An analytical and numerical treatment of the Carter constant for inclined elliptical orbits about a massive Kerr black hole

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Abstract

In an extreme binary black hole system, an orbit will increase its angle of inclination ( ) as it evolves in Kerr spacetime. We focus our attention on the behaviour of the Carter constant (Q) for near-polar orbits.

The value of Q for bound orbits is non-negative; and an increase in Q corresponds to an increase in . For a Schwarzschild black hole, the polar orbit represents the boundary between the prograde and retrograde orbits at which Q is at its maximum value. The introduction of spin ($S = |\mathbf{J}|/M^2$) to the massive black hole causes this boundary, or Abutment, to be moved towards the retrograde orbits. We consider this characteristic to be important for understanding the evolution of Q for near-polar orbits. We have developed analytical formulae for Q for a test-particle in a polar orbit and at the last stable orbit (LSO) for given values of latus rectum (l) and eccentricity ( ). The Abutment is an important analytical and numerical laboratory that allows us to make a detailed investigation of the evolution of Q for a test particle near its LSO.

Introduction

The elliptical orbit of the compact object (CO) about the massive black hole (MBH) is expected to generate gravitational wave (GW) radiation. The shape and energy of the GW is determined by the eccentricity and period of the CO orbit. In turn, the loss of angular momentum cause the orbital characteristics to change over time. Indeed, the orbit will list and, become more circular as its radius decreases, until the CO plunges into the MBH.

Extreme mass ratio inspirals (EMRIs) are expected to emit GW suitable for detection by the LISA space observatory [1]. The emission of GW causes the constants of motion to evolve, which in turn affects the GW power spectrum. Some useful methods have been developed to describe this evolution. For example, the quadrupole formalism [2] and the Teukolsky equation [3] have yielded important results.

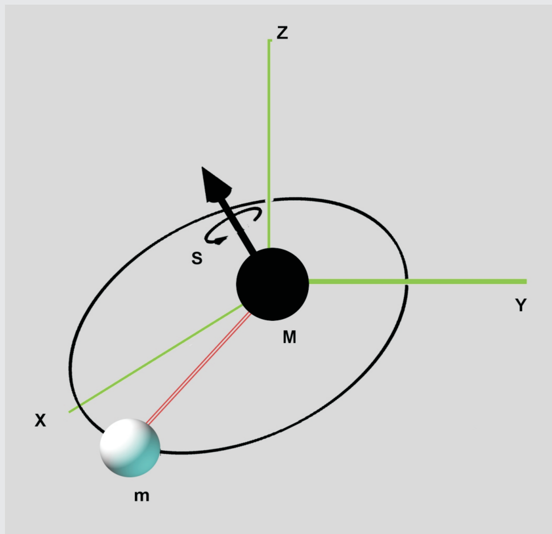

Figure 1. A schematic of an EMRI.

The analytical description of the evolution of Q has been more difficult to achieve than it has for the other two constants of motion; although the use of the Teukolsky equation has yielded promising results in estimating the evolution [4].

In this work we extend our previous work with elliptical equatorial orbits [5] to that of inclined orbits.

The Carter Constant

In his landmark work of 1968, Brandon Carter derived a new constant of motion that pertained to orbital motion in the gravitational field of a Kerr black hole (KBH) [6]. In due course this constant, known as the Carter constant, joined the set of important constants of motion: orbital angular momentum (L_z, z-axis projection), orbital energy (E), and finally the Carter constant (Q).

$$Q \;\; \frac{1}{mM^2} \;\; \frac{\cos^2 \;\; L_z^2}{\sin^2} \;\; L^2 \;\; \cos^2 \;\; S^2 \; m^2 \; E^2 \, ,$$

Method of Calculation

The inverse of the Kerr metric is used to calculate the invariant four-momentum, which represents the energy of the system of an infinitesimal test-particle in orbit around a KBH.

$$P \;\; [\; E, m \; \frac{dr}{d} \;, L \;, L_z \,],$$

$$M^2 \;\; R^2 \;\; 2R \;\; \tilde{S}^2 \qquad M^2 \;\; R^2 \;\; \tilde{S}^2 \cos^2$$

$$g \;|_{Kerr} \quad \begin{matrix} \frac{R^2 \;\; \tilde{S}^2 \;\; 2\tilde{S}^2 R \;\; 2\cos^2 \;\; \tilde{S}^2 R}{R^2 \;\; 2R \;\; \tilde{S}^2} & 0 & 0 & \frac{2\tilde{S}R}{M\,R^2 \;\; 2R \;\; \tilde{S}^2} \\ 0 & R^2 \;\; 2R \;\; \tilde{S}^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2\tilde{S}R}{M\,R^2 \;\; 2R \;\; \tilde{S}^2} & 0 & 0 & \frac{R^2 \;\; 2R \;\; \tilde{S}^2 \cos^2}{R^2 \;\; 2R \;\; \tilde{S}^2 \sin^2} \end{matrix}$$

$$\vec{P} \cdot \vec{P} \;\; P P g \;|_{Kerr} \;\; m^2,$$

The polynomial below arising from the invariant dot product allows us to calculate the effective potential of the test-particle in its LSO. We can then characterise the relationship between the L_z and eccentricity ( ).

$$1 \;\; \tilde{E}^2 \;\; R^4 \;\; 2R^3 \;\; \tilde{L}_z^2 \;\; \tilde{S}^2 \;\; 1 \;\; \tilde{E}^2 \;\; Q \;\; R^2 \;\; 2 \;\; \tilde{L}_z \;\; \tilde{S}\tilde{E}^2 \;\; Q \;\; R \;\; Q\tilde{S}^2 \;\; 0$$

Maple proved to be an essential tool for performing the analytical phase of the work.

Intermediate Results

In Figure 2 Effective potentials are shown for various values of L_z.

The local maximum of each curve corresponds to the LSO of the test-particle at which it tumbles into the black hole. By performing an analysis of the effective potential function, one may obtain an analytical formula of l for an LSO, or make numerical estimates for more complicated cases.

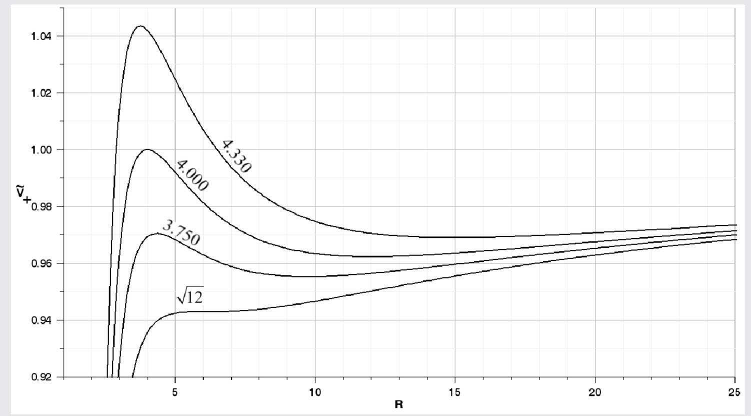

Figure 2. Multiple plots of the effective potential for a test-particle, with various orbital angular momenta, in an equatorial orbit.

By setting the derivative of the effective potential to 0, an analytical formula for the value of L_z can be derived. See Figure 3.

The most important feature of the plot in Figure 3 is the shifting of the Abutment of the prograde and retrograde equations below the horizontal R axis. This effect is observed for KBH systems. Otherwise, the two sets of curves are symmetrical about the R axis.

To extend this concept beyond LSOs to general orbits we must work with a new variable, X, which plays a vital role in the formalism.

$$X \;\; \tilde{L}_z \;\; \tilde{S}\tilde{E}$$

Further analysis of the system allows us to determine an anaytical formula of the location of the Abutment. This will prove to be an important element in our analysis.

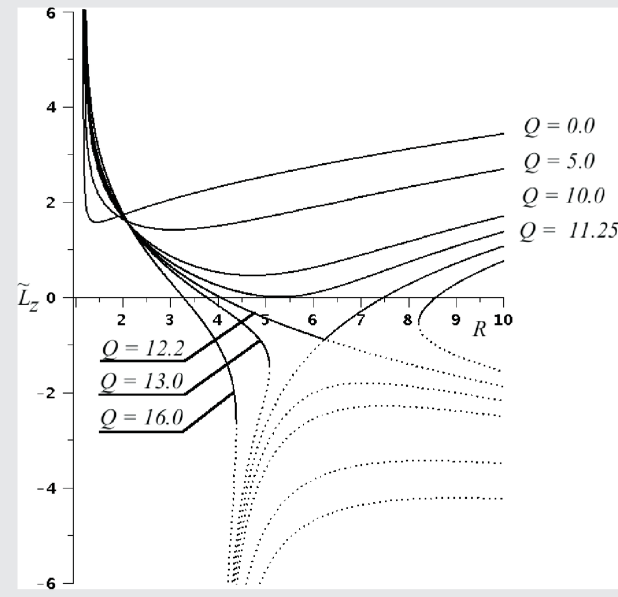

Figure 3. The relationship between L_z and R is affected by the spin (S=0.99) of a KBH.

Advanced Results for the Carter Constant

The LSO

The calculation of l for the inclined LSO required numerical analysis. The observation that the expression for l is second order in Q facilitates an analytical treatment. Thereby, an analytical formula of Q for given spin, S, of the KBH and orbital l and is found.

$$Q_{LSO} \;\; \frac{1}{4} \; Z_a \;\; \tilde{S}^2 Z_b \sqrt{Z_c} \;\; \tilde{S}^4 Z_d \;^{1},$$

$$Z_a \;\; \tilde{l}^4 \;\; 1 \;\; \tilde{l}^2 \;\; 2 \;^2 \;\; 3 \;\tilde{l} \;\; 2 \;\; 1 \;^2 \;2 \;^2 \;\; 3 \;\; \tilde{S}^2 \;\; \tilde{l}^3 \;\; 1 \;^2 \;2 \;^3 \;\;^2 \;\; \tilde{l} \;\; 3 \;\; 1 \;\; \tilde{S}^4,$$

$$Z_b \;\; 3 \;\; 1 \;^2 \tilde{S}^2 \;\; \tilde{l} \;\; 2 \;\; 2 \;\; \tilde{l} \;\; ,$$

$$Z_c \;\; \tilde{l}^5 \;\; 1 \;^3 \;\; 1 \;^2 \tilde{S}^2 \;\; \tilde{l} \;\; \tilde{l} \;\; 2 \;\; 2 \;\; ,$$

$$Z_d \;\; 1 \;^2 \;\; 1 \;^2 \;\; 1 \;^3 \tilde{S}^2 \;\; \tilde{l} \;\; \tilde{l}^2 \;\; 1 \;^2 \;\; 2 \;\; 3 \;\tilde{l} \;\; 2 \;^2 \;\; 4 \;\; 3 \;\; 1 \;^2 \; .$$

The Abutment

The formula for the value of Q at the Abutment is more easily derived. Although the formula does raise some concerns about possible divergence at $S = 0$ and $= 1$, the proper binomial expansion of the formula does show regular behaviour.

$$Q_X \;\; \frac{\tilde{l}^2}{2\tilde{S}^2 \;\; 1 \;^2 \;^2} \;\; \tilde{l} \;\; \tilde{l} \;\;^2 \;\; 3 \;\; \sqrt{\tilde{l}^2 \;\; \tilde{l} \;\;^2 \;\; 3 \;^2 \;\; 4\tilde{l} \;\; 1 \;\;^2 \;^2 \tilde{S}^2} \qquad Q_X \;\; l^2 (l \;\;^2 \;\; 3)^{1}$$

The Polar Orbit

For a polar orbit, $L_z = 0$; therefore, $X = -SE$. This simplification leads to an analytical equation for the value of Q for a polar orbit.

$$Q_{polar} \;\; \tilde{l}^2 B_2 B_1^{\;1}.$$

$$B_1 \;\; \tilde{l}^5 \;\; 3 \;^2 \;\; \tilde{l}^4 \;\; 2\tilde{S}^2 \;\; 1 \;^2 \;\; \tilde{l}^3 \;\; 2\tilde{S}^2 \;\; 1 \;\;^2 \;^2 \tilde{l}^2 \;\; \tilde{S}^4 \;\; 1 \;\;^2 \;^2 \tilde{l} \;\; \tilde{S}^4 \;\; 1 \;\;^2 \;^3$$

$$B_2 \;\; \tilde{l}^4 \;\; 2\tilde{S}^2 \;\; 1 \;^2 \;\; l^2 \;\; 4\tilde{S}^2 \;\; 1 \;^2 \;\; l \;\; \tilde{S}^4 \;\; 1 \;^2 \;^2$$

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[5] P. G. Komorowski, S. R. Valluri, and M. Houde. A study of elliptical last stable orbits about a massive Kerr black hole. Class. Quantum Grav., 26:085001, 2009.
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Q Curve Plots

Figure 4 shows the important interrelation of the three Q formulae we have derived. The LSO and Abutment intersect at a single, tangential point. Further, numerical analysis shows that the points above these two lines is inaccessible to evolving orbits. Only prograde orbits are able to go below the LSO curve; otherwise, both prograde and retrograde orbits can be represented.

The polar curve and Abutment approach a common asymtote with a slope of 1 as l approaches infinity.

In Figure 5, a series of Q curves are plotted for various values of . The boundary of the inaccessible zone is monotonically increasing with respect to l.

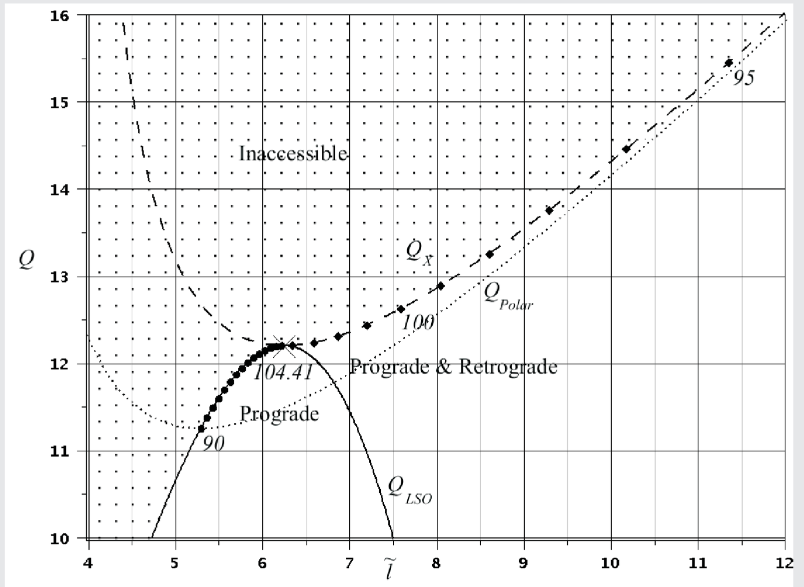

Figure 4. A plot of Q_{LSO}, Q_X, and Q_{polar} for a circular orbit about a KBH with S=0.99.

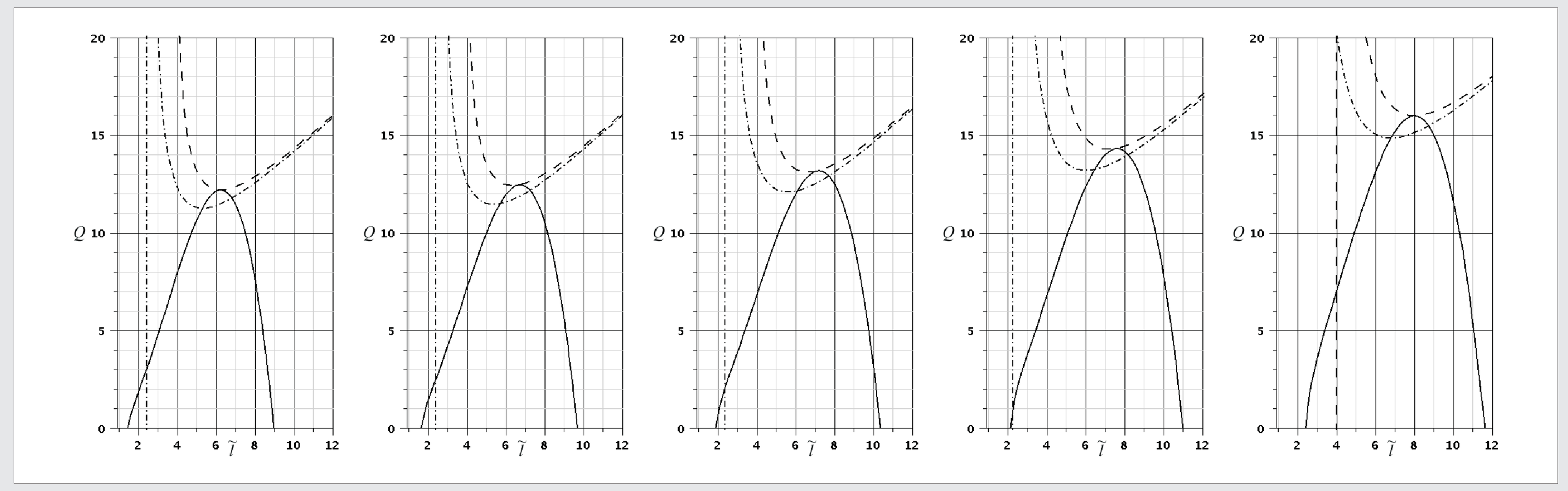

Figure 5. A series of Q curve plots for S = 0.99 and various values of (from left to right = 0.0, 0.25, 0.50, 0.75, and 1.0).

Transition Across the Abutment

An evolving orbit will diminish in terms of l and with respect to time. Radiation back reaction will also increase . The equation below provides an exact relationship between Q and

$$\sin^2(\;) \;\; \frac{1}{2} \; \frac{Q \;\; \tilde{L}_z^2 \;\; \tilde{S}^2 \;\; 1 \;\; \tilde{E}^2 \;\; \sqrt{Q \;\; \tilde{L}_z^2 \;\; \tilde{S}^2 \;\; 1 \;\; \tilde{E}^2 \;^2 \;\; 4Q\tilde{S}^2 \;\; 1 \;\; \tilde{E}^2}}{\tilde{S}^2 \;\; 1 \;\; \tilde{E}^2}$$

The path followed by an evolving orbit (that is crossing the Abutment) will make a tangential intersection with the Abutment curve. Thus the slope of the Abutment curve will give a first order estimate of the change in Q with respect to l and .

In Figure 6 the results are shown for a circular orbit. At = 0 only the change in Q with respect to l need be considered.

Matlab was used for this detailed numerical analysis.

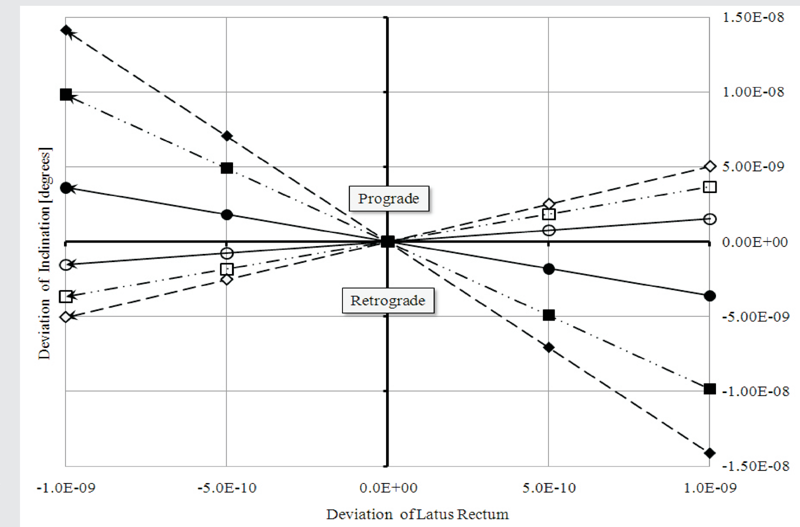

Figure 6. A first order estimate of the change in as a test particle moves across the Abutment. The orbit is circular about a KBH of S = 0.99 with l = 6.25 (diamond), l = 7.0 (square) and l = 10.0 (circle).

Conclusions

We have been able to derive three analytical formulae for the Carter constant (Q) that correspond to the last stable orbit (LSO), the Abutment between the prograde and retrograde forms of X (L_z - SE) and the family of polar orbits.

When plotted with respect to latus rectum (l), for various values of eccentricity (, these curves demonstrate that increasing orbital inclination ( ) can arise from a monotonically decreasing Q with respect to time.

For a particle crossing the Abutment, one may make a first order estimate of the change of Q with respect to l by calculating the derivative of Q_x with respect to l and .

The Abutment provides a useful analytical and numerical laboratory for the analysis of inclined orbits as they evolve.

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