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A Study Of Green's Relations On Algebraic Semigroups

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A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Mathematics

A STUDY OF GREEN'S RELATIONS ON ALGEBRAIC SEMIGROUPS

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by

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It has been a hell of a ride. I cannot wait to see where I go next!

Abstract

The purpose of this work is to enhance the understanding regular algebraic semigroups by considering the structural influence of Green's relations. There will be three chief topics of discussion.

- ∘ Green's relations and the Adherence order on reductive monoids
- ∘ Renner's conjecture on regular irreducible semigroups with zero
- ∘ a Green's relation inspired construction of regular algebraic semigroups

Primarily, we will explore the combinatorial and geometric nature of reductive monoids with zero. Such monoids have a decomposition in terms of a Borel subgroup, called the Bruhat decomposition, which produces a finite monoid, $\mathcal{R}$, the Renner monoid. We will explore the structure of $\mathcal{R}$ by way of Green's relations. In particular, we will be exploring the nature of the Adherence order poset, $(\mathcal{R}, \leq)$ when restricted to $\mathscr{J}$-, $\mathscr{R}$-, $\mathscr{L}$-, and $\mathscr{H}$-classes.

From reductive monoids we broaden the impact of Green's relations and explore regular algebraic semigroups. Specifically, we resolve Renner's conjecture and show that the supports, $\mathbb{X}_\ell = \mathscr{J}/\mathscr{R}$ and $\mathbb{X}_r = \mathscr{J}/\mathscr{L}$ are projective varieties. Spurred on by the result, we use invariant theory to generalise the Rees matrix construction for algebraic semigroups to construct irreducible regular semigroups with 0. Our construction will start with specified maximal classes, R_e, L_e, and H_e and reconstruct an entire semigroup. In a lengthy example, we will use some of our previous combinatorial results to apply the construction to a natural generalisation of determinantal varieties.

Highlights include the unique "vanilla form" decomposition for elements of the Renner monoid (Definition 5.36), a proof of Renner's conjecture on the projectiveness of supports for irreducible regular semigroups with zero (Theorem 8.40), and the construction of irreducible regular semigroups from prespecified maximal $\mathscr{R}$- and $\mathscr{L}$-classes (Definition 9.6).

Keywords: Algebraic semigroups, reductive monoids, Green's relations, Adherence order, Renner monoid, semigroup supports, irreducible regular algebraic semigroups with zero

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1 Introduction

The systematic investigation of linear algebraic semigroups and reductive algebraic monoids was pioneered around 1980 by Mohan Putcha and my supervisor, Lex Renner. Since then, the discipline has blossomed into a coherent branch of algebra involving embedding theory, representation theory and algebraic combinatorics. One of the most important features of a reductive monoid is the existence of the Bruhat Decomposition. More precisely, the Bruhat decomposition, which is much studied for groups, extends to a perfect analogue for reductive monoids.

This monoid Bruhat decomposition allows us to express the monoid as a disjoint union of double cosets (for a given Borel subgroup) indexed by a finite structure called the Renner monoid (the monoid analogue of the Weyl group). The importance of this decomposition is that it allows many questions about the nature and structure of the monoid to become simpler questions about the Renner monoid.

Unlike groups, semigroups and monoids bring an additional structure in the form of Greens relations, which characterise the elements of the semigroup in terms of the ideals they generate. So important are Greens relations that Scottish semigroup theorist, John Mackintosh Howie, once said, "on encountering a new semigroup, almost the first question one asks is 'What are the Green relations like?' ". In reductive monoids, we denote these relations by $\mathscr{J}$, $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{H}$. A natural question is how do these relations interact with the Bruhat decomposition? What additional information can they tell us?

Of particular focus is the $\mathscr{H}$ relation, which has many interesting and desirable properties. $\mathscr{H}$-classes most closely resemble groups (indeed the $\mathscr{H}$-class of an idempotent element forms a group), and so their structure is the one most likely to form a bridge between the Renner monoid and the information we know from the better understood Weyl groups. One way we can investigate this structure is by decomposing our monoid, not in terms of the Bruhat decomposition, which is indexed by elements of the Renner monoid, but in terms of a disjoint union of double cosets of the $\mathscr{H}$-classes of the Renner monoid. These are the so-called fat $\mathscr{H}$-classes.

In the following section of this paper, we will recall some the basic results about regular algebraic semigroups and the Bruhat decomposition, so that readers may proceed with the appropriate amount of background information. We will also take a look at the nature of Green's relations on regular semigroups and Renner monoids, as these structures are the basis of the paper and we require readers to be familiar with their properties.

In Section 3, we will recall some of the results presented by Renner in [28], the paper that first introduced the notion of fat $\mathscr{H}$-class. In [28], Renner presents a decomposition for elements in the Renner monoid, $\mathcal{R}$, which he has dubbed "the trichotomy". We will introduce our own decomposition (Theorem 3.21) that is incredibly similar, but which is more in line with Green's relations and affords us easier analysis later on. Our new trichotomy in particular allows us to better describe the Adherence order on $\mathscr{H}$-, $\mathscr{R}$- and $\mathscr{L}$-classes (Theorem 3.30) which will become an underlying goal in the majority of the paper.

After a number of results with our new trichotomy decomposition, we will move into Section 4, wherein we will deal with the fat $\mathscr{H}$-classes head on. In addition, we will also consider the analogous fat $\mathscr{J}$-classes, fat $\mathscr{L}$-classes and fat $\mathscr{R}$-classes. These structures have been studied at one time or another under different names (for example in the works, [20] and [29]). In this way we will get a more robust picture of the fat $\mathscr{H}$-classes and truly understand where some of the results come from. Our trichotomy will feature prominently in our analogous Bruhat decomposition in terms of fat $\mathscr{H}$-classes, fat $\mathscr{R}$-classes, fat $\mathscr{L}$-classes and fat $\mathscr{J}$-classes. In particular, we will characterize the natural analogue of the Bruhat order, $BT_rB \subseteq \overline{BT_sB}$ for $\mathscr{T} = \mathscr{J}$ (Corollaries 4.11 and 4.20), $= \mathscr{L}, \mathscr{R}$ (Theorem 4.17), and $= \mathscr{H}$ (Theorem 4.25).

In the fifth section we will extend the monumental work of Pennell, Putcha and Renner in [17], where they were able to determine the Adherence order relation between any two elements of the Renner monoid, provided they are in "standard form" (Definition 5.29 and Theorem 5.31). Specifically, we will use our trichotomy decomposition to devise a whole new form (Definition 5.36) for elements of the Renner monoid. This form will allow one to more easily determine the $\mathscr{J}$-, $\mathscr{R}$-, $\mathscr{L}$- and $\mathscr{H}$-classes of the element (Proposition 5.43). We will then show how to use this new form to determine the Adherence order relation (Theorem 5.41), and to glean new information on the structure of the posets given by individual equivalence classes and the Adherence order (Theorem 5.44).

We then triumphantly progress to Section 6, where we put our new structure information to use, and determine maximal and minimal elements in every single $\mathscr{J}$-, $\mathscr{R}$-. $\mathscr{L}$- and $\mathscr{H}$-class (Theorems 6.17 and 6.13). In a remarkable twist of fate, these elements will belong to well-known, well-behaved sets. The decomposition of elements of the Renner monoid allows us to explore some of the structure of the $(\mathcal{R}, \leq)$ poset (Theorem 6.40).

For Section 7, we will generalise many of our previous combinatorial results in terms of new equivalence relations which are based on the standard parabolic subgroups of the Weyl group, W (Definition 7.1). These new equivalence relations will allow us to bridge the gap between the Bruhat decomposition that we are used to (in terms of double cosets of single elements) and the Bruhat decompositions of Section 4 (which are in terms of fat classes). Our newfound relations will also allow us to generalise the Adherence order in the only logical way, by considering containment relations on double cosets involving parabolic subgroups, not just a specified Borel subgroup (Corollary 7.19).

Sections 8 and 9 explore the geometric impact of Green's relations. Section 8 concerns the supports of regular irreducible algebraic semigroups from [24]. In particular, using geometric invariant theory along with Putcha's determinant (Definition 8.20) and the so-named Renner maps (Proposition 8.28), we will show that if such semigroups have a 0, then their supports are projective varieties (Conjecture 8.7 and Theorem 8.40), a strengthening of the quasiprojective-ness shown in [24].

In the final section, Section 9, we will introduce an exciting new way to construct algebraic semigroups by specifying certain Green's equivalence classes ahead of time (Definition 9.6). We will show that certain normal irreducible regular algebraic semigroups with 0 are invariant under this construction (Theorem 9.23). As an example, we will use some of our fat $\mathscr{T}$-class results to show how this construction can recreate a natural generalisation of determinantal varieties (Theorem 9.41 and Corollary 9.43).

2 Background

Readers interested in the results presented in this paper should be familiar with the fundamental results concerning Green's relations, regular semigroups and reductive monoids. This introductory section will refresh the memories of the reader and phrase well-known results in the language presented in this paper.

The results presented here will be assumed background material and will not be explicitly referenced later. There are a few ancillary results to be found in the Appendix. For the most part they will be results basic to semigroup theory or algebraic geometry, but not results that one may typically come across. Proofs of those results are given there.

As they are the primary object of study, we will take the time now to define algebraic semigroups.

Definition 2.1. *We say an affine variety, S, is a* **linear algebraic semigroup** *if it has an associated morphism $\mu : S \times S \to S$ so that (S, μ) forms a semigroup (that is, μ is associative). A linear algebraic semigroup is called a* **linear algebraic monoid** *if it also contains an element, $1 \in S$ so that 1 acts as a two-sided identity element for μ.*

We say that an algebraic semigroup (algebraic monoid) is **irreducible** *if it is irreducible as a variety.*

Example 2.2. *The natural example of an algebraic monoid is the $n \times n$ matrices, $M_n(K)$ with the morphism $\mu(A, B) := AB$, the usual matrix multiplication, and usual identity element, I_n.*

Any finite semigroup (resp. monoid) is an algebraic semigroups (resp. algebraic monoid).

For any algebraically closed field, K, the set $\{(a, b, c) \in K^3 \mid a^2c^3 = b^7\}$ is an algebraic monoid with coordinate-wise multiplication and identity element $(1, 1, 1)$.

Being groups (and hence monoids and semigroups), any algebraic group is an algebraic monoid and an algebraic semigroup.

Both books [20] and [7] have excellent introductory sections concerning the basic properties of algebraic semigroups.

We must note that some of our sources use the term **connected** to refer to an irreducible semigroup (monoid). This is a holdover from algebraic group theory which we will not be continuing in this paper. As such, some of the wording of statements may appear to change between this paper and its references. This is purely cosmetic.

One of the most basic structure theorems about algebraic semigroups is the following result.

Theorem 2.3. *Let S be a linear algebraic semigroup (monoid), then S is isomorphic to a Zariski closed subsemigroup (submonoid) of $M_n(K)$, the set of $n\times n$ matrices over algebraically closed field K, for some n and some K.*

Proof. This is the remarkable Theorem 3.15 and Corollary 3.16 contained in [20]. □

Example 2.4. *With our monoid, $\{(a,b,c) \in K^3 \mid a^2c^3 = b^7\}$ from before, we can write it as the closed subset of the 3×3 matrices, $\{\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \in M_3(K) \mid a^2c^3 - b^7 = 0\}$ and one can observe that the coordinate multiplication we mentioned in Example 2.2 turns into multiplication of 3×3 matrices.*

This is exactly in line with what one would expect as it is well-known that algebraic groups are all closed subgroups of some $GL_n(K)$. Indeed, many of the basic algebraic semigroup theory results have algebraic group counterparts.

One of the main things that separates semigroups and monoids from groups is the potential presence for nonidentity idempotents.

Definition 2.5. *For a semigroup S, the* **set of idempotents** *is $E(S) := \{s \in S \mid ss = s\}$.*

Indeed, algebraic semigroups are known to always have at least one idempotent.

Proposition 2.6. *Let S be an algebraic semigroup. Then $E(S) \neq \emptyset$.*

Proof. This can be found as Proposition 1 in Michel Brion's *On Algebraic Semigroups and Monoids* paper, [6]. □

2.1 Green's Relations

The underlying connecting theme of this paper is the equivalence relations known as Green's relations. These were relations on semigroups (sets with an associative binary operation) introduced in 1951 by James Alexander Green. Before defining the relations, we start with a simple semigroup-theoretic definition.

Definition 2.7. *For a semigroup, S, define the semigroup, S^1, to be S if S has already an identity element, and the semigroup $S \cup \{1\}$ with multiplication, $a \cdot b = \begin{cases} a \cdot b & \text{if } a, b \in S \\ a & \text{if } b = 1 \\ b & \text{otherwise} \end{cases}$ for all $a, b \in S \cup \{1\}$.*

The advantage of S^1 is that unlike an ideal such as, aS, we can guarantee that $a \in aS^1$. This is nice, because one would expect the ideal generated by an element to contain that element. Green's relations on semigroups are defined in terms of ideals using S^1.

Definition 2.8. *Let S be an arbitrary given semigroup. We define the* **Green's relations** *on S, $\mathscr{J}$, $\mathscr{R}$, $\mathscr{L}$, $\mathscr{D}$, and $\mathscr{H}$, as follows. For any two elements, $a, b \in S$,*

$a \mathscr{J} b$ if and only if $S^1 a S^1 = S^1 b S^1$

$a \mathscr{R} b$ if and only if $aS^1 = bS^1$

$a \mathscr{L} b$ if and only if $S^1 a = S^1 b$

$a \mathscr{D} b$ if and only if there exists $c \in S$ so $a \mathscr{R} c$ and $c \mathscr{L} b$

$a \mathscr{H} b$ if and only if $a \mathscr{R} b$ and $a \mathscr{L} b$

Each of Green's relations is an equivalence relation on the elements of S. On a group, the Green's relations become trivial, as there is only one equivalence class.

Proposition 2.9. *For an algebraic semigroup, S, $\mathscr{J} = \mathscr{D}$. That is, for any $a, b \in S$, $a \mathscr{J} b$ if and only if $a \mathscr{D} b$.*

Proof. This is a combination of Theorem 1.4 in [20] which states that $\mathscr{J} = \mathscr{D}$ is S is an $s\pi r$-semigroup, and Theorem 3.18 in the same reference which shows that all algebraic semigroups are $s\pi r$-semigroups. □

This leaves us with just four equivalence relations to investigate.

Example 2.10. *For $n \times n$ matrices over an algebraically closed field, K, the $\mathscr{J}$-classes have a definition based off a well-known property. For a given matrix, $A \in M_n(K)$, if J_A if the $\mathscr{J}$-class of A then we can write $J_A = \{B \in M_n(K) \mid rk(B) = rk(A)\}$, where rk denotes the familiar rank function from linear algebra.*

It would follow from our example that $M_n(K)$ has exactly $n+1$ $\mathscr{J}$-classes, one for each possible matrix rank, $rk(A) = 0, 1, 2, \cdots, n$.

Definition 2.11. *Let $a, b \in S$, and let J_a and J_b be the respective $\mathscr{J}$-classes. We can define a partial order on the $\mathscr{J}$-classes as follows, $J_a \leq J_b$ if and only if $S^1aS^1 \subseteq S^1bS^1$.*

Example 2.12. *For $n \times n$ matrices the partial order on $\mathscr{J}$-classes is identical to the order of the rank. So for matrices $A, B \in M_n(K)$, $J_A \leq J_B$ if and only if $rk(A) \leq rk(B)$.*

In addition to the partial order on $\mathscr{J}$-classes, we can also define a partial order on the idempotents of our semigroups. When we introduce the Adherence order on the Renner monoid we will have a third partial order to contend with. It is extremely fortunate that the work in papers like [17] have demonstrated that these are all compatible with one another. It is something we will touch on again in Section 4.

Definition 2.13. *Let $e, f \in E(S)$ be idempotents. We can define a partial order on the idempotents of S as follows, $e \leq f$ if and only if $ef = e = fe$.*

Proposition 2.14. *For idempotents, $e, f \in E(S)$, $f \leq e$ implies $J_f \leq J_e$.*

Proof. Since $ef = f = fe$ and $f \in S^1$ we can see that $f = ef \in S^1eS^1$. Thus it follows, $S^1fS^1 \subseteq S^1eS^1$. □

The $\mathscr{J}$ relation generalises the notion of rank from $n \times n$ matrices. The classes of the $\mathscr{H}$ relation in some sense provides an analogue of algebraic subgroups. In an amazing theorem, Green showed that for an $\mathscr{H}$-class, H, either $H \cap H^2 = \emptyset$ or H is a group. Indeed, if one takes a look at an idempotent $e \in S$, then H_e, the $\mathscr{H}$-class of e is a group with e as the identity element. As we often wish to relate algebraic semigroups and algebraic monoids back to the much studied algebraic groups, $\mathscr{H}$-classes are a source of particular interest.

As our notation has already hinted at, we will use the script letter ($\mathscr{J}$, $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$) to denote the relation itself, and the common letter (J, R, L, H) to refer to the classes ($\mathscr{L}$-class for example). For the class of a particular element, say $s \in S$ we will use the common letter and a subscript to denote the element (L_s for example).

Many of our results can be applied to each of the four Green's relations. If we wish to make a statement for all of the relations, we will use a stand-in symbol, $\mathscr{T}$. We will often say 'let $\mathscr{T}$ be one of $\mathscr{J}$, $\mathscr{R}$, $\mathscr{L}$, or $\mathscr{H}$.' The $\mathscr{T}$-class of an element $s \in S$ will be denoted T_s.

Frequently in this paper we will use the symbol $\mathscr{T}$ to denote a generic Green's relation. We will employ this notation to save space when a result covers each of $\mathscr{J}$, $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{H}$, though the proofs for each case may vary. For instance, we will later talk about how fat $\mathscr{T}$-classes (sets of the form BT_rB for some $r \in \mathcal{R}$) have their own sort of Bruhat decomposition, but we are getting ahead of ourselves.

One thing we can say concerning Green's relations on linear algebraic semigroups is the following result, showing that they are quasiaffine varieties.

Proposition 2.15. *Let S be a linear algebraic semigroup and let $e \in E(S)$. Then J_e, R_e, L_e, and H_e are relatively open subsets of $\overline{SeS}$, eS, Se, and eSe respectively (here, as in the rest of the paper, $\overline{}$ signifies the closure). In particular, J_e, R_e, and L_e are quasiaffine varieties and H_e is a linear algebraic group, the group of units of the algebraic monoid, eSe.*

Proof. This proposition comes from remarks made in the second section of [24], most notably Theorem 2.2. □

This next result allows us to relate the structure of a $\mathscr{J}$-class for an idempotent to that idempotent's $\mathscr{H}$-, $\mathscr{L}$-, and $\mathscr{R}$-classes.

Lemma 2.16. *Take an algebraic semigroup, S and pick $e \in E(S)$. Fix representatives, $\Gamma = \{\ell_i\}$ of L_e/H_e and $\Delta = \{r_j\}$ of R_e/H_e. Then, $J_e = \bigsqcup_{\ell_i \in \Gamma, r_j \in \Delta} \ell_i H_e r_j$*

Proof. The set $J_e \cup \{0\}$ forms a completely simple semigroup by the multiplication of two elements $a, b \in J$ defined as $a \circ b = \begin{cases} ab & \text{if } ab \in J \\ 0 & \text{otherwise} \end{cases}$. As Putcha remarks in [24], $J \cup \{0\}$ is a Rees matrix semigroup with sandwich map, $P : \Delta \times \Gamma \to H_e \cup \{0\}$. It is from here and Rees' paper, [26] which gives the result. □

2.2 Normality and Regularity

While some of the algebraic geometry and semigroup theory background we will use in this paper is contained in the Appendix, there are two notions we will need in Sections 8 and 9 which enhance the narrative we have laid out in the abstract and introduction. As such they bare explaining early on, and in a prominent position. These concepts are normality and regularity.

Normality is a geometric property coming from the underlying variety of an algebraic semigroup.

Definition 2.17. *A point $x \in X$ in a variety is* **normal** *if the local ring $\mathcal{O}_{X,x}$ is an integral domain which is integrally closed in its field of fractions.*

A variety, X, is called **normal** *if every point of X is normal.*

Example 2.18. *Every linear algebraic group is normal. See §AG.17, §AG.18 and the first Proposition in Chapter I of [3].*

The algebraic semigroup, $M_n(K)$, is normal as a variety. Indeed, $M_n(K) \cong K^{n^2}$, and affine space for any dimension is normal (see [35]'s Example 17.1.6).

Not every variety is a normal one, however to any variety we can associate a unique normal variety.

Definition 2.19. *For any irreducible algebraic variety, X, the* **normalisation** *of X consists of the unique irreducible normal variety, $\tilde{X}$, with finite birational morphism $\eta : \tilde{X} \to X$.*

For affine algebraic varieties, X, at the level of coordinate algebras, if $\tilde{X}$ is the normalisation of X then $\mathcal{O}(\tilde{X})$ is the integral closure of $\mathcal{O}(X)$. When combined with the additional multiplication structure of algebraic semigroups and algebraic monoids, we get an interesting result with the normalisation. Namely the normalisation of an algebraic semigroup is (usually) also an algebraic semigroup.

Proposition 2.20. *If the multiplication morphism $\mu : S \times S \to S$ is dominant, then the normalisation, $\tilde{S}$, of S has a unique algebraic semigroup law $\tilde{\mu}$ such that η is a homomorphism.*

Proof. This result can be found in Section 2.5 of Brion's [6]. □

For algebraic monoids, μ is always dominant, so the normalisation of the monoid turns out to also be a monoid (Proposition 3.15, [30]). However, for algebraic semigroups the result does not always apply (consider the trivial semigroup operation, $a \cdot b = c$ for some constant c, for all a, b which is not a dominant morphism).

Theorem 2.21. *If X and Y are two irreducible normal varieties then $X \times Y$ is also a normal variety.*

Proof. Proposition 17.3.2 from [35]. □

The second property, which will inform our later study is the semigroup theoretic regularity property.

Definition 2.22. *Let S be a semigroup, and let $a \in S$. We say a is* **regular** *if there exists $x \in S$ so that $axa = a$. For $\mathscr{T} = \mathscr{J}$, $\mathscr{R}$, $\mathscr{L}$, or $\mathscr{H}$ we say a $\mathscr{T}$-class, $T \subseteq S$, is* **regular** *if all its elements are regular. S is called* **regular** *if all its elements are regular.*

For a given $a \in S$, the element $x \in S$ in the definition is referred to as a **pseudoinverse**. If additionally, $xax = x$ we say x is an **inverse** of a. If x is just a pseudoinverse of a then xax is an inverse of a. We use the indefinite article as there can potentially be more than one pseudoinverse (inverse). A regular semigroup with unique inverses is known as an **inverse semigroup**. The condition of being an inverse semigroup is equivalent to the set of idempotents $E(S)$ being commutative.

Example 2.23. *The monoid of $M_n(K)$ matrices is regular. In fact, if A is any matrix in $M_n(K)$, then A^+, the Moore-Penrose pseudoinverse, is a well-known matrix which is a semigroup theoretic inverse of A.*

Every group is also regular. Indeed, they are inverse semigroups!

Every semigroup we study in this paper will be regular, unless explicitly stated otherwise. The structure of regular semigroups is highly susceptible to analysis with Green's relations.

Definition 2.24. *$\mathcal{U}(S)$ is the set of all regular $\mathscr{J}$-classes of S. For our choice of S in this paper, this will just be all $\mathscr{J}$-classes of S. We obtain a partial order on $\mathcal{U}(S)$ by defining $J' \leq J$ if $J' \subseteq S^1 J S^1$.*

It is a result of Putcha (5.10 in [20]) that for irreducible S, $\mathcal{U}(S)$ is a finite lattice.

Proposition 2.25. *If S is a regular semigroup then for all $x \in S$, $S^1 x S^1 = SxS$.*

Proof. It is clear that $SxS \subseteq S^1xS^1$. Suppose that $y \in S^1xS^1 \backslash SxS$. Then $y \in Sx$, xS or $\{x\}$. Since x is regular we can find $z \in S$ so $xzx = x$. If $y = sx$ then $y = sxzx \in SxS$. If $y = xs$, then $y = xzxs \in SxS$. If $y = x$ then $y = xzx = xzxzx \in SxS$. By contradiction, $S^1xS^1 = SxS$. □

As a result of this proposition, our partial order on $\mathcal{U}(S)$ simplifies to $J' \leq J$ if and only if $J' \subseteq SJS$. As we head into reductive monoids, we shall see some more equivalent definitions of the partial order on $\mathscr{J}$-classes.

Although regularity is primarily a semigroup-theoretic concept, our last result in this section gives us an important geometric intuition for regular semigroups.

Proposition 2.26. *For any regular algebraic semigroup, S, and any $x \in S$, the set SxS is closed in S.*

Proof. By Corollary 2.4 in [22], every ideal of S is closed. Since $S^1SxSS^1 \subseteq SxS$ we see that SxS is an ideal and hence closed. □

2.3 Reductive Monoids

A particular class of algebraic monoids which has received an enormous amount of attention over the years is the **reductive monoid**, which is a linear algebraic monoid whose group of units is reductive. For interested readers, Solomon's *An Introduction to Reductive Monoids* ([34]) is a superb resource for reductive monoids.

Proposition 2.27. *Let M be an irreducible algebraic monoid.*

(1) If M is reductive then M is regular.

(2) If M has a zero and is regular then M is reductive.

Proof. (1) is a consequence of Theorem 4.4 in [30]. (2) comes from Theorem 4.2 in the same source. □

Through the first several sections of this paper, we shall fix an irreducible reductive algebraic monoid and denote it by the usual letter, M.

Let G denote the group of units of M, an irreducible reductive algebraic group. Recall that M being reductive means exactly that G is reductive (that is, the unipotent radical of G is trivial). Within G, fix a Borel subgroup, B (a maximal closed connected solvable group) and, within B, fix a maximal torus T. By the work of Renner in [27] we know that we can write $M = \bigsqcup_{r \in \overline{N_G(T)}/T} BrB$ (recall from algebraic group theory that $N_G(T)$ denotes the normalizer in G of T). This is the **Bruhat decomposition** for reductive algebraic monoids.

Theorem 2.28. *If M is a reductive algebraic monoid and B is a Borel subgroup of its group of units. Let T be a maximal torus of B. Then,*

(1) $\mathcal{R} := \overline{N_G(T)}/T$ is a finite inverse monoid

(2) $E(\mathcal{R}) = E(\overline{T})$

(3) the group of units of $\mathcal{R}$ is $W := N_G(T)/T$

(4) $M = \bigsqcup_{r \in \mathcal{R}} BrB$

(5) $G = \bigsqcup_{w \in W} BwB$

Proof. This result is stated in many locations, but a good reference would be Chapter 8 of [30]. □

The quotient $\overline{N_G(T)}/T$ is known as the **Renner monoid** which we denote throughout this paper by $\mathcal{R}$. We will often distinguish elements of $\mathcal{R}$ by representatives in M of the cosets of T. We might write $x \in M$ to mean the element associated to the coset $xT = Tx \in \overline{N_G(T)}/T$.

Fixing our choice of B and T to define $\mathcal{R}$ and give the Bruhat decomposition also grants us other fixed structures. The most immediately relatable to the Bruhat decomposition of algebraic groups is the **Weyl group**, W, $N_G(T)/T$. The Weyl group is a finite Coxeter group and has been very well studied (one book we will use for the Weyl group is [2]).

Our distinguished Borel subgroup, B, allows us to give the elements of W a notion of length, $\ell(w) := dim(BwB) - dim(B)$. This length definition has been extended to $\mathcal{R}$ (see Definition 3.4). Using this length function, we can define the set of **simple reflections** $S = \{s \in W \mid \ell(s) = 1\}$. S generates W and for each $w \in W$. As it turns out, ℓ not only has the geometric dimension information from its definition but also coincides with a combinatorial property. $\ell(w)$ is the length of every minimal word (of simple reflections) for $w \in W$.

It is well-known that (W, S) is a Coxeter group, and so has a partial order on it called the Bruhat order. We can define this property geometrically with B and thereby extend it to the whole monoid.

Definition 2.29. *For any two elements, $r, s \in \mathcal{R}$ we say that $r \leq s$ if and only if $BrB \subseteq \overline{BsB}$.*

It does not take too long to determine that this relation makes $(\mathcal{R}, \leq)$ a partial order, which we call the **Adherence order**.

We have already encountered one partial order, but it was on a semigroup's $\mathscr{J}$-classes. It turns out to be related to the idempotents of M and also the Adherence order. Putcha notes the following definition in [20].

Definition 2.30. *A set, $\Psi \subseteq E(\overline{T})$ is called a* **cross sectional lattice** *if*

(i) $|J \cap \Psi| = 1$ for all $J \in \mathcal{U}(S)$

(ii) $e, f \in \Psi$, $J_e \leq J_f$ implies $e \leq f$

With respect to a fixed Borel subgroup, B, we have two natural cross sectional lattices. The sets $\Lambda := \{e \in E(\mathcal{R}) \mid Be \subseteq eB\}$ and $\Lambda^- := \{f \in E(\mathcal{R}) \mid fB \subseteq Bf\}$ are cross sectional lattices.

Theorem 2.31. *The partial order, $(\mathcal{R}, \leq)$ extends the partial orders $(W, \leq)$, $(\Lambda, \leq)$ and $(\Lambda^-, \leq)$.*

Proof. Although we will investigate these properties and the Adherence order in depth in Section 5, it should be noted early on that this result is a consequence of Theorem 1.4 and Corollary 1.5 of [17] and the symmetrical work presented in Appendix A.2. □

As a consequence, if $e, f \in \Lambda$ then $e \leq f$ in the Adherence order if and only if $e \leq f$ in the usual idempotent partial ordering if and only if $J_f \leq J_e$.

While on the subject of $\mathscr{J}$-classes, we can note that our definitions of Green's relations can be somewhat simplified and written in terms of the group of units of our two monoids M and $\mathcal{R}$.

Proposition 2.32. *For any $a, b \in M$,*

(1) $a\mathscr{H}b$ if and only if we can find $e, f \in E(M)$ and $g, h \in G$ so that $a = eg = gf$

and $b = eh = hf$

(2) $a\mathscr{L}b$ if and only if $Ga = Gb$

(3) $a\mathscr{R}b$ if and only if $aG = bG$

(4) $a\mathscr{J}b$ if and only if $GaG = GbG$

For any $a, b \in \mathcal{R}$,

(5) $a\mathscr{H}b$ if and only if we can find $e, f \in E(\mathcal{R})$ and $g, h \in W$ so that $a = eg = gf$ and $b = eh = hf$

(6) $a\mathscr{L}b$ if and only if $Wa = Wb$

(7) $a\mathscr{R}b$ if and only if $aW = bW$

(8) $a\mathscr{J}b$ if and only if $WaW = WbW$

Proof. This is a consequence of both M and $\mathcal{R}$ being unit regular monoids (monoids such that for any $m \in M$ there is a unit $g \in G$ so that $mgm = m$). As a consequence of being unit regular we can write these monoids in terms of their idempotents and group of units: $M = E(M)G = GE(M)$ and $\mathcal{R} = E(\mathcal{R})W = WE(\mathcal{R})$ from which the result follows. □

As it turns out, since Λ provides a cross sectional lattice for $\mathcal{R}$, we can write $\mathcal{R} = \bigsqcup_{e\in\Lambda} WeW$.

The group of units certainly simplifies Green's relations, but the Weyl group in $\mathcal{R}$ affords us more structure. In particular, being a finite Coxeter group there is a **longest element** $w_0 \in W$. This element is maximum in the Bruhat order and thus is maximum in the Adherence order.

(W, S) is a Coxeter system, and considering subsets $I \subseteq S$ of the generating set allows us to generate subgroups of W. These subgroups, owing to their special nature, are given a specific name, **standard parabolic subgroups**. Being finite, each of them also has a longest element as well. For a set of generators, I, we denote the generated subgroup by W_I. Returning back to M, $P_I = BW_IB$ is a standard parabolic subgroup of M in the sense that it is a parabolic subgroup of M containing B. There is a one to one correspondence between the W_I and the subgroups $P \subseteq G$ such that $B \subseteq P$.

At the level of the Borel subgroup, we have a unique **opposite Borel subgroup**, which we denote B^-, such that $B \cap B^- = T$. If $w_0 \in W$ is a longest element of our Weyl group, then $B^- = w_0Bw_0$.

As Renner notes in [27], there exists an antiinvolution, $\tau : M \to M$ with the following properties,

(1) $\tau^2(x) = x$ for all $x \in M$

(2) $\tau(xy) = \tau(y)\tau(x)$ for all $x, y \in M$

(3) $\tau|_T = id$

(4) $\tau(B) = B^-$

(5) τ induces a map $\tau : \mathcal{R} \to \mathcal{R}$ so that $\tau(x) = x^*$ for all $x \in \mathcal{R}$ (where $*$ is the pseudoinverse on $\mathcal{R}$)

Using τ we achieve the last of our background results.

Proposition 2.33.

(1) $\Lambda = \{e \in E(\mathcal{R}) \mid eB^- \subseteq B^- e\}$

(2) $\Lambda^- = \{f \in E(\mathcal{R}) \mid B^- f \subseteq fB^-\}$

(3) $\Lambda = w_0 \Lambda^- w_0$

Proof. Using τ, $eB^- = \tau(e)\tau(B)\tau(Be) \subseteq \tau(eB) = \tau(B)\tau(e) = B^- e$. The Λ^- result follows similarly. For (3), observe $e \in \Lambda$ if and only if $Be \subseteq eB$ if and only if $w_0 Bew_0 \subseteq w_0 eBw_0$ (because $int(w_0)$ is an automorphism) if and only if $w_0 Bw_0 w_0 ew_0 \subseteq w_0 ew_0 w_0 Bw_0$ if and only if $B^- w_0 ew_0 \subseteq w_0 ew_0 B^-$ if and only if $w_0 ew_0 \in \Lambda^-$. □

2.4 Example

The standard example for a reductive algebraic monoid is the matrix monoid, $M_n(K)$, where n is a positive integer and K is an algebraically closed field. The monoid consists of all $n \times n$ matrices over K. The group of units of this monoid is none other than $GL_n(K)$, the invertible $n \times n$ matrices over K. Being reductive it is also an excellent example of a regular algebraic semigroup and one we will use for examples later on.

For an example towards the Bruhat decomposition, one usually takes T to be the invertible diagonal matrices and B to be the invertible upper triangular matrices. It follows that the opposite Borel subgroup, B^-, is the set of invertible lower triangular matrices. The normalizer $N_G(T)$ can then be worked out to be the set of monomial matrices (those that have exactly one nonzero entry in each row and column). Then $\overline{N_G(T)}$ can be seen to be the set of matrices having at most one nonzero entry in each row and column.

Our Weyl group is just $W = N_G(T)/T$ which one can consider as the permutation matrices (and in this way we see $W \cong S_n$). The simple reflections of W are exactly the $n-1$ pairwise transpositions of S_n. As matrices they are, $(k\ k+1) = \begin{pmatrix} I_{k-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-k-1} \end{pmatrix}$ for each $1 \leq k \leq n-1$. The Renner monoid, as one would expect, is the set of matrices with only 0s and 1s for entries (called 0-1 matrices) with the added condition that there is at most one nonzero entry in each row and column. For example, in $M_2(K)$ we get the following Renner monoid,

$$\mathcal{R} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

Notice also that the first 2 elements form our Weyl group. The cross sectional lattices consist of the elements,

$$\Lambda = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \qquad \Lambda^- = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

and in general, $e \in \Lambda$ means $e = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$ and $e \in \Lambda^-$ means $e = \begin{pmatrix} 0 & 0 \\ 0 & I_m \end{pmatrix}$.

When it comes to determining Green's relations on $M_n(K)$, there are not any quick ways to determine the relations, but as was mentioned before, two elements are in the same $\mathscr{J}$-class if they have the same rank. This means that in $M_n(K)$ there are $n + 1$ different $\mathscr{J}$-classes. The Green's relations of the Renner monoid of $M_n(K)$ are a tad easier to determine, as the matrices are 0-1 matrices and hence easier to process at a glance.

For two elements $r, s \in \mathcal{R}$, we have the following shortcuts. To start, $r \mathscr{J} s$ if and only if r and s have the same rank, which is the same as the number of 1s in their expression. So $r \mathscr{J} s$ if and only if r and s have the same number of 1s. $r \mathscr{L} s$ if and only if their nonzero columns are in the same position. Below, the pair of matrices on the left are in the same $\mathscr{L}$-class, whereas the pair on the right are not, since the first column in the first matrix is nonzero, but the first column in the second matrix is zero.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Likewise $r \mathscr{R} s$ if and only if their nonzero rows are in the same position. Below, the left pair of matrices are $\mathscr{R}$ equivalent, but once again the pair on the right are not. This can be seen since the third row is zero in the first matrix, but is nonzero in the second matrix.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Combining these last two results, we see that $r \mathscr{H} s$ if and only if the nonzero columns are in the same positions and the nonzero rows are in the same positions. Another way of phrasing this is to say that the unique invertible minors of r and s that have size $rank(r)$ and $rank(s)$ are formed by considering the same set of entries. For example, the pair of matrices on the left are in the same $\mathscr{H}$-class, whereas the pair on the right are not. The pair on the right are in the same $\mathscr{R}$-class, but are not in the same $\mathscr{L}$-class.

$$\begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&0\\ 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1 \end{pmatrix} \qquad \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&1&0&0 \end{pmatrix}$$

The Adherence order on $M_n(K)$ can be calculated using one of two methods.

For a given matrix, r, take the first k columns from the left, arrange them in the staircase pattern, with 0 columns on the left. Replace the other columns with all zeros. This defines matrix r_k. Similarly we can construct s_k. If the staircase pattern for s_k lies consistently lower than the staircase pattern for r_k we say $r_k \leq s_k$. If $r_k \leq s_k$ for all the choices of $k = 1, \cdots, n$ then we say $r \leq s$.

For example, let us consider the matrices $r = \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0 \end{pmatrix}$ and $s = \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}$ from $M_6(K)$.

Following the procedure, we get the following pairs of matrices,

$$r_1 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix} \quad s_1 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix} \qquad r_2 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix} \quad s_2 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix}$$

$$r_3 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix} \quad s_3 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix} \qquad r_4 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix} \quad s_4 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}$$

$$r_5 = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0 \end{pmatrix} \quad s_5 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0 \end{pmatrix} \qquad r_6 = \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \quad s_6 = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

Since $r_1 \leq s_1$, $r_2 \leq s_2$, $r_3 \leq s_3$, $r_4 \leq s_4$, $r_5 \leq s_5$, and $r_6 \leq s_6$, we can see that $r \leq s$ in the Adherence order.

The other way to determine $\leq$ on $M_n(K)$ is to use the same procedure but order the first k rows from the bottom rather than the first k columns from the left. Our comparison also changes. $r_i \leq s_i$ if and only if the staircase pattern for s_i is consistently further left than the staircase of r_i. Further details of the Adherence order on $M_n(K)$ can be found in the following paper, [8].

With some concrete examples and base information in hand, we may now proceed to our first topic, investigating Green's relations on reductive monoids.

3 A New Trichotomy

The purpose of this section is to introduce a decomposition for elements of the Renner monoid, $\mathcal{R}$, into a product of elements that allow us to easily describe Green's relations for the original element. A similar decomposition was provided by Renner in [28], and is the inspiration for our work, and indeed for our investigation of fat $\mathscr{H}$-classes in Section 4.

3.1 Previous Results

We will begin by recalling some important results from [28] and adding in our own results and notation. To start we note this classic result.

Proposition 3.1. *For all $e \in \Lambda$, there is a unique element, $\nu \in WeW$ such that $B\nu = \nu B$. In particular, $\nu = e\sigma = \sigma f$, where $f \in \Lambda^-$ and σ is the element of minimal length such that $\sigma^{-1}e\sigma = f$.*

Proof. See [28], Proposition 1.2. □

Definition 3.2. $\mathcal{N} = \{r \in \mathcal{R} \mid rB = Br\}$

Phrasing Proposition 3.1 in the language of $\mathscr{J}$-classes gives us the following remark.

Corollary 3.3. *$\mathcal{N} \cong \mathcal{R}/\mathscr{J}$. That is to say, if $r, s \in \mathcal{R}$, $r\mathscr{J}s$ and $r, s \in \mathcal{N}$, then $r = s$, and for all $r \in \mathcal{R}$, there is $s \in \mathcal{N}$ with $r\mathscr{J}s$.*

Proof. This comes straight from Proposition 3.1 and the fact that the sets WeW for $e \in \Lambda$ are exactly the $\mathscr{J}$-classes of the Renner monoid. □

The fact that for each $r \in \mathcal{R}$ there is a unique $\nu \in \mathcal{N}$ so that $r\mathscr{J}\nu$ gives rise to the following definition for the length of an element of the Renner monoid. This definition extends the usual notion of length on the Weyl group.

Definition 3.4. *Define the* **length** *function on the Renner monoid, $\ell : \mathcal{R} \to \mathbb{N}$ to be,*
$\ell(r) = dim(BrB) - dim(B\nu B)$, for each $r \in \mathcal{R}$, where $\nu \in \mathcal{N}$ is unique so that $r\mathscr{J}\nu$.

A weakening of the condition $rB = Br$ has been investigated by Renner in his papers [27] and [29] when he explored the analogue of the Gauss-Jordan matrices.

Definition 3.5. $\mathcal{GJ} = \{r \in \mathcal{R} \mid Br \subseteq rB\}$

Definition 3.6. $\mathcal{JG} = \{r \in \mathcal{R} \mid rB \subseteq Br\}$

The sets can be viewed as an analogue of sorts of Λ and Λ^-. From the definition, the following two results are clear.

Proposition 3.7. $\mathcal{N} = \mathcal{GJ} \cap \mathcal{JG}$.

Proof. $r \in \mathcal{N}$ if and only if $Br = rB$ if and only if $Br \subseteq rB$ and $rB \subseteq Br$ if and only if $r \in GJ$ and $r \in \mathcal{JG}$ if and only if $r \in \mathcal{GJ} \cap \mathcal{JG}$. □

Proposition 3.8. *For $r \in \mathcal{R}$,*

(1) $r \in \mathcal{GJ}$ if and only if $BrB = rB$

(2) $r \in \mathcal{JG}$ if and only if $Br = BrB$

Proof. (1) Suppose that $s \in rB$, then we can find $b \in B$ so that $s = rb$. Since $1 \in B$ it follows that $s = 1s = 1rb \in BrB$. So $rB \subseteq BrB$. So it suffices to show that $r \in \mathcal{GJ}$ if and only if $BrB \subseteq rB$. Suppose that $BrB \subseteq rB$. Take an arbitrary $s \in Br$. Then we can find $b \in B$ so that $s = br$. Since $1 \in B$ it follows that $s = s1 = br1 \in BrB$. So $Br \subseteq BrB \subseteq rB$, or simply, $Br \subseteq rB$.

Conversely, suppose that $r \in \mathcal{GJ}$. If $s \in BrB$ then we can find $b_1, b_2 \in B$ so that $s = b_1rb_2$. Notice that $r \in \mathcal{GJ}$ means that $Br \subseteq rB$, so we can find $b_3 \in B$ so that $b_1r = rb_3$. Then $s = b_1rb_2 = rb_3b_2 \in rB$ as B is closed under multiplication. Thus $BrB \subseteq rB$.

(2) is demonstrated similarly. □

These sets of matrices $\mathcal{GJ}$, for the Gauss-Jordan elements, and $\mathcal{JG}$, which are the analogue of the anti-column reduced matrices, will play a great deal of importance in this paper. This is largely due to the following result, which shows that $\mathcal{GJ}$ is a set of representatives of the $\mathscr{L}$-classes of $\mathcal{R}$, and $\mathcal{JG}$ is a set of representatives of the $\mathscr{R}$-classes of $\mathcal{R}$.

Theorem 3.9.

(1) $\mathcal{GJ} \cong \mathcal{R}/\mathscr{L}$. That is to say, if $r, s \in \mathcal{R}$, $r\mathscr{L}s$ and $r, s \in \mathcal{GJ}$, then $r = s$, and for all $r \in \mathcal{R}$, there is $s \in \mathcal{GJ}$ with $r\mathscr{L}s$.

(2) $\mathcal{JG} \cong \mathcal{R}/\mathscr{R}$. That is to say, if $r, s \in \mathcal{R}$, $r\mathscr{R}s$ and $r, s \in \mathcal{JG}$, then $r = s$, and for all $r \in \mathcal{R}$, there is $s \in \mathcal{JG}$ with $r\mathscr{R}s$.

Proof. The proof of (1) can be given from Corollary 9.4 in [27]. The proof of (2) is done similarly. □

Now we come to the last set that we will need for our trichotomy. This is the set of order preserving elements, $\mathcal{O}$, which was first described in [28].

Definition 3.10. $\mathcal{O} = \{r \in \mathcal{R} \mid rBr^* \subseteq Brr^*\}$

One might guess that, as Green's relations are factoring heavily into our motivation, and we have sets corresponding to $\mathscr{J}$, $\mathscr{L}$ and $\mathscr{R}$, that $\mathcal{O}$ will correspond to the $\mathscr{H}$ relation. This is indeed correct as the next result tells us.

Theorem 3.11. *$\mathcal{O} \cong \mathcal{R}/\mathscr{H}$. That is to say, if $r, s \in \mathcal{R}$, $r\mathscr{H}s$ and $r, s \in \mathcal{O}$, then $r = s$, and for all $r \in \mathcal{R}$, there is $s \in \mathcal{O}$ with $r\mathscr{H}s$.*

Proof. This is a combination of Lemma 2.6 and Theorem 2.8 in [28]. □

Definition 3.10 is not the only way to describe $\mathcal{O}$ as we see in the next proposition.

Proposition 3.12. *$r \in \mathcal{O}$ if and only if $rBr^* \subseteq rr^*B$ if and only if $rBr^* \subseteq rr^*Brr^*$.*

Proof. The result can be found by applying Corollary 2.3 in [28], by Renner. □

Proposition 3.13. $\mathcal{O} = w_0\mathcal{O}w_0$, $\mathcal{O} = \mathcal{O}^*$

Proof. The first result is from Proposition 2.4 in [28]. The second result is Proposition 2.5 in the same paper. □

Corollary 3.14. *$r \in \mathcal{O}$ if and only if $r^*Br \subseteq Br^*r$ if and only if $r^*Br \subseteq r^*rB$ if and only if $r^*Br \subseteq r^*rBr^*r$.*

Proof. By Proposition 3.13, we can just replace r by r^* (and vice versa) in the original definition, Definition 3.10, and Proposition 3.12. □

From the definitions of $\mathcal{GJ}$, $\mathcal{JG}$ and $\mathcal{O}$ we get the following result.

Proposition 3.15. *$\mathcal{JG}, \mathcal{GJ} \subseteq \mathcal{O}$.*

Proof. If $r \in \mathcal{JG}$ then by definition, $rB \subseteq Br$, multiplying on both sides does not change the containment relation, so multiply by r^*. Then we see $rBr^* \subseteq Brr^*$, meaning $r \in \mathcal{O}$. For $r \in \mathcal{GJ}$, we know that $Br \subseteq rB$. Thus, $r^*Br \subseteq r^*rB$, and it follows that $r \in \mathcal{O}$. □

Proposition 3.16. $E(\mathcal{R}) \subseteq \mathcal{O}$

Proof. See [28], Lemma 2.2. The proof given by Renner amounts to using Proposition A.2 and the fact that $e^* = e$ for all idempotents. □

Proposition 3.17. *Let $r = e\sigma = \sigma f \in \mathcal{R}$ for $e, f \in E(\mathcal{R})$ and $\sigma \in W$. The following are equivalent,*

(1) $r \in \mathcal{O}$

(2) $Br \cap rB = eBr$

(3) $Br \cap rB = rBf$

(4) $eBr = rBf$

Proof. See Proposition 2.9 in [28]. □

Proposition 3.18. *Suppose that $r \in \mathcal{R}$ with $r = e\sigma = \sigma f$ for $e, f \in E(\mathcal{R})$ and $\sigma \in W$. Then,*

(1) $r \in \mathcal{JG}$ if and only if $f \in \Lambda^-$ and $r \in \mathcal{O}$

(2) $r \in \mathcal{GJ}$ if and only if $e \in \Lambda$ and $r \in \mathcal{O}$

Proof. We will prove the $\mathcal{JG}$ case, as the other is similar. By Proposition 3.15 it is clear that $r \in \mathcal{JG}$ implies $r \in \mathcal{O}$. If $r \in \mathcal{JG}$ then $rB \subseteq Br$, which we can rewrite as $\sigma f B \subseteq B\sigma f$. It follows that $fB \subseteq \sigma^{-1}B\sigma f \subset Gf$. So if $fb \in fB$ then $fb = gf$ for some $g \in G$. Hence $fb = gf = (gf)f = fbf$. Then $fB \subseteq fBf$. But since f is an idempotent, Proposition A.2 tell us that $fBf \subseteq Bf$. Thus $fB \subseteq Bf$ and so $f \in \Lambda^-$.

For the other direction, suppose that $f \in \Lambda^-$, that is, $fB \subseteq Bf$. This is equivalent to $fBf = fB$. Then, $rB = \sigma fB = \sigma fBf = rBf$. Since $r \in \mathcal{O}$ we know $rB = rBf = Br \cap rB$, by Proposition 3.17, thus $rB \subseteq rB$. □

Corollary 3.19.

(1) Suppose that $r, s \in \mathcal{JG}$, then $r \mathscr{J} s$ if and only if $r \mathscr{L} s$

(2) Suppose that $r, s \in \mathcal{GJ}$, then $r \mathscr{J} s$ if and only if $r \mathscr{R} s$

Proof. (1) Take $r, s \in \mathcal{JG}$. If $r \mathscr{L} s$ then clearly $r \mathscr{J} s$, so we will focus on the only if condition. Suppose that $r \mathscr{J} s$. Then there is a unique element $f \in \Lambda^-$ so that $r \mathscr{J} f \mathscr{J} s$. But by Proposition 3.18 we see that $r, s \in \mathcal{JG}$ means that we can find $\sigma, \tau \in W$ so that $r = \sigma f$ and $s = \tau f$. Thus $r \mathscr{L} s$.

(2) is done similarly. □

The following result tells us something wonderful. The sets we have introduced in this section have got some structure to them. Namely, they are all monoids.

Proposition 3.20. *The sets, $\mathcal{O}$, $\mathcal{GJ}$, $\mathcal{JG}$ and $\mathcal{N}$ are all monoids.*

Proof. Take $r, s \in \mathcal{N}$. Then $rB = Br$ and $sB = Bs$. We see that $rsB = rBs = Brs$, so $rs \in \mathcal{N}$. Suppose $r, s \in \mathcal{GJ}$. Then $Br \subseteq rB$ and $Bs \subseteq sB$. We see that $rsB \subseteq rBs \subseteq Brs$, so $rs \in \mathcal{GJ}$. Let $r, s \in \mathcal{JG}$. Then $rB \subseteq Br$ and $sB \subseteq Bs$. We see that $Brs \subseteq rBs \subseteq rsB$, so $rs \in \mathcal{JG}$. Finally, the case for $\mathcal{O}$ comes from Lemma 2.2 in [28]. □

In particular, this last result along with Proposition 3.13 show us that $\mathcal{O}$ is an inverse monoid.

3.2 The Trichotomy

We have now gathered enough information about our sets $\mathcal{O}$, $\mathcal{JG}$, $\mathcal{GJ}$ and $\mathcal{N}$ to describe our trichotomy. The following trichotomy is similar to the one presented by Renner in [28] but with a twist to allow for our later work with fat $\mathscr{H}$-classes. Our first theorem will state the trichotomy, and its proof will be quite similar to the proof of Renner's trichotomy, as indeed we can derive our trichotomy from his (and vice versa). The proof is included as it makes some use of the results we have just displayed.

Theorem 3.21. *Let $r \in \mathcal{R}$. Then there exist unique elements $r_-, r_0, r_+ \in \mathcal{R}$ such that,*

(1) $r = r_- r_0 r_+$

(2) $r_0 \mathscr{H} v^*$, where $v \mathscr{J} r$ and $v \in \mathcal{N}$

(3) $r \mathscr{R} r_-$ and $r \mathscr{L} r_+$

(4) $r_- \in \mathcal{JG}$ and $r_+ \in \mathcal{GJ}$

Proof. We begin by showing existence. Take an arbitrary $r \in \mathcal{R}$. By Proposition 3.9 we can find elements $s \in \mathcal{JG}$ and $t \in \mathcal{GJ}$ so that $r\mathscr{R}s$ and $r\mathscr{L}t$. That means that we can find $u, w \in W$ so that $su = r = wt$. Now, by Proposition 3.18 we can write $s = vf$, for $v \in W$ and $f \in \Lambda^-$ and $t = ed$, for $d \in W$ and $e \in \Lambda$. Consider $z := v^{-1}rd^{-1} = fud^{-1} = v^{-1}we$.

Let $\nu \in \mathcal{N}$ be the unique element from that set so that $\nu \mathscr{J} r$. Then $\nu = e\sigma = \sigma f$, for some $\sigma \in W$. It follows that $\nu^* = f\sigma^{-1} = \sigma^{-1}e$. It is then clear that $\nu^*\mathscr{R}f\mathscr{R}z\mathscr{L}e\mathscr{L}\nu^*$, so then $z\mathscr{H}\nu^*$. Define $r_0 = z$. Let $r_- = s$ and $r_+ = t$. One can see from our above work that r_-, r_0, r_+ satisfy properties (2), (3) and (4). Now we just note,

$$r = su = vfu = vffud^{-1}d = r_-fud^{-1}d = r_-v^{-1}wed = r_-v^{-1}weed = r_-r_0r_+,$$

and so (1) is also satisfied.

For the uniqueness, suppose that we had two decompositions, $r = r_-r_0r_+ = r'_-r'_0r'_+$. It is clear that by property (3), $r_-\mathscr{R}r\mathscr{R}r'_-$ and $r_+\mathscr{L}r\mathscr{L}r'_+$. But since (4) tells us that $r_-, r'_- \in \mathcal{JG}$ and $r_+, r'_+ \in \mathcal{GJ}$, it follows by Proposition 3.9 that $r_- = r'_-$ and $r_+ = r'_+$.

So, $r = r_-r_0r_+ = r_-r'_0r_+$. We can write $r_- = \sigma_-e_-$, $r_0 = e_-\sigma_0 = e_-\sigma_0e_+ = \sigma_0e_+$ and $r_+ = e_+\sigma_+$ for idempotents, $e_+ \in \Lambda$, $e_- \in \Lambda^-$ and elements $\sigma_-, \sigma_0, \sigma_+ \in W$. Thus, $r_-^* = e_-\sigma_-^{-1}$ and $r_+^* = \sigma_+^{-1}e_+$. It follows that,

$$r_-^*rr_+^* = e_-\sigma_-^{-1}\sigma_-e_-e_-\sigma_0e_+e_+\sigma_+\sigma_+^{-1}e_+ = e_-\sigma_0e_+ = r_0.$$

But also, since $r'_0\mathscr{H}\nu^*$, we can also find a $\tau \in W$ so that $r'_0 = e_-\tau = e_-\tau e_+ = \tau e_+$. Then we see that $r_-^*rr_+^* = e_-\sigma_-^{-1}\sigma_-e_-e_-\tau e_+e_+\sigma_+\sigma_+^{-1}e_+ = e_-\tau e_+ = r'_0$. We conclude that $r_0 = r_-^*rr_+^* = r'_0$, and so the decompositions are identical. □

The first use of our trichotomy is that it allows us to compute the length of r in terms of r_-, r_0 and r_+. This is what we will establish soon, but first we need a few minor results.

Lemma 3.22. *For $r \in \mathcal{R}$, suppose that we can write $r = fu = ve$ for some $u, v \in W$ and some $e, f \in E(\mathcal{R})$. Then, $Br \cap rB = fBr \cap rBe$.*

Proof. A proof is given by Renner, as Proposition 1.6 in [28]. □

Proposition 3.23. *Take $r \in \mathcal{R}$. Decompose it according to our trichotomy, $r = r_-r_0r_+$. Then we can find some $\sigma, \tau \in W$, so that, $Br \cap rB = \sigma(Br_0 \cap r_0B)\tau$.*

Proof. Write $r_0 = f_- w = w e_+$ for some $w \in W$ and idempotents f_-, e_+. Then, by Proposition A.1 we can write, $r_- = f\sigma = \sigma f_-$ and $r_+ = \tau e = e_+\tau$ for some $\sigma, \tau \in W$ and $e, f \in E(\mathcal{R})$. Then since $r\mathscr{L}r_+$ and $r\mathscr{R}r_-$ we can write $r = fu = ve$ for some $u, v \in W$. So by Lemma 3.22,

$$\begin{aligned}
Br \cap rB &= fBr \cap rBe = fBr_-r_0r_+ \cap r_-r_0r_+Be \\
&\subseteq r_-Br_0r_+ \cap r_-r_0Br_+ && \text{by Proposition 3.17, since } r_-, r_+ \in \mathcal{O} \\
&= \sigma f_-Br_0e_+\tau \cap \sigma f_-r_0Be_+\tau \\
&= \sigma(f_-Br_0e_+ \cap f_-r_0Be_+)\tau \\
&= \sigma(f_-Br_0 \cap r_0Be_+)\tau \\
&= \sigma(Br_0 \cap r_0B)\tau && \text{by Lemma 3.22, applied to } r_0. \qquad \square
\end{aligned}$$

We are now in position to demonstrate the length of an element in terms of our new trichotomy.

Proposition 3.24. *Let $r \in \mathcal{R}$, then $\ell(r) = \ell(r_-) - \ell(e_-) + \ell(r_0) - \ell(e_+) + \ell(r_+)$, where $r\mathscr{J}e_+ \in \Lambda$, $r\mathscr{J}e_- \in \Lambda^-$.*

Proof. Due to this result's similarity with Theorem 3.4 in [28], the following proof is understandably similar.

$$\ell(r) = dim(BrB) - dim(BvB) = dim(Br) + dim(rB) - dim(Br \cap rB) - dim(BvB)$$

Since $r_-\mathscr{R}r$, we can find $\sigma_- \in W$ so that $r = r_-\sigma_-$. Likewise, we can find $\sigma_+ \in W$ so that $r = \sigma_+r_+$. Now, because dimension is preserved by automorphism,

$$= dim(Br_-) + dim(r_+B) - dim(Br_0 \cap r_0B) - dim(BvB)$$

On the other hand,

$$\begin{aligned}
\ell(r_-) &= dim(Br_-B) - dim(BvB) = dim(Br_-) - dim(BvB) \\
\ell(e_-) &= dim(Be_-B) - dim(BvB) = dim(Be_-) - dim(BvB) \\
\ell(e_+) &= dim(Be_+B) - dim(BvB) = dim(e_+B) - dim(BvB) \\
\ell(r_+) &= dim(Br_+B) - dim(BvB) = dim(r_+B) - dim(BvB)
\end{aligned}$$

by Proposition 3.8. Since $r_0\mathscr{H}v^*$ we can find elements, $\tau_-, \tau_+ \in W$, such that $e_-\tau_- = r_0 = \tau_+e_+$. Thus,

$$\begin{aligned}
\ell(e_-) &= dim(Br_0) - dim(BvB) \\
\ell(e_+) &= dim(r_0B) - dim(BvB)
\end{aligned}$$

We then notice,

$$\begin{aligned}\ell(r_-) - \ell(e_-) + \ell(r_0) - \ell(e_+) + \ell(r_+) = (dim(Br_-) - dim(BvB)) - (dim(Br_0) - dim(BvB))\\ + (dim(Br_0) + dim(r_0B) - dim(Br_0 \cap r_0B) - dim(BvB))\\ - (dim(r_0B) - dim(BvB)) + (dim(r_+B) - dim(BvB))\\ = dim(Br_-) + dim(r_+B) - dim(Br_0 \cap r_0B) - dim(BvB)\end{aligned}$$

as desired. □

There is a nice characterisation of the elements of $\mathcal{O}$ when we look at the trichotomy of a given element.

Proposition 3.25. *For $r \in \mathcal{R}$, $r \in \mathcal{O}$ if and only if $r_0 \in \mathcal{N}^*$.*

Proof. Suppose that $r_0 = v^*$. It is clear that $v \in \mathcal{O}$, so by Proposition 3.13 then $v^* \in \mathcal{O}$. Since $\mathcal{O}$ is a monoid, then $r = r_- v^* r_+ \in \mathcal{O}$. On the reverse side, suppose $r \in \mathcal{O}$. Then $r_0 = r_-^* r r_+^* \in \mathcal{O}$. But $r_0 \mathscr{H} v^*$. Thus by Theorem 3.11, $r_0 = v^*$. □

Now that we have a condition for $\mathcal{O}$ in terms of our decomposition, we can get conditions for our sets $\mathcal{JG}$ and $\mathcal{GJ}$.

Corollary 3.26.

(1) For $r \in \mathcal{R}$, $r \in \mathcal{JG}$ if and only if $r_0 \in \mathcal{N}^*$ and $r_+ \in \mathcal{N}$. Indeed, for $r \in \mathcal{JG}$,

$r = rv^*v$ is our trichotomy.

(2) For $r \in \mathcal{R}$, $r \in \mathcal{GJ}$ if and only if $r_0 \in \mathcal{N}^*$ and $r_- \in \mathcal{N}$. Indeed, for $r \in \mathcal{GJ}$,

$r = vv^*r$ is our trichotomy.

Proof. (1) Suppose that $r \in \mathcal{JG}$, then since $\mathcal{JG} \subseteq \mathcal{O}$, we see that $r_0 \in \mathcal{N}^*$. By checking our trichotomy, we see that $r_- \in \mathcal{JG}$, and $r \mathscr{R} r_-$, so it follows that $r = r_-$. We also know that $r_+ \in \mathcal{GJ}$ and $r_+ \mathscr{L} r$. Let $v \in \mathcal{N}$ be the unique element so that $r \mathscr{J} v$. Since $r_0 \mathscr{J} r$, it follows that $r_0 = v^*$.

We can find $e \in \Lambda$, $f \in \Lambda^-$ and $\sigma, \tau \in W$ so that $r = r_- = \tau f$, $v^* = f\sigma^{-1} = \sigma^{-1}e$ and $v = e\sigma = \sigma f$. Observe that $rv^*v = (\tau f)(f\sigma^{-1})(\sigma f) = \tau fff = \tau f = r$. By uniqueness of trichotomy, this means that $r = rv^*v$, and thus, $r_+ \in \mathcal{N}$.

Suppose that $r_0 \in \mathcal{N}^*$ and $r_+ \in \mathcal{N}$. Since $r_0 \mathscr{J} r \mathscr{J} r_+$, it is clear that $r_0 = v^*$ and $r_+ = v$, for the unique $v \in \mathcal{N} \cap J_r$. Since $r_- \in \mathcal{JG}$, we can find $e \in \Lambda$, $f \in \Lambda^-$ and $\sigma, \tau \in W$ so that

$r_- = \tau f$, $v^* = f\sigma^{-1} = \sigma^{-1}e$. Then, $r = r_- r_0 r_+ = (\tau f)(f\sigma^{-1})(\sigma f) = \tau fff = \tau f = r = r_-$. Thus, $r \in \mathcal{JG}$.

(2) is done similarly. □

Remark 3.27. *It is clear to see that for $v \in \mathcal{N}$, that we have the trichotomy decomposition, $v = vv^*v$. That is, $r \in \mathcal{N}$ if and only if $r_0 \in \mathcal{N}^*$ and $r_-, r_+ \in \mathcal{N}$.*

While we are on the subject of $\mathcal{JG}$ and $\mathcal{GJ}$, we can also obtain the following two corollaries.

Corollary 3.28. *For $r \in \mathcal{O}$, we can find $s \in \mathcal{JG}^*$ and $t \in \mathcal{GJ}^*$ so that $r = r_- s = tr_+$*

Proof. By Proposition 3.25 we can write $r = r_- v^* r_+$, where $r_+ B \subseteq Br_+$. Let $s = v^* r_+$. We need to show that $s^* B \subseteq Bs^*$. $s^* = r_+^* v$. Let $v = e\sigma$ and $r_+^* = \tau e$ for idempotent e, and $\sigma, \tau \in W$. Then $r_+^* vB = r_+^* Bv = r_+^* Bev \subseteq Br_+^* v$ by Proposition 3.17. Thus $s \in \mathcal{JG}^*$.

A similar proof gives the $\mathcal{GJ}^*$ result. □

Corollary 3.29.

(1) $\mathcal{O}$ is the smallest inverse monoid containing $\mathcal{GJ}$

(2) $\mathcal{O}$ is the smallest inverse monoid containing $\mathcal{JG}$

Proof. We will just prove (1), as (2) is done similarly. We know from Proposition 3.15 that $\mathcal{GJ} \subseteq \mathcal{O}$. We know that $\mathcal{O}^* = \mathcal{O}$, from Proposition 3.13, and $\mathcal{O}$ is a monoid, by Proposition 3.20. Thus $\mathcal{O}$ is an inverse monoid containing $\mathcal{GJ}$. Suppose that $\mathcal{M}$ is an inverse monoid containing $\mathcal{GJ}$. Then $\mathcal{GJ}^* \subseteq \mathcal{M}^* = \mathcal{M}$. Take any $r \in \mathcal{O}$. By Corollary 3.28 for $r \in \mathcal{O}$ we can find $s \in \mathcal{GJ}^*$ so that $r = sr_+$. But by definition, $r_+ \in \mathcal{GJ}$, so it follows that $r \in \mathcal{M}$. Thus $\mathcal{O} \subseteq \mathcal{M}$, and we conclude that $\mathcal{O}$ is the smallest inverse monoid containing $\mathcal{GJ}$. □

The following theorem is the culmination of our trichotomy, as it allows us to describe the Adherence order of the Renner monoid in terms of our trichotomy, when we restrict to particular classes.

Theorem 3.30. *Take elements, $r, s \in \mathcal{R}$.*

(1) If $r \mathscr{H} s$, then $r \leq s$ iff $r_0 \leq s_0$

(2) If $r \mathscr{R} s$, then $r \leq s$ iff $r_0 r_+ \leq s_0 s_+$

(3) If $r \mathscr{L} s$, then $r \leq s$ iff $r_- r_0 \leq s_- s_0$

Proof. (1) It is clear that $r \le s$ is equivalent to $r_-r_0r_+ \in \overline{Bs_-s_0s_+B}$. Since $r\mathscr{H}s$ we see that $r_- = s_-$ and $r_+ = s_+$. So we see $r \le s$ gives $r_-^*r_-r_0r_+r_+^* \in r_-^*\overline{Br_-s_0r_+B}r_+^*$. But by continuity of multiplication, $r_-^*\overline{Br_-s_0r_+B}r_+^* \subseteq \overline{r_-^*Br_-s_0r_+Br_+^*} \subseteq \overline{Br_-^*r_-s_0r_+r_+^*B} = \overline{Bs_0B}$. Thus $r \le s$ implies $r_0 \le s_0$. Suppose that $r_0 \le s_0$, then $r_0 \in \overline{Bs_0B}$, which gives us, $r \in r_-\overline{Bs_0B}r_+ \subseteq \overline{r_-Bs_0Br_+} \subseteq \overline{BsB}$ since $r_- \in \mathcal{JG}$ and $r_+ \in \mathcal{GJ}$. Thus, $r_0 \le s_0$ implies $r \le s$.

(2) and (3) are proven similarly. □

It has been seen throughout this section that our sets $\mathcal{O}$, $\mathcal{JG}$, $\mathcal{GJ}$ and $\mathcal{N}$ are related to Green's relations. This last result, as with other before it, and even the definitions of these sets, also hints that B is also involved in their study. In the next section we will explore these two topics combined in the so called "fat $\mathscr{T}$-classes". We will end this section with the following useful remark, that allows us to name a generic $\mathscr{H}$-class. It will be immensely important, particularly later on in Section 5.

Remark 3.31. *For any $e \in \Lambda$, $f \in \Lambda^-$, with $e\mathscr{J}f$, we know that $w_0e = fw_0$. So for any $m \in \mathcal{JG}$ and $p \in \mathcal{GJ}$ with $m\mathscr{J}p$, we see that $mw_0p \in R_m \cap L_p$ (an $\mathscr{H}$-class). This is useful for distinguishing an element of an $\mathscr{H}$-class that we describe by r_- and r_+ without knowing the element r.*

3.3 Example

We have introduced a lot of sets in this section which will play a major role in this paper, so it would be a good idea to get some concrete examples into our minds. Below, we have plainly listed out our sets $\mathcal{R}$, $\mathcal{O}$, $\mathcal{GJ}$, $\mathcal{JG}$ and $\mathcal{N}$ coming from the monoid, $M_3(K)$.

$$\mathcal{R} = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix}, \right.$$
$$\begin{pmatrix} 1&0&0\\0&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix},$$
$$\begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix},$$
$$\left. \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&1&0\\0&0&0 \end{pmatrix} \right\}$$

$$\mathcal{O} = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&1&0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1&0&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix} \right\}$$

$$\mathcal{GJ} = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix} \right\}$$

$$\mathcal{JG} = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix} \right\}$$

$$\mathcal{N} = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix} \right\}$$

$\mathcal{GJ}$ (for Gauss-Jordan) is the analog of the row reduced matrices. For $\mathcal{JG}$ a small matter needs to be cleared up. They have been stated by Renner in [29] to be the analogue of column reduced matrices (that is a matrix whose transpose is row reduced), but upon closer inspection this is false. In fact they represent the anti-column reduced matrices, which is to say, they are matrices whose anti-transpose is row reduced (anti-transpose being the transpose taken with respect to the anti-diagonal). In fact it is $\mathcal{GJ}^*$ that gives us the column reduced matrices, and $\mathcal{JG}^*$ is the set of so-called anti-row reduced matrices.

The following examples come from $M_4(K)$,

$$\begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \in \mathcal{GJ} \qquad \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&1 \end{pmatrix} \in \mathcal{JG} \qquad \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix} \in \mathcal{GJ}^* \qquad \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&0 \end{pmatrix} \in \mathcal{JG}^*$$

As was mentioned in [17], $\mathcal{O}$ consists of all matrices where, for nonzero entries, the column value is an increasing function of the row value. This is what gives them their name of "order preserving" matrices.

$\mathcal{O}$ provides us an analogue of the monotonic path matrices. The original monotonic path matrices are found in the $M_n(K)$ and have the familiar "staircase" pattern to them, but now all the possible "staircases" from the upper left to the bottom right are fair game. Below we illustrate two such matrices, which belong to $\mathcal{O}$ for $M_6(K)$. Notice that the one on the right has nonzero values on either side of the diagonal, in a departure from $\mathcal{GJ}$ and $\mathcal{JG}$.

$$\begin{pmatrix} 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1 \end{pmatrix} \qquad \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix}$$

When we restrict ourselves to discussing our trichotomy in $M_n(K)$, we get a nice way of deriving the elements r_-, r_0, and r_+.

We will go through the method with the matrix, r = $\begin{pmatrix} 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 1&0&0&0&0 \end{pmatrix}$

To compute r_+ we read the matrix's columns from left to right. Whenever we come across a nonzero column, we put a 1 in the column of r_+, on the next row from the top. That is, if the jth column of r is the ith nonzero column that we have encountered, we place a 1 in the i, j position of r_+.

$$r = \begin{pmatrix} 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 1&0&0&0&0 \end{pmatrix} \qquad r_+ = \begin{pmatrix} 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

In this way, r_+ can be thought of as indicating the nonzero columns of r.

$$r = \begin{pmatrix} 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 1&0&0&0&0 \end{pmatrix} \qquad r_- = \begin{pmatrix} 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&1 \end{pmatrix}$$

Computing r_- is done similarly. To compute r_- we read the matrix's rows from bottom to top. Whenever we come across a nonzero row, we put a 1 in the row of r_-, on the next column from the right. That is, if the ith row of r is the jth nonzero row that we have found, we place a 1 in the i,$(n + 1 - j)$ position of r_-.

In this way, r_- can be thought of as indicating the nonzero rows of r.

Lastly, there is r_0 which describes how the rows and columns that we have indicated in r_- and r_+ act together to produce our element, r. To find r_0, write down the unique minor of r that has $rank(r)$. Then, place this minor in the bottom left corner of r_0, filling the rest of the matrix with 0s.

$$r = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 1&0&0&0&0 \end{pmatrix} \qquad \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix} \qquad \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 1&0&0&0&0 \end{pmatrix} = r_0$$

Thus, we have our decomposition: $\begin{pmatrix} 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 1&0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 1&0&0&0&0 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$

Let us use this decomposition to demonstrate Proposition 3.24. To compute the length of an element in the Renner monoid of $M_n(K)$ we can use the method described by [8]. For a 0-1 matrix in the Renner monoid of $M_n(K)$, A, the length of A can be found by the following computation,

$$\ell(A) = \Sigma_{i=1}^{n}\Sigma_{j=1}^{n}(a_{ij})(n+i-j) - |coinv(A)| - \frac{rk(A)(rk(A)+1)}{2}$$

where $coinv(A) = \{(a_{ij}, a_{kl}) \mid a_{ij} = a_{kl} = 1 \text{ and } i < k, j < l\}$ and $rk(A)$ indicates the rank, or number of nonzero entries. This allows us to compute the lengths of our trichotomy and the associated idempotents,

$$\ell\begin{pmatrix} 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \end{pmatrix} = (5+2-3)+(5+3-4)+(5+5-5)-3-\frac{3(3+1)}{2} = 4+4+5-3-6 = 4$$

$$\ell\begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} = (5+1-1)+(5+2-2)+(5+3-3)-3-\frac{3(3+1)}{2} = 5+5+5-3-6 = 6$$

$$\ell\begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 1&0&0&0&0 \end{pmatrix} = (5+3-2)+(5+4-3)+(5+5-1)-1-\frac{3(3+1)}{2} = 6+6+9-1-6 = 14$$

$$\ell\begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix} = (5+3-3)+(5+4-4)+(5+5-5)-3-\frac{3(3+1)}{2} = 5+5+5-3-6 = 6$$

$$\ell\begin{pmatrix} 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} = (5+1-1)+(5+2-3)+(5+3-5)-3-\frac{3(3+1)}{2} = 5+4+3-3-6 = 3$$

Now Proposition 3.24 tells us that,

$$\ell(r) = \ell(r_-) - \ell(e_-) + \ell(r_0) - \ell(e_+) + \ell(r_+) = 4-6+14-6+3 = 9.$$

Applying this method to the actual element, r, we can verify that,

$$\ell\begin{pmatrix} 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 1&0&0&0&0 \end{pmatrix} = (5+2-3)+(5+3-5)+(5+5-1)-1-\frac{3(3+1)}{2} = 4+3+9-1-6 = 9$$

One might ask whether this is a useful method for computing length. We can see that our elements in $\mathcal{GJ}$ and $\mathcal{JG}$ are written in a way that makes their length computation more apparent, all pairs of nonzero entries are skewed towards the top right corner of the matrix. Additionally, the nonzero entries of r_0 will always be in the bottom left corner, making them easy to read as well. One can reasonably expect that for different reductive monoids, M, that the individual elements of our trichotomy have lengths that are easier to compute in general.

4 Fat Green's Relations

In Section 3 we explored the Renner monoid of our reductive monoid, M, and were able to describe a decomposition for its elements. This decomposition was heavily influenced by the Green's relations on $\mathcal{R}$, and involved elements from $\mathcal{GJ} \cong \mathcal{R}/\mathscr{L}$ and $\mathcal{JG} \cong \mathcal{R}/\mathscr{R}$. What we would like to do is now explore how the Green's relations on $\mathcal{R}$ relate back to M.

We recall that when we wish to discuss a generic Green's relation, we will often denote it by $\mathscr{T}$. In the following section, as with the rest of our paper, unless specified, $\mathscr{T} = \mathscr{J}, \mathscr{L}, \mathscr{R}$ or $\mathscr{H}$.

Definition 4.1. *For an equivalence relation, $\mathscr{T} = \mathscr{J}$, $\mathscr{L}$, $\mathscr{R}$ or $\mathscr{H}$, and an element $r \in \mathcal{R}$, define the* **fat $\mathscr{T}$-class** *of r to be the set $BT_rB = \bigsqcup_{s \in \mathcal{R}, s\mathscr{T}r} BsB$.*

Notationally, depending on our choice of $\mathscr{T}$, we have **fat $\mathscr{J}$-classes***, BJ_rB,* **fat $\mathscr{L}$-classes***, BL_rB,* **fat $\mathscr{R}$-classes***, BR_rB or* **fat $\mathscr{H}$-classes***, BH_rB.*

The fat $\mathscr{T}$-classes give us a natural generalisation of our Bruhat cells, BrB for $r \in \mathcal{R}$. Considering = as an equivalence relation, our Bruhat cells are just BT_rB where $\mathscr{T}$ = "=". Admittedly, $=_r$ only has one element, namely r, but this still shows how considering our Green's relations in the form of fat $\mathscr{T}$-classes can be interesting and is a valid way to seek new information about $\mathcal{R}$.

This section is devoted to answering the following questions which will give us results analogous to the Bruhat decomposition for reductive monoids, established by Renner in [27].

- Do we have a Bruhat-eqsue decomposition, $M = \bigsqcup_{[r] \in \mathcal{R}/\mathscr{T}} BT_rB$, and can we describe the set $\mathcal{R}/\mathscr{T}$ in a nice way?
- Is it true that, $BT_rB \subseteq M$ is an irreducible subvariety for all $r \in \mathcal{R}$?
- Can we describe the partial ordering, $r, s \in \mathcal{R}$ then $BT_rB \subseteq \overline{BT_sB}$ in terms of $[r], [s] \in \mathcal{R}/\mathscr{T}$?
- Can we show the finite union, $\overline{BT_rB} = \bigsqcup_{i=1}^{k} BT_{r_i}B$, and describe the r_i?

In order to address these questions, we will first demonstrate equivalent formulations for our fat $\mathscr{T}$-classes that are more susceptible to manipulation.

4.1 Equivalent Definitions

Definition 4.2. *For $\mathscr{T} = \mathscr{J}, \mathscr{L}, \mathscr{R}$ or $\mathscr{H}$, and $r \in \mathcal{R}$, we define,*

$$T_r = \{s \in \mathcal{R} \mid s\mathscr{T}r\} \qquad T'_r = \{s \in M \mid s\mathscr{T}\dot{r} \text{ where } \dot{r} \text{ is a preimage of } r \text{ in } M\}$$

This gives us two different notions of what the $\mathscr{T}$-class of an element of $\mathcal{R}$ looks like. One may be concerned about the definition of T'_r, that it might vary based on the choice of preimage. The following proposition shows that this is not the case.

Proposition 4.3. *T'_r is well-defined, regardless of our choice of preimage of r in the definition.*

Proof. Unfortunately, there is no general way to prove this, and the result is merely a coincidence based on the definitions of our various Green's relations.

If $\mathscr{T} = \mathscr{J}$, take two preimages of $r \in \mathcal{R}$, say $p, q \in M$. Then we know $pT = qT$, so we can find $t \in T$ such that $p = qt$. Then, since $J'_r = GpG = GqtG = GqG$, as $t \in T \subseteq G$.

If $\mathscr{T} = \mathscr{L}$, take two preimages of $r \in \mathcal{R}$, say $p, q \in M$. Then we know $Tp = Tq$, so we can find $t \in T$ such that $p = tq$. Then, since $L'_r = Gp = Gtq = Gq$, as $t \in T \subseteq G$.

If $\mathscr{T} = \mathscr{R}$, take two preimages of $r \in \mathcal{R}$, say $p, q \in M$. Then we know $pT = qT$, so we can find $t \in T$ such that $p = qt$. Then, since $R'_r = pG = qtG = qG$, as $t \in T \subseteq G$.

If $\mathscr{T} = \mathscr{H}$, our previous work tells us that L'_r and R'_r are well-defined. Since $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$, we see that $H'_r = L'_r \cap R'_r$ is well-defined. □

With our new notion, we see that our fat $\mathscr{T}$-class, $BT_rB = \bigsqcup_{s \in T_r} BsB$ is not only the given definition, but also makes sense as a symbol. An interesting question is, what would happen if we "fattened" the $\mathscr{T}$-class of r in M. That is, what does BT'_rB look like? For $\mathscr{T} = \mathscr{J}, \mathscr{L}, \mathscr{R}$ and $\mathscr{H}$ we will see that $BT_rB = BT'_rB$, and this is very useful for our analysis of fat $\mathscr{T}$-classes. First notice that for any $\mathscr{T}$, $T_r \subseteq T'_r$ by definition, so it is immediately clear that $BT_rB \subseteq BT'_rB$.

To approach the fat $\mathscr{J}$-classes, we will follow along with a suggestion coming from Section 6 in [27] by Renner. To show the independence of definition in this case, we will first need a result coming from that paper. This result presents an analogue to reductive monoids of Tits' axiom.

Proposition 4.4. *Let S be the set of simple reflections of the Weyl group, W. Then if $r \in \mathcal{R}$, we have the following results,*

$$sBr \subseteq BrB \cup BsrB \qquad\qquad rBs \subseteq BrB \cup BrsB$$

Proof. This is Proposition 5.3 and Remark 5.4 in [27]. □

Proposition 4.5. *For $r \in \mathcal{R}$, $BJ_rB = BJ'_rB = J'_r$.*

Proof. As has already been mentioned, it is clear that $BJ_rB \subseteq BJ'_rB$, so it suffices to show the reverse inclusion. Observe that $BJ'_rB = BGrGB = GrG = J'_r$ (taking care of the second equality) and $r \in BJ_rB = \bigsqcup_{t\in J_r} BtB$. So the result will be shown if we can demonstrate that BJ_rB is closed under multiplication on the right by G and closed under multiplication on the left by G. We will just show that $GBJ_rB \subseteq BJ_rB$.

The Bruhat decomposition for our reductive group, G, tells us that $G = \bigsqcup_{w\in W} BwB$. So it suffices to show that $(BwB)(BJ_rB) \subseteq BJ_rB$ for all $w \in W$. Write $w = vs$ for $s \in S$ and $v \in W$ such that $\ell(w) = \ell(v) + 1$. Now,

$$\begin{aligned}
(BwB)(BJ_rB) &= BwB(\textstyle\bigsqcup_{t\in J_r} BtB) = BvBsB(\bigsqcup_{t\in J_r} BtB) \\
&= BvB(\textstyle\bigsqcup_{t\in J_r} sBtB) \subseteq BvB(\bigsqcup_{t\in J_r}(BtB \cup BstB)B) \\
&= BvB(\textstyle\bigsqcup_{t\in J_r} BtB \cup BstB) \subseteq BvB(BJ_rB) && \text{since } t \in J_r \implies st \in J_r. \\
&\subseteq BJ_rB && \text{by induction on the length of } w.
\end{aligned}$$

Our proof is completed upon the statement of our base case, $\ell(w) = 0 \implies w = 1$, and we can clearly see that, $B(BJ_rB) = BJ_rB$. □

Already we can see why the equivalence of $BJ_rB = BJ'_rB$ is useful. Just looking at BJ_rB there does not seem to be a nice way to describe it, but $BJ'_rB = GrG$, which is an orbit of the action of $G \times G$ on M. Orbits of algebraic group actions are well studied and we will make use of them as we go on.

Both fat $\mathscr{L}$-classes and fat $\mathscr{R}$-classes can be handled in a similar manner.

Proposition 4.6. *For $r \in \mathcal{R}$, $BL_rB = BL'_rB$ and $BR_rB = BR'_rB$.*

Proof. We will show this result for fat $\mathscr{R}$-classes, as the proof is similar for fat $\mathscr{L}$-classes. It is clear that $BR_rB \subseteq BR'_rB$, so we just need to show the reverse inclusion. Observe that $BR'_rB = BrGB = BrG$ and $r \in BR_rB = \bigsqcup_{t\in R_r} BtB$. So the result will be shown if we can demonstrate that BR_rB is closed under multiplication on the right by G.

As was remarked in Proposition 4.5, it suffices to show that $(BR_rB)(BwB) \subseteq BR_rB$ for all $w \in W$. Write $w = sv$ for $s \in S$ and $v \in W$ such that $\ell(w) = \ell(v) + 1$. Now,

$$\begin{aligned}
(BR_rB)(BwB) &= (\textstyle\bigsqcup_{t\in R_r} BtB)BwB = (\bigsqcup_{t\in R_r} BtB)BsBvB \\
&= (\textstyle\bigsqcup_{t\in R_r} BtBs)BvB \subseteq (\bigsqcup_{t\in R_r} B(BtB \cup BtsB))BvB \\
&= (\textstyle\bigsqcup_{t\in R_r} BtB \cup BtsB)BvB \subseteq (BR_rB)BvB && \text{since } t \in R_r \implies st \in R_r. \\
&\subseteq BR_rB && \text{by induction on the length of } w.
\end{aligned}$$

Our proof is completed upon the statement of our base case, $\ell(w) = 0 \implies w = 1$, and we can clearly see that, $(BR_rB)B = BR_rB$. □

The problem with $\mathscr{H}$-classes is that they are not defined by a coset or double coset relation (ie $r\mathscr{L}s$ if and only if $Gr = Gs$). So our previous work would not apply to dealing with the fat $\mathscr{H}$-classes. Instead we must take into account the definition of $\mathscr{H}$, namely $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$. It is from here, and with our previous results, that we can address fat $\mathscr{H}$-classes.

Proposition 4.7. *For $r \in \mathcal{R}$, $BL'_rB \cap BR'_rB = BH_rB = BH'_rB$.*

Proof. We will achieve the result by proving the following containments,

$$BL'_rB \cap BR'_rB \subseteq BH_rB \subseteq BH'_rB \subseteq BL'_rB \cap BR'_rB$$

The last two containments are clear, as $H_r \subseteq H'_r$, $H'_r \subseteq L'_r$ and $H'_r \subseteq R'_r$. By Proposition 4.6 we know that $BL'_rB \cap BR'_rB = BL_rB \cap BR_rB$. Suppose that $m \in BL_rB \cap BR_rB$. Then we can find $s, t \in \mathcal{R}$ with $s\mathscr{L}r$ and $t\mathscr{R}r$ so that $m \in BsB$ and $m \in BtB$. But then $BsB \cap BtB \neq \emptyset$. Thus $BsB = BtB$ and so $s = t$. It follows that $s\mathscr{H}r$ and $m \in BsB \subseteq BH_rB$. □

Now that we have a well-defined notion of fat $\mathscr{T}$-classes, we can begin to tackle the four problems listed above.

4.2 Fat $\mathscr{J}$-classes

Fat $\mathscr{J}$-classes have already been studied in detail, under the more familiar expression, $BJ_rB = GrG = J'_r$. Though little in this section is new, we include it for the sake of a complete look at these fat Green's relations, and to acknowledge the work that came before us.

Proposition 4.8. $M = \bigsqcup_{r\in\Lambda} BJ_rB = \bigsqcup_{r\in\mathcal{N}} BJ_rB = \bigsqcup_{r\in\mathcal{N}^*} BJ_rB = \bigsqcup_{r\in\Lambda^-} BJ_rB$

Proof. The case for Λ can be found in [21] in Theorem 3.3. Here we will give a general proof, covering all the cases at once. Let $A \in \{\Lambda, \mathcal{N}, \mathcal{N}^*, \Lambda^-\}$. We know from our standard Bruhat decomposition that $M = \bigsqcup_{r \in \mathcal{R}} BrB$. We also know that $A \cong \mathcal{R}/\mathscr{J}$, that is, for each $\mathscr{J}$-class in $\mathcal{R}$ there is one and only one element in A that is also in that $\mathscr{J}$-class. It follows that, $M = \bigsqcup_{r \in A} \bigsqcup_{s \mathscr{J} r} BsB$ (since $\mathscr{J}$-classes are disjoint). Then we just regroup our result to conclude, $M = \bigsqcup_{r \in A} BJ_rB$. □

Proposition 4.9. *$BJ_rB \subseteq M$ is an irreducible subvariety for all $r \in \mathcal{R}$.*

Proof. As we have made note before, $BJ_rB = GrG$. So our fat $\mathscr{J}$-class is easily seen to be an orbit of the action of $(G \times G) \times M \to M$, given by $((g_1, g_2), m) \mapsto g_1 m g_2^{-1}$ on M. By Proposition A.4, the orbit GrG is an irreducible subvariety of our variety, M, as desired. □

Proposition 4.10. *For $r, s \in \mathcal{R}$, if $r \leq s$ then $BJ_rB \subseteq \overline{BJ_sB}$.*

Proof. $r \leq s$ means that $BrB \subseteq \overline{BsB}$. Then, $GrG = GBrBG \subseteq G\overline{BsB}G \subseteq \overline{GBsBG} = \overline{GsG}$ by continuity of multiplication. Thus, $BJ_rB \subseteq \overline{BJ_sB}$. □

This is the first we have encountered this partial ordering, $BJ_rB \subseteq \overline{BJ_sB}$, which is something we would like to investigate. Rather than a partial order on the elements of $\mathcal{R}$, it can be seen as a partial order on the $\mathscr{J}$-classes of $\mathcal{R}$. The following results will show that it is actually the same partial order on $\mathscr{J}$-classes we know from semigroup theory.

Corollary 4.11. *For $r, s \in \mathcal{R}$, we can find idempotents $e, e' \in \Lambda$ and $f, f' \in \Lambda^-$ so that $r \in WeW = WfW$ and $s \in We'W = Wf'W$. Then $BJ_rB \subseteq \overline{BJ_sB}$ if and only if $e'e = e$ if and only if $f'f = f$.*

Proof. As we remarked in our background material, $e'e = e$ if and only if $e \leq e'$ in the Adherence order. So by Proposition 4.10, $BJ_eB \subseteq \overline{BJ_{e'}B}$. Conversely, if $BJ_rB \subseteq \overline{BJ_sB}$ then $GeG \subseteq \overline{Ge'G}$. By continuity, $MeM \subseteq M\overline{Ge'G}M = \overline{Me'M}$. Now, by Proposition 2.26, since M is reductive (and hence regular), $MeM \subseteq \overline{Me'M} = Me'M$. Thus $J_e \leq J_{e'}$ and hence $e \leq e'$ by definition of cross sectional lattices. □

Theorem 4.12. *Fix a set $A \in \{\Lambda, \mathcal{N}, \mathcal{N}^*, \Lambda^-\}$. For any $r \in \mathcal{R}$, we can find a finite list of $r_1, r_2, \cdots r_s \in A$ so that, $\overline{BJ_rB} = \bigsqcup_{i=1}^{s} BJ_{r_i}B$.*

It follows that $\overline{J'_r} = \bigsqcup_{i=1}^{s} J'_{r_i}$*, so the closure of a* $\mathscr{J}$*-class in* M *is a disjoint union of* $\mathscr{J}$*-classes.*

Proof. The following proof will rely purely on the fact that each A satisfies $A \cong \mathcal{R}/\mathscr{J}$. Since $\mathcal{R}$ is finite we can see, $\overline{BJ_rB} = \bigcup_{s \mathscr{J} r} \overline{BsB} = \bigcup_{s \mathscr{J} r} \bigcup_{t \leq s} BtB$. Recall $BJ_xB = \bigcup_{y \mathscr{J} x} ByB$. So if $BJ_xB \cap \overline{BJ_rB} \neq \emptyset$ we can find $y \mathscr{J} x$ and $s \mathscr{J} r$ with $y \leq s$. It follows, by Proposition 4.10, that $BJ_xB = BJ_yB \subseteq \overline{BJ_sB} = \overline{BJ_rB}$.

Thus the closure of each fat $\mathscr{J}$-class must be a union of fat $\mathscr{J}$-classes, and this union is disjoint since fat $\mathscr{J}$-classes are disjoint (Proposition 4.8). The union itself must be finite, as each $\mathscr{J}$-class can be indexed by a unique element of $A \subseteq \mathcal{R}$, a finite set (Proposition 4.8). □

4.3 Fat $\mathscr{L}$-Classes and Fat $\mathscr{R}$-Classes

Fat $\mathscr{L}$- and $\mathscr{R}$-classes have been studied before, but as with the fat $\mathscr{J}$-classes, it was under a different guise (namely GrB and BrG). Most of the work about them can be found in [29]. As we will find out, fat $\mathscr{L}$- and $\mathscr{R}$-classes are closely related to our sets $\mathcal{GJ}$ and $\mathcal{JG}$, which were first studied by Renner in [27].

Proposition 4.13. $M = \bigsqcup_{r \in \mathcal{GJ}} BL_rB = \bigsqcup_{r \in \mathcal{JG}} BR_rB$

Proof. We know from our standard Bruhat decomposition that $M = \bigsqcup_{r \in \mathcal{R}} BrB$. From Theorem 3.9 it follows that, $M = \bigsqcup_{r \in \mathcal{GJ}} \bigsqcup_{s \mathscr{L} r} BsB$ (since $\mathscr{L}$-classes are disjoint). Then we just regroup our result to conclude, $M = \bigsqcup_{r \in \mathcal{GJ}} BL_rB$.

The case for $\mathcal{JG}$ is done similarly. □

Theorem 4.14. $BL_rB, BR_rB \subseteq M$ *are irreducible subvarieties for all* $r \in \mathcal{R}$

Proof. Notice that $BL_rB = GrB$. So our fat $\mathscr{L}$-class is an orbit of the group action on M, of $(G \times B) \times M \to M$, given by $((g, b), m) \mapsto gmb^{-1}$. By Proposition A.4, the orbit GrB is an irreducible subvariety of our variety, M, as desired.

Likewise BR_rB is the orbit of r with the action of $B \times G$. □

Proposition 4.15.

(1) If $r, s \in \mathcal{GJ}$ *then* $GrB \subseteq \overline{GsB}$ *if and only if* $rB \subseteq \overline{sB}$

(2) If $r, s \in \mathcal{JG}$ *then* $BrG \subseteq \overline{BsG}$ *if and only if* $Br \subseteq \overline{Bs}$

Proof. This result can be found as Proposition 2.8 in [29] for the $\mathcal{JG}$ case. The $\mathcal{GJ}$, as usual, is similar. □

Proposition 4.16. *For $r, s \in \mathcal{R}$, if $r \leq s$ then $BL_rB \subseteq \overline{BL_sB}$ and $BR_rB \subseteq \overline{BR_sB}$.*

Proof. $r \leq s$ means that $BrB \subseteq \overline{BsB}$. Then, $BrBG \subseteq \overline{BsB}G \subseteq \overline{BsBG} = \overline{BsG}$ by continuity of multiplication. Thus, $BR_rB \subseteq \overline{BR_sB}$. As usual, the proof for fat $\mathscr{L}$-classes is similar. □

Theorem 4.17.

(1) If $r, s \in \mathcal{R}$ then $BL_rB \subseteq \overline{BL_sB}$ if and only if $r_+B \subseteq \overline{s_+B}$

(2) If $r, s \in \mathcal{R}$ then $BR_rB \subseteq \overline{BR_sB}$ if and only if $Br_- \subseteq \overline{Bs_-}$

Proof. As usual, we will prove the first result, as the second is handled by symmetry. We know already that $L_x = L_{x_+}$ and $x_+ \in \mathcal{GJ}$. So we just need to show $BL_{r_+}B \subseteq \overline{BL_{s_+}B}$ if and only if $r_+B \subseteq \overline{s_+B}$. But recall that $BL_xB = GxB$, and when we substitute this in, we get $BL_{r_+}B \subseteq \overline{BL_{s_+}B}$ if and only if $Gr_+B \subseteq \overline{Gs_+B}$. The result is concluded with Proposition 4.15. □

The following result is reminiscent of Proposition 3.11 from [28], but with a broader range. It will be useful later on, as we explore the Bruhat order on our different $\mathscr{T}$-classes.

Corollary 4.18. *For $r, s \in \mathcal{R}$, $r \leq s$ implies that $r_- \leq s_-$ and $r_+ \leq s_+$.*

Proof. By Proposition 4.16, $r \leq s$ means that $BR_rB \subseteq \overline{BR_sB}$ and $BL_rB \subseteq \overline{BL_sB}$. By Theorem 4.17, $Br_- \subseteq \overline{Bs_-}$ and $r_+B \subseteq \overline{s_+B}$. By Proposition 3.8, $Br_-B = Br_- \subseteq \overline{Bs_-} = \overline{Bs_-B}$ and $Br_+B = r_+B \subseteq \overline{s_+B} = \overline{Bs_+B}$, or rather $r_- \leq s_-$ and $r_+ \leq s_+$. □

The following example shows that any attempt at a similar result involving r_0 and s_0 is doomed to fail. It is not true, in general, that $r \leq s$ implies $r_0 \leq s_0$.

Example 4.19. *Let us take a look at the elements, $r = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&0&0&0 \end{pmatrix}$ and $s = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \end{pmatrix}$. Using our techniques from the last section, we can decompose these two elements based on our trichotomy.*

$$\begin{pmatrix} 0&1&0&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix} \cdot \begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \end{pmatrix} = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{pmatrix}$$

Knowing that $r \leq s$, we can check out Corollary 4.18 and see $\begin{pmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \leq \begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}$ *and that* $\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \leq \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}$. *However, when we go to compare r_0 and s_0, we actually find that they are in the opposite relationship with respect to order,* $\begin{pmatrix} 0&0&0&0\\0&1&0&0\\1&0&0&0\\0&0&1&0 \end{pmatrix} \leq \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&0 \end{pmatrix}$.

We can now characterise our partial order on the fat $\mathscr{J}$-classes in terms of our representatives coming from $\mathcal{N}$. Since $\mathcal{N} = \mathcal{JG} \cap \mathcal{GJ}$, it is very similar to Theorem 4.17's result.

Corollary 4.20. *If $r, s \in \mathcal{R}$ then $BJ_rB \subseteq \overline{BJ_sB}$ if and only if $B\mu = \mu B \subseteq \overline{\nu B} = \overline{B\nu}$, where μ is the unique element in $\mathcal{N} \cap J_r$ and ν is the unique element in $\mathcal{N} \cap J_s$*

Proof. The equals signs in the statement $B\mu = \mu B \subseteq \overline{\nu B} = \overline{B\nu}$ are clear from our definition of $\mathcal{N}$, so we will just prove that $BJ_rB \subseteq \overline{BJ_sB}$ if and only if $B\mu \subseteq \overline{B\nu}$. Observe that if $B\mu \subseteq \overline{B\nu}$, then by the properties of $\mathcal{N}$ we see that $B\mu B = B\mu \subseteq \overline{B\nu} = \overline{B\nu B}$. Thus, $\mu \leq \nu$ and so by Proposition 4.10, $BJ_rB = BJ_\mu B \subseteq \overline{BJ_\mu B} = \overline{BJ_sB}$.

Conversely, suppose that $BJ_rB \subseteq \overline{BJ_sB}$, then we can find some x, y and $z \in \mathcal{R}$ so that $x \mathscr{J} r$, $y \leq z$ and $z \mathscr{J} s$ with $BxB \cap ByB \neq \emptyset$. Thus, $x = y$, and so we have $x \leq z$. Corollary 4.18 tells us that $x_+ \leq z_+$. But then Corollary 4.18 applied again tells us, $(x_+)_- \leq (z_+)_-$. But by analysing our $\mathscr{J}$-classes and applying Corollary 3.26 this shows us $\mu \leq \nu$, or rather $B\mu = B\mu B \subseteq \overline{B\nu B} = \overline{B\nu}$, as desired. □

Theorem 4.21.

(1) For any $r \in \mathcal{R}$, we can find a finite set of $r_1, r_2, \cdots r_s \in \mathcal{GJ}$ so, $\overline{BL_rB} = \bigsqcup_{i=1}^{s} BL_{r_i}B$

(2) For any $r \in \mathcal{R}$, we can find a finite set of $r_1, r_2, \cdots r_s \in \mathcal{JG}$ so, $\overline{BR_rB} = \bigsqcup_{i=1}^{s} BR_{r_i}B$

Proof. We will just prove (1). Since $\mathcal{R}$ is finite, $\overline{BL_rB} = \bigcup_{s\mathscr{L}r} \overline{BsB} = \bigcup_{s\mathscr{L}r} \bigcup_{t\leq s} BtB$. Recall that $BL_xB = \bigcup_{y\mathscr{L}x} ByB$. So if $BL_xB \cap \overline{BL_rB} \neq \emptyset$ then we can find $y\mathscr{L}x$ and $s\mathscr{L}r$ with $y \leq s$. But then by Proposition 4.16 we see $BL_xB = BL_yB \subseteq \overline{BL_sB} = \overline{BL_rB}$.

Thus the closure of each fat $\mathscr{L}$-class must be a union of fat $\mathscr{L}$-classes, and this union is disjoint since fat $\mathscr{L}$-classes are disjoint (Proposition 4.13). The union itself must be finite, as each $\mathscr{L}$-class can be indexed by a unique element of $\mathcal{GJ}$, a finite monoid. In fact, by our previous work we can characterize these elements, $\overline{BL_rB} = \bigsqcup_{i=1}^{s} BL_{r_i}B$, where $r_i \in \mathcal{GJ}$, $r_iB \subseteq \overline{r_+B}$. □

4.4 Fat $\mathscr{H}$-Classes

Of particular interest are the fat $\mathscr{H}$-classes. One of the reasons is that in monoid theory, we often wish to imitate the results we see in group theory. So we want to get results on our Renner monoid, $\mathcal{R}$, that are similar to those on the Weyl group, W. When Green's relations get involved, the $\mathscr{H}$-class provides us with an intriguing analogue to groups. Indeed, it is a well-known result of Green that for an $\mathscr{H}$-class, H, H is a group if and only if $H \cap H^2 \neq \emptyset$. In particular, this shows that for any idempotent, $e \in \mathcal{R}$, H_e is a group ($H_1 = W$).

So $\mathscr{H}$-classes, and thus fat $\mathscr{H}$-classes are a keen point of interest.

Proposition 4.22. $M = \bigsqcup_{r \in O} BH_rB$

Proof. We know from our standard Bruhat decomposition that $M = \bigsqcup_{r \in \mathcal{R}} BrB$. From Theorem 3.11 it follows that, $M = \bigsqcup_{r \in O} \bigsqcup_{s \mathscr{H} r} BsB$ (since $\mathscr{H}$-classes are disjoint). Then we just regroup our result to conclude, $M = \bigsqcup_{r \in O} BH_rB$. □

The latter half of the following result is interesting, as we cannot use our theory of orbits of algebraic group actions to show irreducibility. In fact, there is no reason why one should expect BH_rB to be irreducible from our previous work, as we have shown $BH_rB = BL_rB \cap BR_rB$ and the intersection of two irreducible sets is not alwas irreducible.

Theorem 4.23. *$BH_rB \subseteq M$ is an irreducible subvariety for all $r \in \mathcal{R}$*

Proof. First, let us recall that BL_rB and BR_rB are subvarieties, by Theorem 4.14. So we see that the fat $\mathscr{L}$-class and fat $\mathscr{R}$-class associated to r are locally closed. The intersection of two locally closed sets is locally closed, and so, by Proposition 4.7, $BH_rB = BL_rB \cap BR_rB$ must be locally closed. Thus BH_rB is a subvariety.

To show irreducibility, first note that we can find an idempotents $e, f \in E(\mathcal{R})$ and $\sigma \in W$ so that $r = e\sigma = \sigma f$. So then $f = \sigma^{-1} e \sigma$. We know that H_e' is an irreducible variety, as it is the group of units of eMe, an irreducible variety. We can recover H_r', by noting that

$H'_r = H'_e\sigma \subseteq eMe\sigma = eMf$. So, since H'_r is the image of H'_e under an automorphism, we see that H'_r is also irreducible.

Recall that B is an irreducible subvariety of M. Consider the map $\varphi : B \times H'_r \times B \to M$, defined by $\varphi(b_1, h, b_2) = b_1 h b_2$. We see that $B \times H'_r \times B$ must also be irreducible, and since BH_rB is the image of $\varphi(B \times H'_r \times B)$, by Proposition 4.7, we conclude that BH_rB must also be irreducible. □

Proposition 4.24. *For $r, s \in \mathcal{R}$, $r \leq s$ implies that $BH_rB \subseteq \overline{BH_sB}$.*

Proof. Proposition 4.16 told us $r \leq s$ implies $BL_rB \subseteq \overline{BL_sB}$ and $BR_rB \subseteq \overline{BR_sB}$. But then we see that $BL_rB \cap BR_rB \subseteq \overline{BL_sB} \cap \overline{BR_sB}$. Proposition 4.7 then shows us our desired result, $BH_rB = BL_rB \cap BR_rB \subseteq \overline{BL_sB} \cap \overline{BR_sB} = \overline{BL_sB \cap BR_sB} = \overline{BH_sB}$. □

We can now combine the two results in Theorem 4.17 to get a similar property for fat $\mathscr{H}$-classes.

Theorem 4.25. *For $r, s \in \mathcal{R}$, $BH_rB \subseteq \overline{BH_sB}$ if and only if $Br_- \subseteq \overline{Bs_-}$ and $r_+B \subseteq \overline{s_+B}$*

Proof. For the 'if' direction, by Theorem 4.17, we know $Br_- \subseteq \overline{Bs_-}$ and $r_+B \subseteq \overline{s_+B}$ implies $BR_rB \subseteq \overline{BR_sB}$ and $BL_rB \subseteq \overline{BL_sB}$. Thus $BL_rB \cap BR_rB \subseteq \overline{BL_sB} \cap \overline{BR_sB} = \overline{BL_sB \cap BR_sB}$. By Proposition 4.7 we see, $BH_rB \subseteq \overline{BH_sB}$.

For the 'only if' direction, suppose that $BH_rB \subseteq \overline{BH_sB}$. Then observe the following containment, $\bigsqcup_{x\mathscr{H}r} BxB = BH_rB \subseteq \overline{BH_sB} = \overline{\bigsqcup_{y\mathscr{H}s} ByB} = \bigcup_{y\mathscr{H}s} \overline{ByB} = \bigcup_{y\mathscr{H}s} \bigcup_{z\leq y} BzB$ (as $\mathcal{R}$ is finite). So we can find $r'\mathscr{H}r$ and $t \leq s'\mathscr{H}s$ so that $Br'B = Bt'B$. Thus $r' \leq s'$. It follows from Proposition 4.16 and Theorem 4.17 that $Br'_- \subseteq \overline{Bs'_-}$ and $r'_+B \subseteq \overline{s'_+B}$.

Now, by our trichotomy, $r'_-\mathscr{R}r'\mathscr{H}r\mathscr{R}r_-$, so $r'_-\mathscr{R}r_-$. Likewise, $r'_+\mathscr{L}r_+$, $s'_-\mathscr{R}s_-$ and $s'_+\mathscr{L}s_+$. And, by the properties of our trichotomy, and Theorem 3.9, we can see that in fact, $r'_- = r_-$, $r'_+ = r_+$, $s'_- = s_-$ and $s'_+ = s_+$. Thus, we have shown that $Br_- \subseteq \overline{Bs_-}$ and $r_+B \subseteq \overline{s_+B}$. □

This is quite a different result from Theorem 4.17 and Corollary 4.20. Those results gave us a way of determining $BT_rB \subseteq \overline{BT_sB}$ from our familiar sets $\mathcal{N}$, $\mathcal{GJ}$ and $\mathcal{JG}$. Each of those sets is of the form $\mathcal{R}/\mathscr{T}$. So what we would like is for this result to be in terms of $\mathcal{O} \cong \mathcal{R}/\mathscr{H}$ as a single containment relation. We do not yet have the tools to describe the result, so we will have to be content with what we have now, and take comfort in the fact that $\mathcal{GJ}, \mathcal{JG} \subseteq O$. The "correct" form of Theorem 4.25 can be uncovered by reading Section 6.

Theorem 4.26. *For any $r \in \mathcal{R}$, we can find a finite collection of $r_1, r_2, \cdots r_s \in O$ so that, $\overline{BH_rB} = \bigsqcup_{i=1}^{s} BH_{r_i}B$*

Proof. Since $\mathcal{R}$ is finite, $\overline{BH_rB} = \bigcup_{s\mathscr{H}r} \overline{BsB} = \bigcup_{s\mathscr{H}r} \bigcup_{t\leq s} BtB$. Recall $BH_xB = \bigcup_{y\mathscr{H}x} ByB$. So if $BH_xB \cap \overline{BH_rB} \neq \emptyset$ then we can find $y\mathscr{H}x$ and $s\mathscr{H}r$ with $y \leq s$. But then by Proposition 4.24 we see $BH_xB = BH_yB \subseteq \overline{BH_sB} = \overline{BH_rB}$.

Thus the closure of each fat $\mathscr{H}$-class must be a union of fat $\mathscr{H}$-classes, and this union is disjoint since fat $\mathscr{H}$-classes are disjoint. The union itself must be finite, as each $\mathscr{H}$-class can be indexed by a unique element of O, a finite monoid. In fact, by our previous work we can characterize these elements, $\overline{BH_rB} = \bigsqcup_{i=1}^{s} BH_{r_i}B$, where $r_i \in O$, $r_{i+}B \subsetneq \overline{r_+B}$ and $Br_{i-} \subseteq \overline{Br_-}$. □

Again, we would like a condition that involves the elements $r_i \in O$, without resorting to our trichotomy. The way we have it now is more taking advantage of Proposition 4.7, rather than using the properties of O in any meaningful way. Just as before, we direct curious readers to Section 6 where this is tackled in an interesting manner.

4.5 Example

Let us compute some of these fat $\mathscr{T}$-classes, so that we can get a sense of what we are talking about, and to inform our examples later on. For starters, let us compute BJ_rB, BL_rB, BR_rB and BH_rB for the element, $r = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. First we compute each of the equivalence classes for each $\mathscr{J}$, $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{H}$.

$$J_r = \{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$$

$$L_r = \{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\}$$

$$R_r = \{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$$

$$H_r = \{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\}$$

Now, our fat classes are disjoint unions, so we can specify BT_rB by writing the general form of BsB for each $s \in T_r$. Using the general form of $B = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} g & h & i \\ 0 & j & k \\ 0 & 0 & l \end{pmatrix}$ with $a, d, f, g, j, l \in K^*$ and $b, c, e, h, i, k \in K$ we can write out our classes.

$$BJ_rB = \{ \begin{pmatrix} bg & bh+aj & bi+ak \\ dg & dh & di \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} bg & bh & bi+al \\ dg & dh & di \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & bj & bk+al \\ 0 & dj & dk \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} cg & ch+aj & ci+ak \\ eg & eh & ei \\ fg & fh & fi \end{pmatrix},$$
$$\begin{pmatrix} cg & ch & ci+al \\ eg & eh & ei \\ fg & fh & fi \end{pmatrix}, \begin{pmatrix} 0 & cj & ck+al \\ 0 & ej & ek \\ 0 & fj & fk \end{pmatrix}, \begin{pmatrix} ag & ah+bj & ai+bk \\ 0 & dj & dk \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} ag & ah & ai+bl \\ 0 & 0 & dl \\ 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & aj & ak+bl \\ 0 & 0 & dl \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} ag & ah+cj & ai+ck \\ 0 & ej & ek \\ 0 & fj & fk \end{pmatrix}, \begin{pmatrix} ag & ah & ai+cl \\ 0 & 0 & el \\ 0 & 0 & fl \end{pmatrix}, \begin{pmatrix} 0 & aj & ak+cl \\ 0 & 0 & el \\ 0 & 0 & fl \end{pmatrix},$$
$$\begin{pmatrix} cg & ch+bj & ci+bk \\ eg & eh+dj & ei+dk \\ fg & fh & fi \end{pmatrix}, \begin{pmatrix} cg & ch & ci+bl \\ eg & eh & ei+dl \\ fg & fh & fi \end{pmatrix}, \begin{pmatrix} 0 & cj & ck+bl \\ 0 & ej & ek+dl \\ 0 & fj & fk \end{pmatrix}, \begin{pmatrix} bg & bh+cj & bi+ck \\ dg & dh+ej & di+ek \\ 0 & fj & fk \end{pmatrix},$$
$$\begin{pmatrix} bg & bh & bi+cl \\ dg & dh & di+el \\ 0 & 0 & fl \end{pmatrix}, \begin{pmatrix} 0 & bj & bk+cl \\ 0 & dj & dk+el \\ 0 & 0 & fl \end{pmatrix} \mid a, d, f, g, j, l \in K^*, b, c, e, h, i, k \in K\}$$

We know that $\mathscr{J}$-classes generalise the notion of rank from $n \times n$ matrices, so another way we could write BJ_rB is $BJ_rB = GrG = \{m \in M_3(K) \mid m \text{ has rank } = 2\}$, since r has rank 2. Sadly, for the remaining fat classes there does not seem to be such a simple way of stating them.

$$BL_rB = \{ \begin{pmatrix} bg & bh+aj & bi+ak \\ dg & dh & di \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} cg & ch+aj & ci+ak \\ eg & eh & ei \\ fg & fh & fi \end{pmatrix}, \begin{pmatrix} ag & ah+bj & ai+bk \\ 0 & dj & dk \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} ag & ah+cj & ai+ck \\ 0 & ej & ek \\ 0 & fj & fk \end{pmatrix},$$
$$\begin{pmatrix} cg & ch+bj & ci+bk \\ eg & eh+dj & ei+dk \\ fg & fh & fi \end{pmatrix}, \begin{pmatrix} bg & bh+cj & bi+ck \\ dg & dh+ej & di+ek \\ 0 & fj & fk \end{pmatrix} \mid a,d,f,g,j,l \in K^*, b,c,e,h,i,k \in K\}$$

$$BR_rB = \{ \begin{pmatrix} cg & ch+aj & ci+ak \\ eg & eh & ei \\ fg & fh & fi \end{pmatrix}, \begin{pmatrix} cg & ch & ci+al \\ eg & eh & ei \\ fg & fh & fi \end{pmatrix}, \begin{pmatrix} 0 & cj & ck+al \\ 0 & ej & ek \\ 0 & fj & fk \end{pmatrix}, \begin{pmatrix} ag & ah+cj & ai+ck \\ 0 & ej & ek \\ 0 & fj & fk \end{pmatrix},$$
$$\begin{pmatrix} ag & ah & ai+cl \\ 0 & 0 & el \\ 0 & 0 & fl \end{pmatrix}, \begin{pmatrix} 0 & aj & ak+cl \\ 0 & 0 & el \\ 0 & 0 & fl \end{pmatrix} \mid a,d,f,g,j,l \in K^*, b,c,e,h,i,k \in K\}$$

$$BH_rB = \{ \begin{pmatrix} cg & ch+aj & ci+ak \\ eg & eh & ei \\ fg & fh & fi \end{pmatrix}, \begin{pmatrix} ag & ah+cj & ai+ck \\ 0 & ej & ek \\ 0 & fj & fk \end{pmatrix} \mid a,d,f,g,j,l \in K^*, b,c,e,h,i,k \in K\}$$

Suppose we wanted to compare BR_rB and BR_sB in terms of the Bruhat order, where $s = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. In the Bruhat order, r and s are incomparable, so we have two options. Either we compute R_s and try to find some $x \in R_r$ and $y \in R_s$ so that we can compare them (if no such x and y then BR_rB and BR_sB are incomparable), or we use Theorem 4.17.

Using Section 3 we can see that $r_- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $s_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. As elements of $\mathcal{JG}$, these are much easier to compare and we see that $r_- < s_-$. Thus $BR_rB \subseteq \overline{BR_sB}$.

5 Vanilla Form

In [17], Pennell, Putcha and Renner introduced standard form for an element of $\mathcal{R}$, which is $r = xey^{-1}$, where $e \in \Lambda, x \in D_*(e), y \in D(e)$ (notation to be reviewed in a bit). This form allows one to describe the Adherence order in terms of the Bruhat order on W and the order on the cross sectional lattice. While the advantage of this unique expression is clear, it has a little to be desired when one wants to talk about Green's relations.

From standard form, we can easily determine if two elements are in the same $\mathscr{J}$-class, or even the same $\mathscr{L}$-class, but that is about it. If we wish to talk about $\mathscr{R}$-classes we are at a loss, until we introduce the "opposite standard form", which easily follows from the same work in [17]. However, neither form is very good at describing the $\mathscr{H}$-class of the element, r. To this end, in this section we will introduce a new unique expression for the elements of $\mathcal{R}$, and investigate how it may also be used to describe the Adherence order.

5.1 Coset Posets

Recall that when we examine a Bruhat decomposition, we first must fix several subgroups, in particular, our Borel subgroup, B and maximal torus, T. From T, we derive the Weyl group, $W = N_G(T)/T$, and ultimately the Renner monoid, $\mathcal{R} = \overline{N_G(T)}/T$. The Borel group allows us to identify a set of "simple reflections" within W, by (among other methods) defining the length function, $\ell : W \to \mathbb{N}$, by $\ell(w) = dim(BwB) - dim(B)$. The simple reflections are exactly those elements with $\ell(w) = 1$.

Throughout this section, let S denote the set of simple reflections for our Coxeter group, W, based on our already fixed B and T.

Definition 5.1. *A subgroup $X \subseteq W$ is called a* **standard parabolic subgroup** *if $X = \langle I \rangle$ for some $I \subseteq S$. We will often denote the subgroup associated to $I \subseteq S$ by W_I.*

As the name would suggest, these standard parabolic subgroups are exactly the subgroups of W corresponding to the parabolic subgroups of G which contain our given B.

Proposition 5.2. *Recall that w_0 is the unique maximum element in W. For any element $w \in W$, $\ell(w) = \ell(w_0ww_0)$.*

Proof. This is from Corollary 2.3.3 in [2]. □

Each of our standard parabolic subgroups is a finite Coxeter group. So each has a longest element, prompting the following notation.

Definition 5.3. *Let $I \subseteq S$. We denote the* **longest element** *of the Coxeter group, W_I by $w_0(I)$. So, $w_0 = w_0(S)$.*

Proposition 5.4. *For a given, $I \subseteq S$, and any two $u, v \in W_I$, the following are equivalent.*

(1) $u \leq v$

(2) $w_0(I)u \leq w_0(I)v$

(3) $uw_0(I) \leq vw_0(I)$

(4) $w_0(I)uw_0(I) \leq w_0(I)vw_0(I)$

Proof. Proposition 2.3.4 from [2]. □

The following collection of results comes largely from [2] and [4]. For brevity, when possible we will be writing our results in terms of double cosets W_IwW_J. Note that our results will also apply for left and right cosets, by taking $I = \emptyset$ or $J = \emptyset$. Our first result comes to us from [4] and [14].

Proposition 5.5. *For all $w \in W$ and any $I, J \subseteq S$, the double coset W_IwW_J has a unique minimal element with respect to the Bruhat order (and hence has minimal length), and W_IwW_J has a unique maximal element with respect to the Bruhat order (and hence has maximal length).*

Proof. A proof is given in [14], as Proposition 23. □

As they will be of great use to us later, we take the time now to distinguish these sets of minimal elements.

Definition 5.6. *For $w \in W$ and $I, J \subseteq S$ we denote the minimal element of W_IwW_J by ${}^Iw^J$. Further, we will denote this collection of minimal length elements by ${}^IW^J$.*

We may often denote ${}^IW^\emptyset$ by ${}^IW$ and likewise denote ${}^\emptyset W^J$ by W^J. This is just notational convenience.

Lemma 5.7. *$w \in {}^IW^J$ and $x \in W_IwW_J$, then there is a decomposition $x = uwv$ where $u \in W_I$, $v \in W_J$ and $\ell(x) = \ell(u) + \ell(w) + \ell(v)$. Furthermore, if $I = \emptyset$ or $J = \emptyset$ then this decomposition is unique.*

Proof. This is noted as Proposition 1.3 in [10] and Proposition 2.4.4 of [2] □

Lemma 5.8. *Let $w \in W$ and $s \in S$. Then $sw < w$ if and only if some reduced word for w starts with s. Likewise, $ws < w$ if and only if some reduced word for w ends with s.*

Proof. This is a restatement of Corollary 1.4.6 from [2]. □

Proposition 5.9. *Suppose that $u, v \in W$ with $u \leq v$ and $s \in S$.*

(1) If $u < su$ then $u \leq sv$.

(2) If $u < us$ then $u \leq vs$.

Proof. This comes from Proposition 11 in [14], which draws upon Proposition 2.2.7 in [2]. □

Much of the proof of this next result is inspired by [14].

Proposition 5.10. *For $I, J \subseteq S$, $w \in {}^IW^J$ if and only if no reduced word for w starts with an element of I or ends in with an element of J.*

Proof. Suppose that $w \in {}^IW^J$. Then for all $s \in I$ and $t \in J$ we see that $w < sw$ and $w < wt$ (since w is minimal in W_IwW_J). Thus no reduced word for w starts with an element of I or ends in with an element of J, by Lemma 5.8.

Now, suppose that $w \in W$ is such that no reduced word for w starts with an element of I or ends in with an element of J. Then by Lemma 5.8 for all $s \in I$ and $t \in J$ we see that $w < sw$ and $w < wt$. We claim that w is the minimal element in W_IwW_J. We know that there exists such an element, call it x. By our previous work, we also know that $s \in I$ and $t \in J$ we see that $x < sx$ and $x < xt$.

We know that we can write $x = s_1s_2 \cdots s_kwt_1t_2 \cdots t_\ell$, where each $s_i \in I$ and each $t_j \in J$, since $x \in W_IwW_J$. But by applying Proposition 5.9 exactly $(k + \ell)$ times, we can then see that $w \leq s_1s_2 \cdots s_kwt_1t_2 \cdots t_\ell = x$. Since x is minimum, it follows that $w = x$, and so $w \in {}^IW^J$. □

Proposition 5.11. *For $I, J \subseteq S$, ${}^IW^J = {}^IW^\emptyset \cap {}^\emptyset W^J$.*

Proof. By using the preceding proposition we see that $w \in {}^I W^\emptyset$ if and only if no reduced word for w starts with an element of I. Likewise, $w \in {}^\emptyset W^J$ if and only if no reduced word for w ends with an element of J. So we conclude that $w \in {}^I W^\emptyset \cap {}^\emptyset W^J$ if and only no reduced word for w starts with an element of I or ends in with an element of J. But by Proposition 5.10 this is equivalent to $w \in {}^I W^J$. □

Corollary 5.12. *$w \in {}^I W^J$ if and only if $w = {}^I u$ for some $u \in W^J$ if and only if $w = v^J$ for some $v \in {}^I W$.*

Proof. The forward implication is clear by taking $u = v = w$. Now suppose $w = {}^I u$ for some $u \in W^J$. By the preceding proposition it suffices to show that $w \in W^J$. Suppose not, then we can find $s \in J$ and a reduced word expression for w that ends in s. But since $w = {}^I u$ we can see from Lemma 5.7 that there is $a \in W_I$ such that $aw = u$ and $\ell(u) = \ell(a) + \ell(w)$. This tells us that any concatenation of reduced words for a and w forms a reduced word for u. Thus we have found a reduced word for u ending in $s \in J$, a contradiction.

The situation with v is shown similarly. □

Proposition 5.13. *For $I, J \subseteq S$ and any $u, v \in W$, if $u \leq v$ then ${}^I u^J \leq {}^I v^J$.*

The proof we give here is basically a copy of Proposition 2.5.1 in [2].

Proof. We will prove the result by induction. First observe ${}^I u^J \leq u \leq v$. If $v = {}^I v^J$ then we are done. If not, by Proposition 5.10 and Lemma 5.8 we can find a $s \in I$ so that $sv < v$ or $t \in J$ so that $vt < v$. Either way, applying Proposition 5.9, we see ${}^I u^J \leq sv < v$ or ${}^I u^J \leq vt < v$ respectively. By induction, ${}^I u^J \leq {}^I (sv)^J = {}^I v^J$ or ${}^I u^J \leq {}^I (vt)^J = {}^I v^J$ respectively. □

Corollary 5.14. *Recall $w_0 \in W$, the unique element of maximal length. Then for any $I, J \subseteq S$, ${}^I w_0{}^J$ is the unique maximal element in ${}^I W^J$ and $1 = {}^I 1^J$ is the unique minimal element in ${}^I W^J$.*

Proof. Since 1 is the minimum element of W it is clear that $1 = {}^I 1^J$ is the unique minimum element of ${}^I W^J$. Additionally, w_0 is the unique maximum element of W. So by Proposition 5.13 we see that for any $w \in W$, $w \leq w_0$ implies ${}^I w^J \leq {}^I w_0{}^J$. Thus, ${}^I w_0{}^J$ is the unique maximal element in ${}^I W^J$. □

We can actually get a stronger result, in that ${}^Iu^J$ is less than or equal to the whole double coset W_IvW_J.

Proposition 5.15. *Take $I, J \subseteq S$, and let $u, v \in W$. If $u = {}^Iu^J$, then $u \leq v$ if and only if $u \leq w$ for all $w \in W_IvW_J$*

Proof. Since $v \in W_IvW_J$ we can see that the "if" direction is clear. Suppose that $u \leq v$. Then $u = {}^Iu^J \leq {}^Iv^J$. It follows that for any $w \in W_IvW_J$, by definition ${}^Iv^J \leq w$, and we conclude that $u \leq w$. □

The last result we will showcase before moving on has connections with Putcha's work in [18]. As we will see later on, this result tells us that $D_*(e) = D(e)W^*(e)$ for $e \in \Lambda$.

Proposition 5.16. *For $K \subseteq I \subseteq S$, suppose that $W_I = W_K \times W_{I\backslash K} = W_{I\backslash K} \times W_K$. Then,*

(1) ${}^KW = W_{I\backslash K}{}^IW$

(2) $W^K = W^IW_{I\backslash K}$

Proof. Both arguments are similar, so we will just prove (1). Suppose that $w \in {}^KW$. Choose simple reflection, $s_1 \in I\backslash K$ so that $sw < w$, if such an s_1 exists. Continue choosing $s_i \in I\backslash K$ so that $s_is_{i-1}\cdots s_1w < s_{i-1}\cdots s_1w$, as long as such s_i exist. This process will terminate after at most $\ell(w)$ steps. If it ends after k steps, let $v = s_k\cdots s_1w$. Then we see that for all $s \in I\backslash K$, $sv > v$. So it follows that $v \in {}^IW$. Thus $w = (s_1\cdots s_k)v \in W_{I\backslash K}{}^IW$. So then ${}^KW \subseteq W_{I\backslash K}{}^IW$

Conversely, consider the sizes of these sets (recall that W is finite). We know $|{}^KW| = \frac{|W|}{|W_K|}$ and $|{}^IW| = \frac{|W|}{|W_I|}$. But since $W_I = W_K \times W_{I\backslash K}$, we see $|W_I| = |W_K||W_{I\backslash K}|$. From there it follows that $|{}^KW| = \frac{|W|}{|W_K|} = \frac{|W||W_{I\backslash K}|}{|W_I|} = |{}^IW||W_{I\backslash K}|$. Now, $|W_{I\backslash K}\,{}^IW| \leq |{}^IW||W_{I\backslash K}|$ and thus our containment must be an equality. ${}^KW = W_{I\backslash K}{}^IW$. □

We have now gathered enough results from Coxeter groups in order to say something meaningful about our Renner monoid and its associated Adherence order.

5.2 Standard Forms

In order to make use of the preceding results, we need to have some standard parabolic subgroups. To that end, we define the following sets of simple reflections and the parabolic subgroups associated to them. Notice that these sets are all defined by properties relating to a given idempotent. This is how we will use Coxeter group theory to understand our monoid.

Definition 5.17. *For an idempotent $e \in \Lambda \cup \Lambda^-$, we define the following sets of simple reflections:*

$$\lambda(e) := \{s \in S \big| se = es\} \quad \lambda_*(e) := \{s \in S \big| se = es = e\} \quad \lambda^*(e) := \{s \in S \big| se = es \neq e\}$$

Using these sets we can also define standard parabolic subgroups associated to our idempotent, and the corresponding collections of minimal elements of the Weyl group, W, with respect to these subgroups, both on the left and the right:

$$W(e) := W_{\lambda(e)} \qquad D(e) := W^{\lambda(e)} \qquad V(e) := {}^{\lambda(e)}W$$
$$W_*(e) := W_{\lambda_*(e)} \quad D_*(e) := W^{\lambda_*(e)} \quad V_*(e) := {}^{\lambda_*(e)}W$$
$$W^*(e) := W_{\lambda^*(e)} \quad D^*(e) := W^{\lambda^*(e)} \quad V^*(e) := {}^{\lambda^*(e)}W$$

Our first result allows us to relate the simple reflections associated to an element of Λ^- to the simple reflections of its counterpart in Λ. The reason we explore this relationship is that the elements of Λ and the simple reflections that interact with them have been studied by Putcha in his book, but there is no corresponding treatment for Λ^-.

Lemma 5.18. *Let $e \in \Lambda$. Define the idempotent $f := w_0 e w_0 \in \Lambda^-$. Then $\lambda(f) = w_0\lambda(e)w_0$, $\lambda_*(f) = w_0\lambda_*(e)w_0$ and $\lambda^*(f) = w_0\lambda^*(e)w_0$.*

Proof. Suppose that $s \in \lambda(e)$. Then $se = es$. Now consider w_0sw_0. One can observe the calculation, $w_0sw_0f = w_0sw_0w_0ew_0 = w_0sew_0 = w_0esw_0 = w_0ew_0w_0sw_0 = fw_0sw_0$. Observe that $\ell(w_0sw_0) = \ell(s) = 1$ by Proposition 5.2, so w_0sw_0 is a simple reflection, and hence $w_0sw_0 \in \lambda(f)$. Thus, $w_0\lambda(e)w_0 \subseteq \lambda(f)$.

For $s \in \lambda(f)$, $w_0sw_0e = w_0sw_0w_0fw_0 = w_0sfw_0 = w_0fsw_0 = w_0fw_0w_0sw_0 = ew_0sw_0$. Thus, $w_0sw_0 \in \lambda(e)$, and $s = w_0(w_0sw_0)w_0$. So, $\lambda(f) \subseteq w_0\lambda(e)w_0$. We conclude from here that $\lambda(f) = w_0\lambda(e)w_0$.

Similar proofs can be given for $\lambda_*(f) = w_0\lambda_*(e)w_0$ and $\lambda^*(f) = w_0\lambda^*(e)w_0$. □

Corollary 5.19. *Let $e \in \Lambda$ and $f = w_0ew_0$. Then $w_0(\lambda(f)) = w_0w_0(\lambda(e))w_0$*

Proof. Take any $v \in W_{\lambda(f)}$. Then we can find $u \in W(e)$ so that $v = w_0uw_0$. Since the longest element of a Coxeter group is also maximal in the Bruhat order, $u \leq w_0(\lambda(e))$. But then by Proposition 5.4, $v = w_0uw_0 \leq w_0w_0(\lambda(e))w_0$. Since v was arbitrary our result is concluded. □

Proposition 5.20. *If $e \in \Lambda \cup \Lambda^-$ then $W_*(e) = \bigcap_{f \leq e} W(f)$ and $W^*(e) = \bigcap_{e \leq f} W(f)$, where these f come from the same cross sectional lattice as e.*

Proof. First suppose that $e \in \Lambda$. For the statement, readers are directed to Lemma 7.15 in [30]. The core of the result can be found (with great notation change) in Chapter 10 of [20], specifically Proposition 10.9.

Now, consider the case when we take idempotent, $e \in \Lambda^-$. One can check that for any idempotent, $g \in \Lambda^-$, $W_*(g) = W_{\lambda_*(g)} = w_0 W_{w_0 \lambda_*(g) w_0} w_0 = w_0 W_*(w_0 g w_0) w_0$. Similarly, we can see that $W^*(g) = w_0 W^*(w_0 g w_0) w_0$ and $W(g) = w_0 W(w_0 g w_0) w_0$. It follows from here that we have $W_*(e) = w_0(\bigcap_{f \leq w_0 e w_0} W(f)) w_0 = \bigcap_{f \leq w_0 e w_0} W(w_0 f w_0) = \bigcap_{f \leq e} W(f)$. Likewise, $W^*(e) = \bigcap_{e \leq f} W(f)$. □

Proposition 5.21. *Suppose that e, f are idempotents such that either $e, f \in \Lambda$ or $e, f \in \Lambda^-$. Then $e \leq f$ implies $W_*(f) \subseteq W_*(e)$ and $W^*(e) \subseteq W^*(f)$.*

Proof. These results are a quick application of the preceding proposition. For example, if $e, f \in \Lambda^-$, then $W_*(f) = \bigcap_{g \leq f} W(g) \subseteq \bigcap_{g \leq e} W(g) = W_*(e)$. □

Proposition 5.22. *For idempotents, $e \in \Lambda \cup \Lambda^-$, we have the following equivalent expressions for our standard parabolic subgroups,*

$$W(e) = \{w \in W \mid we = ew\} \qquad\qquad W_*(e) = \{w \in W \mid we = e = ew\}$$

Proof. If $e \in \Lambda$, applying Lemma 10.15 in [20] we show $W(e) = \{w \in W \mid we = ew\}$, and Lemma 10.16 in the same book, we get $W_*(e) = \{w \in W \mid we = e = ew\}$.

On the flip side, if $e \in \Lambda^-$ observe that $f := w_0 e w_0 \in \Lambda$. So then,

$$\begin{aligned} W(e) = w_0 W(f) w_0 = w_0\{w \in W \mid wf = fw\} w_0 &= \{w \in W \mid w_0 w w_0 f = f w_0 w w_0\} \\ &= \{w \in W \mid w_0 w w_0 w_0 e w_0 = w_0 e w_0 w_0 w w_0\} = \{w \in W \mid w_0 w e w_0 = w_0 e w w_0\} \\ &= \{w \in W \mid we = ew\} \end{aligned}$$

A similar proof holds for $W_*(e) = \{w \in W \mid we = e = ew\}$. □

Proposition 5.23. *Let $e \in \Lambda \cup \Lambda^-$. Then, $W(e) = W_*(e) \times W^*(e) = W^*(e) \times W_*(e)$*

Proof. As with prior results, the case for $e \in \Lambda$ is given in Putcha's book. Specifically this result is part of Proposition 10.9 in [20]. For the second, $e \in \Lambda^-$, situation, let us notice that $W(e) = w_0W(w_0ew_0)w_0 = w_0W_*(w_0ew_0)w_0 \times w_0W^*(w_0ew_0)w_0$, as $w_0ew_0 \in \Lambda$. Thus, $W(e) = W_*(e) \times W^*(e)$, and likewise $W(e) = W^*(e) \times W_*(e)$. □

As a useful side note, these new sets allow us to revisit Proposition 3.1 and talk about the class, H_{v^*} from Theorem 3.21.

Proposition 5.24. *For $v \in \mathcal{N}$, we know that $v^*\mathscr{L}e$, $v^*\mathscr{R}f$ for some $e \in \Lambda$ and $f \in \Lambda^-$. We can write, $H_{v^*} = \{r \in \mathcal{R} \mid r = f\sigma = \sigma e, \sigma \in W(f)w_0W(e)\}$.*

Proof. $r \in H_{v^*}$ if and only if $r = f\sigma = \sigma e$ for some $\sigma \in W$. To prove this result, it suffices to show $f\sigma = \sigma e$ if and only if $\sigma \in W(f)w_0W(e)$. If $\sigma \in W(f)w_0W(e)$ then $\sigma = aw_0b$ with $a \in W(f)$ and $b \in W(e)$. So, $f\sigma = faw_0b = afw_0b = aw_0eb = aw_0be = \sigma e$.

For the "only if" direction, notice that $fw_0 = w_0e$. Thus $w_0fw_0 = e = \sigma^{-1}f\sigma$, and then $\sigma w_0f = f\sigma w_0$, so $\sigma w_0 \in W(f)$. So we can find $b \in W(f)$ so $\sigma = bw_0 \in W(f)w_0W(e)$. □

One major result with these new sets is to describe some familiar sets from Section 3.

Lemma 5.25. *Let $I \subseteq S$, and let $L_I = P_I \cap P_I^-$ be the associated Levi factor of $P_I = BW_IB$. Take $w \in W$*

(1) $w \in W^I$ if and only if $w^{-1}(L_I \cap B)w \subseteq B$

(2) $w \in {}^IW$ if and only if $w(L_I \cap B)w^{-1} \subseteq B$

Proof. (1) is stated in [17], and the reference given there is Proposition 2.3.3 in [9] by Carter.

(2) quickly follows from (1) when we realize that $x \in W^I$ if and only if $x^{-1} \in {}^IW$. □

The Levi factor mentioned in the above lemma might appear to be cause for alert, as this is the first mention in this paper. However, any fear is easily assuage when we consider the specific parabolic subgroups of G that we are considering. For $e \in \Lambda \cap \Lambda^-$, the Levi factor of $P_{\lambda(e)}$ is just $C_G(e)$.

Proposition 5.26. *For $r \in \mathcal{R}$,*

(1) $r \in \mathcal{GJ}$ if and only if $r = ey^{-1}$ for some $e \in \Lambda$ and $y \in D(e)$.

(2) $r \in \mathcal{JG}$ if and only if $r = b^{-1}f$ for some $f \in \Lambda^-$ and $b \in V(f)$.

One can note that Putcha has already demonstrated (1) in his Parabolic Monoids paper ([23]), but as it has such importance to the remainder of the paper, we offer a written proof.

Proof. (1) First, we show $y \in D(e) \Rightarrow ey^{-1} \in \mathcal{GJ}$. $e \in \Lambda$ means $Be = C_B(e) = C_G(e) \cap B$. Thus, $Bey^{-1} = eBey^{-1} = ey^{-1}yBey^{-1} = ey^{-1}y(L_{\lambda(e)} \cap B)y^{-1} \subseteq ey^{-1}B$, by Lemma 5.25. Thus, $ey^{-1} \in \mathcal{GJ}$.

Now, suppose that $r \in \mathcal{GJ}$. By Proposition 3.18, if we write $r = e\sigma$ for $e \in E(\mathcal{R})$ and $\sigma \in W$, then we know that $e \in \Lambda$. Since $\sigma \in W$, we can find $y \in D(e)$ and $v \in W(e)$ so that $\sigma v y^{-1}$. Then $r = evy^{-1} = vey^{-1}$. So $r\mathscr{L}ey^{-1} \in \mathcal{GJ}$ by our previous work. But then, by Proposition 3.9, $r = ey^{-1}$, as desired.

(2) is done similarly. □

Lemma 5.27. *Suppose that $m \in \mathcal{JG}$ and $p \in \mathcal{GJ}$. Then for $r, s \in \mathcal{R}$, $r \leq s$ implies $mrp \leq msp$.*

Proof. Note that, $Bm = BmB$ and $pB = BpB$ since $m \in \mathcal{JG}$ and $p \in \mathcal{GJ}$.

Then $BmrpB = BmBrBpB \subseteq Bm\overline{BsB}pB \subseteq \overline{BmBsBpB} = \overline{BmspB}$. □

Lemma 5.28.

(1) If $e \in \Lambda$, $y \in D(e)$ and $x \in W$ with $x \leq y$, then $xey^{-1} \in \overline{B}$

(2) If $e \in \Lambda^-$, $y \in V(e)$ and $x \in W$ with $x \leq y$, then $y^{-1}ex \in \overline{B}$

Proof. It suffices to prove (1), as (2) is similar by symmetry. Since $y \in D(e)$ the last proposition tells us that $p := ey^{-1} \in \mathcal{GJ}$. Then we see, by the preceding lemma, that $x \leq y$ implies $xp \leq yp$ (just take $m = 1$). But then $xey^{-1} = xp \in \overline{BypB} = \overline{Byey^{-1}B}$. Now, yey^{-1} is an idempotent and thus, $yey^{-1} \in \overline{T} \subseteq \overline{B}$, and so we may conclude that $xey^{-1} \in \overline{B\overline{B}B} = \overline{B}$. □

This decomposition of elements in $\mathcal{GJ}$ and $\mathcal{JG}$ segues nicely into the phenomenal decomposition result presented in [17], called the standard form. This form was instrumental in getting the first real handle at the Adherence order for $\mathcal{R}$.

Definition 5.29. *Let $\sigma \in \mathcal{R}$. We say that $\sigma = xey^{-1}$ is in* **standard form** *if $e \in \Lambda$, $x \in D_*(e)$ and $y \in D(e)$. We say that $\sigma = b^{-1}fa$ is in* **opposite standard form** *if $f \in \Lambda^-$, $a \in V_*(f)$ and $b \in V(f)$.*

An immediate consequence to these definitions from Proposition 5.26 is that we can tell the $\mathscr{L}$-class of $r = xey^{-1}$ by realising that $ey^{-1} \in \mathcal{GJ}$ and $r\mathscr{L}ey^{-1}$. Likewise, opposite standard form readily gives us the $\mathscr{R}$-class of r.

Example 5.30. *For a given element, $r \in \mathcal{R}$, there is a simple procedure to find the standard form and opposite standard form. For example, we will show this procedure for opposite standard form. First, determine the unique element $f \in \Lambda^-$ so that $r \in WfW$.*

Then we can find $u, v \in W$ so that $r = ufv$. Let $b = {}^{\lambda(f)}(u^{-1})$. So then $b \in V(f)$, and we can find $w \in W(f)$ so that $b = wu^{-1}$, and thus $r = (w^{-1}b)^{-1}fv = b^{-1}wfv$. Since $w \in W(f)$ we can rewrite as $r = b^{-1}fwv$. Consider the element wv. Let $a = {}^{\lambda_*(f)}(wv)$. Then we can find $x \in W_*(f)$ so that $xa = wv$, and then $r = b^{-1}fwv = b^{-1}fxa = b^{-1}fa$, which is in opposite standard form.

Similar arguments allow us to see easily that the standard and opposite standard forms are unique for a given element. A similar proof for our new form will be given explicitly, from which more details can be gleaned.

The following result comes from [17] and allow us to showcase the importance of our work with the ${}^IW^J$, as we can now describe the Adherence order. This work by Pennell, Putcha and Renner was the first complete description of the Adherence order in the setting of a general reductive monoid.

Theorem 5.31. *Let $\sigma = xey^{-1}$ and $\tau = sft^{-1}$ with $x, s \in W$, $y \in D(e)$ and $t \in D(e)$. Then the following are equivalent,*

(1) $\sigma \leq \tau$

(2) $ef = e$, and there exists $w \in W(f)W_*(e)$ and $z \in W_*(e)$ such that $x \leq swz$ and $tw \leq y$

Proof. This result comes to us from [17], as Theorem 1.4. □

The following corollary is similar to Corollary 1.5 in [17], but with a minor change to a condition.

Corollary 5.32. *Let $\sigma = xey^{-1}$ and $\tau = sft^{-1}$ be in standard form. Then the following are equivalent,*

(1) $\sigma \leq \tau$

(2) $ef = e$, and there exists $w \in W^*(f)W_*(e)$ such that $x \leq sw$ and $tw \leq y$

Proof. By Theorem 5.31 (1) is equivalent to $ef = e$, and the existence of $w \in W(f)W_*(e)$ and $z \in W_*(e)$ such that $x \leq swz$ and $tw \leq y$. Now, $z \in W_*(e)$ implies by Proposition 5.15 that

$x = x^{\lambda_*(e)} \leq swzz^{-1}$, since xey^{-1} is in standard form. Thus, (1) is equivalent to $ef = e$, and there exists $w \in W(f)W_*(e)$ such that $x \leq sw$ and $tw \leq y$. But, by Proposition 5.23 we can rewrite this with $w \in W^*(f)W_*(f)W_*(e)$, and by Proposition 5.21, this is the same as $w \in W^*(f)W_*(e)$, as desired. □

In the same vein, we can use our standard parabolic groups associated to the opposite cross sectional lattice, Λ^-, and perform similar work to get a characterisation of the Adherence order in terms of elements written in opposite standard form.

Theorem 5.33. *Let $\sigma = b^{-1}fa$ and $\tau = k^{-1}gj$ with $a, j \in W$, $b \in V(f)$ and $k \in V(g)$. Then the following are equivalent,*

(1) $\sigma \leq \tau$

(2) $fg = f$, and there exists $w \in W_*(f)W(g)$ and $z \in W_*(f)$ such that $a \leq zwj$ and $wk \leq b$

Proof. The work behind this result is just a reflection of the work presented by Pennell, Putcha and Renner in [17] for the two results above. Though it is distinct, there is nothing to gain by explicitly writing it here, so it is in the Appendix as Theorem A.9. □

Corollary 5.34. *Let $\sigma = b^{-1}fa$ and $\tau = k^{-1}gj$ be in opposite standard form. Then the following are equivalent,*

(1) $\sigma \leq \tau$

(2) $fg = f$, and there exists $w \in W_*(f)W^*(g)$ such that $a \leq wj$ and $wk \leq b$

Proof. By Theorem 5.33 (1) is equivalent to $fg = f$, and the existence of $w \in W_*(f)W(g)$ and $z \in W_*(f)$ such that $a \leq zwj$ and $wk \leq b$. Now, $z \in W_*(f)$ implies by Proposition 5.15 that $x = {}^{\lambda_*(f)}a \leq z^{-1}zwj$, since $b^{-1}fa$ is in opposite standard form. Thus, (1) is equivalent to $fg = f$, and there exists $w \in W_*(f)W(g)$ such that $a \leq zwj$ and $wk \leq b$. But, by Proposition 5.23 we can rewrite this with $w \in W_*(f)W_*(g)W^*(g)$, and by Proposition 5.21, this is the same as $w \in W_*(f)W^*(g)$, as desired. □

Proposition 5.35. *If $r = xey^{-1}$ is in standard form and $r = b^{-1}fa$ is in opposite standard form, then $\ell(r) = \ell(x) + \ell(e) - \ell(y) = -\ell(b) + \ell(f) + \ell(a)$*

Proof. For standard form, this comes from Section 4 of [17]. Comparing the proofs of Theorem 1.4 in [17] and our work to prove Theorem A.9 in the appendix, one can convince themselves that the same sort of (mindless) symmetry work will show the same length results for opposite standard form. □

Our next work deals with creating a similar decomposition that allows us to tackle things from an $\mathscr{H}$-class perspective.

5.3 Vanilla Form

Having covered the standard form of Pennell, Putcha, and Renner as well as considered its "dual" or "opposite" form, we now combine these two forms to create a new unique decomposition for elements of $\mathcal{R}$. Due to the author's opinion that the terms 'standard', 'normal' and 'canonical' are over used in mathematics, we will now introduce a decomposition for elements of $\mathcal{R}$ which we shall dub 'vanilla'.

It will turn out that this vanilla form will allow us to determine, at a glance, the Green's relations of the given element and also allows us to compute the Adherence order, just as the other two forms did above.

Definition 5.36. *Let $r \in \mathcal{R}$. We say that $r = \sigma_-^{-1} e_- \sigma_0 e_+ \sigma_+^{-1}$ is in* **vanilla form** *if $e_+ \in \Lambda$, $e_- \in \Lambda^-$, $e_- \mathscr{J} r \mathscr{J} e_+$, $\sigma_- \in V(e_-)$, $\sigma_+ \in D(e_+)$, and $\sigma_0 \in V_*(e_-) \cap W(e_-)w_0W(e_+) \cap D_*(e_+)$.*

Proposition 5.37. *For any $r \in \mathcal{R}$, the vanilla form for r exists and is unique.*

Proof. By Theorem 3.21, we can decompose r uniquely as $r = r_- r_0 r_+$. $r_0 \mathscr{H} v^*$ for $v \in \mathcal{N}$, $v \mathscr{J} r$. Now, by Proposition 5.24, $r_0 = f\sigma = \sigma e$ with $e \in \Lambda$, $f \in \Lambda^-$ and $r_0 = f\sigma = \sigma e$ for some $\sigma \in W(f)w_0W(e)$. We will let $e_+ = e$, $e_- = f$ and $\sigma_0 = {}^{\lambda_*(f)}\sigma^{\lambda_*(e)}$.

It is clear, $e_- \mathscr{J} r \mathscr{J} e_+$. Notice that $\sigma_0 \in W(e_-)w_0W(e_+)$, as ${}^{\lambda_*(f)}\sigma^{\lambda_*(e)} \in W_*(f)\sigma W_*(e)$, and $W_*(f) \subseteq W(f)$, $W_*(e) \subseteq W(e)$. So we can find element, $a \in W_*(f)$ and $c \in W_*(e)$ such that $\sigma_0 = a\sigma c$. Thus, $r_0 = e_-\sigma = \sigma e_+$ tells us, $e_-\sigma_0 = e_- a\sigma c = e_-\sigma c = \sigma e_+ c = \sigma e_+ = r_0$. Not only that, but $\sigma_0 = {}^{\lambda_*(f)}\sigma^{\lambda_*(e)}$ implies that $\sigma_0 \in {}^{\lambda_*(f)}W^{\lambda_*(e)}$. By Proposition 5.11 and Definition 5.17 this means $\sigma_0 \in V_*(e_-) \cap D_*(e_+)$. So, $\sigma_0 \in V_*(e_-) \cap W(e_-)w_0W(e_+) \cap D_*(e_+)$, as desired.

We know from Proposition 5.26 that $r_- \in \mathcal{JG}$ means we can find $f' \in \Lambda^-$ and $b \in V(f')$ so that $r_- = b^{-1}f'$. Likewise we can find $e' \in \Lambda$ and $b \in D(e')$ so that $r_+ = e'y^{-1}$. Since $r_- \mathscr{J} r \mathscr{J} r_+$ and $r = r_-r_0r_+$, Proposition A.1 tells us, $f' = f$ and $e' = e$. And so if we let, $\sigma_- = b$, $\sigma_+ = y$, then $\sigma_- \in V(e_-)$ and $\sigma_+ \in D(e_+)$ and $r = r_-r_0r_+ = (\sigma_-^{-1}e_-)(e_-\sigma_0)(e_+\sigma_+^{-1}) = \sigma_-^{-1}e_-\sigma_0e_+\sigma_+^{-1}$.

This shows that we can decompose an element into vanilla form. It remains to show this decomposition is unique. Suppose that $r = \tau_-^{-1}f_-\tau_0f_+\tau_+^{-1}$ is another vanilla decomposition to the one we just determined. By definition, $e_- \mathscr{J} r \mathscr{J} f_-$ and $e_-, f_- \in \Lambda^-$, so we can conclude that $e_- = f_-$ and likewise $e_+ \mathscr{J} r \mathscr{J} f_+$, so $e_+ = f_+$. So, $r = \tau_-^{-1}e_-\tau_0e_+\tau_+^{-1} = (\tau_-^{-1}e_-)(e_-\tau_0)(e_+\tau_+^{-1})$.

Since $\tau_0 \in W(e_-)w_0W(e_+)$ it can be shown that $e_-\tau_0 = \tau_0e_+$, and hence $e_-\tau_0\mathscr{H}v^*$. Observe that, $r = \tau_-^{-1}e_-\tau_0e_+\tau_+^{-1} = \tau_-^{-1}e_-e_-\tau_0\tau_+^{-1} = \tau_-^{-1}e_-\tau_0\tau_+^{-1}$ and $r = \tau_-^{-1}\tau_0e_+\tau_+^{-1}$. This tells us that $r\mathscr{R}(\tau_-^{-1}e_-)$ and $r\mathscr{L}(e_+\tau_+^{-1})$. Finally, by applying Proposition 5.26 we see that $\tau_-^{-1}e_- \in \mathcal{JG}$ and $e_+\tau_+^{-1} \in \mathcal{GJ}$. So we may conclude that $r_- = \tau_-^{-1}e_-$, $r_0 = e_-\tau_0$ and $r_+ = e_+\tau_+^{-1}$ by uniqueness of our trichotomy.

Thus, $e_+\sigma_+^{-1} = r_+ = e_+\tau_+^{-1}$. So it follows that $\sigma_+^{-1}\tau_+ \in W_*(e_+)$, hence $\sigma_+ \in \tau_+W(e_+)$. But since $\sigma_+, \tau_+ \in D(e_+)$ it follows that $\sigma_+ = \tau_+$ as the elements of minimal length in a given coset are unique. Likewise, $\sigma_- = \tau_-$. Similarly, $e_-\sigma_0 = r_0 = e_-\tau_0$, so $\sigma_0 \in W_*(e_-)\tau_0$. But since $\sigma_0, \tau_0 \in V_*(e_-)$ we may conclude that $\sigma_0 = \tau_0$. □

Just as with standard form and opposite standard form, we can compute the length of an element from its vanilla form.

Proposition 5.38. *For $r \in \mathcal{R}$, if $r = \sigma_-^{-1}e_-\sigma_0e_+\sigma_+^{-1}$ is in vanilla form, then we can compute the length, $\ell(r) = -\ell(\sigma_-) + \ell(e_-) + \ell(\sigma_0) - \ell(\sigma_+) = -\ell(\sigma_-) + \ell(\sigma_0) + \ell(e_+) - \ell(\sigma_+)$*

Proof. We will prove the first equality as the second is done similarly. By Proposition 3.24 we know that $\ell(r) = \ell(r_-) + \ell(r_0) + \ell(r_+) - \ell(e_+) - \ell(e_-)$. We know that $r_- = \sigma_-^{-1}e_-$ and $r_0 = e_-\sigma_0$ are in opposite standard form and $r_+ = e_+\sigma_+^{-1}$ is in standard form. So we may use Proposition 5.35 to substitute,

$$\begin{aligned}\ell(r) &= \ell(e_-) - \ell(\sigma_-) + \ell(e_-) + \ell(\sigma_0) + \ell(e_+) - \ell(\sigma_+) - \ell(e_+) - \ell(e_-)\\ &= -\ell(\sigma_-) + \ell(e_-) + \ell(\sigma_0) - \ell(\sigma_+)\end{aligned}$$

as desired. □

Proposition 5.39. *For any, $r \in \mathcal{R}$, if $r = \sigma_-^{-1}e_-\sigma_0 e_+\sigma_+^{-1}$ is in vanilla form, then $\sigma_-^{-1}\sigma_0 \in D_*(e_+)$ and $\sigma_0\sigma_+^{-1} \in V_*(e_-)$*

Proof. Once again, we will just prove the first of the two statements. $r = \sigma_-^{-1}\sigma_0 e_+\sigma_+^{-1}$ and let $r = xe_+\sigma_+^{-1}$ be the standard form for r. Then $x \in D_*(e_+)$ and by using Propositions 5.35 and 5.38 we see that $\ell(x) = \ell(\sigma_0) - \ell(\sigma_-)$.

Now consider $\sigma_- r = \sigma_0 e_+\sigma_+^{-1} = \sigma_- xe_+\sigma_+^{-1}$. We see $\sigma_0 e_+\sigma_+^{-1}$ is in standard form, which tells us that $\sigma_- x \in \sigma_0 W_{e_+}$ and in particular $\ell(\sigma_0) \leq \ell(\sigma_- x)$. By subadditivity of length, $\ell(\sigma_- x) \leq \ell(\sigma_-) + \ell(x) = \ell(\sigma_0)$. Thus $\sigma_0 = \sigma_- x$, or rather, $\sigma_-^{-1}\sigma_0 = x \in D_*(e_+)$. □

The most immediate result from this proposition is that we can easily read off standard form and opposite standard form from our vanilla form.

Corollary 5.40. *For $r \in \mathcal{R}$, if $r = \sigma_-^{-1}e_-\sigma_0 e_+\sigma_+^{-1}$ is in vanilla form, then $r = \sigma_-^{-1}\sigma_0 e_+\sigma_+^{-1}$ is in standard form and $r = \sigma_-^{-1}e_-\sigma_0\sigma_+^{-1}$ is in opposite standard form. As well, $r_- = \sigma_-^{-1}e_-$, $r_0 = e_-\sigma_0 e_+$ and $r_+ = e_+\sigma_+^{-1}$.*

Proof. There is little to prove here, as $e_-\sigma_0 = \sigma_0 e_+$, since $\sigma_0 \in {}^{\lambda_*(e_-)}W(e_-)w_0 W(e_+)^{\lambda_*(e_+)}$. Thus $r = \sigma_-^{-1}(e_-\sigma_0)e_+\sigma_+^{-1} = \sigma_-^{-1}(\sigma_0 e_+)e_+\sigma_+^{-1} = (\sigma_-^{-1}\sigma_0)e_+\sigma_+^{-1}$. By Proposition 5.39 this is in standard form, and by uniqueness of standard form we are done.

A proof for opposite standard form is done similarly. The result for our trichotomy elements comes from the proof of existence and uniqueness of vanilla form (Proposition 5.37), where we derived the vanilla form from our trichotomy decomposition. □

And now we move on to describe the Adherence order in general. One can easily see the influence that standard form and opposite standard form have over our vanilla form.

Theorem 5.41. *For $r, s \in \mathcal{R}$, if $r = \sigma_-^{-1}e_-\sigma_0 e_+\sigma_+^{-1}$ and $s = \tau_-^{-1}f_-\tau_0 f_+\tau_+^{-1}$ are in vanilla form then the following are equivalent,*

(1) $r \leq s$

(2) $e_-, e_+ \leq f_-, f_+$ and $\exists w_- \in W_*(e_-)W^*(f_-)$ and $\exists w_+ \in W^*(f_+)W_*(e_+)$ such that $w_-\tau_- \leq \sigma_-$, $\sigma_0 \leq w_-\tau_0 w_+$ and $\tau_+ w_+ \leq \sigma_+$

Proof. Assume that (2) holds. We see by Lemma 5.28 that $\tau_+ w_+ e_+ \sigma_+^{-1}, \sigma_-^{-1} e_- w_- \tau_- \in \overline{B}$, and hence $r_- w_- s_0 w_+ r_+ = \sigma_-^{-1} e_- w_- \tau_- \tau_-^{-1} f_- \tau_0 f_+ \tau_+^{-1} \tau_+ w_+ e_+ \sigma_+^{-1} \in \overline{B} \tau_-^{-1} f_- \tau_0 f_+ \tau_+^{-1} \overline{B} \subseteq \overline{BsB}$. We may conclude then that $r_- w_- s_0 w_+ r_+ \leq s$.

We know that $w_- = w_-'' w_-'$ and $w_+ = w_+' w_+''$, where $w_-'' \in W_*(e_-)$, $w_-' \in W^*(f_-)$, $w_+' \in W^*(f_+)$ and $w_+'' \in W_*(e_+)$. So then,

$$\begin{aligned} r_- w_- s_0 w_+ r_+ &= \sigma_-^{-1} e_- w_-'' w_-' f_- \tau_0 f_+ w_+' w_+'' e_+ \sigma_+^{-1} = \sigma_-^{-1} e_- w_-' f_- \tau_0 f_+ w_+' e_+ \sigma_+^{-1} \\ &= \sigma_-^{-1} e_- f_- w_-' \tau_0 w_+' f_+ e_+ \sigma_+^{-1} = \sigma_-^{-1} e_- w_-' \tau_0 w_+' e_+ \sigma_+^{-1} = r_- w_- \tau_0 w_+ r_+ \end{aligned}$$

By Lemma 5.27, since $\sigma_0 \leq w_- \tau_0 w_+$, we see that $r_- \sigma_0 r_+ \leq r_- w_- \tau_0 w_+ r_+$. So now we may conclude that, $r \leq r_- w_- \tau_0 w_+ r_+ = r_- w_- s_0 w_+ r_+ \leq s$ as desired.

Now, for the reverse direction, assume that (1) holds. It is clear that $e_+ \leq f_+$ and $e_- \leq f_-$. Rewrite, $r = (\sigma_-^{-1} \sigma_0) e_+ \sigma_+^{-1}$ and $s = (\tau_-^{-1} \tau_0) f_+ \tau_+^{-1}$. By Corollary 5.40 r and s are in standard form. Applying Corollary 5.32 to $r \leq s$ tells us there exists $w_+ \in W^*(f_+) W_*(e_+)$ so that $\tau_+ w_+ \leq \sigma_+$ and $\sigma_-^{-1} \sigma_0 \leq \tau_-^{-1} \tau_0 w_+$. By Proposition 5.39 we know that $\sigma_-^{-1} \sigma_0 \in D_*(e_+)$, so $\sigma_-^{-1} \sigma_0 \leq \tau_-^{-1} \tau_0 w_+$ if and only if $\sigma_-^{-1} \sigma_0 \leq \tau_-^{-1} \tau_0 w_+'$, where $w_+ = w_+' w_+''$, $w_+' \in W^*(f_+)$ and $w_+'' \in W_*(e_+)$ (by Proposition 5.15).

Since $e_+ \leq f_+$ and $\sigma_-^{-1} \sigma_0 \leq \tau_-^{-1} \tau_0 w_+'$, by Theorem 5.31, $\sigma_-^{-1} \sigma_0 e_+ \leq \tau_-^{-1} \tau_0 w_+' f_+$ (just take $w = z = 1$). So, $\sigma_-^{-1} e_- \sigma_0 = \sigma_-^{-1} \sigma_0 e_+ \leq \tau_-^{-1} \tau_0 w_+' f_+ = \tau_-^{-1} \tau_0 f_+ w_+' = \tau_-^{-1} f_- \tau_0 w_+'$. So now, by applying Theorem 5.33 we know there is $w_- \in W_*(e_-) W^*(f_-)$ and $z_- \in W_*(e_-)$ so $w_- \tau_- \leq \sigma_-$ and $\sigma_0 \leq z_- w_- \tau_0 w_+'$. Since $\sigma_0 \in V_*(e_-)$ we see that $\sigma_0 \leq z_- w_- \tau_0 w_+'$ if and only if $\sigma_0 \leq w_- \tau_0 w_+$ (using Proposition 5.15 again). $\square$

We get a slightly stronger condition if r and s belong to the same $\mathscr{J}$-class. Indeed, $r \mathscr{J} s$ allows us to strengthen Corollaries 5.32 and 5.34 as well.

Corollary 5.42. *For $r, s \in \mathcal{R}$, if $r = \sigma_-^{-1} e_- \sigma_0 e_+ \sigma_+^{-1}$ and $s = \tau_-^{-1} e_- \tau_0 e_+ \tau_+^{-1}$ are in vanilla form then the following are equivalent,*

(1) $r \leq s$

(2) $\exists w_- \in W^*(e_-)$ so that $w_- \tau_- \leq \sigma_-$, $\sigma_0 \sigma_+^{-1} \leq w_- \tau_0 \tau_+^{-1}$

(3) $\exists w_+ \in W^*(e_+)$ so that $\sigma_-^{-1} \sigma_0 \leq \tau_-^{-1} \tau_0 w_+$ and $\tau_+ w_+ \leq \sigma_+$

(4) $\exists w_- \in W^*(e_-)$ and $\exists w_+ \in W^*(e_+)$ so that $w_- \tau_- \leq \sigma_-$, $\sigma_0 \leq w_- \tau_0 w_+$ and $\tau_+ w_+ \leq \sigma_+$

Proof. From Theorem 5.41, (1) is equivalent to $e_-, e_+ \leq e_-, e_+$ and $\exists w_- \in W_*(e_-)W^*(e_-)$ and $\exists w_+ \in W^*(e_+)W_*(e_+)$ such that $w_-\tau_- \leq \sigma_-$, $\sigma_0 \leq w_-\tau_0 w_+$ and $\tau_+ w_+ \leq \sigma_+$. The first condition in our statement, $e_-, e_+ \leq e_-, e_+$, is now a tautology, so we can discard it. Thus, (1) is equivalent to $\exists w_- \in W_*(e_-)W^*(e_-)$ and $\exists w_+ \in W^*(e_+)W_*(e_+)$ such that $w_-\tau_- \leq \sigma_-$, $\sigma_0 \leq w_-\tau_0 w_+$ and $\tau_+ w_+ \leq \sigma_+$.

Let us decompose both $w_+ = w'_+ w''_+$ and $w_- = w''_- w'_-$, with, $w'_+ \in W^*(e_+)$, $w''_+ \in W_*(e_+)$, $w'_- \in W^*(e_-)$ and $w''_- \in W_*(e_-)$. By Proposition 5.15,

$$\sigma_0 = {}^{\lambda_*(e_-)}\sigma_0{}^{\lambda_*(e_+)} \leq (w''_-)^{-1} w''_- w'_- \tau_0 w'_+ w''_+ (w''_+)^{-1} = w'_- \tau_0 w'_+.$$

Additionally, since $\tau_- \in V(e_-)$ it follows that $w'_-\tau_- \in V_*(e_-)$ by applying Proposition 5.16. Likewise, we get $\tau_+ w'_+ \in D(e_+)W^*(e_+) = D_*(e_+)$. So by Proposition 5.15 we see that $w'_-\tau_- \leq w''_- w'_- \tau_- = w_-\tau_- \leq \sigma_-$ and $\tau_+ w'_+ \leq \tau_+ w'_+ w''_+ = \tau_+ w_+ \leq \sigma_+$.

By relabelling, we see that (1) is equivalent to $\exists w_- \in W^*(e_-)$ and $\exists w_+ \in W^*(e_+)$ such that $w_-\tau_- \leq \sigma_-$, $\sigma_0 \leq w_-\tau_0 w_+$ and $\tau_+ w_+ \leq \sigma_+$, which is the statement of (4).

Proving (1) is equivalent to (2) and (3) is done similarly. □

Proposition 5.43. *For $r, s \in \mathcal{R}$, if $r = \sigma_-^{-1} e_- \sigma_0 e_+ \sigma_+^{-1}$ and $s = \tau_-^{-1} f_- \tau_0 f_+ \tau_+^{-1}$ are in vanilla form then,*

(1) $r \mathscr{J} s$ if and only if $e_-, e_+ = f_-, f_+$

(2) $r \mathscr{R} s$ if and only if $\sigma_- = \tau_-$ and $e_- = f_-$

(3) $r \mathscr{L} s$ if and only if $e_+ = f_+$ and $\sigma_+ = \tau_+$

(4) $r \mathscr{H} s$ if and only if $\sigma_- = \tau_-$ and $e_- = f_-$ and $e_+ = f_+$ and $\sigma_+ = \tau_+$

Proof. (1) Comes right from the definition, as $e_- \mathscr{J} r \mathscr{J} e_+$.

(2) We know that $r \mathscr{R} s$ if and only if $r_- = s_-$. But, since we have our vanilla forms on hand, $r \mathscr{R} s$ if and only if $\sigma_-^{-1} e_- = \tau_-^{-1} f_-$. Since $r \mathscr{R} s$ implies $r \mathscr{J} s$, we can that $r \mathscr{R} s$ if and only if $r \mathscr{J} s$ and $r \mathscr{R} s$, if and only if $e_-, e_+ = f_-, f_+$ (by (1)) and $\sigma_-^{-1} e_- = \tau_-^{-1} f_-$. Thus, $r \mathscr{R} s$ if and only if $e_- = f_-$ and $\sigma_-^{-1} e_- = \tau_-^{-1} f_-$.

We can rearrange this last equation to get, $\sigma_-^{-1} e_- = \tau_-^{-1} f_-$ if and only if $\tau_- \sigma_-^{-1} e_- = e_-$. So we can say, $\tau_- \sigma_-^{-1} \in W_*(e_-)$. But then $\tau_- \in W_*(e_-)\sigma_- \subseteq W(e_-)\sigma_-$. Since $\tau_-, \sigma_- \in V(e_-)$ we can conclude that $\sigma_- = \tau_-$. Thus, $r \mathscr{R} s$ if and only if $e_- = f_-$ and $\sigma_- = \tau_-$.

(3) is done the same way as (2).

(4) By definition, $r\mathscr{H}s$ if and only $r\mathscr{R}s$ and $r\mathscr{L}s$. So this follows quickly from (2) and (3). □

Now the following results illustrate the power of vanilla form, as when we restrict to an $\mathscr{H}$-, $\mathscr{R}$- or $\mathscr{L}$-class we can determine the Adherence order in terms of the Bruhat order. This is something of an extension to Theorem 3.30 from earlier.

Theorem 5.44. *For $r, s \in \mathcal{R}$, if $r = \sigma_-^{-1}e_-\sigma_0e_+\sigma_+^{-1}$ and $s = \tau_-^{-1}e_-\tau_0e_+\tau_+^{-1}$ are in vanilla form then,*

(1) If $r\mathscr{H}s$, then $r \leq s$ iff $\sigma_0 \leq \tau_0$

(2) If $r\mathscr{R}s$, then $r \leq s$ iff $\sigma_0\sigma_+^{-1} \leq \tau_0\tau_+^{-1}$

(3) If $r\mathscr{L}s$, then $r \leq s$ iff $\sigma_-^{-1}\sigma_0 \leq \tau_-^{-1}\tau_0$

Proof. (1) By Theorem 3.30 we know that $r\mathscr{H}s$ implies that $r \leq s$ if and only if $r_0 \leq s_0$. Since $\sigma_0, \tau_0 \in D_*(e_+)$ we can see that $r_0 = \sigma_0e_+$ and $s_0 = \tau_0e_+$ are in standard form. Then Corollary 5.32, $r_0 \leq s_0$ if and only if there exists $w \in W^*(e_+)W_*(e_+)$ so that $\sigma_0 \leq \tau_0w$ and $w \leq 1$. But since 1 is the minimum element of W, hence $w = 1$ if it exists, we can conclude that $r_0 \leq s_0$ if and only if $\sigma_0 \leq \tau_0$.

(2) By Theorem 3.30 we know that $r\mathscr{R}s$ implies that $r \leq s$ if and only if $r_0r_+ \leq s_0s_+$. Observe that $r_0r_+ = e_-\sigma_0\sigma_+^{-1}$ and $e_-\tau_0\tau_+^{-1}$ are in opposite standard form, by Proposition 5.39. Then by Corollary 5.34, $r_0r_+ \leq s_0s_+$ if and only if there exists $w \in W_*(e_-)W^*(e_-)$ so that $\sigma_0\sigma_+^{-1} \leq w\tau_0\tau_+^{-1}$ and $w \leq 1$. But since 1 is the minimum element of W, hence $w = 1$ if it exists, we can conclude that $r_0r_+ \leq s_0s_+$ if and only if $\sigma_0\sigma_+^{-1} \leq \tau_0\tau_+^{-1}$.

(3) is done in the same manner. □

Corollary 5.45. *For $r \in \mathcal{R}$, let $e_- \in \Lambda^-$, $e_+ \in \Lambda$ with $e_-\mathscr{J}r\mathscr{J}e_+$ and define the collection of minimum elements, $Z(e_-, e_+) = \{{}^{\lambda_*(e_-)}w^{\lambda_*(e_+)} \mid w \in W(e_-)w_0W(e_+)\}$. Then,*

(1) $(Z(e_-, e_+), \leq) \cong (H_r, \leq)$ via the isomorphism, $u \mapsto r_-ur_+$

(2) $(V_*(e_-), \leq) \cong (R_r, \leq)$ via the isomorphism, $v \mapsto r_-v$

(3) $(D_*(e_+), \leq) \cong (L_r, \leq)$ via the isomorphism, $d \mapsto dr_+$

Proof. Let $r = \sigma_-^{-1}e_-\sigma_0e_+\sigma_+^{-1}$ be the vanilla form decomposition for r.

(1) First we will show this is an isomorphism of sets. Suppose that $u_1, u_2 \in Z(e_-, e_+)$, then $r_- u_1 r_+ = \sigma_-^{-1} e_- u_1 e_+ \sigma_+^{-1}$ and $r_- u_2 r_+ = \sigma_-^{-1} e_- u_2 e_+ \sigma_+^{-1}$ are our images under the given map. By Proposition 5.11,

$$Z(e_-, e_+) = {}^{\lambda_*(e_-)}W^{\lambda_*(e_+)} \cap W(e_-)w_0W(e_+) = V_*(e_-) \cap D_*(e_+) \cap W(e_-)w_0W(e_+),$$

so the images are in vanilla form. Uniqueness tells us this map is injective. Suppose $s \in H_r$, with vanilla form, $\sigma_-^{-1} e_- \tau e_+ \sigma_+^{-1}$ for $\tau \in V_*(e_-) \cap D_*(e_+) \cap W(e_-)w_0W(e_+) = Z(e_-, e_+)$. We observe that τ maps to s, and so our map is surjective, hence an isomorphism.

Now we will show that this map preserves the ordering. Suppose that $u_1, u_2 \in Z(e_-, e_+)$. By Theorem 5.44 it follows that $r_- u_1 r_+ \leq r_- u_2 r_+$ if and only if $u_1 \leq u_2$. Thus, the map $u \mapsto r_- u r_+$ is indeed an isomorphism of posets.

(2) First we will show this is an isomorphism of sets. Suppose that $v_1, v_2 \in V_*(e_-)$, then $r_- v_1 = \sigma_-^{-1} e_- v_1$ and $r_- v_2 = \sigma_-^{-1} e_- v_2$ can be seen to be in opposite standard form. The uniqueness of opposite standard form then tells us that the map, $v \mapsto r_- v$ is injective. Suppose that $s \in R_r$, with opposite standard form, $\sigma_-^{-1} e_- \tau$ for $\tau \in V_*(e_-)$. We observe that τ maps to s, and so our map is surjective, hence an isomorphism.

It remains to show that this map preserves the order. Suppose that $v_1, v_2 \in D_*(e_-)$. By Proposition 5.39 and Theorem 5.44 it follows that $r_- v_1 \leq r_- v_2$ if and only if $v_1 \leq v_2$.

(3) is similar to (2). □

Although we did not state it explicitly in the statement of the corollary, the inverses of the maps above are given by taking the vanilla, standard, or opposite standard form decompositions (respectively) and only considering the element that is not already given by our $\mathscr{H}$-, $\mathscr{R}$- or $\mathscr{L}$-class setting.

5.4 Example

The most important results of this section are those involving the vanilla form decomposition, so it would be prudent to show an example of its computation. Let us take an element in the Renner monoid of $M_6(K)$, say $r = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&1&0&0 \end{pmatrix}$.

Procedurally, to compute the vanilla form, it helps to compute the trichotomy first. The idea behind this step comes from our proof of the existence and uniqueness of vanilla order, which relied heavily on our trichotomy work. The steps, and simple tricks, for figuring out the trichotomy decomposition were covered in Section 3, and we will say little more about them. When we perform the decomposition on r we get,

$$r = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix} \cdot \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}$$

The idempotents, e_-, e_+ are easy to determine in $M_n(K)$. $e_- = \begin{pmatrix} 0 & 0 \\ 0 & I_{rk(r)} \end{pmatrix}$ and $e_+ = \begin{pmatrix} I_{rk(r)} & 0 \\ 0 & 0 \end{pmatrix}$, where $rk(r)$ is the rank of r. In this case,

$$e_- = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \qquad e_+ = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}$$

Now we turn our attention to σ_-, σ_0 and σ_+. For σ_- and σ_+ it is much easier to compute σ_-^{-1} and σ_+^{-1} first. Consider all the elements $w \in W$ so that $we_- = r_-$. That is, $w \in \sigma_-^{-1} W_*(e_-)$. Since $\sigma_-^{-1} \in V(e_-)^{-1} \subseteq V_*(e_-)^{-1}$, we just need to minimize the length of any potential w.

No matter which $w \in W$ so that $we_- = r_-$ we are looking at, we must inherit the nonzero entries from r_-. The nonzero elements of r_- along with the zeros that are in their columns and rows, form a minor of r_- of size $rk(r)$. This is the unique minor of $rk(r)$ in r_-. Complementary to this minor, we have a minor of all zeroes of $n - rk(r)$. In order to get a matrix in W, we must replace this complementary minor with a permutation matrix with rank $n - rk(r)$. Now we just need to choose the right permutation matrix in order to minimize the overall length.

Recall from Section 3 that we learned we can compute the length of an arbitrary $n \times n$ matrix, A, by $\ell(A) = \Sigma_{i=1}^n \Sigma_{j=1}^n (a_{ij})(n + i - j) - |coinv(A)| - \frac{rk(A)(rk(A)+1)}{2}$, from [8]. But we are considering elements in W, which are permutation matrices. Thus the rank is constant, and so is the expression involving the summation notation. Thus the length depends on $|coinv(A)|$. To minimize the length of the matrix, we must maximize the number of pairs (a_{ij}, a_{kl}) with $i < k$ and $j < l$, $a_{ij} = a_{kl} = 1$.

Since we must inherit the maximal minor of r_-, and since the remaining 1's must be placed in the complementary minor, the number of coinvariant pairs (see [8]) is maximized exactly when we maximize them in the complementary minor. That is, the complementary minor should be taken to be $I_{n-rk(r)}$.

We illustrate this below, the maximal minor of r_- consists of the gray elements. The dark gray zeroes denote the complementary minor that must be filled in to create an element of W. The rightmost matrix shows this minor swapped with the matrix $I_{n-rk(r)}$.

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \qquad \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \qquad \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

This computes σ_-^{-1}, and to compute σ_-, we take the inverse, which for permutation matrices is just the transpose. It just so happens that in this case, $\sigma_- = \sigma_-^{-1}$.

$$\sigma_- = \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

Likewise, to compute σ_+^{-1} we want to find the minimal length element among those $w \in W$ so that $e_+w = r_+$. The same technique works, replacing the complementary minor of r_+ by the matrix $I_{n-rk(r)}$. This minor will always be in the bottom $n - rk(r)$ rows. So we get,

$$\sigma_+^{-1} = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1 \end{pmatrix} \qquad \text{hence,} \qquad \sigma_+ = \begin{pmatrix} 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

To compute σ_0, we wish to minimize the length of the possible elements $w \in W$ such that $e_-w = r_0 = we_+$. So we will once again fill the complementary minor with $I_{n-rk(r)}$. This will be a little easier to deal with, as we noted in Section 3 the maximal minor of r_0 is in the bottom left corner, so we will simply put $I_{n-rk(r)}$ in the top right corner.

$$\sigma_0 = \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix}$$

This gives us the following vanilla form decomposition.

$$r = \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \cdot \begin{pmatrix} 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{pmatrix}^{-1}$$

Thanks to Corollary 5.40, from here we can perform two simple matrix multiplications and get the standard form of r (on the left) and the opposite standard form (on the right).

$$r = \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \cdot \begin{pmatrix} 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{pmatrix}^{-1} \qquad r = \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}$$

6 Maximum and Minimum Elements

Corollary 5.45 in conjunction with Proposition 5.5 and Corollary 5.14 paint an interesting picture of the Adherence order on our $\mathscr{H}$-, $\mathscr{L}$- and $\mathscr{R}$-classes. In particular they suggest to us that each class is "pointy", in the sense that they have a maximum element and a minimum element with respect to our order. While we will characterise these elements using the aforementioned results, for the moment, they serve to motivate the following section and its results concerning these maximum and minimum elements.

Comments made toward the end of [17] motivate our investigation of what we will call "relative maximum elements", and a paper by Putcha about shellability, [18], motivates our investigation of "relative minimum elements" (and his paper even provides the existence of and expression for the relative minimum element of a $\mathscr{J}$-class). These are elements that are not necessarily maximums and minimums with respect to the whole $\mathscr{T}$-class they are in, but are maximum or minimum with respect to an added condition. For example, if $r \leq s$, is there a unique element $\bar{t} \in R_s$ so that $t \in R_s$ and $r \leq t$ if and only if $\bar{t} \leq t$?

For the following section, as we have done before, we will let T and $\mathscr{T}$ represent a Green's relation. That is, T = H, L, R or J, and $\mathscr{T} = \mathscr{H}, \mathscr{L}, \mathscr{R}$, or $\mathscr{J}$.

6.1 Maximum and Minimum Elements

We will begin our discussion by tackling the maximum elements. However, we will find that when it comes to describing them, a straightforward proof is not exactly evident. This will lead to our discussion of minimum elements, which are easier to describe, even if their existence is not immediately apparent.

Proposition 6.1. *For all $r \in \mathcal{R}$, there exists an element $s \in T_r$ such that BsB is dense and open in BT_rB.*

Proof. We know that $BT_rB = \bigsqcup_{s \in T_r} BsB$. Each BsB is a subvariety, and since T_r is finite, this is a finite disjoint union of subvarieties. By an appropriate choice of Proposition 4.9, Theorem 4.14, or Theorem 4.23 from Section 4, we also know that BT_rB is an irreducible variety. So, by Theorem A.3, there exists an unique $s \in T_r$ so that BsB is open and dense in BT_rB. □

Definition 6.2. *For all $r \in \mathcal{R}$, we denote the unique $s \in T_r$ such that BsB is dense in BT_rB by $\overset{\mathscr{T}}{\lceil r \rceil}$. We say it is the* **maximum element** *of the $\mathscr{T}$-class.*

Corollary 6.3. *For all $r \in \mathcal{R}$, $dim(B\overset{\mathscr{T}}{\lceil r \rceil}B) = dim(BT_rB)$.*

Proof. By the preceding proposition (6.1), $B\overset{\mathscr{T}}{\lceil r \rceil}B$ is a dense subvariety of BT_rB. But by Proposition 14.1.6(iii) of [35] this implies that $dim(B\overset{\mathscr{T}}{\lceil r \rceil}B) = dim(BT_rB)$. □

While we have designated them maximum elements, and shown that they are dense in their respective fat $\mathscr{T}$-classes, it turns out these elements are aptly named with regards to the Adherence order.

Proposition 6.4. *For all $r \in \mathcal{R}$, and all $s \in T_r$, $s \leq \overset{\mathscr{T}}{\lceil r \rceil}$*

Proof. Since $s \in T_r$, then $BsB \subseteq BT_rB$. By definition, $B\overset{\mathscr{T}}{\lceil r \rceil}B$ is the dense orbit in BT_rB, and so $\overline{B\overset{\mathscr{T}}{\lceil r \rceil}B} = \overline{BT_rB}$. So we see, $BsB \subseteq BT_rB \subseteq \overline{BT_rB} = \overline{B\overset{\mathscr{T}}{\lceil r \rceil}B}$. Thus, $BsB \subseteq \overline{B\overset{\mathscr{T}}{\lceil r \rceil}B}$, or rather $s \leq \overset{\mathscr{T}}{\lceil r \rceil}$. □

Proposition 6.5. *For any $r, s \in \mathcal{R}$, $r \leq \overset{\mathscr{T}}{\lceil s \rceil}$ if and only if $\overset{\mathscr{T}}{\lceil r \rceil} \leq \overset{\mathscr{T}}{\lceil s \rceil}$.*

Proof. Let $r \leq \overset{\mathscr{T}}{\lceil s \rceil}$. So, $BT_rB \subseteq \overline{BT_sB}$ and $B\overset{\mathscr{T}}{\lceil r \rceil}B \subseteq \overline{B\overset{\mathscr{T}}{\lceil r \rceil}B} = \overline{BT_rB} \subseteq \overline{BT_sB} = \overline{B\overset{\mathscr{T}}{\lceil s \rceil}B}$. So we can see that $\overset{\mathscr{T}}{\lceil r \rceil} \leq \overset{\mathscr{T}}{\lceil s \rceil}$. Conversely, if $\overset{\mathscr{T}}{\lceil r \rceil} \leq \overset{\mathscr{T}}{\lceil s \rceil}$, then by Proposition 6.4, $r \leq \overset{\mathscr{T}}{\lceil r \rceil}$, so it is clear that $r \leq \overset{\mathscr{T}}{\lceil s \rceil}$. □

Proposition 6.6. *For any $r, s \in \mathcal{R}$, the following are equivalent,*

(1) there exists $a \in T_r$ and $b \in T_s$ so that $a \leq b$

(2) $BT_rB \subseteq \overline{BT_sB}$

(3) $\overset{\mathscr{T}}{\lceil r \rceil} \leq \overset{\mathscr{T}}{\lceil s \rceil}$

Proof. (1) $\implies$ (2) If we can find such a and b, then $BaB \cap \overline{BbB} \neq \emptyset$. But since, $BaB \subseteq BT_rB$ and $\overline{BbB} \subseteq \overline{BT_sB} \neq \emptyset$, we see $BT_rB \cap \overline{BT_sB}$. By an appropriate result from Section 4 we can write $\overline{BT_sB} = \bigsqcup_{i=1}^{n} BT_{s_i}B$ for some s_i's. Thus we can find index i so that $BT_rB \cap BT_{s_i}B \neq \emptyset$. Since fat $\mathscr{T}$-classes are disjoint, we see then that $T_r = T_{s_i}$, and conclude that $BT_rB \subseteq \overline{BT_sB}$.
(2) $\implies$ (3) Since $\overset{\mathscr{T}}{\lceil r \rceil} \in T_r$, we see that $B\overset{\mathscr{T}}{\lceil r \rceil}B \subseteq BT_rB$. By definition, $B\overset{\mathscr{T}}{\lceil s \rceil}B$ is dense in BT_sB. Thus, $\overline{B\overset{\mathscr{T}}{\lceil s \rceil}B} = \overline{BT_sB}$, and we see, $B\overset{\mathscr{T}}{\lceil r \rceil}B \subseteq BT_rB \subseteq \overline{BT_sB} = \overline{B\overset{\mathscr{T}}{\lceil s \rceil}B}$ This tells us, $\overset{\mathscr{T}}{\lceil r \rceil} \leq \overset{\mathscr{T}}{\lceil s \rceil}$.

(3) $\implies$ (1) Notice, $\lceil r \rceil^{\mathscr{T}} \in T_r$ and $\lceil s \rceil^{\mathscr{T}} \in T_s$ by definition. Since $\lceil r \rceil^{\mathscr{T}} \leq \lceil s \rceil^{\mathscr{T}}$, we have found an a and b. □

In order to describe the maximum elements, it turns out it is easier to first describe the all the minimum elements of the $\mathscr{T}$-classes, and then desribe the maximum elements in terms of the minimum elements.

Definition 6.7. *For all $r \in \mathcal{R}$, if $t \in T_r$ is such that for all $s \in T_r$, $t \leq s$, then we say that t is the* **minimum element** *of the $\mathscr{T}$-class. We denote it by writing, $t = \lfloor r \rfloor^{\mathscr{T}}$.*

We can see that any such $\lfloor r \rfloor^{\mathscr{T}}$ is unique by definition. It is not hard to see that, geometrically such elements correspond to closed orbits of $B \times B$ in the irreducible variety, BT_rB. As such, the uniqueness of $\lfloor r \rfloor^{\mathscr{T}}$ shows that each BT_rB has a unique closed orbit.

To show the existence of these minimum elements, we will proceed as we did and find certain dense orbits. The problem is, that with respect to our Adherence order, minimum elements will not generate dense orbits. So we will have to skew our fat $\mathscr{T}$-classes, and look at a different ordering. The aim is to emulate the well-known property of the Bruhat order on Weyl groups,

$$r \leq s \iff w_0 s \leq w_0 r \iff s w_0 \leq r w_0 \iff w_0 r w_0 \leq w_0 s w_0$$

Unfortunately, such a property can easily be seen to not transfer over to the Renner monoids (just consider $0 \leq 1$). But we do have the following results, which will turn out to be enough.

Proposition 6.8. *Let $r, s \in \mathcal{R}$, and suppose that $r \leq s$,*

(1) if $r \mathscr{L} s$, then $w_0 s \leq w_0 r$

(2) if $r \mathscr{R} s$, then $s w_0 \leq r w_0$

(3) if $r \mathscr{H} s$, then $w_0 r w_0 \leq w_0 s w_0$

Proof. (3) clearly follows from applying (1) and (2). (2) is proven similarly to (1), so we will just prove (1). Write r and s in standard form, $r = xey^{-1}$ and $s = zey^{-1}$. By applying Corollary 5.45, we can see that $r \leq s$ if and only if $x \leq z$. But this last ordering relation is between elements of W, so we know that $x \leq z$ if and only if $w_0 z \leq w_0 x$. Then we can see that $w_0 s = (w_0 z) e y^{-1} \leq (w_0 x) e y^{-1} = w_0 r$ by Theorem 5.31. □

These results are really the best we could ask for, as the following example shows, so it is fortunate that they are just enough to set us on our way to find the minimal elements.

Example 6.9. *One might hope to extend the results of Proposition 6.8 to more of $\mathcal{R}$, perhaps to a whole $\mathscr{J}$-class. The following two examples show this is not the case. Here we have two elements in the Renner monoid for $M_3(K)$. In the first case, the two elements are in the same $\mathscr{L}$-class, but we see that $r \le s$ does not imply $sw_0 \le rw_0$. The second case shows a similar counterexample for $\mathscr{R}$.*

(1) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathscr{L} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *with* $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \le \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *but* $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \not\le \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(2) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathscr{R} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *with* $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \le \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *but* $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \not\le \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The following corollary relates our work in Proposition 6.8 to the geometry of the Weyl group property we wish to emulate.

Corollary 6.10. *Let $r, s \in \mathcal{R}$,*

(1) if $r\mathscr{L}s$ then $r \le s$ if and only if $B^-sB \subseteq \overline{B^-rB}$

(2) if $r\mathscr{R}s$ then $r \le s$ if and only if $BsB^- \subseteq \overline{BrB^-}$

(3) if $r\mathscr{H}s$ then $r \le s$ if and only if $B^-rB^- \subseteq \overline{B^-sB^-}$

Proof. The technique for all three is the same, so we will just show (3). By Proposition 6.8, $r \le s$ if and only if $w_0rw_0 \le w_0sw_0$. But by the definition of the Adherence order, this is equivalent to $Bw_0rw_0B \subseteq \overline{Bw_0sw_0B}$. This containment relation is unchanged by multiplying on either side by w_0. So then $r \le s$ if and only if

$$B^-rB^- = w_0Bw_0rw_0Bw_0 \subseteq w_0\overline{Bw_0sw_0B}w_0 = \overline{w_0Bw_0sw_0Bw_0} = \overline{B^-sB^-}. \qquad \square$$

This tells us that if we restrict to an $\mathscr{H}$-class, we get the analogue of a property of the Weyl group Bruhat order,

$$r \le s \iff BrB \subseteq \overline{BsB} \iff B^-sB \subseteq \overline{B^-rB} \iff BsB^- \subseteq \overline{BrB^-} \iff B^-rB^- \subseteq \overline{B^-sB^-},$$

for $r, s \in \mathcal{R}$ such that $r\mathscr{H}s$. If we restrict to an $\mathscr{L}$- or $\mathscr{R}$-class, we get a restricted version of the property, but we will see it is still enough.

Proposition 6.11. *For any sets $C, D \in \{B, B^-\}$, and $\mathscr{T} = \mathscr{H}, \mathscr{L}$, or $\mathscr{R}$, then for any $r \in \mathcal{R}$, $CT_rD = CT_r'D$, and is an irreducible subvariety of M. (Recall T_r' is the $\mathscr{T}$-class of r as an element of M rather than $\mathcal{R}$.)*

Proof. Notice that the case $C = D = B$ has been covered before in Section 4. We will just demonstrate this result for one of the three remaining choices of C and D, say $C = B^-$ and $D = B$.

$$\begin{aligned} B^-R'_rB &= B^-rGB = B^-rG = B^-w_0w_0rG = B^-w_0(w_0r)_-G = w_0Bw_0w_0(w_0r)_-G \\ &= w_0B(w_0r)_-G = w_0B(w_0r)_-BN(T)B = w_0B(w_0r)_-N(T)B = w_0Bw_0rN(T)B \\ &= B^-R_rB \end{aligned}$$

$$B^-L'_rB = B^-GrB = GrB = Gr_+B = B^-N(T)Br_+B = B^-N(T)r_+B = B^-L_rB$$

This establishes the results, $B^-L'_rB = B^-L_rB$ and $B^-R'_rB = B^-R_rB$, and as we did in Section 4, we will now show that $B^-H_rB = B^-H'_rB = B^-L'_rB \cap B^-R'_rB$.

Clearly, $H_r \subseteq H'_r$, so $B^-H_rB \subseteq B^-H'_rB$. Also, $H'_r \subseteq L'_r \cap R'_r$, so $B^-H'_rB \subseteq B^-L'_rB \cap B^-R'_rB$. Now, suppose that $m \in B^-L_rB \cap B^-R_rB$. Then we can find $s\mathscr{L}r$ and $t\mathscr{R}r$ so that $m \in B^-sB$ and $m \in B^-tB$. Thus, $B^-sB \cap B^-tB \neq \emptyset$, and it follows that $s = t$. Thus $s\mathscr{R}r$, and we see that $s\mathscr{H}r$. So $m \in B^-sB \subseteq B^-H_rB$.

Thus,

$$B^-L_rB \cap B^-R_rB \subseteq B^-H_rB \subseteq B^-H'_rB \subseteq B^-L'_rB \cap B^-R'_rB$$

and by our earlier work in this proof, $B^-L_rB \cap B^-R_rB = B^-L'_rB \cap B^-R'_rB$, which squeezes out the remaining result, $B^-H_rB = B^-H'_rB$.

We see that $B^-R_rB = B^-rG$ and $B^-L_rB = GrB$, and so are orbits of the appropriate group actions from $B^- \times G$, $G \times B$ on M. Thus B^-R_rB and B^-L_rB are irreducible subvarieties. Since $B^-H_rB = B^-R_rB \cap B^-L_rB$ it follows that B^-H_rB is also a subvariety.

Lastly, $B^- \times H'_r \times B$ is irreducible, as each of the factors of the Cartesian product is irreducible. Its image under our multiplication map $(c, h, d) \mapsto chd$ in M, must therefore be irreducible. The image is B^-H_rB, which concludes the result. □

This allows us to now establish existence for $\overset{\mathscr{H}}{\lfloor r \rfloor}$, $\overset{\mathscr{R}}{\lfloor r \rfloor}$ and $\overset{\mathscr{L}}{\lfloor r \rfloor}$.

Proposition 6.12. *For any $r \in \mathcal{R}$, the elements $\overset{\mathscr{H}}{\lfloor r \rfloor}$, $\overset{\mathscr{R}}{\lfloor r \rfloor}$ and $\overset{\mathscr{L}}{\lfloor r \rfloor}$ exist.*

Proof. All three results are done similarly, so we will just prove that $\overset{\mathscr{H}}{\lfloor r \rfloor}$ exists. By Proposition 6.11, B^-H_rB is an irreducible subvariety of M. We can decompose $B^-H_rB = \bigsqcup_{s \in H_r} B^-sB$, and so it follows that there exists a unique $s \in H_r$ so that B^-sB is open and dense in B^-H_rB. We claim that $s = \overset{\mathscr{H}}{\lfloor r \rfloor}$.

Take any $t \in H_r$. Since $B^- sB$ is dense, we see $B^- tB \subseteq B^- H_r B \subseteq \overline{B^- H_r B} = \overline{B^- sB}$. By Corollary 6.10 this is equivalent to $s \leq t$ for all $t \in H_r$, which is the definition of $\overset{\mathscr{H}}{\lfloor r \rfloor}$. □

Now that we have existence for all but $\overset{\mathscr{J}}{\lfloor r \rfloor}$ we are in position to describe the minimum elements, which will bring us back to the maximum elements we started with. The following theorem uses our Corollary 5.45 to determine expressions for the minimum elements. As it turns out, the minimum elements all belong to well behaved sets.

Theorem 6.13. *For $r \in \mathcal{R}$ we get the following,*

(1) $r = \overset{\mathscr{H}}{\lfloor r \rfloor}$ iff $r \in \mathcal{O}$

(2) $r = \overset{\mathscr{R}}{\lfloor r \rfloor}$ iff $r \in \mathcal{J}\mathcal{G}$

(3) $r = \overset{\mathscr{L}}{\lfloor r \rfloor}$ iff $r \in \mathcal{G}\mathcal{J}$

(4) $r = \overset{\mathscr{J}}{\lfloor r \rfloor}$ iff $r \in \mathcal{N}$

While the phrasing of (4) might seem to require existence, it really shows existence, as $\mathcal{N} \cong \mathcal{R}/\mathscr{J}$, so there will be exactly one such element in each $\mathscr{J}$-class.

Proof. (1) Let $e_- \in \Lambda^-$, $e_+ \in \Lambda$ with $e_- \mathscr{J} r \mathscr{J} e_+$ and let $\mu \in W$ be minimal such that $e_-\mu = \mu e_+$. Then, by using Corollary 5.45, $(Z(e_-, e_+), \leq) \cong (H_r, \leq)$ by way of the isomorphism, $u \mapsto r_- u r_+$, where $Z(e_-, e_+) = V_*(e_-) \cap W(e_-)w_0W(e_+) \cap D_*(e_+)$. Notice that both ${}^{\lambda(e_-)}w_0{}^{\lambda(e_+)} \in Z(e_-, e_+)$ and ${}^{\lambda(e_-)}w_0{}^{\lambda(e_+)} \leq w$ for all $w \in W(e_-)w_0W(e_+)$. So $Z(e_-, e_+)$ certainly has a minimum element. It is then clear that taking u as the minimum in $Z(e_-, e_+)$ will give us $\overset{\mathscr{H}}{\lfloor r \rfloor}$. So $u = {}^{\lambda(e_-)}w_0{}^{\lambda(e_+)}$.

Thus, $r = \overset{\mathscr{H}}{\lfloor r \rfloor}$ if and only if $r = r_-{}^{\lambda(e_-)}w_0{}^{\lambda(e_+)}r_+ = r_-e_-{}^{\lambda(e_-)}w_0{}^{\lambda(e_+)}e_+r_+$. But now recall that Proposition 3.1 tells us, $e_-{}^{\lambda(e_-)}w_0{}^{\lambda(e_+)}e_+ \in \mathcal{N}^*$. So we conclude, by Proposition 3.25 that $r = \overset{\mathscr{H}}{\lfloor r \rfloor}$ if and only if $r \in \mathcal{O}$.

(2) Let $e_- \in \Lambda^-$, with $e_- \mathscr{J} r$. Then, by Corollary 5.45, $(V_*(e_-), \leq) \cong (R_r, \leq)$ via the isomorphism, $v \mapsto r_-v$. So taking v minimal in $V_*(e_-)$ will give us $\overset{\mathscr{R}}{\lfloor r \rfloor}$. It is clear that 1 is the minimal element of $V_*(e_-)$, as it is the minimum element of W. Thus, $r = \overset{\mathscr{R}}{\lfloor r \rfloor}$ if and only if $r = r_-1$. That is, $r = \overset{\mathscr{R}}{\lfloor r \rfloor}$ if and only if $r \in \mathcal{J}\mathcal{G}$.

(3) is done similarly to (2).

(4) Observe that for $r \in \mathcal{R}$, $L_r, R_r \subseteq J_r$. It is clear that if $r = \overset{\mathscr{J}}{\lfloor r \rfloor}$ then $r = \overset{\mathscr{R}}{\lfloor r \rfloor}$ and $r = \overset{\mathscr{L}}{\lfloor r \rfloor}$. But then by (2) and (3) this tells us $r \in \mathcal{J}\mathcal{G} \cap \mathcal{G}\mathcal{J} = \mathcal{N}$. Thus $r = \overset{\mathscr{J}}{\lfloor r \rfloor}$ implies $r \in \mathcal{N}$. The fact that $\mathcal{N} \cong \mathcal{R}/\mathscr{J}$ completes the result. □

With existence established we can now demonstrate the following result, which allows us to answer a question about the fat $\mathscr{H}$-classes from Section 4.

Proposition 6.14. *For any $r, s \in \mathcal{R}$, the following are equivalent,*

(1) there exists $a \in T_r$ and $b \in T_s$ so that $a \leq b$

(2) $BT_rB \subseteq \overline{BT_sB}$

(3) $\overset{\mathscr{T}}{\lfloor r \rfloor} \leq \overset{\mathscr{T}}{\lfloor s \rfloor}$

Proof. (1) $\iff$ (2) was established in Proposition 6.6.

(2) $\iff$ (3) For $\mathscr{T} = \mathscr{L}, \mathscr{R}$ or $\mathscr{J}$ we observe that the results have already been proven in Theorem 4.17 and Corollary 4.20 (one needs to consider Theorem 6.13 as well). So it remains to tackle the $\mathscr{T} = \mathscr{H}$ case.

For $\mathscr{T} = \mathscr{H}$, we will show (1) $\iff$ (3) instead. If $\overset{\mathscr{H}}{\lfloor r \rfloor} \leq \overset{\mathscr{H}}{\lfloor s \rfloor}$ then since $\overset{\mathscr{H}}{\lfloor r \rfloor} \in H_r$ and $\overset{\mathscr{H}}{\lfloor s \rfloor} \in H_s$ we see that (3) implies (1). Conversely, if $a \in H_r$ and $b \in H_s$ then by definition $\overset{\mathscr{H}}{\lfloor r \rfloor} \leq a \leq b$. By applying Corollary 5.5 from [17] we see that $\overset{\mathscr{H}}{\lfloor r \rfloor} \leq b$ implies $\overset{\mathscr{H}}{\lfloor r \rfloor} \leq \overset{\mathscr{H}}{\lfloor s \rfloor}$, as desired. □

Corollary 6.15. *For $r, s \in \mathcal{R}$, $BH_rB \subseteq \overline{BH_sB}$ if and only if $a \leq b$, where a and b are the unique elements in $O \cap H_r$ and $O \cap H_s$ respectively.*

Proof. This is just (2) and (3) of the above proposition when applied to $\mathscr{T} = \mathscr{H}$. □

Proposition 6.16. *If $r, s \in \mathcal{R}$, then $\overset{\mathscr{L}}{\lfloor r \rfloor} \leq \overset{\mathscr{L}}{\lfloor s \rfloor}$ and $\overset{\mathscr{R}}{\lfloor r \rfloor} \leq \overset{\mathscr{R}}{\lfloor s \rfloor}$ if and only if $\overset{\mathscr{H}}{\lfloor r \rfloor} \leq \overset{\mathscr{H}}{\lfloor s \rfloor}$.*

Proof. It is clear that $t_- = \overset{\mathscr{R}}{\lfloor t \rfloor}$ and $t_+ = \overset{\mathscr{L}}{\lfloor t \rfloor}$ by Theorem 6.13. Now, since $r_-, s_- \in \mathcal{JG}$ and $r_+, s_+ \in \mathcal{GJ}$, we see that $r_- \leq s_-$ and $r_+ \leq s_+$ if and only if $Br_- \subseteq \overline{Bs_-}$ and $r_+B \subseteq \overline{s_+B}$. Theorem 4.25 tells us this is equivalent to $BH_rB \subseteq \overline{BH_sB}$. But Corollary 6.15 says this in turn is equivalent to $a \leq b$, where a and b are the unique elements in $O \cap H_r$ and $O \cap H_s$ respectively. Applying Theorem 6.13 again allows us to conclude that $\overset{\mathscr{L}}{\lfloor r \rfloor} \leq \overset{\mathscr{L}}{\lfloor s \rfloor}$ and $\overset{\mathscr{R}}{\lfloor r \rfloor} \leq \overset{\mathscr{R}}{\lfloor s \rfloor}$ if and only if $\overset{\mathscr{H}}{\lfloor r \rfloor} \leq \overset{\mathscr{H}}{\lfloor s \rfloor}$. □

Now that we have described the minimum elements, we can turn the tables again, using our result, Proposition 6.8, to talk about the maximum elements again. The maximum elements belong to well behaved sets, like the minimum elements.

Theorem 6.17. *For $r \in \mathcal{R}$ we get the following,*

(1) $r = \overset{\mathscr{H}}{\lceil r \rceil}$ *iff* $r \in w_0\mathcal{O} = \mathcal{O}w_0$

(2) $r = \overset{\mathscr{R}}{\lceil r \rceil}$ *iff* $r \in \mathcal{J}\mathcal{G}w_0$

(3) $r = \overset{\mathscr{L}}{\lceil r \rceil}$ *iff* $r \in w_0\mathcal{G}\mathcal{J}$

(4) $r = \overset{\mathscr{J}}{\lceil r \rceil}$ *iff* $r \in w_0\Lambda = \Lambda^- w_0$

Proof. (1) $r = \overset{\mathscr{H}}{\lceil r \rceil}$ if and only if for all $s \in H_r$ we have $s \leq r$. But then, by Proposition 6.8 this is equivalent to $w_0 r \leq w_0 s$ for all $s \in H_r$. By observing that $w_0 H_r = H_{w_0 r}$ we see that $r = \overset{\mathscr{H}}{\lceil r \rceil}$ if and only if $w_0 r \leq t$ for all $t \in H_{w_0 r}$, or rather $w_0 r = \overset{\mathscr{H}}{\lfloor w_0 r \rfloor}$. Thus, $r = \overset{\mathscr{H}}{\lceil r \rceil}$ if and only if $w_0 r \in \mathcal{O}$ if and only if $r \in w_0\mathcal{O}$.

(2) $r = \overset{\mathscr{R}}{\lceil r \rceil}$ if and only if for all $s \in R_r$ we have $s \leq r$. But then, by Proposition 6.8 this is equivalent to $rw_0 \leq sw_0$ for all $s \in R_r$. By observing that $R_r w_0 = R_{rw_0}$ we see that $r = \overset{\mathscr{R}}{\lceil r \rceil}$ if and only if $rw_0 \leq t$ for all $t \in R_{rw_0}$, or rather $rw_0 = \overset{\mathscr{R}}{\lfloor rw_0 \rfloor}$. Thus, $r = \overset{\mathscr{R}}{\lceil r \rceil}$ if and only if $rw_0 \in \mathcal{J}\mathcal{G}$ if and only if $r \in \mathcal{J}\mathcal{G}w_0$.

(3) is done similarly to (2).

(4) Observe that for $r \in \mathcal{R}$, $L_r, R_r \subseteq J_r$. It is clear that if $r = \overset{\mathscr{J}}{\lceil r \rceil}$ then $r = \overset{\mathscr{R}}{\lceil r \rceil}$ and $r = \overset{\mathscr{L}}{\lceil r \rceil}$. But by (2) and (3) this tells us $r \in \mathcal{J}\mathcal{G}w_0 \cap w_0\mathcal{G}\mathcal{J}$. We claim $\mathcal{J}\mathcal{G}w_0 \cap w_0\mathcal{G}\mathcal{J} = w_0\Lambda = \Lambda^- w_0$.

It is clear, since $\Lambda \subseteq \mathcal{G}\mathcal{J}$ and $\Lambda^- \subseteq \mathcal{J}\mathcal{G}$, that $\Lambda^- w_0 \subseteq \mathcal{J}\mathcal{G}w_0 \cap w_0\mathcal{G}\mathcal{J}$. Suppose that $a \in \mathcal{J}\mathcal{G}w_0 \cap w_0\mathcal{G}\mathcal{J}$. By Proposition 5.26 we can write $a = b^{-1}fw_0 = w_0 e y^{-1}$ for $e \in \Lambda$, $f \in \Lambda^-$, $y \in D(e)$ and $b \in V(f)$. Since $e \mathscr{J} f$ we can tell that, $w_0 e = f w_0$. So then we see, $w_0 a = w_0 b^{-1} f w_0 = w_0 b^{-1} w_0 e$. Thus, $w_0 a \mathscr{L} e$. But notice that $a \in w_0\mathcal{G}\mathcal{J}$ means that $w_0 a \in \mathcal{G}\mathcal{J}$. Theorem 3.9 then tells us that $w_0 a = e$, or rather $a \in w_0\Lambda$.

Thus $r = \overset{\mathscr{J}}{\lceil r \rceil}$ implies $r \in \Lambda w_0 = w_0\Lambda^-$. The fact that $\Lambda w_0 \cong \Lambda \cong \mathcal{R}/\mathscr{J}$ completes the result. □

It is interesting to note that this is directly in line with Renner's description of the "big cell" in each $\mathscr{J}$-class, in Section 6 of [27]. Indeed, we can now say that each fat $\mathscr{T}$-class has a "big cell" if we define the concept analogously, and that cell is exactly the orbit $B\overset{\mathscr{T}}{\lceil r \rceil}B$ for the fat $\mathscr{T}$-class, BT_rB.

Corollary 6.18. *Let $e \in \Lambda$ and $f \in \Lambda^-$. Then* $\overset{\mathscr{R}}{\lfloor e \rfloor} = \overset{\mathscr{J}}{\lfloor e \rfloor}$, $\overset{\mathscr{L}}{\lceil e \rceil} = \overset{\mathscr{J}}{\lceil e \rceil}$, $\overset{\mathscr{R}}{\lceil f \rceil} = \overset{\mathscr{J}}{\lceil f \rceil}$, *and* $\overset{\mathscr{L}}{\lfloor f \rfloor} = \overset{\mathscr{J}}{\lfloor f \rfloor}$.

Proof. Observe, $e\mathscr{R}e^{\lambda(e)}{w_0}^{\lambda(w_0ew_0)} \in \mathcal{N}$. So $\nu = \overset{\mathscr{J}}{\lfloor e \rfloor} \leq \overset{\mathscr{R}}{\lfloor e \rfloor} = \overset{\mathscr{R}}{\lfloor \nu \rfloor} \leq \nu$. Thus $\overset{\mathscr{J}}{\lfloor e \rfloor} = \overset{\mathscr{R}}{\lfloor e \rfloor}$. By Theorem 6.17, $\overset{\mathscr{J}}{\lceil e \rceil} \in w_0\Lambda$, so we can see that $\overset{\mathscr{J}}{\lceil e \rceil} = w_0e\mathscr{L}e$. By similar reasoning to the last case, we can conclude that $\overset{\mathscr{L}}{\lceil e \rceil} = \overset{\mathscr{J}}{\lceil e \rceil}$. The cases for $f \in \Lambda^-$ are similar. □

This corollary gives us some useful information about the positions of the elements of Λ Λ^- within their $\mathscr{J}$-classes. Namely, Λ lies in the minimum $\mathscr{R}$-class and maximum $\mathscr{L}$-class, and Λ^- lies in the maximum $\mathscr{R}$-class and minimum $\mathscr{L}$-class.

6.2 Relative Maxima

Having seen the structure of $\mathscr{T}$-classes though the lens of absolute maxima and minima (specifically showing that they exist) we turn to extend these notions to coincide with elements shown to exist in Section 5 of [17]. These elements are unique maximal elements in a $\mathscr{T}$-class, subject to the additional condition that they are less than a given element of $\mathcal{R}$.

Definition 6.19. *For $r, s \in \mathcal{R}$ and $r \leq s$, we define the* **relative maximum** *of T_r with respect to s, as*

$$max_sT_r = \begin{cases} t & \text{if } t \in T_r,\ t \leq s \text{ and } \forall t' \in T_r\ t' \leq s \implies t' \leq t \\ \text{undefined} & \text{otherwise} \end{cases}$$

Remark 6.20. *We can recover our previous work with (absolute) maximums by noting that, $\mathcal{R}$ has a unique maximal element, w_0, so it can be seen that $\overset{\mathscr{T}}{\lceil r \rceil} = max_{w_0}T_r$.*

Proposition 6.21. *For $r, s, t \in \mathcal{R}$ with $r \leq t \leq s$, then (if they exist), $max_tT_r \leq max_sT_r$.*

Proof. By definition we know, $max_tT_r \in T_r$ and $max_tT_r \leq t$. But since, $t \leq s$ it follows that $max_tT_r \leq s$, and so by definition of the relative maximum, $max_tT_r \leq max_sT_r$. □

The next two results will be very helpful in proving the existence of these relative maximum elements. This first one is a kind of strengthening of the condition for two elements to be in the same $\mathscr{J}$-class. It is well-known that $r\mathscr{J}s$ if and only if we can find $t \in \mathcal{R}$ so that $r\mathscr{L}t\mathscr{R}s$. the following lemma shows us that if $r \leq s$ then we can choose t so that $r \leq t \leq s$.

Lemma 6.22. *Suppose that $r, s \in \mathcal{R}$ with $r \leq s$ and $r\mathscr{J}s$. Then we can find $a \in H_{r_-w_0s_+}$ and $b \in H_{s_-w_0r_+}$ so that $r \leq a \leq s$ and $r \leq b \leq s$.*

Proof. Let $r = \sigma_-^{-1}e_-\sigma_0 e_+\sigma_+^{-1}$ and $s = \tau_-^{-1}e_-\tau_0 e_+\tau_+^{-1}$ be in vanilla form. By Corollary 5.42 since $r \leq s$, then $\exists w_- \in W^*(e_-)$ and $\exists w_+ \in W^*(e_+)$ such that $w_-\tau_- \leq \sigma_-$, $\sigma_0\sigma_+^{-1} \leq w_-\tau_0\tau_+^{-1}$ and $\tau_+w_+ \leq \sigma_+$, $\sigma_-^{-1}\sigma_0 \leq \tau_-^{-1}\tau_0 w_+$. Let $a = \sigma_-^{-1}e_-w_-\tau_0\tau_+^{-1}$ and $b = \tau_-^{-1}\tau_0 w_+ e_+\sigma_+^{-1}$.

By Proposition 5.39 we can see that $\tau_0\tau_-^{-1} \in V_*(e_-)$ and $\tau_+^{-1}\tau_0 \in D_*(e_+)$. So by Propositions 5.16 and 5.23, $w_-\tau_0\tau_+^{-1} \in V_*(e_-)$ and $\tau_+^{-1}\tau_0 w_+ \in D_*(e_+)$, so we can see that a is in opposite standard form and b is in standard form.

By comparing their forms, one can easily tell that $r\mathscr{R}a$ and $r\mathscr{L}b$. Observe that since $w_- \in W^*(e_-)$, we can see that $a = \sigma_-^{-1}e_-w_-\tau_0\tau_+^{-1} = \sigma_-^{-1}w_-e_-\tau_0\tau_+^{-1} = \sigma_-^{-1}w_-\tau_0 e_+\tau_+^{-1}$, so $s\mathscr{L}a$, and likewise $s\mathscr{R}b$. So it is clear from here that $a \in H_{r_-w_0s_+}$ and $b \in H_{s_-w_0r_+}$. It remains for us to show that $r \leq a \leq s$ and $r \leq b \leq s$.

Observe that $1 \in W^*(e_-)W_*(e_-)$, $\sigma_- \leq \sigma_-$ and $\sigma_0\sigma_+^{-1} \leq w_-\tau_0\tau_+^{-1}$. So by Corollary 5.34 $r \leq a$. In a similar fashion, $w_- \in W^*(e_-)W_*(e_-)$, $w_-\tau_- \leq \sigma_-$ and $w_-\tau_0\tau_+^{-1} \leq w_-\tau_0\tau_+^{-1}$. So by Corollary 5.34 we can see that $a \leq s$.

$r \leq b \leq s$ is shown similarly. □

Lemma 6.23. *Take $r \in \mathcal{R}$ and let $e, f \in E(\mathcal{R})$ be such that $e\mathscr{R}r$ and $f\mathscr{L}r$. Then,*

(1) H'_r is open and dense in eMf

(2) L'_r is open and dense in Mf

(3) R'_r is open and dense in eM

Proof. (1) We know that $H'_e = eC_G(e)e \subseteq eMe$ is open and dense, as it is the group of units of eMe. Since $e\mathscr{R}r\mathscr{L}f$ we see that $e\mathscr{J}f$. So we can find $u \in G$ so that $u^{-1}fu = e$. Then observe that $H'_r = H'_e u^{-1}$. One can see this, as $s \in H'_r$ if and only if $s = ew = wf$ for some $w \in G$ if and only if $su = ewu = wfu = wuu^{-1}fu = wue$ if and only if $su \in H'_e$ if and only if $s \in H'_e u^{-1}$. Now, since H'_e is open and dense in eMe, we can conclude that $H'_r = H'_e u^{-1} \subseteq eMeu^{-1} = eMueu^{-1} = eMf$ is open and dense.

(2) Observe that $L'_r = Gr = Gf \subseteq Mf$. Since it is the orbit of an element, it is a subvariety of Mf by Proposition A.4. Since it is a subvariety, it is locally closed, and so openness of Gf will follow from density. By continuity of multiplication, $Gf \subseteq Mf = \overline{G}f \subseteq \overline{Gf} \subseteq \overline{Mf} = Mf$, since Mf is closed. Thus, $\overline{Gf} = Mf$, and thus L'_r is dense in Mf.

(3) is done similarly to (2). □

And now we are in position to show the existence of max_sH_r, max_sR_r and max_sL_r. The result comes as a generalisation of work from Section 5 of [17], specifically Corollary 5.5.

Theorem 6.24. *For any $r, s \in \mathcal{R}$, with $r \leq s$, then max_sH_r, max_sL_r and max_sR_r exist.*

Proof. The results are similar, so we will just prove this for max_sR_r. First, pick, $e \in E(\mathcal{R})$, with $e\mathscr{R}r$. Observe that $r \in R'_r$. Also note that $r = er \in eBrB \subseteq e\overline{BsB} \subseteq \overline{eBsB}$. So we see that $r \in \overline{eBsB} \cap R'_r$, hence $\overline{eBsB} \cap R'_r \neq \emptyset$.

Since $\overline{eBsB} \cap R'_r \neq \emptyset$, we know that it is open and dense in $\overline{eBsB}$ (as $\overline{eBsB} \subseteq eM$ and R'_r is open and dense in eM, by Lemma 6.23). We know, $R'_r \subseteq BR_rB$ by Proposition 4.6, so $\overline{eBsB} \cap R'_r \subseteq \overline{eBsB} \cap BR_rB \subseteq \overline{eBsB}$, and hence $\overline{eBsB} \cap BR_rB$ is a dense subvariety of $\overline{eBsB}$.

BsB is the orbit of s under the group action of $B \times B$, and so is irreducible. Then, $eBsB$ is also irreducible, as it is the image of BsB under multiplication by e on the left. It follows that $\overline{eBsB}$ is also irreducible, as the closure of an irreducible is irreducible. Since $\overline{eBsB} \cap BR_rB$ is a dense subvariety of $\overline{eBsB}$, it must be an irreducible variety too.

Now, we see that $\overline{eBsBe} \cap BR_rB = \bigsqcup_{t\mathscr{R}r} \overline{eBsB} \cap BtB$, a finite disjoint union of subvarieties. So by Theorem A.3 we can find a unique $\bar{t}\mathscr{R}r$ so that $\overline{eBsB} \cap B\bar{t}B$ is dense in $\overline{eBsB} \cap BR_rB$ (thus dense in $\overline{eBsB}$). Then, $\overline{eBsB} \subseteq \overline{B\bar{t}B}$, so for any $t \in R_r$, if $BtB \cap \overline{eBsB} \neq \emptyset$, then $BtB \cap \overline{B\bar{t}B} \neq \emptyset$, and we conclude that $BtB \subseteq \overline{B\bar{t}B}$, or rather $t \leq \bar{t}$.

Suppose that $t\mathscr{R}r$. If $t \leq s$, then $t \in \overline{BsB}$, so $t = et \in e\overline{BsB} \subseteq \overline{eBsB}$, by continuity of multiplication. So, $t \leq s$ implies $\overline{eBsB} \cap BtB \neq \emptyset$, hence $t \leq \bar{t}$.

To conclude that $\bar{t} = max_sR_r$, it remains to show that $\bar{t} \leq s$. By our choice of $\bar{t}$ it is clear that $\overline{eBsB} \cap B\bar{t}B \neq \emptyset$. Now, since $e \in \overline{T}$, and $TBsB \subseteq BsB$, we see that $\overline{eBsB} \subseteq \overline{BsB}$. So then $\overline{BsB} \cap B\bar{t}B \neq \emptyset$, and hence $B\bar{t}B \subseteq \overline{BsB}$. We conclude that $\bar{t} \leq s$. □

Applying Lemma 6.22 allows us to describe max_sR_r and max_sL_r in terms of an element that is maximum relative to an $\mathscr{H}$-class.

Proposition 6.25. *For any, $r, s \in \mathcal{R}$, with $r \mathscr{J} s$ and $r \leq s$, then $max_sL_r = max_sH_{s_{-}w_0r_{+}}$ and $max_sR_r = max_sH_{r_{-}w_0s_{+}}$*

Proof. We will just prove this result for $max_sR_r = max_sH_{r_{-}w_0s_{+}}$. Notice that by definition, for all $t \in R_r$, $t \leq s$ if and only if $t \leq max_sR_r$. It is clear that $H_{r_{-}w_0s_{+}} \subseteq R_r$, so then for all $t \in H_{r_{-}w_0s_{+}}$,

$t \leq s$ implies $t \leq max_s R_r$. We can conclude then that $max_s H_{r_- w_0 s_+} \leq max_s R_r$. Now, for any $t \in R_r$ such that $t \leq s$, by Lemma 6.22 we can find $a \in H_{r_- w_0 s_+}$ so that $t \leq a \leq s$. Thus, $t \leq max_s H_{r_- w_0 s_+}$, and so we can conclude that $max_s R_r \leq max_s H_{r_- w_0 s_+}$. □

This is useful, as we now only need to find an expression for $max_s H_r$'s to describe all of our relative maximum elements. Unlike $\mathscr{H}$, $\mathscr{R}$ and $\mathscr{L}$, we cannot guarantee that $\mathscr{J}$-classes always have a unique maximal relative element, as the following example shows us.

Example 6.26. *Consider* $r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *One can check that* $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} < \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} < \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *are both maximal in the* $\mathscr{J}$*-class of rank one matrices, but that neither is greater than the other. So we have no maximum element. Thus,* $max_s J_r$ *does not exist.*

All is not lost, however, as the following proposition allows us to describe all the elements that are relatively maximal (if not relatively maximum).

Proposition 6.27. *For* $r, s, t \in \mathcal{R}$, *if* $r \leq s$, *then there does not exist any element* $t \in J_r$ *such that* $r < t \leq s$ *if and only if* $r = max_s L_r$ *and* $r = max_s R_r$.

Proof. If there does not exist any element $t \in J_r$ such that $r < t \leq s$, then since $L_r, R_r \subseteq J_r$ we easily conclude that $r = max_s L_r$ and $r = max_s R_r$. For the reverse direction, suppose that $r = max_s L_r$ and $r = max_s R_r$. Suppose that $t \in J_r$ so that $r \leq t \leq s$. Then, by Lemma 6.22 we can find an element $a \in J_r$ so that $r \mathscr{R} a \mathscr{L} t$ and $r \leq a \leq t \leq s$. But since $r = max_s R_r$ it follows that $r = a$, and hence $r \mathscr{L} t$. Then, since $r = max_s L_r$, we conclude that $t = r$. Thus, no such t can exist with $r < t$. □

Corollary 6.28. *If* $max_s J_r$ *exists, then* $r = max_s J_r$ *if and only if* $r = max_s L_r$ *and* $r = max_s R_r$.

Proof. If $max_s J_r$ exists, then it is the unique element in J_r such that there does not exist any element $t \in J_r$ such that $r < t \leq s$. So just apply the preceding proposition. □

As it turns out, we can write $max_s J_r$ (when it exists) in terms of a relative maximum of an $\mathscr{H}$-class.

Theorem 6.29. *Suppose that* $r, s \in \mathcal{R}$, *with* $r \leq s$. *Define* $z = (max_s L_{\lfloor r \rfloor^{\mathscr{J}}})_- w_0 (max_s R_{\lfloor r \rfloor^{\mathscr{J}}})_+$. *Then* $max_s J_r$ *exists if and only if* $\lfloor z \rfloor^{\mathscr{H}} \leq s$, *in which case* $max_s J_r = max_s H_z$

Just to make it explicit, note that the element z is always defined by our previous work with absolute minimums and relative maximums for $\mathscr{L}$-classes and $\mathscr{R}$-classes.

Proof. Suppose that $max_s J_r$ exists. By definition, $\lfloor z \rfloor^{\mathscr{H}} \leq s$ if and only if $\lfloor z \rfloor^{\mathscr{H}} \leq max_s J_r$. We know that $L_{\lfloor r \rfloor^{\mathscr{J}}}, R_{\lfloor r \rfloor^{\mathscr{J}}} \subseteq J_r$, so then $max_s L_{\lfloor r \rfloor^{\mathscr{J}}} \leq max_s J_r$ and $max_s R_{\lfloor r \rfloor^{\mathscr{J}}} \leq max_s J_r$. Furthermore, we can see that $(max_s L_{\lfloor r \rfloor^{\mathscr{J}}})_- \leq (max_s J_r)_-$ and $(max_s R_{\lfloor r \rfloor^{\mathscr{J}}})_+ \leq (max_s J_r)_+$. It follows by considering Proposition 6.16, that $\lfloor z \rfloor^{\mathscr{H}} \leq \lfloor max_s J_r \rfloor^{\mathscr{H}} \leq max_s J_r$.

Conversely, suppose that $\lfloor z \rfloor^{\mathscr{H}} \leq s$. It suffices to show that $max_s H_z$ (which we know exists) fits the definition of $max_s J_r$. Suppose that $t \in J_r$ is such that $t \leq s$. By Corollary 3.26 we can see that $t_- \mathscr{L} v = \lfloor r \rfloor^{\mathscr{J}}$ and $t_+ \mathscr{R} v = \lfloor r \rfloor^{\mathscr{J}}$. Thus $t_- \leq max_s L_{\lfloor r \rfloor^{\mathscr{J}}}$ and $t_+ \leq max_s R_{\lfloor r \rfloor^{\mathscr{J}}}$, and it follows that $t_- \leq (max_s L_{\lfloor r \rfloor^{\mathscr{J}}})_- = z_-$ and $t_+ \leq (max_s R_{\lfloor r \rfloor^{\mathscr{J}}})_+ = z_+$. Then we see that $\lfloor z \rfloor^{\mathscr{H}} \leq s$ implies $t_- \leq z_- \leq s$ and $t_+ \leq z_+ \leq s$. So, from Proposition 6.16, $\lfloor t \rfloor^{\mathscr{H}} \leq \lfloor z \rfloor^{\mathscr{H}}$.

Now, Lemma 6.22 tells us there is $a \in J_r$ so that $\lfloor t \rfloor^{\mathscr{H}} \leq a \leq \lfloor z \rfloor^{\mathscr{H}} \leq s$. and $a_+ = z_+$, $a_- = t_-$. Then, $a \in R_t$, and so $a \leq max_s R_t$. It follows that $z_+ = a_+ \leq (max_s R_t)_+ \leq (max_s R_{\lfloor r \rfloor^{\mathscr{J}}})_+ = z_+$, or rather, $(max_s R_t)_+ = z_+$. So $m := max_s R_t \mathscr{L} z$. Consider $max_s L_m = max_s L_z$. Since $\lfloor z \rfloor^{\mathscr{H}} \leq s$ we can see that $z_- \leq (max_s L_m)_-$. But once again, we can see that $(max_s L_m)_- \leq (max_s L_{\lfloor r \rfloor^{\mathscr{J}}})_- = z_-$. Thus, $max_s L_m \in H_z$. And so we have found that, such that $t \leq m \leq max_s L_m \leq max_s H_z$. Thus, $max_s J_r$ exists and is equal to $max_s H_z$ as desired. □

So why do we not always have a relative maximum for a $\mathscr{J}$-class? This last theorem gives us a hint. We know that $max_s J_r$ exists if and only if $\lfloor z \rfloor^{\mathscr{H}} \leq s$. Looking at the level of vanilla forms ($z = \sigma_-^{-1} e_- \sigma_0 e_+ \sigma_+^{-1}$, $s = \tau_-^{-1} f_- \tau_0 f_+ \tau_+^{-1}$), we see that $\lfloor z \rfloor^{\mathscr{H}} = \sigma_-^{-1} e_- \mu e_+ \sigma_+^{-1}$, where $\mu \in W$ is minimal so that $e_- \mu = \mu e_+$. Thus, by Theorem 5.41, $r \leq s$ if and only if we can find $w_- \in W_*(e_-)W^*(f_-)$ and $w_+ \in W^*(f_+)W_*(e_+)$ so that $w_- \tau_- \leq \sigma_-$, $\tau_+ w_+ \leq \sigma_+$ and $\mu \leq w_- \tau_0 w_+$. But since μ is the minimum element in $W(e_-)w_0 W(e_+)$ it follows that this is true if and only if we can find elements $w_- \in W_*(e_-)W^*(f_-)$ and $w_+ \in W^*(f_+)W_*(e_+)$ so that $w_- \tau_- \leq \sigma_-$, $\tau_+ w_+ \leq \sigma_+$ and $w_- \tau_0 w_+ \in W(e_-)w_0 W(e_+)$.

So it seems that with regards to the particular elements $\sigma_+ \in D(e_+)$, $\tau_+ \in D(f_+)$, $\sigma_- \in V(e_-)$, $\tau_- \in V(f_-)$ and $\tau_0 \in V_*(f_-) \cap W(f_-)w_0 W(f_+) \cap D_*(f_+)$, once the set,

$$\mathcal{A} = \left\{ \sigma_0 \in V_*(e_-) \cap W(e_-)w_0 W(e_+) \cap D_*(e_+) \,\middle|\, \begin{array}{c} \exists w_- \in W_*(e_-)W^*(f_-), \exists w_+ \in W^*(f_+)W_*(e_+) \\ w_- \tau_- \leq \sigma_-, \sigma_0 \leq w_- \tau_0 w_+, \tau_+ w_+ \leq \sigma_+ \end{array} \right\}$$

is nonempty, it has a maximum.

One wonders if there is a purely Coxeter group theoretic reason for these results. Something along the lines of the following statement, which would provide an analogue of the existence of $max_s H_r$.

Question 6.30. *Let $K_- \subseteq I_- \subseteq S$, $K_+ \subseteq I_+ \subseteq S$, $L_- \subseteq J_- \subseteq S$, $L_+ \subseteq J_+ \subseteq S$ be sets of simple reflections of the Weyl group such that $W_{I_*} = W_{K_*} \times W_{I_*\backslash K_*} = W_{I_*\backslash K_*} \times W_{K_*}$ and $W_{J_*} = W_{L_*} \times W_{J_*\backslash L_*} = W_{J_*\backslash L_*} \times W_{L_*}$ for all sets $* = +$ or $-$. Suppose also that $L_* \subseteq K_*$ and $I_*\backslash K_* \subseteq J_*\backslash L_*$ for all $* = +$ or $-$ and that $w_0 W_{H_+} w_0 = W_{H_-}$ for all $H = I, J, K, L, I\backslash K$ and $J\backslash L$. For elements, $\sigma_- \in {}^{I_-}W$, $\tau_- \in {}^{J_-}W$, $\sigma_+ \in W^{I_+}$, $\tau_+ \in W^{J_+}$, and $\tau_0 \in {}^{L_-}W \cap W_{J_-} w_0 W_{J_+} \cap W^{L_+}$, define the set,*

$$\mathcal{A} = \left\{ \sigma_0 \in {}^{K_-}W \cap (W_{I_-} w_0 W_{I_+}) \cap W^{K_+} \,\middle|\, \begin{array}{c} \exists w_- \in W_{K_-} W_{J_-\backslash L_-}, \exists w_+ \in W_{J_+\backslash L_+} W_{K_+} \text{ so that} \\ w_- \tau_- \leq \sigma_-, \sigma_0 \leq w_- \tau_0 w_+ \text{ and } \tau_+ w_+ \leq \sigma_+ \end{array} \right\}$$

Is it true that if $\mathcal{A} \neq \emptyset$, then $\mathcal{A}$ is a directed set (a preorder where every pair of elements has an upper bound) with regards to the Bruhat order, $\leq$?

We pose it as a directed set (which in finite cases is equivalent to saying there exists a maximum) so that one may ponder the result for all Coxeter groups, not just finite ones. We will leave this question for readers to consider, and move on to the relative minimal elements, which are equally abundant, but require more work to show existence.

6.3 Relative Minima

It would be nice if we could reproduce our relative maximal element results for relative minimal elements, and for the most part we can, but it requires a more subtle approach. The ease with which we proved the existence of relative maximal elements $max_s H_r$, $max_s L_r$ and $max_s R_r$ relied on the existence of a dense open subvariety that looked like BtB. Unfortunately for minimal elements there are not a lot of algebraic geometry results that talk about minimum orbits. This poses a problem.

What we can do is use results like Proposition 6.8 and Lemma 6.22 to slowly build up our results. First we will show that if $r\mathscr{L}s$ or $r\mathscr{R}s$ then $min_r H_s$ exists. From there we will see that if $r\mathscr{J}s$ then $min_r L_s$ and $min_r R_s$ exist. And so on.

Definition 6.31. *For $r, s \in \mathcal{R}$ and $r \leq s$, we define the* **relative minimum** *of T_s with respect to r, as*

$$min_r T_s = \begin{cases} t & \text{if } t \in T_s,\ r \leq t \text{ and } \forall t' \in T_s\ r \leq t' \implies t \leq t' \\ \text{undefined} & \text{otherwise} \end{cases}$$

Remark 6.32. *We can recover our work with (absolute) minimums, if it happens that $\mathcal{R}$ has a unique minimal element. For instance, if $0 \in \mathcal{R}$ then $\lfloor r \rfloor^{\mathscr{T}} = min_0 T_r$.*

Proposition 6.33. *For $r, s, t \in \mathcal{R}$ with $r \leq t \leq s$, then (if they exist), $min_r T_s \leq min_t T_s$.*

Proof. By definition we know, $min_t T_s \in T_s$ and $t \leq min_t T_s$. But since, $r \leq t$ it follows that $r \leq min_t T_s$, and so by definition of the relative minimum, $min_r T_s \leq min_t T_s$. □

We now begin to show that these $min_r T_s$ exist.

Theorem 6.34. *Suppose that $r, s \in \mathcal{R}$ with $r \leq s$ and either $r\mathscr{L}s$ or $r\mathscr{R}s$. Then $min_r H_s$ exists.*

Proof. Our proof depends on which condition, $r\mathscr{L}s$ or $r\mathscr{R}s$, is satisfied. We will prove the result assuming that $r\mathscr{L}s$ and, as usual, the case for $r\mathscr{R}s$ is proven similarly. First, let us distinguish idempotents, $e, f \in E(\mathcal{R})$, with $e\mathscr{R}s\mathscr{L}f$. Observe that $s \in H'_s$. Also note that $r\mathscr{L}s$, so $r \leq s$ if and only if $B^- sB \subseteq \overline{B^- rB}$. Thus, $s = esf \in eB^- sBf \subseteq e\overline{B^- rB}f \subseteq \overline{eB^- rBf}$. So we see that $s \in \overline{eB^- rBf} \cap H'_s$, hence $\overline{eB^- rBf} \cap H'_s \neq \emptyset$.

Since $\overline{eB^- rBf} \cap H'_s \neq \emptyset$, we know that it is open and dense in $\overline{eB^- rBf}$ (because H'_s is open and dense in eMf, by Lemma 6.23). $H'_s \subseteq B^- H_s B$ by Proposition 6.11, so it follows that $\overline{eB^- rBf} \cap H'_s \subseteq \overline{eB^- rBf} \cap B^- H_s B \subseteq \overline{eB^- rBf}$, and hence $\overline{eB^- rBf} \cap B^- H_s B$ is a dense subvariety of $\overline{eB^- rBf}$.

$B^- rB$ is the orbit of r under the group action of $B^- \times B$, and so is irreducible. Then, $eB^- rBf$ is also irreducible, as it is the image of $B^- rB$ under multiplication by e on the left and f on the right. It follows that $\overline{eB^- rBf}$ is also irreducible, as the closure of an irreducible is irreducible. Since $\overline{eB^- rBf} \cap B^- H_s B$ is a dense subvariety of $\overline{eB^- rBf}$, it must be an irreducible variety too.

Now, $\overline{eB^- rBf} \cap B^- H_s B = \bigsqcup_{t\mathscr{H}s} \overline{eB^- rBf} \cap B^- tB$, a finite disjoint union of subvarieties. So by applying Theorem A.3 we can find a unique $\bar{t}\mathscr{H}s$ so that $\overline{eB^- rBf} \cap B^- \bar{t}B$ is dense in $\overline{eB^- rBf} \cap B^- H_s B$ (thus dense in $\overline{eB^- rBf}$). Then, $\overline{eB^- rBf} \subseteq \overline{B^- \bar{t}B}$, so for any $t \in H_s$, if $B^- tB \cap \overline{eB^- rBf} \neq \emptyset$, then $B^- tB \cap \overline{B^- \bar{t}B} \neq \emptyset$, and we conclude that $B^- tB \subseteq \overline{B^- \bar{t}B}$, or rather $\bar{t} \leq t$ (by Corollary 6.10).

Suppose that $t\mathscr{H}s$, then since $t\mathscr{L}r$, $r \leq t$ is equivalent to $B^{-}tB \subseteq \overline{B^{-}rB}$, by Corollary 6.10. If $r \leq t$, then $t \in \overline{B^{-}rB}$, so $t = etf \in e\overline{B^{-}rB}f \subseteq \overline{eB^{-}rBf}$, by continuity of multiplication. So, $r \leq t$ implies $\overline{eB^{-}rBf} \cap B^{-}tB \neq \emptyset$, hence $\bar{t} \leq t$.

To conclude that $\bar{t} = min_r H_s$, it remains to show that $r \leq \bar{t}$. By our choice of $\bar{t}$ it is clear that $\overline{eB^{-}rBf} \cap B^{-}\bar{t}B \neq \emptyset$. Now, since $e, f \in \overline{T}$, and $TB^{-}rBT \subseteq B^{-}rB$, we see that $\overline{eB^{-}rBf} \subseteq \overline{B^{-}rB}$. So then $\overline{B^{-}rB} \cap B^{-}\bar{t}B \neq \emptyset$, and hence $B^{-}\bar{t}B \subseteq \overline{B^{-}rB}$. But since $r\mathscr{L}\bar{t}$, we conclude that $r \leq \bar{t}$. □

Proposition 6.35. *Suppose that $r, s \in \mathcal{R}$ with $r \leq s$ and either $r \mathscr{J} s$. Then $min_r L_s$ and $min_r R_s$ exists, and we can express them as, $min_r L_s = min_r H_{r_- w_0 s_+}$ and $min_r R_s = min_r H_{s_- w_0 r_+}$.*

Proof. We will just demonstrate this for $min_r R_s$. Since we know that $min_r H_{s_- w_0 r_+}$ exists (by Theorem 6.34), it suffices just to prove that $min_r R_s = min_r H_{s_- w_0 r_+}$. It is clear that $H_{s_- w_0 r_+} \subseteq R_s$, so $min_r H_{s_- w_0 r_+} \in R_s$. Now, for any $t \in R_s$ such that $r \leq t$, by Lemma 6.22 we can find $a \in H_{s_- w_0 r_+}$ so that $r \leq a \leq t$. Thus, $min_r H_{s_- w_0 r_+} \leq t$. So it follows that $min_r R_s = min_r H_{s_- w_0 r_+}$. □

Proposition 6.36. *Suppose that $r, s \in \mathcal{R}$ with $r \leq s$ and $r \mathscr{J} s$. Then $min_r H_s$ exists.*

Proof. Since $r \mathscr{J} s$, by Proposition 6.35, $min_r L_s$ exists. We claim $min_r H_s = min_{min_r L_s} H_s$. We need only prove the equality, as the right hand side exists by Theorem 6.34. $H_s \subseteq L_s$, so by definition, if $t \in H_s$, then $r \leq t$ if and only if $min_r L_s \leq t$. □

Our next result comes to us from [18] by Mohan Putcha. In his paper he describes an order preserving projection map from WeW to WfW for each pair $e \leq f \in \Lambda$. This map turns out to exactly fit the definition of relative minimum for a $\mathscr{J}$-class.

Theorem 6.37. *For any $r, s \in \mathcal{R}$, if $r \leq s$ then $min_r J_s$ exists.*

Proof. This phenomenal result can be found in [18] by Putcha. For informational purposes, we will state here the method to determine $min_r J_s$ from r and s. First put r and s into standard form, $r = xey^{-1}$ and $s = ufv^{-1}$. Let z represent the maximum element of $W_*(e)$ (which exists, as it is a Coxeter group). Then $zy^{-1} \in W = \bigsqcup_{b \in D(f)} W(f)b^{-1}$ so we can find $a \in W(f)$ and some $b \in D(f)$ so that $zy^{-1} = ab^{-1}$. Then $min_r J_s = (xc)fb^{-1}$ where $c \in W$ is chosen such that $c \leq a$ and xc is minimal. □

One can expect that a similar method using opposite standard form exists. Now that we have $min_r J_s$, we can finally show the existence of the other relative minima.

Proposition 6.38. *For any $r, s \in \mathcal{R}$ and any $\mathscr{T} = \mathscr{H}, \mathscr{L}$, or $\mathscr{R}$, if $r \leq s$ then $min_r T_s$ exists, and we can express it as, $min_r T_s = min_{min_r J_s} T_s$*

Proof. First, note that by definition, $r \leq s$ tells us that $min_r J_s \leq s$. So then the expression $min_{min_r J_s} T_s$ makes sense. Second, observe that to prove existence, we need only prove the expression $min_r T_s = min_{min_r J_s} T_s$, as our previous work tells us that the right hand side exists. $T_s \subseteq J_s$, so by definition, if $t \in T_s$, then $r \leq t$ if and only if $min_r J_s \leq t$. This concludes the result. □

The following corollary will be of more use when one looks at a specific monoid, like we will in the Rook monoid momentarily.

Corollary 6.39. *For any $r, s \in \mathcal{R}$, if $r \leq s$ then $min_r J_s = min_r L_{min_r R_s} = min_r R_{min_r L_s}$*

Proof. We will just prove the first equality, as the second is done similarly. It is clear that the elements, $min_r J_s$ and $min_r L_{min_r R_s}$ exist, so we need only establish that they are equal.

It is clear that $r \leq min_r R_s \leq s$ and so $r \leq min_r L_{min_r R_s} \leq min_r R_s \leq s$. It is also true that $min_r L_{min_r R_s} \mathscr{L} min_r R_s \mathscr{R} s$, so $min_r L_{min_r R_s} \mathscr{J} s$. By definition $min_r J_s \leq min_r L_{min_r R_s}$.

By Lemma 6.22 we know that there exists $min_r J_s \leq z \leq s$ with $z \mathscr{R} s$ and $z \mathscr{L} min_r J_s$. Then $min_r R_s \leq z \leq s$. Thus, $(min_r R_s)_+ \leq z_+$. Now, $z \mathscr{L} min_r J_s$ so $min_r L_z = min_r J_s \leq min_r L_{min_r R_s}$. Thus, $z_+ = (min_r L_z)_+ \leq (min_r L_{min_r R_s})_+ = (min_r R_s)_+$. It follows, as desired, that $z \mathscr{L} min_r R_s$ and furthermore, $min_r J_s = min_r L_z = min_r L_{min_r R_s}$. □

One can make a similar statement about $max_s J_r = max_s L_{max_s R_r} = max_s R_{max_s L_{sr}}$ provided that it exists. A proof would proceed like the one above for Corollary 6.39.

Before moving on to the examples of this section, we will conclude with a useful theorem revealing some of the structure of the Adherence order which is achieved with our absolute and relative maxima and minima.

Theorem 6.40. *Let $\mathscr{T} = \mathscr{H}, \mathscr{R}, \mathscr{L}, \mathscr{J}$*

(1) If $r_0 < r_1 < \cdots < r_{k-1} < r_k$ is a chain of elements in $\mathcal{R}$, and $r_0 \mathscr{T} r_k$, then for all indices,

$0 \leq i \leq k$ we have $r_0 \mathscr{T} r_i$.

(2) Suppose $r_0 < r_1 < \cdots < r_{k-1} < r_k$ is a chain of maximum length between $r_0, r_k \in \mathcal{R}$.

For all $1 \leq i \leq k$, if $r_i \not\mathscr{T} r_{i-1}$ then $r_i = min_{r_{i-1}} T_{r_i}$ and if $\mathscr{T} \neq \mathscr{J}$, $r_{i-1} = max_{r_i} T_{r_{i-1}}$.

Proof. (1) By Proposition 6.14, we can see that our given chain, $r_0 < r_1 < \cdots < r_{k-1} < r_k$, implies that $\lfloor r_0 \overset{\mathscr{T}}{\rfloor} \leq \lfloor r_1 \overset{\mathscr{T}}{\rfloor} \leq \cdots \lfloor r_{k-1} \overset{\mathscr{T}}{\rfloor} \leq \lfloor r_k \overset{\mathscr{T}}{\rfloor}$. But $r_0 \mathscr{T} r_k$ means that $\lfloor r_0 \overset{\mathscr{T}}{\rfloor} = \lfloor r_k \overset{\mathscr{T}}{\rfloor}$ and hence for any i, $\lfloor r_0 \overset{\mathscr{T}}{\rfloor} = \lfloor r_i \overset{\mathscr{T}}{\rfloor}$, thus $r_0 \mathscr{T} r_i$.

(2) We shall just show the minimum condition, as the maximum condition follows similarly. Suppose not. Then we see that $r_{i-1} < min_{r_{i-1}} T_{r_i} < r_i$, which contradicts the maximality of the length of the chain. □

6.4 Example

Although we covered the maximum and minimum elements of $\mathscr{T}$-classes first, Remarks 6.20 and 6.32 suggests that we should take a look at the relative maximums and minimums first, as computing them covers the computation of our absolute extremes. Let us take a look at finding a $min_r R_s$. We know from our example in Section 1 that $r < s$ for $r = \begin{pmatrix} 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 1&0&0&0&0&0 \end{pmatrix}$ and $s = \begin{pmatrix} 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \end{pmatrix}$. Now let us find $min_r R_s$. The elements of R_s are obtained by permuting the columns of s.

To compute $min_r R_s$ we begin by looking at each column of r, starting from the left column and moving right, and analysing them in turn. For a specific column of r, pick the column of s that most closely resembles it. If there is not an exact match, we choose the column of s whose nonzero element is as close to the nonzero entry of the column of r from the bottom.

For our specific example, we first look at $\begin{pmatrix} 0\\0\\0\\0\\0\\1 \end{pmatrix}$, the leftmost column of r. One of the columns of s matches this column perfectly, so it becomes the leftmost column of $min_r R_s$. Then, we move right and look at the $\begin{pmatrix} 0\\0\\1\\0\\0\\0 \end{pmatrix}$ column of r. There is no column of s that exactly matches this, so we will have to get as close as we can. Ultimately we chose the $\begin{pmatrix} 0\\0\\0\\1\\0\\0 \end{pmatrix}$ column of s as it is the closest without having its nonzero entry closer to the top than the given column of r. Next we move onto the third column of r, $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. The zero column of s matches this perfectly, and becomes the third column of $min_r R_s$. We proceed on in this way until we finish up with the rightmost column of r. This gives us $min_r R_s$ written below.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As a note, when we have a zero column of r, but no zero columns of s left to match, we choose the column of s that has its nonzero entry as close to the top as possible.

For the relative minimum element of an $\mathscr{L}$-class, we proceed in a similar manner, but instead look at each row of r in turn, starting from the bottom and moving up. We try to match that row of r with a row of s whose nonzero entry is as closefrom the left side without going over. Following these instructions we would get $min_r L_s$ as written below.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As one might suspect, we can derive similar rules for finding $max_s R_r$ and $max_s L_r$. For $max_s R_r$, we look at the columns of s, starting with the leftmost, but this time try to match the column with a column of r as closely as possible, with our nonzero entry being a close from the top as we can. $max_s L_r$ is found by looking at the rows of s, starting from the bottom, and approximating each row's nonzero entry as close as possible from the right without going over.

To compute $min_r J_s$, we perform the operation to find $min_r R_s$ and then perform $min_r L_{min_r R_s}$. By Corollary 6.39 this works out to $min_r J_s$.

We can summarize these methods in the illustrated manner below. This also lends some insight as to why $min_r J_s$ exists, but not $max_s J_r$. The rules for computing $min_r R_s$ and $min_r L_s$ are "compatible" in the sense that they both work together to move the nonzero entries into the top right corner. The rules for $max_s R_r$ and $max_s L_r$ contradict (illustrated as creating a spiral) resulting in conflicting goals and no overall pattern. It matters in what order the approximations of the rows and columns occurs.

$$min_r R_s \circ \rightarrow \quad \uparrow 1 \qquad\qquad max_s R_r \circ \rightarrow \quad \downarrow 1$$

$$min_r L_s \circ \uparrow \quad \rightarrow 1 \qquad\qquad max_s L_r \circ \uparrow \quad \leftarrow 1$$

Here $\circ \rightarrow$ *means begin at the left and move right while looking at the columns,* $\uparrow 1$ *means that in each column try to match the nonzero entry from below.*

While this gives us a vague idea of why $max_s J_r$ might not always exist in $M_n(K)$ it does not give us a general reason that can be applied to all reductive monoids. Let us finish our discussion of relative minima by computing $min_r H_s$.

Computing $min_r H_s$ is the same as computing $min_r L_s$ and then computing $min_{min_r L_s} R_s$ (by Proposition 6.36). Performing either method will give us $min_r H_s$,

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From relative minima for an $\mathscr{H}$-class, we now shift to calculating an absolute maxima. We have calculated r_- and r_+ in Section 3, so we will just calculate $\overset{\mathscr{H}}{\lceil r \rceil}$. To do this, isolate the unique largest invertible submatrix (just as we did for computing r_0). In the case of r we get an element of the Weyl group on $M_4(K)$. Replace this submatrix with the longest element of that same Weyl group (the element whose entries lie wholly on the anti-diagonal). The resulting 6×6 matrix will be the maximum in H_r,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Relating back to Section 4, since $B\overset{\mathscr{H}}{\lceil r \rceil}B$ is dense in BH_rB we can do things like use our length function to compute the dimension of BH_rB.

7 Parabolic Green's Relations

Owing to the use of Coxeter groups and the properties of our simple reflections, S, in Section 5, we elect to further investigate the impact of the simple reflections on the Renner monoid. This section is devoted to exploring a new set of equivalence relations on $\mathcal{R}$. These relations will be based on Green's relations, but will use W_I, rather than W, in their definitions.

Using these new equivalence relations we will extend many of our results from the last four sections, as well as extend a well-known result about the Weyl group (Theorem 7.53). In particular we will show a generalisation of the trichotomy.

7.1 A Series Of Equivalence Relations

We define the parabolic Green's relations similar to Green's relations on the Renner monoid. Instead of W, we will use the standard parabolic subgroups, W_I, in the definitions.

Definition 7.1. *For $I, J \subseteq S$, we define the following equivalence relations on $\mathcal{R}$, $\mathscr{J}^{I,J}$, $\mathscr{R}^J$, $\mathscr{L}^I$ and $\mathscr{H}^{I,J}$ by the following conditions. For $r, s \in \mathcal{R}$,*

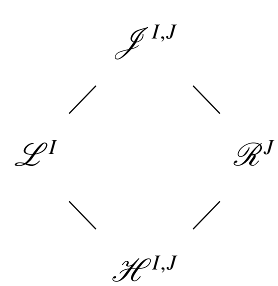
(1) $r\mathscr{L}^I s$ if and only if there exists $w \in W_I$ so that $s = wr$

(2) $r\mathscr{R}^J s$ if and only if there exists $w \in W_J$ so that $s = rw$

(3) $r\mathscr{J}^{I,J} s$ if and only if there exists $t \in \mathcal{R}$ so that $r\mathscr{L}^I t$ and $t\mathscr{R}^J s$

(4) $r\mathscr{H}^{I,J} s$ if and only if $r\mathscr{L}^I s$ and $r\mathscr{R}^J s$

One should note that we can recover our familiar Green's relations by taking $I = J = S$. The reason we wish to investigate these new relations is that they allow us to bridge the gap between the fat $\mathscr{T}$-classes, BT_rB, we have encountered before, and the usual Bruhat cells, BrB for $r \in \mathcal{R}$. We can see this by noting that “ $=$”$= \mathscr{J}^{\emptyset,\emptyset} = \mathscr{L}^{\emptyset} = \mathscr{R}^{\emptyset} = \mathscr{H}^{\emptyset,\emptyset}$. We can also note the familiar implication relationship,



Our first few results will give us properties that we are familiar to using with Green's relations, as well as some immediate and somewhat obvious results.

Proposition 7.2. *For $r, s \in \mathcal{R}$, $r \mathscr{J}^{I,J} s$ if and only if there exists $u \in W_I$ and $v \in W_J$ so that $s = urv$.*

Proof. By definition, if $r \mathscr{J}^{I,J} s$ there is an element, $t \in \mathcal{R}$ so that $r \mathscr{L}^I t$ and $t \mathscr{R}^J s$. Then there exists $u \in W_I$ so that $ur = t$ and $v \in W_J$ so that $tv = s$. Combining these, we get $s = urv$. For the converse, if $s = urv$ then $r \mathscr{L}^I ur$ and $ur \mathscr{R}^J s$. □

Corollary 7.3. *$r \mathscr{J}^{I,J} s$ if and only if there exists $t \in \mathcal{R}$ so that $r \mathscr{R}^J t$ and $t \mathscr{L}^I s$.*

Proof. $r \mathscr{J}^{I,J} s$ if and only if we can find $u \in W_I$ and $v \in W_J$ so that $s = urv$. Then $r \mathscr{R}^J rv$ and $rv \mathscr{L}^J s$. For the converse, if there is $t \in \mathcal{R}$ so that $r \mathscr{R}^J t$ and $t \mathscr{L}^I s$ then we can find $v \in W_J$ so that $t = rv$ and $u \in W_I$ so $s = ut$. Combining them gives us $s = urv$. □

Proposition 7.4. *For $I \subseteq K \subseteq S$ and $J \subseteq L \subseteq S$, then for any $r, s \in \mathcal{R}$, $r \mathscr{J}^{I,J} s \implies r \mathscr{J}^{K,L} s$.*

Proof. If $r \mathscr{L}^I s$ then we can find $w \in W_I$ so that $s = we$. But $I \subseteq J$ implies $W_I \subseteq W_J$ so $w \in W_J$. Thus we conclude that $r \mathscr{L}^J s$. A similar proof gives the other part of this proposition. □

Remark 7.5. *Notice that $\mathscr{L}^I = \mathscr{J}^{I,\emptyset}$ and $\mathscr{R}^J = \mathscr{J}^{\emptyset,J}$. So all our structural questions about Green's relations become questions about the $\mathscr{J}^{I,J}$s and their intersections. We will leave it to other mathematicians to investigate the general $\bigcap_{i=1}^n \mathscr{J}^{I_i,J_i}$, and just focus on $\mathscr{H}^{I,J}$.*

The preceding remark gives us insight to many of the similarities we have seen between $\mathscr{J}$-, $\mathscr{L}$-, and $\mathscr{R}$-classes. With this remark in hand, we get the following corollary.

Corollary 7.6. *If $I \subseteq K \subseteq S$, then for any $r, s \in \mathcal{R}$, $r \mathscr{L}^I s \implies r \mathscr{L}^K s$ and $r \mathscr{R}^I s \implies r \mathscr{R}^K s$.*

Proof. Taking $J = L = \emptyset$, Proposition 7.4 and Remark 7.5 tell us that,

$$r \mathscr{L}^I s \Leftrightarrow r \mathscr{J}^{I,J} s \Rightarrow r \mathscr{J}^{K,L} s \Leftrightarrow r \mathscr{L}^K s.$$

The $\mathscr{R}$ situation is done similarly. □

Our remark also allows us to quickly prove the following result analogous to our descriptions of fat $\mathscr{L}$-, $\mathscr{R}$-, and $\mathscr{J}$-classes.

Proposition 7.7. *For $r \in \mathcal{R}$ and $I, J \subseteq S$, $BJ_r^{I,J}B = P_I r P_J$.*

Proof. Expanding Proposition 7.2, we see $BJ_r^{I,J}B = BW_I r W_J B \subseteq BW_I BrBW_J B = P_I r P_J$. So it remains to show the other inclusion.

The result will be shown if we can demonstrate that $BJ_r^{I,J}B$ is closed under multiplication on the right by P_J and closed under multiplication on the left by P_I. We will just show that $P_I BJ_r^{I,J}B \subseteq BJ_r^{I,J}B$.

The Bruhat decomposition for our parabolic subgroups of G tells us $P_I = \bigsqcup_{w \in W_I} BwB$. So it suffices to show that $(BwB)(BJ_r^{I,J}B) \subseteq BJ_r^{I,J}B$ for all $w \in W_I$. Write $w = vs$ for $s \in I$ and $v \in W_I$ such that $\ell(w) = \ell(v) + 1$. Now,

$$\begin{aligned}(BwB)(BJ_r^{I,J}B) &= BwB(\textstyle\bigsqcup_{t \in J_r^{I,J}} BtB) \subseteq BvBsB(\bigsqcup_{t \in J_r^{I,J}} BtB) \\ &= BvB(\textstyle\bigsqcup_{t \in J_r^{I,J}} sBtB) \subseteq BvB(\bigsqcup_{t \in J_r^{I,J}} (BtB \cup BstB)B) && \text{since } sBt \subseteq BtB \cup BstB \\ &= BvB(\textstyle\bigsqcup_{t \in J_r^{I,J}} BtB \cup BstB) \subseteq BvB(BJ_r^{I,J}B) && \text{since } t \in J_r^{I,J} \implies st \in J_r^{I,J}. \\ &\subseteq BJ_r^{I,J}B && \text{by induction on } \ell(w).\end{aligned}$$

Our proof is completed upon the statement of our base case, $\ell(w) = 0 \implies w = 1$, and we can clearly see that, $B(BJ_r^{I,J}B) = BJ_r^{I,J}B$. □

Corollary 7.8. *For $r \in \mathcal{R}$ and $I, J \subseteq S$, $BJ_r^{I,J}B$ is an irreducible subvariety of M.*

Proof. By Proposition 7.7, $BJ_r^{I,J}B = P_I r P_J$, which is the orbit of r by the action of $P_I \times P_J$ on M given by $((p, q), m) \mapsto pmq^{-1}$. Thus it is an irreducible subvariety of M, since P_I, P_J are irreducible algebraic groups. □

Corollary 7.9. *For $r \in \mathcal{R}$ and $I, J \subseteq S$, there exists a unique element $s \in J_r^{I,J}$ so that BsB is open and dense in $BJ_r^{I,J}B$.*

Proof. Since $BJ_r^{I,J}B$ is an irreducible variety which is closed under the action of $B \times B$, Theorem A.3 tells us that exactly one of the disjoint orbits, BsB, $s \in J_r^{I,J}$ is open and dense in $BJ_r^{I,J}B$. □

This foreshadows our work with absolute maxima for the parabolic Green's relations. But before we can discuss them, we will need to introduce the analogues of our familiar $\mathcal{N}$, $\mathcal{GJ}$, and $\mathcal{JG}$.

Definition 7.10. *Take any $I, J \subseteq S$ and define the following sets,* *(1)* $\mathcal{GJ}^I = \bigsqcup_{e\in\Lambda} \left({}^I W^{\lambda_*(e)}\right) \cdot e \cdot \left({}^{\lambda(e)}W\right)$

(2) $\mathcal{JG}^J = \bigsqcup_{e\in\Lambda^-} \left(W^{\lambda(e)}\right) \cdot e \cdot \left({}^{\lambda_*(e)}W^J\right)$

(3) $\mathcal{N}^{I,J} = \mathcal{GJ}^I \cap \mathcal{JG}^J$

Remark 7.11. *Notice that for (1), if $I = \emptyset$ hen $\mathcal{GJ}^\emptyset$ is just the union of all element in standard form. Hence $\mathcal{GJ}^\emptyset = \mathcal{R}$. Likewise $\mathcal{JG}^\emptyset = \mathcal{R}$. It then follows that $\mathcal{N}^{I,\emptyset} = \mathcal{GJ}^I$ and $\mathcal{N}^{\emptyset,J} = \mathcal{JG}^J$ as one would desire considering Remark 7.5.*

We have begun our definitions with these $\mathcal{GJ}^I$ and $\mathcal{JG}^J$ because they allow us to quickly conclude the following analogue to Theorem 3.9.

Theorem 7.12. *For any $I \subseteq S$,*

(1) $\mathcal{GJ}^I \cong \mathcal{R}/\mathscr{L}^I$. That is to say, if $r, s \in \mathcal{R}$, $r\mathscr{L}^I s$ and $r, s \in \mathcal{GJ}^I$, then $r = s$, and for all $r \in \mathcal{R}$, there is $s \in \mathcal{GJ}^I$ with $r\mathscr{L}^I s$.

(2) $\mathcal{JG}^I \cong \mathcal{R}/\mathscr{R}^I$. That is to say, if $r, s \in \mathcal{R}$, $r\mathscr{R}^I s$ and $r, s \in \mathcal{JG}^I$, then $r = s$, and for all $r \in \mathcal{R}$, there is $s \in \mathcal{JG}^I$ with $r\mathscr{R}^I s$.

Proof. Due to similarity, we will just prove (1). Suppose that $r = ue\sigma^{-1}$, $s = vf\tau^{-1}$ are elements of $\mathcal{GJ}^I$ and are in standard form. If $r\mathscr{L}^I s$ then $r\mathscr{L} s$ and so $e = f$, $\sigma = \tau$. It is clear from the definition of $\mathcal{GJ}^I$ that $u, v \in {}^I W$. But since $r\mathscr{L}^I s$ there exists $w \in W_I$ so that $s = wr$, or rather $ve\sigma^{-1} = wue\sigma^{-1}$. Thus, $v^{-1}wu \in W_*(e)$, and $wu \in vW_*(e)$. Thus $W_I uW_*(e) = W_I vW_*(e)$. But since u and v are both minimal in their double cosets it follows that $u = v$.

Now, let $r \in \mathcal{R}$ with standard form xey^{-1}. Let $u = {}^I x$. Consider $s = uey^{-1}$. It is clear that $r\mathscr{L}^I s$. And since $u = {}^I x$ it follows from a reduced word argument that $u \in D_*(e)$. By Proposition 5.11 we can conclude that $u \in {}^I W^{\lambda_*(e)}$. So $s \in \mathcal{GJ}^I$. □

Definition 7.13. *The unique dense element of Corollary 7.9 will be denote by $\lceil r \rceil^{\mathscr{J}^{I,J}}$ and refer to it as the* **absolute maximum** *of $J_r^{I,J}$. In a similar fashion, we denote by $\lfloor r \rfloor^{\mathscr{J}^{I,J}}$ the unique minimal element of $J_r^{I,J}$ if it exists. Such an element is called an* **absolute minimum**.

In the special cases where $I = \emptyset$ or $J = \emptyset$, we may also choose to use $\mathscr{R}^J$ and $\mathscr{L}^I$ respectively.

Theorem 7.14. *Let $I, J \subseteq S$. For any $r \in \mathcal{R}$,* *(1)* $r = \lfloor r \rfloor^{\mathscr{L}^I}$ *if and only if* $r \in \mathcal{GJ}^I$*.*

(2) $r = \lfloor r \rfloor^{\mathscr{R}^J}$ *if and only if* $r \in \mathcal{JG}^J$*.*

Proof. (1) Since $\mathcal{GJ}^I \cong \mathcal{R}/\mathscr{L}^I$ it suffices to show that $r \in \mathcal{GJ}^I$ implies $r = \lfloor r \rfloor^{\mathscr{L}^I}$. Indeed, let $r = xey^{-1} \in \mathcal{GJ}^I$ in standard form. Consider $ur = uxey^{-1}$ with $u \in W_I$. Then $ur\mathscr{L}^I r$ and $(ux)ey^{-1}$ is in standard form. Thus $r \leq ur$ if and only if $x \leq ux$. But by definition of $\mathcal{GJ}^I$, $x \in {}^I W$ and so $x \leq ux$, and the result follows.

(2) is done similarly to (1). □

Proposition 7.15. *For any* $r, s \in \mathcal{R}$*,* $I \subseteq S$*,*

(1) $r \leq s$ *implies* $\lfloor r \rfloor^{\mathscr{L}^I} \leq \lfloor s \rfloor^{\mathscr{L}^I}$

(2) $r \leq s$ *implies* $\lfloor r \rfloor^{\mathscr{R}^I} \leq \lfloor s \rfloor^{\mathscr{R}^I}$

Proof. Per the norm, we will just show (1). Let $r = xe_+\sigma_+^{-1}$ be the standard form of r, and $s = yf_+\tau_+^{-1}$ the standard form of s. We know that $r \leq s$ if and only if there is $w_+ \in W^*(f_+)W_*(e_+)$ so that $x \leq yw_+$ and $\tau_+ w_+ \leq \sigma_+$. But then ${}^I x \leq yw_+$. We can write $y = u\,{}^I y$ for some $u \in W_I$. It follows, since ${}^I x \in {}^I W$, that ${}^I x \leq {}^I y w_+$ and hence $\lfloor r \rfloor^{\mathscr{L}^I} = ({}^I x)e_+\sigma_+^{-1} \leq ({}^I y)f_+\tau_+^{-1} = \lfloor s \rfloor^{\mathscr{L}^I}$. □

Corollary 7.16. *Let* $I, J \subseteq S$*.*

(1) For any $r \in \mathcal{R}$*,* $r = \lfloor r \rfloor^{\mathscr{J}^{I,J}}$ *if and only if* $r \in \mathcal{N}^{I,J}$*.*

(2) $\mathcal{N}^{I,J} \cong \mathcal{R}/\mathscr{J}^{I,J}$*. That is to say, if* $r, s \in \mathcal{R}$*,* $r\mathscr{J}^{IJ}s$ *and* $r, s \in \mathcal{N}^{I,J}$*, then* $r = s$*, and for all* $r \in \mathcal{R}$*, there is* $s \in \mathcal{N}^{I,J}$ *with* $r\mathscr{J}^{I,J}s$*.*

Proof. We shall prove both results together. Suppose $r = \lfloor r \rfloor^{\mathscr{J}^{I,J}}$. Then $r = \lfloor r \rfloor^{\mathscr{L}^I} = \lfloor r \rfloor^{\mathscr{R}^J}$, since $\mathscr{L}^I, \mathscr{R}^J \Rightarrow \mathscr{J}^{I,J}$. By Theorem 7.14, $r \in \mathcal{GJ}^I \cap \mathcal{JG}^J = \mathcal{N}^{I,J}$. Thus, if $\lfloor r \rfloor^{\mathscr{J}^{I,J}}$ exists, it is an element of $\mathcal{N}^{I,J}$. Take any $r \in \mathcal{R}$. Let $s = \lfloor \lfloor r \rfloor^{\mathscr{L}^I} \rfloor^{\mathscr{R}^J} \in \mathcal{JG}^J$. Consider any $t \in J_r^{I,J}$. Then we know we can find $u \in \mathcal{R}$ so that $\lfloor r \rfloor^{\mathscr{L}^I} \mathscr{L}^I u \mathscr{R}^J t$. It follows by Theorem 7.14 that $\lfloor r \rfloor^{\mathscr{L}^I} \leq u$ and thus $s \leq u$. But then $s = \lfloor s \rfloor^{\mathscr{R}^J} \leq \lfloor u \rfloor^{\mathscr{R}^J} = \lfloor t \rfloor^{\mathscr{R}^J} \leq t$. Thus $s = \lfloor r \rfloor^{\mathscr{J}^{I,J}}$ and is a member of $\mathcal{JG}^J$. A similar argument shows that $s \in \mathcal{GJ}^I$ as well, hence $s \in \mathcal{N}^{I,J}$

So we have shown the minimum elements exist for each $\mathscr{J}^{I,J}$-class, and they must belong to $\mathcal{N}^{I,J}$. It suffices to show for $r, s \in \mathcal{N}^{I,J}$, if $r\mathscr{J}^{I,J}s$ then $r = s$. Suppose the conditions are satisfied. Then, $r = \lfloor r \rfloor^{\mathscr{L}^I} = \lfloor \lfloor r \rfloor^{\mathscr{L}^I} \rfloor^{\mathscr{R}^J} = \lfloor r \rfloor^{\mathscr{J}^{I,J}} = \lfloor s \rfloor^{\mathscr{J}^{I,J}} = \lfloor \lfloor s \rfloor^{\mathscr{L}^I} \rfloor^{\mathscr{R}^J} = \lfloor s \rfloor^{\mathscr{L}^I} = s$, since $\mathcal{N}^{I,J} \subseteq \mathcal{GJ}^I \cap \mathcal{JG}^J$. □

Knowing that $\mathcal{N}^{I,J}$ consists of exactly the minimum elements of $\mathscr{J}^{I,J}$-classes with respect to the Adherence order, we see that this is exactly the analogue of the more familiar set ${}^{I}W^{J}$. One might even choose the alternate notation, ${}^{I}\mathcal{R}^{J}$! As in the preceding section, the nature of the minimum elements leads us to describing the maximum elements.

Proposition 7.17. *Let $I, J \subseteq S$, $r, s \in \mathcal{R}$, and suppose that $r \leq s$,*

(1) if $r\mathscr{L}^{I}s$, then $w_0(I)s \leq w_0(I)r$

(2) if $r\mathscr{R}^{J}s$, then $sw_0(J) \leq rw_0(J)$

(3) if $r\mathscr{H}^{I,J}s$, then $w_0(I)rw_0(J) \leq w_0(I)sw_0(J)$

Proof. (3) clearly follows from applying (1) and (2). (2) is proven similarly to (1), so we will just prove (1). Write r and s in standard form, $r = xey^{-1}$ and $s = zey^{-1}$. By applying Corollary 5.45, we can see that $r \leq s$ if and only if $x \leq z$. Now, since $r\mathscr{L}^{I}s$ then we can write $x = x'y$, $z = z'y$ where $x', z' \in W_I$ and $y \in {}^{I}W$. Then $x \leq z$ if and only if $x' \leq z'$ (by a simple subword argument).

Since $w_0(I)$ is the defined as the longest element of W_I, $x' \leq z'$ implies $w_{0(I)}z' \leq w_0(I)x'$ and thus $w_0(I)z = w_{0(I)}z'y \leq w_0(I)x'y = w_0(I)x$ and $w_0(I)s = (w_0(I)z)ey^{-1} \leq (w_0(I)x)ey^{-1} = w_0(I)r$ by Theorem 5.31. □

Theorem 7.18. *Let $I, J \subseteq S$. For any $r \in \mathcal{R}$,*

(1) $r = \lceil r \rceil^{\mathscr{L}^I}$ if and only if $r \in w_0(I)\mathcal{GJ}^{I}$.

(2) $r = \lceil r \rceil^{\mathscr{R}^J}$ if and only if $r \in \mathcal{JG}^{J}w_0(J)$.

(3) $r = \lceil r \rceil^{\mathscr{J}^{I,J}}$ if and only if $r \in w_0(I)\mathcal{GJ}^{I} \cap \mathcal{JG}^{J}w_0(J)$.

Proof. (1) Suppose $r = \lceil r \rceil^{\mathscr{L}^I}$. Then for all $s \in L^I_r$, $s \leq r$. We know from Proposition 7.17 that $w_0(I)r \leq w_0(I)s$. There is a bijection $L^I_r \to L^I_{w_0(I)r}$ by multiplying by $w_0(I)$ on the left. By Theorem 7.14 it follows that $w_0(I)r \in \mathcal{GJ}^{I}$. So then $r \in w_0(I)\mathcal{GJ}^{I}$.

Conversely, if $r \in w_0(I)\mathcal{GJ}^{I}$ then for all $s \in L^I_{w_0(I)r}$, $w_0(I)r \leq s$. But then $w_0 s \leq r$. By our bijection, $L^I_r \to L^I_{w_0(I)r}$, we can conclude that for all $t \in L^I_r$, $t \leq r$. Thus $r = \lceil r \rceil^{\mathscr{L}^I}$.

(2) is done similarly to (1).

(3) Suppose that $r = \lceil r \rceil^{\mathscr{J}^{I,J}}$. Then $r = \lceil r \rceil^{\mathscr{L}^I} = \lceil r \rceil^{\mathscr{R}^J}$, since $\mathscr{L}^I, \mathscr{R}^J \Rightarrow \mathscr{J}^{I,J}$. And thus, $r \in w_0(I)\mathcal{GJ}^{I} \cap \mathcal{JG}^{J}w_0(J)$ by (1) and (2). For the converse, take any $r \in w_0(I)\mathcal{GJ}^{I} \cap \mathcal{JG}^{J}w_0(J)$.

Then $r = \lceil r \rceil^{\mathscr{L}^I} = \lceil r \rceil^{\mathscr{R}^J}$. Pick any $s \in J_r^{I,J}$. Then we can find $t \in \mathcal{R}$ so that $r\mathscr{L}t\mathscr{R}s$. It follows that $t \le r$ by definition of absolute maximum.

Since $t \le r$ we can see that $BtB \subseteq \overline{BrB}$. Multiplying both sides by P_J on the right we can see that $BR_t^J B \subseteq \overline{BR_r^J B}$. Thus, $BsB \subseteq \overline{BsB} \subseteq \overline{BR_s^J B} = \overline{BR_t^J B} \subseteq \overline{BR_r^J B} = B\lceil r \rceil^{\mathscr{R}^J} B$. This shows us that, $s \le \lceil r \rceil^{\mathscr{R}^J} = r$. So $r = \lceil r \rceil^{\mathscr{J}^{I,J}}$. □

Corollary 7.19. *Suppose that $r, s \in \mathcal{R}$. Then for all $I, J \subseteq S$, the following are equivalent*

(1) $BJ_r^{I,J}B \subseteq \overline{BJ_s^{I,J}B}$

(2) $\lceil r \rceil^{\mathscr{J}^{I,J}} \le \lceil s \rceil^{\mathscr{J}^{I,J}}$

(3) $\lfloor r \rfloor^{\mathscr{J}^{I,J}} \le \lfloor s \rfloor^{\mathscr{J}^{I,J}}$

(4) there exist $a \in J_r^{I,J}$ and $b \in J_s^{I,J}$ with $a \le b$

Proof. (1) ⇒ (2), being dense elements, $\overline{B\lceil r \rceil^{\mathscr{J}^{I,J}} B} = \overline{BJ_r^{I,J}B}$ and $\overline{B\lceil s \rceil^{\mathscr{J}^{I,J}} B} = \overline{BJ_s^{I,J}B}$. Thus, $B\lceil r \rceil^{\mathscr{J}^{I,J}} B \subseteq \overline{B\lceil r \rceil^{\mathscr{J}^{I,J}} B} = \overline{BJ_r^{I,J}B} \subseteq \overline{BJ_s^{I,J}B} = B\lceil s \rceil^{\mathscr{J}^{I,J}} B$, so $\lceil r \rceil^{\mathscr{J}^{I,J}} \le \lceil s \rceil^{\mathscr{J}^{I,J}}$.

(2) ⇒ (3), by Proposition 7.15, $\lceil r \rceil^{\mathscr{J}^{I,J}} \le \lceil s \rceil^{\mathscr{J}^{I,J}}$ implies $\lfloor \lceil r \rceil^{\mathscr{J}^{I,J}} \rfloor^{\mathscr{L}^I} \le \lfloor \lceil s \rceil^{\mathscr{J}^{I,J}} \rfloor^{\mathscr{L}^I}$. Applying again we see that $\lfloor r \rfloor^{\mathscr{J}^{I,J}} = \lfloor \lfloor \lceil r \rceil^{\mathscr{J}^{I,J}} \rfloor^{\mathscr{L}^I} \rfloor^{\mathscr{R}^J} \le \lfloor \lfloor \lceil s \rceil^{\mathscr{J}^{I,J}} \rfloor^{\mathscr{L}^I} \rfloor^{\mathscr{R}^J} = \lfloor s \rfloor^{\mathscr{J}^{I,J}}$.

(3) ⇒ (4), let $a = \lfloor r \rfloor^{\mathscr{J}^{I,J}}$ and $b = \lfloor s \rfloor^{\mathscr{J}^{I,J}}$.

(4) ⇒ (1), suppose we can find such a and b. Then, $BaB \subseteq \overline{BbB}$. Multiplying on the left by P_I and right by P_J we see $BJ_r^{I,J}B = P_IBaBP_J \subseteq P_I\overline{BbB}P_J \subseteq \overline{P_IBbBP_J} = \overline{BJ_s^{I,J}B}$, giving us the result. □

Before moving on we see the general answer to the fourth question from Section 4.

Theorem 7.20. *For $I, J \subseteq S$ and any $\mathscr{T} = \mathscr{J}^{I,J}$, $\mathscr{L}^I$, or $\mathscr{R}^J$, and any $r \in \mathcal{R}$, we can find $r_1, r_2, \cdots, r_s \in \mathcal{R}$ so that, $\overline{BT_rB} = \bigsqcup_{i=1}^{s} BT_{r_i}B$*

Proof. Since $\mathcal{R}$ is finite, $\overline{BT_rB} = \bigcup_{s\mathscr{T}r} \overline{BsB} = \bigcup_{s\mathscr{T}r} \bigcup_{t \le s} BtB$. Recall $BT_xB = \bigcup_{y\mathscr{T}x} ByB$. So if $BT_xB \cap \overline{BT_rB}$ then we can find $y\mathscr{T}x$ and $s\mathscr{T}r$ with $y \le s$. But then we quickly see $BT_xB = BT_yB \subseteq \overline{BT_sB} = \overline{BT_rB}$.

Thus the closure of each fat $\mathscr{T}$-class must be a union of fat $\mathscr{T}$-classes, and this union is disjoint since fat $\mathscr{T}$-classes are disjoint. The union itself is finite, as each $\mathscr{T}$-class can be indexed by a unique element, say the absolute minimum. So then, $\overline{BT_rB} = \bigsqcup_{i=1}^{s} BT_{r_i}B$, where $r_i = \overset{\mathscr{T}}{\lfloor r \rfloor}$, $r_i \leq r$. □

7.2 Generalized Trichotomy

We have seen much of $\mathscr{L}^I$, $\mathscr{R}^J$, and $\mathscr{J}^{I,J}$ but we have not had any meaningful discussion of the other parabolic Green's relation, $\mathscr{H}^{I,J}$. Although a more enigmatic relation, we can still begin an investigation of $\mathscr{H}^{I,J}$ and a generalisation of $\mathcal{O}$. To this end, in this section we shall introduce a new trichotomy, one that takes into account a choice of I and J.

The following lemma is inspired by our proof of Corollary 7.16. It tells us that all elements of $\mathcal{GJ}^I \cap J_r^{I,J}$, for any $r \in \mathcal{R}$, lie within the same $\mathscr{R}^J$-class.

Lemma 7.21. *Fix $I, J \subseteq S$. Suppose $r \in \mathcal{R}$ and $v \in \mathcal{N}^{I,J} \cap J_r^{I,J}$.*

(1) If $r \in \mathcal{GJ}^I$ then $r\mathscr{R}^J v$.

(2) If $r \in \mathcal{JG}^I$ then $r\mathscr{L}^I v$.

Proof. For (1), we know already from the proof of Corollary 7.16 that $v = \overset{\mathscr{L}^I\mathscr{R}^J}{\lfloor\lfloor r \rfloor\rfloor}$. But since $r \in \mathcal{GJ}^I$, Theorem 7.14 says $r = \overset{\mathscr{L}^I}{\lfloor r \rfloor}$. Thus $v = \overset{\mathscr{R}^J}{\lfloor r \rfloor}$ and so $r\mathscr{R}^J v$. (2) can be proven likewise. One needs similar reasoning to Corollary 7.16 to show that $v = \overset{\mathscr{R}^J\mathscr{L}^I}{\lfloor\lfloor r \rfloor\rfloor}$. □

Lemma 7.22. *For any $r, s \in \mathcal{R}$ and any $I, J \subseteq S$,*

(1) $r\mathscr{L}^I s$ if and only if $r^*\mathscr{R}^I s^*$

(2) $r\mathscr{R}^J s$ if and only if $r^*\mathscr{L}^J s^*$

(3) $r\mathscr{H}^{I,J} s$ if and only if $r^*\mathscr{H}^{J,I} s^*$

Proof. (1) $r\mathscr{L}^I s$ if and only if $w \in W_I$ so that $rw = s$, if and only if $w^{-1}r^* = (rw)^* = s^*$ if and only if $r^*\mathscr{R}^I s$. (2) is done similarly, and (3) follows by applying both (1) and (2) together. □

Theorem 7.23. *For any $I, J \subseteq S$, and $r \in \mathcal{R}$. Then there exist unique $r_-^{I,J}, r_0^{I,J}, r_+^{I,J} \in \mathcal{R}$ such that,*

(1) $r = r_-^{I,J} r_0^{I,J} r_+^{I,J}$

(2) $r_0^{I,J} \mathscr{H}^{J,I} v^*$, where $v\mathscr{J}^{I,J} r$ and $v \in \mathcal{N}^{I,J}$

(3) $r\mathscr{R}^J r_-^{I,J}$ and $r\mathscr{L}^I r_+^{I,J}$

(4) $r_-^{I,J} \in \mathcal{JG}^J$ and $r_+^{I,J} \in \mathcal{GJ}^I$

For simplicity, when I, J are clear from context, we may use to our usual trichotomy symbols, r_-, r_0 and r_+.

Proof. Let $r = \sigma_-^{-1}e_-\sigma_0 e_+\sigma_+^{-1}$ be the vanilla decomposition. Let us write $\sigma_0\sigma_+^{-1} = u^J u_J$ where $u^J = (\sigma_0\sigma_+^{-1})^I$ and $u_J = (u^J)^{-1}\sigma_0\sigma_+^{-1} \in W_J$. Likewise let $\sigma_-^{-1}\sigma_0 = v_I{}^I v$ where $v_I \in W_I$ and ${}^I v \in {}^I W$. Now, let $r_- = \sigma_-^{-1}e_- u^J$, $r_0 = (u^J)^{-1}e_-\sigma_0 e_+({}^I v)^{-1}$, and $r_+ = {}^I v e_+\sigma_+^{-1}$.

Clearly r_- and r_+ satisfy conditions (3) and (4), and by construction of these elements we see $r_- r_0 r_+ = \sigma_-^{-1}e_- u^J (u^J)^{-1}e_-\sigma_0 e_+({}^I v)^{-1}{}^I v e_+\sigma_+^{-1} = r$, satisfying (1). It remains to show r_0 satisfies (2). By Lemma 7.22, it suffices to show ${}^I v e_+\sigma_0^{-1}e_- u^J = r_0^* \mathscr{H}^{I,J} v := \lfloor r^{\mathscr{J}^{I,J}} \rfloor$. Notice that $r_0^* u_J = {}^I v e_+\sigma_0^{-1}e_- u^J u_J = {}^I v e_+\sigma_0^{-1}\sigma_0\sigma_+^{-1} = {}^I v e_+\sigma_+ = r_+$. Likewise we can show $v_I r_0^* = r_-$. By Lemma 7.21, since $r_0^* \mathscr{R}^J r_+ \in \mathcal{GJ}^I$ and $r_0^* \mathscr{L}^I r_- \in \mathcal{JG}^J$ we see that $r_0^* \mathscr{H}^{I,J} v$. This concludes the existence part of the proof.

To show uniqueness, suppose that $r = r_- r_0 r_+ = s_- s_0 s_+$ are both decompositions satisfying (1) - (4). Just as with our original trichotomy, we can quickly use (3) and (4) along with Theorem 7.12 that $r_- = s_-$ and $r_+ = s_+$. Consider $r_-^* r r_+^* = r_-^* r_- r_0 r_+ r_+^*$. Since $r_0 \mathscr{H}^{J,I} v^*$ it is not difficult to see that $r_0^* \mathscr{R}^J v \mathscr{R}^J r_+$ and so $r_0 \mathscr{L}^J r_+^*$. That is, we can find $u \in W_J$ so that $r_0 = u r_+^*$. Likewise we find $v \in W_I$ so $r_0 = r_-^* v$. Thus, $r_-^* r_- r_0 r_+ r_+^* = r_-^* r_- u r_+^* r_+ r_+^* = r_-^* r_- r_0 = r_-^* r_- r_-^* v = r_0$.

We show the same result for s_0, allowing us to conclude $r_0 = r_-^* r r_+^* = s_0$. □

For $I = J = S$ we recover our original trichotomy. At the other end of the spectrum, we see that $I = J = \emptyset$ gives the decomposition of $r = rr^*r$.

Among other things, the trichotomy shows us that within a $\mathscr{J}^{I,J}$-class, the $\mathscr{H}^{I,J}$-classes "look the same". Speaking of $\mathscr{H}^{I,J}$-classes, we have given analogues of $\mathcal{GJ}$, $\mathcal{JG}$, and $\mathcal{N}$, but have not yet given a general form of $\mathcal{O}$. We rectify this with the following definition.

Definition 7.24. *For $I, J \subseteq S$, define $\mathcal{O}^{I,J}$ to be the set of all $r \in \mathcal{R}$ so that $(r_0^{I,J})^* \in \mathcal{N}^{I,J}$.*

Notice that in Section 3 we defined $\mathcal{O}$ (3.10) and Theorem 3.25 was an eventual consequence, whereas we have now taken the analogue of Theorem 3.25 as the definition of $\mathcal{O}^{I,J}$.

Example 7.25. *We can, however, exhibit a different definition for $\mathcal{O}^{I,J}$ when we restrict ourselves to the $n \times n$ matrices, $M_n(K)$. For any $I \subseteq S$ we can define the equivalence relation $\sim_I$ on $\{1, 2, \cdots, n\}$ as the closure of the relation defined by $x \sim_I x+1$ if $(x\ x+1) \in I$.*

It is known that $\mathcal{R}$ can be viewed as the set of partial injective functions on n elements, $\{f : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\} \mid f$ is partial and injective$\}$ (a natural generalisation of S_n). With this view of the Renner monoid, Renner remarks in [28] that we can consider $\mathcal{O}$ as the submonoid, $\mathcal{O} = \{f \in \mathcal{R} \mid f(x) < f(y)$ for all $x, y \in dom(f)$ with $x < y\}$ which illustrates why $\mathcal{O}$ is often called the monoid of order-preserving elements. $\mathcal{O}^{I,J}$ generalises this further. Where, $\mathcal{O}^{I,J} = \{f \in \mathcal{R} \mid f(x) < f(y)$ for all $x, y \in dom(f)$ with $x < y$, $x \sim_J y$, and $f(x) \sim_I f(y)\}$ is the set of functions who are order-preserving among $\sim_I$ classes which are in the image of $\sim_J$ classes.

At the level of matrices, we can identify the elements of $\mathcal{O}^{I,J}$ by looking at each of the individual submatrices whose rows are a $\sim_I$ class related and whose columns form a $\sim_J$ class. If each of them has the staircase pattern then the given matrix is an element of $\mathcal{O}^{I,J}$. In the example below, we use gray to distinguish the equivalence classes of $\sim_I$, and light gray for the equivalence classes of $\sim_J$.

$$\begin{pmatrix} 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix} \qquad \begin{pmatrix} 0&0&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0 \end{pmatrix}$$

The matrix on the left is an element of $\mathcal{O}^{I,J}$ because the submatrix given by each small rectangle contains the staircase pattern. The right matrix is not an element of $\mathcal{O}^{I,J}$ because the highlighted submatrix does not exhibit the staircase pattern. It is only this submatrix which fails.

One of the interesting things is that, although $\mathcal{GJ}^I$, $\mathcal{JG}^J$, and $\mathcal{N}^{I,J}$ are monoid generalisations of the minimal elements of the double cosets $W_I w$, wW_J, and $W_I w W_J$ in Coxeter groups (Chapter 2, [2]), there does not appear to be any literature discussing coset intersections $W_I w \cap wW_J$, which are exactly the $\mathscr{H}^{I,J}$-classes. Which means that $\mathcal{O}^{I,J}$ and $\mathscr{H}^{I,J}$ are entirely new concepts!

When we created our generalisations of $\mathcal{GJ}$, $\mathcal{JG}$, and $\mathcal{N}$, we did it by constructing them from standard forms. What we would like is a definition akin to $rB \subseteq Br$. We have already seen the type of structure we need in Lemma 5.25. It is regrettable that it remains at this point a conjecture, though one which can be confidently assumed to fall within the reaches of a discussion of root subgroups.

Conjecture 7.26. *Let $I, J \subseteq S$ and $r \in \mathcal{R}$.*

(1) $r \in \mathcal{GJ}^I$ if and only if $(B \cap L_I)r \subseteq rB$ if and only if $(B \cap L_I)rB = rB$

(2) $r \in \mathcal{JG}^J$ if and only if $r(B \cap L_J) \subseteq Br$ if and only if $Br(B \cap L_J) = Br$

(3) $r \in \mathcal{N}^{I,J}$ if and only if $(B \cap L_I)r(B \cap L_J) \subseteq Br \cap rB$

Unfortunately, if there is a similar intuition to $\mathcal{O}^{I,J}$ it is currently evasive. So rather than a conjecture, $\mathscr{H}^{I,J}$-classes leave us with the following question.

Question 7.27. *For a given $I, J \subseteq S$, can we find a definition for $\mathcal{O}^{I,J}$ that is similar to that given by Definition 3.10 in Section 3?*

Our definition based on the trichotomy still allows us to demonstrate that $\mathcal{O}^{I,J}$ is a set of representatives for $\mathscr{H}^{I,J}$-classes.

Proposition 7.28. *For, $I, J \subseteq S$, $\mathcal{O}^{I,J} \cong \mathcal{R}/\mathscr{H}^{I,J}$. That is to say, if $r, s \in \mathcal{R}$, $r\mathscr{H}^{IJ}s$ and $r, s \in \mathcal{O}^{I,J}$, then $r = s$, and for all $r \in \mathcal{R}$, there is $s \in \mathcal{O}^{I,J}$ with $r\mathscr{H}^{I,J}s$.*

Proof. Suppose that $r, s \in \mathcal{O}^{I,J}$ and $r\mathscr{H}^{I,J}s$. Then $r\mathscr{L}^I s$ and $r\mathscr{R}^J s$. Thus $r_- = s_-$ and $r_+ = s_+$ in our generalised trichotomy. $r\mathscr{J}^{I,J}s$, so $v := \lfloor r \rfloor^{\mathscr{J}^{I,J}} = \lfloor s \rfloor^{\mathscr{J}^{I,J}}$. Thus $r = r_- v^* r_+ = s_- v^* s_+ = s$ as desired. For the second result, let $r \in \mathcal{R}$ be arbitrary. By Theorem 7.23 we can decompose $r = r_- r_0 r_+$. Consider $s := r_- v^* r_+$ where $v = \lfloor r \rfloor^{\mathscr{J}^{I,J}}$. It suffices to show that this is the trichotomy decomposition for s.

If we take $s_- = r_-$, $s_0 = v^*$ and $s_+ = r_+$ we see that by definition, $s = s_- s_0 s_+$ satisfies (1) from Theorem 7.23. Now, by Lemma 7.21, there exists $w \in W_J$ so that $r_+ = vw$ and $u \in W_I$ so that $r_- = uv$. Thus $s = r_- v^* r_+ = r_- v^* v w = r_- r_-^* u^{-1} u r_- w = r_- w$ and we see $s\mathscr{R}^J r_-$. Similarly, $s\mathscr{L}^I r_+$. Thus $s\mathscr{H}^{I,J} r$ and $s\mathscr{J}^{I,J} v$, meaning that (2)-(4) are also satisfied. ◻

Proposition 7.29. *For any $r \in \mathcal{R}$ and any $I, J \subseteq S$,*

(1) $r \in \mathcal{GJ}^I$ if and only if $r_- = r_0^* = v \in \mathcal{N}^{I,J}$

(2) $r \in \mathcal{JG}^J$ if and only if $r_+ = r_0^* = v \in \mathcal{N}^{I,J}$

Proof. By Lemma 7.21 we know that $r\mathscr{R}^J v$. Thus we can find $w \in W_J$ so that $r = vw$. Observe that $r = vw = vv^*vw = vv^*r$. It is not hard to see that $r = vv^*r$ is the trichotomy decomposition with respect to I, J. Thus $r_- = v$ and $r_0^* = (v^*)^* = v$. □

Corollary 7.30. *$\mathcal{N}^{I,J} \subseteq \mathcal{GJ}^I \subseteq \mathcal{O}^{I,J}$ and $\mathcal{N}^{I,J} \subseteq \mathcal{JG}^J \subseteq \mathcal{O}^{I,J}$.*

Proof. That $\mathcal{N}^{I,J} \subseteq \mathcal{GJ}^I$ comes from the definition of $\mathcal{N}^{I,J}$. Since $r \in \mathcal{GJ}^I$ implies $r_0 = v^*$ (by the preceding proposition) we can conclude by definition of $\mathcal{O}^{I,J}$ that $\mathcal{GJ}^I \subseteq \mathcal{O}^{I,J}$. □

At this time, whether $\mathcal{O}^{I,J}$ characterizes the absolute minimal elements of $\mathscr{H}^{I,J}$-classes proves elusive. As it has been verified by computer calculations for $M_n(K)$ from 2 to 6 we include it as a conjecture, rather than a question.

Conjecture 7.31. *For any $r \in \mathcal{R}$ and any $I, J \subseteq S$,*

(1) $r = \lfloor r \rfloor^{\mathscr{H}^{I,J}}$ if and only if $r \in \mathcal{O}^{I,J}$.

(2) $r = \lceil r \rceil^{\mathscr{H}^{I,J}}$ if and only if $r \in w_0(I)\mathcal{O}^{I,J} = \mathcal{O}^{I,J}w_0(J)$.

We can make several remarks however. The first being that (2) will follow from (1) and an application of Proposition 7.17. Upon completion of the conjecture the following corollaries would become apparent.

Corollary 7.32. $w_0(I)\mathcal{O}^{I,J}w_0(J) = \mathcal{O}^{I,J}$

Proof. Since $r \in \mathcal{O}^{I,J}$ if and only if $r = \lfloor r \rfloor^{\mathscr{H}^{I,J}}$, take any $t \in H_r^{I,J}$. It suffices to show that if $r \leq t$ then $w_0(I)rw_0(J) \leq w_0(I)tw_0(J)$. This is the content of (3) in Proposition 7.17, so the result follows. □

Corollary 7.33. $(\mathcal{O}^{I,J})^* = \mathcal{O}^{J,I}$

Proof. We need to recall the involution τ from Section 2. $r \in \mathcal{O}^{I,J}$ then for all $s\mathscr{H}^{I,J}r$, $r \leq s$, or rather $r \in \overline{BsB}$. Then $r^* = \tau(r) \in \tau(\overline{BsB}) = \overline{\tau(BsB)} = \overline{B^-\tau(s)B^-} = \overline{B^-s^*B^-}$. Now, since $r\mathscr{H}^{I,J}s$ we can find $u \in W_I$ and $v \in W_J$ so that $ur = s = rv$. Then $r^*u^{-1} = s^* = v^{-1}r^*$. Thus $r^*\mathscr{H}^{J,I}s^*$.

Since $r^*\mathscr{H}^{J,I}s^*$ we see that $r^*\mathscr{H}s^*$ and so by Corollary 6.10, $r^* \in \overline{B^-s^*B^-}$ if and only if $r^* \leq s^*$. Since τ is an automorphism, it follows that r^* is minimal in $H_{r^*}^{J,I}$ and thus, $r^* \in \mathcal{O}^{J,I}$. □

By taking $I = J$ we would get that $(\mathcal{O}^{I,I})^* = \mathcal{O}^{I,I}$. One might hope that, like $\mathcal{O}$ before, $\mathcal{O}^{I,I}$ is an inverse monoid. This is not case as the following example indicates.

Example 7.34. *Consider the scenario in $M_3(K)$ with $I = (1\ 2)$. Let $r = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. It is not hard to check that $r, s \in \mathcal{O}^{I,I}$, but $rs = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which is not in $\mathcal{O}^{I,I}$.*

Without a definition similar to Definition 3.10 it is very difficult to say when $\mathcal{O}^{I,J}$ is monoid. We can say for certain that $\mathcal{O}^{S,S} = \mathcal{O}$ is, and $\mathcal{O}^{I,\emptyset} = \mathcal{O}^{\emptyset,J} = \mathcal{R}$ as well. Both Corollaries 7.32 and 7.33 give us a little insight into what an answer to what an alternate definition of $\mathcal{O}^{I,J}$ might be, but sadly do not allow us to answer the whole question.

One might worry that, since we were unable to demonstrate Conjecture 7.31, we may not have absolute minima or maxima with respect to $\mathscr{H}^{I,J}$-classes. This will be resolved in the following section, as we can observe, $\lfloor r \rfloor^{\mathscr{H}^{I,J}} = min_{\lfloor r \rfloor^{\mathscr{J}^{I,J}}} H_r^{I,J}$ and $\lceil r \rceil^{\mathscr{H}^{I,J}} = max_{\lceil r \rceil^{\mathscr{J}^{I,J}}} H_r^{I,J}$ which we will show do exist.

7.3 Relative Maxima and Minima

To wrap up our discussion of parabolic Green's relations, it makes sense to ask the question of relative maxima and minima, just as we investigated in Section 6. As it happens, the already established existence of $min_r L_s$ and $min_r R_s$ will make this process very easy. We will only need a few extra results concerning properties of the Weyl group.

Proposition 7.35. *For any $I \subseteq S$, $w_0 W_I = W_{w_0 I w_0} w_0$.*

Proof. Let $s_1 s_2 \cdots s_k \in W_I$. Since $w_0^2 = 1$, then $s_1 s_2 \cdots s_k = s_1 w_0 w_0 s_2 w_0 \cdots w_0 s_k$. So then, $w_0 s_1 s_2 \cdots s_k = w_0 s_1 w_0 w_0 s_2 w_0 \cdots w_0 s_k = (w_0 s_1 w_0)(w_0 s_2 w_0) \cdots (w_0 s_k w_0) w_0 \in W_{w_0 I w_0} w_0$. So $w_0 W_I \subseteq W_{w_0 I w_0} w_0$, and it is clear from our work that the reverse inclusion also holds. □

Proposition 7.36. *Suppose that $s_1 s_2 \cdots s_k$ and $s'_1 s'_2 \cdots s'_k$ are two reduced word expressions for the same element, $w \in W$. Then the set of generators appearing in $s_1 s_2 \cdots s_k$ is the same as the set of generators appearing in $s'_1 s'_2 \cdots s'_k$.*

Proof. Corollary 1.4.8 in [2]. □

Corollary 7.37. *For any $w \in W$, there are only finitely many reduced words for w.*

Proof. By the preceding proposition, let X be the set of generators which appear in any reduced word expression for w. Then every reduced word for w must be among the collection of words, $\{x_1 x_2 \cdots x_{\ell(w)} \mid x_i \in X\}$. But this set is at most $|X|^{\ell(w)} \leq \ell(w)^{\ell(w)}$ which is finite. □

Proposition 7.38. *Let $u, v \in W$. Then the following are equivalent.*

(1) $u \leq v$

(2) every reduced word expression for v has a subword that is a reduced word expression for u

(3) some reduced word expression for v has a subword that is a reduced word expression for u

Proof. [2], Corollary 2.2.3. □

Lemma 7.39. *Fix $I \subseteq S$. Let $s_1, s_2, \cdots s_N$ be any sequence of generators in S (repeats allowed). Let $W(I; s_1, s_2, \cdots s_N)$ denote the elements of W_I which can be written as a product $s_{n_1} s_{n_2} \cdots s_{n_k}$ with $1 \leq n_1 < n_2 < \cdots n_k \leq N$. Then, with respect to the Bruhat order, $W(I; s_1, s_2, \cdots s_N)$ has a unique maximal element. We shall denote this element as $w(I; s_1, s_2, \cdots s_N)$.*

Proof. This result comes to us by way of Lemma 2.1 in [1]. □

Theorem 7.40. *Let $u, v \in W$ with $u \leq v$, and let $I \subseteq S$.*

(1) There exists unique maximal $w \in W_I u$ so that $w \leq v$.

(2) There exists unique maximal $w \in uW_I$ so that $w \leq v$.

Proof. (1) Let $x = \lfloor u \rfloor^{\mathscr{L}^I}$, the minimum element in $W_I u$. Since $u \leq v$ it follows that $x \leq v$. Since there are only finitely many reduced words for v we can pick one $v = s_1 s_2 \cdots s_{\ell(v)}$ with the property that index $1 \leq i \leq \ell(v)$ so that $x \leq s_{i+1} \cdots s_n$ is maximal (among all reduced word expressions for v).

Recall the element, $w(I; s_1, s_2, \cdots s_i) \in W_I$, which we will denote as y. We claim that $w = yx$ is what we are looking for.

Consider any $zx \leq v$ with $z \in W_I$. By minimality of x, $\ell(zx) = \ell(z) + \ell(x)$, so any reduced word of zx is just two reduced words, one for z and one for x, concatenated together. It follows that $s_1 s_2 \cdots s_{\ell(v)}$ contains a reduced word expression for zx. And by maximality of i, it is clear that $s_1, s_2, \cdots s_i$ contains a reduced subword for z. But by Lemma 7.39, $z \leq y$ and so $zx \leq yx$.

Finally, since $\ell(w) = \ell(y) + \ell(x)$ we can see that $s_1 s_2 \cdots s_{\ell(v)}$ also contains a reduced word expression for w, so it follows that $w \leq v$, and we have shown that $w = yx$ is the desired element.

(2) is done similarly. □

We are now in position to begin to show the existence of relative maxima and minima with our new equivalence relations. Although the definition logically extends, we will repeat it for the sake of completeness.

Definition 7.41. *For $r, s \in \mathcal{R}$, $r \leq s$, and any equivalence relation T, we define the* **relative maximum** *of T_r with respect to s, as*

$$max_s T_r = \begin{cases} t & \text{if } t \in T_r,\ t \leq s \text{ and } \forall t' \in T_r\ t' \leq s \implies t' \leq t \\ \text{undefined} & \text{otherwise} \end{cases}$$

We define the **relative minimum** *of T_s with respect to r, as*

$$min_r T_s = \begin{cases} t & \text{if } t \in T_s,\ r \leq t \text{ and } \forall t' \in T_s\ r \leq t' \implies t \leq t' \\ \text{undefined} & \text{otherwise} \end{cases}$$

Corollary 7.42. *For any $r, s \in \mathcal{R}$ and any $I \subseteq S$,*

(1) if $r \mathscr{L} s$ then $max_s L^I_r$ exists

(2) if $r \mathscr{R} s$ then $max_s R^I_r$ exists

Proof. (1) Write $r = \sigma_-^{-1} e_- \sigma_0 e_+ \sigma_+^{-1}$ and $s = \tau_-^{-1} e_- \tau_0 e_+ \sigma_+^{-1}$ in vanilla form. By Theorem 5.44 $r \leq s$ if and only if $u := \sigma_-^{-1}\sigma_0 \leq \tau_-^{-1}\tau_0$. Define $v = \tau_-^{-1}\tau_0 w_0(\lambda_*(e_+))$. Then $u \leq v$. Let be $w \in W_I u$ be the unique maximal element also satisfying $w \leq v$, shown to exist in Theorem 7.40.

Let $t = we_+\sigma_+^{-1}$. We claim t fits the definition of $max_s L^I_r$. Since $u \leq w \leq v$ and $1 \in \mathcal{JG}$, $e_+\sigma_+^{-1} \in GJ$, Lemma 5.27 tells us that $ue_+\sigma_+^{-1} \leq we_+\sigma_+^{-1} \leq ve_+\sigma_+^{-1}$, or rather $r \leq t \leq s$. Observe also that since $u \mathscr{L}^I w$, then $r \mathscr{L}^I t$.

Suppose $t' \mathscr{L}^I r$ and $t' \leq s$. We can find $z \in W_I$ so $t' = zr = z\sigma_-^{-1}\sigma_0 e_+ \sigma_+^{-1} = zue_+\sigma_+^{-1}$. But since $zue_+\sigma_+^{-1} = t' \leq s = \tau_-^{-1}\tau_0 e_+ \sigma_+^{-1}$ a quick application of Theorem 5.31 shows us that there exists $y \in W_*(e_+)$ so that $zu \leq \tau_-^{-1}\tau_0 y \leq \tau_-^{-1}\tau_0 w_0(\lambda_*(e_+))$, since $\tau_-^{-1}\tau_0 \in D_*(e_+)$ by Proposition 5.39. Thus $zu \leq v$. Then $zu \leq w$ and Lemma 5.27 again shows us that $t' \leq t$.

(2) is demonstrated similarly. □

Corollary 7.43. *For any $r, s \in \mathcal{R}$ and any $I \subseteq S$,*

(1) if $r\mathscr{L}s$ then $min_r L_s^I$ exists

(2) if $r\mathscr{R}s$ then $min_r R_s^I$ exists

Proof. (1) We claim that $min_r L_s^I = w_0 max_{w_0 r} L_{w_0 s}^{w_0 I w_0}$. Let $m = max_{w_0 r} L_{w_0 s}^{w_0 I w_0}$. First observe that since $r \leq s$ and $r\mathscr{L}s$, then $w_0 r \mathscr{L} w_0 s$ and $w_0 s \leq w_0 r$, so m exists. Now, $m \leq w_0 r$ by definition, and since they belong to the same $\mathscr{L}$-class, it follows that $r = w_0 w_0 r \leq w_0 m$. Further, $m \mathscr{L}^{w_0 I w_0} w_0 s$, so there exists $w \in W_{w_0 I w_0}$ so that $wm = w_0 s$. But then $w_0 w m = s$. By Proposition 7.35 there exists $v \in W_I$ so that $w_0 w m = v w_0 m$. Thus $w_0 m \mathscr{L}^I s$.

Now, let $t\mathscr{L}^I s$ be arbitrary and suppose that $r \leq t$. Then again by Proposition 7.35 $w_0 t \mathscr{L}^{w_0 I w_0} w_0 s$ and $r\mathscr{L}t$ implies $w_0 t \leq w_0 r$. By definition of m we see that $w_0 t \leq m$. But then $w_0 m \leq t$. So $w_0 m$ satisfies the definition of $min_r L_s^I$ as claimed.

(2) is demonstrated similarly. □

Theorem 7.44. *Let $I \subseteq S$ and take $r, s \in \mathcal{R}$ with $r \leq s$. Then $min_r L_s^I$, $min_r R_s^I$, $min_s L_r^I$, $min_s R_r^I$ all exist.*

Proof. We claim that $min_r L_s^I = min_{min_r L_s} L_s^I$. Let $m = min_{min_r L_s} L_s^I$. First observe that since $r \leq s$, $min_r L_s$ exists. And since $min_r L_s \mathscr{L} s$ and $min_r L_s \leq s$ Corollary 7.43 tells us $min_{min_r L_s} L_s^I$ exists. Now, let $t\mathscr{L}^I s$ be arbitrary and suppose that $r \leq t$. Then $t\mathscr{L}^I s$ implies $t\mathscr{L}s$ and so $min_r L_s \leq t$ and $min_r L_s \mathscr{L} t$. So by definition, $m \leq t$ as desired. The others are shown similarly. □

We can now conclude this discussion with a result about the relative maxima and minima for our $H^{I,J}$ relation.

Corollary 7.45. *For any $r, s \in \mathcal{R}$, with $r \leq s$, if $I, J \subseteq S$, then $min_r H_s^{I,J}$ and $max_s H_r^{I,J}$ exist.*

Proof. We will just prove $min_r H_s^{I,J}$ as $max_s H_r^{I,J}$ is similar. We claim $min_r H_s^{I,J} = min_{min_r R_s^J} L_s^I$. By Theorem 7.44, $min_r R_s^J$ exists, and $r \leq s$ implies that $min_r R_s^J \leq s$. Then Theorem 7.44 again shows us that $min_{min_r R_s^J} L_s^I$ exists and $min_{min_r R_s^J} L_s^I \mathscr{L}^I s$. By definition, $min_r R_s^J \leq min_{min_r R_s^J} L_s^I \leq s$. Now $\lfloor s \rfloor^{\mathscr{R}^J} = \lfloor min_r R_s^J \rfloor^{\mathscr{R}^J} \leq \lfloor min_{min_r R_s^J} L_s^I \rfloor^{\mathscr{R}^J} \leq \lfloor s \rfloor^{\mathscr{R}^J}$, so we can conclude that $min_{min_r R_s^J} L_s^I \mathscr{R}^J s$ as well, hence $min_{min_r R_s^J} L_s^I \mathscr{H}^{I,J} s$. Let $t \in H_s^{I,J}$ with $r \leq t$. Then $t\mathscr{R}^J s$ and thus $min_r R_s^J \leq t$. It the follows that $min_{min_r R_s^J} L_s^I \leq t$ as desired. □

Remark 7.46. *The preceding proof can be generalised for any equivalence classes,* $\mathscr{T}$ *and* $\mathscr{U}$*. If both* $min_r T_s$ *and* $min_r U_s$ *exist for all* $r \leq s \in \mathcal{R}$*, then the relative minimum* $min_r(T \cap U)_s$ *exists and we can show* $min_r(T \cap U)_s = min_{min_r T_s} U_s = min_{min_r U_s} T_s$*.*

Likewise, if $max_s T_r$*,* $max_s U_r$ *exist for all* $r \leq s \in \mathcal{R}$*, then* $max_s(T \cap U)_r$ *exists for all* $r \leq s \in \mathcal{R}$ *and* $max_s(T \cap U)_r = max_{max_s T_r} U_r = max_{max_s U_r} T_r$*.*

Corollary 7.47. *Suppose* $r \mathscr{J}^{I,J} s$ *and* $r \leq s$*, then there exist* $t, u \in \mathcal{R}$ *so that* $r \leq t, u \leq s$ *and* $r \mathscr{L}^I t \mathscr{R}^J s$*,* $r \mathscr{R}^J u \mathscr{L}^I s$*.*

Proof. We shall focus on the existence of t and u will follow by a symmetrical argument. Let $t = min_r R_s^J$, we claim this suffices. It is clear that $t \mathscr{R}^J s$ so we just need to show $t \mathscr{L}^I r$.

Since $t \mathscr{R}^J s$ it follows that $t \mathscr{J}^{I,J} r$ so we can find $v \in \mathcal{R}$ so that $r \mathscr{L}^I v \mathscr{R}^J t$. It is clear that $r = min_r L_v^I$ and $t = min_r R_v^J$. Then it follows that $t = min_{min_r L_v^I} R_v^J$. But by the proof of our previous corollary we see $t = min_r H_v^{I,J}$. So, $t \mathscr{H}^{I,J} v$ and thus $t \mathscr{L}^I v$. We conclude $t \mathscr{L}^I r$. □

Proposition 7.48. *Suppose* $r \leq s \leq t$*,* $I, J \subseteq S$*, and* $\mathscr{T} = \mathscr{J}^{I,J}$*,* $\mathscr{L}^I$*,* $\mathscr{R}^J$*, or* $\mathscr{H}^{I,J}$*. Then, assuming they exist,*

(1) $min_r T_t \leq min_s T_t$

(2) $max_t T_r \leq max_s T_r$

Proof. These proofs will be identical to Propositions 6.21 and 6.33. (1) We know $min_s T_t \in T_t$ and $r \leq s \leq min_s T_t$. By definition, $min_r T_t \leq min_s T_t$. (2) is shown similarly. □

We have investigated the relative maxima and minima for $\mathscr{L}^I$, $\mathscr{R}^J$, and $\mathscr{H}^{I,J}$. As in the previous section, the only thing left to do is consider $\mathscr{J}^{I,J}$. Unfortunately this is where things become more vague.

Example 7.49. *By computer work on the Rook monoids* $M_2(K)$*,* $M_3(K)$*,* $M_4(K)$*,* $M_5(K)$*, and* $M_6(K)$*, one can see that* $min_r J_s^{I,J}$ *exists if and only if there exists some* $e \in \Lambda$ *and* $f \in \Lambda^-$ *so that* $I = \lambda_*(e)$ *and* $J = \lambda_*(f)$*. For those same monoids one can also observe that* $max_s J_r^{I,J}$ *exists if and only if at least one of* I, J *is equal to* $\emptyset$*.*

Given the general expression of these relative minima and maxima one might be inclined to think that this could be a general theorem or at least a conjecture. However, for any reductive group the Renner monoid is also a Weyl group and we get a slightly different story. We can *quickly verify that* $min_r J_s^{I,S}$, $min_r J_s^{S,I}$, $max_s J_r^{I,S}$, *and* $max_s J_r^{S,I}$ *exist regardless of the choice of* I *which does not line up with our* $M_n(K)$ *work.*

If one considers $GL_n(K)$, *then at least for* $GL_2(K)$, $GL_3(K)$, $GL_4(K)$, $GL_5(K)$, *and* $GL_6(K)$, *the idempotents of* $M_n(K)$ *seem to enter the picture again.* $min_r J_s^{I,J}$ *exists if and only if there exists some* $e \in \Lambda$ *and* $f \in \Lambda^-$ *so that either* $I = \lambda_*(e)$ *and* $J = \lambda_*(f)$, *or* $J = \lambda_*(e)$ *and* $I = \lambda_*(f)$, *or one of* $I = \emptyset$, $J = \emptyset$, $I = S$, $J = S$ *holds. Similarly,* $max_s J_r^{I,J}$ *exists if and only if either there exists some* $e, f \in \Lambda$ *so that* $I = \lambda_*(e)$ *and* $J = \lambda_*(f)$, *or there exists some* $e, f \in \Lambda^-$ *so that* $J = \lambda_*(e)$ *and* $I = \lambda_*(f)$, *or one of* $I = \emptyset$, $J = \emptyset$, $I = S$, $J = S$ *holds.*

Although we can not currently approach the general existence problem for maxima and minima, we can describe these relative minima and maxima even when they exist individually.

Proposition 7.50. *Suppose that* $r \leq s$ *and let* $I, J \subseteq S$ *be arbitrary.*

(1) If $min_r J_s^{I,J}$ *exists then* $min_r J_s^{I,J} = min_r R^J_{min_r L_s^I} = min_r L^I_{min_r R_s^J}$.

(2) If $max_s J_r^{I,J}$ *exists then* $max_s J_r^{I,J} = max_s R^J_{max_s L_r^I} = max_s L^I_{max_s R_r^J}$.

Proof. It is clear that $r \leq min_r R_s^J \leq s$ and so $r \leq min_r L^I_{min_r R_s^J} \leq min_r R_s^J \leq s$. It is also true that $min_r L^I_{min_r R_s^J} \mathscr{L}^I min_r R_s^J \mathscr{R}^J s$, so $min_r L^I_{min_r R_s^J} \mathscr{J}^{I,J} s$. By definition $min_r J_s^{I,J} \leq min_r L^I_{min_r R_s^J}$.

By Corollary 7.47 we know there exists $min_r J_s^{I,J} \leq z \leq s$ with $z \mathscr{R}^J s$ and $z \mathscr{L}^I min_r J_s^{I,J}$. Then $min_r R_s^J \leq z \leq s$. Thus, $(min_r R_s^J)_+^{I,J} \leq z_+^{I,J}$ (Proposition 7.15). Now, $z \mathscr{L}^I min_r J_s^{I,J}$ so, $min_r L_z^I = min_r J_s^{I,J} \leq min_r L^I_{min_r R_s^J}$. Thus, $z_+^{I,J} = (min_r L_z^I)_+^{I,J} \leq (min_r L^I_{min_r R_s^J})_+^{I,J} = (min_r R_s^J)_+^{I,J}$. It follows, as desired, that $z \mathscr{L}^I min_r R_s^J$ and furthermore, $min_r J_s^{I,J} = min_r L_z^I = min_r L^I_{min_r R_s^J}$.

The other statements are proven similarly. □

With our new absolute and relative extrema, we can find an analogue to Theorem 6.40.

Theorem 7.51. *Let* $\mathscr{T} = \mathscr{H}^{I,J}, \mathscr{R}^J, \mathscr{L}^I, \mathscr{J}^{I,J}$

(1) If $\mathscr{T} \neq \mathscr{H}^{I,J}$ *and* $r_0 < r_1 < \cdots < r_{k-1} < r_k$ *is a chain in* $\mathcal{R}$ *with* $r_0 \mathscr{T} r_k$, *then for all indices* $0 \leq i \leq k$ *we have* $r_0 \mathscr{T} r_i$

(2) If $\mathscr{T} \neq \mathscr{J}^{I,J}$ *and* $r_0 < r_1 < \cdots < r_{k-1} < r_k$ *is a chain of maximum length between* $r_0, r_k \in \mathcal{R}$. *For all* $1 \leq i \leq k$, *if* $r_i \not\mathscr{T} r_{i-1}$ *then* $r_i = min_{r_{i-1}} T_{r_i}$ *and* $r_{i-1} = max_{r_i} T_{r_{i-1}}$.

Although we stated (1) with $\mathscr{T} \neq \mathscr{H}^{I,J}$ it is strongly suspected that it remains true.

Proof. (1) By Corollary 7.19, we can see that our given chain, $r_0 < r_1 < \cdots < r_{k-1} < r_k$, implies that $\overset{\mathscr{T}}{\lfloor r_0 \rfloor} \leq \overset{\mathscr{T}}{\lfloor r_1 \rfloor} \leq \cdots \overset{\mathscr{T}}{\lfloor r_{k-1} \rfloor} \leq \overset{\mathscr{T}}{\lfloor r_k \rfloor}$. But $r_0 \mathscr{T} r_k$ means that $\overset{\mathscr{T}}{\lfloor r_0 \rfloor} = \overset{\mathscr{T}}{\lfloor r_k \rfloor}$ and hence for any i, $\overset{\mathscr{T}}{\lfloor r_0 \rfloor} = \overset{\mathscr{T}}{\lfloor r_i \rfloor}$, thus $r_0 \mathscr{T} r_i$.

(2) We shall just show the minimum condition, as the maximum condition follows similarly. Suppose not. Then we see that $r_{i-1} < min_{r_{i-1}} T_{r_i} < r_i$, which contradicts the maximality of the length of the chain. □

Our last result will tie in both the relative maxima and absolute minima we have been studying to generalise the following Coxeter group property to Renner monoids. It will generalise the following Weyl group property.

Proposition 7.52. *Let $\{I_\alpha\}_{\alpha \in A}$ be a nonempty family of subsets of S. Define $I = \bigcap_{\alpha \in A} I_\alpha$.*

(1) Let $r \in {}^I W$ and $s \in W$. Then, $r \leq s$ if and only if ${}^{I_\alpha} r \leq {}^{I_\alpha} s$ for all $\alpha \in A$.

(2) Let $r \in W^I$ and $s \in W$. Then, $r \leq s$ if and only if $r^{I_\alpha} \leq s^{I_\alpha}$ for all $\alpha \in A$.

Proof. (2) follows from Theorem 2.6.1 in [2], (1) is done with similar reasoning. □

Theorem 7.53. *Let $\{I_\alpha\}_{\alpha \in A}$ be a nonempty family of subsets of S. Define $I = \bigcap_{\alpha \in A} I_\alpha$.*

(1) Let $r \in \mathcal{G}\mathcal{J}^I$ and $s \in \mathcal{R}$. Then, $r \leq s$ if and only if $r_+^{I_\alpha, \emptyset} \leq s_+^{I_\alpha, \emptyset}$ for all $\alpha \in A$.

(2) Let $r \in \mathcal{J}\mathcal{G}^I$ and $s \in \mathcal{R}$. Then, $r \leq s$ if and only if $r_-^{\emptyset, I_\alpha} \leq s_-^{\emptyset, I_\alpha}$ for all $\alpha \in A$.

Proof. We will just demonstrate the (1) property, as the other follows by mirroring the argument. For the ($\Rightarrow$) direction, this follows directly from Proposition 7.19.

The ($\Leftarrow$) side is done by recalling our work with relative maxima in the preceding section. Since A is nonempty, pick $a \in A$. $r_+^{I_a, \emptyset} \leq s_+^{I_a, \emptyset}$ implies that $r_+^{I_a, \emptyset} \leq s$ and so $m = max_s L_{r_+^{I_a, \emptyset}}$ exists. By Proposition 7.4 for each $\alpha \in A$, $r_+^{I_\alpha, \emptyset} \mathscr{L}^{I_\alpha} r$ implies $r_+^{I_\alpha, \emptyset} \mathscr{L} r$, so $m = max_s L_r$ and $r_+^{I_\alpha, \emptyset} \leq m$.

We can write our $r_+^{I_\alpha, \emptyset}$ in standard form, $r_+^{I_\alpha, \emptyset} = x_\alpha e \sigma^{-1}$, where $x_\alpha \in {}^{I_\alpha} W^{\lambda_*(e)}$. In fact, if we write r in standard form, $xe\sigma^{-1}$, then $x_\alpha = {}^{I_\alpha} x$. Writing $m = ye\sigma^{-1}$ in standard form now allows us to better understand $r_+^{I_\alpha, \emptyset} \leq m$.

By Corollary 5.40 and Theorem 5.44, $r_+^{I_\alpha, \emptyset} \leq m$ if and only if $x_\alpha \leq y$, which as we have noted is ${}^{I_\alpha} x \leq y$. ${}^{I_\alpha} x \leq y$ implies ${}^{I_\alpha} x \leq {}^{I_\alpha} y$ by Proposition 5.13, and so we can see by Proposition 7.52 that $x \leq y$, as $x \in {}^I W^{\lambda_*(e)} \subseteq {}^I W$, since $r \in \mathcal{G}\mathcal{J}^I$. Thus, $xe\sigma^{-1} \leq ye\sigma^{-1}$, or rather $r \leq m$. And by definition, $r \leq m$ if and only if $r \leq s$, which was what we wanted. □

Corollary 7.54. *For $t \in S$, let $I_t := S \setminus \{t\}$. Then, for $r, s \in \mathcal{R}$,*

(1) $r \leq s$ if and only if $\lfloor r \rfloor^{\mathscr{L}^{I_t}} \leq \lfloor s \rfloor^{\mathscr{L}^{I_t}}$ for all $t \in S$

(2) $r \leq s$ if and only if $\lfloor r \rfloor^{\mathscr{R}^{I_t}} \leq \lfloor s \rfloor^{\mathscr{R}^{I_t}}$ for all $t \in S$

Proof. First, observe $\bigcap_{t \in S} I_t = \emptyset$. Then we see $r \in \mathcal{R} = \mathcal{GJ}^{\emptyset} = \mathcal{GJ}^{\bigcap_{t \in S} I_t}$. So by Theorem 7.53, $r \leq s$ if and only if $\lfloor r \rfloor^{\mathscr{L}^{I_t}} \leq \lfloor s \rfloor^{\mathscr{L}^{I_t}}$ for all $t \in S$ if and only if $\lfloor r \rfloor^{\mathscr{R}^{I_t}} \leq \lfloor s \rfloor^{\mathscr{R}^{I_t}}$ for all $t \in S$. □

Corollary 7.55. *For $r, s \in \mathcal{R}$,*

(1) $r \leq s$ if and only if $\lfloor r \rfloor^{\mathscr{L}^{\lambda(e)}} \leq \lfloor s \rfloor^{\mathscr{L}^{\lambda(e)}}$ for all $e \in \Lambda$

(2) $r \leq s$ if and only if $\lfloor r \rfloor^{\mathscr{R}^{\lambda(e)}} \leq \lfloor s \rfloor^{\mathscr{R}^{\lambda(e)}}$ for all $e \in \Lambda$

Proof. Just like in the previous corollary it suffices to show that $\cap_{e \in \Lambda} \lambda(e) = \emptyset$. For any chain $\Gamma \subseteq \Lambda$, $C^r_G(\Gamma) = \{g \in G \mid ge = ege \text{ for all } e \in \Gamma\} = P_{C_S(\Gamma)} = P_{\cap_{e \in \Gamma} \lambda(e)}$. Since Λ is a maximal chain it follows that $C^r_G(\Lambda) = B = P_{\emptyset}$ ([20]). Thus $\bigcap_{e \in \Lambda} \lambda(e) = C_S(\Lambda) = \emptyset$ as desired. □

One can note the similarity of this last application to both the theoretical basis for Young's tableaux (Section 2.6 in [2]) and Problem 3.2 articulated by Renner and Putcha in [25].

7.4 Example

When we restrict ourselves to discussing our trichotomy in $M_n(K)$, we get a nice way of deriving the elements r_-, r_0, and r_+. We will go through the computation method with the matrix, $r = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$ and $I = \{(1\ 2), (2\ 3), (5\ 6)\}$, $J = \{(2\ 3), (3\ 4), (4\ 5)\}$.

To begin, first we determine the equivalence classes on $\{1, 2, \cdots, n\}$ coming from I and J. This notion was touched on briefly in Example 7.25, and we compute it as follows. We define $\sim_I$ to be the smallest equivalence relation on $\{1, 2, 3, \cdots, n\}$ that has the following characteristic. For each simple reflection, $(i\ i+1) \in I$, $i \sim_I i+1$. The relation $\sim_I$ roughly represents belonging to the same connected component of I.

For example, if $n = 10$ and $I = \{(2\ 3), (3\ 4), (5\ 6), (8\ 9), (9\ 10)\}$ then we would get the equivalence relation $\sim_I$ with equivalence classes, $\{1\}$, $\{2, 3, 4\}$, $\{5, 6\}$, $\{7\}$, and $\{8, 9, 10\}$. In our trichotomy construction, the equivalence classes of I and J will be the rows and columns respectively of the matrix that are allowed to be swapped among themselves.

$$r = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$r_- = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

To find r_- first split the matrix up based on the classes of $\sim_J$. $\{(2\ 3),(3\ 4),(4\ 5)\}$ translates to $\{1\},\{2,3,4,5\}$, and $\{6\}$. We group the columns based on these equivalence classes. Any equivalence class of size one will not be affected (as J has no element which can interact with that column, that is, no columns can be swapped in that class). For the submatrices associated to equivalence classes with more than two elements we distinguish them and perform the following column swaps.

For each submatrix, we use column swapping to arrange the columns so that the nonzero columns are on the right and are arranged so the leading ones form our usual staircase pattern. This is exactly what we did for the original trichotomy, we are just confined to submatrices.

After all the columns in each of the relevant submatrices have been arranged we put the submatrices back in their positions and the resulting matrix is r_- for J.

To compute r_+ first split the matrix up based on the classes of $\sim_I$. $\{(1\ 2),(2\ 3),(5\ 6)\}$ translates to $\{1,2,3\}$, $\{4\}$, and $\{5,6\}$ and means we look at the matrix formed by the first three rows, the matrix formed by the fourth row, and the matrix formed by the last two rows. The fourth row, associated to an equivalence class of only one element will not change, so we will ignore it. We distinguish the other two submatrices as light gray and dark gray respectively.

$$r = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$r_+ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

For each of the submatrices, we use row swapping operations to arrange the rows so that the nonzero rows are on the top and are arranged so the leading ones create the staircase pattern. Once the matrices are rearranged, they are then put back in the original r matrix, taking the place of the submatrices we removed.

Notice that the computation of r_+ only used I. This shows us that regardless of choice of J, once we know what I is, the r_+ element of our trichotomy is given. Likewise for r_- and our choice of J.

Once r_- and r_+ have been obtained, a simple computation $r_-^* r r_+^*$ yields r_0.

$$r_0 = r_-^* r r_+^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

There is potentially a way to calculate r_0 for $n \times n$ matrices by pure observation (without first calculating r_- and r_+) as we did in Section 3, however at this time such a method proves elusive. In any case, the ease of computing r_- and r_+ would likely ensure that our $r_-^* r r_+^*$ calculation is faster.

For $M_3(K)$, we get the simple reflections $S = \{(1\ 2), (2\ 3)\} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$. Below is a table showing the relative decompositions, $r = r_-^{I,J} r_0^{I,J} r_+^{I,J}$ for each of the 16 pairs of subsets of S and for the element, $r = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

$I \setminus J$	$\emptyset$	$\{(1\ 2)\}$	$\{(2\ 3)\}$	$\{(1\ 2), (2\ 3)\}$
$\emptyset$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$\{(1\ 2)\}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$\{(2\ 3)\}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\{(1\ 2), (2\ 3)\}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Interested readers can consult the final section in the Appendix for graphs displaying the $\mathscr{J}^{I,J}$-classes in relation to the Bruhat order for $M_3(K)$ and all 16 pairs of $I, J \subseteq S$. As well, readers are encouraged to consider the generalisation of the $M_n(K)$ constructions of the relative minima and maxima from Section 6.

8 Projective Supports

At this point, we are going to shift gears and change from our predominantly combinatorial discussion of reductive monoids and instead investigate some geometric properties of the more general, regular semigroups. In this section we will tackle Renner's conjecture on the projectiveness of supports for irreducible regular algebraic semigroups with zero.

8.1 Rees Theorem And Quotients In Linear Algebraic Semigroups

Rees Theorem And Quotients In Linear Algebraic Semigroups ([24]) is a paper by Mohan Putcha about Rees' theorem on linear algebraic semigroups. Published in 2013, it reexamines the notion of support and studies varieties related to the Rees theorem. To get started, we will need a few results from Putcha's paper. The results we will need will concern a particular kind of semigroup: irreducible regular linear algebraic semigroups with zero.

Proposition 8.1. *Let S be an irreducible regular linear algebraic semigroup. Then S has a unique maximal $\mathscr{J}$-class.*

Proof. This comes from Theorem 5.10 in [20] combined with the fact that finite lattices have a unique maximal element. □

Proposition 8.2. *Let S be an irreducible regular linear algebraic semigroup with zero, and $e \in E(S)$. Then eSe is a reductive linear algebraic monoid.*

Proof. We can see that eSe is an irreducible monoid with zero (irreducibility coming from being the image of S under the morphism, $s \mapsto ese$). It also easily is seen that eSe is regular. Thus, by Theorem 4.2 in [30], eSe is reductive. □

The following definition of supports for a regular semigroup with zero is the true starting point of this section. It is the question of whether supports are projective which will be our overall goal in this section.

Definition 8.3. *Let S be an irreducible regular linear algebraic semigroup with zero. Let J be the unique maximal $\mathscr{J}$-class of S. We define the* **support** *of S to be $\mathbb{X} = J/\mathscr{H}$. Further, we define the* **right (left) support** *of S as $\mathbb{X}_r = J/\mathscr{L}$ ($\mathbb{X}_\ell = J/\mathscr{R}$).*

Theorem 8.4. *Let S be an irreducible regular linear algebraic semigroup with zero. Let J be the unique maximal $\mathscr{J}$-class of S. Fix $e \in E(J)$, and let R, L, H be the respective $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{H}$-classes of e. One can quickly see that H acts on R on the left and L on the right.*

(i) $\mathbb{X}_r \cong R/H$ (as a left action quotient).

(ii) $\mathbb{X}_\ell \cong L/H$ (as a right action quotient).

(iii) $\mathbb{X} \cong \mathbb{X}_r \times \mathbb{X}_\ell$.

Proof. This result comes from [24] as Theorem 2.4. □

Putcha notes in his paper that our supports, $\mathbb{X}$, $\mathbb{X}_r$ and $\mathbb{X}_\ell$ each have the structure of a quasi-projective variety (Lemma 2.3 of [24]) by showing that both $\mathbb{X}_r$ and $\mathbb{X}_\ell$ can be considered as lying inside a suitable Grassmanian space and from there finding an open affine covering. He remarks that if the supports are projective varieties that they have some nice properties. It is that work that motivates this paper.

One may think that singling out idempotents of a maximal $\mathscr{J}$-class may be a bit specific, but as the following proposition demonstrates we can construct this situation at will.

Proposition 8.5. *Let S be an irreducible regular algebraic semigroup. For any idempotent, $e \in E(S)$, the set $\overline{SeS}$ is an irreducible regular algebraic semigroup with J_e as its maximal $\mathscr{J}$-class.*

Proof. By Proposition 2.26 we know that $\overline{SeS} = SeS$, so we instead think in terms of SeS. It is clear that SeS is a subsemigroup and irreducible, since S is irreducible. For any $s \in SeS \subseteq S$ there is an element $\overline{s} \in S$ so that $s\overline{s}s = s$. Thus, $\overline{s}s\overline{s} \in SeS$ and $s(\overline{s}s\overline{s})s = s$, so SeS is also regular.

It is clear that for any $x, y \in SeS$, $x \mathscr{J} y$ in SeS implies $x \mathscr{J} y$ in S. Suppose instead that $x \mathscr{J} y$ in S. Then we can find $a, b, c, d \in S$ so that $axb = y$ and $cyd = x$. Since S is regular we can find $\overline{x}, \overline{y} \in S$ so that $x\overline{x}x = x$ and $y\overline{y}y = y$. Thus $(y\overline{y}a)x(b\overline{y}y) = y\overline{y}y\overline{y}y = y$ and $(x\overline{x}c)y(d\overline{x}x) = x$. Since $x, y \in SeS$ it is clear that $(y\overline{y}a), (b\overline{y}y), (x\overline{x}c), (d\overline{x}x) \in SeS$ so $x \mathscr{J} y$ in SeS. This means we can talk about J_x without having to distinguish between the setting of SeS or S.

For any $x \in SeS$ it follows that $SxS \subseteq SeS$ and so $J_x \leq J_e$. It is also apparent that $J_e \subseteq SeS$. Thus J_e is maximal among $\mathscr{J}$-classes of SeS. Indeed, $SeS = \bigsqcup_{J \in \mathcal{U}(S), J \leq J_e} J$. □

Putcha goes on to prove a result about the structure of S with respect to these $\mathscr{R}$- and $\mathscr{L}$-classes. It was stated in [24] as Renner's conjecture.

Theorem 8.6. *Let S be an irreducible regular linear algebraic semigroup with zero. Let J is the maximal $\mathscr{J}$-class of S and pick $e \in E(J)$. Let R, L be the $\mathscr{R}$, $\mathscr{L}$-classes of e. Then $eS = eSe \cdot R$, $Se = L \cdot eSe$ and $S = L \cdot eSe \cdot R$.*

Proof. This is Theorem 3.2 in [24]. □

In fact, as Renner had pointed out in private communication, his conjecture was that projectivity of the supports would likely follow from Theorem 3.2 of [24]. This will henceforth be called Renner's Conjecture and will be the focus of the first part of this paper.

Conjecture 8.7 (Renner's Conjecture)**.** *Let S be an irreducible regular linear algebraic semigroup with zero. $\mathbb{X}$, $\mathbb{X}_r$ and $\mathbb{X}_\ell$ are projective varieties.*

In this section we will pull together all the necessary results to prove Renner's conjecture about the projectiveness of supports, $\mathbb{X}_r$ and $\mathbb{X}_\ell$. Throughout this section, let S be an irreducible regular linear algebraic semigroup with zero and with maximum $\mathscr{J}$-class, J. Fix $e \in E(J)$ and let H, R, L denote the respective $\mathscr{H}$-, $\mathscr{R}$-, $\mathscr{L}$-classes of e. Then eSe is an irreducible reductive algebraic monoid with unit group H.

8.2 Geometric Invariant Theory

Quotients, like those that define the supports, $\mathbb{X}_r$ and $\mathbb{X}_\ell$, are difficult to deal with in an algebraic geometry sense. So before we move on to deal with the conjecture proper, we will need to cover some basic results in geometric invariant theory. This is the language that is best for handling this problem.

Much of geometric invariant theory concerns itself with actions by reductive algebraic groups. It is fortunate that H is a reductive group (since eSe is a reductive monoid).

Theorem 8.8. *Let A be a finitely generated K-algebra. If G is a reductive group acting on A then A^G is also a finitely generated K-algebra.*

Proof. Theorem 3.4 in [16]. □

From geometric invariant theory we inherit two notions of quotient when a variety is acted upon by a reductive group. We will need to utilize both in order to tackle the question of projectivity of the left and right supports of S.

Definition 8.9. *Let G be a reductive group acting on variety X. Consider a variety, Y and an affine morphism $\phi : X \to Y$. If (Y, ϕ) satisfies the following properties it is called a* **good quotient** *of X by G.*

(i) ϕ is G-invariant

(ii) ϕ is surjective

(iii) if $U \subseteq Y$ is open, then $\phi^* : \mathcal{O}(U) \to \mathcal{O}(\phi^{-1}(U))$ is an isomorphism of $\mathcal{O}(U)$ onto $\mathcal{O}(\phi^{-1}(U))^G$

(iv) if $W \subseteq X$ is closed and G-invariant then $\phi(W)$ is closed

(v) if $W_1, W_2 \subseteq X$ are disjoint, closed, and G-invariant, then $\phi(W_1)$ and $\phi(W_2)$ are disjoint

If (Y, ϕ) is also an orbit space, it is called a **geometric quotient** *of X by G.*

Theorem 8.10. *Let G be a reductive algebraic group and X an affine variety. Then there is an affine variety, Y, and affine morphism, $\phi : X \to Y$, so that (Y, ϕ) is a good quotient of X by G.*

Proof. This is the content of the proof of Theorem 3.5 in [16]. □

Proposition 8.11.

(1) Let (Y, ϕ) be a good (geometric) quotient of projective variety, X, by reductive group, G. If U is open in Y, then (U, ϕ) is a good (geometric) quotient of $\phi^{-1}(U)$ by G.

(2) If $\phi : X \to Y$ is a morphism and $\{U_i\}_{i \in I}$ is an open covering of Y such that (U_i, ϕ) is a good (geometric) quotient of $\phi^{-1}(U_i)$ by G for all $i \in I$, then (Y, ϕ) is a good (geometric) quotient of X by G.

Proof. This is just Proposition 3.10 in [16]. □

Proposition 8.12. *Let (Y, ϕ) be a good quotient of X by G. If the action of G on X is closed, then (Y, ϕ) is a geometric quotient.*

Proof. This is just Proposition 3.11(iii) in [16]. □

For this section of introductory material, we will limit our geometric invariant theory results to those that will explicitly apply to our proof of Renner's conjecture. Although this will require some technical proofs that are basically reproductions of the work in [16], it saves us a discussion of linearisations and ample line bundles. To this end, consider the following scenario:

Let A be an affine variety over K and let K^* act on A. Suppose that A contains a **cone point** a_0. That is, $a_0 \in \overline{K^* y}$ for all $y \in A$. On the level of our coordinate algebra, this turns $\mathcal{O}(A)$ into a nonnegatively graded algebra. A function $f \in \mathcal{O}(A)$ is called **homogeneous** of degree n, for some $n \in \mathbb{N}$, if for all $k \in K^*$, $y \in A$, $f(ky) = k^n f(y)$. If we denote $\mathcal{O}(A)_n$ to be the homogeneous functions of degree n, then $\mathcal{O}(A) = \oplus_{n\in\mathbb{N}} \mathcal{O}(A)_n$.

We let P be the projective variety, $(A\backslash\{a_0\})/K^*$, with projection map, $\pi : A\backslash\{a_0\} \to P$. Suppose that we have a reductive group, G, that acts on A by action σ which commutes with the action of K^*. Then the action of G on P is compatible with our projection map (it follows that $ga_0 = a_0$ for all $g \in G$). Our scenario can be summed up by the following commutative diagram.

$$\begin{array}{ccc} G \times A\backslash\{a_0\} & \xrightarrow{\sigma} & A\backslash\{a_0\} \\ {\scriptstyle id\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times P & \xrightarrow[\overline{\sigma}]{} & P \end{array}$$

Our goal is to consider the supports as geometric quotients arising from exactly this situation involving A and P. So to get there, we now introduce the notions of stable and semi-stable elements in P.

Definition 8.13. *For the situation of varieties A and P which we have defined, we say that a point $x \in P$ is called,*

(i) **semi-stable** *if and only if there is a homogeneous function $f \in \mathcal{O}(A)^G$ of degree ≥ 1 such that $f(x) \neq 0$. By P^{ss} we shall mean the set of all semi-stable elements in P.*

(ii) **stable** *if and only if x there is a homogeneous function $f \in \mathcal{O}(A)^G$ of degree ≥ 1 such that $f(x) \neq 0$ and the action of G on P_f is closed ($P_f = \{x \in P \mid f(x) \neq 0\}$). By P^s we shall mean the set of all stable elements in P.*

(iii) **unstable** *if and only if it is not semi-stable.*

Notice that the conditions of semi-stability and stability rely in part on homogeneous polynomials. So we can choose to show that for some $\hat{x} \in A$ in the fibre associated to $x \in P$ satisfies $f(\hat{x}) \neq 0$ when convenient. These sets of semi-stable and stable elements give us the following theorem, which establishes the existence of good and geometric quotients.

Theorem 8.14. *There exists a good quotient, (Y, ϕ) of P^{ss} by G, and Y is projective. Additionally, there exists an open subset, $Y^s \subseteq Y$ such that $\phi^{-1}(Y^s) = P^s$ and (Y^s, ϕ) is a geometric quotient.*

This proof is essentially a reproduction of Newstead's proof of Theorem 3.14 (see [16]), but is specialized to our particular conditions.

Proof. Since we assumed the action of G commutes with the action of K^* we can see that the action of G on $O(A)$, given by $g \cdot f(x) = f(g \cdot x)$ preserves the degree of homogeneous functions. Indeed, $(g \cdot f)(k \cdot x) = f(g \cdot k \cdot x) = f(k \cdot g \cdot x) = k^n f(g \cdot x) = k^n (g \cdot f)(x)$. Thus, $O(A)^G$ is a homogeneous subalgebra of $O(A) = \oplus_{n \in \mathbb{N}} O(A)_n$.

For homogeneous $f \in O(A)^G$ with $deg(f) \geq 1$, define $P_f = \{x \in P \mid f(x) \neq 0\}$. Notice that $P^{ss} := \bigcup_{f \in O(A)^G, deg(f) \geq 1} P_f$. Each P_f is an open affine subset of P (being the compliment of the zero set of f). Since G is reductive we know that there exists a good quotient of P_f by G. (Theorem 8.10), (Y_f, ϕ_f).

We can glue together all of these good quotients, (Y_f, ϕ_f), to form a projective variety, Y, and also get a map $\phi : \bigcup_{f \in O(A)^G, deg(f) \geq 1} P_f = P^{ss} \to Y$. Our gluing maps should look like, $h_{ff'} : \phi_f(P_{ff'}) \to \phi_{f'}(P_{ff'})$. As Brion notes in [5], this is achieved in a similar way that $\mathbb{P}^n$ is obtained from its open subsets $\mathbb{P}^n_f$. In this way we see that $\phi |_{P_f} = \phi_f$.

Observe that each (Y_f, ϕ_f) is a good quotient of $P_f = \phi^{-1}(Y_f)$. We also note by our gluing that $\{Y_f\}_{f \in O(A)^G, deg(f) \geq 1}$ is an open cover of Y (see 12.6 in [35]). Thus, by Proposition 8.11 it follows that (Y, ϕ) is a good quotient of P^{ss}.

For the second part of our result, let $Y^s := \phi(P^s)$ and let Y^0 be the union of those Y_f for which the action of G on P_f is closed. It is clear that $P^s \subseteq \phi^{-1}(Y^0)$ and thus, $Y^s \subseteq Y^0$. By Proposition 8.11, (Y^0, ϕ) is a good quotient of $P^0 := \phi^{-1}(Y^0)$. Applying Proposition 8.12 then tells us that (Y^0, ϕ) is a geometric quotient of $P^0 := \phi^{-1}(Y^0)$. It follows that $P^s = \phi^{-1}(Y^s)$ and $Y^0 \backslash Y^s = \phi(P^0 \backslash P^s)$.

Thus, $Y^0 \backslash Y^s$ is closed in Y^0 by (*iv*) in our definition of a good quotient. So Y^s is open in Y^0. Applying Proposition 8.11 again, we conclude that (Y^s, ϕ) is a geometric quotient of P^s. □

Now that semistable and stable points give us quotients (as we will see, the quotients we will need to prove the conjecture) we devote the rest of this section to converting our definitions of semistable and stable into more useful forms. These forms are the ones used by C. S. Seshadri in [33] and are equivalent to Newstead's and Mumford's. First we need a lemma.

Lemma 8.15. *Let G be a reductive group acting on affine variety, X. Let X_1, X_2 be two disjoint, closed, G-invariant subsets of X. Then there is a function $f \in \mathcal{O}(X)^G$ so that $f(X_1) = 0$ and $f(X_2) = 1$.*

Proof. This is Lemma 3.3 in [16]. □

Here is our new definition for semistability.

Proposition 8.16. *Let $x \in P$ be an arbitrary element. Then $x \in P^{ss}$ if and only if $a_0 \notin \overline{G\hat{x}}$ (here $\hat{x}$ represents a preimage of x under π).*

Proof. Let $f \in \mathcal{O}(A)$ be a G-invariant homogeneous function of degree ≥ 1 such that $f(x) \neq 0$. Observe that since $deg(f) \geq 1$, then $f(ky) = k^{deg(f)} f(y)$ for all $y \in A$ and $k \in K^*$, and by taking the closure, we can conclude that $0 = f(a_0)$. Then it is clear that $f(\hat{x}) \neq 0$, and by G-invariance $f(y)$ is equal to a non-zero constant for all $y \in G\hat{x}$. Hence $a_0 \notin \overline{G\hat{x}}$.

Conversely, if $a_0 \notin \overline{G\hat{x}}$, then there exists by Lemma 8.15 a G-invariant function, f, such that $f(a_0) = 0$, $f\left(\overline{G\hat{x}}\right) = 1$. Then f has a constant term of 0, and it follows that some homogeneous part of f of degree ≥ 1 must be nonzero at $\hat{x}$, so x is semi-stable. □

In the same vein, we can find an equivalent condition for stability.

Lemma 8.17. *An element $x \in P$ is stable if and only if there exists a homogeneous function $f \in \mathcal{O}(A)^G$ that has degree ≥ 1 such that $f(x) \neq 0$ and the morphism $\tau_f : G \to P_f$ given by $\tau_f(g) = gx$ is proper.*

Proof. This is Remark 3.16 (and Lemma 3.15) of [16]. □

Lemma 8.18. *Let G be a linear group acting on variety, X. Then for $x \in X$, the morphism $\tau : G \to X$, given by $\tau(g) = gx$, is proper if and only if Gx is closed in X and G_x is finite.*

Proof. This is Lemma 3.17 in [16]. □

Proposition 8.19. *Let $x \in P$ be an arbitrary element. Then $x \in P^s$ if and only if $|G_{\hat{x}}| < \infty$ and $G\hat{x}$ is closed in A.*

Proof. By Lemma 8.17 $x \in P$ is stable if and only if there exists a homogeneous function $f \in \mathcal{O}(A)^G$ of degree ≥ 1 such that $f(x) \neq 0$ and the morphism $\tau_f : G \to P_f$ given by $\tau_f(g) = gx$ is proper.

Fix an element, $\hat{x} \in A$ over x. Let $c = f(\hat{x}) \neq 0$, and define $C = \{y \in A \mid f(y) = c\}$, clearly a closed subvariety of A. Consider the morphism $\overline{\tau}_f : G \to C$ given by $\overline{\tau}_f(g) = gx$. It can be seen that $\tau_f = \pi \circ \overline{\tau}_f$. Letting i be the inclusion map of C into A we get,

$$\begin{aligned} \tau_f \text{ is proper } &\iff \pi \circ \overline{\tau}_f \text{ is proper (by equality)} \\ &\iff \overline{\tau}_f \text{ is proper } (\Leftarrow \text{ follows from } \pi \text{ being proper}) \\ &\iff i \circ \overline{\tau}_f : G \to A \text{ is proper (since } C \text{ is closed in } A) \\ &\iff \tau \text{ is proper (by equality)} \end{aligned}$$

So it follows that $x \in P$ is stable if and only if there is a homogeneous function $f \in \mathcal{O}(A)^G$ that has degree ≥ 1 such that $f(x) \neq 0$ and the morphism $\tau : G \to A$ given by $\tau(g) = gx$ is proper. By Proposition 8.16 this is equivalent to $a_0 \notin \overline{G\hat{x}}$ and τ being proper. Then, thanks to Lemma 8.18, we can see that this is equivalent to the conditions that $a_0 \notin \overline{G\hat{x}}$, $|G_{\hat{x}}| < \infty$ and $G\hat{x}$ is closed in A.

Now, if $G\hat{x}$ is closed in A, it follows that $a_0 \notin \overline{G\hat{x}} = G\hat{x}$, since $G\{a_0\} = \{a_0\}$. So then τ being proper implies $a_0 \notin \overline{G\hat{x}}$. Thus, $x \in P$ is stable if and only if $|G_{\hat{x}}| < \infty$ and $G\hat{x}$ is closed in A. □

Geometric invariant theory will come in to play in our proof as our goal is to show that our right are left supports can be represented as P^s/G, and thus is the orbit space we want, but is also projective.

8.3 Putcha's Determinant

By Corollary 3.15 in Putcha's book [20], we can consider S to be a closed subsemigroup of $M_n(K)$ for some n. The following definition from Putcha's book relies on this predetermined embedding of S.

Definition 8.20. *We define the* **determinant with respect to** e *as the map,* $det_e : S \to K$ *given by* $det_e(s) := det(ese + 1 - e)$.

We have defined our determinant by using the ambient $M_n(K)$. It is entirely conceivable that the map would change based on our embedding of S, but that will not affect our proof of Renner's conjecture. Although we defined the determinant on all of S, it only gains the familiar multiplicative property when we restrict to eSe.

Proposition 8.21. *When restricted to* eSe, det_e *is a multiplicative morphism.*

Proof. It is clear that the map is already a morphism in the algebraic geometry sense, just from its definition. It remains to show that it is multiplicative, hence a true algebraic monoid morphism. Let $a, b \in eSe$. Then,

$$\begin{aligned} det_e(a)det_e(b) &= det(a + 1 - e)det(b + 1 - e) \\ &= det(ab + a - ae + b + 1 - e - eb - e + e) = det(ab + 1 - e) = det_e(ab) \end{aligned}$$ □

In light of the previous result and its desired property, we will only consider det_e as restricted to eSe.

Proposition 8.22. $H = \{m \in eSe \mid det_e(m) \neq 0\}$

Proof. This comes from Remark 3.23 in [20]. □

The relative determinant is how Putcha shows in his book that H is an algebraic group. The group, H provides the $GL_n(K)$ to eSe's $M_n(K)$. In the same vein we take the opportunity now to define a useful analogue of the special linear group.

Definition 8.23. $H_1 := \{m \in eSe \mid det_e(m) = 1\}$

While Putcha shows in his paper ([19]) that $H = det_e^{-1}(K^*)$ is independent of the particular embedding in $M_n(K)$, our definition of H_1 is quite dependent on it. However, regardless of which of the possible H_1 we have, the following results show that any such $H_1 = det_e^{-1}(1)$ has the properties we will need to prove Renner's conjecture.

Theorem 8.24. *H_1 is a closed, reductive algebraic group in eSe.*

Proof. That H_1 is a closed algebraic variety comes from its definition as the preimage, $det_e^{-1}(1)$. It is a normal connected subgroup of H, by Proposition 8.22 and the multiplicativity of det_e in Proposition 8.21 and the fact that it is the kernel of det_e. Since H_1 is a connected closed normal subgroup of the reductive group H it follows that H_1 is reductive by 14.2 of [3]. □

Proposition 8.25. *Let $u \in H$ be a unipotent element. Then $u \in H_1$. That is, $det_e(u) = 1$.*

Proof. $det_e : H \to K^*$ is a morphism of algebraic groups. So then for any element, $h \in H$, $det_e(h_u h_s) = det_e(h_u) det_e(h_s)$ preserves the Jordan decomposition. Hence, det_e takes unipotent elements to unipotent elements. But $1 \in K^*$ is the unique unipotent element of K^*. So for any unipotent $u \in H$, $det_e(u) = 1$ as desired. □

We will use H_1 a great deal to reach our end result. First, however, we will move on and introduce Renner's maps and the action of K^* that they provide. We fix a maximal torus of eSe, call it T. The majority of what follows comes from Exercise 5 in Section 4.6 of [30].

Definition 8.26. *Define the set of idempotents of corank 1, to be the set of all idempotents which lies just below e in the Adherence order. $E^1(\overline{T}) = \{f \in E(\overline{T}) \mid e \text{ covers } f\}$.*

For the above definition, remember that e is the identity element of eSe.

Lemma 8.27. *For any $f_0 \in E(\overline{T})$, $\Pi_{f \in E^1(\overline{T}), f_0 \leq f} f = f_0$*

This is a useful way of creating a product of 0 out of our idempotents in $E^1(\overline{T})$.

Proof. This comes from Proposition 3.22 b) in [30], although we wish to acknowledge the typo in the book, as the definition of the set "$E^1(f)$" should contain a "$f \leq e$". □

Proposition 8.28. *For each $f \in E^1(\overline{T})$ there is a unique injective morphism $\alpha_f : K \to \overline{T}$ such that $\alpha_f(0) = f$, $\alpha_f(K^*) = \{t \in T \mid tf = ft = f\}^0$.*

Proof. Exercise 5 b), since $\overline{T}$ is a D-monoid with zero. □

These maps are what we meant when we referred to **Renner's maps**. Taken together they provide the action of K^* that we will be using to prove Renner's conjecture. This action interacts nicely with Putcha's relative determinant.

Definition 8.29. *For $k \in K$ and $m \in S$ we define the product $k \cdot m = \left(\Pi_{f \in E^1(\overline{T})} \alpha_f(k)\right) m$.*

The following proposition shows that our product exhibits the properties of a group action when restricted to K^*. It also behaves like scalar multiplication in the sense that $0 \cdot m = 0$.

Proposition 8.30. *For $m, m' \in S$, and $k, k' \in K$ the following are true,*

(1) $k \cdot (k' \cdot m) = (kk') \cdot m$

(2) $k \cdot (mm') = (k \cdot m)m'$

(3) $1 \cdot m = m$

(4) $0 \cdot m = 0$

Proof. (1) $k \cdot (k' \cdot m) = k \cdot \left(\left(\Pi_{f \in E^1(\overline{T})} \alpha_f(k')\right) m\right) = \left(\Pi_{f \in E^1(\overline{T})} \alpha_f(k)\right)\left(\left(\Pi_{f \in E^1(\overline{T})} \alpha_f(k')\right) m\right)$
But by associativity of our monoid multiplication, we get,

$$= \left(\Pi_{f \in E^1(\overline{T})} \alpha_f(k) \alpha_f(k')\right) m$$

And since each of our α_f is a morphism, and the images of our field elements are in $\overline{T}$, which is commutative

$$= \left(\Pi_{f \in E^1(\overline{T})} \alpha_f(kk')\right) m = (kk') \cdot m$$

(2) This is just an application of the associative law.

(3) For each $f \in E^1(\overline{T})$, $\alpha_f(1) = 1$, the identity of our monoid. So then,

$$1 \cdot m = \left(\Pi_{f \in E^1(\overline{T})} \alpha_f(1)\right) m = \left(\Pi_{f \in E^1(\overline{T})} 1\right) m = m$$

(4) By definition, for each $f \in E^1(\overline{T})$, $\alpha_f(0) = f$. By Lemma 8.27, we can conclude,

$$0 \cdot m = \left(\Pi_{f \in E^1(\overline{T})} \alpha_f(0)\right) m = \left(\Pi_{f \in E^1(\overline{T})} f\right) m = (0)m = 0$$ □

We would like to show that det_e is a homogenous morphism. That is, there exists some q so that $det_e(ka) = k^q det_e(a)$. This is a well-known property of the original determinant function.

Proposition 8.31. *For each $f \in E^1(\overline{T})$, there exists some positive integer, q_f, so that for every $k \in K$, $det_e(\alpha_f(k)) = k^{q_f}$.*

Proof. As it is a composition of a series of morphisms, we can see that $p : x \mapsto det_e(\alpha_f(x))$ is a polynomial, $p(x)$. See that $p(x)$ is nonzero and multiplicative,

$$p(xy) = det_e(\alpha_f(xy)) = det_e(\alpha_f(x)\alpha_f(y)) = det_e(\alpha_f(x))det_e(\alpha_f(y)) = p(x)p(y).$$

It follows that, $p(x) = x^q$ for some $q \in \mathbb{N}$ (by Lemma A.5). And since $f \notin H$, we know $det_e(f) = det_e(\alpha_f(0)) = 0$. But if $q_f = 0$ then $det_e(f) = 1$ a contradiction. Thus, $1 \leq q_f$. □

Corollary 8.32. *There exists some positive integer, q, so that for $m \in eSe$ and $k \in K$, $det_e(km) = k^q det_e(m)$. In fact, $q = \Sigma_{f\in E^1(\overline{T})} q_f$.*

Proof. Since det_e is multiplicative, we see that,

$$\begin{aligned} det_e(km) &= det_e\left((\Pi_{f\in E^1(\overline{T})}\alpha_f(k))m\right) = \left(\Pi_{f\in E^1(\overline{T})} det_e(\alpha_f(k))\right) det_e(m) \\ &= \Pi_{f\in E^1(\overline{T})} k^{q_f} det_e(m) = k^q det_e(m), \end{aligned}$$

where q is defined as in the statement above. □

Thus, det_e is homogeneous of degree ≥ 1. Our final result before attempting the conjecture shows us that our reductive group and our action of K^* commute.

Proposition 8.33. *For any $k \in K$, $h \in H_1$, and $s \in S$, $h(k \cdot s) = k \cdot hs$.*

Proof. This comes from the fact that $\Pi_{f\in E^1(\overline{T})}\alpha_f(k) \in \overline{T} \subseteq Z(eSe)$ and $H_1 \subseteq eSe$. □

Now we have everything we need to discuss Renner's conjecture properly. The group H_1 provides exactly the reductive group we need to show the closed conditions associated to semi-stability and stability. Renner's maps will allow us to make a Bruhat decomposition argument to show the nature of the semi-stable elements.

8.4 The Conjecture

To start things off, let us define our scenario so that we may apply geometric invariant theory to create a good quotient. Let $A = eS$ and let K^* act upon it by our action from the last section. We see that 0 is the cone point of A. Then P is the projective variety defined by $P = \left\{K^* s \;\middle|\; s \in eS\backslash\{0\}\right\}$. By taking $G = H_1$ we get a reductive group acting on A and P as we desired from our Geometric Invariant Theory work.

Proposition 8.34. *Let X, Y be affine varieties over K. Suppose that K^* acts on X and Y as a group action. Suppose also that we have a morphism, $\phi : X \to Y$ such that for all $x \in X$, $k \in K^*$ then $\phi(kx) = k\phi(y)$. Let $x_0 \in X$ and $y_0 \in Y$ be such that for all $x \in X$, $x_0 \in \overline{K^*x}$ and all $y \in Y$, $y_0 \in \overline{K^*y}$.*

If $\phi^{-1}(y_0) = \{x_0\}$, then ϕ is a finite morphism.

Proof. This comes from Remark 2.25(d) in [30]. □

The following lemma will be used for both our discussion of the semi-stable elements of eS and the stable elements of eS.

Lemma 8.35. *For all $r \in R$, $H_1 r$ is closed.*

Proof. Since $r \in R$, we can find $x, y \in S^1$ so that $e = rx$ and $r = ey$. Define the new element $z = ye \in S$. Consider the map, $\phi : eSe \to eS$ given by $\phi(s) = sr$. This map ϕ is one-to-one by observing that if $\phi(s_1) = \phi(s_2)$ then $s_1 = s_1 e = s_1 rz = s_2 rz = s_2 e = s_2$. Thus $0 = \phi^{-1}(0)$.

Observe that for any $k \in K$, $\phi(k \cdot s) = (k \cdot s)r = k \cdot (sr) = k \cdot \phi(s)$. Thus, by Proposition 8.34, ϕ is a finite morphism and hence closed. So then the image of H_1 is closed in eS, and we may conclude that $H_1 r$ is closed. □

We are now in a position to describe the semi-stable elements of $P = eS\backslash\{0\}/K^*$. Putcha's prior work to show that $eS = eSe \cdot R$ comes into play here, along with the familiar Bruhat decomposition.

Proposition 8.36. $(P)^{ss} = \{K^* r \mid r \in R\}$

Proof. Suppose that $s \in R$. By the preceding lemma, $\overline{H_1 s} = H_1 s \subseteq R \subseteq eS\backslash\{0\}$. By our criteria for semi-stability, we can conclude that $\{K^* r \mid r \in R\} \subseteq (P)^{ss}$.

Now, suppose that $s \notin R$. By Theorem 8.6, we can write $s = m \cdot r$ where $m \in eSe$ and $r \in R$. First observe the refinement that $m \in eSe\backslash H$, as otherwise, $m \in H \cdot R = R$.

Since eSe is reductive, we can pick a Borel subgroup, B, containing our T, and perform the Bruhat decomposition. That is, we can find $b, b' \in B$ and $\sigma \in \mathcal{R} = \overline{N_H(T)}/T$ so that, $m = b\sigma b'$. Additionally, $B = UT$, where U is the unipotent subgroup of H. So we can find $u \in U$, $t \in T$ so $b = ut$ and write $m = ut\sigma b' = u\sigma t b'$. However, as pointed out in 8.1 of [30], regularity of $\mathcal{R}$ means that we can write $\sigma = e_0 w$, where $e_0 \in E(\overline{T})$ and $w \in W = N_H(T)/T$. Partition $E^1(\overline{T}) = F \bigsqcup E$, where $F = \{f \in E^1(\overline{T}) \mid e_0 \leq f\}$ and $E = E^1(\overline{T}) \backslash F$. As $m \notin H$, $e_0 \neq 1$, so F is nonempty.

If E is empty, then $F = E^1(\overline{T})$ and Lemma 8.27 tells us $e_0 = 0$ and hence $m = 0$. Thus, $0 \in H_1 s = \{0\}$. Suppose that E is nonempty, and distinguish an element, $f_0 \in F$.

For each $k \in K^*$, consider the tuple, $\left(a_f\right)_{f \in E^1(\overline{T})}$, defined by, $a_f = 1$ for all $f \in F \backslash \{f_0\}$, $a_f = k$ for $f \in E$, and a_{f_0} is a solution to the polynomial $x^{q_{f_0}} - k^{-\Sigma_{f \in E} q_f} = 0$ (recall our constants q_f from Proposition 8.31). Observe that $\Pi_{f \in E^1(\overline{T})} a_f^{q_f} = 1$, so then define $h_k := \left(\Pi_{f \in E^1(\overline{T})} \alpha_f(a_f)\right) u^{-1} \in H_1$.

We have a map $\phi : K^* \to eSe$ given by $\phi(k) = h_k m = h_k u e_0 \sigma t b' = (\Pi_{f \in E} \alpha_f(k)) e_0 \sigma t b'$. We see $\phi(K^*) m \subseteq H_1 m$, and so, $0 = (\Pi_{f \in E} f) e_0 = \left(\Pi_{f \in E} \alpha_f(0)) e_0\right) \sigma t b' \in \overline{\phi(K^*) m} \subseteq \overline{\phi(K^*) m}$ means that $0 \in \overline{\phi(K^*) m} \subseteq \overline{H_1 m}$. Thus, $0 \in \overline{H_1 s}$, and we conclude that s is not semi-stable. Thus $(P)^{ss} \subseteq \{K^* r \mid r \in R\}$.

So we may conclude that R lies above P^{ss} as desired. □

With semistability dealt with, we move on to tackle stability.

Lemma 8.37. *For $s \in R$, $|\{h \in H_1 \mid hs = s\}| < \infty$.*

Proof. Since $s \in R$, then there exists $x, y \in S^1$ so that $e = sx$ and $s = ey$. Let $h \in H_1$ be such that $hs = s$. Then, $h^{-1} = h^{-1} e = h^{-1} sx = h^{-1} hsx = sx = e$, and we see that $h = e$. So, $|\{h \in H_1 \mid hs = h\}| = 1 < \infty$. □

Proposition 8.38. $(P)^s = \{K^* r \mid r \in R\}$

Proof. $s \in eS$ is stable if and only if it is semi-stable, $|\{h \in H_e^1 \mid hs = h\}| < \infty$ and $H_1 s$ is closed. By Proposition 8.36, we know that s is semistable if and only if $s \in R$. Thus, $(P)^s \subseteq \{K^* r \mid r \in R\}$.

If $s \in R$, Lemmas 8.35, 8.37 tell us, in addition to being semistable, s is stable. Thus, $\{K^* r \mid r \in R\} \subseteq (P)^s$. So we conclude that R lies above P^s as desired. □

The preceding proofs allow us to equate the good quotient P^{ss}/H_1, which is projective, with the geometric quotient P^s/H_1. The next lemma will allow us to conclude that it is indeed the support $\mathbb{X}_r = R/H$ we are describing.

Lemma 8.39. *Define the map, $\psi : H_1 \times K^* \to H$, by $\psi(h,k) = hk$. Then ψ is finite-to-one and surjective.*

Proof. We know that $det_e(hk) = k^q det_e(h) = k^q \neq 0$ by commutativity of $\Pi_{f \in E^1(\overline{T})} \alpha_f(k)$ and Corollary 8.32. Now, take an arbitrary element $g \in H$, having $det_e(g) \neq 0$. We observe that $\psi^{-1}(g) = \{(b, b^{-1}g) \mid b \in K^*, b^q = det_e(g)\}$ is finite. □

That was the final piece that we needed, and we conclude with the answer to Renner's conjecture on the projectiveness of the supports.

Theorem 8.40. *Any irreducible regular linear algebraic semigroup with zero has projective supports.*

Proof. $P^s = R/K^* = P^{ss}$ by Propositions 8.36 and 8.38. By Theorem 8.14 there exists a good quotient of P^{ss} by H_1 which is projective and a geometric quotient of P^s by H_1. By uniqueness of categorical quotients these two must coincide. By applying Lemma 8.39 to our geometric quotient, $(R/K^*)/H_1 = \left\{H_1\{kr \mid k \in K^*\} \;\middle|\; r \in R\right\} = \left\{\{hkr \mid h \in H_1, k \in K^*\} \;\middle|\; r \in R\right\} = R/H$. This allows us to conclude that R/H is indeed projective.

A similar proof involving results analogous to the ones created in this section allows us to prove the projectiveness of $\mathbb{X}_\ell$. Lastly we conclude that $\mathbb{X} = \mathbb{X}_r \times \mathbb{X}_\ell$ is projective. □

The following is a direct corollary coming from work by Putcha.

Corollary 8.41. *Let S be an irreducible regular linear algebraic semigroup with zero. If e is an idempotent in the maximal $\mathscr{J}$-class of S then $\overline{SeS} = SeS = S$*

Proof. [24]'s Theorem 2.10 tells us that if $\mathbb{X}$ is projective, then $\overline{SeS} = SeS = L \cdot eSe \cdot R$. As we have just proven in Theorem 8.40, $\mathbb{X}$ is projective. The result follows by combining this with Proposition 8.6 which told us that $S = L \cdot eSe \cdot R$ as well. □

Corollary 8.42. *Let S be an irreducible regular linear algebraic semigroup with zero. If e is an idempotent in the maximal $\mathscr{J}$-class of S. Then J_e is open in S.*

Proof. By Proposition 2.15, we know that J_e is open in $\overline{SeS}$. The preceding corollary tells us that $\overline{SeS} = S$, so J_e is open in S. □

8.5 Examples

Consider $S = \{a \in M_4(K) \mid rk(a) \leq 2\}$ Since $rk(ab) \leq rk(a), rk(b)$ it follows that this is indeed a semigroup with zero. In fact, as we will prove later in the following section, S is an irreducible regular algebraic semigroup with zero, exactly the kind we are interested in. As an idempotent with rank 2, $e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is an idempotent in the maximal $\mathscr{J}$-class of S. Having chosen e we can identify the following sets,

$$eS = \{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a_{ij} \in K\} \cong M_{2\times 4}(K)$$

$$Se = \{\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & 0 & a_{43} & 0 \end{pmatrix} \mid a_{ij} \in K\} \cong M_{4\times 2}(K)$$

$$eSe = \{\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a_{ij} \in K\} \cong M_2(K)$$

$$R = \{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid rk(\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}) = 2\} \cong \text{ the } 2\times 4 \text{ matrices with rank } 2$$

$$L = \{\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & 0 & a_{43} & 0 \end{pmatrix} \mid rk(\begin{pmatrix} a_{11} & a_{13} \\ a_{11} & a_{13} \\ a_{31} & a_{33} \\ a_{41} & a_{43} \end{pmatrix}) = 2\} \cong \text{ the } 4\times 2 \text{ matrices with rank } 2$$

$$H = \{\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid rk(\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}) = 2\} \cong GL_2(K)$$

One can even take time to notice these also nicely illustrate Proposition 2.15.

It is not difficult to figure out the morphism, det_e, as we have the natural embedding of $S \subseteq M_4(K)$. So we see that $det_e(\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}) = a_{11}a_{33} - a_{13}a_{31}$. This allows us to write out the set $H_1 = \{\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a_{11}a_{33} - a_{13}a_{31} = 1\} \cong SL_2(K)$.

If we choose the torus $T = \{\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a_{ij} \in K^*\} \subseteq eSe$ then we can see that our corank 1 idempotents are $E^1(\overline{T}) = \{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\}$. This leads us to the following two Renner maps:

$\alpha_{\begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}}(k) = \begin{pmatrix} k&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}$ and $\alpha_{\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}}(k) = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&k&0\\0&0&0&0 \end{pmatrix}$

Finally we can see that our supports are both isomorphic to the Grassmanian variety $G(2,4)$, the set of 2-dimensional subspaces of a 4-dimensional vector space which is a well-known projective variety. With both $\mathbb{X}_r = \mathbb{X}_\ell = G(2,4)$ we see that the support, $\mathbb{X} = G(2,4) \times G(2,4)$ which is also projective.

9 A New Way To Construct Regular Semigroups

The interaction between Green's relations and regular algebraic semigroups showcased in the previous section leads us to thinking, "are there ways to construct semigroups which imitate the behaviour of regular semigroups when it comes to Green's relations?" A more specific question is, "can we create a general construction of regular semigroups, S that allow us to specify the quasiaffine varieties L_e and R_e beforehand?"

9.1 Affinized Quotients

Rather than start with $\mathscr{L}$-classes and $\mathscr{R}$-classes we choose to start with irreducible quasiaffine varieties, L and R, and respective algebraic group actions, $L \times P \to L$ and $Q \times R \to R$. L and R will stand in for L_e and R_e (recall Proposition 2.15) and P and Q will replace the usual action of H_e. We also suppose the existence of M, an irreducible reductive monoid with zero, which will stand in for eSe. Together we have the ingredients for a result like Theorem 8.6.

If we are trying to construct algebraic semigroups it might be prudent to consider another well-known construction. The Rees construction for algebraic semigroups gives us a nice subclass of the Rees matrix semigroups.

Definition 9.1. *Let X and Y be varieties, and S an algebraic semigroup. Suppose there exists a morphism $\phi : Y \times X \to S$. Then we can define the* **Rees construction** *where $X \times S \times Y$ is a semigroup under the morphism $(x, s, y)(x', s', y') = (x, s\phi(y, x')s', y')$.*

In the case where X and Y are affine, then $X \times S \times Y$ becomes a linear algebraic semigroup.

The morphism, ϕ acts as a matrix with rows indexed by X and columns indexed by Y. From now on, we will suppose there is a morphism from $\phi : R \times L \to M$. Using the Rees construction, we can create a semigroup $L \times M \times R$, but unless R and L are affine, we do not have a linear semigroup yet. This leads us to the following definition.

Definition 9.2. *Let X be a variety with an algebraic group G acting on it in such a way that $\mathcal{O}(X)^G$ is finitely generated. We define the* **affinised quotient** *of X by G to be the affine variety, $X/_{aff}G = Spec\left(\mathcal{O}(X)^G\right)$.*

Proposition 9.3. *The affinised quotient we just introduced has the following universal property. If Z is an affine variety and $f : X \to Z$ is a morphism which is constant on the G-orbits, then there exists a unique morphism, $\widetilde{f} : X/_{aff}G \to Z$ so that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle \pi}\downarrow & \nearrow_{\widetilde{f}} & \\ X/_{aff}G & & \end{array}$$

Proof. The morphism f induces a map, $f^* : \mathcal{O}(Z) \to \mathcal{O}(X)$. Since f is G-invariant, it follows that $f^*(\mathcal{O}(Z)) \subseteq \mathcal{O}(X)^G$. So we have a map, $f^* : \mathcal{O}(Z) \to \mathcal{O}(X)^G = \mathcal{O}(X/_{aff}G)$. By Proposition 3.5 of [11], there exists a morphism $\tilde{f} : X/_{aff}G \to Z$ so that $f(x) = \tilde{f}(\pi(x))$ for all $x \in X$. □

Proposition 9.4. *Suppose that algebraic groups G and H act respectively on varieties X and Y so that $\mathcal{O}(X)^G$ and $\mathcal{O}(Y)^H$ are finitely generated. Then $(X \times Y)/_{aff}(G \times H)$ exists and is isomorphic to $(X/_{aff}G) \times (Y/_{aff}H)$.*

Proof. Observe that, $\mathcal{O}(X \times Y)^{G\times H} \cong (\mathcal{O}(X) \otimes \mathcal{O}(Y))^{G\times H} \cong \mathcal{O}(X)^G \otimes \mathcal{O}(Y)^H$ the latter of which is finitely generated. Thus $(X \times Y)/_{aff}(G \times H)$ exists.

Then,
$$\begin{aligned} \mathcal{O}((X \times Y)/_{aff}(G \times H)) &= \mathcal{O}(X \times Y)^{G\times H} = \mathcal{O}(X)^G \otimes \mathcal{O}(Y)^H \\ &= \mathcal{O}(X/_{aff}G) \otimes \mathcal{O}(Y/_{aff}H) = \mathcal{O}((X/_{aff}G) \times (Y/_{aff}H)). \end{aligned}$$
Being both affine varieties, it follows that they must be isomorphic. □

Remark 9.5. *Our map π is not necessarily surjective. As is noted in [32], if we take X to be a semisimple algebraic group and G to be its maximal unipotent subgroup then $\pi(X)$ is a proper open subset of $X/_{aff}G$.*

Definition 9.6. *Let L be an irreducible quasiaffine variety with algebraic group P acting on the right. Let R be an irreducible quasiaffine variety with algebraic group Q acting on the left. Let M be an irreducible reductive monoid with group of units H.*

Suppose there exist algebraic group morphisms, $\gamma : P \to H$, and $\delta : Q \to H$ and that the action $P \times Q$ on $L \times M \times R$ given by $(p, q) \cdot (\ell, m, r) = (\ell p^{-1}, \gamma(p)m\delta(q^{-1}), qr)$ makes it so that $\mathcal{O}(L \times M \times R)^{P\times Q}$ is finitely generated. Then we define

$$\mathcal{S}(L, P, M, Q, R) = Spec\left(O(L \times M \times R)^{P\times Q}\right) = (L \times M \times R)/_{aff}(P \times Q)$$

When it is understood what L, M, R, P, Q are we may reduce our notation to just $\mathcal{S}$.

How does our Rees sandwich map enter into the picture? Like before, ϕ will let us create a semigroup.

Proposition 9.7. *Suppose there is a map $\phi : R \times L \to M$ so that $\phi(qr, \ell p) = \delta(q)\phi(r, \ell)\gamma(p)$ for all $\ell \in L$, $r \in R$, $p \in P$, and $q \in Q$, and that the natural map $\pi : L \times M \times R \to \mathcal{S}$ is surjective. Then $\mathcal{S}$ is an irreducible algebraic semigroup.*

Proof. That $\mathcal{S}$ is irreducible comes from it being the image of the irreducible variety, $L\times M\times R$. Now, since $P\times Q$ acts on $L\times M\times R$ so that $\mathcal{S}$ exists, Proposition 9.4 tells us $\mathcal{S}\times\mathcal{S}$ is the affinised quotient of variety, $(L \times M \times R) \times (L \times M \times R)$ by algebraic group, $(P \times Q) \times (P \times Q)$. We already have a notion of semigroup on $L \times M \times R$, thanks to the sandwich map, given by the morphism $\mu((\ell_1, m_1, r_1), (\ell_2, m_2, r_2)) = (\ell_1, m_1\phi(r_1, \ell_2)m_2, r_2)$.

Observe that for any $\ell_1, \ell_2 \in L$, $m_1, m_2 \in M$, $r_1, r_2 \in R$, $p_1, p_2 \in P$, and $q_1, q_2 \in Q$,

$$\begin{aligned}
&\pi \circ \mu((\ell_1 p_1^{-1}, \gamma(p_1)m_1\delta(q_1^{-1}), q_1r_1), (\ell_2 p_2^{-1}, \gamma(p_2)m_2\delta(q_2^{-1}), q_2r_2))\\
&\qquad = \pi(\ell_1 p_1^{-1}, \gamma(p_1)m_1\delta(q_1^{-1})\phi(q_1r_1, \ell_2p_2^{-1})\gamma(p_2)m_2\delta(q_2^{-1}), q_2r_2)\\
&\qquad = \pi(\ell_1 p_1^{-1}, \gamma(p_1)m_1\delta(q_1^{-1})\delta(q_1)\phi(r_1, \ell_2)\gamma(p_2^{-1})\gamma(p_2)m_2\delta(q_2^{-1}), q_2r_2)\\
&\qquad = \pi(\ell_1 p_1^{-1}, \gamma(p_1)m_1\phi(r_1, \ell_2)m_2\delta(q_2^{-1}), q_2r_2)\\
&\qquad = \pi(\ell_1, m_1\phi(r_1, \ell_2)m_2, r_2)\\
&\qquad = \pi \circ \mu((\ell_1, m_1, r_1), (\ell_2, m_2, r_2)).
\end{aligned}$$

So it follows that $\pi \circ \mu$ is $(P \times Q) \times (P \times Q)$ invariant, and (by Proposition 9.3) there exists a unique morphism $\widetilde{\pi \circ \mu}$ making the following diagram commute.

$$\begin{array}{ccc}
(L \times M \times R) \times (L \times M \times R) & \xrightarrow{\ \mu\ } & L \times M \times R\\
{\scriptstyle \pi\times\pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi}\\
\mathcal{S}\times\mathcal{S} & \xrightarrow[\widetilde{\pi\circ\mu}]{} & \mathcal{S}
\end{array}$$

By surjectivity of π this is indeed a map on all of $\mathcal{S} \times \mathcal{S}$. It remains to show our morphism is associative.

Take elements, $a, b, c \in \mathcal{S}$. Since π is surjective we find $\ell_a, \ell_b, \ell_c \in L$, $m_a, m_b, m_c \in M$, and $r_a, r_b, r_c \in R$ so that $\pi(\ell_a, m_a, r_a) = a$, $\pi(\ell_b, m_b, r_b) = b$, and $\pi(\ell_c, m_c, r_c) = c$. Then,

$$a(bc) = \pi((\ell_a, m_a, r_a)\,((\ell_b, m_b, r_b)(\ell_c, m_c, r_c)))$$
$$= \pi(((\ell_a, m_a, r_a)(\ell_b, m_b, r_b))\,(\ell_c, m_c, r_c)) = (ab)c$$

since μ is associative. □

As our goal is to discuss $\mathcal{S}$ as a semigroup, we will from now on assume that π is surjective and that a morphism ϕ as in the statement of the proposition exists.

9.2 Constructing Semigroups With Green's Relations

Our other goal is to see how closely we need R and L to imitate R_e and L_e in order to get a result like Theorem 8.6 where $L_e \cdot eSe \cdot R_e = S$. In essence, our construction is an attempt at fusion between the Rees construction and Putcha's fantastic result, $L_e \cdot eSe \cdot R_e = S$.

In a further effort to emulate Theorem 8.6 we will also assume that there exists a pair $(B, A) \in R \times L$ so that $\phi(B, A) = 1$. Our ϕ map acts as a stand in for, $(r, \ell) \mapsto r\ell \in eSe$, the multiplication coming from $L = L_e$, $M = eSe$, and $R = R_e$. In the case we are generalising, $L = L_e$, $M = eSe$, and $R = R_e$, such a pair is already seen to exist, by letting $A = B = e$.

Remark 9.8. *If we know $\gamma(P)\delta(Q) = H$ then this is equivalent to the existence of a pair $(B, A) \in R \times L$ so that $\phi(B, A) \in H$.*

With our pair $(B, A) \in R \times L$ observe that $(A, 1, B)$ is an idempotent in $L \times M \times R$ and by extension, $e := \pi(A, 1, B)$ is an idempotent in $\mathcal{S}$.

Lemma 9.9. *$\theta : M \to \mathcal{S}$ given by $\theta(m) = \pi(A, m, B)$ is a morphism of algebraic monoids and $\theta(M) = e\mathcal{S}e$ with $\theta(H) = H_e$.*

Proof. Let $m, m' \in M$. As we have defined $\theta(m) = \pi(A, m, B)$ we can quickly calculate, $\theta(m)\theta(m') = \pi(A, m, B)\pi(A, m', B) = \pi(A, m\phi(B, A)m', B) = \pi(A, mm', B) = \theta(mm')$. So this is a monoid morphism. Now observe if, $a \in e\mathcal{S}e$ then there exists a tuple, $(\ell, m, r) \in L \times M \times R$ so $a = \pi(A, 1, B)\pi(\ell, m, r)\pi(A, 1, B) = \pi(A, \phi(B, \ell)m\phi(r, A), B)$. We see $\theta(\phi(B, \ell)m\phi(r, B)) = a$. Thus, $\theta(M) = e\mathcal{S}e$.

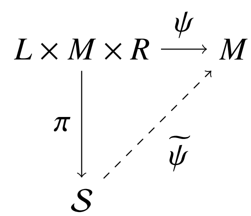
As the group of units, we know $\overline{H} = M$. So, by continuity and the fact that $\theta(H) \leq H_e$, we see $\theta(H) \subseteq H_e \subseteq \theta(\overline{H}) = \theta(M) = e\mathcal{S}e \subseteq \overline{\theta(H)}$. But then it is clear that $\theta(H)$ is a dense subgroup of H_e. This implies $\theta(H) = H_e$ as the image of an algebraic group is a closed subgroup of the algebraic group codomain. □

Theorem 9.10. $e\mathcal{S}e \cong M$

Proof. Consider the morphism, $\psi : L \times M \times R \to M$ given by $\psi(\ell, m, r) = \phi(B, \ell)m\phi(r, A)$. This morphism is $P \times Q$ invariant. Indeed, for $p \in P$, $q \in Q$,

$$\begin{aligned}\psi(\ell p^{-1}, \gamma(p)m\delta(q^{-1}), qr) &= \phi(B, \ell p^{-1})\gamma(p)m\delta(q^{-1})\phi(qr, A)\\ &= \phi(B, \ell)\gamma(p)^{-1}\gamma(p)m\delta(q)^{-1}\delta(q)\phi(r, A) = \phi(B, \ell)m\phi(r, A).\end{aligned}$$

Then there is a unique morphism $\widetilde{\psi} : \mathcal{S} \to M$.



Since ψ is surjective ($\psi(A, m, B) = m$) it follows that $\widetilde{\psi}$ is also surjective. Consider the map $\theta : M \to \mathcal{S}$ given by $\theta(m) = \pi(A, m, B)$. We claim θ and $\widetilde{\psi}$ are inverses.

Consider an element, $m \in M$. Then $\widetilde{\psi}(\theta(m)) = \widetilde{\psi}(\pi(A, m, B)) = \psi(A, m, B) = m$. Additionally, for any element $s \in e\mathcal{S}e$ we know that there exists $m \in M$ so that $\pi(A, m, B)$. Then $\theta(\widetilde{\psi}(s)) = \theta(\widetilde{\psi}(\pi(A, m, B))) = \theta(\psi(A, m, B)) = \theta(m) = \pi(A, m, B) = s$.

It remains to check that these are monoid morphisms. θ is by our previous lemma, so consider $s, s' \in e\mathcal{S}e$. Then there exist m, m' so that $\theta(m) = s$ and $\theta(m') = s'$. And so we see that,

$$\widetilde{\psi}(s)\widetilde{\psi}(s') = \psi(A, m, B)\psi(A, m', B) = \phi(B, A)m\phi(B, A)\phi(B, A)m'\phi(B, A) = \phi(B, A)mm'\phi(B, A)$$

since $\phi(B, A) = 1$. So, $\phi(B, A)mm'\phi(B, A) = \psi(A, mm', B) = \widetilde{\psi}(\pi(A, mm', B)) = \widetilde{\psi}(ss')$. □

Corollary 9.11. $H_e \cong H$.

Proof. This is just a combination of Lemma 9.9 and Theorem 9.10. □

With our ability to identify M and $e\mathcal{S}e$ along with H and H_e, the natural question to ask is what other identifications can we make?

Proposition 9.12. *Define the morphisms,* $\theta_L : L \times M \to \mathcal{S}$ *by* $\theta_L(a, m) = \pi(a, m, B)$ *and* $\theta_R : M \times R \to \mathcal{S}$ *by* $\theta_R(m, b) = \pi(A, m, b)$*. Then,*

(1) $\theta_L(L \times M) = \mathcal{S}e$

(2) $\theta_R(M \times R) = e\mathcal{S}$

Proof. (1) Take any $\pi(\ell, m, r) \in \mathcal{S}$. $\pi(\ell, m, r)\pi(A, 1, B) = \pi(\ell, m\phi(r, A), B)$. So θ_L is onto $\mathcal{S}e$. If $\ell \in L$, $m \in M$, then $\theta_L(\ell, m) = \pi(\ell, m, B) = \pi(\ell, m, B)\pi(A, 1, B) \in \mathcal{S}e$. Thus θ_L's image is contained in $\mathcal{S}e$, hence $\theta_L(L \times M) = \mathcal{S}e$. (2) is done similarly. □

So this settles what $\mathcal{S}e$ and $e\mathcal{S}$ look like, but what about L_e and R_e? Sadly, L and R are beginning to stray away from our original goal. That is, L and R are acting more like Se and eS than like L_e and R_e.

Proposition 9.13. *The following are equivalent for any* $\ell \in L$

(1) $\pi(\ell, h, B) \in L_e$ *for all* $h \in H$

(2) $\pi(\ell, h, B) \in L_e$ *for some* $h \in H$

(3) $\phi(y, \ell) \in H$ *for some* $y \in R$.

The following are equivalent for any $r \in R$

(4) $\pi(A, h, r) \in R_e$ *for all* $h \in H$

(5) $\pi(A, h, r) \in R_e$ *for some* $h \in H$

(6) $\phi(r, x) \in H$ *for some* $x \in L$.

Proof. (1) ⇒ (2) is clear. Suppose that $\pi(\ell, h, B) \in L_e$. Then there exists an element, $\pi(a, m, b)$, so $\pi(a, m, b)\pi(\ell, h, B) = e$. $e = ee = \pi(A, 1, B)\pi(a, m, b)\pi(\ell, h, B) = \pi(A, \phi(B, a)m\phi(b, \ell)h, B)$. By Corollary 9.11 it is clear that $\phi(B, a)m\phi(b, \ell)h = 1$ and it follows that $\phi(b, \ell) \in H$.

Now let $y \in R$ be such that $\phi(y, \ell) \in H$. Consider $\pi(\ell, h, B)$ for arbitrary fixed $h \in H$. Then we can see the following multiplication is correct, $\pi(A, h^{-1}\phi(y, \ell)^{-1}, y)\pi(\ell, h, B) = e$ It is clear that this suffices to show $\pi(\ell, h, B) \in L_e$.

(4) ⇒ (5) ⇒ (6) ⇒ (4) is proven similarly. □

Definition 9.14. *We define the* **quasiclasses** *with respect to* ϕ *to be the sets,*

$$L' = \{\ell \in L \mid \exists r \in R \text{ so that } \phi(r, \ell) \in H\} \text{ and } R' = \{r \in R \mid \exists \ell \in L \text{ so that } \phi(r, \ell) \in H\}$$

The following proposition gives a number of facts about these quasiclasses, showing how closely they approximate the $\mathscr{L}$-classes and $\mathscr{R}$-classes we are trying to imitate.

Proposition 9.15.

(1) $\ell \in L'$ if and only if there exists $r \in R'$ with $\phi(r, \ell) \in H$.

(2) $r \in R'$ if and only if there exists $\ell \in L'$ with $\phi(r, \ell) \in H$.

(3) $L' \times H = \theta_L^{-1}(L_e)$ where θ_L is defined as in Proposition 9.12.

(4) $H \times R' = \theta_R^{-1}(R_e)$ where θ_R is defined as in Proposition 9.12.

(5) L' is open in L, hence a quasiaffine variety.

(6) R' is open in R, hence a quasiaffine variety.

(7) The action of P restricts to L'.

(8) The action of Q restricts to R'.

(9) If γ is surjective, $\pi(L', 1, B) = L_e$.

(10) If δ is surjective, $\pi(A, 1, R') = R_e$.

Proof. We will only prove the odd numbered results.

(1) This comes from considering the definition of both L' and R'.

(3) Suppose that $\pi(\ell, m, B) \in L_e$ for $\ell \in L$ and $m \in M$. Then there exists $\pi(a, n, b)$ so $\pi(a, n, b)\pi(\ell, m, B) = e$. $e = ee = \pi(A, 1, B)\pi(a, n, b)\pi(\ell, m, B) = \pi(A, \phi(B, a)n\phi(b, \ell)m, B)$. By Corollary 9.11 it is clear that $\phi(B, a)n\phi(b, \ell)m = 1$ and it follows that $\phi(b, \ell) \in H$ and $m \in H$. Thus $\ell \in L'$ and $m \in H$ as desired.

Conversely, suppose that $\ell \in L'$ and $h \in H$. So there exists $y \in R$ such that $\phi(y, \ell) \in H$. Then $\pi(A, h^{-1}\phi(y, \ell)^{-1}, y)\pi(\ell, h, B) = e$ shows that $\theta_L(L' \times H) = L_e$.

(5) L_e is open in $\mathcal{S}e$ by Proposition 2.15. So then by (3) and Proposition 9.12 we see that $L' \times H$ is open in $L \times M$. Since projection is an open map we can then conclude that L' is open in L. Quasiaffineness of L' follows, as an open set of a quasiaffine variety is also quasiaffine.

(7) Suppose $\ell \in L'$ and fix $p \in P$. Then there exists $r \in R'$ so that $\phi(r, \ell) \in H$. Observe that $\phi(r, \ell p) = \phi(r, \ell)\gamma(p) \in H$.

(9) Consider $\pi(\ell, m, r) \in L_e$. As we showed in (3) we may assume $r = B$, $m \in H$, and $\ell \in L'$. Since γ is a surjective morphism we know that there exists $p \in P$ so $\gamma(p) = m$. Then $\pi(\ell, m, B) = \pi(\ell p, \gamma(p)^{-1}m, B) = \pi(\ell p, 1, B)$. By (7), $\ell p \in L'$, concluding the result. ◻

Our quasiclasses are now closer to L_e and R_e than L and R are, but they still are not quite there. We would like something closer to Theorem 9.10, with a single isomorphism. However, this is still a noteworthy result and leads us to a discussion of when $\mathcal{S}$ is regular.

Proposition 9.16. *$L' \times H \times R' = \pi^{-1}(J_e)$ and J_e is the unique maximal $\mathscr{J}$-class of $\mathcal{S}$.*

Proof. Suppose $\pi(\ell, m, r) = j \in J_e$. Then it follows that there exist elements $s, s' \in \mathcal{S}^1$ satisfying, $sjs' = e$. Now, since $e = eee = esjs'e$ we may assume that $s = \pi(A, n, y)$ and $s' = \pi(x, n', B)$. Then $e = \pi(A, 1, B) = \pi(A, n\phi(y, \ell)m\phi(r, x)n', B) = sjs'$. By Corollary 9.11 it follows that $\phi(y, \ell), m, \phi(r, x) \in H$ and hence $\ell \in L'$, $r \in R'$. Thus, $\pi^{-1}(J_e) \subseteq L' \times H \times R'$.

For the reverse, take any $\ell \in L'$, $h \in H$, and $r \in R'$. There is $x \in L'$, $y \in R'$ so $\phi(y, \ell), \phi(r, x) \in H$. $e = \pi(A, h^{-1}\phi(y, \ell)^{-1}, y)\pi(\ell, h, r)\pi(x, \phi(r, x)^{-1}, B) \in \mathcal{S}^1\pi(\ell, h, r)\mathcal{S}^1$ and $\pi(\ell, h, r) = \pi(\ell, h, B)\pi(A, 1, B)\pi(A, 1, r) \in \mathcal{S}^1 e\mathcal{S}^1$. So it follows that $L' \times H \times R' = \pi^{-1}(J_e)$.

It remains to show that J_e is the unique maximal $\mathscr{J}$-class of $\mathcal{S}$. Let $J \subseteq \mathcal{S}$ be any other $\mathscr{J}$-class. It suffices to show that for any $s \in J$, $s \in S^1eS^1$. Let $s = \pi(\ell, m, r)$. Then by now it is quick to check that, $\pi(\ell, m, B)\pi(A, 1, B)\pi(A, 1, r) = \pi(\ell, m, r)$. □

This gives us a remarkable similarity to regular semigroups, as we know they have a unique maximal $\mathscr{J}$-class. In fact, as the next theorem indicates, if we were to perform our construction with L' and R' in place of L and R, we would get a regular semigroup for $\mathcal{S}$.

Theorem 9.17. *If $\mathcal{S}(L', P, M, Q, R')$ exists, then it is regular.*

Proof. Consider $(a, m, b) \in L' \times M \times R'$. Let $x \in L'$ and $y \in R'$ be so that $\phi(b, x), \phi(y, a) \in H$. Let $n \in M$ be such that $mnm = m$ (which exists as M is regular). Then,

$$(a, m, b)(x, \phi(b, x)^{-1}n\phi(y, a)^{-1}, y)(a, m, b) = (a, m\phi(b, x)\phi(b, x)^{-1}n\phi(y, a)^{-1}\phi(y, a)m, b),$$

which reduces to (a, m, b). So then $L' \times M \times R'$ is regular. And since we are assuming that π is surjective then $\mathcal{S}$, as the image of a regular semigroup, is also regular. □

With this theorem we now see that our idempotent, $e = \pi(A, 1, B)$ is shown to be an idempotent of the maximal $\mathscr{J}$-class of our regular semigroup, which is further imitation of the setup in [24]. This is what we have been shooting for since the start of our section. So when does $\mathcal{S}(L', P, M, Q, R')$ exist?

The trivial answer is: when $\mathcal{O}(L' \times M \times R')^{P\times Q}$ is finitely generated. Obviously we would like a more substantial answer and one that perhaps relates to our original choice of varieties L and R. So when does $\mathcal{S}(L, P, M, Q, R)$ exist?

Another naive answer would be when L and R are affine and $P \times Q$ is reductive (recall Theorem 8.10). Suppose that L and R are affine and $P\times Q$ is reductive. When can we show that $\mathcal{S}(L', P, M, Q, R')$ exists? And let us not forget, we will also need π to be surjective, as many of our previous results have employed this assumption. The following section gives us a possible direction to pursue.

9.3 Normality

Lemma 9.18. *Suppose that X and Y are affine varieties, with X normal. Suppose also that X has an open subset U, so that the subvariety $X\backslash U$ has codimension at least 2 in X. Then any morphism $U \to Y$ can be uniquely extended to a morphism $X \to Y$*

Proof. Lemma 2.2 in [31]. □

Corollary 9.19. *Let X be a normal affine variety with an open subset $U \subseteq X$, such that $codim_X(X\backslash U) \geq 2$. Then $\mathcal{O}(U) \cong \mathcal{O}(X)$.*

Proof. This is another application of Proposition 3.5 from [11]. □

Proposition 9.20. *Suppose that X is a normal affine variety. Let U be an open subset with $codim_X(X\backslash U) \geq 2$ and an algebraic group G acting on it. Then,*

(1) the action of G on U extends uniquely to an action on X

(2) if $X/_{aff}G$ exists then so does $U/_{aff}G$ and $U/_{aff}G = X/_{aff}G$.

Proof. (1) Algebraic groups are known to be normal and by applying Theorem 2.21, $G \times X$ is also seen to be normal. Now, since we have assumed that $codim_X(X\backslash U) \geq 2$ it follows that $codim_{G\times X}(G \times X\backslash G \times U) = codim_{G\times X}(G \times (X\backslash U)) = codim_X(X\backslash U) \geq 2$. Our action of $\sigma : G \times U \to U \to X$ can then be extended uniquely to a morphism $\overline{\sigma} : G \times X \to X$.

It remains to show that $G \times X \to X$ is a group action. Consider the morphism $x \mapsto \overline{\sigma}(1, x)$. Restricted to U we see that this map must be the inclusion map $U \to X$. By normality of X and codimension of $X\backslash U$ this extends uniquely to a map $X \to X$, namely id_X. But by uniqueness, since $x \mapsto \overline{\sigma}(1, x)$ also extends the inclusion map we see that $\overline{\sigma}(1, x) = x$.

Take any two elements $g, h \in G$. Then $g : X \to X$ and $h : X \to X$ are given by $x \mapsto \overline{\sigma}(g, x)$ and $x \mapsto \overline{\sigma}(h, x)$ respectively. Consider as well, the map $gh : X \to X$ given by $x \mapsto \overline{\sigma}(gh, x)$ Another uniqueness of the extension argument shows us that indeed $\overline{\sigma}(g, \overline{\sigma}(h, x)) = \overline{\sigma}(gh, x)$.

(2) By Corollary 9.19, $U/_{aff}G = Spec\left(O(U)^G\right) \cong Spec\left(O(X)^G\right) = X/_{aff}G$. □

Proposition 9.21. *Suppose we assume that L and R are affine and also that $L \times M \times R$ is a normal variety. As well, suppose $codim_L(L\backslash L'), codim_R(R\backslash R') \geq 2$. Then if $\mathcal{S}(L, P, M, Q, R)$ exists, $\mathcal{S}(L', P, M, Q, R')$ will also exist and $\mathcal{S}(L', P, M, Q, R') \cong \mathcal{S}(L, P, M, Q, R)$.*

Proof. We already know, from Proposition 9.15, that $L' \times M \times R'$ is open in $L \times M \times R$. It is not difficult to conclude from our conditions that $codim_{L\times M\times R}(L \times M \times R\backslash L' \times M \times R') \geq 2$. Then $X = L \times M \times R$, $U = L' \times M \times R'$, and $G = P \times Q$ satisfies the conditions of Proposition 9.20. So $\mathcal{S}(L, P, M, Q, R) = X/_{aff}G \cong U/_{aff}G = \mathcal{S}(L', P, M, Q, R')$ as desired. □

Our hope is that if we start with a regular semigroup S and choose the $\mathscr{L}$- and $\mathscr{R}$-classes of e, an idempotent in the maximal $\mathscr{J}$-class, then $\mathcal{S}(L_e, H_e, eSe, H_e, R_e) \cong S$. Let us investigate this now.

Lemma 9.22. *Let $\psi : X \to Y$ be a surjective birational morphism of irreducible affine varieties and suppose that Y is normal. Then ψ is in fact an isomorphism.*

Proof. This result is exactly Lemma 2.1 in [15]. □

The following theorem shows us that when we begin with the case described in [24] (a normal irreducible semigroup with zero) and apply this process we get the same semigroup as our output.

Theorem 9.23. *Suppose that S is an irreducible, regular, normal, affine semigroup with zero. Let $e \in E(S)$ be an idempotent in the unique maximal $\mathscr{J}$-class of S.*

Then $S = \mathcal{S}(Se, H_e, eSe, H_e, eS)$.

Proof. Since S is affine it follows that Se, eSe, and eS are, and thus so is $Se \times eSe \times eS$. Since S is regular and has a zero, eSe is regular and has a zero. Thus, eSe is a reductive monoid, and it follows that H_e is a reductive group, as is $H_e \times H_e$. So we may conclude that $Se \times eSe \times eS/_{aff}H_e \times H_e$ is a good quotient, and so it follows that the natural map, $\pi : Se \times eSe \times eS \to Se \times eSe \times eS/_{aff}H_e \times H_e$ is surjective.

Consider the morphism $f : Se \times eSe \times eS \to S$ given by $f(\ell, h, r) = \ell hr$. A consequence of Theorem 8.6 is that f is surjective. By the universal property of affinised quotients there exists a unique morphism $\tilde{f}$ making the following diagram commute.

$$\begin{array}{ccc} Se \times eSe \times eS & \xrightarrow{f} & S \\ \pi \big\downarrow & \nearrow_{\tilde{f}} & \\ \mathcal{S} & & \end{array}$$

The morphism $\tilde{f}$ is surjective since f is. Now, by applying Theorem 2.16 we can see that $J_e = f(L_e, H_e, R_e) = \tilde{f} \circ \pi(L_e, H_e, R_e)$. Consider $x \in \tilde{f}^{-1}(J_e)$. Then since π is surjective there exists $(\ell, h, r) \in Se \times eSe \times eS$ so that $\pi(\ell, h, r) = x$. But then $\ell hr = \tilde{f}(x) \in J_e$ and since J_e is maximum this implies $h \in H_e$, $\ell \in L_e$, and $r \in R_e$. Thus, $\tilde{f}^{-1}(J_e) = \pi(L_e \times H_e \times R_e)$.

Now, suppose that $\tilde{f}(\pi(\ell, h, r)) = \tilde{f}(\pi(\ell', h', r'))$. Then $\ell hr = \ell' h' r' \in J_e$. So by using Theorem 2.16 we conclude they are in the same $H_e \times H_e$ orbit, and hence $\pi(\ell, h, r) = \pi(\ell', h', r')$. Thus $\tilde{f}$ restricted to $\pi(L_e \times H_e \times R_e)$ is injective.

So we see that $\tilde{f}$ is an isomorphism from $\pi(L_e \times H_e \times R_e)$ to J_e. Since J_e is open in S (recall Corollary 8.42) we see that $\tilde{f}^{-1}(J_e) = \pi(L_e \times H_e \times R_e)$ must also be open. So $\tilde{f}$ is a surjective morphism which is an isomorphism between two open sets. In other words it is a birational morphism. By Lemma 9.22, since S is normal, it follows that $\tilde{f}$ is an isomorphism. □

Corollary 9.24. *Let S be a normal irreducible regular algebraic semigroup with zero. Let $e \in E(S)$ be an idempotent in the unique maximal $\mathscr{J}$-class of S. If Se, eSe, and eS are normal varieties, $codim_{Se}(Se \backslash L_e) \geq 2$ and $codim_{eS}(eS \backslash R_e) \geq 2$, then $\mathcal{S}(L_e, H_e, eSe, H_e, R_e)$ exists and is isomorphic to S.*

Proof. By Theorem 9.23 above we know that $\mathcal{S}(Se, H_e, eSe, H_e, eS)$ exists and also satisfies, $\mathcal{S}(Se, H_e, eSe, H_e, eS) \cong S$. So it suffices to show that the conditions of Proposition 9.21 are satisfied. By our assumptions in the statement of this corollary it remains only to show that for $\phi : eS \times Se \to eSe$ given by $\phi(r, \ell) = r\ell$, $L_e = (Se)'$ and $R_e = (eS)'$.

Suppose $\ell \in L_e$. Then there exists $y \in S$ so that $y\ell = e$. But then $e = ee = ey\ell = (ey)\ell$, so we may assume that $y \in eS$. Thus $\ell \in (Se)'$. Suppose that $\ell \in (Se)'$. Since $\ell \in Se$ we can see that there exists $x \in S$ so that $xe = \ell$ (namely, $x = \ell$). By definition of $(Se)'$ there exists $y \in eS$ so that $y\ell = e$. So it follows that $\ell \in L_e$. Thus $L_e = (Se)'$.

$R_e = (eS)'$ is shown similarly, completing the proof. □

So we have almost shown $S \cong \mathcal{S}(L_e, H_e, eSe, H_e, R_e)$. Unfortunately, we had to make some normality and dimension assumptions, which leads us to the following question, which we will not pursue in the remainder of this paper.

Question 9.25. *Is $S \cong \mathcal{S}(L_e, H_e, eSe, H_e, R_e)$ for all irreducible regular algebraic semigroups with zero, S?*

One can also wonder whether the normality conditions can be simplified. It seems that, as retracts of S, the normality of eS, Se, and eSe should follow. However, such a result (if true) is currently elusive. As such, we are left to ponder the necessity of all the normality assumptions of Corollary 9.24.

We will showcase two examples here, with the first one leading into our next discussion.

Example 9.26. *Our first example is very similar to the example which came at the very end of Section 8. Consider the determinantal variety, $S = \{$the matrices of rank ≤ 2 in $M_3(K)\}$, and let $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Let us observe that $Se = \{\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} \mid a_{ij} \in K\}$ and $eS = \{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \mid a_{ij} \in K\}$, so $Se \cong eS \cong K^6$. Likewise $eSe = \{\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a_{ij} \in K\} \cong K^4$, so Se, eSe, and eS are all normal.*

S is also normal (since determinantal rings are Cohen-Macaulay, [12], and hence normal) so by Corollary 9.24 it remains to show R_e, L_e satisfy the codimension 2 condition. We can see that, $R_e = \{A \in eS \mid rk(A) = 2\}$. So it follows that $eS \setminus R_e$ is the set of all matrices in eS which rank 0 or 1. We see $eS \setminus R_e = \{\begin{pmatrix} ka & kb & kc \\ a & b & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c, k \in K\} \cup \{\begin{pmatrix} a & b & c \\ ka & kb & kc \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c, k \in K\}$ which has dimension 4. Hence, $codim_{eS}(eS \setminus R_e) = 2$, likewise for the codimension of L_e in Se. Thus $\mathcal{S}(L_e, H_e, eSe, H_e, R_e)$ exists and is isomorphic to the matrices of rank 2 or less.

Example 9.27. *The second example shows that we need not take $P = Q = H$. Consider the same S and e as the last example. Let $P = C^{\ell}_{GL_3(K)}(e)$, $Q = C^{r}_{GL_3(K)}(e)$. We can quickly see that* *maps* $\gamma : P \to H_e$ *given by* $x \mapsto ex$ *and* $\delta : Q \to H_e$ *given by* $x \mapsto xe$ *are surjective. Along with* $\phi : eS \times Se \to eSe$ *given by* $(r, \ell) \mapsto r\ell$ *they satisfy all the conditions for Definition 9.6.*

Now, for $f \in \mathcal{O}(Se \times eSe \times eS)^{P\times Q}$ *we see that for any* $(\ell, m, r) \in Se \times eSe \times eS$ *and* $p \in P$, $q \in Q$, $f(\ell, m, r) = f(\ell p^{-1}, epmq^{-1}e, qr) = f(\ell ep^{-1}, epmq^{-1}e, qer) = f(\ell g^{-1}, gmh^{-1}, hr)$ *for* $g = ep, h = qe \in H_e$*. So* $f \in \mathcal{O}(Se \times eSe \times eS)^{H_e\times H_e}$ *since* γ, δ *are surjective. Likewise we can show the reverse containment,* $\mathcal{O}(Se \times eSe \times eS)^{H_e\times H_e} \subseteq \mathcal{O}(Se \times eSe \times eS)^{P\times Q}$*. It follows that* $\mathcal{O}(Se \times eSe \times eS)^{P\times Q} = \mathcal{O}(Se \times eSe \times eS)^{H_e\times H_e}$ *the latter of which is finitely generated since* H_e *is reductive. So it follows that* $\mathcal{S}(Se, C^\ell_G(e), eSe, C^r_G(e), eS)$ *exists.*

In fact, we can generalise the former example. Let us take a look at a broader example of Corollary 9.24 in action, as we apply it to normal determinantal varieties.

9.4 Determinantal Varieties

In algebraic geometry, **determinantal varieties** are spaces of matrices which have an upper bound on their ranks. Their significance comes from the fact that many examples in algebraic geometry are of this form, such as the Segre embedding of a product of two projective spaces. For our purposes, the usual notion of determinantal variety takes the form,

$$\mathfrak{D}_{n,r} = \{A \in M_n(K) \mid rank(A) \leq r\}$$

However, they can also be written in a different manner,

$$\mathfrak{D}_{n,r} = \overline{GL_n(K)\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}GL_n(K)}$$

Of course, $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ can be replaced by any rank r matrix, but it is the fact that $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ is an idempotent which draws our attention and allows us to make a generalisation.

Definition 9.28. *A* **(generalised) determinantal variety** *is an algebraic variety of the form* $\mathfrak{D}_{M,e} = \overline{GeG}$ *for some* $e \in E(M)$*, where* M *is an irreducible reductive algebraic monoid with zero and* G *is its group of units.*

Proposition 9.29. *Fix a cross sectional lattice,* Λ*, containing* e *then,* $\mathfrak{D}_{M,e} = \bigsqcup_{f\leq e, f\in\Lambda} GfG$

Proof. This is an immediate consequence of Corollary 4.11. □

Proposition 9.30. $\mathfrak{D}_{M,e}$ *is an irreducible regular algebraic semigroups with zero.*

Proof. Since M has a zero and $0 \leq e$, $0 \in G0G \subseteq \overline{GeG}$. Since G is irreducible it follows that GeG is irreducible and so $\overline{GeG}$ is irreducible.

For any elements, $x, y \in \mathfrak{D}_{M,e}$, $J_{xy} \leq J_x, J_y$ ($\mathscr{J}$-classes in M). By Proposition 9.29, $\mathfrak{D}_{M,e}$ is a union of all $\mathscr{J}$-classes in M below $J_e = GeG$. $x \in \mathfrak{D}_{M,e}$ implies $J_x \subseteq \mathfrak{D}_{M,e}$. Since $J_{xy} \leq J_x$ we then see $J_{xy} \subseteq \mathfrak{D}_{M,e}$. Hence $xy \in \mathfrak{D}_{M,e}$, making it a semigroup.

For any $x \in \mathfrak{D}_{M,e} \subseteq M$ we can find $y \in M$ so that $xyx = x$. Then $x(yxy)x = x$ and $yxy \in J_{yxy} \leq J_{xy} \leq J_x \leq J_e$, since $x \in \mathfrak{D}_{M,e}$. This shows us that $\mathfrak{D}_{M,e}$ is also regular. □

One of the advantages of determinantal varieties is that no matter our choice of idempotent, e, the $\mathscr{T}$-class of e in $\mathfrak{D}_{M,e}$ is the same as the $\mathscr{T}$-class of e in M. This is the content of the following proposition.

Proposition 9.31. *For any irreducible reductive monoid with zero, M, a given idempotent, $e \in E(M)$ and any idempotent, $f \leq e$,*

(1) The $\mathscr{J}$-class of f in $\mathfrak{D}_{M,e}$ is GfG

(2) The $\mathscr{L}$-class of f in $\mathfrak{D}_{M,e}$ is Gf

(3) The $\mathscr{R}$-class of f in $\mathfrak{D}_{M,e}$ is fG

(4) The $\mathscr{H}$-class of f in $\mathfrak{D}_{M,e}$ is $Gf \cap fG$

Proof. (1) As a consequence of Proposition 9.29 $GfG \subseteq \mathfrak{D}_{M,e}$. For any $gfh \in GfG$ we can see that $gf1, 1fh, 1fg^{-1}, h^{-1}f1 \in GfG \subseteq \mathfrak{D}_{M,e}$. So then, $gfh = gfffh = gf1f1fh$ and $f = fff = 1fg^{-1}gfhh^{-1}f1$. So $f \mathscr{J} gfh$. Thus, $GfG \subseteq J_f$. Since $\mathfrak{D}_{M,e}$ is a submonoid of M we can see that $J_f \subseteq GfG$, which completes the result.

(2) The proof is similar for R_f, so we will just show L_f. Since Gf is the $\mathscr{L}$-class of f in M, and $Gf \subseteq GfG \subseteq \overline{GeG} = \mathfrak{D}_{M,e}$ it follows that $L_f \subseteq Gf$. Suppose that $x \in Gf$. Then we can find $g \in G$ so $x = gf$. Observe that $fg^{-1} \in fG \subseteq \mathfrak{D}_{M,e}$, so $f = (fg^{-1})x$ and $x = xf$. Thus, $x \in L_f$ and our result is concluded.

(4) follows from the $\mathscr{L}$ and $\mathscr{R}$ cases. □

Corollary 9.32. *GeG is the maximal $\mathscr{J}$-class of $\mathfrak{D}_{M,e}$.*

Proof. By Proposition 9.31 (1) we see GeG is a $\mathscr{J}$-class of $\mathfrak{D}_{M,e}$. Since $\mathfrak{D}_{M,e} = \bigcup_{f \leq e} GfG$ is is clear that every other $\mathscr{J}$-class of $\mathfrak{D}_{M,e}$ is of the form GfG for some idempotent $f \leq e$. But then $GfG \subseteq \overline{GeG}$, hence GeG is maximal. □

This next proposition gives an alternate definition for determinantal varieties and also describes some of the subsemigroups of $\mathfrak{D}_{M,e}$.

Proposition 9.33. *For any irreducible reductive monoid with zero, M, a given idempotent, $e \in E(M)$ and any idempotent, $f \le e$,*

(1) $\mathfrak{D}_{M,e} = MeM$

(2) $\mathfrak{D}_{M,e}f = Mf$

(3) $f\mathfrak{D}_{M,e} = fM$

Proof. (1) Since M is a monoid, MeM consists of exactly the $\mathscr{J}$-classes, J' with $J' \le J_e$. But, by the definition of the cross-sectional lattice, there exists $f \in \Lambda$ so that $J' = J_f$, and $J_f \le J_e$ implies $f \le e$. Thus $MeM \subseteq \bigsqcup_{f \le e, f \in \Lambda} J_f$ But for any idempotent $f \le e$, we know $ef = f$, and so $J_f = GfG \subseteq MfM = Me(fM) \subseteq MeM$ and so by Proposition 9.29, $MeM \supseteq \bigsqcup_{f \le e, f \in \Lambda} J_f$. Thus $MeM = \overline{GeG} = \mathfrak{D}_{M,e}$.

(2) Since $f \le e$ if and only if $fe = f = ef$ the general statement will follow from showing $\mathfrak{D}_{M,e}e = Me$. By (1) we know that $\mathfrak{D}_{M,e}e = MeMe$. But Me is a subsemigroup of M, so it follows that $\mathfrak{D}_{M,e}e = MeMe = Me$.

(3) can be shown similarly to (2). □

To showcase determinantal varieties as an example of our semigroup construction, we will need to show that $codim_{eM}(eM \backslash eG)$, $codim_{Me}(Me \backslash Ge) \ge 2$. To do this we will need some combinatorial facts coming from the Bruhat decomposition for reductive monoids.

We would like to know if $\mathcal{S}(L_e, H_e, e\mathfrak{D}_{M,e}e, H_e, R_e)$ exists, so we will rely on Corollary 9.24. The codimension 2 condition is unlike anything we have seen before in this paper, so can be difficult to get a handle on. Fortunately, Putcha's paper, [24], already contains a structure that will let us tackle this condition.

Definition 9.34. *Let S be a regular irreducible algebraic semigroup. For any $e', e'' \in E(S)$ and $\mathscr{J}$-class $J' \in \mathcal{U}(S)$ define $e' \star J'$ to be $e'S \cap J'$, $J' \star e''$ to be $Se'' \cap J'$, and $e' \star J' \star e''$ to be $e'Se'' \cap J'$.*

Lemma 9.35. *Let S be a regular irreducible algebraic semigroup. Let e be an idempotent in its maximal $\mathscr{J}$-classes, J. Let R, L, and H be the respective classes associated to e. For an idempotent $e' \in E(S)$ and $\mathscr{J}$-class $J' \in \mathcal{U}(S)$, with $e' \in E(J')$ and $e' \le e$,* *(1)* $e \star J' = He'R$

(2) $J' \star e = Le'H$

Proof. We will just prove (1) as (2) is achieved similarly. This whole result is an application of Lemma 3.1 from [24]. By (i) of Lemma 3.1, $e \star J' = (He')(e' \star J')$. (iv) tells us that $e' \star J' = e'R$. By combining these two results, we conclude,

$$e \star J' = (He')(e' \star J') = (He')(e'R) = He'R.$$ □

In fact, Lemma 3.1 of [24] gives us a couple of formulas to compute the dimension of $e' \star J''$, which we will not need in this paper, but which can be of great use to anyone who wants to show the codimension 2 condition for situations other than determinantal varieties.

Proposition 9.36. *Suppose that* $e' \leq e \in \Lambda$. *Then* $e \star J_{e'} = P_{\lambda(e)}e'G$.

Proof. By Lemma 9.35, $e \star J_{e'} = He'R$, where $H = H_e$ and $R = R_e$. By recalling Section 4.2 of [30], we note that $P_{\lambda(e)} = \{x \in G \mid xe = exe\}$ and it quickly follows that, $H = P_{\lambda(e)}e$ and Proposition 9.31 tells us, $R = eG$. Combining these results we see, $He'R = P_{\lambda(e)}ee'eG$. Since $e' \leq e$ we can see that $ee'e = e'$. Thus $e \star J_{e'} = P_{\lambda(e)}e'G$. □

A similar statement can be made about $J_{e'} \star e$, but it will involve the standard parabolic subgroups relative to B^-, rather than B. In what follows, similar statements can be made about $J_{e'} \star e$, but one will need to make them involving the Bruhat order, the parabolic subgroups, and cross sectional lattices, with respect to B^- rather than B. Notice that this will end up being acceptable as B^-, T produce the same Renner monoid, only the Bruhat order really changes.

Corollary 9.37. $e \star J_{e'}$ *is an irreducible variety.*

Proof. This comes to us by way of Corollary 7.8, since Proposition 9.36 now tells us that $e \star J_{e'}$ is a fat $\mathscr{J}^{\lambda(e),S}$-class □

Corollary 9.38. $dim(eS \backslash R_e) = max\{dim(e \star J') \mid J' \in \mathcal{U}(S), J' < J_e\}$

Proof. Notice that $eS = S \cap eS = (\bigsqcup_{J' \in \mathcal{U}(S)} J') \cap eS = \bigsqcup_{J' \in \mathcal{U}(S)} (J' \cap eS) = \bigsqcup_{J' \in \mathcal{U}(S)} (e \star J')$. We can see $eS \backslash R_e = Se \backslash e \star J_e$ (by Lemma 9.35), so it follows, $eS \backslash R_e = \bigsqcup_{J' \in \mathcal{U}(S) \backslash \{J_e\}} e \star J'$. Since J_e is the maximum element of the lattice, $\mathcal{U}(S)$, we see $eS \backslash R_e = \bigsqcup_{J' < J_e} (e \star J')$.

Thus we see $eS\backslash R_e$ is a finite disjoint union of irreducible subvarieties, so from the definition of dimension (as the maximum dimension of the irreducible components of the variety), $dim(eS\backslash R_e) = max\{dim(e \star J') \mid J' \in \mathcal{U}(S), J' < J_e\}$ as desired. □

Corollary 9.39. $\overline{Bw_0(\lambda(e))e'B} = \overline{e \star J_{e'}}$

Proof. By Proposition 9.36, $e \star J_{e'} = P_{\lambda(e)}e'G$. So it follows that $B\lceil e' \rceil^{\mathscr{J}^{\lambda(e),S}} B$ is dense in $e \star J_{e'}$. To finish the proof, it remains to show that $\lceil e' \rceil^{\mathscr{J}^{\lambda(e),S}} = w_0(\lambda(e))e'$.

It is clear that $w_0(\lambda(e))e' \mathscr{L}^{\lambda(e)} e'$, and hence $w_0(\lambda(e))e' \in J_{e'}^{\lambda(e),S}$. We can see that $w_0(\lambda(e))e' \in w_0(\lambda(e))\Lambda \subseteq w_0(\lambda(e))\mathcal{G}\mathcal{J} \subseteq w_0(\lambda(e))\mathcal{G}\mathcal{J}^{\lambda(e)}$. Multiplying by w_0, $w_0(\lambda(e))e'w_0 = w_0(\lambda(e))w_0 f'$ where $f' = w_0 e' w_0 \in \Lambda^-$. By Corollary 5.19 we have $w_0(\lambda(e))w_0 f' = w_0 w_0(\lambda(f))f'$. And so from there it follows that $w_0 w_0(\lambda(f))f' = w_0^{\lambda(f)} w_0(\lambda(f)) w_0(\lambda(f)) f' = w_0^{\lambda(f)} f' = w_0^{\lambda(f)\cup\lambda_*(f')} f'$. Now, we can see, $\lambda(f') = \lambda^*(f') \cup \lambda_*(f')$ which is a subset of $\lambda^*(f) \cup \lambda_*(f')$, since $f' \leq f$. Thus, $\lambda(f') \subseteq \lambda(f) \cup \lambda_*(f')$ and it follows that $w_0(\lambda(e))e'w_0 \in \mathcal{J}\mathcal{G}$. Thus, $w_0(\lambda(e))e' \in \mathcal{J}\mathcal{G}w_0$.

By Theorem 7.18 it follows that $\lceil e' \rceil^{\mathscr{J}^{\lambda(e),S}} = w_0(\lambda(e))e'$. □

With the dense $B \times B$ orbit of $e \star J_{e'}$ identified we are in position to tackle the codimension 2 condition. We just need the following lemma which follows from work in [2].

Lemma 9.40. *Suppose that $I \subsetneq J \subseteq S$. Then,*

(1) $^J w_0 < {}^I w_0$

(2) $w_0^J < w_0^I$

Proof. By Proposition 5.10, $^J w_0 \in {}^J W \subseteq {}^I W$. Since $^I w_0$ is the maximum of $^I W$ it suffices to show $^J w_0 \neq {}^I w_0$. By Proposition 2.4.4 in [2], $\ell(^I w_0) + \ell(w_0(I)) = \ell(w_0) = \ell(^J w_0) + \ell(w_0(J))$. By Corollary 1.4.8(ii) in [2], $I \neq J$ implies $w_0(J) \neq w_0(I)$. But since $I \subsetneq J$, $w_0(I) \leq w_0(J)$. From here we conclude that $\ell(w_0(I)) < \ell(w_0(J))$. Hence, $\ell(^I w_0) > \ell(^J w_0)$ and from there we conclude (1). (2) is done similarly. □

Theorem 9.41. *Let M be an irreducible reductive algebraic monoid with zero, with group of units, G, and let $e \in \Lambda$ (a cross sectional lattice). Suppose that for all $e' \in \Lambda$ covered by e, we have $\lambda_*(e) \subsetneq \lambda_*(e')$. Then $codim_{e\mathfrak{D}_{M,e}}(e\mathfrak{D}_{M,e}\backslash R_e) \geq 2$.*

Proof. By Corollary 9.38, $codim_{e\mathfrak{D}_{M,e}}(e\mathfrak{D}_{M,e}\backslash R_e) = max\{dim(e \star J_{e'}) \mid e' < e, e' \in \Lambda\}$. For any variety, X, $dimX = dim\overline{X}$. So it follows by applying Proposition 9.36 and Corollary 9.39,

$$\begin{aligned} codim_{e\mathfrak{D}_{M,e}}(e\mathfrak{D}_{M,e}\backslash R_e) &= max\{dim(P_{\lambda(e)}e'G) \mid e' < e, e' \in \Lambda\} \\ &= max\{dim(\overline{P_{\lambda(e)}e'G}) \mid e' < e, e' \in \Lambda\} \\ &= max\{dim(\overline{Bw_0(\lambda(e))e'B}) \mid e' < e, e' \in \Lambda\} \end{aligned}$$

Observe that $\overline{Bw_0(\lambda(e))eB} = \overline{P_{\lambda(e)}eG} = \overline{eG} = eM = e\mathfrak{D}_{M,e}$. Suppose there exists, $r \in \mathcal{R}$, so that $w_0(\lambda(e))e' < r < w_0(\lambda(e))e$. Then, $\overline{Bw_0(\lambda(e))e'B} \subsetneq \overline{BrB} \subsetneq \overline{Bw_0(\lambda(e))eB} = e\mathfrak{D}_{M,e}$. Thus, at the level of dimensions, $dim\overline{Bw_0(\lambda(e))e'B} < dim\overline{BrB} < dim\overline{Bw_0(\lambda(e))eB} = dime\mathfrak{D}_{M,e}$. Then it follows that, $dim(e \star J_{e'}) \leq dim(e\mathfrak{D}_{M,e}) - 2$. So to show $codim_{e\mathfrak{D}_{M,e}}(e\mathfrak{D}_{M,e}\backslash R_e) \geq 2$ it suffices to show for each $e' \in \Lambda$, $e' < e$, there exists $r \in \mathcal{R}$ so that $w_0(\lambda(e))e' < r < w_0(\lambda(e))e$.

First notice that if e does not cover e' then we can find $e'' \in \Lambda$ so that $e' < e'' < e$ and hence $\lceil e' \rceil^{\mathscr{J}^{\lambda(e),S}} < \lceil e'' \rceil^{\mathscr{J}^{\lambda(e),S}} < \lceil e \rceil^{\mathscr{J}^{\lambda(e),S}}$. We need only show that such an r exists for e' covered by e.

Written in standard form, $\lceil e \rceil^{\mathscr{J}^{\lambda(e),S}} = w_0(\lambda(e))^{\lambda_*(e)}e$ and $\lceil e' \rceil^{\mathscr{J}^{\lambda(e),S}} = w_0(\lambda(e))^{\lambda_*(e')}e'$. Consider $r = min_{w_0(\lambda(e))^{\lambda_*(e')}e'}J_{w_0(\lambda(e))^{\lambda_*(e)}e}$. Recalling Theorem 6.37, we know can write r in standard form, $r = xey^{-1}$ with $y \in D(e)$ such that there exists element, $z \in W(e)$, with $zy^{-1} = w_0(\lambda_*(e'))$ and $x = min\{w_0(\lambda(e))^{\lambda_*(e')}c \mid c \in W, c \leq z\}$.

Now, regardless of z, $1 \leq z$, and so $x \leq w_0(\lambda(e))^{\lambda_*(e')}1 = w_0(\lambda(e))^{\lambda_*(e')}$. By assumption, $\lambda_*(e) \subsetneq \lambda_*(e')$, so by Lemma 9.40, $x \leq w_0(\lambda(e))^{\lambda_*(e')} < w_0(\lambda(e))^{\lambda_*(e)}$. Since both $r = xey^{-1}$ and $w_0(\lambda(e))^{\lambda_*(e)}e$ are in standard form, yet different we can conclude that $r \neq \lceil e \rceil^{\mathscr{J}^{\lambda(e),S}}$ and hence $w_0(\lambda(e))e' < r < w_0(\lambda(e))e$ as desired. □

Recall that although our last few results have been strictly stated in terms of the the right side (i.e. involving R_e, $P_{\lambda(e)}rG$, eM) there are analogues to each statement on the left side, but involving dense orbits of $B^- \times B^-$ and the corresponding 'opposite' Bruhat order on $\mathcal{R}$. So by analogy, we have also proven the statement, *Let M be an irreducible reductive algebraic monoid with zero, with group of units, G, and let $e \in \Lambda$ (a cross sectional lattice). Suppose that for all $e' \in \Lambda$ covered by e, we have $\lambda_*(e) \subsetneq \lambda_*(e')$. Then $codim_{\mathfrak{D}_{M,e}e}(\mathfrak{D}_{M,e}e\backslash L_e) \geq 2$.*

Before we see the fruits of the theorem, let us note this condition does not always hold.

Example 9.42. *Consider the $n \times n$ matrices, $M_n(K)$. If we let $e = 1$ then $\mathfrak{D}_{M_n(K),1} = M_n(K)$. Take the idempotent, $e' = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$. Then with respect to the simple reflections, S, obtained with* *the usual Borel subgroup of invertible upper triangular matrices, $\lambda_*(1) = \lambda_*(e') = \emptyset$. As a result, our theorem cannot be directly applied to this situation.*

Corollary 9.43. *If $\mathfrak{D}_{M,e}$, $\mathfrak{D}_{M,e}e$, $\mathfrak{D}_{M,e}e$, and $e\mathfrak{D}_{M,e}e$ are normal and for all $e' \in \Lambda$ covered by e, we have $\lambda_*(e) \subsetneq \lambda_*(e')$, then $\overline{GeG} = \mathcal{S}(Ge, H_e, eMe, H_e, eG)$.*

Proof. By Proposition 9.30 we know that $\mathfrak{D}_{M,e}$ is an irreducible regular algebraic semigroup with zero. We have also shown that e is belongs to the maximal $\mathscr{J}$-class of $\mathfrak{D}_{M,e}$ (Corollary 9.32). Furthermore, we have by assumption that $\mathfrak{D}_{M,e}$, $\mathfrak{D}_{M,e}e$, $\mathfrak{D}_{M,e}e$, and $e\mathfrak{D}_{M,e}e$ are normal.

By Corollary 9.24 it suffices to show that the conditions $codim_{e\mathfrak{D}_{M,e}}(e\mathfrak{D}_{M,e}\backslash R_e) \geq 2$ and $codim_{\mathfrak{D}_{M,e}e}(\mathfrak{D}_{M,e}e\backslash L_e) \geq 2$ are satisfied. But these are satisfied by Theorem 9.41 and its analogous statement for L_e which we have already remarked on. From here the result follows. □

Example 9.44. *Let $n > 2$ and consider $M = M_n(K)$ with the usual Borel subgroup of invertible upper triangular matrices. If we fix a nontrivial idempotent of the cross sectional lattice given to us by B, $0 < e < 1$, $e \in \Lambda$, we can say that $e = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ for some $0 < r < n$. Since $n > 2$ and $0 < r < n$ we compute $\lambda_*(\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}) = \{(r\ r+1), (r+1\ r+2), \cdots, (n-1\ n)\}$.*

Suppose $e' \in \Lambda$ is covered by e. Then $e' = \begin{pmatrix} I_{r-1} & 0 \\ 0 & 0 \end{pmatrix}$, and just as above it is not hard to compute that $\lambda_*(\begin{pmatrix} I_{r-1} & 0 \\ 0 & 0 \end{pmatrix}) = \{(r-1\ r), (r\ r+1), \cdots, (n-1\ n)\}$. It is clear $(r-1\ r) \in \lambda_*(\begin{pmatrix} I_{r-1} & 0 \\ 0 & 0 \end{pmatrix})\backslash\lambda_*(\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix})$. Now observe that,

$$\mathfrak{D}_{M,e}e = M_n(K)\begin{pmatrix} I_{r-1} & 0 \\ 0 & 0 \end{pmatrix} = \text{all } n \times n \text{ matrices with the right } n-r \text{ rows all zero} \cong K^{nr}$$

$$e\mathfrak{D}_{M,e} = \begin{pmatrix} I_{r-1} & 0 \\ 0 & 0 \end{pmatrix} M_n(K) = \text{all } n \times n \text{ matrices with the bottom } n-r \text{ rows all zero} \cong K^{nr}$$

$$e\mathfrak{D}_{M,e}e = \begin{pmatrix} I_{r-1} & 0 \\ 0 & 0 \end{pmatrix} M_n(K) \begin{pmatrix} I_{r-1} & 0 \\ 0 & 0 \end{pmatrix} \cong M_r(K) \cong K^{r^2}$$

are all normal varieties as they are each isomorphic to affine space. Lastly, recall from our motivation that, $\mathfrak{D}_{M,e} = \{$all matrices of rank $\leq r\}$ which is known to be a normal variety [12]. Since the requirements of Corollary 9.43 are satisfied we may conclude that,

$$\mathfrak{D}_{M,e} \cong \mathcal{S}(GL_n(K)e, GL_r(K), M_r(K), GL_r(K), eGL_n(K)).$$

It is entirely possible that all determinantal varieties (as defined in Definition 9.28) are normal (perhaps even Cohen-Macaulay) in which case this result would hold in a much broader sense.

10 Concluding Remarks

It was quite unexpected that the absolute maxima elements of our parabolic Green's relations (which was constructed as a purely combinatorial generalisation of Green's relations in $\mathcal{R}$) was able to answer a deep geometric problem (the codimension 2 condition of Theorem 9.41).

There is the sense that the surface has not even been scratched on the possible use of maximal and minimal elements. Indeed, the entire theory presented here still has some gaps. We take the time to reiterate these unanswered questions and conjectures of this paper.

The first open question was born from the general nonexistence of relative maxima of $\mathcal{J}$-classes. This problem was also encountered with our parabolic relations, with the added wrinkle that now $min_r J_s^{I,J}$ and $max_s J_r^{I,J}$ were both nonexistent.

Question. *Let $K_- \subseteq I_- \subseteq S$, $K_+ \subseteq I_+ \subseteq S$, $L_- \subseteq J_- \subseteq S$, $L_+ \subseteq J_+ \subseteq S$ be sets of simple reflections such that $W_{I_*} = W_{K_*} \times W_{I_*\backslash K_*} = W_{I_*\backslash K_*} \times W_{K_*}$ and $W_{J_*} = W_{L_*} \times W_{J_*\backslash L_*} = W_{J_*\backslash L_*} \times W_{L_*}$ for all sets $* = +$ or $-$. Suppose also that $L_* \subseteq K_*$ and $I_*\backslash K_* \subseteq J_*\backslash L_*$ for all $* = +$ or $-$ and that $w_0 W_{H_+} w_0 = W_{H_-}$ for all $H = I, J, K, L, I\backslash K$ and $J\backslash L$. For elements, $\sigma_- \in {}^{I_-}W$, $\tau_- \in {}^{J_-}W$, $\sigma_+ \in W^{I_+}$, $\tau_+ \in W^{J_+}$, and $\tau_0 \in {}^{L_-}W \cap W_{J_-} w_0 W_{J_+} \cap W^{L_+}$, define the set,*

$$\mathcal{A} = \left\{ \sigma_0 \in {}^{K_-}W \cap (W_{I_-} w_0 W_{I_+}) \cap W^{K_+} \;\middle|\; \begin{array}{c} \exists w_- \in W_{K_-} W_{J_-\backslash L_-}, \exists w_+ \in W_{J_+\backslash L_+} W_{K_+} \text{ so that} \\ w_-\tau_- \leq \sigma_-, \sigma_0 \leq w_-\tau_0 w_+ \text{ and } \tau_+ w_+ \leq \sigma_+ \end{array} \right\}$$

Is it true that if $\mathcal{A} \neq \emptyset$, then $\mathcal{A}$ is a directed set (a preorder where every pair of elements has an upper bound) with regards to the Bruhat order, $\leq$?

The following conjecture and question concern geometric (Borel subgroup-centric) definitions for our generalised sets, $\mathcal{O}^{I,J}$, $\mathcal{GJ}^I$, $\mathcal{JG}^J$, and $\mathcal{N}^{I,J}$.

Conjecture. *Let $I, J \subseteq S$ and $r \in \mathcal{R}$.*

(1) $r \in \mathcal{GJ}^I$ if and only if $(B \cap L_I)r \subseteq rB$ if and only if $(B \cap L_I)rB = rB$

(2) $r \in \mathcal{JG}^J$ if and only if $r(B \cap L_J) \subseteq Br$ if and only if $Br(B \cap L_J) = Br$

(3) $r \in \mathcal{N}^{I,J}$ if and only if $(B \cap L_I)r(B \cap L_J) \subseteq Br \cap rB$

Question. *For a given $I, J \subseteq S$, can we find a definition for $\mathcal{O}^{I,J}$ that is similar to that given by Definition 3.10 in Section 3?*

Although our later theory of relative maxima and minima allowed us to answer the existence of the absolute maxima and minima for any $\mathscr{H}^{I,J}$-class, it remains to describe these elements in a meaningful way. As we noted, the set $\mathcal{O}^{I,J}$ is the most likely candidate, however a proof proves elusive for now.

Conjecture. *For any $r \in \mathcal{R}$ and any $I, J \subseteq S$,*
(1) $r = \lfloor r \rfloor^{\mathscr{H}^{I,J}}$ if and only if $r \in \mathcal{O}^{I,J}$.
(2) $r = \lceil r \rceil^{\mathscr{H}^{I,J}}$ if and only if $r \in w_0(I)\mathcal{O}^{I,J} = \mathcal{O}^{I,J} w_0(J)$.

Our final open question's positive answer would add a lot of meaning to our exciting new construction of irreducible regular algebraic semigroups with zero. At the moment our stumbling block is that the only affirmative answers rely on conditions of normality which are not presently bypassable.

Question. *Is $S \cong \mathcal{S}(L_e, H_e, eSe, H_e, R_e)$ for all irreducible regular algebraic semigroups with zero, S?*

As we can see, the bulk of the outstanding results and questions come from our new parabolic Green's relations. As a new concept this is understandable, and hopefully sufficient use of these concepts has been demonstrated in this paper to warrant their further study. Indeed, for both the combinatorial investigations of the Renner monoid and the Green's building constructions there appear to be many possibilities for subsequent investigation.

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Appendix

A.1 Results From Other Sources

Proposition A.1. *Let S be a regular semigroup. Suppose that $a \in S$ and $a = a_1 a_2 \cdots a_k$. Then $a \mathscr{J} a_i$ for all i if and only if we can find $e_1, e_2, \cdots e_{k-1} \in E(S)$ so that $a_i \mathscr{L} e_i \mathscr{R} a_{i+1}$ for all $1 \leq i \leq k-1$.*

Proof. The case for $k = 2$ follows from [20] Theorem 1.4(vi). Now suppose $k \geq 3$. Suppose that $a \mathscr{J} a_i$. For any i, $SaS \subseteq Sa_ia_{i+1}S \subseteq Sa_iS = Sa_{i+1}S = SaS$ so $a_ia_{i+1} \mathscr{J} a_i \mathscr{J} a_{i+1}$. By the $k = 2$ case we than see that there must also exist e_i so that $a_i \mathscr{L} e_i \mathscr{R} a_{i+1}$.

For the reverse direction, suppose that such $e_1, e_2, \cdots e_{k-1} \in E(S)$ exist. Then we can see that $Sa_1a_2 = Se_1a_2 = Sa_2 = Se_2$, so $a_1a_2 \mathscr{L} e_2 \mathscr{R} a_3$ and so by induction, $a \mathscr{J} a_1a_2$, $a \mathscr{J} a_i$ for all $i \geq 3$. We conclude by noting that $a_1a_2 \mathscr{J} a_1 \mathscr{J} a_2$ (by the existance of e_1), which shows that $a \mathscr{J} a_1a_2 \mathscr{J} a_1 \mathscr{J} a_2$. □

Proposition A.2. *For any idempotent $e \in E(\mathcal{R})$, we get the following, $eBe \subseteq eB$ and $eBe \subseteq Be$.*

Proof. Putcha notes in Corollary 7.2(ii) of his book ([20]), for any $e \in E(\overline{B})$, $eBe = eC_B(e)$, where $C_B(e)$ is the centralizer, $\{g \in B \mid eg = ge\}$. From there we see $eBe = eC_B(e) \subseteq eB$. Since this subgroup commutes with e we also have, $eBe = C_B(e)e$ and may conclude $eBe \subseteq Be$. □

Theorem A.3. *Suppose that X is an irreducible variety. Then, if we can decompose X into a finite disjoint union of subvarieties $X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_m$, then there exists a unique i with $1 \leq i \leq m$ so that X_i is open and dense in X.*

Proof. We can see that $X = \overline{X} = \overline{\bigsqcup_{i=1}^m X_i} = \bigcup_{i=1}^m \overline{X_i}$ since it is a finite union. Suppose that none of the $\overline{X_i} = \overline{X}$, then $X = \overline{X_1} \cup \overline{\bigsqcup_{i=2}^m X_i}$, both closed sets, which gives us a contradiction to the irreducibility of X. Thus there exists an i so that $\overline{X_i} = \overline{X}$. It follows that $dim(X_i) = dim(X)$. By Proposition 14.1.6(iv) of [35], since X is irreducible X_i must be open.

To see that i is unique, suppose there is another, X_j. It must also be open, so being dense sets, $X_i \cap X_j \neq \emptyset$, contradicting our assumption of disjointness. □

Proposition A.4. *Suppose that an irreducible algebraic group, G, acts on a variety, X. Then for any element $x \in X$, the orbit Gx is an irreducible subvariety of X.*

Proof. Since G is irreducible and $\{x\}$ is a singleton set (hence irreducible) we can see that $G \times \{x\}$ is an irreducible variety. So its image under the morphism of our group action must be irreducible. But this image is exactly $Gx \subseteq X$. Thus Gx is an irreducible subvariety of X. □

Lemma A.5. *The nonzero multiplicative single-variable polynomials are monic monomials.*

Proof. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with coefficients in K, and particularly, $a_n \neq 0$. Suppose that p is multiplicative. Then $p(x)p(y) = p(xy)$. Expanding we see that,

$$a_n^2 x^n y^n + a_n a_{n-1} x^n y^{n-1} + a_{n-1} a_n x^{n-1} y^n + a_{n-1}^2 x^{n-1} y^{n-1} + \cdots + a_1 a_0 x + a_0 a_1 y + a_0^2$$
$$= a_n x^n y^n + \cdots + a_1 xy + a_0$$

Comparing coefficients we can derive the following conditions on the coefficients, $a_n a_{n-1} = 0$, $a_n a_{n-2} = 0$, $a_n a_{n-3} = 0, \cdots, a_n a_0 = 0$. By assumption, $a_n \neq 0$, so $a_{n-1} = a_{n-2} = \cdots = a_0 = 0$. This simplifies the equation to,

$$a_n^2 x^n y^n = a_n x^n y^n$$

Thus, $a_n^2 = a_n$ and we can conclude that $a_n = 1$, yielding $p(x) = x^n$, a monic monomial. □

A.2 Opposite Standard Form

In this part of the appendix, we will lay out a number of results that are similar to those given at the start of [17]. Ultimately, these will culminate in our proof of Theorem 5.33, as Theorem A.9.

Lemma A.6.

(1) If $e, f \in \Lambda$ and $e \leq f$ then $W(e) = W_*(e)\,(W(e) \cap W(f))$.

(2) If $e, f \in \Lambda^-$ and $e \leq f$ then $W(e) = (W(e) \cap W(f))\,W_*(e)$.

Proof. (1) is noted in [17] and is included for a sense of completeness. Now, to prove (2) recall that, by Proposition 5.21, $W^*(e) \subseteq W^*(f)$, since $e \leq f$. Proposition 5.23 tells us that $W(e) = W^*(e)W_*(e)$. Since $W^*(e) \subseteq W(e)$ and $W^*(e) \subseteq W^*(f) \subseteq W(f)$, we can see that $W(e) \subseteq (W(e) \cap W(f))\,W_*(e)$. Conversely, $W(e) \cap W(f) \subseteq W(e)$ and $W_*(e) \subseteq W(e)$, so then $(W(e) \cap W(f))\,W_*(e) \subseteq W(e)$. □

Lemma A.7. *Let $x, y \in W$. The following are equivalent.*

(1) $x \leq y$

(2) $xBy^{-1} \cap B^{-}B \neq \emptyset$

(3) $yBx^{-1} \cap BB^{-} \neq \emptyset$

Proof. The equivalent of (1) and (2) is established in [17] by Lemma 1.2. Recall that in W, $x \leq y$ if and only if $w_0 y \leq w_0 x$. Then our equivalence between (1) and (2) tells us that $w_0 yB(w_0 x)^{-1} \cap B^{-}B \neq \emptyset$. Recall that $w_0^{-1} = w_0$ and $w_0 B w_0 = B^{-}$. So, by multiplying both sides by w_0 does not change the emptiness of a set, and we can see,

$$yBx^{-1} \cap BB^{-} = w_0(w_0 y)B(w_0 x)^{-1} w_0 \cap w_0 B^{-} B w_0 \neq \emptyset$$

if and only if $(w_0 y)B(w_0 x)^{-1} \cap B^{-}B \neq \emptyset$ if and only if (1). □

Lemma A.8. *For all* $w \in W$

(1) $B^{-}xB \subseteq B^{-}Bx \cap xB^{-}B$

(2) $BxB^{-} \subseteq BB^{-}x \cap xBB^{-}$

Proof. The proof of (1) is given in [17] as Lemma 1.3. To prove (2), consider the element $w_0 x w_0$. Then by (1), $B^{-} w_0 x w_0 B \subseteq B^{-} B w_0 x w_0 \cap w_0 x w_0 B^{-} B$. The containment is unchanged if we multiply on the left and right by w_0. So we have,

$$w_0 B^{-} w_0 x w_0 B w_0 \subseteq w_0 (B^{-} B w_0 x w_0 \cap w_0 x w_0 B^{-} B) w_0.$$

Recall that $B^{-} = w_0 B w_0$, so then, $w_0 B^{-} w_0 x w_0 B w_0 = BxB^{-}$ and,

$$w_0 (B^{-} B w_0 x w_0 \cap w_0 x w_0 B^{-} B) w_0 = BB^{-}x \cap xBB^{-},$$

yielding the result. □

Theorem A.9. *Let* $e, f \in \Lambda^{-}$*,* $x, s \in W$*,* $y \in V(e)$ *and* $t \in V(f)$*. Then the following are equivalent,*

(1) $y^{-1}ex \leq t^{-1}fs$

(2) $ef = e$ *and there exist* $w \in W_{*}(e)W(f)$*,* $z \in W_{*}(e)$ *so that* $wt \leq y$ *and* $x \leq zws$ *in* W

Proof. Assume that (2) holds. By Lemma 5.28 we can see that $y^{-1}ewt \in \overline{B}$. It follows that $y^{-1}ewfs = yewtt^{-1}fs \in \overline{B}t^{-1}fs \subseteq \overline{Bt^{-1}fsB}$. By assumption, $w = w_1 w_2$ with $w_1 \in W_{*}(e)$ and $w_2 \in W(f)$. Thus, $y^{-1}ewfs = y^{-1}ew_1 w_2 fs = y^{-1}ew_2 fs = y^{-1}ew_2 s = y^{-1}ews$. But, by Lemma 5.27 $y^{-1}ex \leq y^{-1}ezws = y^{-1}ews = y^{-1}ewfs \leq t^{-1}fs$.

Conversely, suppose $y^{-1}ex \leq t^{-1}fs$. Then clearly, $e \leq f$, and $y^{-1}ex \in \overline{Bt^{-1}fsB}$. Hence, $e \in \overline{yBt^{-1}fsBx^{-1}}$. Now, for $w \in W$ let,

$$A_w = sBx^{-1} \cap BwB^-$$

It is clear that $sBx^{-1} = \bigsqcup_{w \in W} A_w$, since $sBx^{-1} \subseteq G = \bigsqcup_{w \in W} BwB^-$. Since this is a finite disjoint union of subvarieties, Theorem A.3 tells us we can find a unique element, $w \in W$, so that A_w is open and dense in sBx^{-1}.

It follows that , $e \in \overline{eyBt^{-1}fA_w}e \subseteq \overline{eyBt^{-1}fA_we} \subseteq \overline{eyBt^{-1}fBwB^-e} = \overline{eyBt^{-1}fBfweB^-e}$, since $fB = fBf$ and $B^-e = eB^-e$, as $e, f \in \Lambda^-$. Hence, $fwe \mathscr{J} e$. So $e \leq w^{-1}fw$. But $e \leq f$, and thus there exists $v \in C_W(e)$ such that $v^{-1}fv = w^{-1}fw$. But then, $v \in W(f) \cap W(e) \neq \emptyset$, so we quickly see that we can write $w = cv$, with $c \in W(f)$. It then follows, by Lemma A.6, $w = cv \in W(f)W(e) \subseteq W(f)\,(W(f) \cap W(e))\,W_*(e) \subseteq W(f)W_*(e)$. We conclude $w = w_1w_2$ for some $w_1 \in W(f)$ and $w_2 \in W_*(e)$.

Since $A_w \neq \emptyset$ we see by Lemma A.8 that $\emptyset \neq sBx^{-1} \cap BwB^- \subseteq sBx^{-1} \cap wBB^-$. Thus, we see $w^{-1}sBx^{-1} \cap BB^- \neq \emptyset$. So by Lemma A.7, $x \leq w^{-1}s = w_2^{-1}w_1^{-1}s$.

Then we see, $e \in \overline{yBt^{-1}fA_w} \subseteq \overline{yBt^{-1}fBw_1w_2B^-} = \overline{yBt^{-1}fw_1w_2B^-} = \overline{yBt^{-1}w_1fw_2B^-}$, since $w_1 \in W(f)$ and $t^{-1}f \in \mathcal{J}\mathcal{G}$. For $u \in W$ let,

$$C_u = yBt^{-1}w_1 \cap BuB^-$$

As before, we see that $yBt^{-1}w_1 = \bigsqcup_{u \in W} C_u$, and there exists a unique u so $C_u \subseteq yBt^{-1}w_1$ is open and dense. Thus, $e \in \overline{C_ufw_2B^-}$. It follows from there that we get a short chain of inclusions, $e \in \overline{eC_ufw_2B^-}e \subseteq \overline{eC_ufw_2B^-e} \subseteq \overline{eBuB^-fw_2eB^-e}$. And from there we see that $e \in \overline{eBuB^-feB^-e} = \overline{eBuB^-eB^-e} = \overline{eBuB^-e} = \overline{eBeueB^-e}$, since $e \in \Lambda^-$.

Thus, $eue \mathscr{J} e$, and hence $u \in W(e)$. So in $eC_G(e)$, we see $e \in \overline{eBeueB^-e}$. But $eC_G(e)$ is a reductive algebraic group, with a Borel subgroup, eBe that has opposite, $(eBe)^- = eB^-e$. Thus $e \in \overline{(eBe)eu(eBe)^-}$ means that $eu \leq e$ in $W(eC_G(e))$. But e is the identity of $eC_G(e)$, so it is the minimum element of the Weyl group, and we can see $eu = e$. So $u \in W_*(e)$.

Since $C_u \neq \emptyset$, we see $\emptyset \neq yBt^{-1}w_1 \cap BuB^- \subseteq yBt^{-1}w_1 \cap BB^-u$. So $yBt^{-1}w_1u^{-1} \cap BB^- \neq \emptyset$. Thus, $uw_1^{-1}t \leq y$ by Lemma A.7. Let $\overline{w} = uw_1^{-1} \in W_*(e)W(f)$, $z = w_2^{-1}u^{-1} \in W_*(e)$. We may then conclude that,

$$x \leq w_2^{-1}w_1^{-1}s = w_2^{-1}u^{-1}uw_1^{-1}s = z\overline{w}s \qquad \overline{w}t = uw_1^{-1}t \leq y$$

as desired. □

A.3 Pointed Parabolic $\mathscr{J}$-classes on $M_3(K)$

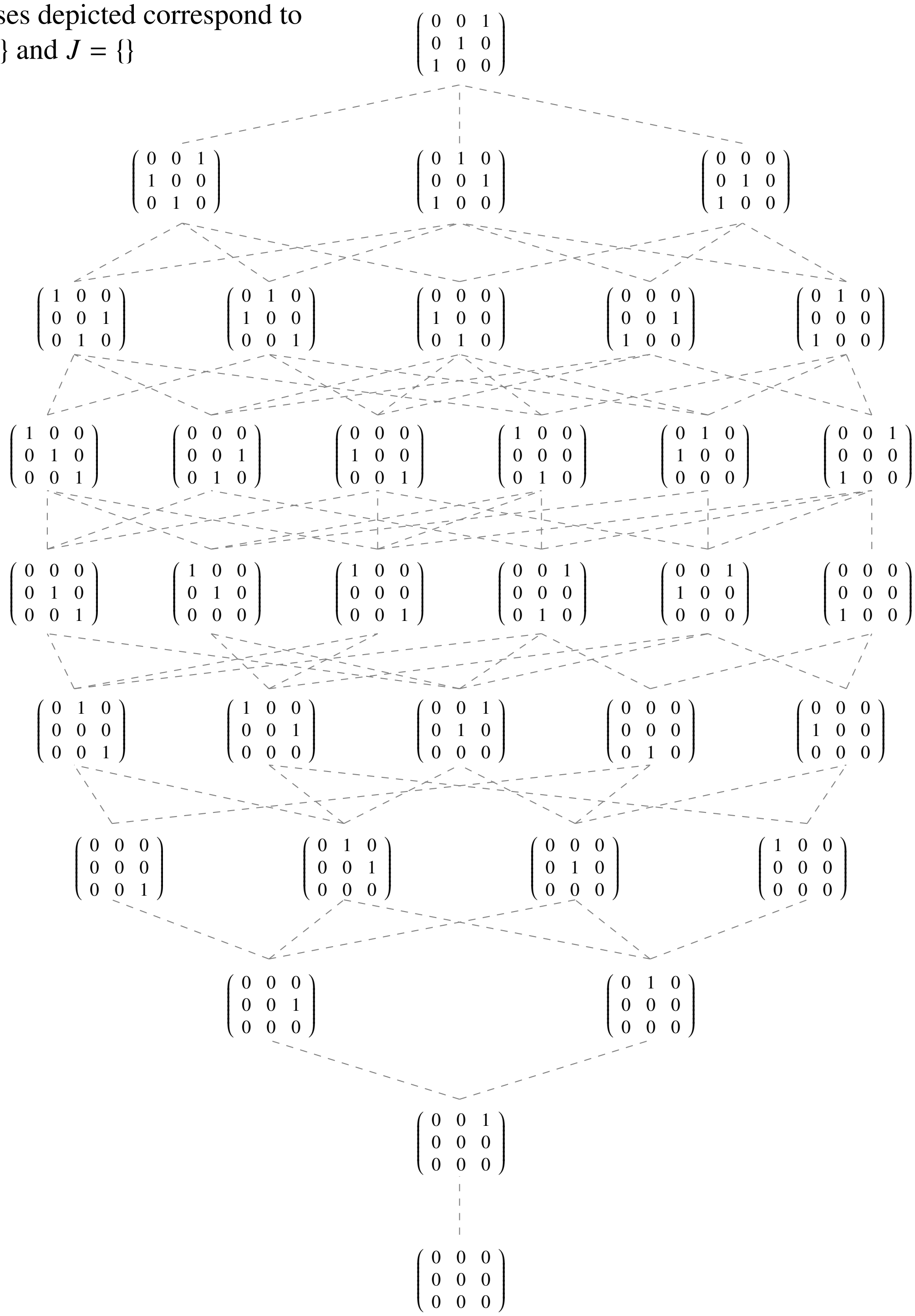
Recall the simple reflections for the Weyl group $M_3(K)$ with B, the group of upper triangular matrices. $S = \{(1\ 2), (2\ 3)\} = \{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\}$. The next 16 pages show the equivalence classes defined in Section 7 in relation to the Bruhat order. Specifically, for each of the pairs $(I, J) \in P(S) \times P(S)$, we look at the graph of the covering relation of the Bruhat order and highlight the $\mathscr{J}^{I,J}$-classes. This allows us to illustrate concepts like $min_r J_s^{I,J}$ and $\lceil r \rceil^{\mathscr{J}^{I,J}}$.

Each graph contains the 34 matrices of the Renner monoid of $M_3(K)$ two matrices are connected by a line if the matrix lower down the page is covered by the higher matrix with respect to the Bruhat order. That is, r and s are connected if there exists no matrix t so that $r < t < s$. A matrix lower on the page is smaller with respect to $\leq$ than a matrix higher up if there is an upward path in the graph connecting the two matrices.

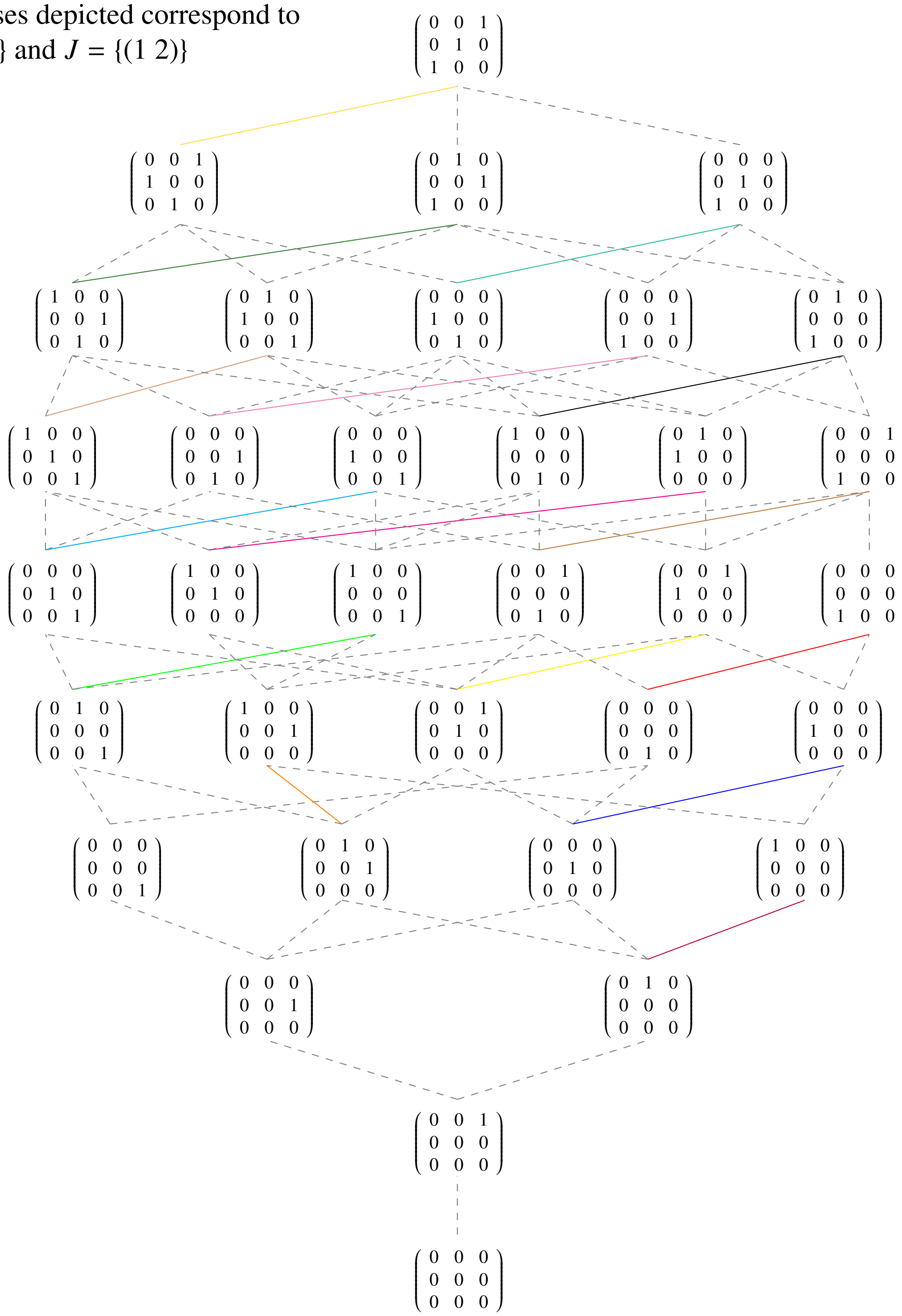
Matrices connected by a gray, dashed line are in different $\mathscr{J}^{I,J}$-classes. Each pair of matrices connected by a solid, coloured line are members of the same $\mathscr{J}^{I,J}$-class. Each $\mathscr{J}^{I,J}$-class is granted its own colour, although in the case of size one equivalence classes this colour does not appear. The colours are for visual convenience only, and do not have any particular meaning.

The first of these charts corresponds to $\mathscr{J}^{\emptyset,\emptyset}$, or rather the equality relation. As such every equivalence class has only one element, so there are no solid coloured lines. The fourth, thirteenth, and sixteenth graphs represent the $\mathscr{R}$-, $\mathscr{L}$-, and $\mathscr{J}$-classes. So confused readers are encouraged to view those charts first to get a better sense of reading the others.

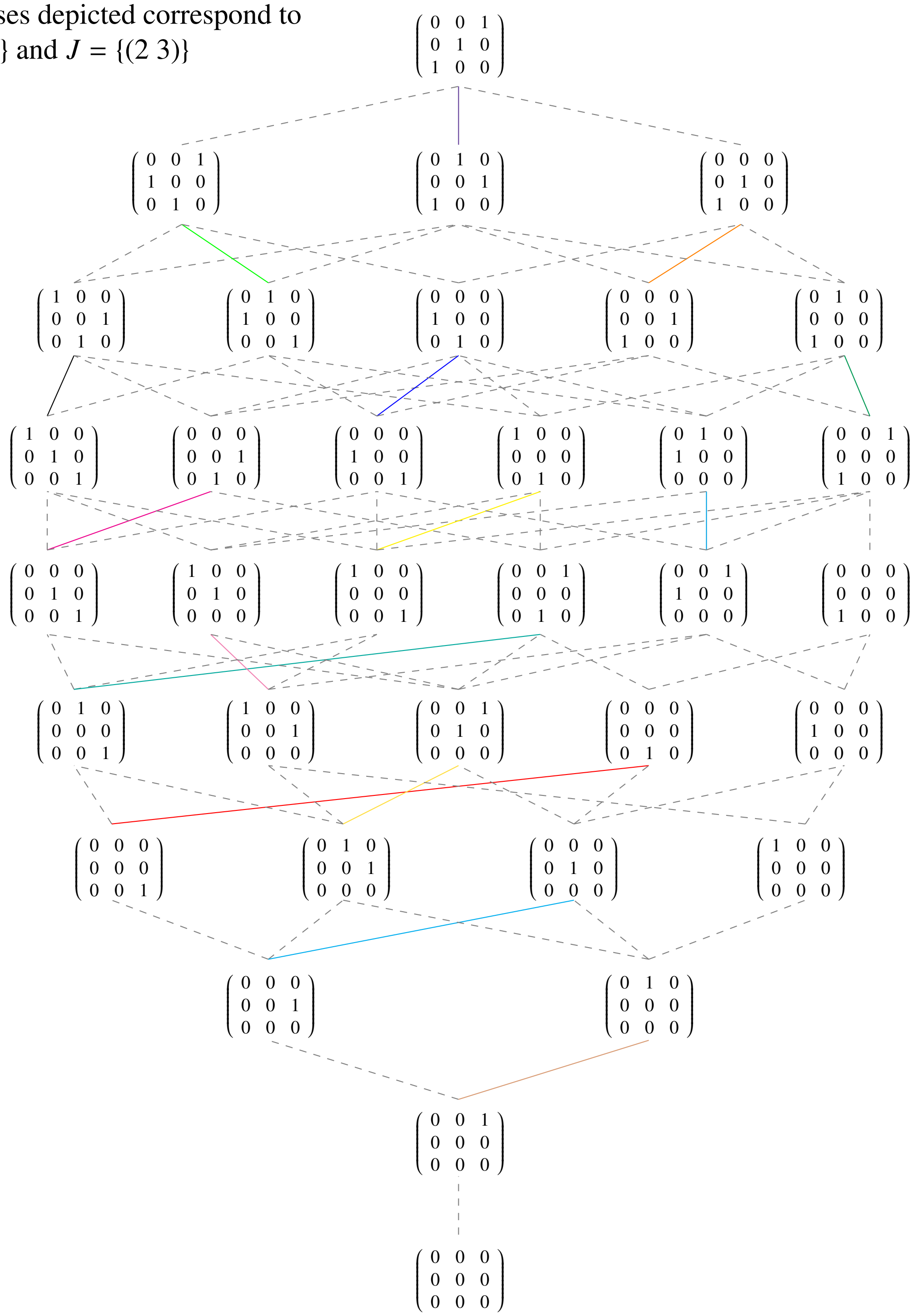
Classes depicted correspond to
$I = \{\}$ and $J = \{\}$



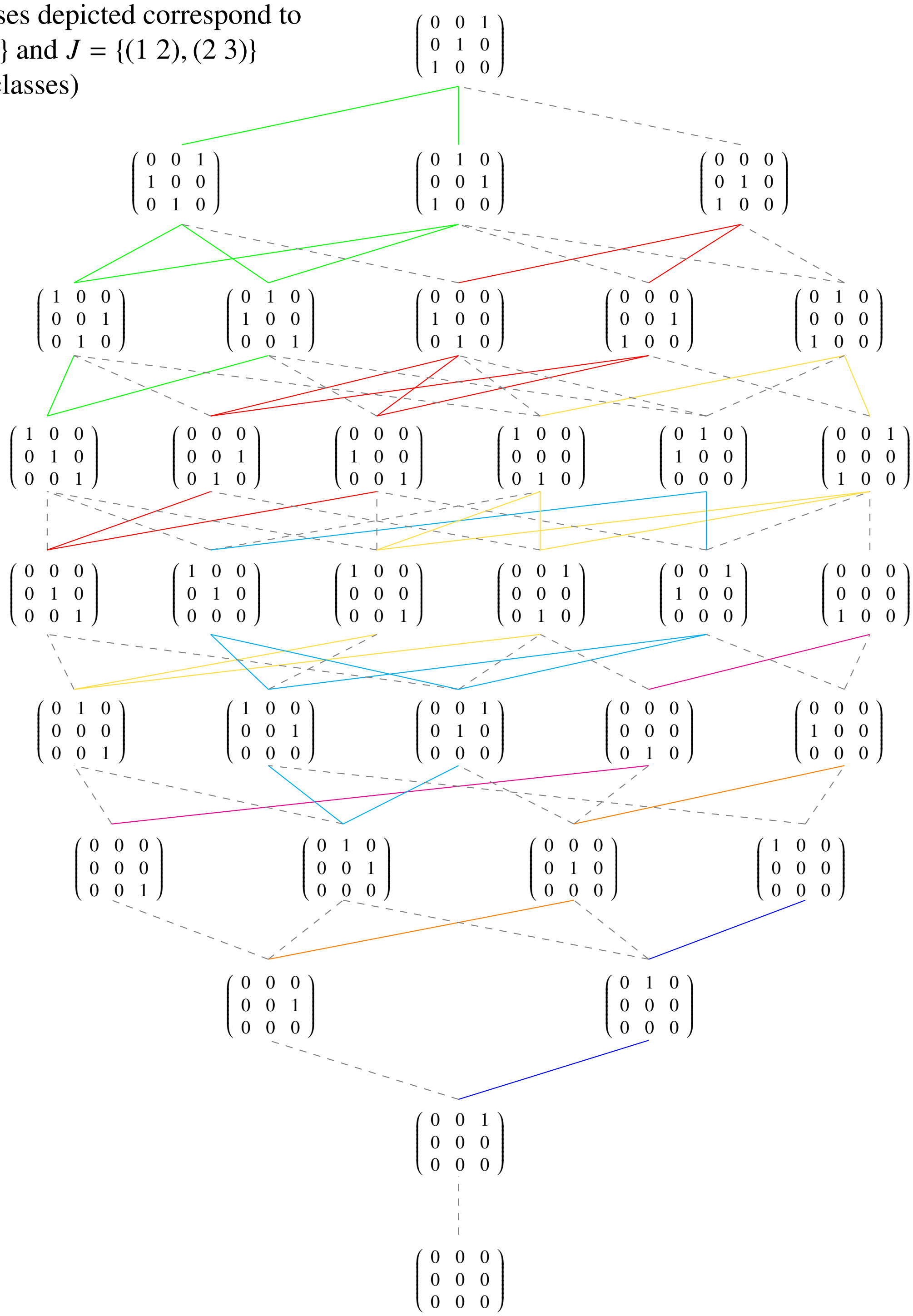
Classes depicted correspond to
$I = \{\}$ and $J = \{(1\ 2)\}$



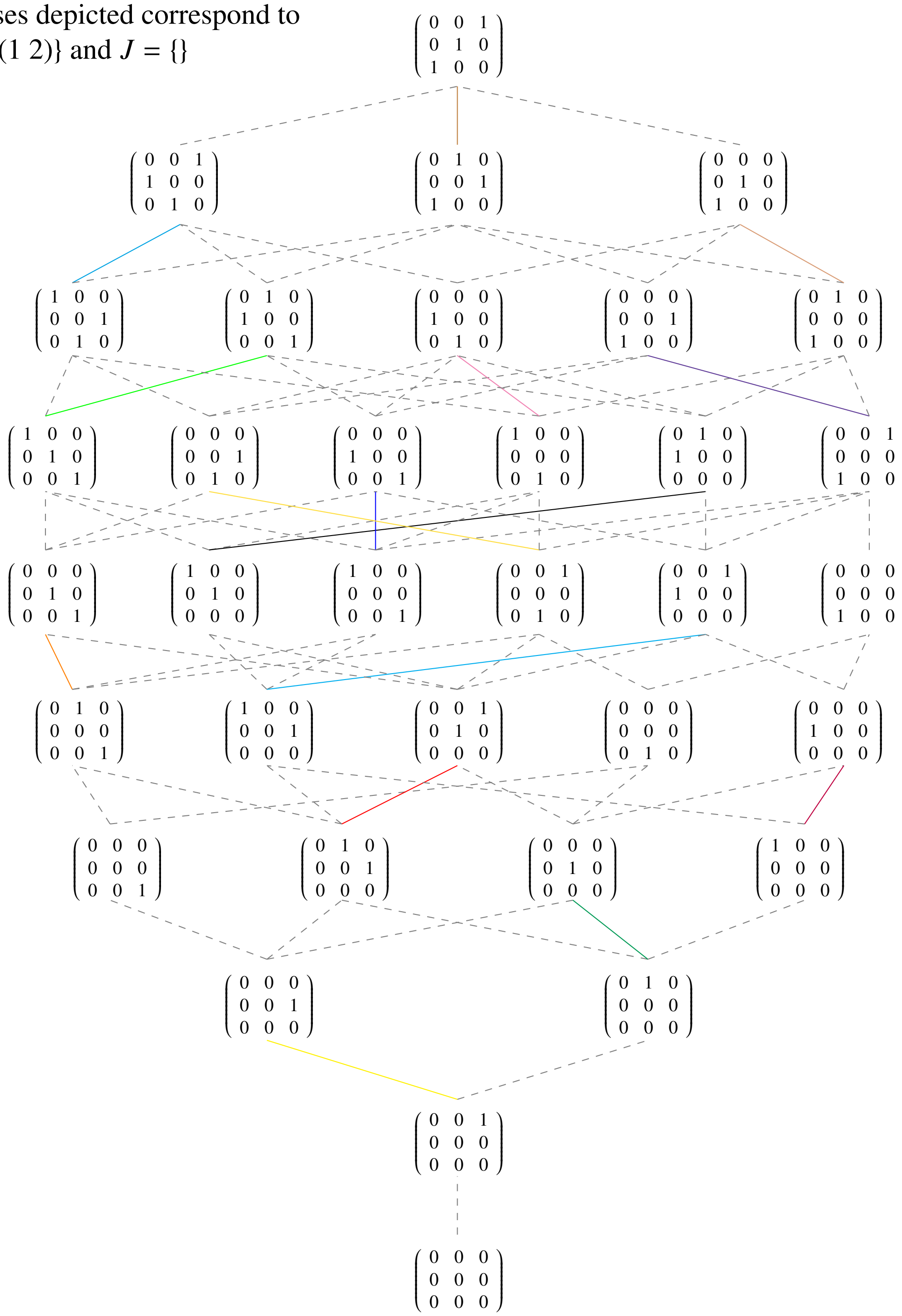
Classes depicted correspond to
$I = \{\}$ and $J = \{(2\ 3)\}$



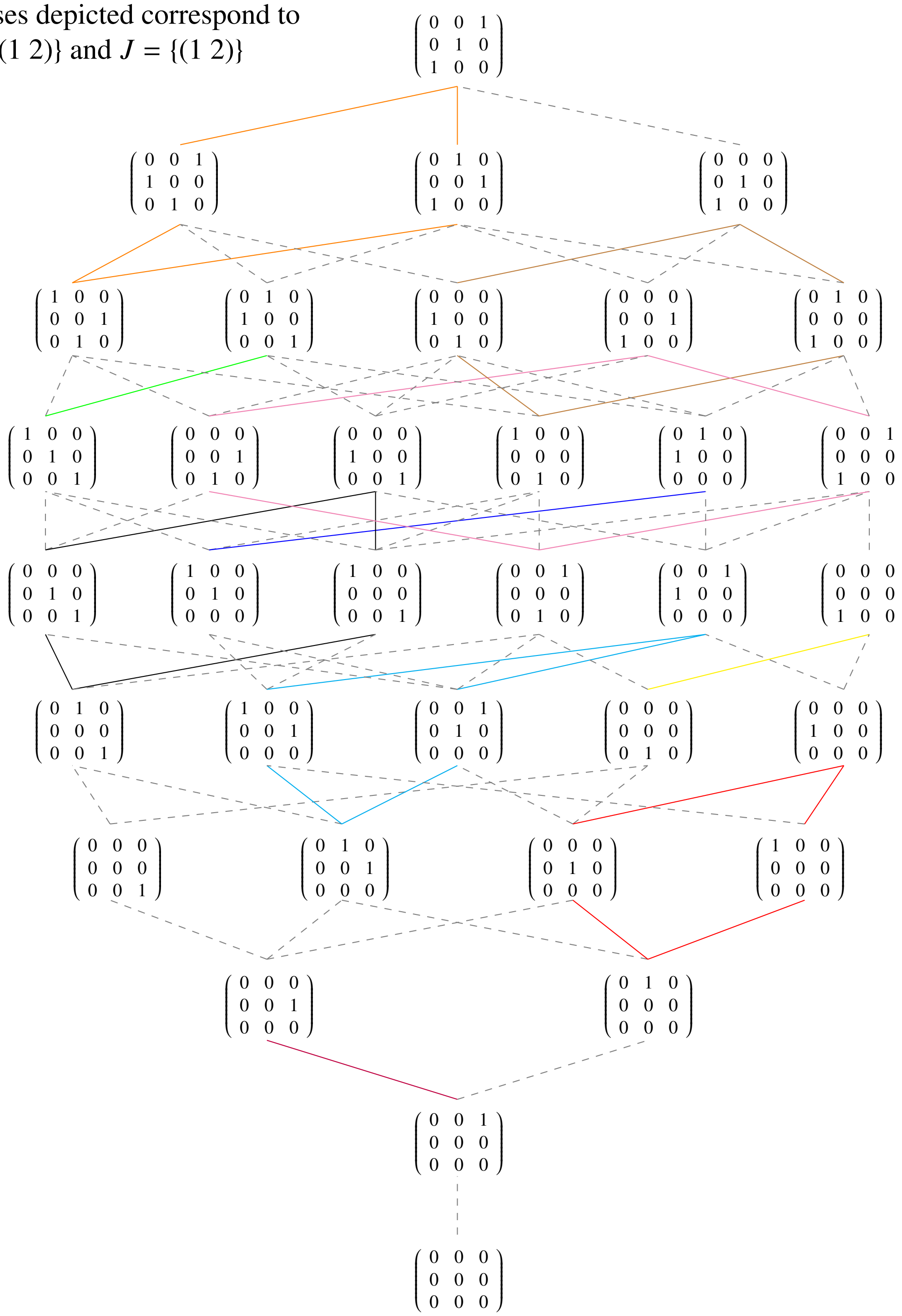
Classes depicted correspond to
$I = \{\}$ and $J = \{(1\ 2), (2\ 3)\}$
($\mathscr{R}$-classes)



Classes depicted correspond to
$I = \{(1\ 2)\}$ and $J = \{\}$



Classes depicted correspond to
$I = \{(1\ 2)\}$ and $J = \{(1\ 2)\}$



Classes depicted correspond to
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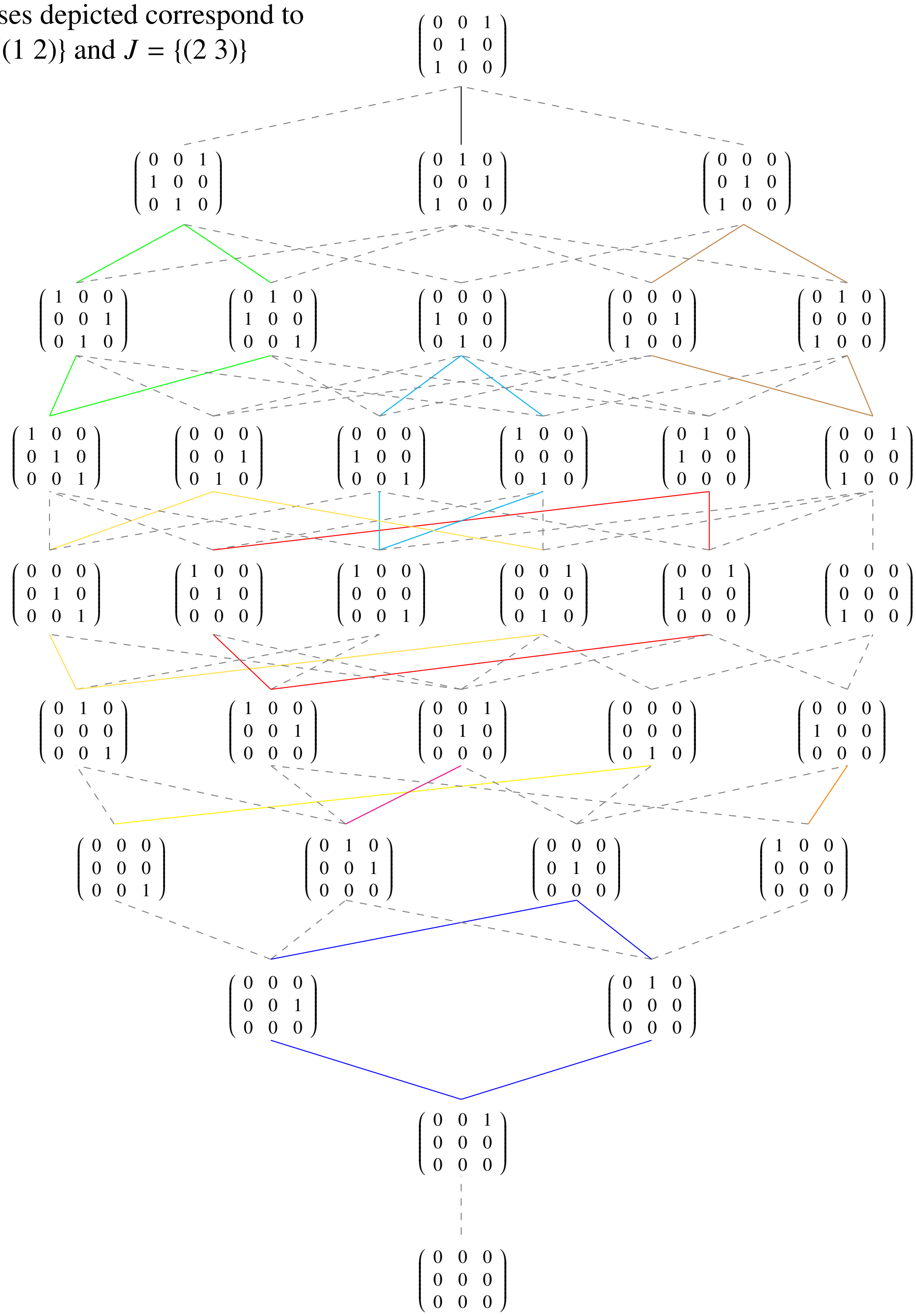
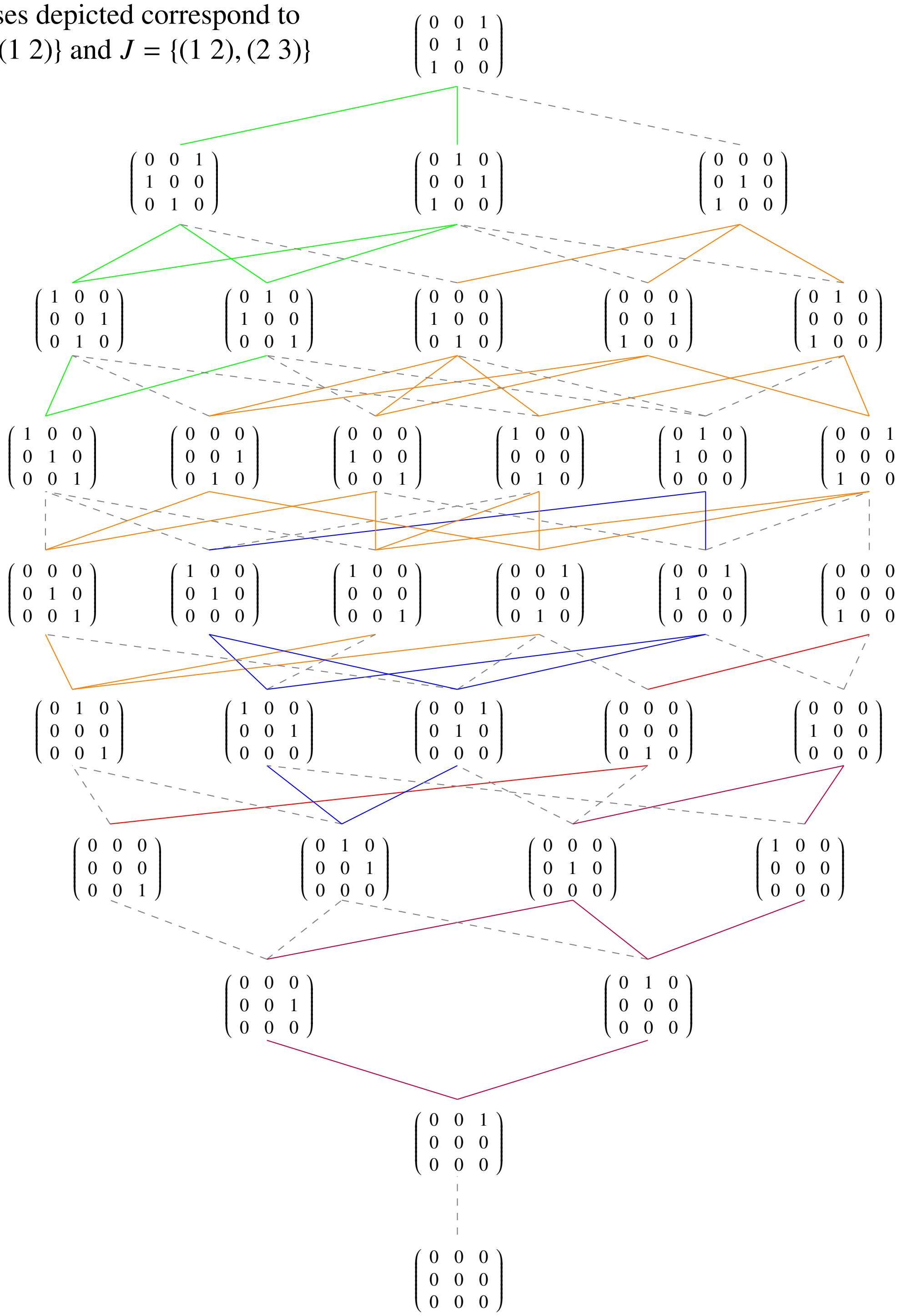


Classes depicted correspond to
$I = \{(1\ 2)\}$ and $J = \{(1\ 2), (2\ 3)\}$



Classes depicted correspond to
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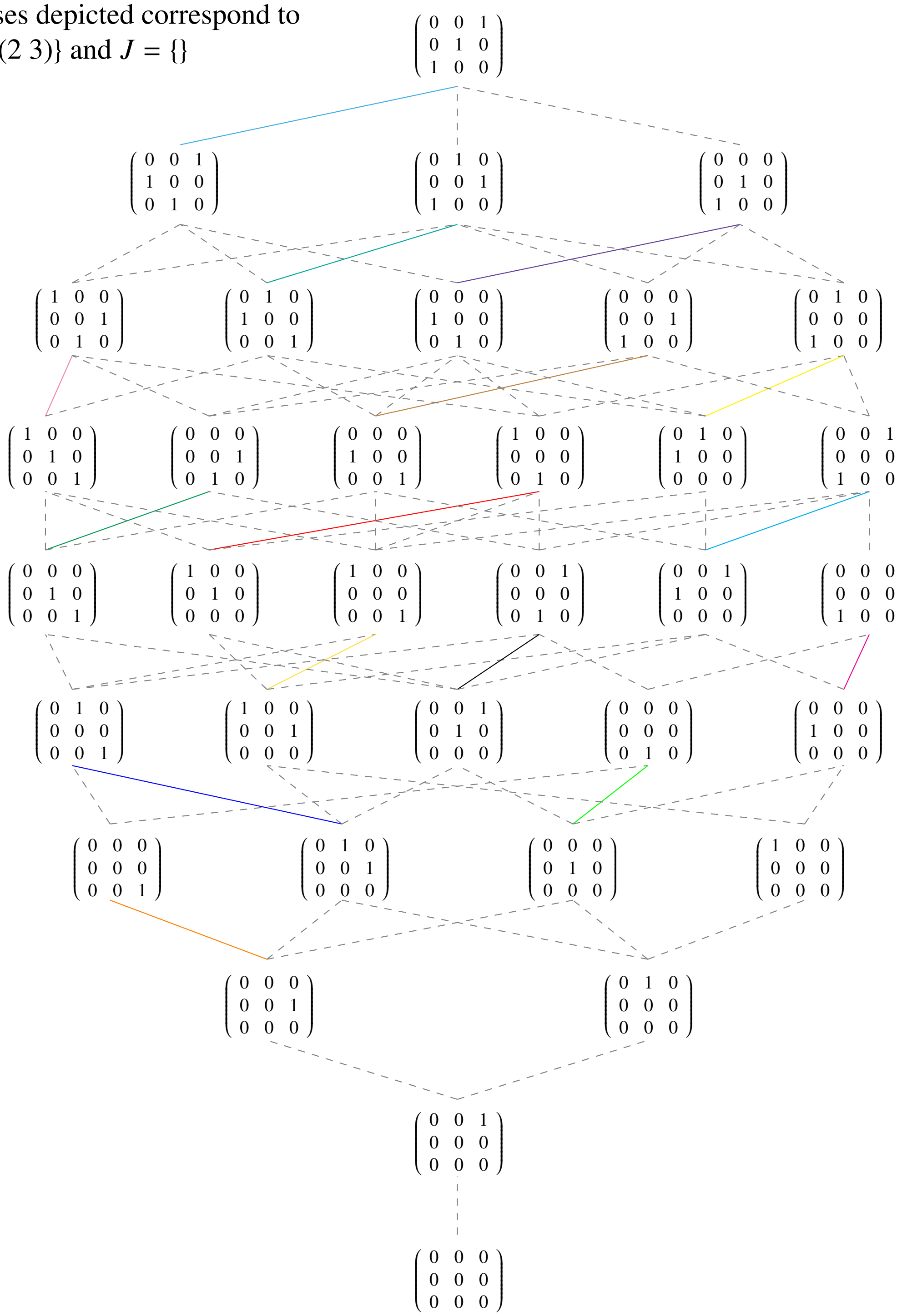

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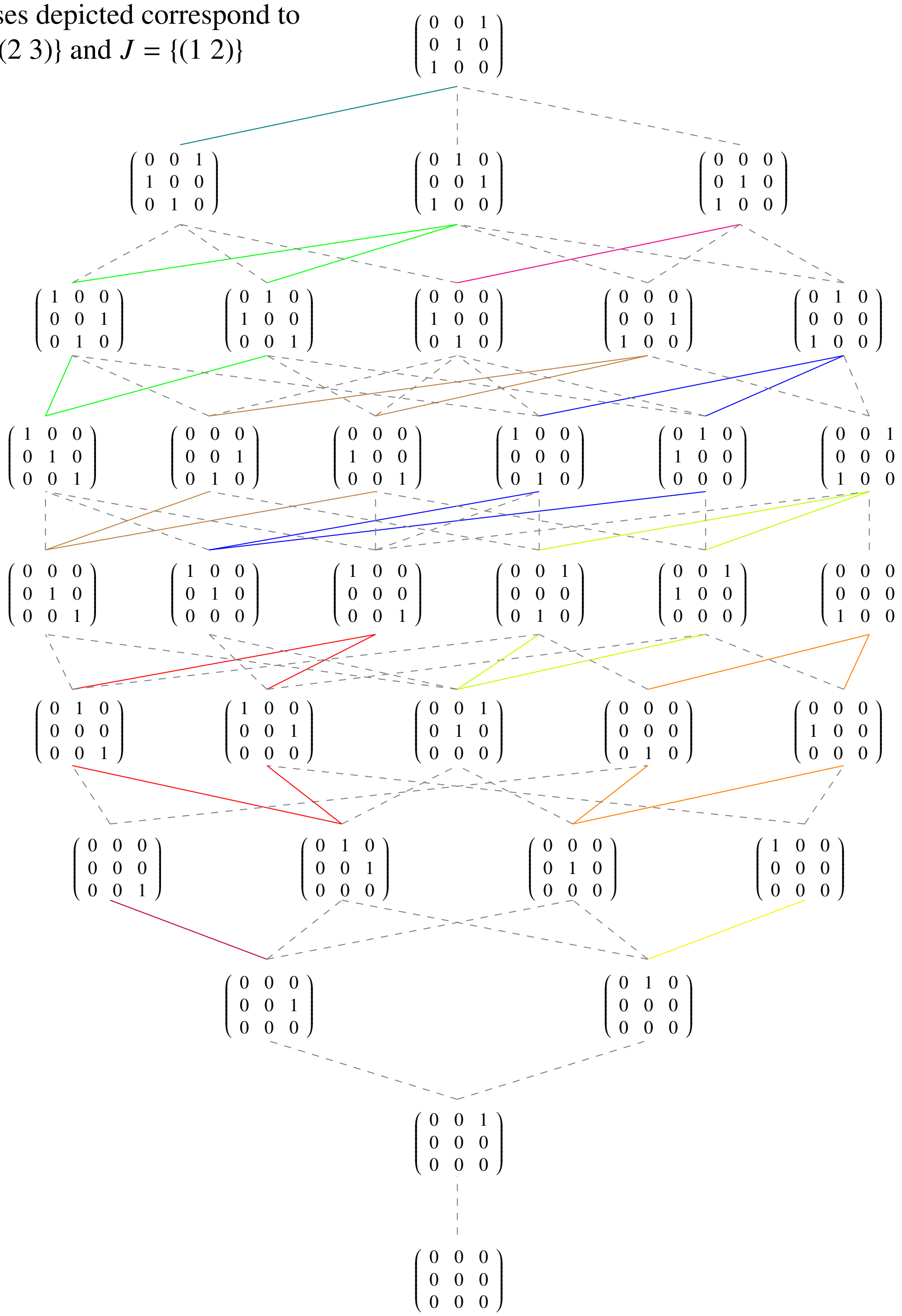

Classes depicted correspond to
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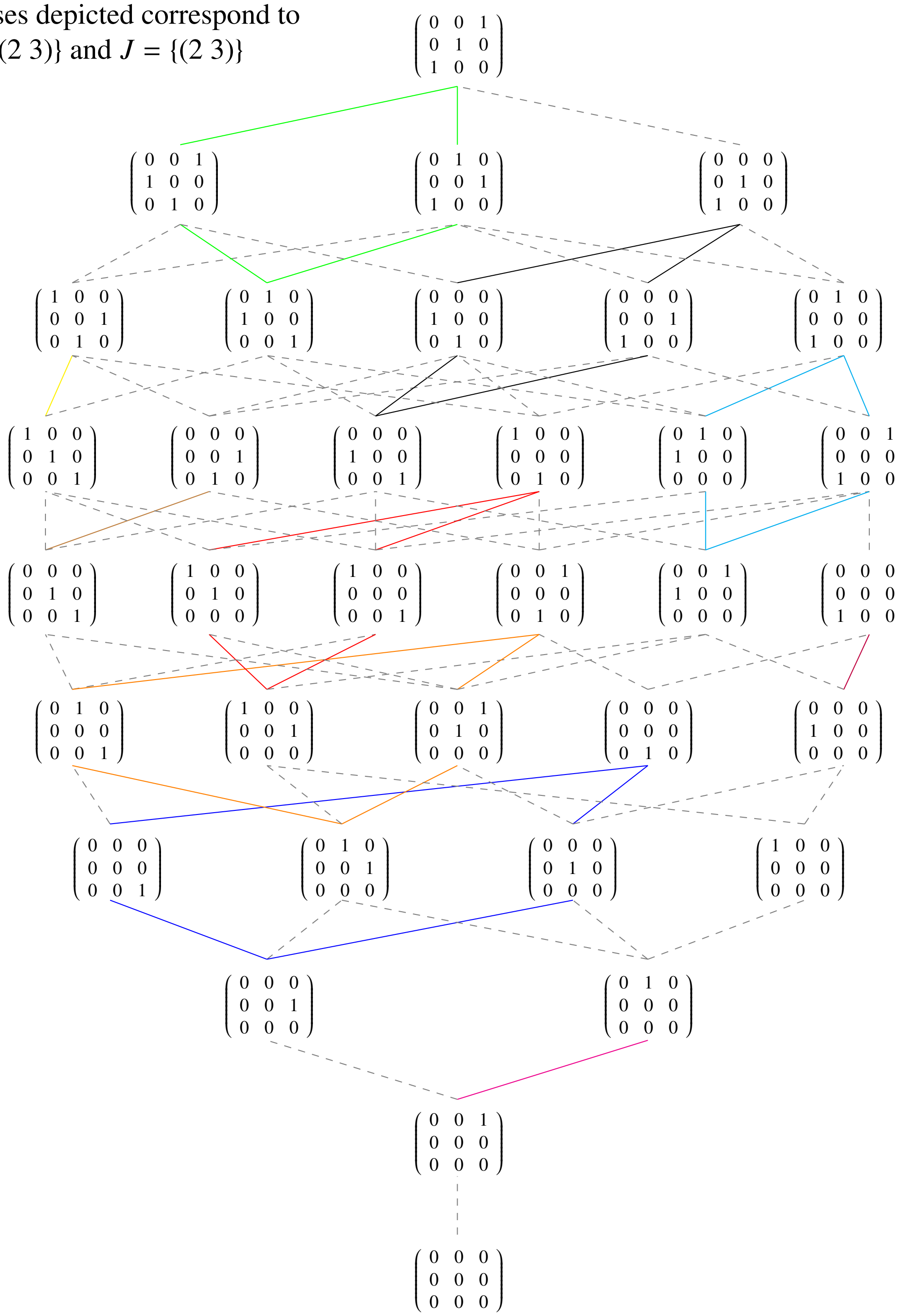
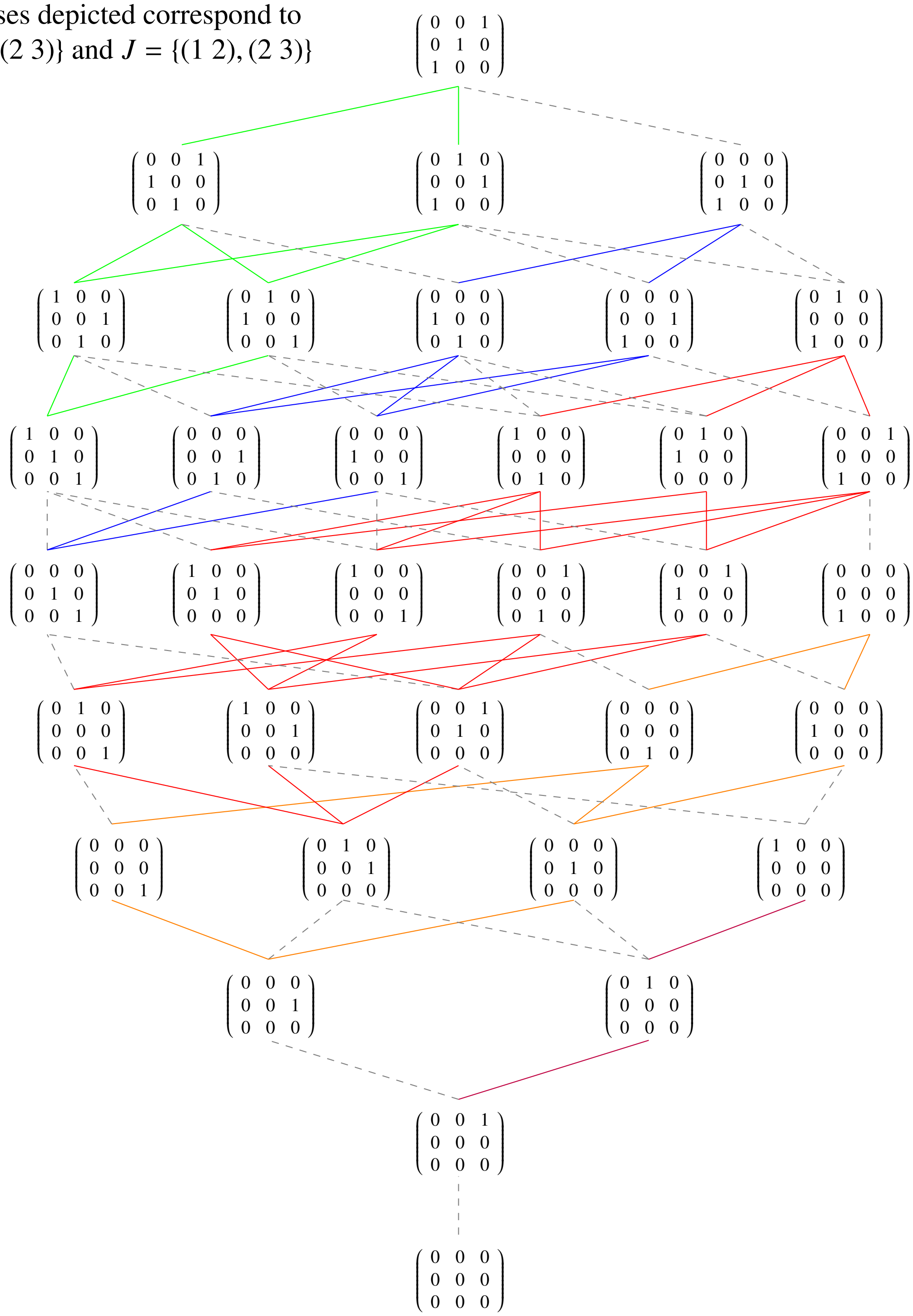
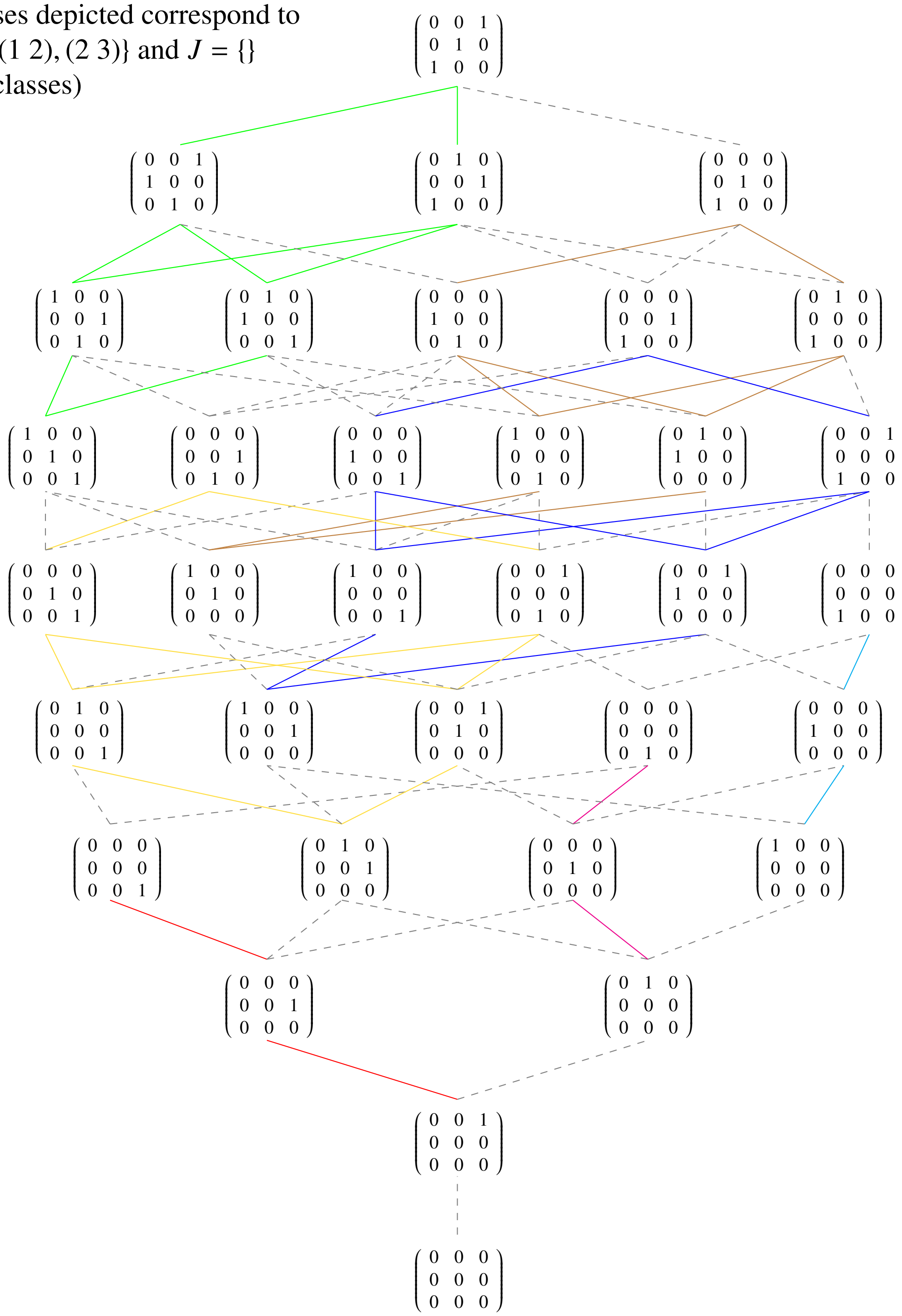


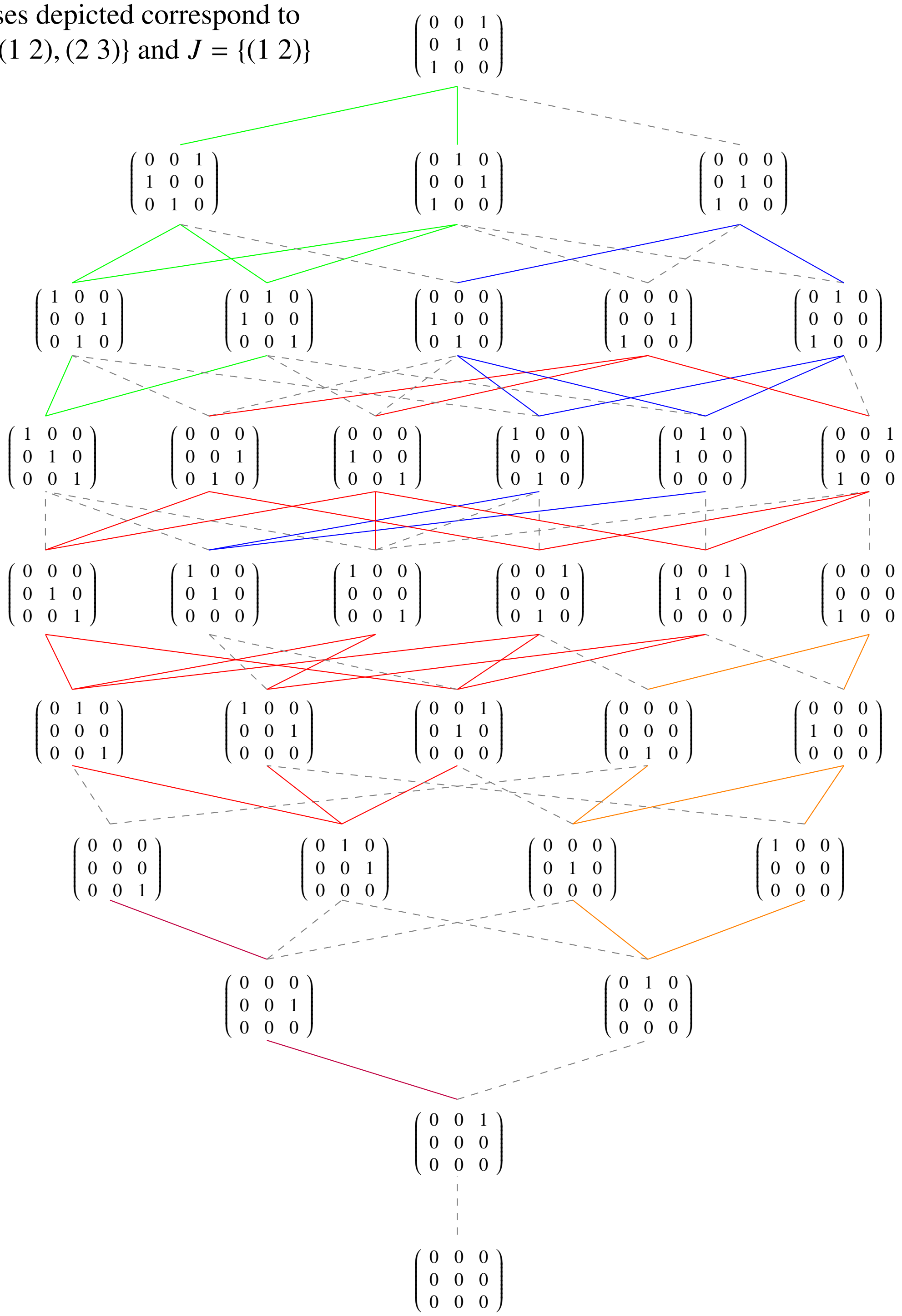
Classes depicted correspond to
$I = \{(2\ 3)\}$ and $J = \{(1\ 2), (2\ 3)\}$



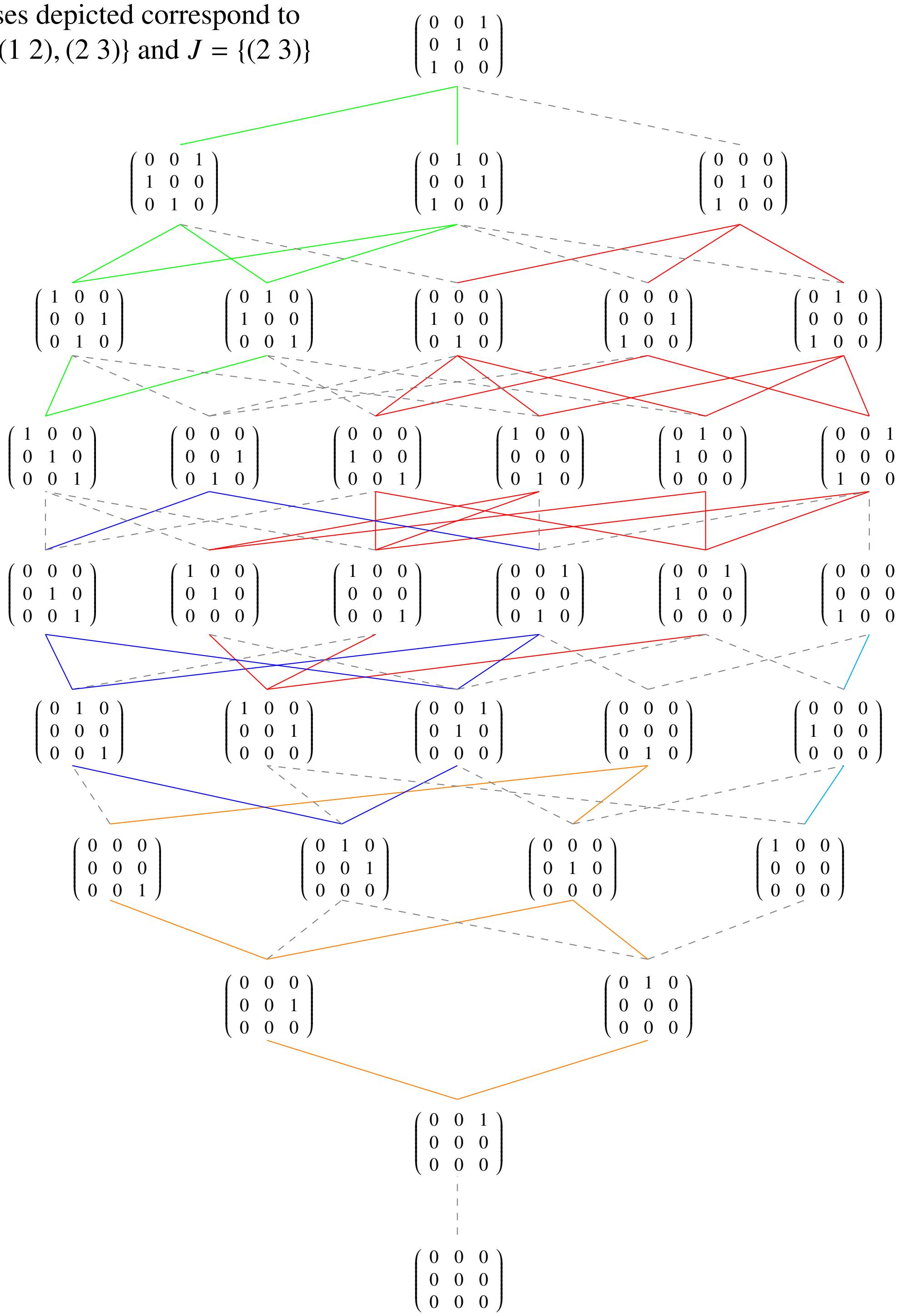
Classes depicted correspond to
$I = \{(1\ 2), (2\ 3)\}$ and $J = \{\}$
($\mathscr{L}$-classes)



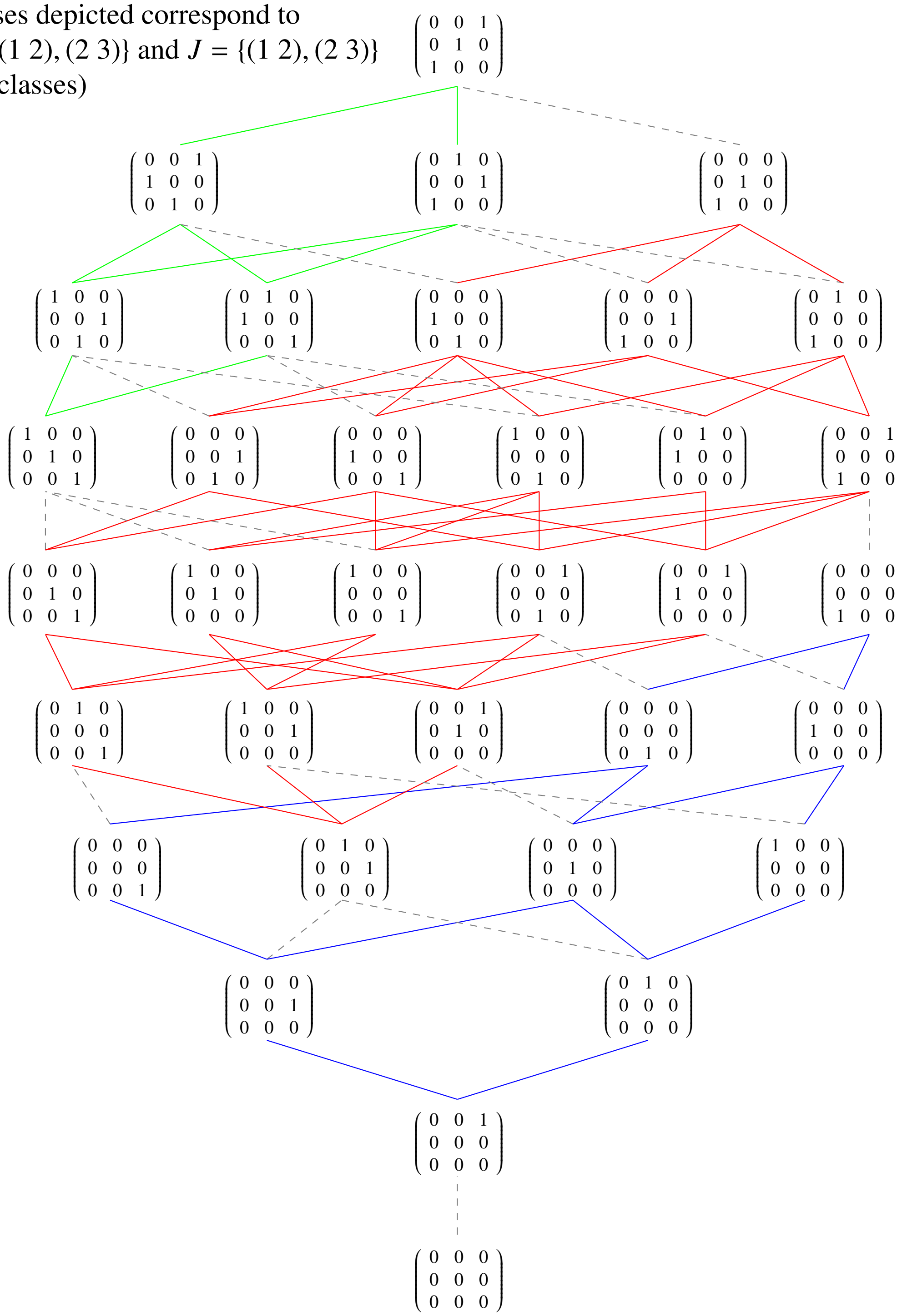
Classes depicted correspond to
$I = \{(1\ 2), (2\ 3)\}$ and $J = \{(1\ 2)\}$



Classes depicted correspond to
$I = \{(1\ 2), (2\ 3)\}$ and $J = \{(2\ 3)\}$



Classes depicted correspond to
$I = \{(1\ 2), (2\ 3)\}$ and $J = \{(1\ 2), (2\ 3)\}$
($\mathscr{J}$-classes)



Curriculum Vitae

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