat{\delta}} m_p = v_l^o$ in both case (8) and (14).[3] Then

[3]See problem (B.1) for the definition of v_l^o.

all sellers are better off in (15) than in case (8) and (14). When π is small, all sellers are better off in (15) than in (11) as well because undistressed sellers no longer have to prevent distress low-quality sellers from deviating. Hence, (15) is the unique undefeated equilibrium. Lastly, if π is large and a is large (asset trade is constrained by buyers' money), then distressed sellers are better off in (14) than in (8) because they can pool. However, undistressed low-quality sellers may be better off in (8) than in (14) because in (8), m_p approaches to m_l if η^d is large. Recall that when η^d is large so asset trade is constrained by money, m_p in (14) is equal to m_l in (8). In other words, undistressed low-quality sellers are better off in (8) than in (14), and so are undistressed high-quality sellers because (B.59) binds. This means (14) and (8) are defeated by each other. Since similar to before (11) is defeated by (8) and (15) is defeated by (14), there are no undefeated equilibria.

Part II. Suppose that $z_s \geq z_s'$. First, consider cases (1)-(4). It must be that $(\psi_h^u, s_h^u) = (\psi_h^d, s_h^d)$ so that undistressed low-quality sellers do not have the incentive to deviate. This means only cases (3) and (4) are possible, and we have case (4) if low-quality sellers are constrained by either money or assets without the shock and (3) if they are not. Without loss of generality, in what follows I focus on case (3). Because $z_s \geq z_s'$, undistressed sellers prefer to separate. Hence, cases (5), (7), and (15) is defeated by (3). Next, in (6), distressed high-quality sellers only pool with distressed low-quality sellers. That is, we have

$$u(z_s + m_l) - m_l \geq u(z_s + m_p^d) - \frac{\delta_l}{\bar{\delta}} m_p^d. \tag{B.63}$$

A necessary condition for (B.63) to not bind is that $u'(z_s + m_p) < 1$. Hence, η^d, a and z_b need to be large. In such case, (3) is defeated by (6). In what follows, assume η^d, a and z_b are large. Consider cases (8), (11), and (14). Suppose π is small so that $u(z_s + m_p) - \frac{\delta_l}{\bar{\delta}} m_p = v_l^o$ in case (8) and (14). Then distressed sellers are better off in (6) while undistressed sellers are as well off as in (8) and (14). When π is small, all sellers are better off in (6) than in (11) because in (11), undistressed sellers have to pool while distressed high-quality sellers have to separate from distressed low-quality sellers. Hence, (6) is the unique undefeated equilibrium. If π is large so that $u(z_s + m_p) - \frac{\delta_l}{\bar{\delta}} m_p > v_l^o$, then undistressed sellers are better off in either (8) or (14), while distressed sellers are better off in (6). Distressed sellers are also better off in (6) than in (11). Hence, there are no undefeated equilibria.

Now assume a is small so (B.63) binds. We are left with cases (3), (6), (8), (11), and (14). First, (6) cannot exist because if (B.63) binds, distressed low-quality sellers' participation constraint is violated. If η^d is large so that $\eta^d u(z_s + m_l) - \frac{\delta_h}{\delta_l} m_l > \eta^d u(z_s + m_h) - m_h$ where m_l and m_h are given by Proposition B.2, then (3) cannot exist because distressed high-quality sellers would prefer pooling with low quality sellers to any separating offer. Next, suppose π is small so that $u(z_s + m_p) - \frac{\delta_l}{\bar{\delta}} m_p = v_l^o$ in case (8) and (14). Undistressed high-quality sellers are better off in (8) and (14) than in (11) because they do not pool with undistressed low-quality sellers in (8) and (14). Distressed high-quality sellers and undistressed low-quality sellers are better off in (8) than in (14) since they do not have to pool with distressed low-quality sellers. Since (B.59) binds, undistressed high-quality sellers are better off in (8) than in (14) as well. Hence, (8) is the only undefeated equilibrium. Lastly, assume (i) π is large, (ii) a is small, and (iii) $2 - \delta_l/\delta_h < \bar{\delta}/\delta_l$. Following the arguments in part I, (14) is the only undefeated equilibrium. If π is large and a is large (asset trade is constrained by buyers' money), then similar to part I, distressed high-quality sellers are better off in (14), while undistressed high-quality sellers are better off in (8). So (14) and (8) are defeated by each other. Since (11) is defeated by (8), there are no undefeated equilibria. □

Proof of Proposition 3.4: First, suppose $z_s < z_s'$. When the distress statuses are revealed, all sellers pool. It is easy to see that in the semi-pooling equilibrium, undistressed low-quality sellers are worse off because they are pooling with distressed sellers, and the average asset quality is below the average asset quality of the market. For the same reason, distressed sellers are also worse off. Undistressed high-quality sellers, however, are better off because case (15) is defeated by (14) (see the proof of Proposition 3.3).

Second, suppose $z_s \geq z_s'$. In the semi-pooling equilibrium, undistressed low-quality sellers are better off because they pool with distressed sellers. For the same reason, distressed sellers' surplus is lower. In the semi-pooling equilibrium, undistressed high-quality sellers' surplus is

$$u(z_s + m_h) - m_h \tag{B.64}$$

where m_h solves

$$u(z_s + m_h) - \frac{\delta_l}{\delta_h} m_h = u(z_s + m_p) - \frac{\delta_l}{\hat{\delta}} m_p. \tag{B.65}$$

Note that $m_p = \hat{\delta} a$ (asset trade is constrained by a) and $\hat{\delta} = [\Delta \delta_l + (1-\Delta)\pi\delta_h]/[\Delta + (1-\Delta)\pi]$. When the distress statuses are revealed, let $m_l = \delta_l a$ and m_h' be given by

$$u(z_s + m_h') - \frac{\delta_l}{\delta_h} m_h' = u(z_s + m_l) - m_l. \tag{B.66}$$

Then undistressed high-quality sellers' surplus is

$$u(z_s + m_h') - m_h'. \tag{B.67}$$

Since $m_h > m_h'$, undistressed sellers' surplus is higher in the semi-pooling equilibrium. □

B.4 Proofs for Section 3.4

Proof of Proposition 3.5: First, note that because $z_s \geq z_s'$, for any amount sellers borrow from the discount window, the equilibrium in the asset market must be separating. It is clear that low-quality sellers do not benefit from discount window loans as long as $r^D > 1$, and they are indifferent if $r^D = 1$. Note that if low-quality sellers choose mimic high-quality sellers' strategy in the asset market, the former will also borrow the same amount from the discount window as the latter even though borrowing from the discount window is unobservable. This is because all sellers' preferences over money are the same and their assets have the same capacity when serving as collateral.

Now, let $v_0 = u(z_s + m^*) - u(z_s) - m^*$ where m^* solves $u'(z_s + m^*) = 1$. Let m_s^D denote the amount of discount window loan by high-quality sellers. Let m_h^D denote the amount of real balances high-quality sellers obtain from the asset market. High-quality sellers solve

$$\max_{m_s^D, m_h^D} u(z_s + m_s^D + m_h^D) - R^D m_s^D - m_h^D \tag{B.68}$$

$$\text{s.t. } u(z_s + m_s^D + m_h^D) - u(z_s) - R^D m_s^D - \frac{\delta_l}{\delta_h} m_h^D \leq v_0. \tag{B.69}$$

Constraint (B.69) must bind, otherwise high-quality sellers can decrease m_s^D and increase m_h^D to increase their surplus. Suppose that $m_s^D = 0$. If $u'(z_s + m_s^D + m_h^D) > R^D$, high-quality sellers can increase m_s^D whiling decreasing m_h^D so that constraint (B.69) still holds. This increases high-quality sellers' surplus because for any $m_h^{D\prime} < m_h^{D\prime\prime}$, we have

$$u(z_s + m_s^{D\prime} + m_h^{D\prime}) - u(z_s) - R^D m_s^{D\prime} - \frac{\delta_l}{\delta_h} m_h^{D\prime} = u(z_s + m_s^{D\prime\prime} + m_h^{D\prime\prime}) - u(z_s) - R^D m_s^{D\prime\prime} - \frac{\delta_l}{\delta_h} m_h^{D\prime\prime}. \tag{B.70}$$

So

$$\begin{aligned} &u(z_s + m_s^{D\prime} + m_h^{D\prime}) - u(z_s) - R^D m_s^{D\prime} - m_h^{D\prime} \\ &> u(z_s + m_s^{D\prime\prime} + m_h^{D\prime\prime}) - u(z_s) - R^D m_s^{D\prime\prime} - m_h^{D\prime\prime}. \end{aligned} \tag{B.71}$$

Hence, the optimal m_s^D and m_h^D solve

$$u'(z_s + m_s^D + m_h^D) = R^D, \tag{B.72}$$

$$u(z_s + m_s^D + m_h^D) - u(z_s) - R^D m_s^D - \frac{\delta_l}{\delta_h} m_h^D = v_0. \tag{B.73}$$

It is easy to see that if $R^D = 1$, $m_h^D = 0$. Next, if $R^D \geq R^{D\prime} \equiv u'(z_s + m_h^D)$ where m_h^D solves $u(z_s + m_h^D) - u(z_s) - \frac{\delta_l}{\delta_h} m_h^D = v_0$, then high-quality sellers do not borrow from the discount window.

Lastly, suppose a is small and the AM equilibrium is pooling. Let m^D denote the amount of discount window loan, and let m_p^D denote the amount of real balances obtained by sellers from the asset market. Consider the problem below.

$$\max_{m^D, m_p^D} u(z_s + m^D + m_p^D) - R^D m^D - \frac{\delta_h}{\bar{\delta}} m_p^D \tag{B.74}$$

$$\text{s.t. } u(z_s + m^D + m_p^D) - u(z_s) - R^D m^D - \frac{\delta_l}{\bar{\delta}} m_p^D \geq v_0; \tag{B.75}$$

$$\frac{R^D m^D}{\delta_l} + \frac{m_p^D}{\bar{\delta}} \leq a. \tag{B.76}$$

I assume a is sufficiently small so that (B.76) binds. Suppose that (1) $R^D = 1$; (2) $z_s + m^\dagger < z_s'(a)$ where $m^\dagger \equiv \delta_l a$; and (3) $u'(z_s + m^\dagger) > \frac{\delta_h - \delta_l}{\bar{\delta} - \delta_l}$. Now, assume $m^D = m^\dagger$. Substitute (B.76) into the objective function and take the derivative of the objective function with respect to m_p^D at $m_p^D = 0$. We have

$$(1 - \delta_l/\bar{\delta}) u'(z_s + m^\dagger) - (\delta_h - \delta_l)/\bar{\delta} > 0 \tag{B.77}$$

because of assumption (3). Then as long as $\frac{\delta_h}{\bar{\delta}} < u'(z_s + \bar{\delta} a) < \frac{\delta_h - \delta_l}{\bar{\delta} - \delta_l}$, the optimal m^D and m_p^D satisfy $0 < m^D < m^\dagger$ and $m_p^D > 0$. Because $z_s + m^\dagger < z_s'(a)$, $z_s + m^D$ is small enough that high-quality sellers will want to pool in the asset market. It is also clear that high-quality sellers will prefer borrowing from the discount window and then pooling in the asset market to only borrowing from the discount window. Then, it is easy to that the total liquidity shoppers bring to the GM, $z_s + m^D + m_p^D$, is less than what sellers would have brought to the GM if there was no discount window and the equilibrium was pooling. $\square$

Proof of Proposition 3.6: First, let $z_s^\diamond$ be such that $u'(z_s^\diamond) = 1/(1 - \Delta)$. Let $v_h^s = u(z_s + \psi_h s_h) - u(z_s) - \delta_h s_h$ denote high-quality sellers' surplus in a separating equilibrium and let $v_h^p = u(z_s + \psi_p s_p) - u(z_s) - \delta_h s_p$ denote high-quality sellers' surplus in a pooling equilibrium. The terms of trade are given by Proposition 3.1. It is easy to see that both v_h^s and v_h^p are increasing in δ_l. Next, note that

$$\lim_{\delta_l \to 0} v_h^s = 0 \tag{B.78}$$

$$\lim_{\delta_l \to 0} v_h^p = \begin{cases} 0, \text{ if } z_s \geq z_s^\diamond; \\ u(z_s + m_p) - \frac{m_p}{1-\Delta} > 0, \text{ if } z_s < z_s^\diamond. \end{cases} \tag{B.79}$$

where $u'(z_s + m_p) = 1/(1 - \Delta)$. If $z_s < z_s^\diamond$, as $\delta_l \to 0$, v_h^p goes to a strictly positive value while v_h^s goes to zero. Hence, there exists a δ_l' such that for all $\delta_l < \delta_l^\diamond$, $v_h^p > v_h^s$ and the pooling equilibrium is undefeated. If $z_s > z_s^{\diamond}$, let $\delta_l^{\diamond\diamond}$ solve

$$u'(z_s) = \frac{\delta_h}{\Delta\delta_l^{\diamond\diamond} + (1-\Delta)\delta_h}. \tag{B.80}$$

Then $v_h^p = 0$ for all $\delta_l < \delta_l^{\diamond\diamond}$ while $v_h^s > 0$, so the separating equilibrium is undefeated. In this case, when $\delta_l = 0$, there is no trading in the AM and $v_h^s = v_h^p = 0$. □

Proof of Proposition 3.7: Let ϵ denote the amount of bad assets purchased by the government as a percentage of the total assets. First, the increase in welfare as a result of the program is given by

$$\mathcal{W} = u(z_s + m_p) - (z_s + m_p) - [u(z_s) - z_s], \tag{B.81}$$

where m_p is given by

$$u'(z_s + m_p) = \frac{\delta_h}{\frac{\Delta-\epsilon}{1-\epsilon}\delta_l + \frac{1-\Delta}{1-\epsilon}\delta_h} = \frac{1-\epsilon}{1-\Delta}. \tag{B.82}$$

We also have

$$\frac{\mathrm{d}m_p}{d\epsilon} = \begin{cases} -\frac{1}{(1-\Delta)u''(z_s+m_p)}, & \text{if } \epsilon < \Delta; \\ 0, & \text{if } \epsilon = \Delta. \end{cases} \tag{B.83}$$

$$\frac{\mathrm{d}\mathcal{W}}{d\epsilon} = -\frac{\Delta-\epsilon}{1-\Delta}\frac{1}{(1-\Delta)u''(z_s+m_p)}. \tag{B.84}$$

To derive the cost of purchasing assets, first note that sellers with bad assets must be indifferent between selling to the government and selling in the asset market. This means the government pays m_p to obtain the bad assets from each seller. Hence, the total cost of the program is.

$$\mathcal{C} = m_p\epsilon. \tag{B.85}$$

The marginal cost is

$$\frac{\mathrm{d}\mathcal{C}}{d\epsilon} = \begin{cases} m_p - \frac{\epsilon}{(1-\Delta)u''(z_s+m_p)}, & \text{if } m_p = m_p^* \text{ and } \epsilon < \Delta; \\ m_p, & \text{if } \epsilon = \Delta. \end{cases} \tag{B.86}$$

Since $u''(z_s + m_p)$ is finite, $\frac{\mathrm{d}\mathcal{W}}{d\epsilon}$ goes to zero as $\epsilon \to \Delta$. However, $\frac{\mathrm{d}\mathcal{C}}{d\epsilon}$ is always strictly positive. Hence, the optimal proportion of bad assets purchased must be strictly less than Δ.

Lastly, note that the government offers m_p to purchase all assets from a seller. In the asset market, the sale quantity s_p is determined by high-quality sellers who solve $\bar{\delta}u'(z_s + \bar{\delta}s_p) = \delta_h$. Because it is not optimal for the government to purchase all bad assets, we have $\bar{\delta} < \delta_h$ so high-quality sellers do not sell all assets. Since sellers sell less asset in the market but obtain the same amount of money, the market price must be higher. □

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Appendix C

Appendices to Chapter 4

C.1 Proofs for Section 4.3

In this appendix, I show how to solve the competitive equilibrium in period 2. Following Guerrieri and Shimer (2014), I first take buyer's search value k^* as given. I then endogenize k^* to solve for the full equilibrium.

To solve the partial equilibrium where k^* is given, I follow Guerrieri et al. (2010) and solve a set of problems $\{P(\delta_j)\}_{j=1}^J$ first. And then I prove any solution to $\{P(\delta_j)\}_{j=1}^J$ is a partial equilibrium and any partial equilibrium is a solution to $\{P(\delta_j)\}_{j=1}^J$.

Let

$$\bar{v}_b = u(c^*) - u(b) - (c^* - b)$$

denote highest possible trade surplus. For any $j \in \mathcal{J}$ and $k \in [0, \bar{v}_b]$, problem $P(\delta_j)$ is given by

$$v_{s,j} = \max_{\theta,z,s} \{p(\theta)[u(b+z) - u(b) - \delta_j s]\} \tag{C.1}$$

$$\text{s.t. } k \leq q(\theta)(s\delta_j - z) \tag{C.2}$$

$$v_{s,j'} \geq p(\theta)[u(b+z) - u(b) - \delta_{j'} s] \text{ for all } j' < j. \tag{C.3}$$

The first constraint says (z, s) must satisfy buyer's participation constraint. The second constraint requires that no sellers with quality worse than j have the incentive to deviate.

Before solving the problem let me first prove a lemma that will simplify the problem.

Lemma C.1 *Constraint (C.2) binds for all j.*

Proof: First it is easy to see that constraint (C.2) binds for $j = 1$ where constraint (C.3) disappears. Next, suppose constraint (C.2) binds for all $j' = 2, ..., j-1$ and suppose the inequality is strict for $P(\delta_j)$ with the solution being $\{z_j, s_j, \theta_j\}$. Now pick $\{z', s', \theta'\}$ such that

$$\begin{aligned}
&p(\theta_j)(u(b+z_j) - u(b) - \delta_{j-1}s_j) = p(\theta')(u(b+z') - u(b) - \delta_{j-1}s') \\
&k \leq q(\theta')(s'\delta_j - z') \\
&z' = z_j \\
&s' < s_j \\
&\theta' < \theta_j.
\end{aligned}$$

Now let us check constraint (C.3). For any $\delta_{j'} < \delta_{j-1}$, we have

$$\begin{aligned}&p(\theta_j)(u(b+z_j) - u(b) - \delta_{j'} s_j) + p(\theta_j) s_j(\delta_{j'} - \delta_{j-1})\\ =&p(\theta')(u(b+z') - u(b) - \delta_{j'} s') + p(\theta') s'(\delta_{j'} - \delta_{j-1}).\end{aligned}$$

Since $s' < s_j$ and $\theta' < \theta_j$, we have

$$p(\theta_j)(u(b+z_j) - u(b) - \delta_{j'} s_j) > p(\theta')(u(b+z') - u(b) - \delta_{j'} s')$$

so constraint (C.3) is still satisfied. But similarly we have

$$p(\theta')(u(b+z') - u(b) - \delta_j s') > p(\theta_j)(u(b+z_j) - u(b) - \delta_j s_j),$$

a contradiction. □

The next proposition characterizes the solution to this set of problems.

Proposition C.1 *Assume $k < \bar{v}_b$. The unique solution $\{(\theta_j, z_j, s_j)\}_{j=1}^J$ to $\{P(\delta_j)\}_{j=1}^J$ is given by (1) $z_1 = z^*$ where $u'(b+z^*) = 1$; θ_1 solves*

$$q(\theta) + \theta q'(\theta) = \frac{k}{u(b+z_1) - u(b) - z_1}; \tag{C.4}$$

and $s_1 = (k/q(\theta_1) + z_1)/\delta_1$. (2) For all $j > 1$, z_j and θ_j solve

$$u'(b+z_j)[(\theta_j q'(\theta_j) + q(\theta_j))z_j + k] = (\theta_j q'(\theta_j) + q(\theta_j))[u(b+z_j) - u(b)], \tag{C.5}$$

$$v_{s,j-1} = \theta_j q(\theta_j)\left[u(b+z) - u(b) - \frac{\delta_{j-1}}{\delta_j} z_j\right] - \frac{\delta_{j-1}}{\delta_j}\theta_j k; \tag{C.6}$$

and $s_j = (k/q(\theta_j) + z_j)/\delta_j$.

Proof: First, let us consider $j = 1$. Lemma C.1 allows Problem $P(\delta_1)$ to be simplified to

$$v_{s,1} = \max_{\theta,z} \theta q(\theta)\left[u(b+z) - u(b) - z\right] - \theta k. \tag{C.7}$$

It is straightforward to formulate the solution: $z_1 = z^*$, and θ_1 solves

$$q(\theta) + \theta q'(\theta) = \frac{k}{u(b+z_1) - u(b) - z_1}. \tag{C.8}$$

Since $p'(\theta) < 0$ and $\frac{k}{u(b+z_1)-z_1-u(b)} < 1$, there exists a unique solution of $\theta_1 > 0$.

Next let us look at $j = 2$. It must be that constraint (C.3) binds. Suppose not, then problem $P(\delta_2)$ is the same as $P(\delta_1)$ except for the difference in tree quality. Then $j = 2$ sellers will choose $z_2 = z_1$ and $\theta_2 = \theta_1$. But this is strictly better than what $j = 1$ sellers have: if they deviate they receive $\theta_1 q(\theta_1)\left[u(b+z_1) - u(b) - \frac{\delta_1}{\delta_2} z_1\right] - \frac{\delta_1}{\delta_2}\theta_1 k$. So it is a contradiction. Using the binding constraint (C.2) we can rewrite the problem as

$$v_{s,2} = \max_{\theta,z} \theta q(\theta)\left[u(b+z) - u(b) - z\right] - \theta k \tag{C.9}$$

$$\text{s.t. } v_{s,1} = \theta q(\theta)\left[u(b+z) - u(b) - \frac{\delta_1}{\delta_2} z\right] - \frac{\delta_1}{\delta_2}\theta k. \tag{C.10}$$

The solution is given by

$$u'(b+z)[(\theta q'(\theta) + q(\theta))z + k] = (\theta q'(\theta) + q(\theta))[u(b+z) - u(b)], \tag{C.11}$$

$$v_{s,1} = \theta q(\theta)\left[u(b+z) - u(b) - \frac{\delta_1}{\delta_2} z\right] - \frac{\delta_1}{\delta_2}\theta k. \tag{C.12}$$

Now, use (C.11) to take the implicit derivative of θ with respect to z.

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2q'(\theta)+\theta q''(\theta)} \frac{u''(b+z)[(\theta q'(\theta)+q(\theta))z+k]}{u(b+z)-u(b)-u'(b+z)z} > 0, \tag{C.13}$$

because $2q'(\theta)+\theta q''(\theta) = p''(\theta) < 0$. Next, the RHS of (C.12) is strictly increasing in θ because

$$(\theta q'(\theta)+q(\theta))[u(b+z)-u(b)-\delta_1/\delta_2 z] - \delta_1/\delta_2 k > (\theta q'(\theta)+q(\theta))[u(b+z)-u(b)-z]-k > 0. \tag{C.14}$$

It is also increasing in z because $z \leq z^*$. It is then easy to check that there exists a unique solution to (C.11) and (C.12), and that $\theta_1 > \theta_2$ and $z_1 > z_2$.

For cases where $j > 2$, the following claim gives the solution.

Claim A.2: For all $j > 2$, constraint (C.3) binds for $j' = j-1$ and is slack for all the other j''s.

Proof: I start with $j = 3$ and proceed by induction.

$$v_{s,3} = \max_{\theta,z} \theta q(\theta)\left[u(b+z)-u(b)-z\right] - \theta k \tag{C.15}$$

$$\text{s.t. } v_{s,1} = \theta q(\theta)\left[u(b+z)-u(b)-\frac{\delta_1}{\delta_3}z\right] - \frac{\delta_1}{\delta_3}\theta k, \tag{C.16}$$

$$v_{s,2} = \theta q(\theta)\left[u(b+z)-u(b)-\frac{\delta_2}{\delta_3}z\right] - \frac{\delta_2}{\delta_3}\theta k. \tag{C.17}$$

First, by the same reasoning in $P(\delta_2)$, at least one of the two resource constraints must bind. Now suppose constraint (C.16) binds but constraint (C.17) is slack. Then the problem is very similar to $P(\delta_2)$ but $\delta_2 > \delta_3$. Hence, we have $\theta_1 > \theta_2 > \theta_3$ and $z_1 \geq z_2 > z_3$. However, (C.16) implies

$$\theta_2 q(\theta_2)\left[u(b+z_2)-u(b)-\frac{\delta_1}{\delta_2}z_2\right] - \frac{\delta_1}{\delta_2}\theta_2 k = \theta_3 q(\theta_3)\left[u(b+z_3)-u(b)-\frac{\delta_1}{\delta_3}z_3\right] - \frac{\delta_1}{\delta_3}\theta_3 k, \tag{C.18}$$

which then implies

$$\begin{aligned}
v_{s,2} &= \theta_2 q(\theta_2)\left[u(b+z_2)-u(b)-\frac{\delta_2}{\delta_2}z_2\right] - \frac{\delta_2}{\delta_2}\theta_2 k \\
&= \theta_2 q(\theta_2)\left[u(b+z_2)-u(b)-\frac{\delta_1}{\delta_2}z_2\right] - \frac{\delta_1}{\delta_2}\theta_2 k + \theta_2[z_2 q(\theta_2)+k]\frac{\delta_1-\delta_2}{\delta_2} \\
&= \theta_3 q(\theta_3)\left[u(b+z_3)-u(b)-\frac{\delta_1}{\delta_3}z_3\right] - \frac{\delta_1}{\delta_3}\theta_3 k + \theta_2[z_2 q(\theta_2)+k]\frac{\delta_1-\delta_2}{\delta_2} \\
&= \theta_3 q(\theta_3)\left[u(b+z_3)-u(b)-\frac{\delta_2}{\delta_3}z_3\right] - \frac{\delta_2}{\delta_3}\theta_3 k + \theta_2[z_2 q(\theta_2)+k]\frac{\delta_1-\delta_2}{\delta_2} + \theta_3[z_3 q(\theta_3)+k]\frac{\delta_2-\delta_1}{\delta_3} \\
&< \theta_3 q(\theta_3)\left[u(b+z_3)-u(b)-\frac{\delta_2}{\delta_3}z_3\right] - \frac{\delta_2}{\delta_3}\theta_3 k,
\end{aligned}$$

where the last inequality is because $\theta_2 > \theta_3$ and $\theta_2 q(\theta_2)z_2 > \theta_3 q(\theta_3)z_3$. This contradicts with the assumption that (C.17) is slack. Now, suppose constraint (C.17) binds but constraint (C.16) is slack. Follow similar arguments and it is easy to see that all the constraints are satisfied and θ_3 is now given by a binding constraint (C.17) and (C.11). Lastly, it is not possible that both constraints bind. Hence, only $j' = 2$ constraint binds.

Now suppose that the claim holds for all $j' < j$. For $P(\delta_j)$ we can rewrite constraint (C.3) as

$$\theta_{j-1}q(\theta_{j-1})\left[u(b+z_{j-1})-u(b)-\frac{\delta_{j-1}}{\delta_{j-1}}z_{j-1}\right]-\frac{\delta_{j-1}}{\delta_{j-1}}\theta_{j-1}k\geq\theta q(\theta)\left[u(b+z)-u(b)-\frac{\delta_{j-1}}{\delta_j}z\right]-\frac{\delta_{j-1}}{\delta_j}\theta k$$

$$\theta_{j-1}q(\theta_{j-1})\left[u(b+z_{j-1})-u(b)-\frac{\delta_{j-2}}{\delta_{j-1}}z_{j-1}\right]-\frac{\delta_{j-2}}{\delta_{j-1}}\theta_{j-1}k\geq\theta q(\theta)\left[u(b+z)-u(b)-\frac{\delta_{j-2}}{\delta_j}z\right]-\frac{\delta_{j-2}}{\delta_j}\theta k$$

...

$$\theta_{j-k+1}q(\theta_{j-k+1})\left[u(b+z_{j-k+1})-u(b)-\frac{\delta_{j-k}}{\delta_{j-k+1}}z_{j-k+1}\right]-\frac{\delta_{j-k+1}}{\delta_{j-k+1}}\theta_{j-k}k$$

$$\geq\theta q(\theta)\left[u(b+z)-u(b)-\frac{\delta_{j-k}}{\delta_j}z\right]-\frac{\delta_{j-k+1}}{\delta_j}\theta k$$

...

Again, at least one of the $j-1$ constraints must be binding. Let one of the binding constraints be $j'\in\{1,...,j-1\}$. Now suppose $j'<j-1$. We have

$$\begin{aligned}&\theta_{j'+1}q(\theta_{j'+1})\left[u(b+z_{j'+1})-u(b)-\frac{\delta_{j'}}{\delta_{j'+1}}z_{j'+1}\right]-\frac{\delta_{j'}}{\delta_{j'+1}}\theta_{j'+1}k\\&=\theta_jq(\theta_j)\left[u(b+z_j)-u(b)-\frac{\delta_{j'}}{\delta_j}z_j\right]-\frac{\delta_{j'}}{\delta_j}\theta_jk\end{aligned}\tag{C.19}$$

and again $\theta_j<\theta_{j'+1}$ and $z_j<z_{j'+1}$. Similar to the $j=3$ case, this implies that

$$\begin{aligned}v_{s,j'+1}&=\theta_{j'+1}q(\theta_{j'+1})\left[u(b+z_{j'+1})-u(b)-\frac{\delta_{j'+1}}{\delta_{j'+1}}z_{j'+1}\right]-\frac{\delta_{j'+1}}{\delta_{j'+1}}\theta_{j'+1}k\\&=\theta_{j'+1}q(\theta_{j'+1})\left[u(b+z_{j'+1})-u(b)-\frac{\delta_{j'}}{\delta_{j'+1}}z_{j'+1}\right]-\frac{\delta_{j'}}{\delta_{j'+1}}\theta_{j'+1}k+\theta_{j'+1}[z_{j'+1}q(\theta_{j'+1})+k]\frac{\delta_{j'}-\delta_{j'+1}}{\delta_{j'+1}}\\&=\theta_jq(\theta_j)\left[u(b+z_j)-u(b)-\frac{\delta_{j'}}{\delta_j}z_j\right]-\frac{\delta_{j'}}{\delta_j}\theta_jk+\theta_{j'+1}[z_{j'+1}q(\theta_{j'+1})+k]\frac{\delta_{j'}-\delta_{j'+1}}{\delta_{j'+1}}\\&=\theta_jq(\theta_j)\left[u(b+z_j)-u(b)-\frac{\delta_{j'+1}}{\delta_j}z_j\right]-\frac{\delta_{j'+1}}{\delta_j}\theta_jk\\&\quad+\theta_{j'+1}[z_{j'+1}q(\theta_{j'+1})+k]\frac{\delta_{j'}-\delta_{j'+1}}{\delta_{j'+1}}+\theta_j[z_jq(\theta_j)+k]\frac{\delta_{j'+1}-\delta_{j'}}{\delta_j}\\&<\theta_jq(\theta_j)\left[u(b+z_j)-u(b)-\frac{\delta_{j'+1}}{\delta_j}z_j\right]-\frac{\delta_{j'+1}}{\delta_j}\theta_jk,\end{aligned}$$

contradicting with the assumption that constraint $j'+1$ does not bind. Hence, only constraint $j-1$ binds. □

The next proposition shows the connection between $\{P(\delta_j)\}_{j=1}^J$ and the equilibrium defined in Definition 4.1. In short, any solution to $\{P(\delta_j)\}_{j=1}^J$ is an equilibrium and any equilibrium is a solution to $\{P(\delta_j)\}_{j=1}^J$.

Proposition C.2 (1) *For any solution to $\{P(\delta_j)\}_{j=1}^J$, for all j, let $\Psi=\{(z_j,s_j)\}_{j=1}^J$ and $\theta(z_j,s_j)=\theta_j$; let $\gamma(z_j,s_j;\delta)=1$ if and only if $\delta=\delta_j$; and let $v^*_{s,j}=v_{s,j}$. Then $\{\Psi,\{v^*_{s,j}\}_{j=1}^J,\theta(.),\gamma(.),F(.)\}$ is an equilibrium;*
(2) *For any equilibrium, let (z_j,s_j) be such that $\gamma(z,s;\delta_j)>0$ and let $\theta_j=\theta(z_j,s_j)$. Then*

$\{(z_j, s_j, \theta_j)\}_{j=1}^J$ *solves* $\{P(\delta_j)\}_{j=1}^J$.

Proof: Note that because the above solution to $\{P(\delta_j)\}_{j=1}^J$ is unique, this proof implies that the equilibrium is unique.

Part 1: To show that the solution to $\{P(\delta_j)\}_{j=1}^J$ is an equilibrium, I look for for an equilibrium characterized by (θ_j, z_j, s_j). For all j, let $\Psi = \{(z_j, s_j)\}_{j=1}^J$ and $\theta(z_j, s_j) = \theta_j$; let $\gamma((z_j, s_j); \delta) = 1$ if and only if $\delta = \delta_j$; let $v^*_{s,j} = v_{s,j}$; and let $dF(z_j, s_j) = \Delta_j$.

Let us check Condition (1) to (4). Condition (1) and (2) hold by construction. Buyer's optimal behavior also holds because for all j, $(z_j, s_j, \theta(z_j, s_j))$ offers utility k to buyers. We need to show that Seller's optimal behavior is satisfied and Condition (4) satisfied.

By construction, for all j and for all $j' < j$, j' will not deviate to j. We only need to prove $j' > j$ will not deviate to j either. Let us proceed by induction. First note that from $\{P(\delta)\}$ we have

$$\theta_j q(\theta_j)\left[u(b+z_j) - u(b) - \delta_j s_j\right] = \theta_{j+1} q(\theta_{j+1})\left[u(b+z_{j+1}) - u(b) - \delta_j s_{j+1}\right]. \tag{C.20}$$

Since $\theta_j \geq \theta_{j+1}$ and $s_j \geq s_{j+1}$ imply that $\theta_j q(\theta_j) s_j \geq \theta_{j+1} q(\theta_{j+1}) s_{j+1}$, we have

$$\theta_j q(\theta_j)[u(b+z_j) - u(b) - \delta_{j+1} s_j] < \theta_{j+1} q(\theta_{j+1})[u(b+z_{j+1}) - u(b) - \delta_{j+1} s_{j+1}]. \tag{C.21}$$

Now suppose for some $j' > j+1$, the above equation is also true for $j'-1$:

$$\theta_j q(\theta_j)[u(b+z_j) - u(b) - \delta_{j'-1} s_j] < \theta_{j'-1} q(\theta_{j'-1})[u(b+z_{j'-1}) - u(b) - \delta_{j'-1} s_{j'-1}]. \tag{C.22}$$

Note that $\theta_j q(\theta_j) s_j \geq \theta_{j'-1} q(\theta_{j'-1}) s_{j'-1}$ and $\delta_{j'} > \delta_{j'-1}$, which imply that

$$\theta_j q(\theta_j)[u(b+z_j) - u(b) - \delta_{j'} s_j] < \theta_{j'-1} q(\theta_{j'-1})[u(b+z_{j'-1}) - u(b) - \delta_{j'} s_{j'-1}]. \tag{C.23}$$

Apply (C.21) again to get

$$\theta_{j'-1} q(\theta_{j'-1})[u(b+z_{j'-1}) - u(b) - \delta_{j'} s_{j'-1}] < \theta_{j'} q(\theta_{j'})[u(b+z_{j'}) - u(b) - \delta_{j'} s_{j'}]. \tag{C.24}$$

The last two inequalities imply

$$\theta_j q(\theta_j)[u(b+z_j) - u(b) - \delta_{j'} s_j] < \theta_{j'} q(\theta_{j'})[u(b+z_{j'}) - u(b) - \delta_{j'} s_{j'}], \tag{C.25}$$

which is what we want.

Now suppose there exists a set S that satisfies condition (4.10) to (4.12). Let $j^* = \min S$. Then for all $j < j^*$, sellers have no incentive to deviate. Now consider the belief system that assigns all the weights to δ_{j^*}. That is, $\gamma(z', s'; \delta_{j^*}) = 1$. By assumption, (z, s) must be accepted by buyers under γ. Then we have a contradiction: δ_{j^*} seller is strictly better off while not violating any constraints in $P(\delta_{j^*})$.

Part 2: In this part I show that any equilibrium defined by Definition 4.1 is a solution to $\{P(\delta_j)\}_{j=1}^J$. Aggregate consistency implies that for all j there exists (z, s) such that $\gamma(z, s; \delta_j) > 0$. Denote such (z, s) as (z_j, s_j). Let $\theta_j = \theta(z_j, s_j)$. I first show that (z_j, s_j, θ_j) satisfies constraints (C.2) and (C.3) for all j. Then I show that $\{(z_j, s_j, \theta_j)\}_{j=1}^J$ solves $\{P(\delta)\}$.

First, note that Buyer's optimal behavior and Active markets together imply that

$$k = q(\theta_j)(s_j \delta_j - z_j)$$

for all j and that resource constraints are satisfied. That is, constraint (C.2) and (C.3) are satisfied.

Next, Equilibrium beliefs and Seller's optimal behavior imply that

$$v^*_{s,j} = p(\theta_j)[u(b+z_j) - u(b) - \delta_j s_j]$$

and that

$$v^*_{s,j} \geq p(\theta_{j'})[u(b+z_{j'} s_{j'}) - u(b) - \delta_j s_{j'}] \text{ for all } j'.$$

Hence, constraint (C.3) is satisfied as long as $v_{s,j} = v^*_{s,j}$.

Lastly, I prove that $v_{s,j} = v^*_{s,j}$ for all j. That is, $\{(z_j, s_j, \theta_j)\}_{j=1}^J$ is a solution to $\{P(\delta)\}$. Let me proceed by induction. First, it is easy to see that $v_{s,1} = v^*_{s,1}$. Next, suppose $v_{s,j'} = v^*_{s,j'}$ for all $j' < j$. Now suppose $v_{s,j} > v^*_{s,j}$. That is, some (θ, z) satisfies the constraints of $P(\delta_j)$ and delivers higher utility to sellers. That is,

$$v^*_{s,j} < p(\theta)[u(b+z) - u(b) - (k+z)] \tag{C.26}$$

$$v^*_{s,j'} \geq p(\theta)[u(b+z) - u(b) - (k+z)\delta_{j'}/\delta_j] \text{ for } j' = j-1 \tag{C.27}$$

where (C.27) uses the induction assumption. Now let the off-equilibrium offer be (z', s', θ). Then there exists S such that $j \in S$ and $j' \notin S$ for all $j' < j$. Then (4.10) and (4.11) are satisfied by definition. (4.12) is satisfied because for all $j' \in S$, $\delta_{j'} \geq \delta_j$. Hence, Condition (5) is violated. That is, it must be that $v_{s,j} = v^*_{s,j}$. $\square$

Proof to Proposition 4.2: Recall that $\frac{\mathrm{d}\theta}{\mathrm{d}z}$ is given by (C.13).

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{p''(\theta)} \frac{u''(b+z)[(\theta q'(\theta) + q(\theta))z + k]}{u(b+z) - u(b) - u'(b+z)z}. \tag{C.28}$$

Now assume $\frac{p'(\theta)}{p(\theta)p''(\theta)}$ is increasing in θ. Assume also that $\frac{-zU'(z)}{U(z)}$ is increasing in z. We have

$$\frac{\mathrm{d}p(\theta)}{\mathrm{d}z}\frac{z}{p(\theta)} = \frac{p'(\theta)}{p(\theta)p''(\theta)}\left(\frac{z^2u''(b+z)(\theta q'(\theta) + q(\theta))}{u(b+z) - u(b) - u'(b+z)z} + \frac{kzu''(b+z)}{u(b+z) - u(b) - u'(b+z)z}\right). \tag{C.29}$$

Notice that (1) $\frac{p'(\theta)}{p(\theta)p''(\theta)} < 0$ and is increasing in θ and therefore increasing in z; (2) $\frac{z^2u''(b+z)(\theta q'(\theta)+q(\theta))}{u(b+z)-u(b)-u'(b+z)z} < 0$ is increasing in z because $\theta q'(\theta) + q(\theta) > 0$ is decreasing in θ and θ is increasing in z; and (3) $\frac{kzu''(b+z)}{u(b+z)-u(b)-u'(b+z)z} < 0$ is increasing in z because $\frac{z^2u''(b+z)}{u(b+z)-u(b)-u'(b+z)z} = \frac{-zU'(z)}{U(z)} < 0$ is increasing in z. In conclusion, $\frac{\mathrm{d}p(\theta)}{\mathrm{d}z}\frac{z}{p(\theta)}$ is decreasing in z. $\square$

C.2 Proofs for Section 4.4

Proof to Proposition 4.3: First, recall that z_1 is given by $z_1 = z^*$ where $\eta\hat{u}'(b+z^*) = 1$. Hence, z_1 is increasing in η. θ_1 is given by

$$q(\theta) + \theta q'(\theta) = \frac{k}{\eta\hat{u}(b+z_1) - \eta\hat{u}(b) - z_1}. \tag{C.30}$$

It is therefore easy to see that θ_1 is also weakly increasing in η. Second, θ_2 and z_2 are given by

$$\hat{u}'(b+z)[(\theta q'(\theta) + q(\theta))z + k] = (\theta q'(\theta) + q(\theta))[\hat{u}(b+z) - \hat{u}(b)], \tag{C.31}$$

$$\theta_1 q(\theta_1)\left[\eta\hat{u}(b+z_1) - \eta\hat{u}(b) - z_1\right] - \theta_1 k = \theta q(\theta)\left[\eta\hat{u}(b+z) - \eta\hat{u}(b) - \frac{\delta_1}{\delta_2}z\right] - \frac{\delta_1}{\delta_2}\theta k. \tag{C.32}$$

If θ_2 and z_2 are unchanged after the increase in η, then the RHS of (C.32) will be smaller than the LHS because $\theta_1 q(\theta_1)\left[\eta\hat{u}(b+z_1) - \eta\hat{u}(b)\right] > \theta_2 q(\theta_2)\left[\eta\hat{u}(b+z_2) - \eta\hat{u}(b)\right]$. This means the θ_2 and z_2 must be strictly increasing in η. Following the same argument, it is easy to show that θ_j and z_j are increasing in η for all $1 < j \leq J$. $\square$

Proof to Proposition 4.4: First, recall that z_1 is given by $z_1 = z^*$ where $\eta\hat{u}'(b+z^*) = 1$. Hence, z_1 does not change after the shock. θ_1 is given by

$$q(\theta) + \theta q'(\theta) = \frac{k}{\eta\hat{u}(b+z_1) - \eta\hat{u}(b) - z_1}. \tag{C.33}$$

It is therefore easy to see that θ_1 also does not change after the shock. Second, θ_2 and z_2 are given by

$$\hat{u}'(b+z)[(\theta q'(\theta)+q(\theta))z+k] = (\theta q'(\theta)+q(\theta))[\hat{u}(b+z)-\hat{u}(b)], \tag{C.34}$$

$$\theta_1 q(\theta_1)\left[\eta\hat{u}(b+z_1)-\eta\hat{u}(b)-z_1\right]-\theta_1 k = \theta q(\theta)\left[\eta\hat{u}(b+z)-\eta\hat{u}(b)-\frac{\delta_1^\dagger}{\delta_2^\dagger}z\right]-\frac{\delta_1^\dagger}{\delta_2^\dagger}\theta k. \tag{C.35}$$

If θ_2 and z_2 are unchanged after the increase in η, then the RHS of (C.35) will be larger than the LHS because

$$\frac{\delta_1^\dagger}{\delta_2^\dagger} = \frac{h(\delta_1)\delta_1}{h(\delta_2)\delta_2} < \frac{h(\delta_1)\delta_1}{h(\delta_2)\delta_2}. \tag{C.36}$$

This means the θ_2 and z_2 must be smaller in η. Following the same argument, it is easy to show that θ_j and z_j are smaller for all $1 < j \leq J$. $\square$

C.3 Proofs for Section 4.5

Proof to Proposition 4.5: The process of solving the partial equilibrium is almost identical to the process of solving the equilibrium in Section 4.3 and therefore is omitted. I repeat the partial equilibrium. Assume $v_b < \bar{v}_b$. The unique partial equilibrium $\{(\theta_j, z_j, s_j)\}_{j=1}^J$ is given by (1) $z_1 = \min\{z^*, b^l\}$ where $u'(b^e+z^*)=1$; θ_1 solves

$$q(\theta)+\theta q'(\theta) = \frac{v_b}{u(b^e+z_1)-u(b^e)-z_1}. \tag{C.37}$$

(2) For all $j>1$, z_j and θ_j solve

$$u'(b^e+z_j)[(\theta_j q'(\theta_j)+q(\theta_j))z_j+v_b] = (\theta_j q'(\theta_j)+q(\theta_j))[u(b^e+z_j)-u(b^e)], \tag{C.38}$$

$$v_{s,j-1}^* = \theta_j q(\theta_j)\left[u(b^e+z)-u(b^e)-\frac{\delta_{j-1}}{\delta_j}z_j\right]-\frac{\delta_{j-1}}{\delta_j}\theta_j v_b. \tag{C.39}$$

It is easy to see that both θ_1 and $v_{s,1}^*$ are decreasing in v_b. Now, consider θ_2. It is given by

$$u'(b^e+z)[(\theta q'(\theta)+q(\theta))z+v_b] = (\theta q'(\theta)+q(\theta))[u(b^e+z)-u(b^e)], \tag{C.40}$$

$$\theta_1 q(\theta_1)\left[u(b^e+z_1)-u(b^e)-z_1\right]-\theta_1 v_b = \theta q(\theta)\left[u(b^e+z)-u(b^e)-\delta_1/\delta_2 z\right]-\delta_1/\delta_2\theta v_b. \tag{C.41}$$

To see how v_b affects θ_2, consider $v_b' > v_b$. Let θ_2^a solve

$$u'(b^e+z)[(\theta q'(\theta)+q(\theta))z+v_b'] = (\theta q'(\theta)+q(\theta))[u(b^e+z)-u(b^e)], \tag{C.42}$$

$$\theta_1(v_b)q(\theta_1(v_b))\left[u(b^e+z_1)-u(b^e)-z_1\right]-\theta_1(v_b)v_b = \theta q(\theta)\left[u(b^e+z)-u(b^e)-\delta_1/\delta_2 z\right]-\delta_1/\delta_2\theta v_b. \tag{C.43}$$

It must be that $\theta_2^a < \theta_2$, because otherwise θ_2 is (weakly) larger and z_2 is strictly larger, which mean (C.43) does not hold. In addition, it must be that

$$\theta_2^a q(\theta_2^a)\left[u(b^e+z_2^a)-u(b^e)-z_2^a\right]-\theta_2^a v_b < \theta_2 q(\theta_2)\left[u(b^e+z_2)-u(b^e)-z_2\right]-\theta_2 v_b, \tag{C.44}$$

because otherwise (θ_2^a, z_2^a) offers higher utility than (θ_2, z_2) given v_b. Since $\theta_2^a < \theta_2$, we have

$$\theta_2^a q(\theta_2^a)\left[u(b^e+z_2^a)-u(b^e)-z_2^a\right] < \theta_2 q(\theta_2)\left[u(b^e+z_2)-u(b^e)-z_2\right], \tag{C.45}$$

Next, let θ_2^b solve

$$u'(b^e+z)[(\theta q'(\theta)+q(\theta))z+v_b'] = (\theta q'(\theta)+q(\theta))[u(b^e+z)-u(b^e)], \tag{C.46}$$

$$\theta_1(v_b')q(\theta_1(v_b))\left[u(b^e+z_1)-u(b^e)-z_1\right]-\theta_1(v_b')v_b = \theta q(\theta)\left[u(b^e+z)-u(b^e)-\delta_1/\delta_2 z\right]-\delta_1/\delta_2\theta v_b. \tag{C.47}$$

Compared to the LHS of (C.43), the LHS of (C.47) is smaller. Therefore, $\theta_2^b < \theta_2^a$ and $z_2^b < z_2^a$. Hence,

$$\theta_2^b q(\theta_2^b)\left[u(b^e + z_2^b) - u(b^e) - z_2^b\right] < \theta_2^a q(\theta_2^a)\left[u(b^e + z_2^a) - u(b^e) - z_2^a\right], \tag{C.48}$$

Lastly, let $\theta_2(v_b')$ solve

$$u'(b^e + z)[(\theta q'(\theta) + q(\theta))z + v_b'] = (\theta q'(\theta) + q(\theta))[u(b^e + z) - u(b^e)], \tag{C.49}$$

$$\begin{aligned}&\theta_1(v_b')q(\theta_1(v_b))\left[u(b^e + z_1) - u(b^e) - z_1\right] - \theta_1(v_b')v_b' \\ &= \theta q(\theta)\left[u(b^e + z) - u(b^e) - \delta_1/\delta_2 z\right] - \delta_1/\delta_2\theta v_b'.\end{aligned} \tag{C.50}$$

Then $\theta_2(v_b') < \theta_2^b$ because $\theta_1(v_b') > \theta_2^b$, and hence θ_2 must be smaller so (C.50) holds. In addition, because $z_2(v_b') < z_2^b$,

$$\theta_2(v_b')q(\theta_2(v_b'))\left[u(b^e + z_2(v_b')) - u(b^e) - z_2(v_b')\right] < \theta_2^b q(\theta_2^b)\left[u(b^e + z_2^b) - u(b^e) - z_2^b\right]. \tag{C.51}$$

In conclusion, θ_2 and $\theta_2 q(\theta_2)\left[u(b^e + z_2) - u(b^e) - z_2\right]$ are decreasing in v_b.

Now, suppose that for all $j' < j$, $\theta_{j'}$ and $\theta_{j'}q(\theta_{j'})\left[u(b^e + z_{j'}) - u(b^e) - z_{j'}\right]$ are decreasing in v_b. For j, the proof follows closely the case with $j = 2$. Let θ_j^a solve

$$u'(b^e + z)[(\theta q'(\theta) + q(\theta))z + v_b'] = (\theta q'(\theta) + q(\theta))[u(b^e + z) - u(b^e)], \tag{C.52}$$

$$\begin{aligned}&\theta_{j-1}(v_b)q(\theta_{j-1}(v_b))\left[u(b^e + z_{j-1}) - u(b^e) - z_{j-1}\right] - \theta_{j-1}(v_b)v_b \\ &\qquad = \theta q(\theta)\left[u(b^e + z) - u(b^e) - \delta_{j-1}/\delta_j z\right] - \delta_{j-1}/\delta_j\theta v_b.\end{aligned} \tag{C.53}$$

It must be that $\theta_j^a < \theta_j$. In addition, it must be that

$$\theta_j^a q(\theta_j^a)\left[u(b^e + z_j^a) - u(b^e) - z_j^a\right] - \theta_j^a v_b < \theta_j q(\theta_j)\left[u(b^e + z_j) - u(b^e) - z_j\right] - \theta_j v_b. \tag{C.54}$$

Since $\theta_j^a < \theta_j$, it must be that

$$\theta_j^a q(\theta_j^a)\left[u(b^e + z_j^a) - u(b^e) - z_j^a\right] < \theta_j q(\theta_j)\left[u(b^e + z_j) - u(b^e) - z_j\right], \tag{C.55}$$

Next, let θ_j^b solve

$$u'(b^e + z)[(\theta q'(\theta) + q(\theta))z + v_b'] = (\theta q'(\theta) + q(\theta))[u(b^e + z) - u(b^e)], \tag{C.56}$$

$$\begin{aligned}&\theta_{j-1}(v_b')q(\theta_{j-1}(v_b))\left[u(b^e + z_{j-1}) - u(b^e) - z_{j-1}\right] - \theta_{j-1}(v_b')v_b \\ &\qquad = \theta q(\theta)\left[u(b^e + z) - u(b^e) - \delta_{j-1}/\delta_j z\right] - \delta_{j-1}/\delta_j\theta v_b.\end{aligned} \tag{C.57}$$

According to the induction assumption, θ_{j-1} and $\theta_{j-1}q(\theta_{j-1})\left[u(b^e + z_{j-1}) - u(b^e) - z_{j-1}\right]$ are decreasing in v_b. Hence, the LHS of (C.57) is smaller, and therefore $\theta_j^b < \theta_j^a$ and $z_j^b < z_j^a$. We have

$$\theta_j^b q(\theta_j^b)\left[u(b^e + z_j^b) - u(b^e) - z_j^b\right] < \theta_j^a q(\theta_j^a)\left[u(b^e + z_j^a) - u(b^e) - z_j^a\right], \tag{C.58}$$

Lastly, let $\theta_j(v_b')$ solve

$$u'(b^e + z)[(\theta q'(\theta) + q(\theta))z + v_b'] = (\theta q'(\theta) + q(\theta))[u(b^e + z) - u(b^e)], \tag{C.59}$$

$$\begin{aligned}&\theta_{j-1}(v_b')q(\theta_{j-1}(v_b))\left[u(b^e + z_{j-1}) - u(b^e) - z_{j-1}\right] - \theta_{j-1}(v_b')v_b' \\ &\qquad = \theta q(\theta)\left[u(b^e + z) - u(b^e) - \delta_{j-1}/\delta_j z\right] - \delta_{j-1}/\delta_j\theta v_b'.\end{aligned} \tag{C.60}$$

Then $\theta_j(v_b') < \theta_j^b$ and $z_j(v_b') < z_j^b$. Hence,

$$\theta_j(v_b')q(\theta_j(v_b'))\left[u(b^e + z_j(v_b')) - u(b^e) - z_j(v_b')\right] < \theta_j^b q(\theta_j^b)\left[u(b^e + z_j^b) - u(b^e) - z_j^b\right]. \tag{C.61}$$

In conclusion, θ_j and $\theta_j q(\theta_j)\left[u(b^e + z_j) - u(b^e) - z_j\right]$ are decreasing in v_b. Hence, I have established that $\{\theta_j\}_{j=1}^{\infty}$ is decreasing in v_b.

Now, define

$$g(v_b) = \sum_{j=1}^{J} \theta_j(v_b)\Delta_j. \tag{C.62}$$

Then $g(v_b)$ denotes the overall buyer-to-seller ratio of the asset market given v_b. The equilibrium v_b^* solves

$$g(v_b) = \frac{1-\alpha}{\alpha}. \tag{C.63}$$

Now consider $v_b = 0$. Let $\bar{\theta}$ be such that $p(\bar{\theta}) = 1$. It is easy to show that solution to the $P(\theta)$ problem is that for all j, $\theta_j = \bar{\theta}$, and z_j is given by binding constraint $j-1$ (see (C.3)). If $\bar{\theta} < \infty$, then $g(0) < \infty$ and it is possible that $g(0) < \frac{1-\alpha}{\alpha}$. In such case some late consumers do not participate in the asset market. If $\bar{\theta} = \infty$, $v_b^* > 0$ as long as $\alpha > 0$.

Next, consider when $v_b = \bar{v}_b$. Then we have $q(\underline{\theta}) = 1$. Then all sellers are indifferent between selling or not, and the incentive constraints (see (C.3)) are satisfied automatically. Hence, $g(\bar{v}_b) = \underline{\theta}$, and it constitutes as an equilibrium only when $\alpha \geq 1/(1+\underline{\theta})$. This means that as long as $\alpha < 1/(1+\underline{\theta})$, $v_b^* < \bar{v}_b$. $\square$

Proof to Proposition 4.6: The proof is identical to the proof of Proposition 4.2 and therefore is omitted.

Proof to Proposition 4.7: First, consider an increase in η. Recall that z_1 is given by $z_1 = \min\{z^*, b^l\}$ where $\eta\hat{u}'(b^e + z^*) = 1$. Hence, z_1 is increasing in η. θ_1 is given by

$$q(\theta) + \theta q'(\theta) = \frac{v_b}{\eta\hat{u}(b^e + z_1) - \eta\hat{u}(b^e) - z_1}. \tag{C.64}$$

It is easy to see that θ_1 is also weakly increasing in η. Second, θ_2 and z_2 are given by

$$\hat{u}'(b^e + z)[(\theta q'(\theta) + q(\theta))z + v_b] = (\theta q'(\theta) + q(\theta))[\hat{u}(b^e + z) - \hat{u}(b^e)], \tag{C.65}$$

$$\theta_1 q(\theta_1)\left[\eta\hat{u}(b^e + z_1) - \eta\hat{u}(b^e) - z_1\right] - \theta_1 v_b = \theta q(\theta)\left[\eta\hat{u}(b^e + z) - \eta\hat{u}(b^e) - \frac{\delta_1}{\delta_2}z\right] - \frac{\delta_1}{\delta_2}\theta v_b. \tag{C.66}$$

If θ_2 and z_2 are unchanged after the increase in η, then the RHS of (C.66) will be smaller than the LHS because $\theta_1 q(\theta_1)\left[\eta\hat{u}(b^e + z_1) - \eta\hat{u}(b^e)\right] > \theta_2 q(\theta_2)\left[\eta\hat{u}(b^e + z_2) - \eta\hat{u}(b^e)\right]$. This means the θ_2 and z_2 must be strictly increasing in η. Following the same arguments, it is easy to show that θ_j and z_j are increasing in η for all $1 < j \leq J$. From the proof to Proposition 4.5, we know that v_b^* must increase so that condition (4) of Definition 4.2 is satisfied. Next, consider an increase in α. It is clear that v_b^* must increase so that condition (4) of Definition 4.2 is satisfied. Lastly, it is easy to see that z_1 and θ_1 are weakly increasing in b^l. This means that if θ_2 and z_2 are unchanged, then the RHS of (C.66) will be larger than the LHS. Following the same arguments, it is easy to show that θ_j and z_j are increasing in b^l for all $1 < j \leq J$. Hence, v_b^* must increase so that condition (4) of Definition 4.2 is satisfied. $\square$

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