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K-Theory Of Root Stacks And Its Application To Equivariant K-Theory

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Abstract

We give a definition of a root stack and describe its most basic properties. Then we recall the necessary background (Abhyankar's lemma, Chevalley-Shephard-Todd theorem, Luna's étale slice theorem) and prove that under some conditions a quotient stack is a root stack. Then we compute G-theory and K-theory of a root stack. These results are used to formulate the theorem on equivariant algebraic K-theory of schemes.

Keywords: Algebraic stacks, root stacks, quotient stacks, algebraic K-theory, equivariant K-theory.

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Contents

Chapter 1

Introduction

Let X be an algebraic variety equipped with an action of a finite group G. In algebraic geometry one frequently needs to consider equivariant objects on X with respect to the action of G. These objects correspond to objects over the quotient stack $[X/G]$. However, it can happen that $[X/H] \cong [X'/H']$ for seemingly unrelated X and X'. In such situation, it is useful to have a canonical description of the quotient stack $[X/H]$, perhaps in terms of its coarse moduli space Y. This may not always be possible but sometimes it is. In this paper we will describe a situation in which this occurs, see (6.2.8). When our hypotheses are satisfied the quotient stack becomes a root stack over its coarse moduli space Y.

Using the description of quasi-coherent sheaves on a root stack (see Theorem 3.2.1 and [BV]), we give a description of the algebraic G-theory of a root stack (see section 7.4). The main tool is the localisation sequence associated to a quotient category (see Theorem 7.1.3).

These results have immediate applications to equivariant K-theory (see Theorem 8.1.1). We obtain a generalization of the main result of ([EL]). This paper studies the equivariant Grothendieck group of a smooth curve. Combining (7.4.3) and (6.2.8) yields a generalization of this theorem by noting that the hypothesis of (6.2.1) will always be satisfied for tame actions of groups on smooth projective curves. It should be noted that other approaches to equivariant K-theory using a top down description of the stack $[X/G]$ exist in the literature, see [Vi].

This thesis is structured as follows:

Chapter 2. Here we give the important definitions and constructions we are going to use.

First we define symmetric monoidal functors (see Definition 2.1.1). These are functors from monoids to monoidal categories which preserve tensor structures (we consider a monoid as a category with only identity arrows). Then we prove a classical lemma (see Lemma 2.1.10) which is a generalization of a simple fact that there is an equivalence between $\mathbb{G}_m$-torsors over $\operatorname{Spec} R$ and $\mathbb{Z}$-graded R-algebras. This result helps us identify a monoid of special symmetric monoidal functors and a monoid of points of a quotient stack (see Corollary 2.1.12).

After the preparation work is done we define a root stack (Definition 2.2.4). The precise definition is quite involved, but the rough idea is the following. Assume we have a scheme X and a Cartier divisor D. Then a root stack is the same as giving this scheme an orbifold structure along the divisor. Proposition 2.2.6 allows us to formulate an equivalent definition of a root stack as a fiber product. If X is a scheme and D is an irreducible Cartier divisor, then D provides us with a map: $X \to [\mathbb{A}^1/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication. Choosing a natural number r, we can form a fiber product $X \times_{[\mathbb{A}^1/\mathbb{G}_m],\theta_r} [\mathbb{A}^1/\mathbb{G}_m]$, where $\theta_r : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ is raising to r-th power. This fiber product is a stack, and we call it a root stack $X_{D,r}$. (Notation in section 2.2 is a bit different).

Then we give two descriptions of a root stack which are important for the rest of the monograph. Proposition 2.2.7 states that a root stack is globally a $\mathbb{G}_m$-quotient stack. This observation helps us see that a root stack $X_{D,r}$ is Deligne-Mumford, if the number r is invertible in X (see Proposition 2.2.9). Proposition 2.2.12 shows that a root stack is globally a μ_r-quotient stack. Example 2.2.13 is a local description of a root stack.

None of these results is original and could be found in many papers on the subject, for example [BV] or [C].

Chapter 3. First we define a parabolic sheaf (see Definition 3.1.1). If we have a scheme X and an irreducible Cartier divisor D, then a parabolic sheaf on (X, D) with a denominator r is a symmetric monoidal functor $E : \mathbb{Z} \to \mathfrak{Qcoh}\, X$, such that $E_r \cong E_0(D)$ with some additional properties.

Theorem 3.2.1 shows the equivalence of two abelian categories: the category of parabolic sheaves on (X, D) with a denominator r and a category of quasi-coherent sheaves on a root stack $X_{D,r}$. The main tool in the proof is a description of a root stack as a $\mathbb{G}_m$-quotient stack. Locally a sheaf on a $\mathbb{G}_m$-quotient can be identified with $\mathbb{Z}$-graded O_X-algebra with some additional properties, and it turns out that this is the same as a parabolic sheaf. Corollary 3.2.2 states that we can impose the finiteness condition and formulate the analogue of Theorem 3.2.1 about coherent sheaves.

This is a result of [BV].

Chapter 4. Firstly we recall the main facts from SGA I on ramification theory. We consider a situation where a finite group G acts on a scheme X admissibly (see Definition 4.1.1). Basically this means that the quotient X/G exists. We give a definition of decomposition and inertia groups (Definition 4.1.4) in this situation.

Then we give a definition of an unramified morphism of schemes (Definition 4.1.5). In the situation of discrete valuation rings (see Lemma 4.1.6 for a precise formulation), an unramified extension $\mathfrak{o} \subset \mathfrak{O}$ implies that all inertia groups are trivial. It allows us to define a tamely ramified extension of DVRs, when inertia groups are tame, i.e. their orders are invertible in the residue field $\mathfrak{o}/\mathfrak{m}$ (Definition 4.1.7).

We define a normal crossing divisor (Definition 4.2.2). It is an effective divisor, such that locally its irreducible components form a part of a regular system. If D is a normal crossing divisor in a scheme Y, we define a morphism $V \to Y$ to be tamely ramified along D if for all maximal points of supp D we have a tamely ramified extension of DVRs (see Definition 4.2.3).

The important result of the chapter is Abhyankar's Lemma (Proposition 4.2.4). It lets us "turn" a tamely ramified cover into an unramified cover. Assume that we have a tamely ramified cover $V \to Y$ along a divisor D. Then there is a base change $Y' \to Y$ of a special type, such that the morphism $V \times_Y Y' \to (Y - D) \times_Y Y'$ extends uniquely to an étale cover $V \times_Y Y \to Y'$. Using descent theory and purity theorem we reduce the proof to the situation of DVRs (see 4.1.9) and now it is easy.

Chapter 5. First we explain what it means for the inertia group to be generated in codimension one (see Definition 5.1.1). Basically it is the situation in which the inertia group of each point $x \in X$ is finite abelian and each factor (a cyclic group) comes from the inertia group of a general point of an irreducible component of a ramification divisor which contains x.

Then we recall the notion of pseudo-reflections (Definition 5.2.1) and prove an important Lemma 5.2.6 which states that in the tame situation a group G generated by pseudo-reflections is abelian, if the divisor of fixed points of each pseudo-reflection in G is a strict normal crossing. Then we formulate Chevalley-Sheppard-Todd Theorem (5.2.7). If we have a finite dimensional vector space V and a finite group $G \subset \mathrm{GL}(V)$, then in the tame situation the invariant algebra $k[V]^G$ is regular if and only if G is generated by pseudo-reflections.

Then we give a formulation of Luna's étale slice Theorem (5.3.4). Basically it says that in the tame situation a quotient X/G is étale locally around a closed point $x \in X$ isomorphic to a linear quotient N_x/G_x.

The last section of the chapter is the proof of Theorem 5.1.2 which states that if we have a "good" action of a finite group G on a regular variety X, such that $X \to X/G$ ramifies along a normal crossing divisor, then inertia groups of all points are abelian. We proceed by a series of reductions. First using some geometric invariant theory we reduce everything to an affine situation over a separably closed field. Then we apply Luna's étale slice Theorem and reduce the problem to a question about a linear action on a vector space. Then using the Chevalley-Sheppard-Todd Theorem (5.2.7) we can see that the inertia group of each point must be generated by pseudo-reflections. Finally, using Lemma 5.2.6 we can see that the inertia groups must be abelian.

This result is not in the literature, but probably well-known to experts.

Chapter 6. We address the issue of when a quotient stack is a root stack. First we consider an action of a finite group G on a scheme X, such that the Assumption 6.2.1 satisfies. In Lemma 6.2.5 we describe the branch locus in this situation. This description naturally gives us a map from X to a root stack $Y_{D,r}$, where $Y := X/G$, D is a ramification divisor and r is the order of the inertia group. Proposition 6.2.6 shows that if D is a normal crossing divisor, then this map is étale. The main tool in the proof is the description of a root stack as a μ_r-quotient (Proposition 2.2.12) and Abhyankar's Lemma (Proposition 4.2.4)

Because the morphism $X \to Y_{D,r}$ is equivariant, it induces a map from a quotient stack to a root stack: $[X/G] \to Y_{D,r}$. In the remainder of the chapter we are proving that this morphism is an isomorphism (see the final result in Theorem 6.2.8). It is enough to show that $X \times G \to$

$X \times_{Y_{D,r}} X$ is an isomorphism. By Proposition 2.2.12 this map is étale. So it is enough to see that it is a bijection on points (see 6.1.1 for the explanation). The bijectivity now follows from the easy Proposition 6.2.7.

Theorem 6.2.8 is the first main result of our monograph. It is original, but we should mention that it follows from the paper [GS], though we used different methods to prove it.

Chapter 7. In a short preliminary section (§7.1) we mention an important Theorem 7.1.3. It says that if we have an exact functor between abelian categories $F : \mathbf{A} \to \mathbf{B}$ under some condition which resembles surjectivity, there is an equivalence of categories $\mathbf{A}/\mathbf{ker}(F) \simeq \mathbf{B}$. It is called a localization of abelian categories. This result is used throughout the whole chapter.

We define an extendable pair (Definition 7.2.1), which is basically a "cut" of a parabolic sheaf. It turns out that there is an equivalence of abelian categories between extendable pairs and parabolic sheaves (Corollary 7.2.5).

Next we do a first localization. We construct a functor π_* from the category of extendable pairs $\mathcal{EP}$ on X to $\mathfrak{Coh}\, X$ which satisfies the conditions of 7.1.3, so there is an equivalence $\mathcal{EP}/\mathbf{ker}(\pi_*) \simeq \mathfrak{Coh}\, X$ (see Corollary 7.3.3). Then we describe the abelian category $\mathbf{ker}(\pi_*)$ (see Lemma 7.3.4).

In the rest of the chapter we are localizing the category $\mathbf{ker}(\pi_*)$. For that we introduce the series of functors Face^k (Definition 7.3.6) and describe their kernels (Lemma 7.3.8). Theorem 7.3.11 states that these functors satisfy the conditions of 7.1.3, hence we can localize.

In the final section we study the G-theory of a root stack. By G-theory of a root stack we mean the K-theory of the abelian category of coherent sheaves on the stack. We have a series of localizations from the previous section, so applying the localization property of K-theory, we obtain the second main result of our thesis (see Lemmas 7.4.1 and 7.4.3). Then we notice that if the root stack is regular then its K-theory is the same as G-theory.

The result is new.

Chapter 8. We combine the results of Chapters 6 and 7 to study equivariant K-theory of a scheme (see Theorem 8.1.1).

This thesis is an extended version of our paper [DK].

Chapter 2

Root stacks: Definition and Local description

In this chapter we will carefully repeat the main constructions and theorems from the papers [BV] and [C].

2.1 Symmetric monoidal functors

Let X be a scheme. Denote by $\mathfrak{Div}X$ the groupoid of line bundles over X with sections. It has the structure of a symmetric monoidal category with tensor product given by

$$(L, s) \otimes (L', s') = (L \otimes L', s \otimes s').$$

Definition 2.1.1. Let A be a monoid, $\mathcal{M}$ a symmetric monoidal category. A *symmetric monoidal functor* $L : A \to \mathcal{M}$ consists of the data:

(a) A function $L : A \to \mathrm{Obj}\mathcal{M}$.

(b) An isomorphism $\epsilon^L : 1 \cong L(0)$

(c) For each a and $b \in A$, an isomorphism $\mu^L_{a,b} : L(a) \otimes L(b) \cong L(a + b)$ in $\mathcal{M}$.

For any $a, b, c \in A$ the diagrams

$$
\begin{tikzcd}
L(a)\otimes(L(b)\otimes L(c)) \arrow[r, "id\otimes\mu^L"] \arrow[dd, "\alpha"] & L(a)\otimes L(b+c) \arrow[dr, "\mu^L"] & \\
 & & L(a+b+c), \\
(L(a)\otimes L(b))\otimes L(c) \arrow[r, "\mu^L\otimes id"] & L(a+b)\otimes L(c) \arrow[ur, "\mu^L"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
L(a)\otimes L(b) \arrow[r, "\mu^L"] \arrow[d, "\sigma"] & L(a+b) \arrow[d, "="] \\
L(b)\otimes L(a) \arrow[r, "\mu^L"] & L(b+a)
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
1\otimes L(a) \arrow[r, "\lambda"] \arrow[d, "\epsilon^L\otimes id"] & L(a) \arrow[d, "="] \\
L(0)\otimes L(a) \arrow[r, "\mu^L"] & L(0+a)
\end{tikzcd}
$$

are required to be commutative.

Definition 2.1.2. If $L_1 : A \to \mathcal{M}$ and $L_2 : A \to \mathcal{M}$ are symmetric monoidal functors, a *morphism* $\phi : L_1 \to L_2$ is a collection of arrows $\phi_a : L_1(a) \to L_2(a)$ in $\mathcal{M}$ for each $a \in A$, such that the diagram:

$$
\begin{tikzcd}
L_1(a)\otimes L_1(b) \arrow[r, "\mu^{L_1}"] \arrow[d, "\phi_a\otimes\phi_b"] & L_1(a+b) \arrow[d, "\phi_{a+b}"] \\
L_2(a)\otimes L_2(b) \arrow[r, "\mu^{L_2}"] & L_2(a+b)
\end{tikzcd}
$$

commutes for each $a, b \in A$.

Remark 2.1.3. Obviously, if $\mathcal{M}$ is a groupoid, a category of symmetric monoidal functors $A \to \mathcal{M}$ is a groupoid.

Let us fix the target category $\mathcal{M}$ to be $\mathfrak{Div}X$. We will be most interested in the following example.

Example 2.1.4. Choose n objects $(L_1, s_1), ...(L_n, s_n)$ of $\mathfrak{Div}X$. Then define a symmetric monoidal functor by:

$$L : \mathbb{N}^n \longrightarrow \mathfrak{Div}X$$
$$(k_1, ..., k_n) \longmapsto (L_1, s_1)^{\otimes k_1} \otimes ... \otimes (L_n, s_n)^{\otimes k_n}.$$

Lemma 2.1.5. *Every functor $M : \mathbb{N}^n \longrightarrow \mathfrak{Div}X$ is isomorphic to one given by the rule in Example 2.1.4.*

Proof. Indeed take the basis $\{e_i\}_{i=1}^n$ of the monoid $\mathbb{N}^n$ and consider its image $(L_i, s_i) := M(e_i) \in \mathfrak{Div}X$. Then we can define new symmetric monoidal functor L associated to (L_i, s_i) by the rule 2.1.4.

Let's describe a morphism $M \to L$. Take an element $a \in \mathbb{N}^n$ and write it in the basis as $a = \sum_{i=1}^n a_i e_i$. Then consider a composition of isomorphisms:

$$M(a) \xrightarrow{(\mu^M)^{-1}} M(\sum_{i=1}^{n-1} a_i e_i) \otimes M(e_n)^{\otimes a_n} \xrightarrow{(\mu^M)^{-1} \otimes \mathrm{id}} \dots \xrightarrow{(\mu^M)^{-1} \otimes \mathrm{id}} \otimes_{i=1}^n M(e_i)^{\otimes a_i} = L(a)$$

which we call ϕ_a. The commutativity of the diagram in Definition 2.1.2 follows from the commutativity of the first diagram in definition 2.1.1. □

Remark 2.1.6. Recall that if M is a commutative monoid then $\hat{M} = \mathrm{Hom}(M, \mathbb{G}_m)$ is its dual.

Functors from Example 2.1.4 arise from morphisms $X \to [\mathrm{Spec}\,\mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$. Let us explain how.

First we should recall some standard facts about torsors.

Proposition 2.1.7. *GL_n-torsors (and in particular $\mathbb{G}_m$-torsors) are locally trivial in Zariski topology.*

Proof. The proof basically follows from descent theory. See [M, III, Lemma 4.10]. □

Remark 2.1.8. We will consider only $\mathbb{G}_m^n$-torsors in the present work. However in the original paper [BV] the **fppf** topology is needed. The setting in *loc. cit.*, is more general and the monoids in question may have torsion so that the torsor P is a torsor over μ_n. Such a torsor may not be trivial in the Zariski topology.

Example 2.1.9. Let's take a commutative ring R and consider a trivial $\mathbb{G}_m$-torsor T over $X = \operatorname{Spec} R$:

$$\operatorname{Spec} R[x, x^{-1}] \cong \mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{p_2} \operatorname{Spec} R.$$

It corresponds to the inclusion of rings $i : R \hookrightarrow R[x, x^{-1}]$.

Consider a quasi-coherent sheaf $\mathcal{F}$ on T. It corresponds to a $R[x, x^{-1}]$-module M. The pushforward $p_{2*}\mathcal{F}$ corresponds to the module M considered as an R-module via the inclusion i. In particular if we take the structure sheaf $\mathcal{O}_T$, then $p_{2*}\mathcal{O}_T$ corresponds to $R[x, x^{-1}]$ considered as R-module which is the same as the $\mathbb{Z}$-graded R-module $R[x, x^{-1}] \cong \oplus_{l \in \mathbb{Z}} R x^l$.

In other words $p_{2*}\mathcal{O}_T \cong \oplus_{l\in\mathbb{Z}} \mathcal{O}_X x^l$. This is a weight grading induced by the characters $\mathbb{G}_m \to \mathbb{G}_m$.

Let us consider the more general situation.

Lemma 2.1.10. *Let* $\mathbf{A}$ *be the groupoid whose objects are quasi-coherent* $\mathcal{O}_X$*-algebras* $\mathcal{A}$ *with a* $\mathbb{Z}^n = \widehat{\widehat{\mathbb{N}^n}}$*-grading* $\mathcal{A} = \oplus_{u\in\mathbb{Z}^n}\mathcal{A}_u$ *such that each summand* $\mathcal{A}_u$ *is an invertible sheaf. The morphisms are graded algebra isomorphisms. Then there is an equivalence of categories between* $\mathbf{A}^{\mathrm{op}}$ *and the groupoid of* $\widehat{\mathbb{N}^n}$*-torsors* $P \to X$.

Proof. Consider a $\widehat{\mathbb{N}^n}$-torsor $\pi : T \to X$. It gives us a sheaf of $\mathcal{O}_X$-algebras $\pi_*(\mathcal{O}_T)$. This sheaf is a $\widehat{\mathbb{N}^n}$-equivariant sheaf (see [FKM, Ch. 1, §3]). That means that there exists an isomorphism of sheaves $\phi : p_2^*\pi_*(\mathcal{O}_T) \to p_2^*\pi_*(\mathcal{O}_T)$, where $p_2 : \widehat{\mathbb{N}^n} \times X \to X$ is a projection, and ϕ satisfies the co-cycle condition. It follows that ϕ induces a $\widehat{\mathbb{N}^n}$-co-action on global sections:

$$\Gamma(X, \pi_*(\mathcal{O}_T)) \xrightarrow{p_2^*} \Gamma(\widehat{\mathbb{N}^n} \times X, p_2^*\pi_*(\mathcal{O}_T)) \xrightarrow{\phi} \Gamma(\widehat{\mathbb{N}^n} \times X, p_2^*\pi_*(\mathcal{O}_T)) \cong \mathbb{Z}[\mathbb{Z}^n] \otimes \Gamma(X, \pi_*(\mathcal{O}_T)).$$

Let's say that the section $s \in \Gamma(X, \pi_*(\mathcal{O}_T))$ has weight $u \in \mathbb{Z}^n$, if under this action the image of s is $x^u \otimes s$. So we can define a subsheaf $\mathcal{A}_u \subset \pi_*\mathcal{O}_T$ of sections of weight u.

Now we have a morphism of sheaves: $\sum_{u\in\mathbb{Z}^n} \mathcal{A}_u \to \pi_*\mathcal{O}_T$. We claim that the sum in the left hand side is a direct sum and the morphism is an isomorphism. To prove these claims it is enough to prove them over any closed point of $x \in X$. But over a point, ϕ defines a $\widehat{\mathbb{N}^n}$-action on a vector space and $(\pi_* \mathcal{O}_T)_x \cong \oplus_{u\in\mathbb{Z}^n}(\mathcal{A}_u)_x$ is just a usual weight decomposition. Also we can see that each $\mathcal{A}_u$ is locally isomorphic to $\mathcal{O}_X x^u$ and so is an invertible sheaf.

An isomorphism of X-torsors $T_1 \xrightarrow{\pi_1} X$ and $T_2 \xrightarrow{\pi_2} X$ corresponds to a morphism of sheaves of $\mathcal{O}_X$-algebras $\pi_{2*}\mathcal{O}_{T_2} \to \pi_{1*}\mathcal{O}_{T_1}$. The construction described above is natural, so we have maps of $\mathcal{O}_X$-modules: $\mathcal{A}_{2u} \to \mathcal{A}_{1u}$ for each $u \in \mathbb{Z}^n$. Thus this is $\mathbb{Z}^n$-graded isomorphism $\pi_{2*}\mathcal{O}_{T_2} \to \pi_{1*}\mathcal{O}_{T_1}$ and so we have defined a functor from $\widehat{\mathbb{N}^n}$-torsors $P \to X$ to $\mathbf{A}^{\mathrm{op}}$.

Given an object $\mathcal{A}$ in $\mathbf{A}$ we can consider a relative spectrum $\underline{\mathrm{Spec}}_X(\mathcal{A})$ with a $\widehat{\mathbb{N}^n}$-action defined by the grading. It will be locally trivial (and so a torsor), because each summand in the grading is an invertible sheaf. A graded map of algebras $\mathcal{A} \to \mathcal{B}$ in $\mathbf{A}$ gives a morphism of torsors $\underline{\mathrm{Spec}}_X(\mathcal{B}) \to \underline{\mathrm{Spec}}_X(\mathcal{A})$ by the relative spectrum construction.

These functors are quasi-inverse. From the construction of the relative spectrum we have an isomorphism of algebras $\mathcal{A} \to \pi_*\mathcal{O}_{\underline{\mathrm{Spec}}_X(\mathcal{A})}$ which is locally an isomorphism of $\mathbb{Z}^n$-graded algebras. Also there is a canonical map $P \to \underline{\mathrm{Spec}}_X(\pi_*\mathcal{O}_P)$ which is $\widehat{\mathbb{N}^n}$-equivariant (again by local consideration) and so an isomorphism of torsors. □

Proposition 2.1.11. *Let* $\mathbf{B}$ *be the groupoid whose objects are pairs* $(\mathcal{A}, \alpha)$ *where* $\mathcal{A}$ *is a sheaf of algebras satisfying the conditions in Lemma 2.1.10 and*

$$\alpha : \mathcal{O}_X[\mathbb{N}^n] \to \mathcal{A}$$

is a morphism respecting the grading. The morphisms in the category $\mathbf{B}$ *are graded algebra morphisms commuting with the structure maps. Then there is an equivalence of categories between* $\mathbf{B}^{\mathrm{op}}$ *and the groupoid of morphisms* $X \to [\mathrm{Spec}\,\mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$

Proof. This proposition is a summary of the discussion in [BV, p. 1343-1344].

An object of the groupoid of X-points of a quotient stack $[\mathrm{Spec}\,\mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$ consists of:

- A $\widehat{\mathbb{N}^n}$-torsor $\pi : E \to X$
- A $\widehat{\mathbb{N}^n}$-equivariant map $E \to \mathrm{Spec}\,\mathbb{Z}[\mathbb{N}^n]$.

A morphism is an isomorphism of X-torsors: $E_1 \to E_2$, such that the triangle

$$\begin{tikzcd} E_1 \arrow[rr] \arrow[dr] & & E_2 \arrow[dl] \\ & \operatorname{Spec} \mathbb{Z}[\mathbb{N}^n] & \end{tikzcd}$$

commutes.

By Lemma 2.1.10, an object gives us a sheaf of $O_X(\mathbb{N}^n)$-algebras $\pi_*(O_P)$ which is compatible with the $\mathbb{Z}^n$-grading of $\mathbb{Z}[\mathbb{N}^n]$. And morphisms corresponds to morphisms in **B** under this identification.

Now let's take an object $(\mathcal{A}, \alpha) \in \mathbf{B}$. Then by Lemma 2.1.10 we obtain a torsor $\underline{\operatorname{Spec}}_X(\mathcal{A})$. Then α induces a $\widehat{\mathbb{N}^n}$-equivariant map $\underline{\operatorname{Spec}}_X(\mathcal{A}) \to \operatorname{Spec} \mathbb{Z}[\mathbb{N}^n]$. It is clear by the construction of the relative spectrum that morphisms are sent to morphism.

It follows from Lemma 2.1.10 that these functors are quasi-inverse.

□

Corollary 2.1.12. *There is an equivalence of categories between the groupoid of symmetric monoidal functors* $\mathbb{N}^n \to \mathfrak{D}ivX$ *and the groupoid of* X*-points of* $[\operatorname{Spec} \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$.

Proof. This is [BV, Proposition 3.25].

Take the symmetric monoidal functor $L : \mathbb{N}^n \to \mathfrak{D}ivX$. Then if $\{e_i\}_{i=1}^n$ is a standard basis of $\mathbb{N}^n$, define $(L_i, s_i) := L(e_i)$. Then we can produce the graded sheaf of O_X-algebras:

$$\mathcal{A}_L = \bigoplus_{\vec{u} \in \mathbb{Z}^n} L_1^{u_1} \otimes \dots L_n^{u_n}.$$

The sections produce an algebra map

$$O_X[\mathbb{N}^n] \to \mathcal{A}.$$

Take a morphism $X \to [\operatorname{Spec} \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$. By Proposition 2.1.11 it gives a sheaf of $\mathbb{Z}^n$-graded O_X-algebras $\mathcal{A} = \oplus_{\vec{u} \in \mathbb{Z}^n} \mathcal{A}_{\vec{u}}$ and a morphism $\alpha : O_X[\mathbb{N}^n] \to \mathcal{A}$ which respects the grading. Then we can define a symmetric monoidal functor $L_{\mathcal{A}} : \mathbb{N}^n \to \mathfrak{D}ivX$ by the rule:

$$\forall a \in \mathbb{N}^n, \quad L(a) := (\mathcal{A}_{\iota(a)}, x^a),$$

where $\iota : \mathbb{N}^n \hookrightarrow \mathbb{Z}^n$ is an inclusion of monoids and $x^a \in \mathbb{Z}[\mathbb{N}^n]$ is identified with its image in $\mathcal{A}(X)$.

We should check that these functors are quasi-inverse to each other. The isomorphism of symmetric monoidal functors $L \to L_{\mathcal{A}_L}$ follows Lemma 2.1.5. The isomorphism of $\mathbb{Z}^n$-graded $\mathcal{O}_X$-algebras $\mathcal{A} \to \mathcal{A}_{L_{\mathcal{A}}}$ follows from the isomorphism of $\mathcal{O}_X$-algebras: $\mathcal{A}_a \otimes_{\mathcal{O}_x} \mathcal{A}_b \cong \mathcal{A}_{a+b}$.

□

2.2 Root stacks

Definition 2.2.1. Let $\vec{r} = (r_1, r_2, ..., r_n)$ be a collection of positive natural numbers. We denote by $r_i\mathbb{N}$ the monoid $\{vr_i | v \in \mathbb{N}\}$. We denote by $\vec{r}\mathbb{N}^n$ the monoid

$$\vec{r}\mathbb{N}^n = r_1\mathbb{N} \times r_2\mathbb{N} \times \ldots r_n\mathbb{N}.$$

We will view our symmetric monoidal functor above as a functor

$$L : \vec{r}\mathbb{N}^n \to \mathfrak{D}ivX$$

in the following way :

$$(r_1\alpha_1, r_2\alpha_2, \ldots, r_n\alpha_n) \longmapsto (L_1, s_1)^{\otimes \alpha_1} \otimes \ldots (L_n, s_n)^{\otimes \alpha_n}.$$

Consider the natural inclusion of monoids $j_{\vec{r}} : \vec{r}\mathbb{N}^n \hookrightarrow \mathbb{N}^n$. The *category of $\vec{r}$th roots of L* denoted by $(L)_{\vec{r}}$, is defined as follows:

Its objects are pairs (M, α), where $M : \mathbb{N}^n \to \mathfrak{Div}X$ is a symmetric monoidal functor, and $\alpha : L \to M \circ j$ is an isomorphism of symmetric monoidal functors.

An arrow from (M, α) to (M', α') is an isomorphism h: $M \to M'$ of symmetric monoidal functors $\mathbb{N}^n \to \mathfrak{Div}X$, such that the diagram

$$\begin{tikzcd} & L \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\ M \circ j \arrow[rr, "h \circ j"] & & M' \circ j \end{tikzcd}$$

commutes.

Remark 2.2.2. This category is in fact a groupoid as a morhpism ϕ in $\mathfrak{Div}X$ whose tensor power $\phi^{\otimes k}$ is an isomorphism must be an isomorphism to begin with.

Given a morphism of schemes $t : T \to X$ there is pullback functor

$$t^* : \mathfrak{D}ivX \to \mathfrak{D}ivT.$$

Hence we can form the category of roots $(t^* \circ L)_{\vec{r}}$.

Assume that we have a morphism of X-schemes:

$$\begin{tikzcd} T' \arrow[rr, "f"] \arrow[dr, "t'"] & & T \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

Take an object (M, α) in the category of roots $(t^* \circ L)_{\vec{r}}$. Then define

$$M' := f^* \circ M : \quad \mathbb{N}^n \to \mathfrak{D}ivT'$$

a symmetric monoidal functor. We can also pull back the isomorphism α:

$$f^*\alpha : \quad f^*t^*L \to f^*M$$

and applying the natural isomorphisms of the compositions of pullbacks we obtain a new isomorphism:

$$\alpha' : \quad t'^*L \to M' \circ j.$$

So (M', α') is an object in the category of roots $(t'^* \circ L)_{\vec{r}}$.

Take a morphism $h : (M_1, \alpha_1) \to (M_2, \alpha_2)$ in $(t^* \circ L)_{\vec{r}}$. Then there is a morphism of symmetric monoidal functors: $h' := f^* \circ h : \quad M'_1 \to M'_2$. Consider the diagram:

$$\begin{tikzcd}
 & & t'^*L \arrow[ddll, "\alpha'_1"'] \arrow[d, "\cong"] \arrow[ddrr, "\alpha'_2"] & & \\
 & & f^*t^*L \arrow[dl, "f^* \circ \alpha_1"'] \arrow[dr, "f^* \circ \alpha_2"] & & \\
M'_1 \circ j \arrow[r, "="] & f^* \circ M_1 \circ j \arrow[rr, "f^* \circ h \circ j"] & & f^* \circ M_2 \circ j & M'_2 \circ j. \arrow[l, "="']
\end{tikzcd}$$

The commutativity of the left and right squares follows from the definition of α', the inner triangle commutes as a pullback of a commutative triangle. The commutativity of the outer triangle proves that $h' : (M'_1, \alpha'_1) \to (M'_2, \alpha'_2)$ is a morphism in $(t'^* \circ L)_{\vec{r}}$.

Corollary 2.2.3. *The rule:*

$$\{t : T \to X\} \mapsto (t^* \circ L)_{\vec{r}}$$

described above is a pseudo-functor from Sch$/X$ *to the 2-category of categories.*

Definition 2.2.4. In the above situation, the fibered category associated to this pseudo-functor is called *the stack of roots* associated to L and $\vec{r}$. It is denoted by $X_{L,\vec{r}}$.

Remark 2.2.5. The pseudo-functor gives us a category fibered in groupoids and one needs to check that it is a stack. For that it is enough to check that all descent data is effective. Consider an fppf-cover $U \to T$ of a scheme T and a monoidal functor $M : \mathbb{N}^n \to \mathfrak{D}ivU$, such that $M|_{U\times_T U} \simeq M|_{U\times_T U}$. This functor is defined by the objects $(M_i, s_i) \in \mathfrak{D}ivU$. Each of this divisors can be glued to $(N_i, t_i) \in \mathfrak{D}ivT$, and they provide us with a monoidal functor $N : \mathbb{N}^n \to \mathfrak{D}ivT$, such that $N|_T \simeq M$. In a similar way we can glue an isomorphism α.

We will often denote the stack of roots by

$$X_{L,\vec{r}} = X_{(L_1,s_1,r_1),\dots,(L_n,s_n,r_n)}.$$

There are also two equivalent definitions of the stack $X_{L,\vec{r}}$ and the equivalence is proved in [BV, Proposition 4.13] and [BV, Remark 4.14]. Let's give the description of this stack as a fibered product.

According to (a slightly modified version of) Corollary 2.1.12 a symmetric monoidal functor $L : \vec{r}\mathbb{N}^n \to \mathfrak{Div}X$ corresponds to a morphism $X \to [\operatorname{Spec}\mathbb{Z}[\vec{r}\mathbb{N}^n]/\widehat{\vec{r}\mathbb{N}^n}]$.

Proposition 2.2.6. *The stack $X_{L,\vec{r}}$ is isomorphic to the fibered product*

$$X \times_{[\operatorname{Spec}\mathbb{Z}[\vec{r}\mathbb{N}^n]/\widehat{\vec{r}\mathbb{N}^n}]} [\operatorname{Spec}\mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}],$$

where the map $X \to [\operatorname{Spec}\mathbb{Z}[\vec{r}\mathbb{N}^n]/\widehat{\vec{r}\mathbb{N}^n}]$ is given by L and the map of quotient stacks

$$[\operatorname{Spec}\mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}] \to [\operatorname{Spec}\mathbb{Z}[\vec{r}\mathbb{N}^n]/\widehat{\vec{r}\mathbb{N}^n}]$$

is induced by the inclusion of monoids: $j : \mathbb{Z}[\vec{r}\mathbb{N}^n] \hookrightarrow \mathbb{Z}[\mathbb{N}^n]$.

Proof. Two stacks in groupoids are equivalent if and only if their fibers are equivalent groupoids. So it is enough to prove that T-points of both stacks are equivalent groupoids, where T is a scheme over X.

First consider $X_{L,\vec{r}}(T)$. By construction it is equivalent to the category of roots $(t^* \circ L)_{\vec{r}}$. Its objects are pairs (M, α), where $M : \mathbb{N}^n \to \mathfrak{Div}T$ is a symmetric monoidal functor, and $\alpha : t^*L \to M \circ j$ is an isomorphism of symmetric monoidal functors. By Corollary 2.1.12 M corresponds to a morphism $T \to [\operatorname{Spec}\mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$ which we will also denote by M. Then the pair (M, α) gives an object over T in the fiber product $X \times_{[\operatorname{Spec}\mathbb{Z}[\vec{r}\mathbb{N}^n]/\widehat{\vec{r}\mathbb{N}^n}]} [\operatorname{Spec}\mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$.

In a similar way by Corollary 2.1.12 the object of a fiber product over T gives an object in the groupoid $X_{L,\vec{r}}(T)$. □

Let us slightly reformulate this description. $L : \vec{r}\mathbb{N}^n \to \mathfrak{Div}X$ gives a morphism $X \to [\operatorname{Spec}\mathbb{Z}[\vec{r}\mathbb{N}^n]/\widehat{\vec{r}\mathbb{N}^n}]$ which in turn corresponds to a $\widehat{\vec{r}\mathbb{N}^n}$-torsor $\pi : P \to X$ and a $\widehat{\vec{r}\mathbb{N}^n}$-equivariant morphism $P \to \operatorname{Spec}\mathbb{Z}[\vec{r}\mathbb{N}^n]$. This gives

Proposition 2.2.7. *The stack $X_{L,\vec{r}}$ is isomorphic to the quotient stack*

$$[P \times_{\operatorname{Spec}\mathbb{Z}[\vec{r}\mathbb{N}^n]} \operatorname{Spec}\mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}],$$

where the action on the first factor is defined through the dual of the inclusion $j_{\vec{r}} : \vec{r}\mathbb{N}^n \hookrightarrow \mathbb{N}^n$.

Proof. As above we just need to prove the equivalence of groupoids of points over any X-scheme T. By the previous proposition an object $X_{L,\vec{r}}(T)$ is given by the pair (M, α), where $M : T \to [\operatorname{Spec}\mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$ and α is an isomorphism coming from the definition of a fiber product. But M provides us a $\widehat{\mathbb{N}^n}$-torsor E over T and a $\widehat{\mathbb{N}^n}$-equivariant map $E \to \operatorname{Spec}\mathbb{Z}[\mathbb{N}^n]$. The isomorphism α induces a $\mathbb{Z}[\mathbb{N}^n]$-equivariant map $E \to P \times_{\operatorname{Spec}\mathbb{Z}[\vec{r}\mathbb{N}^n]} \operatorname{Spec}\mathbb{Z}[\mathbb{N}^n]$, so we get a T-point of a quotient stack. This functor is clearly an equivalence. □

Corollary 2.2.8. *The root stack $X_{L,\vec{r}}$ is an algebraic stack.*

Proposition 2.2.9. *If each r_i is invertible in X then the root stack $X_{L,\vec{r}}$ is a Deligne-Mumford stack.*

Proof. This is a combination of proofs from [C, Theorem 2.3.3] and [BV, Proposition 4.19].

Let's consider a surjective étale map $p : U \to X$, such that p^*L_i is a trivial bundle for each $i = 1, \dots, n$. Then from Proposition 2.2.6 it is clear that $U \times_X X_{L,\vec{r}} \cong U_{p^*L,\vec{r}}$.

Because of the triviality of p^*L_i the composition $U \to X \to [\operatorname{Spec}\mathbb{Z}[\vec{r}\mathbb{N}^n]/\widehat{\vec{r}\mathbb{N}^n}]$ must factor through $\operatorname{Spec}\mathbb{Z}[\vec{r}\mathbb{N}^n]$.

There is a cartesian diagram:

$$\begin{tikzcd}
{[\operatorname{Spec}\mathbb{Z}[\mathbb{N}^n]/\mu_{r_1} \times \cdots \times \mu_{r_n}]} \arrow[r] \arrow[d] & {[\operatorname{Spec}\mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]} \arrow[d] \\
\operatorname{Spec}\mathbb{Z}[\vec{r}\mathbb{N}^n] \arrow[r] & {[\operatorname{Spec}\mathbb{Z}[\vec{r}\mathbb{N}^n]/\widehat{\vec{r}\mathbb{N}^n}]}
\end{tikzcd}$$

so using the property of a fiber product we have that

$$U \times_X X_{L,\vec{r}} \cong U \times_{\operatorname{Spec}(\mathbb{Z}[\vec{r}\mathbb{N}^n])} [\operatorname{Spec}\mathbb{Z}[\mathbb{N}^n]/\mu_{r_1} \times \cdots \times \mu_{r_n}].$$

A stack $[\operatorname{Spec}\mathbb{Z}[\mathbb{N}^n]/\mu_{r_1} \times \cdots \times \mu_{r_n}]$ is a Deligne-Mumford stack if each r_i is invertible in X (see, for example, [LMB]). Also the 2-category of Deligne-Mumford stacks is closed under fiber products. Finally using étale descent one gets the result. □

We want to give one more description of a root stack which is a rephrasing of a construction from [C].

Remark 2.2.10. A μ_r-bundle P on a scheme Z is equivalent to the data of an invertible sheaf $\mathcal{K}$ and an isomorphism $\phi : \mathcal{K}^r \to \mathcal{O}_Z$. To construct P explicitly consider the sheaf of algebras $\mathrm{Sym}^\bullet \mathcal{K}^{-1}$. There is a distinguished global section $T \in \mathcal{K}^{-r}$ given by $(\phi \otimes 1_{\mathcal{K}^{-r}})(1)$. Then

$$P = \mathrm{Spec}\left(\mathrm{Sym}^\bullet \mathcal{K}^{-1}/(T-1)\right).$$

Notation 2.2.11. If X is a scheme, $\mathcal{L}$ is an invertible sheaf and $s \in H^0(X, \mathcal{L})$ is a section, let's denote by $X_{\mathcal{L},s,r}$ a root stack associated to a symmetric monoidal functor induced by $(\mathcal{L}, s) \in \mathfrak{D}iv X$.

Proposition 2.2.12. *Suppose that there is an invertible sheaf $\mathcal{N}$ on X and an isomorphism $\mathcal{N}^r \to \mathcal{L}$. Then $X_{\mathcal{L},s,r}$ is a global quotient stack.*

Proof. This is a summary of [C, 2.3.1 and 2.4.1] and [B, 3.4].

The coherent sheaf

$$\mathcal{A} = \mathcal{O}_X \oplus \mathcal{N}^{-1} \oplus \ldots \oplus \mathcal{N}^{-(r-1)}$$

can be given the structure of an $\mathcal{O}_X$-algebra via the composition

$$\mathcal{N}^{-r} \xrightarrow{\sim} \mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_X.$$

There is an action of μ_r on this sheaf via the action of μ_r on $\mathcal{N}^{-1}$ given by scalar multiplication. Then there is an isomorphism of stacks:

$$X_{\mathcal{L},s,r} \to [\mathrm{Spec}(\mathcal{A})/\mu_r].$$

Let's describe it. Consider a morphism $a : T \to X$. A morphism $Y \to X_{\mathcal{L},s,r}$, lifting a, is a triple $(\mathcal{M}, t, \phi)$, where $\mathcal{M}$ is an invertible sheaf on Y, t is a section of $\mathcal{M}$ and ϕ is an isomorphism of sheaves $\mathcal{M}^{\otimes r} \cong a^*L$, such that $\phi(t) = a^*s$. As per the previous remark the sheaf

$\mathcal{M}^{-1} \otimes a^*\mathcal{N}$ gives a μ_r-torsor on Y. The torsor comes from the algebra

$$\mathcal{B} = \mathrm{Sym}^\bullet(\mathcal{M} \otimes a^*\mathcal{N}^{-1})/(T-1).$$

To produce an Y-point of $[\mathrm{Spec}(\mathcal{A})/\mu_r]$ we need to describe a μ_r-equivariant map

$$a^*\mathcal{A} \to \mathcal{B}.$$

This map comes from the section t via :

$$t \in \mathrm{Hom}(\mathcal{O}, \mathcal{M}) = \mathrm{Hom}(a^*\mathcal{N}^{-1}, \mathcal{M} \otimes a^*\mathcal{N}^{-1}).$$

This construction generalizes in the obvious way to a finite list of invertible sheaves with section. □

Example 2.2.13. Let us consider an example where $X = \mathrm{Spec}(R)$. Let $f_1, f_2, \dots f_n$ be elements of R. Consider Cartier divisors $L_i = (\mathcal{O}_X, f_i)$ and an n-tuple of natural numbers $\vec{r}$. By the construction in Proposition 2.2.12, we get that the root stack is a quotient stack:

$$X_{L,\vec{r}} = [\mathrm{Spec}(A)/\mu_{r_1} \times \dots \times \mu_{r_n}],$$

where $A = R[t_1, \dots, t_n]/(t_1^{r_1} - f_1, \dots, t_n^{r_n} - f_n)$.

Let us state an easy lemma from commutative algebra.

Lemma 2.2.14. *Let R be a local ring, f_i a part of a regular sequence of R. Let A be a R-algebra defined in the example above, such that each r_i is invertible in R. Then A is regular.*

Proof. The proof will be given in Step 1 of Abhyankar's lemma (Proposition 4.2.4). □

Proposition 2.2.15. *Let X be a regular scheme over a field k. Let $D = \sum_{i=1}^n D_i$ be a normal crossing divisor (see Definition 4.2.2). Assume that $\vec{r}$ is an n-tuple of natural numbers, such that each r_i is coprime to the characteristic of k. Then a root stack $X_{D,\vec{r}}$ is regular.*

Proof. By definition a stack is regular if its presentation is a regular scheme. The question is local, so we can assume that $X = \mathrm{Spec}(R)$ and a divisor D is a strict normal crossing divisor.

If we localize further, we can assume that R is a local ring, $D_i = (f_i)$ and f_i form a part of a regular sequence.

By Example 2.2.13, the presentation of a root stack $X_{D,\vec{r}}$ is an affine scheme $A = R[t_1, \ldots, t_n]/(t_1^{r_1} - f_1, \ldots, t_n^{r_n} - f_n)$. By the previous lemma this scheme is regular.

$\square$

Chapter 3

Root stacks: (Quasi)-coherent sheaves

In this chapter we will repeat the proof of the main result in [BV] about the characterization of sheaves on a root stack in terms of parabolic sheaves.

3.1 Parabolic sheaves

First we give the definition of a parabolic sheaf from [BV, Definition 5.6].

Definition 3.1.1. Consider a scheme X, an inclusion $\vec{r}\mathbb{Z}^n \subseteq \mathbb{Z}^n$, divisors $(L_i, s_i) \in \mathfrak{Div}X$ for $1 \leq i \leq n$ and the symmetric monoidal functor $L : \vec{r}\mathbb{Z}^n \to \mathfrak{Div}X$ which is defined by

$$L_u = L(u) = L_1^{\alpha_1} \otimes \dots L_n^{\alpha_n},$$

where $u = (r_1\alpha_1, \dots, r_n\alpha_n)$ and $\alpha_i \in \mathbb{Z}$. A *parabolic sheaf* (E, ρ) on (X, L) with denominators $\vec{r}$ consists of the following data :

(a) A functor $E : \mathbb{Z}^n \to \mathfrak{QCoh}X$, denoted by $v \mapsto E_v$ on objects and $b \mapsto E_b$ on arrows.

(b) For any $u \in \vec{r}\mathbb{Z}^n$ and $v \in \mathbb{Z}^n$, an isomorphism of O_X-modules:

$$\rho^E_{u,v} : E_{u+v} \simeq L_u \otimes_{O_X} E_v.$$

This map is called the *pseudo-period isomorphism*.

This data are required to satisfy the following conditions. Let $u, u' \in \vec{r}\mathbb{Z}^n$, $a = (r_1\alpha_1, \ldots, r_n\alpha_n) \in \vec{r}\mathbb{N}^n$, $b \in \mathbb{N}^n$, $v \in \mathbb{Z}^n$. Then the following diagrams commute.

(i)

$$\begin{tikzcd}
E_v \arrow[r, "E_a"] \arrow[d, "\simeq"] & E_{a+v} \arrow[d, "\rho^E_{a,v}"] \\
\mathcal{O}_X \otimes E_v \arrow[r, "\sigma^L_a \otimes id_{E_v}"] & L_a \otimes E_v
\end{tikzcd}$$

where $\sigma_a = \sigma_1^{\alpha_1} \otimes \ldots \sigma_n^{\alpha_n}$ and each σ_i is a multiplication by a section $s_i \in \mathrm{H}^0(X, L_a)$

(ii)

$$\begin{tikzcd}
E_{u+v} \arrow[r, "\rho^E_{u,v}"] \arrow[d, "E_b"] & L_u \otimes E_v \arrow[d, "id \otimes E_b"] \\
E_{u+b+v} \arrow[r, "\rho^E_{u,b+v}"] & L_u \otimes E_{b+v}
\end{tikzcd}$$

(iii)

$$\begin{tikzcd}
E_{u+u'+v} \arrow[r, "\rho^E_{u+u',v}"] \arrow[d, "\rho^E_{u,u'+v}"] & L_{u+u'} \otimes E_v \arrow[d, "\mu \otimes id"] \\
L_u \otimes E_{u'+v} \arrow[r, "id \otimes \rho^E_{u',v}"] & L_u \otimes L_{u'} \otimes E_v
\end{tikzcd}$$

(iv) The map

$$E_v = E_{0+v} \xrightarrow{\rho^E_{0,v}} \mathcal{O}_X \otimes E_v$$

is the natural isomorphism.

Definition 3.1.2. A parabolic sheaf (E, ρ) is said to be *coherent* if for each $v \in \mathbb{Z}^n$ the sheaf E_v is a coherent sheaf on X.

3.2 Coherent sheaves on a root stack

Theorem 3.2.1 (Borne, Vistoli). *Let X be a scheme and L is a monoidal functor defined as in Example 2.1.4. Then there is a canonical tensor equivalence of abelian categories between the category $\mathfrak{QCoh} X_{L,\vec{r}}$ and the category of parabolic sheaves on X, associated with L with denominators $\vec{r}$.*

Proof. This is a proof from [BV, Proposition 5.10, Theorem 6.1].

The proof relies on the description of the stack as a quotient, (2.2.7). From this description, sheaves on the stack are equivariant sheaves on

$$P \times_{\operatorname{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]} \operatorname{Spec} \mathbb{Z}[\mathbb{N}^n].$$

As remarked in the proof of (2.1.11), the torsor P is obtained from a sheaf of algebras on X. The sheaf of algebras $\mathcal{A}$ is constructed from the functor L by taking a direct sum construction, it has a natural grading. There is an isomorphism of schemes:

$$P \times_{\operatorname{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]} \operatorname{Spec} \mathbb{Z}[\mathbb{N}^n] \cong \operatorname{Spec}(\mathcal{A} \otimes_{\mathbb{Z}[\vec{r}\mathbb{N}^n]} \mathbb{Z}[\mathbb{N}^n]).$$

The algebra on the right has a natural $\mathbb{Z}^n$-grading. So we obtained the first identification (equivalence of categories):

$$\mathfrak{QCoh} X_{L,\vec{r}} \simeq \mathbb{Z}^n - \text{graded } \mathcal{A} \otimes_{\mathbb{Z}[\vec{r}\mathbb{N}^n]} \mathbb{Z}[\mathbb{N}^n] - \text{modules}.$$

Let's now prove the equivalence of the latter category and the category of parabolic sheaves.

The question is local on X, so we may assume that X is an affine scheme Spec(R). By further restrictions we can assume that all the line bundles L_i are in fact trivial, and we identify them with R. In this situation the symmetric monoidal functor corresponds to a graded homomorphism

$$\mathbb{Z}[X_1, X_2, \dots, X_n] \to R[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}]$$

sending X_i to $x_i t_i$ with $x_i \in R$. Further the morphism

$$\operatorname{Spec}(\mathbb{Z}[\mathbb{N}^n]) \to \operatorname{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]$$

comes from an integral extension of algebras

$$\mathbb{Z}[X_1, X_2, \dots, X_n][Y_1, \dots, Y_n]/(Y_1^{r_1} - X_1, \dots Y_n^{r_n} - X_n)$$

Then taking tensor products yields a $\mathbb{Z}^n$-graded algebra

$$B := R[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}][s_1, \dots, s_n]/(s_1^{r_1} - x_1 t_1, \dots, s_n^{r_n} - x_n t_n)$$

where s_i has degree $(0, \dots, 0, 1, 0 \dots, 0) = e_i$.

We claim that the category of $\mathbb{Z}^n$-graded B-modules is equivalent to the category of parabolic sheaves on X.

Let M be a graded B-module, then we should define a parabolic sheaf $\Phi(M)$. First of all it is a functor $\mathbb{Z}^n \to \mathfrak{QCoh}X$. For $a \in \mathbb{Z}^n$ define $\Phi(M)(a) := M_a$. For $a' = a + e_i$ the morphism $M_a \to M_{a'}$ is a multiplication by s_i. Because t_i is invertible in B the multiplication by t_i induces isomorphisms $M_v \cong M_{v+r_i e_i}$. This induces a pseudo-period isomorphism. Because $s_i^{r_i} = x_i t_i$ in B the condition (i) of parabolic sheaf is satisfied. Other conditions are obvious in this context.

A map of two graded modules $M \to N$ is given by maps $M_u \to N_u$ for $u \in \mathbb{Z}^n$ so it induces a natural transformation of functors $\mathbb{Z}^n \to \mathfrak{QCoh}X$. It is clear that this map is a morphism of parabolic sheaves $\Phi(M) \to \Phi(M)$.

Now let us take a parabolic sheaf E and construct a R-module as $M := \oplus_{u \in \mathbb{Z}^n} E_u$. It will be in fact a B-module where s_i acts as a structure morphism $E_u \to E_{u+e_i}$ and multiplication by t_i is given by $L_i \otimes E_u \xrightarrow{\rho^{-1}} E_{u+r_i e_i}$.

These two functors are quasi-inverse. This is more or less clear, but the rigorous proof will be given in chapter 6, where we introduce the notion of extendable pairs.

□

Actually we can add the finiteness condition to the previous theorem and get the following

Corollary 3.2.2. *There is a canonical tensor equivalence of abelian categories between the category* $\mathfrak{Coh} X_{L,\vec{r}}$ *and the category of coherent parabolic sheaves on* X, *associated with* L.

Proof. We will use the construction from the proof above.

Consider now M a finitely generated $\mathbb{Z}^n$-graded B-module. We can assume that the generators of M are in fact homogeneous and hence there is an epimorphism

$$\oplus_{i=1}^{p} A(n_i) \to M.$$

The graded pieces of the module on the left are free of rank p and hence the graded pieces of M are finitely generated. It follows that a finitely generated A-module gives rise to a parabolic sheaf with values in the category of finitely generated R-modules, in other words coherent sheaves on X.

Conversely suppose that each we have a graded A-module M with each graded piece a finitely generated R-module. We can find finitely many elements of M, lets say $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ of degrees

$$\deg(\alpha_i) = (\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in}) \in \mathbb{Z}^n$$

with $0 \leq \lambda_{ij} \leq r_j$ such that the associated morphism

$$\phi : \oplus_{i=1}^{p} A(\deg(\alpha_i)) \to M$$

is an epimorphism in degrees

$$(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{Z}^n$$

whenever $0 \leq \mu_i \leq r_i$. It follows that ϕ is an epimorphism as multiplication by t_i induces an isomorphism $M_v \xrightarrow{\sim} M_{v+e_i}$. □

Chapter 4

Abhyankar's lemma

In this chapter we want to talk about a very important construction from [SGA1] which can be used to "replace" (tamely) ramified cover by a very special étale cover.

4.1 Unramified and tamely ramified morphisms

Let us recall some definitions first.

Definition 4.1.1. Let X be a scheme with an action of a finite group G. This action is called *admissible* if there exists an affine morphism $p : X \to Y$, such that $O_Y \cong p_*(O_X)^G$.

Remark 4.1.2. Basically, admissibility of an action means that X/G exists and is isomorphic to Y. Also notice that the map p is surjective.

Proposition 4.1.3. *G acts admissibly on X if and only if X is a union of affine open subschemes invariant under the G-action.*

Proof. $\Rightarrow$) Follows from the condition that p is affine. We just find an affine open cover of Y and then take its preimage. It will be an affine open cover of X. Each element of this cover is invariant under the G-action.

$\Leftarrow$) Let $X = \bigcup_i X_i$, where X_i are affine open and invariant under G-action. We can construct affine schemes $Y_i = X_i/G$. Finally, because $Y_i \bigcap Y_j = (X_i \bigcap X_j)/G$, we can glue the schemes Y_i together and get Y.

□

Definition 4.1.4. If $x \in X$ is a point (not necessarily closed) the subgroup of G stabilising x is called the *decomposition group* and we denote it by $D(x, G)$.

This group acts on the residue field $k(x)$ in a canonical way. The subgroup of the decomposition group acting trivially on $k(x)$ is called the *inertia group* of x and we denote it by $I(x, G)$.

Definition 4.1.5. Let $f : X \to Y$ be a morphism of schemes of finite type, $x \in X$ is a point and $y = f(x) \in Y$. It is called *unramified at* x, if one of the two equivalent conditions holds:

1. $O_x/\mathfrak{m}_y O_x$ is a finite separable extension of $k(y)$.

2. $(\Omega_{X/Y})_x = 0$.

If f is unramified at every point $x \in X$, it is called unramified.

Proof. Let's recall the proof of the equivalence of (i) and (ii).

(i)⇒(ii) By Nakayama Lemma, it is enough to prove that $(\Omega_{X/Y})_x \otimes_{O_x} k(x) = 0$. The sheaf of differentials behaves well under base change, so we just need to prove that $\Omega_{k(x)/k(y)} = 0$. But it is clear, because the extension $k(y) \subset k(x)$ is finite separable.

(ii)⇒(i) Using base change we can reduce the theorem to the case where $y = \mathrm{Spec}(k(y))$ is a point and $X = \mathrm{Spec}(O_x)$. Let's consider the sequence of morphisms $k(y) \to O_x \to O_x/\mathfrak{m}_x$. There is an exact sequence: $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \Omega_{O_x/k(y)} \otimes_{O_x} k(x) \to \Omega_{k(x)/k(y)} \to 0$. The term in the middle is given to be zero. Hence we obtain that $\Omega_{k(x)/k(y)} = 0$, and this proves that the extension $k(y) \subset k(x)$ is separable.

Now we need to prove that O_x is a field. Because the morphism $k(y) \to O_x$ is of finite type, O_x is finitely generated $k(y)$-algebra. We can assume without loss of generality that $k(y)$ is algebraically closed. Then $k(x) = k(y)$. In this situation the map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \Omega_{O_x/k(x)} \otimes_{O_x} k(x)$ is an isomorphism. Surjectivity is clear. To see the injectivity we need to prove the surjectivity of the dual map: $\mathrm{Hom}_{k(x)}(\Omega_{O_x/k(x)} \otimes_{O_x} k(x), k(x)) \to \mathrm{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x))$. By the universal property the latter map is the same as the map: $\mathrm{Der}_{k(x)}(O_x, k(x)) \to \mathrm{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x))$. Let's take a $k(x)$-linear morphism $\mu : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k(x)$. Any element $a \in O_x$ we can write as a sum $a = a_0 + a_1$, where $a_0 \in k(x)$ and $a_1 \in \mathfrak{m}_x$. Then define a derivation $\delta(a) = \mu(a_1)$. It is well defined.

Because $\Omega_{O_x/k(y)} = 0$, the previous isomorphism gives us $\mathfrak{m}_x/\mathfrak{m}_x^2 = 0$. So by Nakayama, $O_x \cong k(x)$.

□

Let $\mathfrak{o}$ be a DVR with a maximal ideal $\mathfrak{m}$ and a field of fractions K. $K \subset L$ is a Galois extension with a group G. Let $\mathfrak{O}$ be a normalization of $\mathfrak{o}$ in L. It is well known that $\mathfrak{O}$ is a finite $\mathfrak{o}$-module of rank $[L : K]$. Denote by $\mathfrak{M}$ its maximal ideal. As in definition 4.1.4 we have a decomposition group G_d which is a subgroup of G stabilizing $\mathfrak{M}$. Also there is an inertia group G_i which is a subgroup of G_d, such that it acts trivially on $\mathfrak{O}/\mathfrak{M}$.

Lemma 4.1.6. *In the situation above if $\mathfrak{o} \to \mathfrak{O}$ is unramified then the order of inertia group is equal to one.*

Proof. Indeed, if the map $\mathfrak{o} \to \mathfrak{O}$ is unramified, the extension of residue fields $\mathfrak{o}/\mathfrak{m} \subset \mathfrak{O}/\mathfrak{M}$ is separable. Hence by Proposition 9.6 of [N, I§9] the order of inertia group $|G_i| = e$, where e is a ramification index. But because $\mathfrak{m}\mathfrak{O} = \mathfrak{M}$, we have that $e = 1$. □

We want to slightly relax this condition and give a definition of tamely ramified map following [SGA1].

Definition 4.1.7. One says that the extension $K \subset L$ is *tamely ramified over* $\mathfrak{o}$, if the order of the inertia group $|G_i|$ is not divisible by the characteristic of the residue field $\mathfrak{o}/\mathfrak{m}$.

Proposition 4.1.8. *If the extension $K \subset L$ is tamely ramified over $\mathfrak{o}$, then the inertia group G_i is cyclic.*

Proof. Let us denote by $\mathfrak{O}$ the ring of integers of L and by $\mathfrak{M}$ its maximal ideal. Choose an uniformizing parameter t. Then one can consider a map:

$$G_i \to (\mathfrak{O}/\mathfrak{M})^*,$$

$$\sigma \mapsto \frac{\sigma t}{t} \bmod \mathfrak{M}.$$

This map turns a homomorphism of group which does not depend on the choice of t. Moreover it is an injection. See [CF, Ch.1, §8, Th. 1] for the proofs. Hence G_i is a subgroup of a cyclic group, so it is cyclic. □

Lemma 4.1.9. *(Abhyankar's lemma I) Let $\mathfrak{o}$ be DVR with a field of fractions K. Consider L and K' be two Galois extension of K tamely ramified with respect to $\mathfrak{o}$. Let n, m be the orders of their inertia groups. Denote by L' a composite of L and K' over K. If m is a multiple of n, then L' is unramified over localizations of the normal closure $\mathfrak{o}'$ of $\mathfrak{o}$ in K'.*

Proof. From [SGA1, X, Lemma 3.6]. Denote by G, H and M Galois groups of extensions $K \subset L$, $K \subset K'$ and $K \subset L'$ respectively. Consider a maximal ideal $\mathfrak{m}$ of $\mathfrak{o}'$ lying over the maximal ideal $\mathfrak{p} \subset \mathfrak{o}$. Let's denote by $\mathfrak{o}''$ a normal closure of $\mathfrak{o}'$ in L' and consider any maximal ideal $\mathfrak{m}'$ lying over $\mathfrak{m}$. Also take a normal closure of $\mathfrak{o}$ in L and consider its maximal ideal $\mathfrak{n}$ over $\mathfrak{p}$.

Let's denote the inertia groups corresponding to ideals $\mathfrak{n}$, $\mathfrak{m}$ and $\mathfrak{m}'$ by G_i, H_i and M_i. It is clear that we must have a diagram:

$$\begin{tikzcd}
 & G_i \\
M_i \arrow[ur, two heads] \arrow[r, tail] \arrow[dr, two heads] & G_i \times H_i \arrow[u] \arrow[d] \\
 & H_i
\end{tikzcd}$$

Because H_i and G_i are cyclic groups of orders prime to p, the order of M_i must be also prime to p (as it is a subgroup of $G_i \times H_i$). Because m is a multiple of n any element of $G_i \times H_i$ (and thus M_i) has order which divides m. But because the map $M_i \twoheadrightarrow H_i$ is surjective, the order of M_i is exactly m. So the morphism $M_i \to H_i$ is in fact an isomorphism.

Clearly, inertia group associated to $\mathfrak{m}'$ over $\mathfrak{m}$ is isomorphic to the kernel of $M_i \to H_i$. Different choice of ideals will give the same answer, because the extension is Galois. So lemma is proved.

□

4.2 Abhyankar's lemma

Let's present two generalizations of tame ramification. First consider the case where L is not just a field, but an étale K-algebra.

Definition 4.2.1. Let $\mathfrak{o}$ be a DVR with a maximal ideal $\mathfrak{m}$ and a field of fractions K. L is an étale K-algebra, so it is a finite product of fields L_n each of which is a separable extension of K. Let's denote by L'_n a Galois extension of K generated by L_n in an algebraic closure $\overline{L_n}$. Then L is *tamely ramified over* $\mathfrak{o}$, if each extension $K \subset L'_n$ is tamely ramified in the sense of Definition 4.1.7.

Now let's give a definition of tame ramification for schemes. First let's recall a notion of normal crossing divisors.

Definition 4.2.2. (from [SGA5, I.3]).

Let Y be a regular scheme and D is an effective divisor. A divisor D is called a *strict normal crossing* if there is a finite family of sections $\{f_i\}_1^n$, where $f_i \in \Gamma(Y, O_Y)$, such that:

1. $D = \sum_{i=1}^n \operatorname{div}(f_i)$
2. For each $y \in \operatorname{supp}(D)$ the elements $(f_i)_y$ that satisfy $(f_i)_y \in \mathfrak{m}_y$ form a part of a regular system of parameters for the local ring $O_{Y,y}$.

D is called *a divisor with normal crossings* if it is strict normal crossing locally in étale topology.

Definition 4.2.3. Let $Y = \operatorname{Spec}(A)$ be a regular scheme and D a divisor with normal crossing. $Y_0 = \operatorname{supp}(D)$, $U = Y \setminus Y_0$. Consider an étale morphism $f : V \to U$. Let $y \in Y_0$ be a maximal point (i.e. a generic point of an irreducible component), denote by K the fraction field of $O_{Y,y}$. One says that the morphism f is *tamely ramified along D* if for all maximal points y of Y_0 we have that $V|_K$ is a spectrum of K-algebra tamely ramified over $O_{Y,y}$.

Proposition 4.2.4. *(Abhyankar's lemma II) Let* $Y = \operatorname{Spec}(A)$ *be a regular local scheme and* $D = \sum_{1\le i\le r} \operatorname{div}(f_i)$ *a divisor with normal crossings. Set* $Y_0 = \operatorname{supp}(D)$ *and let* $U = Y \setminus Y_0$*. Consider* $V \to U$*, a cover tamely ramified along D. If* y_i *are the generic points of* $\operatorname{supp}(\operatorname{div}(f_i))$

then O_{Y,y_i} is a discrete valulation ring. If we let K_i be its field of fractions then as V ramifies tamely we have that

$$V|_{K_i} = \mathrm{Spec}(\prod_{j\in J_i} L_{ji})$$

where the L_{ji} are finite separable extensions of K_i. We let n_{ji} be the order of the inertia group of the Galois extension generated by L_{ji} and let

$$n_i = \mathrm{lcm}_{j\in J_i} n_{ji},$$

and set

$$A' = A[T_1, \ldots, T_r]/(T_1^{n_1} - f_1, \ldots, T_r^{n_r} - f_r) \qquad Y' = \mathrm{Spec}(A').$$

Then the étale cover $V' = V \times_Y Y'$ of $U' = U \times_Y Y'$ extends uniquely up to isomorphism to an étale cover of Y'.

Proof. This is [SGA1, Expose XIII, proposition 5.2]. Let's prove it in several steps.

Step 1. First we prove that the ring A' is regular. We can assume that the dimension of A is exactly r. It is clear that A is a local ring with a maximal ideal generated by $T_1, T_2, \ldots, T_n$. Because the map $A \to A'$ is finite and flat, $\dim(A') = \dim(A) = r$. Hence by definition, A' is regular.

Step 2. Let's denote $D' = \sum_{1\leq i\leq r} \mathrm{div}(T_i)$, then $U' = Y' \setminus \mathrm{supp}(D')$. From the local description of étale maps (see, for example, [M, Ch. I, Corollary 3.16]) it follows that the map $U' \to U$ is étale (because n_i are not divisible by the characteristic of the residue field). We claim that the map $U' \to Y$ is tamely ramified over D. Indeed, choose y to be a generic point of $\mathrm{supp}(\mathrm{div}(f_i))$. Denote $R = O_{Y,y}$ and by $\overline{R}$ its strict localization. $\overline{K}$ is a field of fractions of $\overline{R}$. Then $\overline{K}$-algebra $U'|_{\overline{K}}$ is isomorphic to a field $\overline{K}[T]/(T^{r_i} - f_i)$ and the extension is clearly tamely ramified over $\overline{R}$.

Step 3. From step 1 we know that Y' is regular. So we can use the result [SGA1, I, Corollary 10.2] that if X' is an étale cover of Y', it must be a normalization of Y' in the ring of rational functions of X'. But if X' is a cover which extends V', the ring of rational functions of X' is isomorphic to the fiber of V' at the generic point of Y'. So if there exists an étale cover X', it must be unique.

Step 4. For any geometric point $\bar{y} \in Y'_0$ let's denote by $\overline{Y}'$ the strict localization of Y' at $\bar{y}$. By $\overline{U}'$ and $\overline{V}'$ we denote the base change to $\overline{Y}'$. We claim that it is enough to show that $\overline{V}' \to \overline{U}'$ can be extended to the étale cover $X' \to \overline{Y}'$. Indeed let's consider a map $\overline{Y}' \to Y'$. It is open and $Y' = \mathrm{Spec}(A')$ is quasi-compact, so it is enough to choose a finite number of geometric points, such that the union of $\overline{Y}'$ will give a fppf cover of $\overline{Y}'$. If we construct morphisms $X' \to \overline{Y}'$ for each element in the cover, we can glue them together as far as they agree on the intersections. But they will agree because of the uniqueness proved in step 3.

Step 5. We will need the homotopy purity theorem. Let us state the result here.

Theorem 4.2.5. *From [SGA2, X, Theorem 3.4]. Let R be local Noetherian regular ring of dimension $\geqslant 2$. Then for any open subset $U \subset \mathrm{Spec}(R)$ there is an equivalence of categories of étale covers: $\acute{E}t(X) \to \acute{E}t(U)$.*

So for any geometric points $\bar{y} \in Y'_0$, such that $\dim(O_{Y,y}) \geqslant 2$, we can extend the étale cover.

Step 6. We just need to prove our result for $Y' = \mathrm{Spec}(O_{Y',\bar{y}_i})$, where $\bar{y}_i$ lies over a maximal point of $\mathrm{div}(T_i)$. But this is exactly the situation of Lemma 4.1.9. Indeed if K'_i is a field of fractions of Y', we have a tamely ramified extension $K_i \subset K'_i$ with the order of inertia group n_i (this follows from step 2). At the same time $K_i \subset L_{ji}$ is a tamely ramified extension with the order of inertia group n_{ji}. Also by definition, n_i is a multiple of n_{ji}. Hence we can apply Lemma 4.1.9.

□

Remark 4.2.6. The proof given shows how to construct the extension of V', we will need this in Chapter 5. The extension can be constructed as the normalization of Y' in the generic point of $V \times_Y Y'$.

Chapter 5

The structure of inertia groups and Chevalley-Shephard-Todd theorem

In this chapter we want to describe the inertia groups of a quotient morphism under some assumptions.

5.1 Inertia groups. Generation in codimension one

Consider a scheme X and an admissible action of a finite group G on it. For any point $x \in X$ we defined a decomposition group $D(x, G)$ and inertia group $I(x, G)$ in 4.1.4. There is an induced action of $D(x, G)$ on the closure of the point x, and $I(x, G)$ acts trivially on this closure. Hence if $y \in \bar{x}$ then there is an inclusion $I(x, G) \hookrightarrow I(y, G)$.

Definition 5.1.1. In the situation above we will say that *the inertia groups are generated in codimension one* if for each point $y \in X$ we have that

$$I(y, G) = \prod_{y \in \bar{x}} I(x, G)$$

where the product is over all points of codimension one containing y and the identification is via the inclusions above.

For a group acting generically free on a smooth curve all inertia groups will be generated in codimension one. We are going to prove that under special assumption this will be also true in higher dimensions.

Theorem 5.1.2. *Let X be a regular, separated, noetherian scheme over a field k. Assume that G is a finite group with cardinality coprime to the characteristic of k and that G acts admissibly and generically freely on X with quotient $\phi : X \to Y$ and Y is regular. Assume that the map ϕ is ramified along a normal crossing divisor. For any point $x \in X$ the inertia group $I(x, G)$ is generated in codimension one.*

We will give a proof at the end of the chapter.

5.2 Pseudo-reflections

Let us repeat some important facts from invariant theory.

Definition 5.2.1. Let k be a field, V - a finite dimensional vector space over k. An element $g \in \mathrm{GL}(V)$ is called a pseudo-reflection, if the image $\mathrm{Im}(1 - g)$ is of dimension one.

Notation 5.2.2. If g is a pseudo-reflection on V, denote by V_g the hyperplane of stable vectors: $\{v \in V | \, gv = v\}$.

Definition 5.2.3. Let k be a field, V - a vector space. Let g and h be pseudo-reflections. Suppose that the group $G := \langle g, h\rangle$ generated by them is of finite order coprime to the characteristic of k. Let us call this group *of tame order*.

We will need the following lemmas.

Lemma 5.2.4. *Let g and h be pseudo-reflections and let $G = \langle g, h\rangle$ be of tame order. Also suppose that $V_g = V_h$. Then g and h commute.*

Proof. Because $V_g = V_h$ is fixed by G, we can find a G-invariant complement W. As $\dim(W) = 1$ it is a subspace of common eigenvectors for G. □

Lemma 5.2.5. *Let g and h be pseudo-reflections and $G = \langle g, h\rangle$ is of the tame order. If $V_g = V_{hgh^{-1}}$, then g and h commute.*

Proof. Note that g and hgh^{-1} are pseudoreflections with the same characteristic polynomial. The previous lemma tells us they are simultaneously diagonalizable. As their eigenvalues are the same, $g = hgh^{-1}$.

□

Lemma 5.2.6. *Let g and h be pseudo-reflections and $G = \langle g, h\rangle$ is of the tame order. Denote by Λ the set of all pseudo-reflections in G. If $\bigcup_{\tau\in\Lambda} V_\tau$ is a support of a strict normal crossing divisor, then the group G is abelian. Moreover $G \cong C_{ord(g)} \times C_{ord(h)}$, where C_n is a cyclic group of order n.*

Proof. It is enough to show that g and h commute. We have three pseudo-reflections $g,\ h,\ hgh^{-1}$ and three hyperplanes $V_g,\ V_h,\ V_{hgh^{-1}}$. Firstly, observe that $V_g \cap V_h = V_g \cap V_h \cap V_{hgh^{-1}}$, because $V_{hgh^{-1}} = hV_g$.

The normal crossing condition forces $V_{hgh^{-1}} = V_g$ or $V_{hgh^{-1}} = V_h$.

If $V_{hgh^{-1}} = V_g$, then the result follows from Lemma 5.2.5.

The condition $V_{hgh^{-1}} = V_h$ is equivalent to $hV_g = V_h$. Let's apply h to both sides $(\mathrm{ord}(h) - 1)$ times, then we get $V_h = V_g$. The result follows from Lemma 5.2.4.

The isomorophism $G \cong C_{\mathrm{ord}(g)} \times C_{\mathrm{ord}(h)}$ is clear.

□

Let us state an important theorem we want to use.

Theorem 5.2.7 (Chevalley-Sheppard-Todd)**.** *Let k be a field, V a finite dimensional vector space over k, and $G \subset GL(V)$ a finite group of order coprime to the characteristic of k. Denote by $S = Symm(V)$ the symmetric algebra and $R = S^G$ the algebra of invariants. Then the following three conditions are equivalent:*

(i) G is generated by pseudo-reflections.

(ii) R is a graded polynomial k-algebra.

(iii) R is a regular ring.

Proof. $(i) \Leftrightarrow (ii)$ is in [Bou, Ch.V, §5, 5].

$(ii) \Rightarrow (iii)$ obvious

$(iii) \Rightarrow (i)$ Let $H \triangleleft G$ be a normal subgroup generated by pseudo-reflections. Then $X := \mathrm{Spec}(S)/H$ is an affine space and G/H acts on it, such that $X/(G/H) = \mathrm{Spec}(R)$. Because G/H doesn't contain pseudo-reflections, its action on X is free in codimension one. Hence the map $X \to \mathrm{Spec}(R)$ is étale in codimension one and so, by purity of the branch locus (see Theorem 6.2.3 in the next chapter), is étale. So G/H must act freely on X, but at the same time it is a linear action, where the origin is fixed. That means that G/H must be a trivial group, hence $G = H$ is generated by pseudo-reflections.

□

5.3 Luna's étale slice theorem

Definition 5.3.1. Assume that G is an algebraic group acting on affine varieties X and Y. Let $\phi : X \to Y$ be G-morphism. One says that ϕ is *strongly étale*, if

1. $\phi/G : X/G \to Y/G$ is étale.
2. The diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d] & Y \arrow[d] \\ X/G \arrow[r, "\phi/G"] & Y/G \end{tikzcd}$$

   is cartesian.

Notation 5.3.2. Let G be a finite group and k a field, such that its characteristic is coprime to $|G|$. Let $H \subset G$ be a subgroup. Then we can define an H-action on G by the rule:

$$G \times H \to G$$
$$(g, h) \mapsto gh^{-1}.$$

Suppose that H acts on an affine variety Y. Then this action and the described action of H on G provides the H-action on $G \times Y$ in such a way that $G \times Y$ is a principal H-bundle. The action is: $h \cdot (g, y) = (gh^{-1}, hy)$. Denote:

$$G \times_H Y = (G \times Y)/H.$$

The G-action on $G \times_H Y$ is induced by the left action on the first components.

If $(g, y) \in G \times Y$, then denote by $\overline{(g, y)}$ its image in $G \times_H Y$.

We will need the following easy Lemma.

Lemma 5.3.3. *If $y \in Y$, $g \in G$, $n = \overline{(g, y)} \in G \times_H Y$, then $G_n = gH_y g^{-1}$. Here G_n is a decomposition group $D(n, G)$ (or an isotropy group) and H_y is a decomposition group $D(y, H)$.*

Proof. Take $s \in H_y$, then by definition $sy = y$. Then

$$gsg^{-1}(g, y) = (gs, y) \sim_H (g, sy) = (g, y).$$

Hence $gsg^{-1} \in G_n$.

Conversely, take $t \in G_n$. Then $t(g, y) \sim_H (g, y)$. So there is $h \in H$, such that $(tg, y) = (gh, h^{-1}y)$. This implies that $tg = gh$, or $t = ghg^{-1}$. Also we have that $h^{-1}y = y$, or $h \in H_y$. □

Let us formulate the important theorem which will be heavily used in the last part of the chapter.

Theorem 5.3.4 (Luna's étale slice)**.** *Let k be a field, X an affine regular variety, and G a finite group acting on X, such that its order is coprime to the characteristic of k. Consider a closed point $x \in X$. Then there exists a locally closed subscheme V of X, such that:*

1. *V is affine and contains x.*
2. *V is G_x-invariant.*
3. *The image of the G-morphism $\psi : G \times_{G_x} V \to X$ is a saturated open subset U of X.*
4. *The restriction of $\psi : G \times_{G_x} V \to U$ is strongly étale.*

Also there exists an étale G_x-invariant morphism $\phi : V \to T_x V$, such that $\phi(x) = 0$, $T\phi_x = Id$, and the following properties are satisfied:

5. *$T_xX = T_x(Gx) \oplus T_xV$. We will denote $N_x := T_xV$ - the normal space to Gx at x.*

6. *The image of ϕ is a saturated open subset W of T_xV.*

7. *The restriction of $\phi : V \to W$ is a strongly étale G_x-morphism.*

Proof. It is a slightly modified version of [D, Th. 5.3 and Th. 5.4]. □

5.4 Proof of Theorem 5.1.2

Let us recall the following fundamental result by Mumford.

Proposition 5.4.1. *Let X be an affine scheme over a field k. Assume that G is a finite group with cardinality coprime to the characteristic of k and that G acts generically freely on X. Then the quotient $\phi : X \to Y$ is a universal geometric quotient.*

Proof. Can be proved similarly to [FKM, Theorem 1.1]. □

We will use the fact that the inertia group is preserved under arbitrary base change.

Proposition 5.4.2. *Consider a scheme X and an admissable action of a finite group G on it. Denote $Y = X/G$. Take a surjective map $f : Y' \to Y$ and form a pull-back diagram:*

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d] & X \arrow[d, "\phi"] \\
Y' \arrow[r, "f"] & Y
\end{tikzcd}.$$

Take a point $x \in X$ and a point lying over it: $x' \in X'$. Then $I(x, G) = I(x', G)$.

Proof. The inclusion $I(x, G) \subset I(x', G)$ is clear. Indeed, if we consider a geometric closure $\bar{x}$ of x and take $g \in I(\overline{x}, G)$, then g acts as identity on $\bar{x}$. Hence it acts as identity on $h^{-1}(\bar{x})$, and so on $\bar{x}'$.

Let's prove the other inclusion. Firstly, we can assume that X is affine, because the action of G is admissible. Take $y := \phi(x)$. Because ϕ is finite, X_y is a disjoint union of fields. We have a cartesian diagram:

$$\begin{tikzcd} X_y \arrow[r] \arrow[d, "\hat{\phi}"] & X \arrow[d, "\phi"] \\ \mathrm{Spec}(k(y)) \arrow[r] & Y \end{tikzcd},$$

By universality of the quotient (Proposition 5.4.1), the morphism on the left is also a G-quotient. If we replace G with a decomposition group $D(x,G)$ and Y with $\mathrm{Spec}(k(y))$, we reduce the problem to finding the inertia group of $D(x,G)$-action on $\mathrm{Spec}(k(x))$. The quotient by this action is $\hat{\phi} : \mathrm{Spec}(k(x)) \to \mathrm{Spec}(k(y))$, where the extension $k(y) \subset k(x)$ is Galois.

Consider the base change diagram:

$$\begin{tikzcd} \mathrm{Spec}(O_{x'}) \arrow[r] \arrow[d] & \mathrm{Spec}(k(x)) \arrow[d] \\ \mathrm{Spec}(O_{y'}) \arrow[r] & \mathrm{Spec}(k(y)) \end{tikzcd},$$

We can now replace G with $G/I(x,G)$. Then $\mathrm{Gal}(k(x)/k(y)) = G$. So the morphism on the right is a G-torsor. But the base change of a torsor is a torsor again. So the action of G on $\mathrm{Spec}(O_{x'})$ has no inertia.

□

We will also need few basic facts about étale maps.

Proposition 5.4.3. *Let $\phi : Y' \to Y$ be étale. Then Y' is regular if Y is regular.*

Proof. See [M, I, 3.17]. □

Proposition 5.4.4. *If $\phi : Y' \to Y$ is unramified, and D is a normal crossing divisor in Y, then $D' = \phi^{-1}(D)$ is a normal crossing divisor in Y'.*

Proof. By definition of a normal crossing divisor it is enough to prove everything in a local case. The problem will become the following. Let $O_y \subset O_{y'}$ be an unramified extension. If $\mathfrak{m}_y = (f_1, f_2, \dots, f_n)$, then $\mathfrak{m}_{y'} = (f_1, f_2, \dots, f_n)$. This is true by definition. □

Before proving Theorem 5.1.2 let's make a series of reductions.

Reduction 1: Take a point $x \in X$ not necessary closed. If we localize at x using the universality of the quotient (Proposition 5.4.1), we can reduce the theorem to the situation where X and Y are affine schemes and x is a closed point.

Reduction 2: We can assume that the field k is separably closed. Indeed, consider a base change $k \subset k^{\text{sep}}$. Denote $\overline{X} = X \times \text{Spec}(k^{\text{sep}})$, $\overline{Y} = Y \times \text{Spec}(k^{\text{sep}})$. Then the map $\overline{\phi} : \overline{X} \to \overline{Y}$ will satisfy the assumptions of Theorem 5.1.2. Indeed, $\overline{\phi}$ ramifies at simple normal crossing divisor, because the map $\overline{Y} \to Y$ is unramified (by Proposition 5.4.4).

The map $\overline{Y} \to Y$ is surjective. If $x \in X$ is a closed point and $\overline{x} \in \overline{X}$ is a point over x, then $I(x, G) = I(\overline{x}, G)$ by Proposition 5.4.2. So if we prove that $I(\overline{x}, G)$ is generated in codimension one, the same thing will be true for $I(x, G)$.

This reduction is useful, because for $\overline{X}$ the inertia group is the same as the decomposition group, which is easier to analyze.

Reduction 3: Here we will apply Luna's slice theorem. First we can replace X with an open subset U from part (3) of Theorem 5.3.4. Then we can make an étale base change and replace U with $G \times_{G_x} V$ from part (4). We will get a cartesian diagram:

$$\begin{tikzcd}
G \times_{G_x} V \arrow[r] \arrow[d] & U \subset X \arrow[d] \\
V/G_x \arrow[r] & U/G
\end{tikzcd},$$

where the horizontal maps are étale. The inertia group will remain unchanged.

Finally, using part (7) we obtain a cartesian diagram:

$$\begin{tikzcd}
G \times_{G_x} V \arrow[r] \arrow[d] & G \times_{G_x} N_x \arrow[d] \\
V/G_x \arrow[r] & N_x/G_x
\end{tikzcd},$$

where the horizontal maps are étale and N_x is a vector space. Because the inertia group is again unchanged, we just need to compute the inertia on the right.

Proof. We reduced the Theorem 5.1.2 to the statement that the decomposition subgroups of G-action on $G \times_{G_x} N_x$ are generated in codimension one, where N_x is a vector space.

First notice that $(G \times_{G_x} N_x)/G \cong N_x/G_x$. Also N_x/G_x is regular as étale cover of a regular scheme. For any element $u := \overline{(g, n)} \in G \times_{G_x} N_x$ the decomposition group G_u is equal to $gH_n g^{-1}$ - the conjugate of the decomposition group of n under G_x-action on N_x (by Lemma 5.3.3).

So finally, we have a linear action of G_x on a vector space N_x , the quotient N_x/G_x is regular and it ramifies along a simple normal crossing divisor. Applying Theorem 5.2.7, we get that G_x is generated by pseudo-reflections. Using Lemma 5.2.6 we get that G_x is an abelian group. Moreover it is a product of cyclic groups $C_{i_1} \times \ldots \times C_{i_n}$, where n is a number of irreducible components in the normal crossing divisor and i_j is the order of stabilizing group of each component. By definition, this means that the decomposition group of each point of N_x is generated in codimension one.

$\square$

Chapter 6

Quotient stacks as root stacks

In this chapter we will provide sufficient conditions for a quotient stack to be a root stack.

6.1 Quotient stack in case of discrete valuation rings

To illustrate the procedure we will use in the general situation, let us start with an example.

Example 6.1.1. Let k be an algebraically closed field, such that $\gcd(r, \mathrm{char}(k)) = 1$. Let $\mathcal{O}$ be a discrete valuation ring containing k with an action of μ_r.

The ring of invariants $\mathcal{O}^{\mu_r}$ is also a discrete valuation ring. Because we assumed that $\mathcal{O}$ contains a field, its completion $\hat{\mathcal{O}}$ is a power series ring in one variable over residue field. Note that μ_r must preserve the maximal ideal of $\mathcal{O}$. If we further assume that the action is generically free and inertial, i.e μ_r acts trivially on the residue field then if s is a local parameter for $\mathcal{O}$ we can conclude that $t = s^r$ is a local parameter for $R = \mathcal{O}^{\mu_r}$.

We set $Y = \mathrm{Spec}(R)$ and consider the root stack

$$\mathfrak{Y} = Y_{R,t,r} \to Y.$$

The parameter s induces a μ_r-equivariant morphism

$$X \to \mathfrak{Y}$$

corresponding to the triple (O, s, m) where m is the canonical isomorphism $O^r \to O$. We will show later (6.2.6) that this morphism is in fact étale. Using the 2 out of 3 property (see [M, I, Cor.3.6]) for étale maps we get that the natural morphism

$$X \times \mu_r \to X \times_{\mathfrak{Y}} X$$

is étale. To show that $[X/\mu_r] \cong \mathfrak{Y}$ is equivalent to show that the morphism above is an isomorphism (because X is a presentation of a quotient stack $[X/\mu_r]$ - see [stacks, Tag 04T5]). To understand that it is indeed an isomorphism let us recall some basic facts from SGA.

Definition 6.1.2. A morphism $f : X \to Y$ of schemes over a field k is called *radicial* (or *universally injective*), if for all fields $K \supset k$ we have that the morphism $f(K) : X(K) \to Y(K)$ is injective.

Theorem 6.1.3. *Let $f : X \to Y$ be a morphism of finite type. Then f is an open immersion if and only if f is étale and radicial.*

Proof. See [SGA1, I, Theorem 5.1] □

Also we want to use the following easy Lemma.

Lemma 6.1.4. *If a morphism of scheme is an open immersion and surjective then it is an isomorphism.*

To prove that the morphism $X \times \mu_r \to X \times_{\mathfrak{Y}} X$ it suffices to show that this morphism is radicial and surjective. In other words we need to show that it is a bijection on K-points for each field K.

Given a pair of K-points a and b of X that give a K-point of $X \times_Y X$, the fiber of

$$X \times_{\mathfrak{Y}} X \to X \times_Y X$$

over this point consists of the space of isomorphisms between $a^*(O, s, m)$ and $b^*(O, s, m)$ in $\mathfrak{Y}$. If the support of the K-points is the generic point of O this is just a singleton and if the support is the closed point then the space is a bitorsor (i.e. simultaneously left and right torsor such that the actions commute) over μ_r. At any rate the fiber of the map $X \times_{\mathfrak{Y}} X \to X \times_Y X$ over K is in bijection with the fiber of the map $X \times \mu_r \to X \times_Y X$ over K, so the morphism $X \times \mu_r \to X \times_{\mathfrak{Y}} X$ is an isomorphism. Hence in this case we have that

$$[X/\mu_r] \cong \mathfrak{Y}.$$

6.2 General case

Assumption 6.2.1. We will assume that X and Y are regular, separated, noetherian schemes over a field k. Let G be a finite group with cardinality coprime to the characteristic of k. We will assume that G acts admissibly and generically freely on X with quotient $\phi : X \to Y$. Note that by [GW, Theorem 14.126] our hypothesis imply that the quotient map $X \to Y$ is flat.

Definition 6.2.2. Consider the map $\phi : X \to Y$ which is locally of finite type. The set of points of X where ϕ is ramified is called *the branch locus*. It has a natural closed subscheme structure defined by $\operatorname{supp}(\Omega_{X/Y})$.

Theorem 6.2.3. *If a map $\phi : X \to Y$ is faithfully flat and finite, then the branch locus (if nonempty) has pure codimension one in X.*

Proof. This is called *purity of branch locus*, see [AK, VI, Thm 6.8]. □

Corollary 6.2.4. *Under assumption 6.2.1 the conditions of the purity theorem are satisfied. So the branch locus is in fact an effective Cartier divisor.*

Lemma 6.2.5. *We can write the branch divisor as*

$$D = \sum_{i=1}^{n} (r_i - 1) \left(\sum_{g \in G} g^* D_i \right),$$

where each D_i is a prime divisor. We can view the D_i as points of the scheme X. The multiplicities r_i are related to the inertia groups of D_i via

$$r_i = |I(D_i, G)|.$$

Proof. Let's consider the image $\phi(B) =: E$ which is again an effective divisor. Let's write E as a sum of prime divisors: $E = \sum_{i=1}^{n} E_i$. As G acts generically freely, passing to generic points of our regular variety produces a Galois extension with Galois group G. The branch locus behaves well with respect to base change so we can localize everything at E_i. As the morphism ϕ is affine, the question reduces to the question in commutative algebra.

Let $\mathfrak{o}$ be DVR and $\mathfrak{o} \subset O$ be a Galois extension of Dedekind domains. Then we have: $\mathfrak{p}O = (\prod_{g\in G} g\mathfrak{P})^e$ for some natural number e, which is called the ramification index. We also know that $|G|$ is a multiple of e and so e is coprime to $\mathrm{char}(k)$. See, for example, [N, Ch. I.9]. We want to compute $\mathrm{supp}(\Omega_{O/\mathfrak{o}})$. For that let's again localize at $\mathfrak{P}$. So we can assume that $\mathfrak{o}$ and O are DVRs. Also $\mathfrak{o}$ and O are k-algebras and their residue fields contain k. Hence $\Omega_{O/\mathfrak{o}}$ is generated by $d(x^e) = ex^{e-1}dx$ and because e is invertible in k, $\mathrm{supp}(\Omega_{O/\mathfrak{o}}) = \mathfrak{P}^{e-1}$.

We just need to prove that the ramification index is equal to the order of inertia group. But this is Proposition (9.6) of [N, Ch. I.9].

□

We let E_i be the image of D_i under ϕ. It is called the ramification divisor. We form the root stack

$$\mathfrak{Y} = Y_{((E_1,r_1),\ldots,(E_n,r_n))}.$$

Note that we have assumed that the characteristic of our ground field is coprime to G and hence to each r_i. It follows, via a local calculation along the ring extension $O_{X,D_i}/O_{Y,E_i}$ that we have $\phi^*(E_i) = r_i(\sum_{g\in G} g^* D_i)$. This allows us to lift ϕ to produce a diagram

$$\begin{tikzcd} X \arrow[d, "\phi"'] \arrow[dr, "\psi"] & \\ Y & \mathfrak{Y}. \arrow[l, "\pi"'] \end{tikzcd}$$

The morphism ψ is equivariant in the sense that precomposition with $g \in G$ produces a two-commuting diagram. This gives us a morphism

$$[X/G] \to \mathfrak{Y}$$

that we would like to show is an isomorphism under our assumption (6.2.1).

In the proof below we will need to make use of Abhyankar's lemma (Proposition 4.2.4).

Proposition 6.2.6. *Suppose that $\phi : X \to Y$ is ramified along a simple normal crossings divisor. The morphism $\psi : X \to \mathfrak{Y}$ constructed above is étale.*

Proof. Étale maps are local on the source so we can assume that $Y = \mathrm{Spec}(S)$, and all E_i are trivial line bundles so that $s_i \in S$. Further, by shrinking X we can assume that the morphism $X \to \mathfrak{Y}$ is defined by trivial bundles on X. Because the map ϕ is finite we can write $X = \mathrm{Spec}(T)$. Here T and S are local regular Noetherian k-algebras, T is a finite S-module, s_i is part of a regular system of parameters and there are elements $t_i \in T$, such that $t_i^{r_i} = s_i$.

We may check étaleness after a faithfully flat base extension of the base field and hence may assume that the ground field k contains r_i-th roots of unity for all $1 \leq i \leq n$.

Using (2.2.12) the stack $\mathfrak{Y}$ is isomorphic to the quotient stack

$$[\mathrm{Spec}(S')/\mu_{r_1} \times \cdots \times \mu_{r_n}],$$

where $S' = S[y_1, \ldots, y_n]/(y_1^{r_1} - s_1, \ldots, y_n^{r_n} - s_n)$.

We want to show that the map $\mathrm{Spec}(T) \to [\mathrm{Spec}(S')/\mu_{r_1} \times \cdots \times \mu_{r_n}]$ is étale. Denote by T' the ring $T[x_1, \ldots, x_n]/(x^{r_1} - 1, \ldots, x^{r_n} - 1)$. Using (2.2.12) again we have see that we have a Cartesian diagram :

$$\begin{tikzcd}
\mathrm{Spec}(T') \arrow[r] \arrow[d] & \mathrm{Spec}(S') \arrow[d] \\
\mathrm{Spec}(T) \arrow[r] & {[\,\mathrm{Spec}(S')/\mu_{r_1} \times \cdots \times \mu_{r_n}]}
\end{tikzcd}$$

Because $\mathrm{Spec}(S')$ is a presentation of a quotient stack it is enough to show that the map $S' \to T'$ given by $y_i \mapsto t_i x_i$ is étale.

The morphism $S_{s_1 \ldots s_n} \to T_{t_1 \ldots t_n}$ is flat and unramified by assumption, hence it is étale. By Abhyankar's lemma, (4.2.4), this morphism extends after base change to an étale cover of S'. By the proof of Abhyankar's lemma it suffices to show that T' is normal and the map $S' \to T'$ is integral. Both of these facts are easily checked and the result follows. □

For a point $p \in Y$ we define

$$I(p, Y) = \prod_{p \in \mathrm{supp}(E_i)} \mu_{r_i}.$$

Proposition 6.2.7. *Let K be a field and consider the morphism of K-points*

$$\pi_K : X \times_{\mathfrak{Y}} X(K) \to X \times_Y X(K).$$

The fiber $\pi_K^{-1}(x_1, x_2)$ over a K-point (x_1, x_2) is a bitorsor under the inertia group $I(\phi(x_1), Y)$.

Proof. In what follows, we will use the shorthand G^* when we mean $\sum_{g \in G} g^*$. Recall that the morphism ψ is defined by $(\mathcal{O}(G^* E_i), s_{G^* E_i}, \alpha_i)$ where α_i are isomorphisms coming from the fact that

$$r_i G^* E_i = r_i \phi^*(D_i).$$

The fiber over (x_1, x_2) is exactly the set of isomorphisms from $x_1^* \mathcal{O}(G^* E_i)$ to $x_2^* \mathcal{O}(G^* E_i)$ as i varies. As in (6.1.1) this depends on whether the section $x_1^* s_{G^* E_i}$ vanishes or not. The vanishing condition precisely depends on $\phi(x_1)$ and the result follows. □

Theorem 6.2.8. *If the assumption 6.2.1 is satisfied and if additionally the ramification divisor D is a normal crossing divisor then we have the isomorphism of stacks $[X/G] \cong \mathfrak{Y}$.*

Proof. To prove this all we need to show is that the map

$$\chi : \quad X \times G \to X \times_{\mathfrak{Y}} X$$

$$(x, g) \mapsto (x, gx)$$

is an isomorphism.

Using (6.2.6), the map $\psi : X \to \mathfrak{Y}$ is étale, and so the map $X \times_{\mathfrak{Y}} X \to X$ is étale as a pullback. Clearly the two maps $X \times G \to X$ given by $(x, g) \mapsto x$ and $(x, g) \mapsto gx$ are étale and so the map χ must be étale.

We are going to show that the map

$$\chi(K) : X(K) \times G \to X \times_{\mathfrak{Y}} X(K)$$

is bijective for any field extension of the ground field $k \subset K$. The points of the scheme on the left is a pair (x, g), where $g \in G$ and $x : \mathrm{Spec}(K) \to X$ a K-point.

Consider the morphism $\Psi : X \times G \to X \times_Y X$. This morphism is surjective as we have a geometric quotient, see [FKM, Definition 0.4]. Consider a K-point $(x_1, x_2) \in X \times_Y X(K)$. Using the properties of geometric quotients we have that $x_2 = gx_1$ for some $g \in G$. Using this we see the fiber $\Psi^{-1}(x_1, x_2)$ is a torsor over the inertia group $I(\mathrm{supp}(x_1), G)$. Under our assumptions by Theorem 5.1.2 the inertia groups are generated in codimension one we see that we have an identification

$$I(\mathrm{supp}(x_1), G) = \mu_{r_{i_1}} \times \ldots \times \mu_{r_{i_l}}$$

as in the previous proposition. It follows that the morphism χ is étale and universally injective (radicial). This implies that it is an open immersion. As it is also surjective it is an isomorphism and the result follows. $\square$

Chapter 7

The K-theory of a root stack

In this chapter we will describe G-theory and K-theory of a root stack.

7.1 Localization via Serre subcategories

Let $\mathbf{A}$ be an abelian category. Recall that a *Serre subcategory* S of $\mathbf{A}$ is a non-empty full subcategory that is closed under extensions, subobjects and quotients. When $\mathbf{A}$ is well-powered the quotient category $\mathbf{A}/S$ exists, see [S, pg. 44, Theorem 2.1]. Let's recall that "well-powered" means that the subobjects of each object $a \in \mathbf{A}$ can be indexed by a small set.

We will need the following result to identify quotient categories.

Theorem 7.1.1. *Let $F : \mathbf{A} \to \mathbf{B}$ be an exact functor between abelian categories. Denote by S the full subcategory whose objects are x with $F(x) \cong 0$. Then S is a Serre subcategory and we have a factorisation*

$$\begin{tikzcd} \mathbf{A} \arrow[r] \arrow[d, "F"] & \mathbf{A}/S \arrow[dl] \\ \mathbf{B} & \end{tikzcd}$$

Proof. See [S, page 114] ◻

Definition 7.1.2. The category S is called the *kernel of the functor F* and is denoted by $\mathbf{ker}(F)$.

Theorem 7.1.3. *In the situation of the previous theorem suppose that we have*

1. *for every object $y \in \mathbf{B}$ there is a $x \in \mathbf{A}$ such that $F(x)$ is isomorphic to y and*

2. *for every morphism $f : F(x) \to F(x')$ there is $x'' \in \mathbf{A}$ with $h : x'' \to x$ and $g : x'' \to x'$ such that $F(h)$ is an isomorphism and the following diagram commutes*

$$\begin{tikzcd} F(x'') \arrow[d, "F(h)"'] \arrow[dr, "F(g)"] & \\ F(x) \arrow[r, "f"] & F(x'). \end{tikzcd}$$

Then there is an equivalence of categories $\mathbf{A}/S \cong \mathbf{B}$.

Proof. See [S, pg. 114, theorem 5.11]. □

Consider n-tuples of integers $\vec{r} = (r_1, r_2, \ldots, r_n)$ and $\vec{s} = (s_1, s_2, \ldots, s_n)$. We denote by $[\vec{r}, \vec{s}]$ the poset of n-tuples $(x_1, \ldots, x_n)$ with

$$x_i \in \mathbb{Z} \quad \text{and} \quad r_i \leq x_i \leq s_i.$$

We will make use of the following shorthand notation :

$$rI = [0, r] \quad \text{and} \quad \vec{r}I^n = [0, \vec{r}].$$

These intervals are naturally posets with

$$(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n) \quad \text{if and only if} \quad x_i \leq y_i \text{ for all } i.$$

This poset structure allows us to view them as categories in the usual way.

Fix an abelian category $\mathbf{A}$ and consider the functor category

$$\mathrm{Func}(\vec{r}I^n, \mathbf{A}).$$

This category is an abelian category with kernels, cokernels formed pointwise. We will be interested in the K-theory of such categories. In this subsection we will try to understand some of their quotient categories. Given an object $\mathcal{F}$ in this category and an object u of $\vec{r}I^n$ we denote by $\mathcal{F}_u \in \mathbf{A}$ the value of the functor $\mathcal{F}$ on this object and if $u \leq v$ the arrow from F_u to F_v will be denoted by

$$F_{+(v-u)} : F_u \to F_v.$$

In particular, we take $e_i = (0, 0, \dots, 1, 0, \dots, 0)$ to be a standard basis vector so that we have a morphism

$$F_{+e_i} : F_{(u_1,\dots,u_n)} \to F_{u_1,\dots,u_{i-1},u_i+1,u_{i+1}\dots,u_n}.$$

Lemma 7.1.4. *To give an object F of* $\mathrm{Func}(\vec{r}I^n, \mathbf{A})$ *is the same as providing the following data :*

(D1) objects $F_{(u_1,u_2,\dots,u_n)} \in \mathbf{A}$

(D2) arrows

$$F_{+e_i} : F_u \to F_{u+e_i},$$

such that all diagrams of the form

$$\begin{tikzcd} F_u \arrow[r] \arrow[d] & F_{u+e_j} \arrow[d] \\ F_{u+e_i} \arrow[r] & F_{u+e_i+e_j} \end{tikzcd}$$

Proof. The hypothesis insure that if $u \leq v$ in $\vec{r}I^n$ then there is a well defined map $F_u \to F_v$ which produces our functor. □

Proposition 7.1.5. *(i) Let* $\mathrm{tr}_{n-1}(\vec{r}) = (r_1, r_2, \dots, r_{n-1})$. *There is an exact functor*

$$\pi : \mathrm{Func}(\vec{r}I^n, \mathbf{A}) \to \mathrm{Func}(\mathrm{tr}_{n-1}(\vec{r})I^{n-1}, \mathbf{A})$$

defined on objects by

$$\pi(G)_{(u_1,u_2,\dots,u_{n-1})} = (G)_{(u_1,\dots,u_{n-1},0)}$$

(ii) The functor π has a left adjoint denoted π^*. We have $\pi \circ \pi^* \simeq 1$.

(iii) The functor π^* is fully faithful.

Proof. (i) There is an inclusion functor $\mathrm{tr}_{n-1}(\vec{r})I^{n-1} \hookrightarrow \vec{r}I^n$ defined by

$$(x_1, x_2, \ldots, x_{n-1}) \mapsto (x_1, x_2, \ldots, x_{n-1}, 0).$$

The functor π is just the restriction along this inclusion. The exactness follows from the fact that in functor categories, limits and colimits are computed pointwise.

(ii) Given $F \in \mathrm{Func}(\mathrm{tr}_{n-1}(\vec{r})I^{n-1}, \mathbf{A})$, we need to construct an object $\pi^*(F) \in \mathrm{Func}(\vec{r}I^n, \mathbf{A})$. We set

$$\pi^*(F)_{(u_1,u_2,\ldots,u_n)} = F_{(u_1,u_2,\ldots,u_{n-1})}.$$

To produce a functor, we need maps

$$\lambda^i_{(u_1,\ldots,u_n)} : \pi^*(\mathcal{F})_{(u_1,\ldots,u_i,\ldots,u_n)} \to \pi^*(F)_{(u_1,\ldots,u_i+1,\ldots,u_n)}$$

We define

$$\lambda^i_{(u_1,\ldots,u_n)} = \begin{cases} F_{(u_1,\ldots,u_i,\ldots,u_{n-1})} \to F_{(u_1,\ldots,u_i+1,\ldots,u_{n-1})} & \text{if } i < n \\ \text{identity} & \text{if } i = n. \end{cases}$$

One checks that the hypothesis of (7.1.4) are satisfied. Observe that $\pi \circ \pi^* = 1$. This produces a natural map

$$\mathrm{Hom}(\pi^*(F), G) \to \mathrm{Hom}(F, \pi(G)).$$

To see that this is a bijection, suppose that we are given a morphism $\beta : F \to \pi(G)$. There is a diagram, where the dashed arrow is defined to be the composition,

$$\begin{tikzcd} \pi^*(F)_{(u_1,\ldots,u_n)} \arrow[r, dashed] \arrow[d, equal] & G_{(u_1,\ldots,u_n)} \\ F_{(u_1,\ldots,u_{n-1})} \arrow[r, "\beta"'] & G_{(u_1,\ldots,u_{n-1},0)} \arrow[u] \end{tikzcd}$$

This produces a natural morphism

$$\mathrm{Hom}(\pi^*(F), G) \leftarrow \mathrm{Hom}(F, \pi(G))$$

and we check that it is inverse to the previous map.

(iii) We have

$$\mathrm{Hom}(\pi^*(F), \pi^*(F')) = \mathrm{Hom}(F, \pi\pi^*(F')) = \mathrm{Hom}(F, F').$$

$\square$

Theorem 7.1.6. *1. The functor*

$$\pi : \mathrm{Func}(\vec{r}I^n, \mathbf{A}) \to \mathrm{Func}(\mathrm{tr}_{n-1}(\vec{r})I^{n-1}, \mathbf{A})$$

satisfies the hypothesis of (7.1.3).

2. *Let* $\vec{s} = (r_1, r_2, \ldots, r_{n-1}, r_n - 1)$. *If* $r_n > 0$ *then the kernel of this functor is equivalent to* $\mathrm{Func}(\vec{s}I^n, \mathbf{A})$.

3. *If* $r_n = 0$ *then there is an equivalence of categories*

$$\mathrm{Func}(\vec{r}I^n, \mathbf{A}) \cong \mathrm{Func}(\mathrm{tr}_{n-1}(\vec{r})I^{n-1}, \mathbf{A}).$$

Proof. (1) The functor π is exact so it remains to check the two conditions of the theorem. The first condition follows from the fact that $\pi \circ \pi^*$ is the identity. Now suppose that we have a morphism $\pi(F) \to \pi(F')$. By adjointness we obtain a diagram

$$\begin{tikzcd} \pi^*\pi(F) \arrow[d] \arrow[dr] & \\ F & F' \end{tikzcd}$$

Applying π to this picture shows that the second condition holds.

(2) The functor π was defined on objects by the rule $\pi(G)_{(u_1,u_2,\ldots,u_{n-1})} = (G)_{(u_1,\ldots,u_{n-1},0)}$. So it is clear that if $\pi G \cong 0$ then $(G)_{(u_1,\ldots,u_{n-1},0)} \cong 0$ and to give an object G of $\mathbf{ker}\,\pi$ is the same (up

to isomorphism) as giving the objects $(G)_{(u_1,\dots,u_n)} \in \mathbf{A}$ for all $u \in \vec{r}I^n, u_n \neq 0$. And according to Lemma 7.1.4 it is the same as providing an object of the category $\mathrm{Func}(\vec{s}I^n, \mathbf{A})$

(3) If $r_n = 0$ then we have an equivalence of categories $\mathrm{tr}_{n-1}(\vec{r}) \cong \vec{r}$.

□

7.2 Extension Lemma

We want to slightly simplify the formulation of a parabolic sheaf in the present context using the pseudo-periodicity condition. We let

$$e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^n,$$

where the 1 is in the ith spot.

Definition 7.2.1. Recall that L is a symmetric monoidal functor

$$L : \vec{r}\mathbb{Z}^n \to \mathfrak{D}iv_X,$$

determined by n divisors (L_i, s_i). An *extendable pair* (F, ρ) on (X, L) with denominators $\vec{r}$ consists of the following data:

(a) A functor $F_\bullet : \vec{r}I^n \to \mathfrak{QCoh}(X)$.

(b) For any $\alpha \in \vec{r}I^n$ such that $\alpha_i = r_i$, an isomorphism of $\mathcal{O}_X$-modules

$$\rho_{\alpha,\alpha-r_ie_i} : F_\alpha \xrightarrow{\sim} L_i \otimes F_{\alpha-r_ie_i}.$$

We will frequently drop the subscripts from the notation involving ρ, when they are clear from the context.

This data is required to satisfy the following three conditions :

(EX1) For all $i \in \{1, \dots, n\}$ and $\alpha \in \vec{r}I^n$ the following diagram commutes

$$\begin{tikzcd}
F_\alpha \arrow[r, "F_{+(r_i-\alpha_i)\vec{e}_i}"] \arrow[d, "\sigma_i"] & F_{\alpha+(r_i-\alpha_i)e_i} \arrow[d, "\rho"] \\
L_i \otimes F_\alpha & L_i \otimes F_{\alpha-\alpha_i\vec{e}_i} \arrow[l, "L_i \otimes F_{+\alpha_i\vec{e}_i}"]
\end{tikzcd}$$

where σ_i is multiplication by the section s_i

(EX2) For all $i \neq j$ and α with $\alpha_i = r_i$ the following diagram commutes

$$\begin{tikzcd}
F_\alpha \arrow[r, "\rho"] \arrow[d, "F_{\vec{e}_j}"'] & L_i \otimes F_{\alpha-r_ie_i} \arrow[d, "F_{\vec{e}_j}"] \\
F_{\alpha+e_j} \arrow[r, "\rho"] & L_i \otimes F_{\alpha+e_j-r_ie_i}
\end{tikzcd}$$

(EX3) For all i and j and $\alpha \in \vec{r}I^n$ with $\alpha_i = r_i$ and $\alpha_j = r_j$ the following diagram commutes

$$\begin{tikzcd}
F_\alpha \arrow[r, "\rho"] \arrow[d, "\rho"'] & L_i \otimes F_{\alpha-r_ie_i} \arrow[d, "\rho"] \\
L_j \otimes F_{\alpha-r_je_j} \arrow[r, "\rho"] & L_i \otimes L_j \otimes F_{\alpha-r_ie_i-r_je_j}
\end{tikzcd}$$

Definition 7.2.2. An extendable pair (F,ρ) is called *coherent* if for each $v \in \vec{r}I^n$ the sheaf F_v is a coherent sheaf on X.

Proposition 7.2.3. *Let (E,ρ) be a parabolic sheaf on (X,L) with denominators $\vec{r}$. Then the restricted functor $E|_{\vec{r}I^n}$ produces an extendable pair on (X,L).*

Proof. Note that the restricted functor has all the required data for an extendable pair by restricting the collection $\rho_{\alpha,\beta}$. We need to check that the axioms of an extendable pair are satisfied.

(1) We have that the composition

$$E_{\alpha+(r_i-\alpha_i)e_i} \xrightarrow{\rho} E_{\alpha-\alpha_ie_i} \otimes L_i \to E_\alpha \otimes L_i \xrightarrow{\rho^{-1}} E_{\alpha+r_ie_i}$$

is just the morphism $E_{+\alpha_i e_i}$ using axiom (ii) of parabolic sheaf. Precomposing with the map

$$E_{+(r_i-\alpha_i)e_i} : E_\alpha \to E_{\alpha+(r_i-\alpha_i)e_i}$$

gives the morphism $E_{+r_i e_i}$. The result now follows from axiom (i).

(2) This follows directly from axiom (ii).

(3) This follows directly from axiom (iii). □

Proposition 7.2.4. *Given an extendable pair (F,ρ) we can extend it to a parabolic sheaf $(\hat{F},\rho)$ and the extension is unique up to a canonical isomorphism. A coherent extendable pair extends to a coherent parabolic sheaf.*

Proof. For $v \in \mathbb{Z}^n$ we need to define its extension $\hat{F}_v$. We can write $v_i = r_i u_i + q_i$ with $0 \le q_i < r_i$ and $u_i \in \mathbb{Z}$. As before we denote $L_u = \otimes_{i=1}^n L^{\otimes u_i}$ and $q = (q_1, \ldots, q_n)$. Set $\hat{F}_v = L_u \otimes F_q$.

We need to construct maps $\hat{F}_{+e_i} : \hat{F}_v \to \hat{F}_{v+e_i}$. If $q_i < r_i - 1$ then the map is obtained by tensoring the map $F_q \to F_{q+e_i}$ with L_u. If $q_i = r_i - 1$ then the map is defined by

$$\begin{tikzcd}
\hat{F}_v = L_u \otimes F_q \arrow[rr, "\hat{F}_{e_i}"] \arrow[dr, "1 \otimes F_{e_i}"'] & & \hat{F}_{v+e_i} = L_u \otimes L_i \otimes F_{q'} \\
& L_u \otimes F_{q+e_i} \arrow[ur, "1 \otimes \rho"'] &
\end{tikzcd}$$

where $q'_j = q_j$ for all $j \neq i$ and $q'_i = 0$.

In order to show that the construction above indeed produces a functor we need to show that all diagrams of (7.1.4) commute. If both $q_i < r_i - 1$ and $q_j < r_j - 1$ then this is straightforward. Suppose that $q_i = r_i - 1$ and $q_j < r_j - 1$ then this follows from (EX2). This leaves the case $q_i = r_i - 1$ and $q_j = r_j - 1$. We have a diagram

$$\begin{tikzcd}
L_u \otimes F_q \arrow[r] \arrow[d] & L_u \otimes F_{q+e_i} \arrow[r] \arrow[d] & L_u \otimes L_i \otimes F_{q-q_ie_i} \arrow[d] \\
L_u \otimes F_{q+e_j} \arrow[r] \arrow[d] & L_u \otimes F_{q+e_i+e_j} \arrow[r] \arrow[d] & L_u \otimes L_i \otimes F_{q-q_ie_i+e_j} \arrow[d] \\
L_u \otimes L_j \otimes F_{q-q_je_j} \arrow[r] & L_u \otimes L_j \otimes F_{q-q_je_j+e_i} \arrow[r] & L_u \otimes L_i \otimes L_j \otimes F_{q-q_ie_i-q_je_j}
\end{tikzcd}$$

The top left square commutes using the fact that F is a functor. The top right and bottom left squares commute using axiom (EX2). The bottom right square commutes using axiom (EX3). So indeed $\hat{F}_\bullet$ is a functor.

Note that we have canonical isomorphisms $L_u \otimes L_v \cong L_{u+v}$ for $u, v \in \vec{r}\mathbb{Z}$. These isomorphisms induce our pseudo-period isomorphisms.

Finally we need to check the conditions (i) to (iv) of a parabolic sheaf.

Condition (i): For $\vec{r}\alpha, \vec{r}\alpha' \in \vec{r}\mathbb{N}^n$ the following diagram commutes

$$\begin{tikzcd}
\hat{F}_v \arrow[r, "\hat{F}_{+\vec{r}\alpha}"] & \hat{F}_{v+\vec{r}\alpha} \arrow[r, "\hat{F}_{+\vec{r}\alpha'}"] \arrow[d] & \hat{F}_{v+\vec{r}\alpha+\vec{r}\alpha'} \arrow[d] \\
 & L_\alpha \otimes \hat{F}_v \arrow[r] & L_\alpha \otimes L_{\alpha'} \otimes \hat{F}_v
\end{tikzcd}$$

This follows by the definition of the functor $\hat{F}_\bullet$ and the symmetric monoidal structure of L.

This allows us to make the following reduction: in order to check axiom (i) it suffices to check that the following diagram commutes

$$\begin{tikzcd}
\hat{F}_v \arrow[r] \arrow[dr, "\sigma_i"'] & \hat{F}_{v+r_ie_i} \arrow[d, "\rho"] \\
 & L_i \otimes \hat{F}_v
\end{tikzcd}$$

And this follows directly from (EX1).

Condition (ii): Once again we reduce to showing that

$$\begin{tikzcd} \hat{F}_{v+r_ie_i} \arrow[r] \arrow[d, "\hat{F}_{+b}"] & L_i \otimes \hat{F}_v \arrow[d, "L_i \otimes \hat{F}_{+b}"] \\ \hat{F}_{v+b+r_ie_i} \arrow[r] & L_i \otimes \hat{F}_{v+b} \end{tikzcd}$$

commutes. If we write $v = \vec{r}u + q$ then this diagram will become

$$\begin{tikzcd} L_{u+e_i} \otimes F_q \arrow[r] \arrow[d, "L_{u+e_i} \otimes \hat{F}_{+b}"] & L_i \otimes (L_u \otimes F_q) \arrow[d, "L_i \otimes L_u \otimes \hat{F}_{+b}"] \\ L_{u+e_i} \otimes \hat{F}_{q+b} \arrow[r] & L_i \otimes (L_u \otimes \hat{F}_{q+b}) \end{tikzcd}$$

We can use the symmetric monoidal structure of L to show that this diagram indeed commutes.

Condition (iii): We reduce to showing the commutativity of the following diagram

$$\begin{tikzcd} \hat{F}_{v+r_ie_i+r_je_j} \arrow[r] \arrow[d] & L_i \otimes \hat{F}_{v+r_je_j} \arrow[d] \\ L_j \otimes \hat{F}_{v+r_ie_i} \arrow[r] & L_i \otimes L_j \otimes \hat{F}_v \end{tikzcd}$$

which follows from the monoidal structure of L.

Condition (iv) is by definition.

Finally, let $E_\bullet$ be another extension of $F_\bullet$. Again we can again write $v_i = r_iu_i + q_i$ with $0 \leq q_i < r_i$ and $u_i \in \mathbb{Z}$. By pseudo-periodicity, $E_v \simeq L(u) \otimes E_q$, and $F_q = E_q$ because $E_\bullet$ is an extension. So, $E_v \cong \hat{F}_v$ for any $v \in \mathbb{Z}^n$.

It is clear from the construction that the finitely generated condition is preserved under extension. □

Corollary 7.2.5. *The category of parabolic sheaves (coherent parabolic sheaves) on (X, L) with denominators $\vec{r}$ is equivalent to the category of extendable pairs (resp. coherent extendable pairs) on (X, L) with denominators $\vec{r}$.*

Proof. There is a pair of functors between these categories. The truncation functor sends a parabolic sheaf (E, ρ) to an extendable pair by forgetting all E_v when $v \notin \vec{r}I^n$. The extension functor from extendable pairs to parabolic sheaves was defined in the previous Proposition on objects by the rule $F_\bullet \mapsto \hat{F}_\bullet$. It is easy to see that these functors are mutually inverse and preserve the finitely-generation condition. □

Remark 7.2.6. We will denote the category of coherent extendable pairs by $\mathcal{EP}(X, L, \vec{r})$.

7.3 The localization sequence

In this section we will localize the category of finitely-generated extendable pairs so that it will be glued from simpler parts.

First let us consider the functor $\pi_*^{L,\vec{r}} : \mathcal{EP}(X, L, \vec{r}) \to \mathfrak{Coh}X$, given by $F_\bullet \mapsto F_0$ on objects. It is an exact functor because exact sequences in diagram categories are defined point-wise.

Lemma 7.3.1. *The functor $\pi_*^{L,\vec{r}}$ has a left adjoint denoted $\pi^*_{L,\vec{r}}$ and there is a natural isomorphism $\pi_*^{L,\vec{r}} \circ \pi^*_{L,\vec{r}} \simeq 1$.*

Proof. In what follows, we will omit, the superscripts (resp. subscripts) L and $\vec{r}$ in the notation for the appropriate functors. For any $0 \le i \le n$ consider functions $\epsilon_i : \vec{r}I \to \{0, 1\}$, defined by $\epsilon_i(u) = 1$ if $u_i = r_i$ and zero otherwise. We define the functor π^* on a sheaf $F \in \mathfrak{Coh}X$ by the rule:

$$(\pi^*(F))_u = (\otimes_{i=1}^n L_i^{\epsilon_i(u)}) \otimes F.$$

This forms a functor via the maps

$$(\pi^*(F))_u \to (\pi^*(F))_{u+e_i} = \begin{cases} \text{identity} & \text{if } u_i \in [0, r_i - 2] \\ \sigma_i & \text{if } u_i = r_i - 1, \end{cases}$$

where σ_i is the multiplication by the section s_i.

Define ρ to be identity map. It is easy to see that all axioms of extendable pair are satisfied.

Now let's take a coherent sheaf F and an extendable pair $E_\bullet$ and consider a map

$$\mathrm{Hom}_{\mathfrak{Coh}X}(F, \pi_* E) \to \mathrm{Hom}_{\mathcal{EP}}(\pi^* F, E)$$

given by sending $\phi \in \mathrm{Hom}_{\mathfrak{Coh}X}(F, \pi_* E)$ to precomposition of the structure maps of the extendable pair E with ϕ. It's obviously an injection. Surjectivity will follow from commutativity of the squares in $\mathrm{Hom}_{\mathcal{EP}}(\pi^* F, E)$ and because all structure maps in $\pi^* F$ are identity.

$\square$

Proposition 7.3.2. *The functor $\pi_*^{L,\vec{r}} : \mathcal{EP}(X, L, \vec{r}) \to \mathfrak{Coh}X$ satisfies the hypothesis of (7.1.3).*

Proof. The only thing which is not completely obvious is the second condition. Consider two extendable pairs $E_\bullet$ and $F_\bullet$. Suppose that we have a morphism $\pi_*(E_\bullet) \to \pi_*(F_\bullet)$. By adjointness we obtain a diagram

$$\begin{tikzcd} \pi^*\pi_*(E_\bullet) \arrow[d] \arrow[dr] & \\ E_\bullet & F_\bullet \end{tikzcd}$$

Applying π to this picture shows that the second condition holds. $\square$

Using the Theorem 7.1.3 we obtain the following

Corollary 7.3.3. *There is an equivalence of abelian categories:*

$$\mathcal{EP}(X, L, \vec{r})\big/\mathbf{ker}(\pi_*^{L,\vec{r}}) \to \mathfrak{Coh}X,$$

In the rest of this subsection we would like to give a description of the category $\mathbf{ker}(\pi_*^{L,\vec{r}})$. Let us study the objects first. Let $F_\bullet$ be an extendable pair. Then $\pi_*(F_\bullet) = F_0$, and if $F_\bullet \in \mathbf{ker}(\pi_*^{L,\vec{r}})$ then $F_0 \cong 0$. The pseudo-period isomorphism imply in turn that $F_u \cong 0$ if all $u_i \in \{0, r_i\}$.

Let us consider the sheaves F_u such that, for any $j \neq i$, $u_j \in \{0, r_j\}$ (we can imagine them as sheaves on the edges of the cubical diagram $F_\bullet \in \mathrm{Func}(\vec{r}I^n, \mathbf{A})$). Using the axiom (EX 1) we get that the multiplication by section map $s_i : F_u \to L_i \otimes F_u$ must factor through $F_{u+(r_i-u_i)e_i}$ which is a zero sheaf if $F_\bullet \in \mathbf{ker}(\pi_*^{L,\vec{r}})$. This implies the following:

Lemma 7.3.4. *If $F_\bullet \in \mathbf{ker}(\pi_*^{L,\vec{r}})$ and $u \in \vec{r}I^n$ is such that $\forall j \neq i$, $u_j \in \{0, r_j\}$, then* $\mathrm{supp}(F_u)$ *is contained in the divisor of zeroes of the section $s_i \in H^0(L_i)$.*

If $s_i = 0$ for some i, we will say that $\mathrm{div}(s_i) = X$.

We will apply the localization method (7.1.3), to this partial description of the kernel.

Let's fix some notation. Denote by

$$S(k) = \{T \subset \{1, \dots, n\} \mid |T| = k\}.$$

We will view each interval $[0, r_i]$ as a pointed set, pointed at 0. It follows that we have order preserving inclusions

$$\iota_T : \prod_{i \in T} [0, r_i] \to \prod_{i=1}^{n} [0, r_i] := \vec{r} I^n.$$

Ignoring the pointed structure produces order preserving ($\leq$) projection maps

$$\pi_T : \vec{r} I^n \to \prod_{i \in T} [0, r_i].$$

Definition 7.3.5. Assume that $(L_1, s_1), \dots, (L_n, s_n)$ are objects of $\mathfrak{Div} X$ and $L : \mathbb{N}^n \to \mathfrak{Div} X$ is the corresponding symmetric monoidal functor as in section 3.1.

If $1 \leq k \leq n$ and $T \in S(k)$ then we will define a symmetric monoidal functor $L_T : \mathbb{N}^k \to \mathfrak{Div} X$ as a composition:

$$\mathbb{N}^k \xrightarrow{\iota_T} \mathbb{N}^n \xrightarrow{L} \mathfrak{Div} X$$

We will say that L_T is obtained from L by the *restriction along ι_T*.

Now for $T \in S(k)$ let's consider the functor

$$\iota_T^* : \mathcal{EP}(X, L, \vec{r}) \longrightarrow \mathcal{EP}(X, L_T, \pi_T(\vec{r}))$$

which is the restriction of an extendable pair $F_\bullet$ along the inclusion ι_T. The pseudo-period isomorphism is just obtained by restriction.

Definition 7.3.6. For any $1 \leq k \leq n$ we define functors

$$\mathrm{Face}^k := \prod_{T \in S(k)} \iota_T^* : \mathcal{EP}(X, L, \vec{r}) \longrightarrow \prod_{T \in S(k)} \mathcal{EP}(X, L_T, \pi_T(\vec{r}))$$

Definition 7.3.7. For any $1 \leq k \leq n$ we denote by $\mathbf{ker}^k = \mathbf{ker}(\mathrm{Face}^k)$.

Also denote $\mathbf{ker}^0 = \mathbf{ker}(\pi_*)$.

Lemma 7.3.8. *For any $1 \leq k \leq n$, any $F_\bullet \in \mathbf{ker}^{k-1}$ and any $T \in S(k)$ we can consider $(\iota_T^*(F_\bullet))_\bullet$ as an element of*

$$\mathrm{Func}(\prod_{i \in T}[1, r_i - 1],\ \ \mathfrak{Coh}(\cap_{i \in T} \operatorname{div}(s_i))).$$

As in Lemma 7.3.4 we will say that if $s_i = 0$, then $\operatorname{div}(s_i) = X$.

Proof. If $k = 1$ then the result is proved in the Lemma 7.3.4 and the observation before it.

Let's take any $2 \leq k \leq n$ and an extendable pair $F_\bullet \in \mathbf{ker}^{k-1}$.

If we consider an extendable pair $(\iota_T^*(F_\bullet))_\bullet \in \mathcal{EP}(X, L_T, \pi_T(\vec{r}))$ then for any $v \in \prod_{i \in T}[0, r_i]$ we will have isomorphisms of sheaves: $(\iota_T^*(F_\bullet))_v \cong 0$, whenever $v_i = 0$ for some $i \in T$. Because of the pseudo-periodicity isomorphism we also have that $(\iota_T(F_\bullet))_v \cong 0$, whenever $v_i = r_i$ for some $i \in T$.

The last step is an application of the axiom EX1 to the extendable pair $(\iota_T^*(F_\bullet))_\bullet$. Because $(\iota_T^*(F_\bullet))_v \cong 0$ if $v_i = r_i$ for some $i \in T$ that implies that for any $w \in \prod_{i \in T}[1, r_i - 1]$ the multiplication of the sheaf $(\iota_T^*(F_\bullet))_w$ by the sections $s_i \in H^0(X, L_i)$ for all $i \in T$ must factor through zero. So the support of the sheaf $(\iota_T^*(F_\bullet))_w$ is contained in $\cap_{i \in T} \operatorname{div}(s_i)$.

□

Lemma 7.3.9. *If we restrict the domain of the functor Face^k to the full subcategory $\mathbf{ker}^{k-1}$ for any $1 \leqslant k \leqslant n$, then we will obtain functors:*

$$\mathrm{Face}^k \big|_{\mathbf{ker}^{k-1}} : \mathbf{ker}^{k-1} \longrightarrow \prod_{T \in S(k)} \mathrm{Func}(\prod_{i \in T}[1, r_i - 1],\ \ \mathfrak{Coh}(\cap_{i \in T} \operatorname{div}(s_i))).$$

There is an equivalence of categories $\mathbf{ker}^k$ and $\mathbf{ker}(\mathrm{Face}^k \big|_{\mathbf{ker}^{k-1}})$.

Proof. The first part follows directly from the Lemma before. The proof of the second part is straightforward and follows from the fact that $\mathbf{ker}^k$ is a full subcategory of $\mathbf{ker}^{k-1}$. □

Remark 7.3.10. In order to apply localization procedure to the category $\mathbf{ker}^{k-1}$ we need to show that the functor $\mathrm{Face}^k \big|_{\mathbf{ker}^{k-1}}$ has a left adjoint. The existence of a left adjoint follows from special adjoint functor theorem. Indeed, the category of extendable pairs is equivalent to a locally-presentable subcategory of a category of sheaves on a root stack. Being a reflexive subcategory, the category $\mathbf{ker}^{k-1}$ has small hom-sets and is well-powered. Because limits and morphisms in $\mathcal{EP}(X, L, \vec{r})$ are defined point-wise, it follows that the category $\mathbf{ker}^{k-1}$ has small limits and a small cogenerating set, and also the restriction ι_T^* preserves small limits.

But for the purpose of splitting of the corresponding short exact sequence of K-groups (see section 7.4 for details) we need the unit of the adjunction to be the natural isomorphism. This doesn't follow from abstract nonsense, so we need an explicit construction of a left adjoint functor. It is given in the following theorem.

Theorem 7.3.11. *(i) For any $1 \leq k \leq n$ there is an exact functor*

$$\mathrm{Face}^k\big|_{\mathbf{ker}^{k-1}} : \mathbf{ker}^{k-1} \longrightarrow \prod_{T \in S(k)} \mathrm{Func}(\prod_{i \in T}[1, r_i - 1],\ \mathfrak{Coh}(\cap_{i \in T} \operatorname{div}(s_i))),$$

where $\mathbf{ker}^k$ is a kernel of the functor Face^k and $\mathbf{ker}^0 := \mathbf{ker}(\pi_*^{L,\vec{r}})$

(ii) The functors $\mathrm{Face}^k\big|_{\mathbf{ker}^{k-1}}$ have left adjoints D^k such that

$$\mathrm{Face}^k\big|_{\mathbf{ker}^{k-1}} \circ D^k \simeq 1$$

.

(iii) $\mathrm{Face}^k\big|_{\mathbf{ker}^{k-1}}$ satisfies the condition of 7.1.3

(iv) The functor

$$\mathrm{Face}^n\big|_{\mathbf{ker}^{n-1}} : \mathbf{ker}^{n-1} \longrightarrow \mathrm{Func}(\prod_{i=1}^{n}[1, r_i - 1],\ \mathfrak{Coh}(\cap_{i=1}^{n} \operatorname{div}(s_i)))$$

is an equivalence of categories.

Proof. (i) This follows from the fact that restriction or pullback functors are exact in general.

(ii) Given a functor $G_\bullet^T \in \mathrm{Func}(\prod_{i \in T}[1, r_i - 1],\ \mathfrak{Coh}(\cap_{i \in T} \operatorname{div}(s_i)))$ for each $T \in S(k)$, we will denote the corresponding object:

$$(G^T_\bullet)_{T\in S(k)} \in \prod_{T\in S(k)} \mathrm{Func}(\prod_{i\in T}[1, r_i - 1], \quad \mathfrak{Coh}(\cap_{i\in T}\operatorname{div}(s_i))).$$

Further we will view $G^T_\bullet$ as a functor $\prod_{i\in T}[0, r_i] \to \mathfrak{Coh}(\cap_{i\in T}\operatorname{div}(s_i))$ by taking $G^T_u = 0$ if for some $i \in T$ we have $u_i \in \{0, r_i\}$, where 0 is some fixed zero object in $\mathfrak{Coh}(X)$. Also for $i \in \{1, \ldots, k\}$ if $u_i \in \{0, r_i - 1\}$ we define the morphisms $G^T_{+e_i} : G^T_u \to G^T_{u+e_i}$ as the initial and terminal map correspondingly.

Let's recall the definition of ϵ from the Lemma 7.3.1. For any $0 \le i \le n$ we have functions $\epsilon_i : \vec{r}I \to \{0, 1\}$, such that for any $u \in \vec{r}I^n$: $\epsilon_i(u) = 1$ if $u_i = r_i$ and zero otherwise.

We define the functor D^k on objects as follows:

$$(D^k((G^T_\bullet)_{T\in S(k)}))_u = (\otimes_{i=1}^n L_i^{\epsilon_i(u)}) \otimes (\oplus_{T\in S(k)} G^T_{\pi_T(u)})$$

Let's denote $(D^k((G^T_\bullet)_{T\in S(k)}))_\bullet$ by $D^k_\bullet$ for simplicity of notation. First of all we want to view it as a functor $\vec{r}I^n \to \mathfrak{Coh}(X)$. For that we have to define the morphisms:

$$D^k_{+e_i} : D^k_u \to D^k_{u+e_i}.$$

If $0 \leqslant u_i < r_i - 1$, then this map is induced by $\oplus_{\substack{T\in S(k)\\ \text{s.t. } i\in T}} G^T_{+1}$. If $u_i = r_i - 1$, then it is induced by the terminal maps $\oplus_{\substack{T\in S(k)\\ \text{s.t. } i\in T}} G^T_{+1}$. and also by multiplication by the section s_i.

The pseudo-period isomorphisms ρ are defined by the symmetric monoidal structure of the functor L. The proof of the axioms EX2 and EX3 is automatic. And the proof of EX1 will follow from the comutativity of the diagram:

$$\begin{tikzcd}
D_u \arrow[r, "D_{+(r_i-u_i)e_i}"] \arrow[d, "\sigma_i"] & D_{u+(r_i-u_i)e_i} \arrow[d, "\rho"] \\
L_i \otimes D_u & L_i \otimes D_{u-u_ie_i}. \arrow[l, "L_i \otimes D_{+u_ie_i}"]
\end{tikzcd}$$

This diagram will commute because of the definition of $D_{+(r_i-\alpha_i)\vec{e}_i}$ and because $\operatorname{supp}(G^T_u) \subseteq \cap_{i\in T}\operatorname{div}(s_i)$ for any $u \in \prod_{i\in T}[0, r_i]$.

So we have shown that $D^k_\bullet$ is an extendable pair. If $k = 1$ then it's clear that $D^1_\bullet$ is in $\mathbf{ker}^0$, because $D^1_0 \cong 0$.

If $2 \leqslant k \leqslant n$, we want to see that $D^k_\bullet$ is in $\mathbf{ker}^{k-1}$. For that we have to see that for any $W \in S(k-1)$ and any $v \in \prod_{i\in W}[0, r_i]$ the sheaf $(\iota^*_W(D^k_\bullet))_v$ is isomorphic to zero. But this is true because for any $T \in S(k)$ we have that $G^T_u = 0$ if $u_i \in \{0, r_i\}$ for some $i \in T$.

Clearly, $\mathrm{Face}^k\big|_{\mathbf{ker}^{k-1}} \circ D^k = 1$.

Next we would like to show that D^k is indeed a left adjoint. Suppose that we have a morphism

$$(G^T_\bullet)_{T\in S(k)} \to \mathrm{Face}^k(F_\bullet).$$

Such a morphism consists of an $\binom{n}{k}$-tuple of morphisms

$$\phi_T : G^T_\bullet \to \iota^*_T(F_\bullet).$$

We wish to describe the adjoint map

$$\tilde{\phi} : D^k_\bullet \to F_\bullet.$$

Using the universal property of coproduct, this morphism is determined by maps

$$\tilde{\phi}(u)_T : \otimes^n_{i=1} L^{\epsilon(u)}_i \otimes G^T_{\pi_T(u)} \to F_u.$$

If u is such that $\epsilon_i(u) = 0$ for all $1 \leq i \leq n$, then these maps are just the compositions of ϕ_T with the morphisms $F_{+\alpha}$. If there are l's, such that $u_l = r_l$, then $\tilde{\phi}(u)_T$ is induced by the composition of ϕ_T with ρ^{-1}_F and with $F_{+\alpha}$.

We want to check that the map $\tilde{\phi}$ is indeed a natural transformation of functors. It's enough to check that the diagram commutes:

$$\begin{tikzcd}
D_u \arrow[r, "\tilde{\phi}(u)"] \arrow[d, "D_{+e_i}"] & F_u \arrow[d, "F_{+e_i}"] \\
D_{u+e_i} \arrow[r, "\tilde{\phi}(u+e_i)"] & F_{u+\vec{e}_i}
\end{tikzcd}$$

If $\epsilon_k(u) = 0$ for all $1 \leq k \leq n$ and also $u_i < r_i - 1$, then it commutes directly from the construction of the maps $\tilde{\phi}(u)$. Otherwise the commutativity will follow from EX1, EX2 and EX3 for $F_\bullet$.

Finally we have obtained the map:

$$\mathrm{Hom}((G^T_\bullet)_{T\in S(k)}, \mathrm{Face}^k(F_\bullet)) \to \mathrm{Hom}(D^k((G^T_\bullet)_{T\in S(k)}), F_\bullet).$$

It's easy to see that this map is bijective, because the right Hom is uniquely defined by the restriction to k-faces.

(iii) Follows from (ii).

(iv) Because for $S(n)$ there is only one element, the set $\{1, \ldots, n\}$ itself, we have that $\iota_{\{1,\ldots,n\}} = id$ and $\pi_{\{1,\ldots,n\}} = id$. So $\mathrm{Face}\big|^n_{\mathbf{ker}^{n-1}}$ and D^n are identity functors.

□

7.4 The G-theory and K-theory of a root stack

In this subsection we will finally describe the G-theory of a root stack $X_{L,\vec{r}}$.

According to the Corollary 3.2.2 and the Corollary 7.2.5 there is an equivalence of categories:

$$\mathfrak{Coh} X_{L,\vec{r}} \simeq \mathcal{EP}(X, L, \vec{r})$$

so we reduced the problem to describing the K-theory of the (abelian) category of extendable pairs $\mathcal{EP}(X, L, \vec{r})$:

$$G(X_{L,\vec{r}}) \cong K(\mathcal{EP}(X, L, \vec{r})).$$

We are going to use different splittings of the category of extendable pairs to simplify the latter K-theory. The first step is this

Lemma 7.4.1.

$$K_i(\mathcal{EP}(X, L, \vec{r})) \cong G_i(X) \oplus K_i(\mathbf{ker}(\pi_*^{L,\vec{r}})) \text{ for any } i \in \mathbb{Z}_+$$

Proof. Using the Corollary 7.3.3 and the localization property of K-theory (see for example [Q]) we have the long exact sequence of groups:

$$\dots \to K_i(\mathbf{ker}(\pi_*^{L,\vec{r}})) \to K_i(\mathcal{EP}(X, L, \vec{r})) \to G_i(X) \to \dots$$

But this sequence splits because of the property $\pi_*^{L,\vec{r}} \circ \pi_{L,\vec{r}}^* \simeq 1$ proved in the Lemma 7.3.1. □

Also we want to state the following

Lemma 7.4.2. *If* $\mathbf{A}$ *is an abelian category then*

$$K_i(\mathrm{Func}(\vec{r}I^n, \mathbf{A})) \cong K_i(\mathbf{A})^{\oplus \prod_{j=1}^n r_j}.$$

Proof. The proof follows from the iterated application of the Theorem 7.1.6 and localization property of the K-theory. □

Now we want to proceed with $K_\bullet(\mathbf{ker}(\pi_*^{L,\vec{r}}))$ as in the previous lemmas. By combining the localization property of the K-theory, Theorem 7.3.11 and the previous lemma one can easily obtain the proof of the final

Lemma 7.4.3. *For any* $i \in \mathbb{Z}_+$

$$K_i(\mathbf{ker}(\pi_*^{L,\vec{r}})) \cong \oplus_{k=1}^n \oplus_{T \in S(k)} G_i(\cap_{l \in T} \operatorname{div}(s_l))^{\oplus \prod_{l \in T}(r_l - 1)}.$$

For the sake of completeness let us also give a description of K-theory of root stacks.

Proposition 7.4.4. *If* X *is a regular scheme over a field* k*,* $D = \sum_{i=1}^n D_i$ *is a normal crossing divisor, and* $\vec{r}$ *is an* n*-tuple of natural numbers, such that each* r_i *is coprime to the characteristic of* k*. Then* $K_\bullet(X_{D,\vec{r}}) = G_\bullet(X_{D,\vec{r}})$*.*

Proof. By Proposition 2.2.15, the stack $X_{D,\vec{r}}$ is regular. The result follows from [J, Cor.2.2]. □

Chapter 8

Application to equivariant K-theory

In this chapter we we will prove our main result.

8.1 Main result

As an application of the theorems proved in the chapter 6 and 7 we can formulate the main result about equivariant K-theory.

Theorem 8.1.1. *Let X be a regular, separated, noetherian scheme over the field k with a generically free admissible action of a finite group G, such that the order of G is coprime to the characteristic of k. Let's denote by Y the quotient X/G and assume all the conditions from Assumption 6.2.1. Also assume that $\phi : X \to Y$ is ramified along a normal crossing divisor.*

We will use notation from Lemma 6.2.5: $E = \sum_{i=1}^{n} E_i$ is a ramification divisor in Y, D is a branch divisor in X, and $r_i = |I(D_i, G)|$.

Then there is an isomorphism of groups:

$$K_G^\bullet(X) \cong K^\bullet(Y) \oplus (\oplus_{i=1}^{n} Z_i^\bullet),$$

where $Z_i^\bullet$ comes from ramification data:

$$Z_i^\bullet = \oplus_{T \in S(i)} G^\bullet(\cap_{l \in T} E_l)^{\oplus \prod_{l \in T}(r_l - 1)},$$

where $S(i) = \{T \subset \{1, \ldots, n\} \mid |T| = i\}$.

Proof. By the assumptions X is a regular scheme and the group G is finite so for any G-equivariant sheaf we can always construct an equivariant locally free resolution by averaging the usual locally free resolution. This simple argument shows that equivariant K-theory of X should be the same as equivariant G-theory.

The category of G-equivariant sheaves on X is equivalent to the category of sheaves on the quotient stack $[X/G]$ so we can see that

$$K_G(X) \cong G([X/G]).$$

In Theorem 6.2.8 we proved that under our assumptions there is an isomorphism of stacks $[X/G] \cong \mathfrak{Y}$, so we have an isomorphism of their G-theories:

$$G([X/G]) \cong G(\mathfrak{Y}).$$

Finally the application of Lemma 7.4.1 and Lemma 7.4.3 gives the formula we wanted to prove.

□

Let us give some examples.

Example 8.1.2. Let's consider $\mathbb{A}^1$ over a field k with an action of μ_3 (it acts by multiplication). Assume that $\mathrm{char}(k) \neq 3$. Then $\mathbb{A}^1/\mu_3 \cong \mathbb{A}^1$ and ramification divisor is $\mathrm{div}(0)$. The inertia group is μ_3. So by Theorem 8.1.1:

$$K^\bullet_{\mu_3}(\mathbb{A}^1) \cong K^\bullet(\mathbb{A}^1) \oplus K^\bullet(k) \oplus K^\bullet(k) \cong K^\bullet(k)^{\oplus 3}.$$

Example 8.1.3. This example was inspired by the paper [AO]. Burniat surface X with $K_X^2 = 6$ is a Galois $G := C_2 \times C_2$-cover of $\mathrm{Bl}_3\mathbb{P}^2$ (del Pezzo surface of degree 6). Let's assume that the ground field k is algebraically closed and $\mathrm{char}(k) \neq 2$. Ramification divisor is given in Figure 1 *loc. cit*: it is denoted by A_l, B_l, C_l, where $0 \leq l \leq 4$. Inertia group of each component is

C_2, inertia group of an intersection point of any two components is G. An intersection of three components is empty. Also $A_l \cong B_l \cong C_l \cong \mathbb{P}^1$, for all $l = 0, \ldots, 3$.

Applying Theorem 8.1.1 one gets:

$$K_G^\bullet(X) \cong K^\bullet(\mathrm{Bl}_3\mathbb{P}^2) \oplus (\oplus_{i=1}^2 Z_i^\bullet),$$

$$Z_1^\bullet = K^\bullet(\mathbb{P}^1)^{\oplus 12},$$

$$Z_2^\bullet = K^\bullet(k)^{\oplus 30}.$$

Example 8.1.4. Consider the action of a symmetric group S_3 on $\mathbb{P}^1$, induced by the permutation of coordinates (x, y, z), subject to a relation $x + y + z = 0$. The ground field k is chosen such that $\mathrm{char}(k) \neq 2,\ 3$ and it contains a primitive root of unity of order 3. Easy computation shows that the branch locus consists of five points. Two of them have inertia group C_3 and three of them have inertia group C_2. The quotient by this action is again $\mathbb{P}^1$ (this follows, for example, from Chevalley-Shephard-Todd theorem (5.2.7) as S_3 is generated by reflections). So Theorem 8.1.1 gives us:

$$K_{S_3}^\bullet(\mathbb{P}^1) \cong K^\bullet(\mathbb{P}^1) \oplus K^\bullet(k)^{\oplus 7}$$

Chapter 9

Conclusions and Summary

This thesis contains two new results:

1. The G-theory and K-theory of root stacks were computed.
2. Sufficient conditions when a quotient stack is a root stack were provided.

As a corollary we get a description of equivariant K-theory of schemes under some assumptions.

Further directions:

1. Give a description of the derived category of a root stack. Find its generator and compute its A_∞-algebra in some situations.
2. Formulate and prove an analogue of the theorems in Chapter 6 in the case of any reductive algebraic group (not necessary finite).
3. Compute other homological theories of root stacks (cyclic homology, Chow groups etc).

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