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Use of Natural Instability For Enhancement of Flow Mixing in Annuli

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Abstract

A technique has been proposed to increase flow mixing in annuli by means of vortex generation. Corrugations in the form of axisymmetric ribs are placed at the walls of annulus to modulate the axial flow which can potentially induce vortex instabilities. Unlike other vortex generation methods which suffer from relatively high pressure losses, this technique is expected to cause less pressure drop. Spectral algorithm based on Fourier and Chebyshev expansions has been used to study the stationary state and its stability. Due to the irregularities of the boundaries, the immersed boundary conditions (IBC) method is used to enforce the flow boundary conditions. The effect of geometric and flow parameters on pressure losses and stability have been thoroughly investigated. Characteristics of vortex mode and travelling wave instabilities as well as region of dominance in each case are also determined. Moreover, it has been shown that effect of arbitrary ribs can be accurately captured using reduced geometry model.

Keywords

Annulus, flow control, surface roughness, pressure losses, vortex instability, travelling wave instability, critical Reynolds number, immersed boundary conditions, linear stability analysis
Co-Authorship Statement

The following dissertation is presented in the monograph format. Chapters 2 and 3 are based on manuscript that is finalized for submission. I, Amirreza Seddighi, am the first author of all these manuscripts with Dr. H. V. Moradi and Prof. J. M. Floryan as the co-authors.
To my parents and sisters

for their endless love, encouragement and support.
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List of Abbreviations, Symbols, Nomenclature

Abbreviations

APF  Annular Poiseuille flow
FFT  Fast Fourier transform
IBC  Immersed boundary conditions
OS   Orr-Sommerfeld
PPF  Planar Poiseuille flow
TS   Tollmien-Schlichting

Nomenclature

\( A \) Pressure gradient correction
\( A_{\text{in}}, A_{\text{out}} \) Rib locations at the inner and outer cylinders in the transformed domain
\( C \) Pressure normalization constant
\( C_{\text{OS}}^{(n)}, C_{\text{SQ}}^{(n)} \) Orr-Sommerfeld and Squire coupling operators
\( D \) Derivative with respect to the transverse direction
\( F_{\text{in}}^{(n)}, F_{\text{out}}^{(n)} \) Coefficients of Fourier expansions representing velocity of the reference flow at the inner and outer cylinders
\( G_{3}, G_{p} \) Amplitude functions
\( G_{k}^{(n)} \) Coefficients of the Chebyshev expansions representing modal functions in the Fourier expansion of the stream function
\( G_{rr,k}^{(n)}, G_{zr,k}^{(n)}, G_{zz,k}^{(n)} \) Coefficients of the Chebyshev expansions of the modal functions in the Fourier expansions representing velocity products
\( H_{\text{in}}^{(n)}, H_{\text{out}}^{(n)} \) Fourier coefficients of the rib geometries at the inner and outer cylinders
\( L \) Average annulus opening
\( L_{\text{OS}}^{(n)} \) Orr-Sommerfeld operator
\( L_{\text{SQ}}^{(n)} \) Squire operator
\( M \) Disturbance wave number in the circumferential direction

\( N_A \) Number of Fourier modes used in description of geometry of the ribs

\( N_M \) Number of Fourier modes used for discretization in the direction associated with periodicity of the ribs

\( N_j \) Number of stationary state Fourier modes retained in the stability equations

\( N_R \) Number of Fourier modes used to represent the function \( r^{-1} \) evaluated at the inner and outer cylinders

\( N_S \) Number of Fourier modes used to describe the Chebyshev polynomials and their derivatives evaluated at the inner and outer cylinders

\( N_T \) Number of Chebyshev polynomials used for discretization of the modal functions in the transverse direction

\( p^{(n)} \) Modal functions in the Fourier expansion of the pressure field

\( Q \) Volume flow rate

\( Q_{total}, Q_{ref}, Q_{mod} \) Volume flow rates of the total, reference, and modification flows

\( R_{in}, R_{out} \) Locations of the rib extremities at the inner and outer cylinders

\( R_1 \) Average radius of the inner annulus

\( Re \) Reynolds number

\( S \) Rib amplitude

\( T_k \) Chebyshev polynomials of the \( k^{th} \) order

\( U_{max} \) Maximum of the streamwise velocity component of the reference flow

\( U, L \) Upper and lower walls (as subscript)

\( Z_{in}^{(n)}, Z_{out}^{(n)} \) Coefficients of the Fourier expansion of the function \( r^{-1} \) evaluated at the inner and outer cylinders

\( c \) Constant of transformation for the IBC method

\( d_{in,k}^{(n)}, d_{out,k}^{(n)} \) Coefficients of the Fourier expansions of the first derivative of the Chebyshev polynomials evaluated at the inner and outer cylinders
\( e_{ij} \) Strain rate tensor
\( g_u^{(n)}, g_v^{(n)}, g_w^{(n)}, g_p^{(n)} \) Modal functions in the Fourier expansions of the disturbance field
\( g_\eta^{(n)} \) Modal functions in the Fourier expansions of the radial vorticity
\( i \) Imaginary unit
\( in,out \) Inner and outer cylinders (as subscript)
\( p \) Pressure
\( r_{in}, r_{out} \) Locations of the inner and outer cylinders
\( \bar{r} \) Transformed radial coordinate
\( \bar{r}_{in}, \bar{r}_{out} \) Geometries of the inner and outer cylinders expressed in the transformed radial coordinate
\( v_z, v_r \) Axial and radial components of velocity vector
\( \{v_z v_z, v_z v_r, v_r v_r\} \) Velocity products in physical space
\( v_r^{(n)}, v_{rz}^{(n)}, v_{zz}^{(n)} \) Modal functions in the Fourier expansions of the velocity products
\( w_{in,k}^{(n)}, w_{out,k}^{(n)} \) Coefficients of the Fourier expansions representing the Chebyshev polynomials evaluated at the inner and outer cylinders
\( z, r, \theta \) Axial, radial, and circumferential directions
\( \alpha \) Rib wave number in the axial direction
\( \delta \) Disturbance wave number in the axial direction
\( \sigma_i \) Rate of growth of disturbances
\( \sigma_r \) Frequency of the disturbances
\( \Gamma \) Factor in the coordinate transformation for the IBC method
\( \lambda \) Rib wavelength in the axial direction
\( \phi^{(n)} \) Modal function in the Fourier expansion representing the stream function
\( \psi \) Stokes stream function
\( \omega \) Weight function
\( \vec{\omega} \) Vorticity vector
\( z, r, \theta \) Axial, radial, and circumferential directions
\( \eta \) Radial vorticity component
\( \xi \) Axial vorticity component
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$\phi^{(n)}$</td>
<td>Modal function in the Fourier expansions expressing the axial velocity</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Dynamic viscosity</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Kinematic viscosity</td>
</tr>
<tr>
<td>$\langle f, g \rangle$</td>
<td>Inner product of two functions</td>
</tr>
<tr>
<td>0, 1, 2, 3</td>
<td>Reference flow, flow modifications, total stationary state, disturbance (as subscript)</td>
</tr>
<tr>
<td>$*$</td>
<td>Complex conjugate (as superscript)</td>
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Chapter 1

1 Introduction

1.1 Objective

The issue of interest in this work is the identification of conditions leading to vortex instabilities in an annulus by studying the response of flow to various surface topographies. We are particularly interested in this type of instabilities because they are able to increase mixing within laminar flow without transition to turbulence; in other words, they make efficient mixing possible while pressure losses are kept the minimum.

The roughness considered on the walls of the annulus will be in the form of axisymmetric ribs of arbitrary shapes. By modulating the annular flow, these ribs create centrifugal forces which can potentially result in the formation of vortices. Experiments conducted by Gschwind, Regele & Kottke (1995) and Nishimura et al. (1990a) quantitatively confirm the existence of such centrifugally-driven vortices. While vortex instability is of interest due to enhancement of mixing, one may want to avoid travelling wave instabilities as they can lead to transition from laminar flow to turbulent. Thus, we are also interested in determination of conditions where each of these instabilities is dominant.

The study focuses on the analysis of flow characteristics and investigates the effect of ribs on the pressure losses and flow stability. The mass flow rate has been considered to be the same through the rib-fitted and the smooth annuli and drag variations are assessed by determining the additional pressure gradient required to maintain this mass flow rate. The stability characteristics are subsequently studied using the linear stability theory which can account for spatial flow modulations due to corrugated surfaces.

1.2 Motivations

The pressure driven axial flow between two concentric cylindrical shells is widely encountered in applications, e.g. flow in tubular heat exchangers, plastic extrusion operations, movement of drilling mud in oil and gas wells, fuel cells, turbomachinery,
various chemical installations and many others. Since the character of this flow has profound effects on the efficiency of the relevant devices, enhancement of efficiency requires thorough analysis of this type of flow.

Ideal annular flow can rarely be found in practice as the surface finish of the conduit depends on the manufacturing process used. This surface degrades during the lifetime of any particular device due to fouling, corrosion, erosion and similar processes. Presence of surface roughness will have direct effect on the pressure losses as well as the indirect effects either through the delay or through the acceleration of the laminar-turbulent transition; thus, investigation of the effects of such surface modifications on the flow characteristics is of particular interest. Surface modifications could be also intentionally applied to the walls in the form of arbitrary shape ribs either to represent a certain class of roughness shape or with the goal of mixing enhancement.

While turbulent flow enhances mixing within the system, it has the disadvantage of high pressure loss through the domain. On the other hand, enhanced mixing can be also acquired in a flow field with a complex laminar state (but not yet turbulent) without a high cost in terms of pressure loss. This can be achieved by selecting surface topographies which are able to destabilize the flow with respect to the streamwise vortices. Such vortices are expected to increase the radial mixing by an order of magnitudes. Compared with conventional vortex generation methods which may result in pressure drop of up to 150% (Lograni et al., 2003), techniques that rely on flow instabilities to create vortices are expected to produce much smaller pressure losses (typically around 1-2% of the reference flow) and thus, are of particular interest in design of more efficient tubular heat exchangers.

1.3 Literature review

A number of variations of the annular flow have been studied. Effects of cylinder eccentricity are discussed in Walton et al. (2010). The core-annular flow (S. Ghosh et al., 2009; G. Moritis 1995) is formed when the motion of two immiscible liquids in a pipe is considered, one forming the core and the other one wrapping around it and forming an annular domain. The character of the flow in the annular zone could be similar to that in
the annulus formed by two solid cylinders depending on the assumptions used to model the interfacial conditions. Reducing the required pumping power in heavy oil extraction (D. D. Joseph, Y. Renardy, 1993) where there exist the possibility of laminar-turbulent transition due to the pipe surface roughness, drives the interest in such flows and forms the basis of the lubricated pipelining technique (Moradi, H. V. & Floryan 2015). In this approach a low-viscosity liquid is injected into the pipe to wet the wall to reduce the shear stress that the oil flowing through the core region is exposed to (D. D. Joseph et. al., 1999). Other special cases studied by Blyth et al. (2006), Frei et al. (2000), Howard & Patankar (1999), Preziosi et al. (1989), Walton (2003, 2005) and Wei & Rumschitzki (2002a, b) bring in applications ranging from pipeline lubrication through modeling of needle injection in minimally invasive surgery to modeling of lung dynamics.

Recent interest in flows in micro-channels and the inability to control the quality of surface finishes has led to numerous studies of the effects of surface roughness on laminar flows (Papautsky et al., 1999; Sobhan & Garimella, 2001; Morini, 2004; Sharp & Adrian, 2004; Gamrat et al., 2008; Valdez et al., 2007). The common feature of these analyses is the case-study approach, where a number of very specific configurations are studied and an extrapolation to arbitrary surface irregularities is attempted. A conceptually different approach, based on the reduced geometry model (Floryan, 1997, 2007), projects the surface geometry onto a Fourier space and focuses the analysis on the identification of the segment of this space which is hydraulically relevant. This concept decouples the analysis from the details of the surface geometry and identifies geometry features which are hydraulically relevant. The reduced geometry model has been successfully applied to analyses of losses in channel (Mohammadi & Floryan, 2013a, b; 2014 a, b; Moradi & Floryan, 2013b) and annular flows (Moradi & Floryan, 2013a).

Although there is a large amount of experimental data, meaningful progress in the understanding of the flow mechanics associated with the effects of surface modifications have received less attention. This is not because these effects are less important, but rather due to the wealth of possible forms of surface topography and different roles played by these topographies in flow dynamics. Progress in this area depends on the development of algorithms which are sufficiently flexible to deal with a large number of
geometric configurations with limited interaction with the user. The unstructured grid method and the immersed boundary method (IB) are different approaches that have been used. The latter method was originally developed by Peskin in the context of modelling the flow around heart valves (C. S. Peskin, 1977). IB has been successfully applied to a variety of problems in biological fluid mechanics (C. S. Peskin, D. M. McQueen, 1995), for flows with suspended particles (J. M. Stockie & S. I. Green, 1998), in heat transfer problems (Z. Wang et. al., 2009), etc. The IB method has an advantage over the unstructured grid method in low memory requirements, CPU saving and ease of grid generation (see detailed reviews in R. Mittal, G. Iaccarino, 2005), but suffers from numerical stability problems and low accuracy of the spatial discretization (C. S. Peskin, 2002; J. M. Stockie et. al., 1999). As stability properties are of interest, these algorithms need to be able to deliver near machine accuracy if required, and this focuses our attention on spectral methods.

Spectral methods are well suited for the analysis of flows in regular domains but have difficulty describing complex domains (Moradi & Floryan, 2015). This difficulty can be addressed using the immersed boundary conditions (IBC) method (Szumbarski & Floryan, 1999). A version of this method suitable for the analysis of stationary flows in annuli with axisymmetric ribs is described by Moradi & Floryan, 2012. The geometry of the ribs is expressed using Fourier expansions, the field equations are discretized using Fourier expansions in the axial direction and Chebyshev expansions in the radial direction, and the boundary conditions are enforced using the specially developed boundary relations. As the information about the geometry enters the solution process in the form of a finite number of Fourier coefficients, the resulting algorithm provides the ability to study a large number of geometries with little manual effort.

To develop a spectrally accurate IBC method suitable for the linear stability analysis, one requires developing a formulation of the field equations convenient for the numerical work, accounting for the rib-induced spatial modulations of the basic state, generalization of the IBC method to three-dimensions and development of an efficient solution strategy for the resulting eigenvalue problem (Moradi & Floryan, 2015). The stability formulation
and the IBC method follows the formulation by Floryan, 1997, 2002 which was expressed in Cartesian coordinates.

Spectral finite-element method is, of course, considered as a technique which is able to provide the accuracy required for analysis of flow stability in complex geometries. This method, however, requires numerical grid generation for modelling of geometry and, thus, is very labor intensive as each geometry needs to be modelled separately (C.D. Cantwell et al., 2015).

For completeness purposes, one should mention the limitations of the IBC method as well. The method losses accuracy in the case of too extreme geometries, i.e. ribs with large amplitudes and wave numbers. The effect of nonlinearities increases with an increase of the Reynolds number which reduces the rate of convergence of the iterative solution method. These problems can be overcome by increasing the number of Fourier modes and Chebyshev polynomials used in the computations.

In recent years, IBC method has been successfully used for the investigation of flow instabilities in geometries with surface roughness. The two-dimensional distributed roughness transverse to the flow direction has been shown to destabilize traveling waves (Floryan, 2005; Asai & Floryan, 2006) with the two-dimensional waves playing the critical role (Floryan, 2007). The same roughness can amplify disturbances in the form of streamwise vortices (Floryan, 2007). Similar flow responses have been found in Couette flow (Floryan, 2002) and in the pressure driven flow in a converging-diverging channel (Floryan, 2003; Floryan & Floryan, 2010). Predictions regarding the onset of two-dimensional travelling waves have been confirmed experimentally (Asai & Floryan, 2006). Rotation of the roughness system by 90 degrees so that it becomes parallel to the flow direction (longitudinal grooves) results in flow stabilization when long wavelength grooves are used and flow destabilization in the presence of short wavelength grooves (Moradi & Floryan, 2014). In all cases two-dimensional waves play the critical role.

In the current work, the stationary state is discussed in Section 2. Section 2.1 gives the problem formulation, Section 2.2 presents the numerical method and Section 2.3 discusses mean flow properties of the rib-modified flow. Section 3 is focused on the
stability analysis. The general formulation for arbitrary disturbances is given in Section 3.1 while discretization of the linear stability problem is presented in Section 3.2. The numerical solution is discussed in Section 3.3. Results for stability characteristics of the flow and properties of different disturbance types are presented in Section 3.4. Stability of flow in smooth annuli is discussed in Section 3.4.1 as the reference case. Vortex and travelling wave instabilities are then presented in Sections 3.4.2 and 3.4.3, respectively. To investigate conditions where each type of instability is dominant, analysis has been carried out comparing vortices and travelling waves and the results are illustrated in Section 3.4.4. Finally, conclusion and summary of the entire work is presented in Section 4.
Chapter 2

2 Stationary state

2.1 Problem Formulation

Steady, pressure-gradient-driven axial flow of viscous incompressible fluid in an annulus extending to $\pm \infty$ in the $z$-direction is investigated. The geometric characteristics of the annulus and structure of the flow field are described using the cylindrical system of coordinates with $z$, $r$ and $\theta$ describing the axial, radial and circumferential directions. The inner and outer walls of the annulus are modified by introducing axisymmetric surface ribs of arbitrary cross-section (see Figure 2-1). The geometry of the cylinders is described using Fourier expansions of the form

$$
\begin{align}
    r_{in}(z) &= R_1 + \sum_{l=-N_A}^{N_A} H^{(l)}_{in} e^{il\alpha z}, \\
    r_{out}(z) &= 1 + R_1 + \sum_{l=-N_A}^{N_A} H^{(l)}_{out} e^{il\alpha z},
\end{align}
(2.1 a, b)
$$

All modal functions must satisfy the reality conditions of the form $H^{(l)}_{in} = H^{(-l)*}_{in}$ and $H^{(l)}_{out} = H^{(-l)*}_{out}$ where stars denote the complex conjugate, $R_1$ stands for the average radius of the inner cylinder, subscripts “out” and “in” show the outer and inner cylinders, respectively, $N_A$ is the number of Fourier modes used for description of the geometry and the average annulus opening $L$ has been used as the length scale. The ribs are periodic in the $z$-direction with wavelength $\lambda = 2\pi/\alpha$.

Annular Poiseuille flow (APF, i.e. flow in a smooth annulus with walls located at $r = R_1$ and $r = 1 + R_1$) is used as the reference flow which has the form

$$
\begin{align}
    u_0(r) &= \frac{R_1^2}{k_1} \left[ 1 - \left( \frac{r}{R_1} \right)^2 \right] + \frac{k_2}{k_1} \ln \left( \frac{r}{R_1} \right), \\
    \frac{dp_0}{dz} &= -\frac{4}{k_1 Re}
\end{align}
(2.2, 2.3)
\[ Q_0 = \frac{2\pi}{k_1} \left[ \frac{(1 + R_1)^2}{4} \left( R_1^2 - 2R_1 - 1 - k_2 \right) + \frac{k_2(1 + R_1)^2}{2} \ln \left( \frac{1 + R_1}{R_1} \right) \right. \]
\[ \left. + \frac{R_1^2}{4} (k_2 - 1) \right] \]

where \( u_0 \) stands for the axial velocity component, \( p_0 \) denotes pressure, \( Q_0 \) stands for the volume flow rate and \( k_1 = R_1^2 - k_2 \ln R_1 + k_2 / 2 \left[ \ln \left( \frac{k_2}{2} \right) - 1 \right] \), \( k_2 = (1 + 2R_1) / \ln \left( (1 + R_1) / R_1 \right) \). Velocity scale and pressure scale have been defined as the maximum of the axial velocity component \( U_{\text{max}} \) and \( \rho U_{\text{max}}^2 \), respectively. \( \rho \) stands for the density and the Reynolds number is defined as \( Re = U_{\text{max}} L / \nu \) where \( \nu \) denotes the kinematic viscosity. The maximum velocity \( U_{\text{max}} \) occurs at \( r = \sqrt{k_2/2} \).

Figure 2-1: axisymmetric annulus with transverse ribs of arbitrary cross-section.

Flow in the ribbed annuli can be considered as the superposition of the APF and modifications, i.e.

\[ \vec{V}_2(z,r) = [u_2(z,r), v_2(z,r)] = \vec{V}_0(r) + \vec{V}_1(z,r) \]
\[ = [u_0(r), 0] + [u_1(z,r), v_1(z,r)], \]
\[ p_2(z,r) = p_0(z) + p_1(z,r) \]
where \( \vec{V} \) denotes the velocity vector, subscripts 0, 1, 2 denote the reference flow, the rib-induced modifications and the complete flow quantities, respectively, and the flow modification are assumed to be axisymmetric. The axisymmetric form of the governing equations can be expressed as follows

\[
\frac{\partial u_1}{\partial z} + \frac{1}{r} \frac{\partial (ru_1)}{\partial r} = 0,
\]

\[
- \frac{\partial p_1}{\partial z} + \frac{1}{Re} \left( \frac{1}{r} \frac{\partial u_1}{\partial r} + \frac{\partial^2 u_1}{\partial r^2} + \frac{\partial^2 u_1}{\partial z^2} \right) - u_0 \frac{\partial u_1}{\partial z} - D u_0 v_1,
\]

\[
- \frac{\partial p_1}{\partial r} + \frac{1}{Re} \left( \frac{1}{r} \frac{\partial v_1}{\partial r} + \frac{\partial^2 v_1}{\partial r^2} + \frac{\partial^2 v_1}{\partial z^2} - \frac{v_1}{r^2} \right) - u_0 \frac{\partial v_1}{\partial z},
\]

Where the hat over the symbol identifies the velocity product and \( D \equiv d/dr \). At the inner and outer walls, no-slip and no-penetration conditions have been imposed as the boundary conditions which have the form

\[
\begin{align*}
&u_1[z, r_{in}(z)] = -u_0[z, r_{in}(z)], v_1[z, r_{in}(z)] = 0, \\
u_1[z, r_{out}(z)] = -u_0[z, r_{out}(z)], v_1[z, r_{out}(z)] = 0.
\end{align*}
\]

(2.9a-d)

Flow rates in ribbed and smooth annuli have been assumed to be the same which leads to the constant flow rate constraint of the form

\[
\int_{r_{in}}^{r_{out}} 2\pi r u_2(z, r) dr = Q_0.
\]

(2.10)

To keep the same flow rate as APF in ribbed annuli, pressure gradient correction of \( A \) would be required which has been used as the measure of flow losses induced by the ribs.

For simplicity of the solution, the field equations have been expressed in terms of the Stokes stream functions \( \psi_0, \psi_1, \psi_2 \) defined as
\[ \psi_2 = \psi_0 + \psi_1, \quad u_1 = -\frac{1}{r} \frac{\partial \psi_1}{\partial r}, \quad v_1 = \frac{1}{r} \frac{\partial \psi_1}{\partial z}, \quad u_0 = -\frac{1}{r} \frac{\partial \psi_0}{\partial r} \]

\[ \psi_0(r) = \frac{1}{2k_1} \left[ \left( \frac{k_2}{2} - R_1^2 - k_2 \ln \frac{r}{R_1} \right) \frac{r^2}{2} + \frac{r^4}{4} + \frac{R_1^2}{4} (R_1^2 - k_2) \right]. \] (2.11a,e)

These equations reduce to a single equation for \( \psi_1 \) of the form

\[ \frac{E^4 \psi_1}{Re} - u_0 \frac{\partial}{\partial z} \left( \frac{\partial^2 \psi_1}{\partial z^2} + \frac{\partial^2 \psi_1}{\partial r^2} \right) + \frac{u_0}{r} \frac{\partial^2 \psi_1}{\partial z \partial r} + \left[ D^2 u_0 - \frac{D u_0}{r} \right] \frac{\partial \psi_1}{\partial z} \]

\[ = \left\{ r \frac{\partial}{\partial z} \left( \frac{\partial v_1}{\partial r} + \frac{\partial u_1}{\partial z} \right) - r \frac{\partial}{\partial r} \left( \frac{\partial u_1}{\partial r} + \frac{\partial u_1}{\partial z} \right) + \frac{\partial v_1}{\partial z} + \frac{u_1 v_1}{r} \right\} \] (2.12)

where

\[ E^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{r} \frac{\partial}{\partial r}. \] (2.13)

And the boundary conditions are of the form

\[ \frac{1}{r} \frac{\partial \psi_1}{\partial r} = u_0(z, r), \quad \frac{1}{r} \frac{\partial \psi_1}{\partial z} = 0 \quad \text{at} \quad r = r_{\text{in}}(z) \text{ and } r = r_{\text{out}}(z). \] (2.14a-d)

The stream function normalization condition is selected by taking \( \psi_2 = 0 \) at the inner cylinder, i.e.

\[ \psi_1[z, r_{\text{in}}(z)] = -\psi_0[z, r_{\text{in}}(z)] \] (2.15)

and the flow rate constraint is expressed as

\[ \psi_1[z, r_{\text{out}}(z)] = -\psi_0[z, r_{\text{out}}(z)] - Q_0 / 2\pi. \] (2.16)

### 2.2 Numerical Discretization

We wish to determine spectrally accurate solution of the flow problem described by Equations (2-12, 2-14, 2-15, 2-16). Because of the irregularities of the boundaries, the immersed boundary conditions (IBC) method is used to enforce the flow boundary
conditions. In this method the physical domain is immersed inside the cylindrical computational box and the boundary conditions are imposed as the internal constraints (Szumbarski & Floryan, 1999; Husain & Floryan, 2010; Moradi & Floryan, 2012). The IBC method has been selected due to its relative simplicity as well as ability to provide the desired accuracy.

2.2.1 Field equation

Figure 2-1 illustrates the computational domain which is bounded by \(-R_{in} + R_1\) from below and by \(1 + R_1 + R_{out}\) from above in the radial direction, where \(R_{in}\) and \(R_{out}\) denote locations of the rib extremities. The radial domain is mapped onto \([-1, 1]\) in order to use the standard definition of Chebyshev polynomials. Transformation used in this study has the form

\[
\bar{r} = \Gamma (r + C) \text{ where } \Gamma = 2(1 + R_{in} + R_{out})^{-1}, \quad C = 0.5(R_{in} - R_{out} - 2R_1 - 1),
\]

with \(\bar{r} \in [-1, 1]\). The rib geometry, field equation and the boundary conditions in the \((z, \bar{r})\) coordinate system are of the form

\[
\bar{r}_{in}(z) = \sum_{n=-N_A}^{N_A} A_{in}^{(n)} e^{inz}, \quad \bar{r}_{out}(z) = \sum_{n=-N_A}^{N_A} A_{out}^{(n)} e^{inz}
\]

\[
1 \left( \frac{\partial^4 \psi_1}{\partial z^4} + 2\Gamma^2 \frac{\partial^4 \psi_1}{\partial z^2 \partial r^2} + \Gamma^4 \frac{\partial^4 \psi_1}{\partial r^4} - \frac{2\Gamma}{f} \frac{\partial^3 \psi_1}{\partial z^3} - \frac{2\Gamma^3}{f} \frac{\partial^3 \psi_1}{\partial z \partial r^2} + \frac{3\Gamma^2}{f^2} \frac{\partial^2 \psi_1}{\partial r^2} \right) - \frac{3\Gamma}{f^3} \frac{\partial \psi_1}{\partial r} = u_0 \left( \frac{\partial^3 \psi_1}{\partial z^3} + \Gamma^2 \frac{\partial^3 \psi_1}{\partial z \partial r^2} \right) + \Gamma \frac{u_0}{f} \frac{\partial^2 \psi_1}{\partial z \partial r} + \frac{\Gamma}{f} \frac{\partial^2 \psi_1}{\partial z^2} \quad (2.17)
\]

\[
\left[ \Gamma^2 D^2 u_0 \right] = f \left( \frac{\partial^2 \bar{u}_1 \bar{v}_1}{\partial z^2} + \Gamma f \left( \frac{\partial^2 \bar{v}_1 \bar{v}_1}{\partial z \partial \bar{r}} - \frac{\partial^2 \bar{u}_1 \bar{u}_1}{\partial z \partial \bar{r}} \right) \right)
\]

\[
\Gamma^2 f \frac{\partial^2 \bar{u}_1 \bar{v}_1}{\partial \bar{r}^2} + \frac{\partial \bar{v}_1 \bar{v}_1}{\partial \bar{z}} = - \Gamma \frac{\partial \bar{u}_1 \bar{v}_1}{\partial \bar{r}} + \bar{u}_1 \bar{v}_1 \quad (2.18)
\]

\[
\frac{\Gamma}{f} \frac{\partial \psi_1}{\partial \bar{r}} = u_0(z, \bar{r}), \quad \frac{1}{f} \frac{\partial \psi_1}{\partial \bar{z}} = 0 \text{ at } \bar{r} = \bar{r}_{in}(z) \text{ and } \bar{r} = \bar{r}_{out}(z)
\]
where

\[ A_{in}^{(n)} = \begin{cases} \Gamma(R_1 + C + H_{in}^{(n)}), & n = 0 \\ \Gamma H_{in}^{(n)}, & n \neq 0 \end{cases}, \quad A_{out}^{(n)} = \begin{cases} \Gamma(1 + R_1 + C + H_{out}^{(n)}), & n = 0 \\ \Gamma H_{out}^{(n)}, & n \neq 0 \end{cases}, \]  

\[ f(\tilde{r}) = \Gamma^{-1} \tilde{r} - C \]

where \( \bar{D} \equiv d/d\tilde{r} \). Equation (2.19) is solved iteratively with the nonlinear terms on the right hand side considered to be known and their values being updated from iteration to iteration. The solution process begins with expressing the unknown \( \psi_i \) and known \( \bar{u}_1 \bar{u}_1, \bar{v}_1 \bar{v}_1 \) in terms of Fourier expansions in the \( z \)-direction of the form

\[ \psi_1(z, \tilde{r}) \approx \sum_{n=-N_M}^{N_M} \phi^{(n)}(\tilde{r}) e^{inaz}, \quad \bar{u}_1 \bar{u}_1(z, \tilde{r}) \approx \sum_{n=-N_M}^{N_M} \bar{u}u^{(n)}(\tilde{r}) e^{inaz}, \]

\[ \bar{u}_1 \bar{v}_1(z, \tilde{r}) \approx \sum_{n=-N_M}^{N_M} \bar{u}v^{(n)}(\tilde{r}) e^{inaz}, \quad \bar{v}_1 \bar{v}_1(z, \tilde{r}) \approx \sum_{n=-N_M}^{N_M} \bar{v}v^{(n)}(\tilde{r}) e^{inaz} \]  

(2.22)

where \( \phi^{(n)} \) denotes the unknown modal functions and \( \bar{u}u^{(n)}, \bar{u}v^{(n)}, \bar{v}v^{(n)} \) stand for the known modal functions. All modal functions need to satisfy the reality conditions of the form \( \phi^{(n)*} = \phi^{(-n)}, \bar{u}u^{(n)*} = \bar{u}u^{(-n)*}, \bar{u}v^{(n)*} = \bar{u}v^{(-n)*}, \bar{v}v^{(n)*} = \bar{v}v^{(-n)*} \). Substitution of (2.22) into (2.19) and separation of Fourier components lead to a system of ordinary differential equations for \( \phi^{(n)} \) of the form

\[ \left\{ \begin{array}{l} \frac{1}{Re} \left[ (n\alpha)^4 - 2\Gamma^2 (n\alpha)^2 \bar{D}^2 + \Gamma^4 \bar{D}^4 + \frac{2\Gamma(n\alpha)^2}{f} \bar{D} - \frac{2\Gamma^3}{f^2} \bar{D}^2 + \frac{3\Gamma^2}{f^3} \bar{D} - \frac{3\Gamma}{f^4} \right] \\
-ina\Gamma^2 u_0 \bar{D}^2 + ina\Gamma^2 \bar{D} + ina \left[ (n\alpha)^2 u_0 + \Gamma^2 \bar{D} u_0 - \Gamma \frac{\bar{D} u_0}{f} \right] \end{array} \right\} \phi^{(n)} \]

\[ = ina \bar{v}v^{(n)} - \Gamma \bar{D} \bar{u}v^{(n)} + f^{-1} \bar{u}v^{(n)} - (n\alpha)^2 f \bar{u}v^{(n)} \]

\[ + ina \Gamma f \bar{D} \left[ \bar{v}v^{(n)} - \bar{u}v^{(n)} \right] - \Gamma^2 f \bar{D}^2 \bar{u}v^{(n)}. \]  

The next step involves expressing all modal functions in terms of Chebyshev expansions of the form
\[
\phi^{(n)}(\tilde{r}) \approx \sum_{k=0}^{N_T} G_k^{(n)} T_k(\tilde{r}), \quad \bar{\phi}^{(n)}(\tilde{r}) \approx \sum_{k=0}^{N_T} G_{uu,k}^{(n)} T_k(\tilde{r}), \\
\bar{u}^{(n)}(\tilde{r}) \approx \sum_{k=0}^{N_T} G_{uv,k}^{(n)} T_k(\tilde{r}), \quad \bar{v}^{(n)}(\tilde{r}) \approx \sum_{k=0}^{N_T} G_{vv,k}^{(n)} T_k(\tilde{r})
\]

(2.24)

where \( T_k \) denotes Chebyshev polynomial of the \( k \)th order, \( G_k^{(n)} \) denote the unknown coefficients of expansion expressing \( \phi^{(n)} \) and \( G_{uu,k}^{(n)}, G_{uv,k}^{(n)}, G_{vv,k}^{(n)} \) stand for the known coefficients of expansions expressing \( \bar{u}^{(n)}, \bar{u}^{(n)}, \bar{v}^{(n)} \), respectively.

Substitution of (2.24) into (2.23) and use of the Galerkin projection method lead to a set of algebraic equations for \( G_k^{(n)} \), that is

\[
\sum_{k=0}^{N_T} \left\{ (n\alpha)^4 R e^{-1} \langle T_j, T_k \rangle - 2(n\alpha)^2 R e^{-1} \langle T_j, \vec{D}^2 T_k \rangle + \Gamma^4 R e^{-1} \langle T_j, \vec{D}^4 T_k \rangle \right. \\
+ \sum_{l=0}^{N_T} \left[ (2(n\alpha)^2 \langle T_j, T_l \vec{D} T_k \rangle - 2\Gamma^2 \langle T_j, T_l \vec{D}^3 T_k \rangle) \right] \Gamma G_{l,1} R e^{-1} \\
+ 3\Gamma^2 R e^{-1} G_{l,2} \langle T_j, T_l \vec{D}^2 T_k \rangle - 3\Gamma R e^{-1} G_{l,3} \langle T_j, T_l \vec{D} T_k \rangle \\
+ in\alpha(n\alpha)^2 \langle T_j, T_l T_k \rangle - \Gamma^2 \langle T_j, T_l \vec{D}^2 T_k \rangle) \Gamma G_{l,u} + \Gamma in\alpha G_{l,uf} \langle T_j, T_l \vec{D} T_k \rangle \\
+ \Gamma^2 in\alpha G_{l,D2u} \langle T_j, T_l T_k \rangle - \Gamma in\alpha G_{l,Duf} \langle T_j, T_l T_k \rangle \bigg\} G_k^{(n)} \\
= in\alpha G_{uu,k}^{(n)} \langle T_j, T_k \rangle - \Gamma G_{uv,k}^{(n)} \langle T_j, \vec{D} T_k \rangle \\
+ \sum_{l=0}^{N_T} \left\{ \left\{ -(n\alpha)^2 G_{uv,k}^{(n)} \langle T_j, T_l T_k \rangle + \Gamma in\alpha \left( G_{vv,k}^{(n)} - G_{uu,k}^{(n)} \right) \langle T_j, T_l \vec{D} T_k \rangle \\
- \Gamma^2 G_{uv,k}^{(n)} \langle T_j, T_l \vec{D}^2 T_k \rangle \right\} G_{l,f0} + G_{l,1} G_{uv,k}^{(n)} \langle T_j, T_l T_k \rangle \right\}
\]

(2.25)

where \( G_{l,f0}, G_{l,f1}, G_{l,f2}, G_{l,f3}, G_{l,u}, G_{l,uf}, G_{l,Duf}, G_{l,D2u} \) is a vector of Chebyshev coefficients defined as

\[
[f, f^{-1}, f^{-2}, f^{-3}, u_0, u_0 f^{-1}, \vec{D} u_0 f^{-1}, \vec{D}^2 u_0] = \\
\sum_{l=1}^{N_T} \left[ G_{l,f0}, G_{l,f1}, G_{l,f2}, G_{l,f3}, G_{l,u}, G_{l,uf}, G_{l,Duf}, G_{l,D2u} \right] T_k(\tilde{r}).
\]

(2.26)
The inner product \( \langle T_j, \bar{D}^n T_l \bar{D}^m T_k \rangle \) is defined as

\[
\langle T_j, \bar{D}^n T_l \bar{D}^m T_k \rangle = \int_{-1}^{1} T_j(\bar{r}) \bar{D}^n T_l(\bar{r}) \bar{D}^m T_k(\bar{r}) \omega(\bar{r}) d\bar{r}, \tag{2.27}
\]

where \( \omega(\bar{r}) = 1/\sqrt{1 - \bar{r}^2} \). These products can be evaluated efficiently by taking advantage of the orthogonality properties of the Chebyshev polynomials (Moradi & Floryan, 2012).

The algebraic equations for each modal function are coupled through the nonlinear terms which are evaluated on the basis of information available from the previous iteration. Four equations corresponding to the highest Chebyshev polynomials are removed to accommodate the boundary conditions. Details of discretization are the same as used by Moradi & Floryan (2012) for the total flow quantities; these details are not repeated here. The boundary conditions provide another coupling between different modes resulting in a very large global system. This system has been solved using specialized very efficient linear solver (Husain & Floryan, 2013, 2014).

Since we are using a spectral algorithm, the solution error decreases exponentially as a function of the number \( N_T \) of Chebyshev polynomials used in the solution as well as the number \( N_M \) of Fourier modes used in the solution. This error has been defined as

\[
Er = \text{Max}_{\text{Solution Domain}} |u(x, y) - u_{ref}(x, y)| \tag{2.28}
\]

where \( u_{ref}(x, y) \) stands for the reference solution determined numerically with high accuracy, i.e. with \( N_T = 80 \) and \( N_M = 20 \). Although this solution is not exact, the relevant numerical error is smaller than the machine accuracy which justifies use of this solution as the actual solution.

Determination of changes of the pressure gradient represents a post-processing step. The flow field is expressed as
\[ \overline{V}_2(z, \bar{r}) = \overline{V}_0(\bar{r}) + \overline{V}_1(z, \bar{r}) = [u_0(\bar{r}), 0] + \sum_{n=-N_M}^{N_M} \left[ f^{(n)}_u(\bar{r}), f^{(n)}_v(\bar{r}), 0 \right] e^{in\alpha z} \] (2.29)

where \( f_u^{(n)} = f_u^{(-n)*} \) and \( f_v^{(n)} = f_v^{(-n)*} \).

The pressure field has the form

\[ p_1(z, \bar{r}) = Az + \sum_{n=-N_M}^{N_M} p^{(n)}(\bar{r}) e^{in\alpha z} \] (2.30)

where constant \( A \) denotes the mean pressure gradient correction. Evaluation of \( A \) begins with the \( z \)-momentum equation, i.e.,

\[
\begin{align*}
\frac{\partial \bar{u}_1 \bar{u}_1}{\partial z} + \Gamma \frac{\partial \bar{u}_1 \bar{v}_1}{\partial \bar{r}} + \frac{\bar{u}_1 \bar{v}_1}{f} &= - \frac{\partial p_1}{\partial z} + \Gamma \frac{u_0 \partial^2 \psi_1}{f \partial z \partial \bar{r}} - \frac{\Gamma D u_0 \partial \psi_1}{f} \\
+ \frac{1}{Re} \left[ \frac{\Gamma \partial \psi_1}{f^3} + \frac{\Gamma^2 \partial^2 \psi_1}{f^2} - \frac{\Gamma^3 \partial^3 \psi_1}{f} - \frac{\Gamma \partial^3 \psi_1}{f \partial \bar{r}^2} \right].
\end{align*}
\] (2.31)

Substitution of (2.22) and (2.30) into (2.31), separation of Fourier modes, and extraction of mode zero lead to

\[ A = \frac{1}{Re} \left[ \frac{\Gamma D \phi^{(0)}}{f^3} + \frac{\Gamma^2 D^2 \phi^{(0)}}{f^2} - \frac{\Gamma^3 D^3 \phi^{(0)}}{f} \right] - \frac{\Gamma D \bar{u} \bar{v}^{(0)}}{f} - \frac{\bar{u} \bar{v}^{(0)}}{f}. \] (2.32)

### 2.2.2 Boundary conditions

Boundary conditions used in the computations have the form

\[ \frac{\Gamma \partial \psi_1}{f \partial \bar{r}} = u_0(z, \bar{r}), \quad \frac{1}{f} \frac{\partial \psi_1}{\partial z} = 0 \quad \text{at} \quad \bar{r} = \bar{r}_{in}(z) \quad \text{and} \quad \bar{r} = \bar{r}_{out}(z) \] (2.33)

Where
\[ \bar{r} = \Gamma(r + c), \Gamma = 2(1 + R_{in} + R_{out})^{-1}, \]
\[ c = 0.5(R_{in} - R_{out} - 2 R_1 - 1), \quad f(\bar{r}) = \Gamma^{-1}\bar{r} - c. \quad (2.34) \]

Substitution of (2.22) and (2.24) into above equations leads to boundary conditions of the form

\[ \left( \frac{\Gamma}{f} \sum_{n=-N_M}^{+N_M} \sum_{k=0}^{N_T} G_k^{(n)} D T_k(\bar{r}) \right) e^{in\alpha z} = u_0(z, \bar{r}), \]  
\[ (2.35a, b) \]
\[ \frac{1}{f} \sum_{n=-N_M}^{+N_M} \sum_{k=0}^{N_T} \text{ima} G_k^{(n)} T_k(\bar{r}) e^{in\alpha z} = 0 \quad \text{at} \quad \bar{r} = \bar{r}_{in}(z) \quad \text{and} \quad \bar{r} = \bar{r}_{out}(z). \]  

Chebyshev polynomials and their derivatives evaluated at the cylinder surface are periodic functions of \( z \) and can be expressed in terms of Fourier series of the form

\[ T_k(\bar{r}(z)) = \sum_{m=-N_s}^{+N_s} w_k^{(m)} e^{im\alpha z}, \quad DT_k(\bar{r}(z)) = \sum_{m=-N_s}^{+N_s} d_k^{(m)} e^{im\alpha z} \quad (2.36a, b) \]

Where \( N_s = N_T N_A \). Method for determination of \( w_{in,k}^{(m)} \) and \( d_{in,k}^{(m)} \) is described in Moradi & Floryan (2012). Substitution of (2.36a, b) into (2.35a, b) leads to

\[ \left( \frac{\Gamma}{f} \sum_{n=-N_s-N_M}^{N_s+N_M} \sum_{m=-N_M}^{N_M} \sum_{k=0}^{N_T} G_k^{(m)} d_k^{(n-m)} e^{in\alpha z} \right) = u_0(z, \bar{r}) \]
\[ \text{at} \quad \bar{r} = \bar{r}_{in}(z) \quad \text{and} \quad \bar{r} = \bar{r}_{out}(z) \quad (2.37a, b) \]
\[ \frac{1}{f} \sum_{n=-N_s-N_M}^{N_s+N_M} \sum_{m=-N_M}^{N_M} \sum_{k=0}^{N_T} \text{ima} G_k^{(m)} w_k^{(n-m)} e^{in\alpha z} = 0 \]
\[ \text{at} \quad \bar{r} = \bar{r}_{in}(z) \quad \text{and} \quad \bar{r} = \bar{r}_{out}(z) \]

Function \( 1/f \) evaluated at the cylinder can be represented as Fourier series of the form

\[ \frac{1}{f} = \sum_{p=-N_R}^{N_R} Z^{(p)} e^{ip\alpha z} \quad \text{at} \quad \bar{r} = \bar{r}_{in}(z) \quad \text{and} \quad \bar{r} = \bar{r}_{out}(z) \quad (2.38) \]
Coefficients of the above series need to be determined numerically using FFT. The required length of this series depends on the character of \( \frac{1}{f} \), however, in most cases \( N_R = N_M \) was found to be sufficient, in agreement with Moradi, H. V. & Floryan (2012).

Substitution of (2.38) into (2.37a,b) leads to the boundary relations of the form

\[
\Gamma \sum_{n=-N_M}^{N_M} \sum_{m=-N_M}^{N_M} \sum_{k=0}^{N_T} G_k^{(m)} d_k^{(n-m)} \sum_{p=-N_M}^{N_M} d_k^{(p-m)} Z^{(n-p)} e^{inaz} = u_0(z, \bar{r}),
\]

\[
\sum_{n=-N_M}^{N_M} \sum_{m=-N_M}^{N_M} \sum_{k=0}^{N_T} i\alpha G_k^{(m)} d_k^{(n-m)} \sum_{p=-N_M}^{N_M} w_k^{(p-m)} Z^{(n-p)} e^{inaz} = 0,
\]

at \( \bar{r} = \bar{r}_{\text{in}}(z) \) and \( \bar{r} = \bar{r}_{\text{out}}(z) \)

The reference flow velocity \( u_0 \) needs to be evaluated at the inner cylinder and its values represent a periodic function which can be expressed in terms of Fourier series of the form

\[
u_0[z, \bar{r}(z)] = \sum_{n=-N_M}^{N_M} F^{(n)} e^{inaz} \quad \text{at} \quad \bar{r} = \bar{r}_{\text{in}}(z) \quad \text{and} \quad \bar{r} = \bar{r}_{\text{out}}(z)
\]

where the coefficients \( F^{(n)} \) need to be evaluated using FFT. Substitution of (2.40) into (2.39a,b) and separation of Fourier modes lead to the discretization form of the boundary conditions, i.e.

\[
\sum_{m=-N_M}^{N_M} \sum_{k=0}^{N_T} G_k^{(m)} d_k^{(n-m)} \sum_{p=-N_M}^{N_M} d_k^{(p-m)} Z^{(n-p)} = \frac{F^{(n)}}{\Gamma},
\]

\[
\sum_{m=-N_M}^{N_M} \sum_{k=0}^{N_T} i\alpha G_k^{(m)} d_k^{(n-m)} \sum_{p=-N_M}^{N_M} w_k^{(p-m)} Z^{(n-p)} e^{inaz} = 0,
\]

at \( \bar{r} = \bar{r}_{\text{in}}(z) \) and \( \bar{r} = \bar{r}_{\text{out}}(z) \)
2.2.3 Flow constraints

In this section we will describe the discretization of the flow constraints. To begin with, $\psi_0$ has been discretized as follows

$$\psi_0[z, \bar{r}(z)] = \sum_{n=-N_M}^{+N_M} \xi_{in}^{(n)} e^{inaz},$$  \hspace{1cm} (2.42)

where $\xi_{in}^{(n)}$ denote the coefficients of Fourier expansion describing $\psi_0$ and need to be evaluated using FFT. Substituting (2.36a,b) and (2.42) into (2.15) gives

$$\sum_{m=-N_M}^{+N_M} \sum_{k=0}^{N_T} G_k^{(m)} w_{in,k} (-m) = -\xi_{in}^{(0)}$$  \hspace{1cm} (2.43)

The total mass flow rate in the annulus can be expressed as a superposition of the flow rate of the reference flow $Q_{ref}$ given by equation (2.4) and a change in the flow rate due to presence of the ribs $Q_{mod}$, that is,

$$Q_{total} = Q_{ref} + Q_{mod}$$  \hspace{1cm} (2.44)

Substitution of the relation mentioned earlier into (2.16), leads to the following form of the flow constraint

$$\sum_{m=-N_M}^{+N_M} \sum_{k=0}^{N_T} G_k^{(m)} w_{out,k} (-m) = -\xi_{out}^{(0)} - \frac{Q_{ref} + Q_{mod}}{2\pi}$$  \hspace{1cm} (2.45)

It had been assumed in all tests discussed in this study that addition of the ribs did not alter the mass flow through the annulus, that is, $Q_{mod} = 0$.

Discretization of the fixed pressure gradient constraint begins with the $z$-momentum equation expressed in the $(z, \bar{r})$ coordinate system. Using (2.22) and (2.30) and separation of Fourier modes leads to
\[
\begin{align*}
\in \alpha P^{(n)}(\bar{r}) &= -\in \alpha \bar{v}^{(n)}(\bar{r}) - \Gamma D \bar{u}^{(n)}(\bar{r}) - \frac{1}{f} \bar{u}^{(n)}(\bar{r}) \\
&\quad + \in \alpha t \frac{u_0}{f} D\phi^{(n)}(\bar{r}) - \in \alpha \Gamma \frac{\bar{D}u_0}{f} \phi^{(n)}(\bar{r}) \\
&\quad + \frac{1}{Re} \left[ -\Gamma \frac{D\phi^{(n)}(\bar{r})}{f^3} + \Gamma^2 \frac{D^2\phi^{(n)}(\bar{r})}{f^2} - \Gamma^3 \frac{D^3\phi^{(n)}(\bar{r})}{f} + (\Gamma \alpha)^2 \frac{D\phi^{(n)}(\bar{r})}{f} \right]
\end{align*}
\] (2.46)

When \( n \neq 0 \) and

\[
A = \frac{1}{Re} \left[ -\Gamma \frac{\bar{D}\phi^{(0)}}{f^3} + \Gamma^2 \frac{\bar{D}^2\phi^{(0)}}{f^2} - \Gamma^3 \frac{\bar{D}^3\phi^{(0)}}{f} \right] - \Gamma \bar{D}\bar{u}^{(0)} - \frac{\bar{u}^{(0)}}{f}.
\] (2.47)

When \( n = 0 \). Substitution of the relevant Chebyshev expansions into equation (2.47) and enforcement of the resulting relation at any \( \bar{r} \)-location provides form of the fixed pressure gradient constraint suitable for computations, for example,

\[
\sum_{k=0}^{N_T} \left[ -\Gamma \frac{D T_k(\bar{r})}{f^3} + \Gamma^2 \frac{D^2 T_k(\bar{r})}{f^2} + \Gamma^3 \frac{D^3 T_k(\bar{r})}{f} \right] G_k^{(0)} = Re \left[ A + \Gamma \bar{D}\bar{u}^{(0)} + \frac{\bar{u}^{(0)}}{f} \right]
\] (2.48)

2.3 Description of the stationary state

A constant pressure gradient needs to be applied along the annulus in order to produce a desired flow rate. We shall refer to this pressure gradient as a pressure loss. Introduction of ribs changes flow resistance and the pressure gradient correction \( A \) provides a measure of additional pressure loss. The main objective of this section is to determine dependence of \( A \) on the geometric and flow parameters.

In order to reduce the number of geometric parameters we focus our attention on the sinusoidal ribs placed on cylinders with phase difference of \( \phi \) between the ribs at the inner and outer cylinder, i.e.

\[
r_{in}(z) = R_1 + S \cos(\alpha z), \quad r_{out}(z) = 1 + R_1 + S \cos(\alpha z + \phi)
\] (2.49)
which is completely characterized in terms of three geometric parameters, i.e. $R_1$, $\alpha$, $S$ and $\varphi$.

Pressure gradient in smooth annulus as a function of $R_1$ has been determined as shown in Figure 2-2 and is used as the reference case. The ratio of pressure gradient correction in ribbed annulus is then determined with respect to this reference case.

To investigate the effect of Re number, pressure losses have been calculated as a function of Re. It is evident from Figure 2-3 that significant increase of Re number from 1 to 10000 increases ratio of pressure gradient correction only less than 1%. A close look at this figure reveals that effect of Re number is almost negligible in case of small rib wave numbers where pressure loses are effectively independent of Re number.

![Figure 2-2: Variation of the pressure gradient correction $A.Re$ as a function of $R_1$ in smooth annulus. Thin dotted line identifies asymptotes for $R_1 \to \infty$.](image-url)
Figure 2-3: Variation of the ratio of pressure gradient correction $A$ to reference flow pressure gradient as a function of $Re$ when ribs are placed on inner cylinder only with $S=0.015$. Solid, dashed lines indicate $R_1=1, 10$, respectively.

Variations of the pressure gradient correction as a function of the rib amplitude have been also determined. Figure 2-4 shows that as long as the amplitude is small enough, the losses are proportionate to square of the amplitude.

Figure 2-4: Variation of the pressure gradient correction $A.Re$ as a function of the rib amplitude $S$ in an annulus with geometry described by Eq. (2-49) with $\phi = 0$ and $Re=1$. Solid, dashed lines indicate $R_1=1, 10$, respectively.
In Figure 2-5 effect of the curvature has been investigated for different values of rib wave numbers and rib amplitudes when the inner cylinder is corrugated only. Bigger losses are associated with bigger rib wave numbers and rib amplitudes. Moreover, the results suggest that pressure losses increase with $R_1$ and approach their asymptotic values as $R_1 \to \infty$. Figure 2-6 compares the pressure losses when ribs are located only on inner or outer cylinder. In both cases the losses approach to the same value when $R_1 \to \infty$. It is observed, however, that increase of $R_1$ can decrease or increase these losses depending on which cylinder is corrugated. Losses decrease as $R_1$ increases if the ribs are placed on the outer cylinder but increase if the ribs are placed on the inner cylinder.

![Figure 2-5: Variation of the ratio of pressure gradient correction $A$ to reference flow pressure gradient as a function of $R_1$ when ribs are only placed on inner cylinder for $Re=1000$. Solid, dashed, dotted lines indicate $S=0.005, 0.01, 0.015$, respectively. Thin dotted lines identify asymptotes for $R_1 \to \infty$](image)

Figure 2-7 shows the variation of pressure losses as a function of rib wave number and phase shift. The results have been obtained for different values of $R_1$ and $Re$ numbers and as it is observed, increase of $\alpha$ or $\varphi$ result in greater pressure losses. The effect of phase shift, however, is considerable only for small rib wave numbers in all cases.
Figure 2-6: Variation of the ratio of pressure gradient correction $A$ to reference flow pressure gradient as a function of $R_1$ when ribs are only placed on one cylinder for $S=0.015$ and $Re=1000$. Solid, dashed lines indicate ribs placed on inner, outer cylinder, respectively. Thin dotted lines identify asymptotes for $R_1 \to \infty$.

Figure 2-7: Variation of the ratio of pressure gradient correction $A$ to reference flow pressure gradient as a function of $\alpha$ and $\varphi$ in an annulus with geometry described by Eq. (2-49) with $S=0.015$, $Re=1$ (2-7A), $Re=1000$ (2-7B) and $Re=5000$ (2-7C). Solid, dashed lines indicate $R_1=1$, 10, respectively.
Variation of pressure losses as a function of rib amplitude and rib wave number are presented in Figure 2-8. It can be seen that pressure losses increase consistently with $S$ and $\alpha$. It is also evident that regardless of value of $R_1$, effect of the phase shift is merely important for small wavenumbers, that is, for ribs with high wave numbers, the phase shift does not have noticeable influence on the pressure loss.

Figure 2-8: Variation of the ratio of pressure gradient correction $A$ to reference flow pressure gradient as a function of $\alpha$ and $S$ in an annulus with geometry described by Eq. (2-49) with $R_1=1$ (2-8 A), $R_1=10$ (2-8 B) and $Re=1000$. Solid, dashed, dot lines indicate $\varphi=0, \pi/2, \pi$, respectively.

Variation of pressure losses as a function of $R_1$ and $\alpha$ presented in Figure 2-9 also confirms that effect of phase shift is considerable only for small rib wave numbers and that the losses become independent of curvature when value of $R_1$ is large. Moreover, results presented in this figure serve as the validation of the method used in this study since the pressure gradient corrections approach the values determined in case of channel flow as $R_1 \to \infty$. 
Figure 2-9: Variation of the ratio of pressure gradient correction $A$ to reference flow pressure gradient as a function of $\alpha$ and $R_1$ in an annulus with geometry described by Eq. (2-49) with $S=0.015$ and $Re=1000$. Solid, dashed, dotted lines indicate $\varphi=0$, $\pi/2$, $\pi$, respectively. Thin dot lines identify asymptotes for $R_1 \to \infty$. 
Chapter 3

3 Linear Stability

3.1 Formulation of the Linear Stability Analysis

We begin the description of the algorithm by considering arbitrary three-dimensional disturbances and then follow up with the special case of axisymmetric disturbances.

3.1.1 Three-dimensional disturbances

The analysis begins with the momentum and continuity equations written in terms of the primitive variables of the form

\[
\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{V},
\]

(3-1)

\[
\nabla \cdot \mathbf{V} = 0,
\]

(3-2)

where \( \mathbf{V} = (u, v, w) \) is the velocity vector in the \((z, r, \theta)\)-coordinates and \( p \) denotes pressure. Unsteady, three-dimensional disturbances are superimposed on the mean part leading to the flow quantities of the form

\[
\mathbf{V} = \mathbf{V}_2(z, r) + \mathbf{V}_3(z, r, \theta, t) = [u_2, v_2, 0] + [u_3, v_3, w_3],
\]

(3-3)

\[
p = p_2(z, r) + p_3(z, r, \theta, t),
\]

where subscripts 2 and 3 refer to the stationary state and the disturbance fields, respectively. Substitution of (3-3) into (3-1)-(3-2), subtraction of the mean part and linearization lead to the disturbance equations of the form

\[
\nabla \cdot \mathbf{V}_3 = 0,
\]

(3-4)

\[
\frac{\partial \mathbf{V}_3}{\partial t} + (\mathbf{V}_2 \cdot \nabla) \mathbf{V}_3 + (\mathbf{V}_3 \cdot \nabla) \mathbf{V}_2 = -\nabla p_3 + \frac{1}{Re} \nabla^2 \mathbf{V}_3
\]

(3-5)

subject to the no-slip and no-penetration conditions at the inner and outer cylinders of the form
\[ \vec{V}_3 = 0 \text{ at } r = r_{in}(z) \text{ and } r = r_{out}(z). \] (3-6)

We focus our attention on the asymptotic temporal stability which permits separation of the time-dependence. Since (3-4)-(3-5) have coefficients that are functions of \( z \) and \( r \) only, the disturbance flow quantities can be written as

\[
\begin{aligned}
\left[ \vec{V}_3(z, r, \theta, t), p_3(z, r, \theta, t) \right] &= \left[ \tilde{G}_3(z, r), G_p(z, r) \right] e^{i(\delta z + M \theta - \sigma t)} + c.c.
\end{aligned}
\] (3-7)

where \( \delta \) is the real wave number in the axial direction, \( M \) is an integer wave number in the circumferential direction, \( \sigma = \sigma_r + i \sigma_i \) is the complex frequency, \( \sigma_i \) describes the rate of growth of disturbances, \( \sigma_r \) describes their frequency, and \( c.c. \) stands for complex conjugate. \( \tilde{G}_3(z, r) \) and \( G_p(z, r) \) are the \( z \)-periodic amplitude functions and, thus, they can be expressed in terms of Fourier expansions of the form

\[
\begin{aligned}
\left[ \tilde{G}_3(z, r), G_p(z, r) \right] &= \left[ h_u(z, r), h_v(z, r), i h_w(z, r), h_p(z, r) \right] \\
&= \sum_{m=-\infty}^{+\infty} \left[ g_u^{(m)}(r), g_v^{(m)}(r), g_w^{(m)}(r), g_p^{(m)}(r) \right] e^{im\alpha z} + c.c.
\end{aligned}
\] (3-8)

Substitution of (3-8) into (3-7) leads to the disturbance flow quantities of the form

\[
\begin{aligned}
\left[ \vec{V}_3(z, r, \theta, t), p(z, r, \theta, t) \right] &= \sum_{n=-\infty}^{+\infty} \left[ g_u^{(n)}(r), g_v^{(n)}(r), g_w^{(n)}(r), g_p^{(n)}(r) \right] e^{i[(\delta + n\alpha)z + M \theta - \sigma t]} + c.c.
\end{aligned}
\] (3-9)

Substitution of (3-9) into (3-4) and (3-5) and separation of Fourier modes lead to a system of homogeneous ordinary differential equations for \( g_u^{(n)}, g_v^{(n)}, g_w^{(n)}, g_p^{(n)} \) of the form
\[ it_n g_u^{(n)} + r^{-1} D \left( r g_v^{(n)} \right) + ir^{-1} M g_w^{(n)} = 0, \]  
(3-10)

\[ -i \sigma g_u^{(n)} + it_n u_0 g_u^{(n)} + Du_0 g_v^{(n)} + \sum_{m=-N_f}^{N_f} \{ \left[ i t_n f_u^{(m)} + f_v^{(m)} D \right] g_u^{(n-m)} + D f_u^{(m)} g_u^{(n-m)} \} \]
(3-11)

\[ -i \sigma g_v^{(n)} + it_n u_0 g_v^{(n)} + \sum_{m=-N_f}^{N_f} \{ im \alpha f_v^{(m)} g_u^{(n-m)} + \left[ i t_n - m f_u^{(m)} + D f_v^{(m)} + f_v^{(m)} D \right] g_v^{(n-m)} \} \]
(3-12)

\[ -i \sigma g_w^{(n)} + it_n u_0 g_w^{(n)} + \sum_{m=-N_f}^{N_f} \left[ i t_n - m f_u^{(m)} + r^{-1} f_v^{(m)} + f_v^{(m)} D \right] g_w^{(n-m)} \]
(3-13)

and subject to the homogeneous boundary conditions of the form

\[ \vec{V}_3(z, r, \theta, t) = 0 \text{ at } r = r_{in}(z) \text{ and } r = r_{out}(z). \]
(3-14)

In the above, \( t_n = \delta + n \alpha, k_n^2 = t_n^2 + M^2 / r^2, D^p = d^p / dr^p \) and \( N_f \) denotes the number of stationary state Fourier modes retained in the stability equations.

The presence of two periodic directions allows us to further reduce the equations to a more convenient form involving the radial velocity and vorticity components. Radial vorticity is defined as

\[ \eta_3 = \frac{1}{r} \frac{\partial u_3}{\partial \theta} - \frac{\partial w_3}{\partial z} \approx \sum_{n=-N_N}^{N_N} g_n^{(n)}(r) e^{i[(\delta + n \alpha)z + M \theta - \sigma t]} + c.c. \]
(3-15)

\[ = \sum_{n=-N_N}^{N_N} \left( i Mr^{-1} g_u^{(n)} - it_n g_w^{(n)} \right) e^{i[(\delta + n \alpha)z + M \theta - \sigma t]} + c.c. \]
where \( g_{\eta}^{(n)}(r) \) stands for the modal function. The reduction process results in a set of two equations of the form

\[
L^{(n)}_{OS} G_v^{(n)}(n) + C^{(n)}_{OS} \Omega^{(n)} = \sum_{m=-N_j}^{N_j} \left( E^{(n,m)}_{\Omega} \Omega^{(n-m)} + E^{(n,m)}_{v} G_v^{(n-m)} \right),
\]

(3-16)

\[
L^{(n)}_{SQ} \Omega^{(n)} + C^{(n)}_{SQ} G_v^{(n)} = \sum_{m=-N_j}^{N_j} \left( F^{(n,m)}_{\Omega} \Omega^{(n-m)} + F^{(n,m)}_{v} G_v^{(n-m)} \right),
\]

(3-17)

where \( L^{(n)}_{OS} \), \( L^{(n)}_{SQ} \), \( C^{(n)}_{OS} \), and \( C^{(n)}_{SQ} \) are the Orr-Sommerfeld, Squire and coupling operators defined as

\[
L^{(n)}_{OS} = (-i\sigma + i t_n u_0) \mathcal{T}^{(n)} G_v^{(n)} - i r t_n k_n^2 D \left( \frac{D u_0}{r k_n^2} \right) G_v^{(n)} - Re^{-1} \mathcal{T}^{(n)} \mathcal{J}^{(n)} G_v^{(n)},
\]

\[
C^{(n)}_{OS} = 2 M t_n Re^{-1} \mathcal{J}^{(n)},
\]

(3-18)

\[
L^{(n)}_{SQ} = (-i\sigma + i t_n u_0) \Omega^{(n)} - Re^{-1} \mathcal{S}^{(n)} \Omega^{(n)},
\]

\[
C^{(n)}_{SQ} = -i M D u_0 G_v^{(n)} - \frac{2 M t_n}{r^4 k_n^4 Re} \mathcal{T}^{(n)} G_v^{(n)}
\]

(3-19)

In the above,

\[
G_v^{(n)}(r) = -i r g_v^{(n)}, \quad \Omega^{(n)}(r) = \frac{i g_{\eta}^{(n)}}{r k_n^2} = \frac{r t_n g_w^{(n)} - M g_u^{(n)}}{r^2 k_n^2}
\]

(3-20)

and the operators \( \mathcal{T}^{(n)} \) and \( \mathcal{S}^{(n)} \) are defined as

\[
\mathcal{T}^{(n)}(r) = r k_n^2 D \left( \frac{D}{r k_n^2} \right) - k_n^2, \quad \mathcal{S}^{(n)}(r) = \frac{1}{r^3 k_n^2 D(r^3 k_n^2 D) - k_n^2}.
\]

(3-21)

The explicit forms of the operators \( E \) and \( F \) are
\[ E_{\Omega}^{(n,m)} = -2iM^3 \alpha (\alpha a + 2\alpha n - m) \frac{f_u^{(m)}}{r^2 k_n^2} - iM \alpha (\alpha a + 2\alpha n - m) rD f_u^{(m)} \]

\[ -M \left[ \alpha a r k_n^2 + \frac{2\alpha n - m}{r k_n^2} \left( t_n^2 - \frac{M^2}{r^2} \right) \right] f_v^{(m)} + 2Mt_n-mD f_v^{(m)} \]

\[ -M \left[ \alpha a (\alpha a + 2\alpha n - m) r f_u^{(m)} + 2 \left( \frac{mM^2 \alpha}{r^2 k_n^2} - t_n - m \right) f_v^{(m)} + \alpha a D f_v^{(m)} \right] D \]

\[ -M \alpha a r f_v^{(m)} D^2, \]

\[ E_v^{(n,m)} = it_n k_n^2 f_u^{(m)} - it_n \left( t_n^2 - \frac{M^2}{r^2} \right) \frac{D f_u^{(m)}}{r k_n^2} + it_n D^2 f_u^{(m)} + 2k_n D f_v^{(m)} \]

\[ + \left\{ \frac{it_n - m}{r k_n^2} \right\} \left[ -2M^2 \alpha \frac{\alpha a + 2\alpha n - m}{r^2 k_n^2} + \alpha a (\alpha a + 2\alpha n - m) \left( t_n^2 - \frac{M^2}{r^2} \right) \frac{k_n^2}{k_n^2 - m} \right. \]

\[ + \left. \left( t_n^2 - \frac{M^2}{r^2} \right) \frac{f_v^{(m)}}{r k_n^2} \right] + iM \left[ 1 - t_n - m \frac{\alpha a + 2\alpha n - m}{k_n^2 - m} \right] D f_u^{(m)} \]

\[ + \left[ k_n^2 + \frac{2t_n^2}{r^2 k_n^2} - \alpha a t_n - m \frac{k_n^2}{k_n^2 - m} \right] + \frac{2\alpha a t_n - m}{r^2 k_n^2} + \frac{2\alpha a t_n - m}{r^2 k_n^2} - \frac{2t_n^2}{r^2 k_n^2} - \left( t_n^2 - \frac{M^2}{r^2} \right) \]

\[ \left( t_n^2 - \frac{M^2}{r^2} \right) \left( 1 + \frac{\alpha a t_n - m}{k_n^2 - m} \right) D f_v^{(m)} \]

\[ + \left\{ -it_n \left[ 1 + \frac{\alpha a (\alpha a + 2\alpha n - m)}{k_n^2 - m} \right] f_u^{(m)} \right\} \]
\[+2 \left[ \frac{mat_{n-m}}{rk_{n-m}^4} \left( t_{n-m} - \frac{M^2}{r^2} \right) - \frac{mM^2at_{n-m}}{r^3k_{n-m}^2} + \frac{t_{n-m}^2}{rk_{n-m}^2} - \frac{M^2}{r^3k_{n-m}^2} \right] f^{(m)}_v - \left( 1 + \frac{mat_{n-m}}{k_{n-m}^2} \right) D f^{(m)}_v \left\{ D^2 - \left( 1 + \frac{mat_{n-m}}{k_{n-m}^2} \right) f^{(m)}_v D^3, \right.\]

\[
F^{(n,m)}_\Omega = - \frac{t_n}{k_n^2} \left[ ik_{n-m}^2 f^{(m)}_u + 2t_{n-m} \frac{f^{(m)}_v}{r} \right] - \frac{k_{n-m} + mat_{n-m}}{k_n^2} f^{(m)}_v D, \tag{3-24}
\]

\[
F^{(n,m)}_v = iM \frac{Df_u^{(m)}}{r^3k_n^2} - mM \alpha \left( \frac{t_{n-m}^2}{r^2} - \frac{M^2}{r^2} \right) F_v^{(m)} D + mM \alpha \frac{f_v^{(m)}}{r^3k_{n-m}^2} D^2, \tag{3-25}
\]

where \( t_{n-m} = \delta + (n-m)\alpha, k_{n-m}^2 = t_{n-m}^2 + M^2/r^2, \) and \( D^n = d^n/dr^n. \)

The modal functions for the velocity components can be computed \textit{a posteriori} from \( G_v^{(n)} \) and \( \Omega^{(n)} \) using relations of the form

\[
g^{(n)}_u = - \frac{t_nDG_v^{(n)}}{r^2k_n^2} - M\Omega^{(n)}, \quad g^{(n)}_v = \frac{ig_v^{(n)}}{r}, \quad g^{(n)}_\omega = - \frac{MDG_v^{(n)}}{r^2k_n^2} + rt_n\Omega^{(n)}. \tag{3-26}
\]

Terms grouped on the left hand side of (3-16)-(3-17) represent a \( 6^{th} \) order system for mode \( n \) while terms on the right hand side provide the inter-modal coupling associated with the ribs. In the limit of smooth cylinders, the right hand sides of (3-16)-(3-17) disappear, the modal equations decouple and each of them reduces to a coupled system consisting of one fourth-order and one second-order equation for \( G_v^{(n)} \) and \( \Omega^{(n)} \) in the form

\[
(-i\sigma + it_nu_0)\mathcal{T}^{(n)}G_v^{(n)} - irt_nk_n^2 D \left( \frac{Du_0}{r^3k_n^2} \right) G_v^{(n)} = \frac{1}{Re} \mathcal{T}^{(n)}\mathcal{T}^{(n)}G_v^{(n)} - \frac{2Mt_n}{Re} \mathcal{T}^{(n)}\Omega^{(n)}, \tag{3-27}
\]

\[
(-i\sigma + it_nu_0)\Omega^{(n)} - \frac{iMDu_0}{r^3k_n^2} G_v^{(n)} = \frac{1}{Re} S^{(n)}\Omega^{(n)} + \frac{2Mt_n}{r^4k_n^2Re} \mathcal{T}^{(n)}G_v^{(n)} \tag{3-28}
\]
The boundary conditions (3-14) expressed in terms of the modal function are of the form

\[
\begin{align*}
g_u^{(n)} &= -\frac{t_n D \zeta_v^{(n)}}{r k_n^2} - M \Omega^{(n)} = 0, \\
g_v^{(n)} &= \frac{i \zeta_v^{(n)}}{r} = 0, \\
g_w^{(n)} &= -\frac{M D \zeta_v^{(n)}}{r^2 k_n^2} + r t_n \Omega^{(n)} = 0, \\
\end{align*}
\]

(3-29a-c)

at \( r = r_{in}(z) \) and \( r_{out}(z) \).

3.1.2 Axisymmetric disturbances

Axisymmetric disturbances represent a special case of the three-dimensional disturbances with \( M = 0 \). The governing equations for the three-dimensional disturbances discussed above do not reduce to the axisymmetric disturbances and this necessitates the development of a different formulation. The simplest formulation results from the use of the Stokes stream function i.e.

\[
\frac{E^4 \psi}{Re} = \frac{\partial}{\partial t} (E^2 \psi) + \frac{1}{r} \frac{\partial \psi}{\partial z} \left( \frac{\partial}{\partial r} (E^2 \psi) \right) - \frac{1}{r} \frac{\partial \psi}{\partial r} \left( \frac{\partial}{\partial z} (E^2 \psi) \right) - \frac{2}{r^2} \frac{\partial^2 \psi}{\partial z^2} (E^2 \psi) \tag{3-30}
\]

where \( E^2 \) is the operator defined by (2.13). Disturbances are superimposed on the mean part leading to the stream functions of the form

\[
\psi(z, r, t) = \psi_2(z, r) + \psi_3(z, r, t) \tag{3-31}
\]

where subscripts 2 and 3 refer to the stationary state and the disturbance fields, respectively. Equation (3-31) is substituted into (3-30), the mean part is subtracted, and the remainder is linearized. The resulting linear disturbance equation has the form

\[
\frac{E^4 \psi_3}{Re} = \frac{\partial}{\partial t} (E^2 \psi_3) + \frac{1}{r} \frac{\partial \psi_2}{\partial z} \left( \frac{\partial}{\partial r} (E^2 \psi_3) \right) + \frac{1}{r} \frac{\partial \psi_3}{\partial z} \left( \frac{\partial}{\partial r} (E^2 \psi_2) \right) - \frac{1}{r} \frac{\partial \psi_2}{\partial r} \left( \frac{\partial}{\partial z} (E^2 \psi_3) \right) - \frac{1}{r} \frac{\partial \psi_3}{\partial r} \left( \frac{\partial}{\partial z} (E^2 \psi_2) \right) - \frac{2}{r^2} \frac{\partial^2 \psi_2}{\partial z^2} (E^2 \psi_3) - \frac{2}{r^2} \frac{\partial \psi_2}{\partial r} \left( \frac{\partial}{\partial z} (E^2 \psi_3) \right) - \frac{2}{r^2} \frac{\partial \psi_3}{\partial r} \left( \frac{\partial}{\partial z} (E^2 \psi_2) \right) \tag{3-32}
\]
and is subject to the homogeneous boundary conditions of the form

\[
\frac{1}{r} \frac{\partial \psi_3}{\partial r} = 0, \quad \frac{1}{r} \frac{\partial \psi_3}{\partial z} = 0 \quad \text{at} \quad r = r_{in}(z) \text{ and } r = r_{out}(z). \tag{3-33}
\]

Since our interest is in the temporal stability problem and (3-32) has coefficients that are functions of \( z \) and \( r \) only, the disturbance stream function is written in the form

\[
\psi_3(z, r, t) = \Lambda_3(z, r)e^{i(\delta z - \sigma t)} + c.c. \tag{3-34}
\]

where \( \delta \) is the real axial wave number, \( \sigma = \sigma_r + i\sigma_i \) is the complex frequency, \( \sigma_i \) describes the rate of growth of disturbances, \( \sigma_r \) describes their frequency, and c.c. stands for the complex conjugate. Function \( \Lambda_3(z, r) \) is the \( z \)-periodic amplitude function and, thus, it can be expressed in terms of the Fourier expansion of the form

\[
\Lambda_3(z, r) = \sum_{m=-\infty}^{+\infty} g^{(m)}(r)e^{im\alpha z}. \tag{3-35}
\]

Substitution of (3-35) into (3-34) leads to the final form of the disturbance stream function, i.e.

\[
\psi_3(z, r, t) = \sum_{m=-\infty}^{+\infty} g^{(m)}(r)e^{i(\delta z + m\alpha z - \sigma t)} + c.c. \tag{3-36}
\]

\[
\approx \sum_{m=-N}^{N} g^{(m)}(r)e^{i(\delta + m\alpha z - \sigma t)} + c.c. 
\]

and, eventually, to the ordinary differential equations for the modal functions \( g^{(m)}(r) \) of the form

\[
\frac{1}{Re} \left\{ (D^2 - t_n^2)g^{(n)} + \frac{1}{r} \left[ 2t_n^2 - \frac{3}{r^2} \right] D + \frac{3D^2}{r^2} - \frac{2D^3}{r} \right\} g^{(n)} + it_n \left[ t_n^2u_0 - \frac{Du_0}{r} + D^2u_0 + \frac{u_0}{r}D - u_0D^2 \right] g^{(n)} + i\sigma \left[ D^2 - t_n^2 - \frac{D}{r} \right] g^{(n)} = 0. \tag{3-37}
\]
\[ \sum_{m=-N}^{N} \left\{ i t_{n-m} \left[ 2 m a t_n \frac{\phi^{(m)}}{r^2} + \left( \frac{3}{r^2} + t_{n-2m} t_n \right) \frac{D\phi^{(m)}}{r} - 3 \frac{D^2\phi^{(m)}}{r^2} \right. \right. + \left. \frac{D^3\phi^{(m)}}{r^3} \right] \right. \]

\[ = \frac{1}{Re} \left[ \left( D^2 - \delta^2 \right)^2 + \frac{1}{r} \left( 2 \delta^2 - \frac{3}{r^2} \right) D + \frac{3D^2}{r^2} - \frac{2D^3}{r} \right] g \]

\[ + i \delta \left( \delta^2 u_0 - \frac{Du_0}{r} + D^2 u_0 + \frac{u_0 r}{r} D - u_0 D^2 \right) g(r) = -i \sigma \left( D^2 - t_n^2 - \frac{D}{r} \right) g \] (3-38)

given for the first time by Corcos & Sellars (1959) and being equivalent to the Orr–Sommerfeld equation. The complete problem consists of an infinite set of equations of type (3-37) supplemented with the homogeneous boundary conditions.

### 3.2 Discretization and Numerical Solution of the Linear Stability Problem

The stability problem represents an eigenvalue problem which has a nontrivial solution only for certain combinations of \((\delta, M, \sigma)\) for the flow conditions \((Re)\) and the rib geometry \((R_1, \alpha, H_{in}^{(n)}, H_{out}^{(n)})\) of interest. The required dispersion relation has to be determined numerically and the relevant methodology is described in this Section.
We wish to determine a spectrally accurate solution of the eigenvalue problem described by (3-16)-(3-17) with the boundary conditions (3-14). We also wish to solve the simpler problem for axisymmetric disturbances consisting of (3-37) with (3-33). Since the boundary conditions need to be applied at the surfaces of cylinders with a complex geometry described by (2.1), we wish to develop an algorithm which is able to deal with all types of surfaces with minimal effort. Here we take advantage of the immersed boundary conditions (IBC) concept where the physical domain is immersed inside the cylindrical computational box and the boundary conditions are imposed as the internal constraints.

Figure 2-1 illustrates the computational domain which is bounded by \(-R_{in} + R_1\) from below and by \(1 + R_1 + R_{out}\) from above in the radial direction, where \(R_{in}\) and \(R_{out}\) denote locations of the rib extremities at the inner and outer cylinders, respectively. The radial domain is mapped onto \([-1,1]\) in order to use the standard definition of Chebyshev polynomials. The transformation used in this study has the form

\[
\tilde{r} = \Gamma (r + C) \quad \text{where} \quad \Gamma = 2(1 + R_{in} + R_{out})^{-1}, \quad C = 0.5(R_{in} - R_{out} - 2R_1 - 1),
\]

with \(\tilde{r} \in [-1,1]\). The cylinder geometry in the \((z, \tilde{r})\)-coordinate system has the form

\[
\tilde{r}_{in}(z) = \sum_{n=-N_A}^{N_A} A_{in}^{(n)} e^{inz}, \quad \tilde{r}_{out}(z) = \sum_{n=-N_A}^{N_A} A_{out}^{(n)} e^{inz}
\]

(3-40)

where

\[
A_{in}^{(n)} = \begin{cases} \Gamma \left( R_1 + C + H_{in}^{(n)} \right), & n = 0 \\ \Gamma H_{in}^{(n)}, & n \neq 0 \end{cases}, \quad A_{out}^{(n)} = \begin{cases} \Gamma \left( 1 + R_1 + C + H_{out}^{(n)} \right), & n = 0 \\ \Gamma H_{out}^{(n)}, & n \neq 0 \end{cases},
\]

(3-41)

\[
f(\tilde{r}) = \Gamma^{-1} \tilde{r} - C
\]
3.2.1 Three-dimensional disturbances

Equations (3-16) and (3-17) expressed in the \((z, \bar{r})\)-coordinates have the form

\[
\begin{align*}
(-i\sigma + it_n u_0)\tilde{f}^{(n)}(\bar{r})v^{(n)} - if t_n \bar{k}_n^2 \Gamma \bar{D} \left( \frac{\Gamma \bar{D} u_0}{f^2 \bar{k}_n^2} \right) v^{(n)} \\
- Re^{-1}\tilde{f}^{(n)}(\bar{r})\bar{f}^{(n)}(\bar{r})G_v^{(n)} + 2Mt_n Re^{-1}\tilde{f}^{(n)}(\bar{r})\Omega^{(n)} &= \sum_{m=-N_J}^{N_J} \left( \tilde{E}^{(n,m)}_{\Omega} \Omega^{(n-m)} + \tilde{E}^{(n,m)}_v G_v^{(n-m)} \right), \\
(3-42)
\end{align*}
\]

\[
\begin{align*}
(-i\sigma + it_n u_0)\Omega^{(n)} - \frac{iM\Gamma \bar{D} u_0}{f^2 \bar{k}_n^2} G_v^{(n)} - Re^{-1}\tilde{\delta}^{(n)}(\bar{r})\Omega^{(n)} &= \sum_{m=-N_J}^{N_J} \left( \tilde{F}^{(n,m)}_{\Omega} \Omega^{(n-m)} + \tilde{F}^{(n,m)}_v G_v^{(n-m)} \right), \\
(3-43)
\end{align*}
\]

where \(\bar{k}_n^2 = \delta^2 + M^2/f^2\), \(f(\bar{r})\) is given by (3-41), and \(\bar{D} = d/d\bar{r}\). Operators \(\tilde{f}^{(n)}(\bar{r})\), \(\tilde{\delta}^{(n)}(\bar{r})\), \(\tilde{E}^{(n,m)}_{\Omega}\), \(\tilde{E}^{(n,m)}_v\), \(\tilde{F}^{(n,m)}_{\Omega}\), \(\tilde{F}^{(n,m)}_v\) have the form

\[
\begin{align*}
\tilde{f}^{(n)}(\bar{r}) &= \Gamma \bar{D} \frac{\bar{D}}{f^2 \bar{k}_n^2} - \bar{k}_n^2, \\
\tilde{\delta}^{(n)}(\bar{r}) &= \frac{\Gamma^2}{f^2 \bar{k}_n^2} \bar{D} \left( f^2 \bar{k}_n^2 \bar{D} \right) - \bar{k}_n^2. \\
(3-44)
\end{align*}
\]

\[
\begin{align*}
\tilde{E}^{(n,m)}_{\Omega} &= -2imM^3 \alpha (m\alpha + 2t_{n-m}) f_u^{(m)} \frac{f_u^{(m)}}{f^2 k_n^2} - \Gamma mM^3 \alpha (m\alpha + 2t_{n-m}) f_u^{(m)} \\
- M \left[ m\alpha \bar{k}_n^2 + 2t_{n-m} \left( t_n^2 - \frac{M^2}{f^2} \right) \right] f_u^{(m)} + 2\Gamma Mt_{n-m} \bar{D} f_u^{(m)} \\
- M \left[ t_n \left( t_n^2 - \frac{M^2}{f^2} \right) \right] f_u^{(m)} + \Gamma m\alpha \bar{D} f_u^{(m)} \\
- \Gamma^2 mM^3 \alpha f_u^{(m)} \bar{D}^2, \\
(3-45)
\end{align*}
\]

\[
\begin{align*}
\tilde{E}^{(n,m)}_v &= it_n \bar{k}_n^2 f_u^{(m)} - \Gamma it_n \left( t_n^2 - \frac{M^2}{f^2} \right) \bar{D} f_u^{(m)} + \Gamma^2 it_n \bar{D}^2 f_u^{(m)} + 2\Gamma \bar{k}_n^2 \bar{D} f_u^{(m)} + (3-46)
\end{align*}
\]
\[
\Gamma \left\{ \frac{it_{n-m}}{f k_n^2} \right\} \left[ -2mM^2 \alpha \frac{ma + 2t_{n-m}}{f^2 k_n^2 - m} + ma(ma + 2t_{n-m}) \left( t_{n-m}^2 - M^2 f^2 \right) \frac{k_n^2}{k_{n-m}^2} \right]
\]

\+
\frac{t_n^2}{f^2} - \frac{M^2}{f^2} f_u^{(m)} + \Gamma m^2 \left[ 1 - t_{n-m} \frac{ma + 2t_{n-m}}{k_n^2 - m} \right] D f_u^{(m)} + \left[ \frac{k_n^2}{k_{n-m}^2} + \frac{2t_n^2}{f^2 k_n^2} \right]

\-
mat_{n-m} \frac{k_n^2}{k_{n-m}^2} - \frac{2t_{n-m}}{f^2 k_n^2 k_{n-m}^2} \left( t_n^2 - \frac{M^2}{f^2} \right) + \frac{2mM^2 a t_{n-m}}{f^4 k_n^2 k_{n-m}^4} \left( t_{n-m}^2 - \frac{M^2}{f^2} \right)

\-
\frac{2t_{n-m}}{f^2 k_{n-m}^4} \left( t_{n-m}^2 - \frac{M^2}{f^2} \right) + \frac{2mat_{n-m}}{f^2 k_{n-m}^4} \left( t_{n-m}^2 - \frac{2t_{n-m}}{k_{n-m}^2} + \frac{2M^2 t_{n-m}}{f^2 k_{n-m}^4} \right) f_v^{(m)}

\+
\Gamma \left( t_{n-m}^2 - \frac{M^2}{f^2} \right) \left( 1 + \frac{mat_{n-m}}{k_n^2} \right) \frac{D f_v^{(m)}}{f k_{n-m}^2} \bar{D}

\+
\Gamma^2 \left\{ -it_{n-m} \left[ 1 + \frac{ma(ma + 2t_{n-m})}{k_n^2 - m} \right] f_u^{(m)} + 2 \left[ \frac{mat_{n-m}}{f k_{n-m}^4} \left( t_{n-m}^2 - \frac{M^2}{f^2} \right) \right]

\-
mM^2 a t_{n-m} \frac{f^3 k_n^2 k_{n-m}^2}{f^3 k_n^2} + \frac{t_{n-m}}{f^3 k_n^2 k_{n-m}^2} - \frac{M^2}{f^3 k_n^2 k_{n-m}^2} f_v^{(m)} - \Gamma \left( 1 + \frac{mat_{n-m}}{k_n^2} \right) \bar{D} \right\}

\bar{D}

\-
\Gamma^3 \left( 1 + \frac{mat_{n-m}}{k_n^2} \right) f_v^{(m)} \bar{D}^3,

\bar{F}_\Omega^{(n,m)} = - \frac{t_n}{k_n^2} \left[ i k_{n-m} f_u^{(m)} + 2t_{n-m} f_v^{(m)} \right] - \Gamma \frac{k_{n-m}^2 + mat_{n-m}}{k_n^2} f_v^{(m)} \bar{D}, \quad (3-47)

\bar{F}_v^{(n,m)} = \Gamma iM \frac{D f_u^{(m)}}{f^3 k_n^2} - \Gamma m M_a \left( t_{n-m}^2 - \frac{M^2}{f^2} \right) f_v^{(m)} \frac{f_v^{(m)}}{f^3 k_n^2 k_{n-m}^2} \bar{D}

\quad + \Gamma^2 m M_a \frac{f_v^{(m)}}{f^3 k_n^2 k_{n-m}^2} \bar{D}^2,

\quad (3-48)

The modal functions are expressed in terms of Chebyshev expansions of the form

\[
\left[ G_v^{(n)}(\bar{r}), \Omega^{(n)}(\bar{r}) \right] = \sum_{k=0}^{+\infty} \left[ G_{k,v}^{(n)}, G_{k,\Omega}^{(n)} \right] T_k(\bar{r}) \approx \sum_{k=0}^{N_C} \left[ G_{k,v}^{(n)}, G_{k,\Omega}^{(n)} \right] T_k(\bar{r}) \quad (3-49)
\]
where $T_k$ denotes the Chebyshev polynomial of the $k^{th}$ order, and $G_{k,v}^{(m)}$ and $G_{k,\Omega}^{(m)}$ stand for the unknown coefficients of the expansions.

The Chebyshev collocation technique based on the Gauss-Chebyshev-Lobatto points is used to form a system of algebraic equations. Locations of the Chebyshev collocation points in the $\bar{r}$-direction are given as

$$\bar{r}_j = \cos \left( \frac{\pi j}{N_C} \right), j \in [0, N_C].$$  \hfill (3-50)

Substitution of (3-49) into (3-43) and evaluation of the resulting equations at points given by (3-50) lead to a system of homogeneous equations for $G_{k,v}^{(m)}$ and $G_{k,\Omega}^{(m)}$ of the form

$$\sum_{k=0}^{N_C} \left[ (-i\sigma + it_n u_0) \bar{\mathcal{F}}^{(n)}(\bar{r}_j) - i f t_n k_n^2 \Gamma D \left( \frac{\Gamma D u_0}{f k_n^2} \right) - \frac{1}{Re} \bar{\mathcal{F}}^{(n)}(\bar{r}_j) \right] T_k(\bar{r}_j) G_{k,v}^{(n)}$$

$$+ 2M \sum_{k=0}^{N_C} \frac{t_n}{Re} \bar{\mathcal{F}}^{(n)}(\bar{r}_j) T_k(\bar{r}_j) G_{k,\Omega}^{(n)}$$

$$= \sum_{m=-N_J}^{N_J} \sum_{k=0}^{N_C} \bar{E}_v^{(n,m)} T_k(\bar{r}_j) G_{k,v}^{(n)} + \sum_{m=-N_J}^{N_J} \sum_{k=0}^{N_C} \bar{E}_\Omega^{(n,m)} T_k(\bar{r}_j) G_{k,\Omega}^{(n)}$$

$$= \sum_{k=0}^{N_C} \left[ (-i\sigma + it_n u_0) \bar{\mathcal{F}}^{(n)}(\bar{r}_j) - \frac{1}{Re} \bar{\mathcal{F}}^{(n)}(\bar{r}_j) \right] T_k(\bar{r}_j) G_{k,\Omega}^{(n)}$$

$$- \sum_{k=0}^{N_C} \left[ i M \Gamma D u_0 \right] f^3 k_n^2 + 2M t_n \bar{\mathcal{F}}^{(n)}(\bar{r}_j) T_k(\bar{r}_j) G_{k,v}^{(n)}$$

$$= \sum_{m=-N_J}^{N_J} \sum_{k=0}^{N_C} \bar{E}_v^{(n,m)} T_k(\bar{r}_j) G_{k,v}^{(n)} + \sum_{m=-N_J}^{N_J} \sum_{k=0}^{N_C} \bar{E}_\Omega^{(n,m)} T_k(\bar{r}_j) G_{k,\Omega}^{(n)}.$$  \hfill (3-52)

Points closest to the edges of the computational domain are omitted for both equations to provide space for the boundary conditions given by (3-29a-c). The methods used for the discretization of the boundary conditions at the inner and outer cylinders are identical
and, thus, we shall describe only the inner cylinder. Conditions (3-6) for the inner cylinder expressed in terms of the radial component of the disturbance velocity and the radial component of disturbance vorticity have the following form

$$u_3[z, \tilde{r}_{\text{in}}(z), \theta, t] = \sum_{n=-N_N}^{N_N} \left( -\frac{\Gamma n \bar{D} G_v^{(n)}}{f k_n^2} - M \Omega^{(n)} \right) e^{i[(\delta + n \alpha)z + M \theta - \sigma t]}_{r=\tilde{r}_{\text{in}}(z)} = 0, \quad (3-53)$$

$$v_3[z, \tilde{r}_{\text{in}}(z), \theta, t] = \sum_{n=-N_N}^{N_N} \left( \frac{iG_v^{(n)}}{f} \right) e^{i[(\delta + n \alpha)z + M \theta - \sigma t]}_{r=\tilde{r}_{\text{in}}(z)} = 0, \quad (3-54)$$

$$w_3[z, \tilde{r}_{\text{in}}(z), \theta, t] = \sum_{n=-N_N}^{N_N} \left( -\frac{\Gamma M \bar{D} G_v^{(n)}}{f^2 k_n^2} + t_n f \Omega^{(n)} \right) e^{i[(\delta + n \alpha)z + M \theta - \sigma t]}_{r=\tilde{r}_{\text{in}}(z)} = 0 \quad (3-55)$$

where the location of the cylinder is given by (3-40) and $f[\tilde{r}_{\text{in}}(z)]$ is given by (3-41). Use of (3-49) brings (3-53)-(3-55) to the following form

$$\sum_{n=-N_N}^{N_N} \sum_{k=0}^{N_C} \left( -\frac{\Gamma n \bar{D} T_k G_{k,v}^{(n)}}{f k_n^2} - M T_k G_{k,\Omega}^{(n)} \right) e^{i[(\delta + n \alpha)z + M \theta - \sigma t]}_{r=\tilde{r}_{\text{in}}(z)} = 0, \quad (3-56)$$

$$\sum_{n=-N_N}^{N_N} \sum_{k=0}^{N_C} \left( \frac{iT_k G_{k,v}^{(n)}}{f} \right) e^{i[(\delta + n \alpha)z + M \theta - \sigma t]}_{r=\tilde{r}_{\text{in}}(z)} = 0, \quad (3-57)$$

$$\sum_{n=-N_N}^{N_N} \sum_{k=0}^{N_C} \left( -\frac{\Gamma M \bar{D} T_k G_{k,v}^{(n)}}{f^2 k_n^2} + t_n f T_k G_{k,\Omega}^{(n)} \right) e^{i[(\delta + n \alpha)z + M \theta - \sigma t]}_{r=\tilde{r}_{\text{in}}(z)} = 0. \quad (3-58)$$

The above relations require evaluation of the Chebyshev polynomials and their derivatives along the cylinder surface. The resulting values represent periodic functions of $z$ and, thus, can be expressed using Fourier expansions of the form

$$T_k[\tilde{r}_{\text{in}}(z)] = \sum_{m=-N_S}^{N_S} (w_{in})_k^{(m)} e^{im\alpha z}, \quad \bar{D} T_k[\tilde{r}_{\text{in}}(z)] = \sum_{m=-N_S}^{N_S} (d_{in})_k^{(m)} e^{im\alpha z} \quad (3-59)$$
where $N_S = N_C N_A$. Substitution of (3-59) into (3-56)-(3-58) leads to boundary conditions of the form

$$
\sum_{n=-N_S-N_N}^{N_S+N_N} \sum_{m=-N_N}^{N_N} \sum_{k=0}^{N_C} \left( - \Gamma \frac{t \Gamma G^{(m)}_{k,\nu} (d_{in})^{(n-m)}_k}{f k_m^2} - MG^{(m)}_{k,\Omega} (w_{in})^{(n-m)}_k \right) \bigg|_{\bar{r} = \bar{r}_{in}(z)} e^{i[(\delta + n\alpha)z + M\theta - \sigma t]} = 0,
$$

(3-60)

$$
\sum_{n=-N_S-N_N}^{N_S+N_N} \sum_{m=-N_N}^{N_N} \sum_{k=0}^{N_C} \left( i \frac{G^{(m)}_{k,\nu} (w_{in})^{(n-m)}_k}{f} \right) \bigg|_{\bar{r} = \bar{r}_{in}(z)} e^{i[(\delta + n\alpha)z + M\theta - \sigma t]} = 0,
$$

(3-61)

$$
\sum_{n=-N_S-N_N}^{N_S+N_N} \sum_{m=-N_N}^{N_N} \sum_{k=0}^{N_C} \left( - \frac{t \Gamma G^{(n)}_{k,\nu} (d_{in})^{(n-m)}_k}{f^2 k_m^2} + \Gamma MG^{(n)}_{k,\Omega} (w_{in})^{(n-m)}_k \right) \bigg|_{\bar{r} = \bar{r}_{in}(z)} e^{i[(\delta + n\alpha)z + M\theta - \sigma t]} = 0.
$$

(3-62)

Functions $1/(f k_m^2)$, $1/f$, $1/(f^2 k_m^2)$ are periodic in the $z$-direction and, thus, can be expressed in terms of Fourier expansions of the form

$$
\frac{1}{f k_m^2} \approx \sum_{p=-\infty}^{+\infty} F_{u,m}^{(p)} e^{ip\alpha z} \approx \sum_{p=-N_R}^{N_R} F_{u,m}^{(p)} e^{ip\alpha z},
$$

(3-63)

$$
\frac{1}{f} \approx \sum_{p=-\infty}^{+\infty} F_{v}^{(p)} e^{ip\alpha z} \approx \sum_{p=-N_R}^{N_R} F_{v}^{(p)} e^{ip\alpha z},
$$

(3-64)

$$
\frac{1}{f^2 k_m^2} \approx \sum_{p=-\infty}^{+\infty} F_{w,m}^{(p)} e^{ip\alpha z} \approx \sum_{p=-N_R}^{N_R} F_{w,m}^{(p)} e^{ip\alpha z},
$$

(3-65)

$$
f \approx \sum_{p=-\infty}^{+\infty} F_{f}^{(p)} e^{ip\alpha z} \approx \sum_{p=-N_R}^{N_R} F_{f}^{(p)} e^{ip\alpha z}
$$

(3-66)
where \( F_{u,m}^{(p)}, F_v^{(p)}, F_w^{(p)} \) need to be computed numerically using FFT, \( F_f^{(p)} = A_{in}^{(p)}/\Gamma \) for \( p \neq 0 \), \( F_f^{(p)} = A_{in}^{(p)}/\Gamma - C \) for \( p = 0 \) and Fourier coefficients \( A_{in}^{(p)} \) are given by (3-41).

The reader may note that functions \( 1/f k_m^2 \) and \( 1/f^2 k_m^2 \) need to be evaluated separately for each Fourier mode \( m \). This expansion is truncated after \( N_R \) terms. The exact value of \( N_R \) depends on the geometry, however experience shows that typically \( N_R = N_N \) is sufficient. Substitution of (3-63)-(3-65) into (3-60)-(3-62) results in

\[
\sum_{n=-N_S-N_R-N_N}^{N_S+N_R+N_N} \sum_{m=-N_N}^{N_N} \sum_{p=-N_N-N_S}^{N_N-N_S} \sum_{k=0}^{N_C} \left[ -\Gamma t_m G_{k,v}^{(m)} (d_{in})_k^{(p-m)} F_{u,m}^{(n-p)} - M G_{k,\Omega}^{(m)} (w_{in})_k^{(n-m)} \right] e^{i[(\delta+n\alpha)z+M\theta-\sigma t]} = 0, \tag{3-67}
\]

\[
\sum_{n=-N_S-N_R-N_N}^{N_S+N_R+N_N} \sum_{m=-N_N}^{N_N} \sum_{p=-N_N-N_S}^{N_N-N_S} \sum_{k=0}^{N_C} i G_{k,v}^{(m)} (w_{in})_k^{(p-m)} F_v^{(n-p)} e^{i[(\delta+n\alpha)z+M\theta-\sigma t]} = 0, \tag{3-68}
\]

\[
\sum_{n=-N_S-N_R-N_N}^{N_S+N_R+N_N} \sum_{m=-N_N}^{N_N} \sum_{p=-N_N-N_S}^{N_N-N_S} \sum_{k=0}^{N_C} \left[ -\Gamma M G_{k,v}^{(m)} (d_{in})_k^{(p-m)} F_{w,m}^{(n-p)} + t_m G_{k,\Omega}^{(m)} (w_{in})_k^{(p-m)} F_f^{(n-p)} \right] e^{i[(\delta+n\alpha)z+M\theta-\sigma t]} = 0, \tag{3-69}
\]

Separation of the Fourier modes results in boundary relations whose enforcement is equivalent to the enforcement of the flow boundary conditions and whose form is suitable for the numerical implementation. These boundary relations are of the form

\[
\sum_{m=-N_N}^{N_N} \sum_{k=0}^{N_C} \Gamma t_m G_{k,v}^{(m)} (d_{in})_k^{(p-m)} F_{u,m}^{(n-p)} + \sum_{m=-N_N}^{N_N} \sum_{k=0}^{N_C} M G_{k,\Omega}^{(m)} (w_{in})_k^{(n-m)} \tag{3-70}
\]

\[
= 0,
\]

\[
\sum_{m=-N_N}^{N_N} \sum_{k=0}^{N_C} G_{k,v}^{(m)} (w_{in})_k^{(p-m)} F_v^{(n-p)} = 0, \tag{3-71}
\]
\[\begin{align*}
\sum_{m=-N_N}^{N_N} \sum_{k=0}^{N_C} -\Gamma M G_{k,\nu}^{(m)} & \sum_{p=-N_N-NS}^{N_N+NS} (d_{in})_k^{(p-m)} f_{w,m}^{(n-p)} \\
+ \sum_{m=-N_N}^{N_N} \sum_{k=0}^{N_C} t_m G_{k,\Omega}^{(m)} & \sum_{p=-N_N-NS}^{N_N+NS} (w_{in})_k^{(p-m)} f_f^{(n-p)} = 0.
\end{align*}\] (3-72)

3.2.2 Axisymmetric disturbances

The numerical discretization of the field equation for the axisymmetric disturbances follows the same procedure as used for the three-dimensional disturbances. It begins with the replacement of \( r \) in (3-37) with \( \bar{r} \) defined by (3-39) to bring the disturbance equations to a form suitable for the discretization using expansions in terms of the classical Chebyshev polynomials defined on \((-1,1)\), i.e.

\[
\frac{1}{Re} \left\{ (\Gamma^2 \bar{D}^2 - t_n^2) + \frac{\Gamma}{f} \left[ 2t_n^2 - \frac{3}{f^2} \right] \bar{D} + \frac{3\Gamma^2 \bar{D}^2}{f^2} - \frac{2\Gamma^3 \bar{D}^3}{f^2} \right\} g^{(n)} + it_n
\]

\[
[ t_n^2 u_0 - \frac{\Gamma \bar{D} u_0}{f} + \Gamma \bar{D} u_0 + \Gamma \frac{u_0}{f} \bar{D} - \Gamma^2 u_0 \bar{D}^2 ] g^{(n)} + i\sigma \left[ \Gamma^2 \bar{D}^2 - t_n^2 - \frac{\Gamma \bar{D}}{f} \right] g^{(n)}
\]

\[
= \sum_{m=-N_J}^{N_J} \left\{ it_{n-m} \left[ 2m\alpha t_n \frac{\phi^{(m)}}{f^2} + \Gamma \left( \frac{3}{f^2} + t_n - 2m t_n \right) \frac{\bar{D} \phi^{(m)}}{f} - 3 \frac{\Gamma^2 \bar{D}^2 \phi^{(m)}}{f^2} \right] + \frac{\Gamma^3 \bar{D}^3 \phi^{(m)}}{f^3} \right\} g^{(n-m)} + i\sigma \left[ m\alpha \left( \frac{3}{f^2} - t_n - 2m t_n \right) \frac{\phi^{(m)}}{f} + t_n \frac{\Gamma \bar{D} \phi^{(m)}}{f^2} \right.
\]

\[
- m\alpha \left( \frac{\Gamma^2 \bar{D}^2 \phi^{(m)}}{f} \right) \bar{D} g^{(n-m)} - i\Gamma^2 \left[ t_{n-m} \frac{\Gamma \bar{D} \phi^{(m)}}{f} + 3m\alpha \frac{\phi^{(m)}}{f^2} \right] \bar{D}^2 g^{(n-m)}
\]

\[
+ ima \frac{\Gamma^3 \phi^{(m)}}{f} \bar{D}^3 g^{(n-m)} \right\}
\] (3-73)

The modal functions are expressed using Chebyshev expansions of the form
\[ g^{(m)}(\tilde{r}) = \sum_{k=0}^{N_C} F^{(m)}_k T_k(\tilde{r}) \]  

(3-74)

where \( F^{(m)}_k \) stands for the unknown expansion coefficient. Substitution of (3-74) into (3-73) leads to a system of ordinary differential equations of the form

\[
\sum_{k=0}^{N_C} \frac{1}{f} \left\{ \left( \Gamma^2 D^2 - t_n^2 \right)^2 + \frac{\Gamma}{f} \left[ 2t_n^2 - \frac{3}{f^2} \right] D + \frac{3\Gamma^2 D^2}{f^2} - \frac{2\Gamma^3 D^3}{f^2} \right\} T_k F^{(n)}_k + it_n t_n^2 u_0 \\
+ \frac{\Gamma D u_0}{f} + \Gamma^2 D^2 u_0 + \frac{\Gamma u_0}{f} D - \Gamma^2 u_0 D^2 \right\} T_k F^{(n)}_k + \left[ \Gamma^2 D^2 - t_n^2 - \frac{\Gamma D}{f} \right] T_k F^{(n)}_k \\
= \sum_{m=-N_J}^{N_J} \sum_{k=0}^{N_C} \left\{ i t_{n-m} \left[ 2m \alpha t_n \frac{\phi^{(m)}_k}{f^2} + \Gamma \left( \frac{3}{f^2} + t_{n-2m} t_n \right) \frac{\phi^{(m)}_k}{f} - 3 \frac{\Gamma^2 D^2 \phi^{(m)}_k}{f^2(\tilde{r})} + \frac{\Gamma^3 D^3 \phi^{(m)}_k}{f^3(\tilde{r})} \right] T_k F^{(n-m)}_k \\
\right. \\
\left. + \Gamma m \alpha \left( \frac{3}{f^2} - t_{n-2m} t_n \right) \frac{\phi^{(m)}_k}{f} + t_n \frac{\Gamma D \phi^{(m)}_k}{f^2} \right\} T_k F^{(n-m)}_k \\
- \frac{\Gamma^2 D^2 \phi^{(m)}_k}{f} \right\} T_k F^{(n-m)}_k - i \Gamma^2 \left[ t_{n-m} \frac{\Gamma D \phi^{(m)}_k}{f} + 3m \alpha \frac{\phi^{(m)}_k}{f^2} \right] D^2 T_k F^{(n-m)}_k \\
+ im \alpha \frac{\Gamma^3 \phi^{(m)}_k}{f} D^3 T_k F^{(n-m)}_k \right\}
\]

(3-75)

whose evaluation on the Gauss-Lobatto collocation points results in a system of algebraic equations for the coefficients \( F^{(m)}_k \). Equations corresponding to the first two and the last two points are replaced with the boundary relations.

Boundary conditions at the inner and outer cylinders are given by Eq. (3-33). Since the treatment of boundary conditions at both cylinders is identical, only boundary conditions at the inner cylinder are discussed. Substitution of (3-36) into (3-33) and introduction of transformation (3-39) expresses these conditions in terms of the modal functions, i.e.
\[
\frac{1}{r} \sum_{m=-N_N}^{N_N} D g^{(m)} e^{i[(\delta+m\alpha)z-\sigma t]} = 0, \quad \text{(3-76a, b)}
\]

\[
\frac{1}{r} \sum_{m=-N_N}^{N_N} it_m g^{(m)} e^{i[(\delta+m\alpha)z-\sigma t]} = 0 \quad \text{at} \quad r = r_{in}(z)
\]

Substitution of (3-74) into (3-76a, b) expresses these conditions in terms of the computational coordinates

\[
\frac{1}{f(\tilde{r})} \sum_{m=-N_N}^{N_N} \sum_{k=0}^{N_C} F_{k}^{(m)} DT_k(\tilde{r}) e^{i[(\delta+m\alpha)z-\sigma t]} = 0 \quad \text{at} \quad \tilde{r} = \tilde{r}_{in}(z), \quad \text{(3-77a, b)}
\]

where \(\tilde{r}_{in}(z)\) is given by (3-40) and \(f(\tilde{r})\) is given by (3-41). Chebyshev polynomials and their derivatives evaluated along the cylinder surface represent periodic functions of \(z\) and can be expressed in terms of Fourier series of the form given by (3-59). Function \(1/f(\tilde{r}_{in})\) in (3-76a, b) is a periodic function of \(z\) and, thus, it can be expressed in terms of Fourier expansion of the form

\[\frac{1}{f[\tilde{r}_{in}(z)]} = \sum_{p=-\infty}^{+\infty} Z_{in}^{(p)} e^{ip\alpha z} \approx \sum_{p=-N_R}^{N_R} Z_{in}^{(p)} e^{ip\alpha z} \quad \text{(3-78)}\]

whose coefficients need to be evaluated numerically. The maximum truncation \(N_R\) depends on the geometry being considered, however, various tests carried out as a part of this study demonstrated that \(N_R = N_N\) is sufficient. Substitution of (3-59) and (3-78) into (3-76a, b) lead to

\[
\sum_{n=-N_N-N_S-N_R}^{N_N+N_S+N_R} \sum_{m=-N_N}^{N_N} \sum_{k=0}^{N_C} F_{k}^{(m)} Z_{in}^{(n-p)} (d_{in})_k^{(p-m)} e^{i[(\delta+n\alpha)z-\sigma t]} = 0, \quad \text{(3-79)}
\]
Separation of Fourier modes results in boundary relations equivalent to the boundary conditions and suitable for the numerical implementation, i.e.

\[ \sum_{n=-N_N-N_S-N_R}^{N_N+N_S+N_R} \sum_{m=-N_N}^{N_N} \sum_{k=0}^{N_C} i t_m F_k^{(m)} \sum_{p=-N_N-N_S}^{N_N+N_S} Z_{ln}^{(n-p)} (w_{ln})_k^{(p-m)} e^{i(\delta+n\alpha)z-\sigma t} = 0. \] (3-80)

The same procedure can be used for the enforcement of boundary conditions at the outer cylinder.

The form of the boundary conditions (3-76a, b) suggests that the factor $1/r$ could have been eliminated. The resulting boundary relations would have been based on the elimination of the first $N_N$ of the leading Fourier modes from $\partial \psi_3 / \partial r$ and $\partial \psi_3 / \partial z$. The eliminated factor represents a $z$-periodic function and its product with the derivatives of the stream function produces another periodic function. Use of boundary relations based on this product, i.e. use of the "primitive" form of boundary conditions (3-76a, b), results in a more accurate representation of these conditions in the IBC method.

### 3.3 Numerical Solution

The homogeneous algebraic system resulting from the spectral discretization of the differential system either for the three-dimensional disturbances or for the axisymmetric disturbances is posed as a general algebraic eigenvalue problem of the form

\[ AE = \sigma BE \] (3-83)

where $E$ denotes the eigenvectors. The $\sigma$-spectrum needs to be determined numerically. The numerical solution is computationally expensive and suffers from accuracy problems when large matrices are involved. Efficiencies can be gained by using the Arnoldi
method (Saad; 2003) which permits evaluation of only a selected part of the spectrum. Local solutions are still more computationally efficient and more accurate but produce a limited number of eigenvalues, mostly just one eigenvalue. In this case, the solution process starts with an initial guess either for the eigenvalue or for the eigenvector and iterations are used to converge to the true eigenvalue and/or eigenvector.

Three methods for eigenvalue tracing have been tested. In the first method, one of the homogeneous boundary conditions is replaced by an inhomogeneous boundary condition imposed on a different quantity resulting in an inhomogeneous system which can be easily solved. The true eigenvalue is found if the solution of the inhomogeneous system happens to satisfy the eliminated boundary condition. Since this is not true in general, the eigenvalue is searched for by looking for the zero of the replaced boundary condition using the Newton-Raphson procedure. The boundary condition for the vertical velocity component at the inner cylinder has been replaced in this study with a condition for the second derivative of the vertical velocity component. A good initial guess for $\sigma$ significantly accelerates convergence.

In the second method the eigenvalue is searched for by looking for zeros of the determinant of $(A - \sigma B)$ where the system is posed as

$$ (A - \sigma B)E = 0 $$  \hspace{1cm} (3-84)

In the third method, the inverse iterations method, we compute an approximation for the eigenvector $E_a$ corresponding to the unknown eigenvalue $\sigma_a$ using an iterative process in the form

$$ (A - \sigma_0 B)E^{(n+1)} = BE^{(n)} $$  \hspace{1cm} (3-85)

where $\sigma_0$ and $E^{(0)}$ are the eigenvalue and the eigenvector (an eigenpair) corresponding to the current state. If $\sigma_a$ is the eigenvalue closest to $\sigma_0$, $E^{(n)}$ converges to $E_a$. The eigenvalue $\sigma_a$ is evaluated using

$$ \sigma_a = E_a^{(n)\tau} A E_a^{(n)\tau} / E_a^{(n)\tau} B E_a^{(n)} $$  \hspace{1cm} (3-86)

where $T$ denotes the complex conjugate transpose.
3.4 Results and discussions

Two classes of instabilities have been investigated in ribbed annulus. Vortex mode instability (characterized by $\delta = 0$) is discussed in Section 3.4.2 while results for travelling waves (including axisymmetric and oblique waves) are presented in Section 3.4.3. For the convenience, however, we shall start discussion of stability results with the description of stability characteristics of flow in a smooth annulus.

3.4.1 Smooth annuli

The unstable disturbances have form of travelling waves whose characteristics at the onset depend on the radius of the annulus. The critical conditions computed as a part of this study are summarized in Figure 3-1 which displays variations of $Re_c$, $\delta_c$, $\sigma_{r,c}$ as a function of $R_1$. It can be seen that the axisymmetric disturbances ($M = 0$) play the critical role for $R_1 > 3.37$ with $Re_c \rightarrow 11544$, $\delta_c \rightarrow 2.041$ and $\sigma_{r,c} \rightarrow 0.5388$ as $R_1 \rightarrow \infty$, with the limiting values corresponding to the critical conditions for the plane Poiseuille flow. Reduction of $R_1$ makes oblique waves critical. In particular, waves with $M = 1$ play the critical role for $3.184 < R_1 < 3.372$, waves with $M = 2$ become critical for $2.362 < R_1 < 3.184$, waves with $M = 3$ become critical for $1.944 < R_1 < 2.362$, waves with $M = 2$ become critical again for $0.7157 < R_1 < 1.944$, waves with $M = 1$ also become critical again for $0.1323 < R_1 < 0.7157$ and, finally, the axisymmetric waves become critical again for $R_1 < 0.1323$ (Cotrell & Pearlstein, 2006). It can be seen that $Re_c \rightarrow \infty$ as $R_1 \rightarrow 0$ for $M = 0$ as predicted analytically by Heaton (2008). Waves with $M \neq 0$ become absolutely stable at a finite $R_1$. The axial critical wave number $\delta_c$ decreases monotonically from the limiting value of $\delta_c = 2.041$ at large enough $R_1$ with a small downward jump occurring at $R_1 = 3.372$ (change to $M = 1$), a larger downward jump taking place at $R_1 = 3.184$ (change to $M = 2$), a still larger downward jump taking place at $R_1 = 2.362$ (change to $M = 3$), an upward jump taking place at $R_1 = 1.944$ (change back to $M = 2$), an almost 50% reduction occurring between this point and $R_1 = 0.7157$ where a large upward jump takes place (change back to $M = 1$), an almost 80% reduction between this point and $R_1 = 0.13233$ where a huge upward jump takes place (change back to $M = 0$) which is followed by a rapid increase with further reduction of $R_1$ (Figure 3-1...
A similar evolution of $\sigma_{r,c}$ interspersed with jumps associated with changing $M$ can be followed in the same figure with a very large reduction taking place in the interval $0.7157 < R_1 < 1.944$ ($M = 2$) followed by a large upward jump at $R_1 = 0.7157$ (change from $M = 2$ to $M = 1$), then followed by large reduction in the interval $0.1323 < R_1 < 0.7157$, and a large upward jump at $R_1 = 0.13233$ followed by a rapid increase with further reduction of $R_1$. These results illustrate a rather smooth evolution of the critical disturbance as a function of $R_1$ if $R_1$ is large enough but very rapid changes of their characteristics for small $R_1$.

Figure 3-1: Variations of the critical conditions as a function of $R_1$ for smooth annulus. Thick solid and regular solid, dashed, dotted, dashed-dotted lines correspond to $M = 0$, 1, 2, 3, 4, respectively. Figure 3-1A displays the critical Reynolds number $Re_c$, Fig. 3-1B displays enlargement of the bottom section of Fig. 3-1A, Fig. 3-1C displays the critical axial wave number $\delta_c$, and Fig. 3-1D displays the critical frequency $\sigma_{r,c} = Real(\sigma_c)$. Vertical dashed lines mark borders between zones where disturbances with different $M$ play the critical role. Dotted lines identify different asymptotes.
3.4.2 Vortex mode instability

3.4.2.1 Sinusoidal ribs

A typical disturbance spectrum is displayed in Figure 3-2A for the radius $R_1 = 1$, the rib wave number $\alpha = 4$, the rib amplitude $S=0.015$, phase shift $\varphi=0$, the Reynolds number $Re=5000$ and vortex wave number $M=5$. The dominant eigenvalue is located at the peak of the main column. Spectra for the smooth annuli displayed in the same figure demonstrate that addition of ribs causes the dominant eigenvalue to move upward along the amplification rate axis. Figure 3-2B displays the evolution of the main eigenvalue as the rib amplitude decreases ($S \to 0$). The OS and Squire spectra in the smooth annuli are coupled for arbitrary disturbances but decouple for vortices (See Appendix). The Squire spectrum has been computed separately and its least attenuated eigenvalue provides the limit point for unstable vortex eigenvalue as $S \to 0$.

![Figure 3-2: Spectrum of $\sigma$ (Fig. 3-2A) for ribs described by Eq. (2-49) with $R_1 = 1$, $\alpha = 4$, $S = 0.015$, $\varphi = 0$ for $Re = 5000$ and $M = 5$. Black rectangles identify the Squire spectrum for the smooth annulus. Figure 3-2B shows variation of the disturbance growth rate $\sigma_1$ as a function of $S$.]

Eigenfunctions for the dominant eigenvalues are illustrated in Figure 3-3 for the $R_1 = 1$, $\alpha = 5$, $S=0.015$, phase shift $\varphi=0$ and the Reynolds numbers corresponding to the onset conditions, i.e. $Re=Re_c=3401.5$. The axial velocity component is by an order of magnitude larger than the remaining components which is a characteristic feature of the vortex instability. The primary, vortex-like motion occurs in the $(r,\theta)$ plane and it forces
fluid elements to move either towards or away from the walls. These elements preserve their axial velocity producing either a local velocity increase (at locations where high-velocity fluid approaches the wall; downwash) or a local velocity decrease (at locations where the low velocity fluid moves away from the wall; upwash). As a result, the axial component of the disturbance velocity is associated with the stationary state velocity deficit created by the disturbance motion rather than directly with this motion itself, i.e. a very weak vortex motion is able to create a large velocity deficit. This fact is captured by the large difference in the magnitude of the axial velocity component and the remaining two components (See Figure 3-3). The distribution of the radial velocity component $g_v^{(0)}$ suggests existence of a single layer of vortices in the bulk of the flow with complex structures forming closed to the inner wall.

![Figure 3-3: Eigenfunctions for flow in an annulus with geometry described by Eq. (2-49) with $R_1 = 1$, $\alpha = 5$, $S = 0.015$, $\varphi = 0$ for $Re = 3401.5$ and $M = 5$. Figures 3-3A-C display $g_u^{(n)}$, $g_v^{(n)}$, $g_w^{(n)}$, $n = 0, 1$, respectively. $\max g_u^{(0)}(r) = 1$ is used as the normalization condition. Solid and dashed lines identify the real and imaginary parts, respectively. Thin dashed-dotted and dotted lines identify the real and imaginary parts of the eigenfunctions for the smooth annuli with the same $R_1, M, Re.$]
The topology of the flow field and mechanism driving the instability can be deduced from the distribution of the radial and circumferential amplitude functions $h_v$ and $h_w$ displayed in Figure 3-4, and for the circumferential cuts through the disturbance velocity field displayed in Figure 3-5 starting at $z=0$ every $\lambda/4$. The topology consists of a layer of large vortices occupying most of the annular volume and a fairly thin zone of complex structures attached to the inner cylinder. The zeros of $h_v$ and $h_w$ indicate that a significant flow re-adjustment occurs in the axial direction (Figure 3-4). As it is indicated in Figure 3-5B, Two layers of vortices exist at $z=0$. Small vortex attached to the inner cylinder weakens as flow travels along the $z$ axis, resulting in the appearance of a saddle point (Figure 3-5D). Further through the $z$ axis, a single large vortex with two centers separated with a saddle point is observed (Figure 3-5F) which finally splits into two vortex layers (Figure 3-5H) with a thin, distinct layer formed at the inner cylinder.

![Figure 3-4: The velocity amplitude functions $h_v(z,r) \times 10^3$ (Fig. 3-4A) and $i h_w(z,r) \times 10^2$ (Fig. 3-4B) defined by Eq. (3-8) for the same conditions as in Fig. 3-3.](image)

The flow topology can be illustrated by using the disturbance axial vorticity component defined as

$$\xi_3 = \frac{1}{r} \frac{\partial (r w_3)}{\partial r} - \frac{1}{r} \frac{\partial v_3}{\partial \theta}$$

(3-87)

The formation of streamwise streaks attributed to the vortices is well illustrated in Figure 3-6 which shows iso-surfaces of disturbance axial vorticity component.
Figure 3-5: Distributions of the \((r, \theta)\)-component of the disturbance velocity vector for the same conditions as in Fig. 3-4. Figures 3-5A, 3-5C, 3-5E, 3-5G display data at \(z = 0\), \(\lambda/4\), \(\lambda/2\), \(3\lambda/4\), respectively, with the enlargements of the zones next to the inner cylinder for one circumferential wavelength shown in Figures 3-5B, 3-5D, 3-5F, 3-5H, respectively. Field lines are marked using grey color.

Figure 3-6: Iso-surfaces of the disturbance axial vorticity component (see Eq. (3-87)) for the same conditions as in Fig. 3-3.
Changes of the system response can be deduced from tracing of the dominant eigenvalue (as identified in Figure 3-2) through the parameter space. Figure 3-7 illustrates results of the simplest tracing where one fixes the annulus geometry and then determines the flow conditions required to destabilize vortices of different sizes. The tip of the relevant neutral curve determines the critical Reynolds number $Re_c$ and the critical wave number $M_c$ for this particular geometry.

Figure 3-7: The neutral curves in the ($Re, M$)-plane for an annulus described by Eq. (2-49) with $S = 0.015$ and $\varphi = 0$. Figures 3-6A-B display results for $R_1 = 1, 10$, respectively. Tip of each curve identifies the critical Reynolds number $Re_c$ and the critical wave number $M_c$ for this particular geometry.

Assessment of effects of different geometries parameters requires a more complex tracing. Figure 3-8 illustrates neutral curves in the ($R_1, Re$)-plane for annulus with ribs of $S=0.015$ and $\alpha = 2$. While in Figure 3-8A the inner and outer ribs are in-phase ($\varphi = 0$), Figure 3-8B illustrates the results for the case where $\varphi = \pi/2$. It is evident that applying phase shift between ribs and deforming the in-phase configuration contributes to instability of the flow since it shifts the neutral curves to the left. Since the annulus geometry allows only discrete (integer) values of the vortex wave number, each $M$ has to be traced separately producing distinct neutral curves. The boundary of the union of all of the unstable zones for all $M$'s plays the role of the critical curve and is marked by the dashed line in Figure 3-8.
Figure 3-8: Neutral curves in the \((Re, R_1)\)-plane for the instability in an annulus described by Eq. (2-49) with \(\alpha = 2, S=0.015\). Figure 3-8 A-B display results for \(\varphi = 0, \pi/2\), respectively. Dashed line identifies the critical curve which defines the onset conditions.

Figure 3-9 demonstrate critical stability curves in the \((R_1, Re)\)-plane. Each figure indicates three curves associated with different phase shifts between ribs of the inner and outer cylinders. It can be inferred from these figures that increasing phase shift from 0 to \(\pi\) decreases the critical Reynolds number, resulting in a less stable system. Comparing these three plots, it is also evident that the effect of \(\varphi\) becomes insignificant as \(\alpha\) increases. Moreover, one may notice that the critical Reynolds number approaches asymptotic value found in a plane ribbed channel when \(R_1\) is sufficiently big. The asymptotes for \(Re\) number have been shown by dotted lines in the figures.

The above analysis has been repeated for several rib wave numbers producing critical curves displayed in Figure 3-10, each representing a certain phase shift \(\varphi\). Same pattern is observed in each plot with the critical \(M\) increasing consistently with \(R_1\). Moreover, it can be seen that effect of rib wave number depends on its value, i.e. Critical Reynolds number changes significantly with rib wave number when \(\alpha\) is rather small \((\alpha<5)\); not a considerable change is observed, however, when higher ranges of \(\alpha\) are applied to the system.
Figure 3-9: Critical stability curves in the \((Re, R_1)\)-plane in an annulus described by Eq. \((2-49)\) with \(S=0.015\). Figure 3-9A-C display results for \(\alpha = 1, 3, 4.5\), respectively.

Effect of rib wave number and vortex wave number are illustrated vividly in Figure 3-11 displaying the neutral curves in \((\alpha, Re)\)-plane for representative values of \(M\)’s. It can be seen that in all cases, there exists a certain value of \(\alpha\) at which critical Reynolds number reaches a minimum. While different \(M\)’s play the critical role for different \(\alpha\)’s, the border of the union of all unstable \(M\)’s defines the critical conditions.
Figure 3-10: Critical stability curves in the \((Re, R_1)\)-plane in an annulus described by Eq. (2-49) with \(S=0.015\). Figure 3-10 A-C display results for \(\varphi = 0, \pi/2, \pi\), respectively.

The same analysis has been repeated for several \(R_1\) and the resulting critical curves displayed in Figure 3-12 illustrate very well the role played by both \(R_1\) and \(\alpha\). In general, reduction of \(\alpha\) stabilizes the flow. This is associated with the fact that the use of the long wavelength ribs reduces the streamwise flow modulation which, in turn, weakens the centrifugal forces until it is unable to support the instability. Increase of \(\alpha\) also stabilized the flow but the mechanics of the process is different. Use of very short wavelength ribs lifts up the stream above the rib’s peak and reduces flow modulation. As a result, the fluid movement becomes nearly rectilinear which weakens the centrifugal force field until it is unable to support the instability. Further, it can be seen that increase of \(R_1\) may have stabilizing or destabilizing effect regarding the value of \(\alpha\). For high rib wave numbers \((\alpha>8)\), however, increase of \(R_1\) results in consistent increase of critical Reynolds number. It is evident that stability curves become effectively independent of \(R_1\) as this
parameter becomes sufficiently high ($R_1 > 5$). Comparing the figures, one may also note that increasing $\varphi$ from 0 to $\pi$, decreases critical Reynolds number, resulting in a less stable system.

![Figure 3-11](image1.png)

Figure 3-11: Neutral curves in the ($Re, \alpha$)-plane for selected $M$'s for an annulus described by Eq. (2-49) with $S=0.015$ and $\varphi = \pi/2$. Figure 3-11 A-B display results for $R_1 = 1, 10$, respectively.

![Figure 3-12](image2.png)

Figure 3-12: The critical stability curves in the ($Re, \alpha$)-plane for an annulus described by Eq. (2-49) with $S = 0.015$. Figure 3-12 A-C display results for $\varphi = 0, \pi/2, \pi$, respectively.
Results presented in Figure 3-13 illustrate the effect of increase of the rib amplitude. Use of taller ribs amplifies the flow modulation, increases the centrifugal force and, as a result, reduces the critical Reynolds number. This reduction occurs almost uniformly over the whole range of the rib wave numbers.

3.4.2.2 Arbitrary ribs

Results discussed so far dealt with sinusoidal ribs, i.e. ribs with shape represented by one Fourier mode from expansions (2.1a, b). We shall now discuss stability characteristics of flows modified by ribs with arbitrary shapes. Such ribs can be replaced by the leading Fourier mode from the expansion representing their shape and the error associated with such approximation is likely acceptable for most of applications. This gives rise to the so-called reduced geometry model and demonstrates the generality of the results obtained for sinusoidal ribs. We shall now demonstrate the validity of the reduced geometry model for stability analysis.

We select ribs with various shapes and carry out stability analysis expressing their shapes using different number of Fourier modes from the relevant Fourier expansions. Figure 3-14 shows four shapes selected for analysis, i.e. rectangular, trapezoidal, triangular and rectified ribs. The relevant Fourier expansions are of the form
\[ H_{in}^{(n)} = \begin{cases} \frac{2iS}{n\pi} \left\{ \cos \left( \frac{nad}{2} \right) \left[ \frac{1 - \cos(naa)}{a} + \frac{1 - \cos(nab)}{b} \right] \right\} & \text{rectangular,} \\ \frac{S}{a\pi n^2} \left\{ \sin \left( \frac{nad}{2} \right) \left[ \frac{\sin(naa)}{a} + \frac{\sin(nab)}{b} \right] \right\} & \text{trapezoidal,} \\ \frac{2S}{(n\pi)^2} \left[ 1 - (-1)^n \right] & \text{triangular,} \\ \frac{4S(-1)^n}{\pi(1 - 4n^2)} & \text{rectified} \end{cases} \] (3-88)

Figure 3-14: Rib shapes used in the present study: A- rectangular ribs, B- trapezoidal ribs, C- triangular ribs, D- rectified ribs. In the above \( \lambda \) denotes the rib wavelength. Fourier representation of each shape is given by (3-88).

Figure 3-15 displays the neutral curves in the \((Re, M)\)-plane for rib shapes shown in Figure 3-14. It can be seen that replacement of the actual rib shape with the first mode of its Fourier expansion results in just a few percent error in the determination of the critical Reynolds number for all shapes considered. Even in the case of rectangular ribs where
the Fourier representation suffers from the Gibb’s phenomenon the error is no worse than 10%. This demonstrates that the reduced geometry model is valid for the stability and provides fairly accurate information about the stability properties of flow in an annulus modified using ribs of any shape.

Figure 3-15: The neutral curves in the \((Re, \delta)\)-plane for ribs of various shapes with \(R_1 = 2, \alpha = 3, S = 0.015\) and \(\varphi = 0\). The solid lines correspond to the rectangular ribs with \(a = b = \lambda/2\) (see Figure 3-14A for definition), the dashed lines correspond to the trapezoidal ribs with \(a = b = \lambda/6, c = d = \lambda/3\) (see Figure 3-14B for definition), the dotted lines correspond to the triangular ribs (see Figure 3-14C), and the dashed-dotted lines correspond to the rectified shapes (see Figure 3-14D).

3.4.3 Travelling wave instability

3.4.3.1 Sinusoidal ribs

We start discussion with the axisymmetric disturbances. Equations (3-16) and (3-17) demonstrate that the OS and Squire operators are decoupled and, thus, one needs to look at separate OS and Squire spectra. Figure 3-16A illustrates a typical OS spectrum which has been computed for \(R_1 = 10, \alpha = 4, S = 0.015, \varphi = 0, Re = 12500\) and wave with \(\delta = 2.1, M = 0\). The spectrum consists of vertical “columns” that are associated with different Fourier modes used in the solution. The spectrum for the same wave in a smooth annulus is shown for comparison purposes. Only one unstable eigenvalue exists and its tracing for \(S \to 0\) shown in Figure 3-16B demonstrates that it connects to the eigenvalue describing the axisymmetric wave in a smooth annulus. This suggests that the unstable
disturbances in the smooth and ribbed annuli are qualitatively similar, i.e., they have the form of travelling waves which are destabilized by the ribs.

Figure 3-17A illustrates the Squire spectrum for the same conditions as Figure 3-16. It has structure similar to the OS spectrum discussed above. The least attenuated eigenvalue remains stable in the range of rib amplitudes considered. Tracing of this eigenvalue for $S \to 0$ shown in Figure 3-17B demonstrates that it connects to the eigenvalue describing the least attenuated axisymmetric Squire mode in the smooth annulus.

Figure 3-16: OS spectrum of $\sigma$ (Figure 3-16A) for ribs described by Eq. (2-49) with $R_1 = 10$, $\alpha = 4$, $S = 0.015$, and $\varphi = 0$ for $Re = 12500$, and axisymmetric waves with $\delta = 2.1$. Black rectangles identify the OS spectrum for the smooth annulus. Figure 3-16B shows variation of the disturbance growth rate $\sigma_i$ as a function of $S$.

Figure 3-17: Squire spectrum of $\sigma$ (Figure 3-17A) for the same conditions as Figure 3-16. Black rectangles identify the Squire spectrum for the smooth annulus. Figure 3-17B shows variation of the disturbance growth rate $\sigma_i$ as a function of $S$. 
Figure 3-18A illustrates spectrum for the oblique waves. It accounts for all possible modes as the OS and Squire spectra do not separate. This particular spectrum has been computed for \( R_1 = 1, \alpha = 4, S = 0.015, \varphi = 0, Re = 12500 \) and wave with \( \delta = 1.6, M = 2, \) and has structure very similar to that found in the case of axisymmetric disturbance. The unstable eigenvalue tracing for \( S \to 0 \) shown in Figure 3-18B demonstrates that it connects to the eigenvalue describing the oblique wave in a smooth annulus and this underlines conclusion that ribs just amplify these waves without affecting their qualitative properties.

![Figure 3-18: Spectrum of \( \sigma \) (Figure 3-18A) for the rib geometry described by Eq. (2-49) with \( R_1 = 1, \alpha = 4, S = 0.015, \varphi = 0 \) for \( Re = 12500, \) and oblique waves with \( \delta = 1.6, M = 2. \) Black rectangles identify the complete spectrum for the smooth annulus (OS and Squire spectra do not separate). Figure 3-18B shows variation of the disturbance growth rate \( \sigma_i \) as a function of \( S \).](image)

The form of the eigenfunctions for axisymmetric waves at the onset is illustrated in Figure 3-19. In the case of smooth annulus, this form is nearly symmetric with respect to the center line. In the case of ribbed annulus, however, the symmetry is eroded. The eigenfunctions for the higher modal functions are significant only close to the ribs. Figure 3-20 illustrates the eigenfunctions for the oblique waves at the onset. The dominant eigenfunction, i.e. eigenfunction for mode 0, is much bigger in the inner half of the annulus; its peak in this half is about 200% higher than the peak in the outer half for \( R_1 = 1. \) The eigenfunctions for higher modal functions play significant role only close to the ribs. One may conclude on the basis of data displayed in Figure 3-19 and 3-20 that the
character of the disturbance velocity field is dictated by the dominant modal function in (3-9) and (3-35), i.e. it is not too different from the case of smooth annulus, with deviations confined to the zone next to the ribs.

Figure 3-19: Eigenfunctions $g_u^{(n)}(r)$, $n = 0, 1$, describing the axisymmetric waves for the annulus described by Eq. (2-49) with $R_1 = 10$, $\alpha = 5$, $S = 0.015$ and $\varphi = 0$ at the onset, i.e. $(Re_c, \delta_c) = (7064, 2.21)$. The eigenfunctions are normalized with $\max g_u^{(0)}(r) = 1$. Thick solid and dashed lines correspond to the real and imaginary parts, respectively. Thin lines identify eigenfunctions for the smooth annuli.

Figure 3-20: Eigenfunctions $g_u^{(n)}(r)$, $n = 0, 1$, describing oblique waves with $M = 2$ for the annulus described by Eq. (2-49) with $R_1 = 10$, $\alpha = 5$, $S = 0.015$ and $\varphi = 0$ at the onset, i.e. $(Re_c, \delta_c) = (10009, 1.67)$. The eigenfunctions are normalized with $\max g_u^{(0)}(r) = 1$. Thick solid and dashed lines correspond to the real and imaginary parts, respectively. Thin lines identify eigenfunctions for the smooth annuli.
The second invariant of the velocity gradient tensor has been used to illustrate the topology of the flow field. It is referred to $q$-criterion (Dubief and Delcayre; 2000) defined as

$$ q = \frac{1}{4} (|\bar{\omega}|^2 - e_{ij}e_{ij}) \quad (3-89) $$

where $\bar{\omega}$ and $e_{ij}$ are vorticity vector and the strain rate tensor, respectively. The explicit form of $q$ in cylindrical coordinate can be written as

$$ q = \frac{1}{4} \left[ \left( \frac{1}{r} \frac{\partial u_D}{\partial \theta} - \frac{\partial w_D}{\partial z} \right)^2 + \left( \frac{\partial v_D}{\partial z} - \frac{\partial u_D}{\partial r} \right)^2 + \left( w_D + \frac{\partial w_D}{\partial r} - \frac{1}{r} \frac{\partial v_D}{\partial \theta} \right)^2 \right] $$

$$ - \frac{1}{2} \left[ \left( \frac{\partial v_D}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial w_D}{\partial \theta} + \frac{v_D}{r} \right)^2 + \left( \frac{\partial u_D}{\partial z} \right)^2 \right] $$

$$ - \frac{1}{4} \left[ \left( \frac{1}{r} \frac{\partial u_D}{\partial \theta} + \frac{\partial w_D}{\partial z} \right)^2 + \left( \frac{\partial v_D}{\partial z} + \frac{\partial u_D}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial w_D}{\partial \theta} + \frac{\partial w_D}{\partial r} - \frac{w_D}{r} \right)^2 \right]. \quad (3-90) $$

Figure 3-21 and Figure 3-22 illustrate the formation of structures aligned in the axial direction for oblique and axisymmetric waves, respectively.
Figure 3-21: Structure of disturbance flow shown using $q$-criterion (see Eq. (3-90)) with $q = 0.2$ in an annulus with ribs described by Eq. (2-49) with $R_1 = 1$, $S = 0.015$, $\alpha = 5$, and $\varphi = 0$ for oblique disturbance with $M = 2$ at the onset condition, i.e. $(Re_c, \delta_c) = (10009, 1.67)$.

Figure 3-22: Structure of disturbance flow shown using $q$-criterion (see Eq. (3-90)) with $q = 0.2$ in an annulus with ribs described by Eq. (2-49) with $R_1 = 10$, $S = 0.015$, $\alpha = 5$, and $\varphi = 0$ for axisymmetric disturbance at the onset condition, i.e. $(Re_c, \delta_c) = (7064, 2.21)$. 
Tracing of the unstable eigenvalue as a function of the flow conditions (Re), the rib geometry (R₁, α, S, φ) and the wave characteristics (δ, 𝑀) provides information about the flow stability characteristics.

Figure 3-23 illustrates the neutral curves for axisymmetric and oblique waves in the (Re, δ)-plane. Data displayed in Figure 3-23A are for R₁ = 1 and M = 2 while Figure 3-23B shows results for R₁ = 10 and M = 0. Effect of α and φ can be seen on each figure. Higher rib wave number is associated with lower critical Reynolds and a less stable system. Moreover, as the configuration of ribs changes from wavy (φ = 0) to converging-diverging (φ = π), Reₐ decreases monotonically for all rib wave numbers.

Figure 3-23: Neutral curves in the (Re, δ)-plane for ribs described by Eq. (2-49) with S = 0.015. Results for the (R₁, M) = (1, 2) and (10, 0) are displayed in Figure 3-23A, B, respectively. Solid, dashed and dashed-dotted lines correspond to α = 1, 2, 5, respectively.

Figure 3-24 displays summary plots illustrating variations of the critical Reynolds number as a function of the rib wave number α and the annulus’ radius R₁ with the largest rib amplitude considered in this study, i.e. S = 0.015. Figure 3-24A-D display data for M = 0, 1, 2, 3, respectively. It can be seen that the flow is stabilized when both R₁ and α decrease. Stabilizing effect of decreasing α is more significant for big R₁’s and the same is true in for R₁, i.e. decreasing R₁ stabilizes the flow more effectively when α becomes higher. Further investigation of the results shows that increase of phase shift
between inner and outer ribs has a destabilizing effect; this effect, however, becomes rather negligible for $\alpha > 5$.

Figure 3-25 presents another set of summary plots which illustrates variations of $Re_c$ for axisymmetric waves as a function of $\alpha$ and $S$. Generally, higher rib amplitude and rib wave number are shown to be associated with smaller critical Reynolds number and to have destabilizing effect. One may also note that effect of phase shift becomes negligible for $\alpha > 5$ as illustrated in previous figures. Results presented in this figure also show that critical Reynolds numbers approach to the value determined in case of smooth annulus as $S \to 0$

Figure 3-24: Variations $Re_c$ as a function of $R_1$ and $\alpha$ for ribs described by Eq. (2-49) with $S = 0.015$. Results for waves with $M = 0, 1, 2, 3$ are displayed in Figure 3-24A, B, C, D, respectively. Solid, dashed, dashed-dotted lines indicate $\phi = 0, \pi/2, \pi$, respectively.
Figure 3-25: Variations of $Re_c$ as a function of $\alpha$ and $S$ in an annulus with ribs described by Eq. (2-49) with $R_1 = 10$ for axisymmetric disturbance. Solid, dashed and dotted lines indicate $\phi = 0, \pi/2, \pi$, respectively. $Re_c$ for the smooth annuli with $R_1 = 10$ and $M = 0$ is $Re_c = 11760$

Variations of $Re_c$ for axisymmetric and oblique waves as a function of $\alpha$ and $\phi$ are presented in Figure 3-26. These figures explicitly demonstrate that increase of phase shift from 0 to $\pi$ destabilizes the flow and reduces the critical Reynolds number. It also shows that the effect of $\phi$ on the stability of the flow diminishes for $\alpha > 5$.

Figure 3-26: Variations of $Re_c$ for axisymmetric and oblique waves as a function of $\alpha$ and $\phi$ for ribs described by Eq. (2-49) and $S = 0.015$. Results for the $(R_1, M) = (1, 2)$ and $(10, 0)$ are displayed in Figure 3-26A, B, respectively.
3.4.3.2 Arbitrary ribs

Figure 3-27 displays the neutral curves in the \((Re, \delta)\)-plane for the axisymmetric and oblique waves for rib shapes shown in Figure 3-14. As it was discussed for vortex instability, replacement of the actual rib shape with the first mode of its Fourier expansion results in just a few percent error in the determination of the critical Reynolds number for all shapes considered and it verifies the fact that the reduced geometry model is valid for the stability analysis.

![Figure 3-27](image)

Figure 3-27: The neutral curves in the \((Re, \delta)\)-plane for ribs of various shapes with \(\alpha = 1\), \(S = 0.015\) and \(\varphi = 0\). Results for the \((R_1, M) = (1, 2)\) and \((10, 0)\) are displayed in Figure 3-27A, B, respectively. Dotted lines correspond to the triangular ribs (see Figure 3-14C for notation), the dashed lines correspond to the trapezoidal ribs with \(a = b = \lambda/6\), \(c = d = \lambda/3\) (see Figure 3-14B for notation), the solid lines correspond to the rectangular ribs with \(a = b = \lambda/2\) (see Figure 3-14A for notation), and the dashed-dotted lines correspond to the rectified shapes (see Figure 3-14D).

3.4.4 Competition between vortex and travelling wave instabilities

The presence of two instability mechanisms raises the question of when each of them will dominate the instability process (Floryan, 2015). To approach this problem, we have defined the global critical Reynolds number \((Re_{g,cr})\) as the minimum critical Reynolds number for a fixed \(S\) over all \(\alpha\)’s. Figure 3-28 illustrates variation of \(Re_{g,cr}\) as a function of \(S\) for travelling wave and vortex instabilities. The curves cross each other at \((S, Re_{g,cr})=(0.0045,14740)\) for \(R_1 = 1\) and at \((0.006,10600)\) for \(R_1 = 10\).
The critical Reynolds number in the case of flows in smooth channels is \( Re_c = 11544 \) (S. A. Orszag 1971). Comparing Figure 3-28A and 3-28B, it can be seen that \( Re_c \) for the TS waves in case of small rib amplitude \( (S \rightarrow 0) \) approaches the smooth channel value for \( R_1 \rightarrow \infty \) and this serves as one of validations of the method used in the stability analysis.

![Diagram](image)

Figure 3-28: Variations of the global critical Reynolds number \( Re_{c,global} \) describing the traveling-wave instability and the vortex instability as a function of the rib amplitude \( S \). Figure 3-28A-B show results for \( R_1 = 1 \) and \( R_1 = 10 \), respectively.

Figure 3-29 and Figure 3-30 clearly illustrate the competition between travelling wave and vortex instabilities. One may note that although the critical Reynolds number for both modes decreases with an increase of \( S \), this increase is more rapid in case of vortex mode than travelling waves. The changes in the critical Reynolds number are such that \( Re_c \) for both modes become equal at certain points. The position of these points is presented by a thick curve in the figures which separates the zones where vortex and TS wave instabilities are dominant. It can be seen that for geometries with high rib amplitudes which are located in the upper zone, the critical Reynolds number for vortices is significantly lower than TS waves. This would therefore suggest a rib geometry which leads to formation of vortices, yet, is not affected by travelling wave instabilities. In other words, accidental occurrence of TS waves can be avoided due to large difference between \( Re_c \) of the two modes. Comparison between Figure 3-29 and Figure 3-30 also indicates
that increase of \( R_1 \) leads to significant decrease of \( Re_c \) for TS waves while \( Re_c \) of vortex mode is not considerably affected by change in \( R_1 \).

Figure 3-29: Variations of \( Re_c \) as a function of \( \alpha \) and \( S \) for the onset of the travelling wave (solid lines) and the vortex instabilities (dashed lines) for ribs described by Eq. (2-49) with \( R_1 = 1 \). Figure 3-29A,B display results for \( \varphi = 0, \pi/2 \), respectively. The thick line separates zones of dominance of the travelling wave and the vortex instabilities.

Figure 3-30: The same as in Figure 3-29, but for \( R_1 = 10 \). Figure 3-29A,B display results for \( \varphi = 0, \pi/2 \), respectively. The thick line separates zones of dominance of the travelling wave and the vortex instabilities.

Variations of \( Re_c \) as a function of \( R_1 \) and \( \alpha \) presented in Figure 3-31 also confirms previous results. Unlike vortex mode, TS waves are sensitive to change of annuli radii, showing a significant decrease in \( Re_c \) with an increase in \( R_1 \). Moreover, increase of \( \alpha \) or \( \varphi \) destabilizes the flow by reducing \( Re_c \).
Figure 3-31: Variations of $Re_c$ as a function of $R_1$ and $\alpha$ for the onset of the travelling wave (solid lines) and the vortex instabilities (dashed lines) for ribs described by Eq. (2-49) with $S = 0.015$. Figure 3-31A,B display results for $\varphi = 0, \pi/2$, respectively.
Chapter 4

4 Conclusions

An analysis of flows in ribbed annuli has been carried out to determine conditions which can lead to generation of vortices. The annuli have been modified by axisymmetric ribs of arbitrary shapes placed at the cylinders.

Investigation of the stationary state suggests that addition of ribs results in an increase of the pressure losses for all rib wave numbers. Reduction of the rib amplitude or the rib wave number results in the reduction of these losses. Depending on which cylinders the ribs are placed at, change of the annulus’ radius $R_1$ can affect the losses differently. Losses increase as $R_1$ increases if the ribs are placed at the inner cylinder; they decrease, however, if the ribs are placed at the outer cylinder. Increase of phase shift $\varphi$ was shown to increase pressure losses with small rib wave numbers being more effective.

To determine circumstances under which vortices are created, vortex instability analysis has been carried out. Since we would like to avoid transition to turbulence, travelling wave instabilities have been also studied. It has been demonstrated that the addition of ribs always results in flow destabilization and an increase of the rib amplitude $S$ reduces the critical Reynolds number. Moreover, increasing phase shift between ribs of the inner and the outer cylinders contributes to instability of the flow in both types of instabilities considered. To assess validation of method used for stability analysis, results for the limiting cases of $R_1 \to \infty$ and $S \to 0$ were shown to match with results in channel flow and smooth annulus, respectively.

In case of vortex instability, centrifugal effect due to flow modulation results in formation of axial vortices. Use of the long wavelength ribs reduces the streamwise flow modulation which, in turn, weakens the vortex instability. Use of high values of $\alpha$ also stabilizes the flow by lifting up the stream above the rib’s peak and reducing flow modulation. It was shown that increase of $R_1$ may have slight stabilizing or destabilizing effect regarding the value of $\alpha$. On the other hand, in case of travelling wave instabilities
which can occur in the form of axisymmetric or oblique waves, it was found that critical Reynolds number decreases significantly with an increase of $R_1$.

Investigation and comparison of the critical Reynolds numbers in travelling waves and vortex instabilities provided useful information about conditions where each of these instabilities is dominant. Generally, use of ribs with higher amplitudes will contribute to formation of vortices without interference with travelling waves. The minimum rib amplitude required for formation of vortices decreases with an increase in rib wave number. Although better mixing is expected under such conditions, one should notice that increase of $S$ and $\alpha$ also results in higher pressure losses. So, proper specifications should be selected depending on the design requirements.
References


Dubief, Delcayre, 2000 On coherent-vortex identification in turbulence, J. Turbulence. 1, 11


Floryan, J. M. 2015 Flow in a meandering channel, J. Fluid Mech. 770, 52-84


Appendix

Stability of flow in a smooth annulus

The linear disturbance equations (3-16) – (3-17) reduce in the limit of $S_{in} \to 0$ to the form of

\[ L_{OS}^{(n)} G_v^{(n)} + C_{OS}^{(n)} \Omega^{(n)} = 0, \quad L_{SQ}^{(n)} \Omega^{(n)} + C_{SQ}^{(n)} G_v^{(n)} = 0. \] (A-1a, b)

Each pair of the above equations describes disturbances of periodicity $2\pi/(\delta + n\alpha)$ in the axial direction and $2\pi/M$ in the circumferential direction. In the case of vortices ($\delta = 0, n = 0$) the above operators reduce to

\[ L_{OS}^{(0)} = -i\sigma \left( D^2 + \frac{D}{r} - \frac{M^2}{r^2} \right) \]
\[ - \frac{1}{Re} \left[ D^4 + \frac{2}{r} D^3 - \frac{(2M^2 + 1)}{r^2} D^2 + \frac{(2M^2 + 1)}{r^3} D \right. \]
\[ + \frac{M^2(M^2 - 4)}{r^4} \left. \right], \quad C_{OS}^{(0)} = 0, \]

\[ L_{SQ}^{(0)} = -i\sigma - \frac{1}{Re} \left( D^2 + \frac{D}{r} - \frac{M^2}{r^2} \right), \quad C_{SQ}^{(0)} = -\frac{iDu_0}{rM}, \] (A-2a, b)

with $G_v^{(0)} = -irg_v^{(0)}$, $\Omega^{(0)} = -g_u^{(0)}/M$, and (A-1a) and (A-1b) assume the following forms

\[ Re^{-1} \left[ D^4 + \frac{6D^3}{r} + \frac{(5 - 2M^2)}{r^2} D^2 - \frac{(2M^2 + 1)D}{r^3} + \left( \frac{M^2 - 1}{r^2} \right)^2 \right] \]
\[ - i\sigma \left( D^2 + \frac{3D}{r} - \frac{M^2 - 1}{r^2} \right) g_v^{(0)} = 0, \] (A-3)

\[ \left( D^2 + \frac{D}{r} - \frac{M^2}{r^2} + iRe \sigma \right) g_u^{(0)} = ReDu_0 g_v^{(0)}. \] (A-4)

The relevant boundary conditions have the form
\[ G_v^{(0)} = DG_v^{(0)} = 0, \quad G_u^{(0)} = 0 \quad \text{at} \ r = R_1 \text{ and } r = 1 + R_1. \]  

The original coupled problem (A-1) separates into two independent eigenvalue problems with (A-3)-(A-5a) describing the OS modes and the homogeneous part of (A-4) and (A-5b) describing the Squire modes.

Solution for the Squire mode can be expressed in terms of Bessel functions in the form of

\[ g_u^{(0)}(r) = c_1 J_M(\sqrt{-Re \sigma_i r}) + c_2 Y_M(\sqrt{-Re \sigma_i r}) \]  

where \( J_M \) and \( Y_M \) denote the Bessel functions of the first and second kinds with index \( M \), respectively, and \( c_1 \) and \( c_2 \) are arbitrary constants. The amplification rate \( \sigma_i \) could be positive, zero, or negative. Positive \( \sigma_i \) leads to the Bessel functions with imaginary arguments, these functions become complex and, thus, cannot represent real \( g_u^{(0)} \). It can be concluded that a positive \( \sigma_i \) is not physically relevant. Zero \( \sigma_i \) leads to infinite value of \( Y_M \) and, thus, is not admissible. Negative \( \sigma_i \) provides an ability to construct a real solution which can satisfy the boundary conditions. This demonstrates that only attenuated disturbances are admissible. Imposition of homogeneous boundary conditions on (A-6) leads to nontrivial solution only if

\[
\begin{align*}
J_M(\sqrt{-Re \sigma_i R_1})Y_M[\sqrt{-Re \sigma_i(1 + R_1)}] \\
- J_M[\sqrt{-Re \sigma_i(1 + R_1)}]Y_M(\sqrt{-Re \sigma_i R_1}) = 0.
\end{align*}
\]  

Numerical solution of (A-7) leads to the determination of the amplification rate for the specified \( R_1, Re \) and \( M \).
Curriculum Vitae

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