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Quantization of two types of Multisymplectic manifolds

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by

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Abstract

This thesis is concerned with quantization of two types of multisymplectic manifolds that have multisymplectic forms coming from a Kähler form. In chapter 2 we investigate how Berezin-Toeplitz quantization can be used to quantize them and we study their properties in the semiclassical limit.

In the last chapter of this work, we obtain two additional results. The first concerns the deformation quantization of the $(2n-1)$-plectic structure that we examine in chapter 2, we make the first step toward the definition of a star product on the Nambu-Poisson algebra $(C^\infty(M), \{., \ldots, .\})$. The second result concerns the algebraic properties of the generalized commutator.

Keywords: Differential Geometry, Quantization, Nambu-Poisson structures, Berezin-Toeplitz Quantization, n-plectic Quantization, Star products, Deformation Quantization, Generalized Commutator, n-plectic Manifolds, Hamiltonian mechanics, Nambu mechanics, Geometric Quantization, Kähler Quantization, Projective embedding.

To my parents

Elahe and Siamak

and my sister

Sareh.

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List of symbols and abbreviations

This list is organized by the order of appearance of the symbol or abbreviation in the text of the thesis.

(M,ω) denotes a symplectic manifold

$C^{\infty}(M)$ denotes the smooth functions on the manifold M

$\{\cdot,\cdot\}$ denotes a Poisson bracket

$[\cdot,\cdot]$ denotes the commutator of operators

$\mathcal{H}$ denotes a Hilbert space

$End(\mathcal{H})$ denotes the algebra of endomorphisms on the Hilbert space $\mathcal{H}$

$const(x)$ denotes a constant that depends on x

$\bigotimes^m$ denotes the mth tensor power

Ω always denotes an m-plectic form

C^i denotes the i times differentiable functions

$\mathbb{CP}^q$ denotes the q dimensional complex projective space

$T_f^{(k)}$ denotes the Toeplitz operator of level k for the function f

TM denotes the tangent bundle of the manifold M

T^*M denotes the cotangent bundle of the manifold M

df denotes the differential of the function f, it is a 1-form

ω_{can} denotes the canonical symplectic 2-form that can be defined on $T*M$

∇V denotes the gradient of the function V

$\dot{f}$ denotes the time derivative of the function f

$\frac{\partial f}{\partial x}$ denotes the partial derivative of the function f with respect to x

X_f denotes the Hamiltonian vector field for the function f

$\iota(X_H)\omega$ denotes the insertion of the vector field X_H into the 2-form ω

$\mathcal{L}_X$ denotes the Lie derivative alond the vector field X

$\Gamma_\infty(TM)$ denotes the smooth sections of TM, which are smooth vector fields on M

$\Box$ is used to indicate the end of a proof

$Aut(\mathcal{H})$ denotes the automorpisms of the Hilbert space $\mathcal{H}$

$Y_{\hat{F}}$ denotes the Schrödinger vector field associated to the operator $\hat{F}$

$\{\cdot,\cdot\}_\Omega$ denotes the Poisson bracket defined on any complex Hilbert space $\mathcal{H}$

$\mathcal{PH}$ denotes the projectivization of the Hilber space $\mathcal{H}$

$|\psi|$ denotes the absolute value of the wave function ψ

$L^2(\mathbb{R}^n)$ denotes the square integrable functions defined on $\mathbb{R}^n$

S^1 denotes the unit circle in $\mathbb{C}$

σ_p denotes the principal symbol of a Toeplitz operator

Π denotes a projection operator

$\mathcal{I}$ denotes the identity operator

$c_1(L)$ denotes the first Chern class of the line bundle L

L^m denotes the mth tensor power of the line bundle L

$L^{\bigotimes k}$ denotes the kth tensor power of the line bundle L

$H^0(M, L^k)$ denotes the space of holomorphic sections of the line bundle L^k

$|f|_\infty$ denotes the sup norm of the function f

$||A||$ denotes the operator norm of the operator O

Q_f^k denotes the geometric quantization operator of level k

inf is a shorthand for the word infimum

sup is a shorthand for the word supremum

$Alt\cdot$ denotes the alternating sum of an expression

$\epsilon(\sigma)$ denotes the sign (+ for even and - for odd) of the permuation σ

$\bigoplus$ denotes a direct sum

$\bigotimes$ denotes a tensor product

$sign(i, j, k, l)$ stands for the sign of a particular permutation of 4 indices

$\star$ denotes a formal star product

$\star\{\cdot, \cdots, \cdot\}$ denotes the generalized star product

$[\cdot,\cdot]_Q$ denotes the quantum Lie bracket that comes from the star product $\star$

$K_N(f,g)$ denotes a constant that may depend on the functions f and g or the index N but does not depend on the index m

A^* denotes the adjoint of the operator A

dim is a shorthand for the word dimension

Contents

Introduction

A lot of rich mathematics (in differential geometry, algebra, complex analysis, topology) comes from physics or is motivated by physics, as a result a lot of the language is borrowed from physics as well. The set of results referred to as"quantization" encompasses a number of powerful techniques in complex geometry and representation theory and goes back to the 1950-60s when well known representation theorists, including Kostant and Kirillov became interested in problems related to the correspondence between classical mechanics and quantum mechanics. Kirillov became interested in quantization when he observed that the coadjoint orbits of a Lie group G carry a natural symplectic structure which is invariant under the action of G. In this context the axioms of quantization (section 1.3) say that one should look for the irredulcile unitary representations of the Lie algebra $\mathfrak{g}$. This thesis deals with asymptotic questions in analysis on line bundles on compact Kähler manifolds that arise in certain problems motivated by physics.

Hamiltonian mechanics is a formulation of classical mechanics that allows for the phase space to be any symplectic manifold, rather than just $\mathbb{R}^{2n}$. In 1973 Yoichiro Nambu [N] introduced a generalization of Hamiltonian mechanics which later became known as Nambu mechanics. Nambu mechanics is based on a generalization of the Poisson bracket of Hamiltonian mechanics to a bracket involving n functions, for $n \geq 3$. In Hamiltonian mechanics, the state of a physical system is given by a point in a symplectic manifold and observables are represented by functions on this phase space. In the Hilbert space formulations of quantum mechanics states are given by points in a Hilbert space and observables are represented by endomorphisms on the Hilbert space (this is summarized in the table below). Quantization is a mapping between the algebra of observables on the classical side and the algebra of observables on the quantum side that satisfies certain properties, we will be interested in applications of a particular realization of this mapping known as Berezin-Toeplitz quantization.

	Hamiltonian mechanics	Quantum mechanics
Phase space	(M, ω)	$\mathcal{H}$
Observables	$C^\infty(M)$	$End(\mathcal{H})$
Assoc algebra	$\cdot$ (commutative)	$\circ$ (noncommutative)
Lie algebra	$\{\cdot, \cdot\}$	$[\cdot, \cdot]$

While Berezin-Toeplitz quantization is interesting to study by itself, it has also turned out to have applications in several areas of mathematics. Over the years Berezin-Toeplitz quantization was found to have applications to deformation quantization (see e.g. [**?**], [KS]), to study of the Hitchin connection and TQFT (work of J. Andersen, see in particular [A1], [A2]), L. Polterovich's work on rigidity of Poisson brackets [P], and work of Y. Rubinstein and S. Zelditch [RZ] on homogeneous complex Monge-Ampère equation, in connection to geodesics on the space of Kähler metrics. T. Foth (T. Barron) and A. Uribe applied Berezin-Toeplitz quantization to give another proof of Donaldson's"scalar curvature is a moment map" statement [FU].

Both geometric quantization and Kähler/Berezin-Toeplitz quantization are operator quantizations in the sense of section 1.3, they associate a Hilbert space $\mathcal{H}$ and operators on it to a symplectic manifold (M, ω) and its algebra of functions

$C^\infty(M)$. In physics' terminology this is a way to pass from classical Hamiltonian mechanics to a quantum system. Let $C^\infty(M)$ denote the space of complex-valued smooth functions on M. Quantization is a linear map $C^\infty(M) \to \{\text{operators on } \mathcal{H}\}$, $f \mapsto \hat{f}$, satisfying a version of Dirac's quantization conditions:

$$1 \mapsto const(\hbar)I,$$

$$\{f, g\} \mapsto const(\hbar)[\hat{f}, \hat{g}].$$

Berezin-Toeplitz quantization can be understood as a modified[1] version of geometric quantization. The groundwork for Berezin-Toeplitz quantization was laid in [Ber], [BG]. Theorem 1.5.3(i) below shows that the $\{.,.\} \rightsquigarrow [.,.]$ quantization condition is satisfied in the semiclassical limit $\hbar = \frac{1}{k} \to 0$, which is essentially the best one can get, due to no-go theorems [GM],[Go].

We will be interested in quantization in a setting where the algebraic structure on $C^\infty(M)$ is given by an m-ary bracket $\{.,\ldots,.\} : \otimes^m C^\infty(M) \to C^\infty(M)$. Quantization in this context is the same as in the symplectic case, where we have a bracket of just two functions except that now we are interested in a correspondence $\{.,\ldots,.\} \to [.,\ldots,.]$, between an m-ary bracket and a generalization of the commutator. The symplectic (operator) quantization axioms will be discussed in section 1.3. Their generalization to the m-ary case will be discussed in 2.2. In particular we will be interested in two situations where the m-ary bracket comes from an $(m-1)$-plectic form Ω defined on M (i.e. a closed non-degenerate m-form), for $m \geq 1$. The

[1] The Hilbert space of Berezin-Toeplitz quantization is exactly the Hilbert space of geomertic quantization, the Kähler structure provides a natural polarization known as the Kähler polarization. The quantization maps of geometric quantization and Berezin-Toeplitz quantization are different. The two quantization maps are however related by the Tuynman relation [Tu], which implies that the these operators become equal up to multiplication by i for sufficiently large k.

case $m = 1$ is when Ω is symplectic. Let (M, ω) be a compact integral[2] Kähler manifold of complex dimension n. In both of the cases that we will be looking into, the m-plectic form Ω on (M, ω) is constructed from a Kähler form (or forms):

(I) $m = 2n - 1$, $\Omega = \frac{\omega^n}{n!}$

(II) M is, moreover, hyperkähler, $m = 3$,

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$$

where $\omega_1, \omega_2, \omega_3$ are the three Kähler forms on M given by the hyperkähler structure. It is well-known (and easy to prove) that a volume form on an oriented N-dimensional manifold is an $(N-1)$-plectic form, and that the 4-form above is a 3-plectic form on a hyperkähler manifold. See, for example, [CIL], [R1].

It is intuitively clear that in these two cases the classical multisymplectic system is essentially built from Hamiltonian system(s) and it should be possible to quantize (M, Ω) using the (Berezin-Toeplitz) quantization of (M, ω). In both cases there are natural multisymplectic analogues of the Poisson bracket and the commutator: an almost Poisson bracket $\{., ..., .\}$ and the generalized commutator $[., ..., .]$. Our discussion mainly revolves around the $\{., ..., .\} \rightsquigarrow [., ..., .]$ quantization condition.

We note that while, for simplicity, the exposition throughout the thesis is for C^∞ symbols, - all our results hold, in fact, for C^4 symbols. To modify the proofs in order to get the same statements for C^4 symbols, the estimates from [BMS] should be replaced by estimates from [BMMP] - see subsection 1.5. Results from [BMMP] allow to tackle the case of C^2 and C^3 symbols as well, but we do not include the corresponding version of our results (the asymptotics will differ from the C^∞ case).

There are physical systems whose behaviour is encoded by an *m-plectic* form on M. Specific examples from physics, with $m \geq 2$, are discussed in [N], [CT],

[2] This terminology is explained in section 1.4.3.

[BHR]. See also discussion and references in [CIL]. Multisymplectic geometry has been thoroughly studied by mathematicians. See, in particular, [M], [CIL], [MSw], [BCI], [T], [BHR], [BR], [R1]. There has been extensive discussion of *quantization* of n-plectic manifolds in physics literature, and substantial amount of work has been done by mathematicians too. See, for example, [N], [T], [CT], [DFST], [CZ1], [CZ2], [DSZ], [SS], [R2], [V]. Work of C. Rogers [R2] addresses quantization of 2-plectic manifolds. It seems that the appropriate quantum-mechanical setting there involves a category, instead of a vector space, and intuitively this makes sense because an (integral) 2-plectic form corresponds to a gerbe and sections of a gerbe form a category, not a vector space.

There have been attempts, informally speaking,“to embed a multisymplectic physical system into Hamiltonian system” [BF], [MSu], [DSZ]. As far as we know, there is no known canonical way of doing this.

DeBellis, Sämann and Szabo [DSZ] used Berezin-Toeplitz quantization for multisymplectic spheres via embedding them in a certain explicit way into complex projective spaces $\mathbb{CP}^q$ and using Berezin-Toeplitz quantization on $\mathbb{CP}^q$. This is somewhat related to our results in Section 2.3, only for $M = S^2$ (because among spheres only S^2 admits a Kähler form).

Thesis Organization

Chapter 1: This chapter is meant to provide the context and much of the background for the investigations in chapters 2 and 3. We begin in section 1.1 by explaining the role of symplectic geometry in Hamiltonian mechanics. In section 1.2 we introduce quantum mechanics and its Hilbert space (mathematical) formulations.

We also discuss the geometric formulation of quantum mechanics in this section. In section 1.3 we discuss the meaning of quantization and introduce the quantization axioms. In section 1.4 we introduce the theory of Toeplitz operators. Finally in section 1.5 we describe the Berezin-Toeplitz quantization mapping and the main theorems that relate to it.

Chapter 2: The main result of section 2.3 is Theorem 2.3.4. It is an analogue, for brackets of order $2n$, of well-known Theorem 1.5.3(i), and of its C^l analogue ($l \in \mathbb{N}$) from [BMMP].

In section 2.4 we work on a hyperkähler manifold M. For a smooth function f on M we have three Berezin-Toeplitz operators $T^{(k)}_{f;1}$, $T^{(k)}_{f;2}$, $T^{(k)}_{f;3}$, and to four smooth functions f, g, h, t on M we associate three brackets of order 4: $\{f, g, h, t\}_r$, $r = 1, 2, 3$. In subsection 2.4.1 we show that the direct sum of generalized commutators is asymptotic to

$$T^{(k)}_{\{f,g,h,t\}_1;1} \oplus T^{(k)}_{\{f,g,h,t\}_2;2} \oplus T^{(k)}_{\{f,g,h,t\}_3;3}$$

(Theorem 2.4.3). In subsection 2.4.2 we show that the attempt to formulate everything on *one* vector space (not three), by taking direct sums, goes through all the way in the case when M is the 4-torus with three linear complex structures, where we get a straightforward analogue of Theorem 1.5.3(i) - see Example 2.4.6 (2.9). In subsection 2.4.3 we take the tensor product of the three operators, instead. The tensor product of generalized commutators is asymptotic to

$$T^{(k)}_{\{f,g,h,t\}_1;1} \otimes T^{(k)}_{\{f,g,h,t\}_2;2} \otimes T^{(k)}_{\{f,g,h,t\}_3;3}$$

(Proposition 2.4.13). Asymptotic properties of commutators and generalized commutators of operators $\mathbb{T}^{(k)}_f = T^{(k)}_{f;1} \otimes T^{(k)}_{f;2} \otimes T^{(k)}_{f;3}$ are captured in Prop. 2.4.10 and Theorem 2.4.14.

Chapter 3: The Berezin-Toeplitz operator quantization leads also to a deformation quantization (Berezin-Toeplitz star product)[**?**]. In section 3.2 we make use of that result to make a first step toward defining a star product (deformation quantization) for Nambu-Poisson algebra of a Kähler manifold with the Nambu-Poisson bracket defined using the volume form. In section 3.3 we use a result that falls out of the geometrical formulation of quantum mechanics to look further into the algebraic properties of the generalized commutator.

To summarize, the main results of the thesis are: Theorem 2.3.4, Theorem 2.4.3, Proposition 2.4.4, Theorem 2.4.5, Proposition 2.4.10, Proposition 2.4.13, Theorem 2.4.14, Proposition 3.3.1, Proposition 3.4.4.

Chapter 1

From classical to quantum mechanics: 1-plectic quantization via Toeplitz operators

1.1 Hamiltonian mechanics

In this section we will describe how the formalism of symplectic geometry may be used to extend classical mechanics from the phase space $\mathbb{R}^n \times \mathbb{R}^n$, which carries a natural symplectic structure, to more general symplectic manifolds. This more general formulation of classical mechanics is known as Hamiltonian mechanics. Historically Hamilton's formulation of classical mechanics has been important because it had a strong influence on the early mathematical formulations for quantum mechanics, this will be the subject of the next section. In this section we will be concerned only with those aspects of symplectic geometry that we will need later on or that elucidate the role of symplectic manifolds in Hamiltonian mechanics.

1.1.1 Remarks on Hamiltonian mechanics

In classical mechanics to describe a system consisting of a single point mass moving in n dimensions it is sufficient to use the flat manifold $\mathbb{R}^n \times \mathbb{R}^n \cong T^*\mathbb{R}^n$, referred to as its phase space.

Remark 1.1.1. The **configuration space** of a (dynamical[1]) system is a manifold with one coordinate q_j for each of the possible positional degrees of freedom[2]. This is in contrast to the **phase space** (sometimes referred to as the state space) of a classical mechanical system, which must have one coordinate q_j for each positional degree of freedom in the system and another coordinate $p_j = \dot{q}_j$ for it's time derivative (in classical mechanics the positions and their time derivatives constitute all of the degrees of freedom). Consequently, the phase space is always even dimensional (when the configuration space is N, the phase space is T^*N). Furthermore, knowing the state of a system (the point in phase space that it occupies) is necesssary and sufficient to determine its past and future time evolution. This is a consequence of the second order nature of Newton's (second) law of motion, and to first approximation (that is, when effects related to quantum mechanics and relativity can be

[1]The concept of a dynamical system is a mathematical formalisation which unifies any fixed rules that describe how a point moves with time inside it's ambient space. Many different types of rules have emerged, with different choices for the ambient space as well as different choices for how time is measured. Time can be measured by integers, by real or complex numbers or can be a more general algebraic object, losing the memory of its physical origin, and the ambient space may be simply a set, without the need of a smooth space-time structure defined on it. For the rest of this thesis, whenever we mention the word "system", it will refer to a dynamical system in the domain of classical mechanics, although sometimes we will be discussing the quantum mechanical description of the system.

[2]A system is said to have m degrees of freedom if in order to know its condition for certain one must specify the value of exactly m variables. For a point mass in $\mathbb{R}^n$ this is 2n.

ignored), this leads to an accurate description of time evolution for real physical systems [Ar].

Remark 1.1.2. Let N be a smooth manifold, then T^*N carries a natural 2-form. First of all, one may canonically define a 1-form on T^*N. Let $v \in T_{(p,q)}(T^*N)$ and define a 1-form α by the formula

$$\alpha_{(p,q)}(v) = q(d\pi(v)),$$

where $(p,q) \in T^*N$, so that $p \in N$ and $q : T_pN \to \mathbb{R}$ is a linear mapping. Now the natural 2-form is defined to be

$$\omega_{can} := d\alpha.$$

Let $(q_1, \ldots, q_n)$ be local coordinates for a neighbourhood $U \subset M$. These can be extended to local coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ on T^*U by the condition:

$$\forall \xi \in {T^*}_x U, p_i(\xi) = \xi(\frac{\partial}{\partial q_i}).$$

In these coordinates we have

$$\alpha = \sum_{i=1}^{n} p_i \, dq_i$$

so that

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

Every state has an energy associated with it, these energies fit together to give a smooth[3] function $H \in C^\infty(T^*\mathbb{R}^n)$ called the Hamiltonian function.

For our point mass moving in $\mathbb{R}^n$ its state may be specified by giving its position and its momentum[4], that is, by giving a point in the phase space $M = \mathbb{R}^n \times \mathbb{R}^n$. If

[3] Although non-smooth Hamiltonians have been considered as well [Ma].

[4] The momentum is defined by $p = m\dot{q}$ so that $\frac{p^2}{2m} = \frac{mv^2}{2}$ is the usual kenetic energy term.

the point mass is under the influence of a potential V its Hamiltonian is given by

$$H(p,q) = \frac{P^2}{2m} + V(q),$$

in Newton's formulation it's time evolution is given by Newton's (second) Law of Motion:

$$\frac{dp}{dt} := \dot{p} = -\nabla V.$$

By combining the definition of momentum with Newton's second law of motion Hamilton was able to treat the q and the p variables symmetrically and to rewrite Newton's second order relation for time evolution in configuration space as a system of two first order equations that give the time evolution of a point (q,p) in phase space:

$$\dot{q} = \frac{p}{m} \qquad \dot{p} = -\nabla V.$$

These can be expressed in terms of the Hamiltonian as

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q},$$

these are Hamilton's equations. In Hamiltonian mechanics it is Hamilton's equations that are taken as the equations of motion rather than Newton's law. That is, it was Hamilton's postulate that the time evolution of a classical system with phase space (T^*N, ω_{can}) for which the energy is given by the Hamiltonian $H \in C^\infty(M)$ is determined by the vector field:

$$X_H = (\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}),$$

X_H is called the Hamiltonian vector field and its flow is called the Hamiltonian flow. The canonical 2-form ω_{can} can be used to encode the equations of motion[5] in a coordinate-independant way, X_H may be defined in terms of the symplectic structure by the formula

[5]The relationship between the Hamiltonian function H and the Hamiltonian vector field X_H.

$$dH(.) = \omega(X_H, .) = \iota(X_H)\omega. \tag{1.1}$$

Remark 1.1.3. It should be noted that Hamilton's equations form a system of two first order equations. The dynamics depend only on the gradient of the energy function H, the coordinate free expression makes this evident.

To read this definition recall that being a 2-form on the phase space $M = T^*N$, ω_{can} may be regarded as a bundle map, $\omega_{can} : TM \to T^*M$ by insertion into the first argument. This means that at each point (q, p) of T^*N, ω_{can} restricts to a maping of linear spaces, $\omega_{can}|_{(q,p)} : T_{(q,p)}M \to T^*_{(q,p)}M$. In order for the vector field X_H to be well defined at each point (q, p) of T^*N, these linear mappings must be injective[6], in that case ω_{can} is called **non-degenerate**. In fact, ω_{can} is non-degenerate, which can easily be checked using the expression in local coordinates. Because $\omega_{can} = d\alpha$ is globally exact, it is closed, $d\omega_{can} = dd\alpha = 0$. Consider Cartan's formula for the Lie derivative

$$\mathcal{L}_X = \iota(X) \circ d + d \circ \iota(X).$$

Along the vector field X_H, the Lie derivative of ω_{can} is

$$\mathcal{L}_{X_H}\omega_{can} = ddH = 0,$$

where we have applied equation 1.1. We have demonstrated that for any function $f \in C^\infty(M)$ it's Hamiltonian vector field X_f generates a flow which leaves ω_{can} invariant.

This brings us to the concept of a symplectic manifold, this notion generalizes the structure on T^*N that we have been looking at. The need to consider these

[6]Because the domain and target are finite dimentional, the mappings will be surjective as well and so they will be isomorphisms.

more general phase spaces in physics arises for example when some classical systems are subject to constraint.

1.1.2 Symplectic geometry

Definition 1.1.4. A **symplectic manifold** is a pair (M, ω), where M is a smooth manifold and ω is a 2-form on M which is both closed ($d\omega = 0$) and non-degenerate.

Example 1.1.5. T^*N with the natural 2-form ω_{can} is a symplectic manifold. ω_{can} is exact ($\omega_{can} = d\alpha$), so $d^2 = 0$, it is closed. The easiest way to see that ω_{can} is non-degenerate is to use the formula in local coordinates given in remark 1.1.2. This is in some sense[7] the canonical example of a symplectic manifold.

Because the symplectic form ω is non-degenerate every symplectic manifold M will have even dimension $dimM = 2n$. ω^n is a nowhere vanishing 2n-form and so it defines a (canonical) orientation on M. $\frac{\omega^n}{n!}$ is called **Liouville's volume form**.

With the discussion of (T^*N, ω_{can}) in mind, it is easy to see that given any symplectic manifold M and a real valued function $f \in C^\infty(M)$ one may define a dynamical system in a parallel fashion. We now proceed to make this explicit. After an introduction to the notion of quantization in section 1.3, the discussion in section 1.5 will be about the quantization of this dynamical system. The discussion in the last two chapters will revolve around the quantization of one of its generalizations.

Definition 1.1.6. (Symplectomorphism)

An isomorphism in the category of symplectic manifolds is called a **symplectomorphism**. A symplectomorphism is a diffeomorphism $\phi : (M, \omega) \to (N, \omega')$

[7]See for example the Weinstein Tubular Neighborhood Theorem [CD]page 17.

between two symplectic manifolds where the pull back of ω' under ϕ is ω,

$$\phi^*\omega' = \omega.$$

The infinitesimal version (which consists of the collection of all tangent vectors to the trajectories of ϕ) leads to the concept of symplectic vector fields. A vector field $X \in \Gamma^\infty(TM)$ is a **symplectic vector field** if its flow generates a symplectomorphism, $\mathcal{L}_X\omega = 0$. The Hamiltonian vector fields are symplectic. Examples of symplectomorphisms include the flow generated by a Hamiltonian function and the map of cotangent bundles $T^*M \to T^*N$ induced by a diffeomorphism $M \to N$.

Definition 1.1.7. (Hamiltonian dynamical system)

The triple (M, ω, f) of a smooth manifold M, a symplectic 2-form ω defined on M and a real valued function $f \in C^\infty(M; \mathbb{R})$ define a dynamical system whose time evolution is given by the flow of the vector field X_f defined by

$$df(.) = \omega(X_f, .) = \iota(X_f)\omega.$$

X_f is referred to as the **Hamiltonian vector field** associated to f, its flow is referred to as the **Hamiltonian flow**. The manifold M is called the phase space of the system and the set $C^\infty(M; \mathbb{R})$ is the **Poisson algebra of observables** (see theorem 1.1.13). The Poisson bracket is given by $\{f, g\} := \omega(X_f, X_g)$. The time evolution of an observable $g \in C^\infty(M; \mathbb{R})$ is by definition given by

$$\frac{dg}{dt} = \mathcal{L}_{X_f}g = dg(X_f) = \omega(X_f, X_g) = \{f, g\}. \tag{1.2}$$

For any function $f \in C^\infty(M; \mathbb{R})$, it's Hamiltonian vector field X_f generates a symplectomorphism, $\mathcal{L}_{X_f}\omega = 0$, this can be established in exactly the same way as it was above for T^*N.

Remark 1.1.8. In the general context of symplectic geometry the vector field X_f is referred to as the **symplectic gradient** of f. While flow according to the gradient (df) of f results in the fastest possible change in f, the symplectic gradient is tangent to the level sets of f, so flow along X_f preserves f,

$$df(X_f) = \omega(X_f, X_f) = 0$$

when f is the Hamiltonian of a system, this statement expresses the fact that energy is conserved.

Example 1.1.9. Some examples of observables include: For a system of point masses, the position and momentum coordinate functions, the total angular momentum L (about the center of mass) and the energy function H.

Example 1.1.10. (Geodesic flow on T^*N)[Bo]
A particle moving freely on a Riemannian manifold (N,g) has phase space $M = T^*N$ as above, the Hamiltonian has only the kinetic energy term and is given by

$$H = \frac{1}{2}|\xi|_g{}^2.$$

One can compute that

$$dH = \frac{1}{2}\frac{\partial g^{ij}}{\partial x^k}\xi_i\xi_j dx_k + g^{ij}\xi_i d\xi_j,$$

so that

$$X_H = g^{ij}\xi_i\frac{\partial}{\partial x^j} - \frac{1}{2}\frac{\partial g^{ij}}{\partial x^k}\xi_i\xi_j\frac{\partial}{\partial \xi_k}.$$

The resulting tragectory when projected onto N satisfies the defining equation for a geodesic (a path which is a local extremum of the distance functional [Mil]Section 10) on N,

$$\ddot{x}^j = \frac{d}{dt}(g^{jk}\xi_k) = -g^{jl}\frac{\partial g^{li}}{\partial x^l}\dot{x}^l\dot{x}^i + \frac{1}{2}g^{jk}\frac{\partial g_{li}}{\partial x^k}\dot{x}^j\dot{x}^i = -\Gamma^j{}_{li}\dot{x}^l\dot{x}^i,$$

where Γ is the Christoffel symbol for g. That is, a free particle will travel along a trajectory that is from its point of view a straight line.

Remark 1.1.11. In the Darboux coordinates 1.1.15 where $\omega = \sum_i dq^i \wedge dp^i$, the bracket $\{.,.\}$ is given by the formula

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}.$$

Lemma 1.1.12. *For $f, g \in C^\infty(M)$,*

$$[X_f, X_g] = -X_{\{f,g\}}.$$

Proof. From Cartan's formula

$$\mathcal{L}_{X_f}(\iota(X_g)\omega) = \iota(\mathcal{L}_{X_f}X_g)\omega + \iota(X_g)\mathcal{L}_{X_f}\omega,$$

and $[X_f, X_g] = \mathcal{L}_{X_f}X_g$, so we have

$$\iota([X_f, X_g])\omega = \mathcal{L}_{X_f}(\iota(X_g)\omega) - \iota(X_g)\mathcal{L}_{X_f}\omega.$$

Now $\iota(X_g)\omega = dg$ by definition, and because the Hamiltonian flow is a symplectomorphism, $\mathcal{L}_{X_f}\omega = 0$, we have

$$\iota([X_f, X_g])\omega = dg(X_f) = \{g, f\}.$$

□

Theorem 1.1.13. *Let (M, ω) be a symplectic manifold, the bracket $\{.,.\}$ gives the associative algebra of observables $(C^\infty(M), \cdot)$ the structure of a Poisson algebra.*

Proof. Bilinearity and skew-symmetry of the bracket are evident from the definition. The Leibniz rule follows from the formula

$$\{f, g\} = \omega(X_f, X_g) = df(X_g).$$

To prove the Jacobi identiy Cartan's formula for the exterior derivative is useful.

$$d\omega(X,Y,Z) = \frac{1}{3}(\mathcal{L}_X\omega(Y,Z) - \mathcal{L}_Y\omega(X,Z) + \mathcal{L}_Z\omega(X,Y) \\ - \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X)),$$

where $X, Y, Z \in Vect(M)$. After substituting $X = X_f$, $Y = X_g$, $Z = X_h$ and using the fact that $d\omega = 0$, this becomes

$$\begin{aligned} d\omega(X_f, X_g, X_h) &= \frac{-1}{3}(\omega([X_f, X_g], X_h) + \omega([X_h, X_f,]X_g) + \omega([X_g, X_h], X_f) \\ &= \frac{1}{3}(\omega(X_h, X_{\{f,g\}}) + \omega(X_f, X_{\{g,h\}}) + \omega(X_g, X_{\{h,f\}})) \\ &= \frac{1}{3}(\{h, \{f,g\}\} + \{f, \{g,h\}\} + \{g, \{h,f\}\}) \\ &= 0 \end{aligned}$$

where we have used the lemma and the fact that each of the vector fields X_f, X_g, X_h generates a 1 parameter family of symplectomorphisms.

□

Remark 1.1.14. The coordinate positions and momenta form a **complete set of observables**, this means that any observable that Poisson commutes with all of them is constant (since all of its directional derivatives at all points of the phase space will be constant). As far as we are aware this term was coined by M. Blau[B].

Theorem 1.1.15. *(Darboux) All symplectic manifolds are locally diffeomorphic.*

Let (M, ω) be a symplectic manifold, then for each $m \in M$ there exists a neibourhood $U_m \in M$ with a coordinate chart $x = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n) : U_m \to \mathbb{R}^n \times \mathbb{R}^n$ such that $\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$. Every symplectic 2-form can be written in this local canonical form.

1.2 Quantum mechanics

Quantum mechanics describes the behaviour of systems that are very very small, on the scale of an atom. These systems are so easily perturbed that the effect of a measurement on the state of the system is not negligible. Classical mechanics does not take into account the disturbance caused by interacting with the system during the measurement process (this is manifested by the commutative nature of the pointwise product of functions as opposed to the noncommutative nature of the corresponding operators), so it is only applicable to systems where the disturbance is negligible. In the classical (Hamiltonian) theory the phase space (state space) is (T^*N, ω_{can}) or more generally any symplectic manifold (M, ω) and the set of observable quantities is represented by functions $f \in C^\infty(M; \mathbb{R})$. When the system is in the state $(q, p) \in M$, the result of a measurement of the observable $f \in C^\infty(M; \mathbb{R})$ will always be $f(q, p)$. With the pointwise multiplication of functions $C^\infty(M)$ is a commutative associative algebra. The bracket $\{f, g\} := \omega(X_f, X_g)$ satisfies the Leibnitz rule (with respect to the pointwise multiplication) as well as the Jacobi identity, making it a Poisson bracket and this makes $C^\infty(M)$ into a Poisson algebra. To every observable f there is associated a Hamiltonian vector field X_f, the dynamics are determined by a special observable H (which represents the energy) called the Hamiltonian and the dynamical equation is, expressed in terms of this natural algebraic structure by,

$$\frac{df}{dt} = \{H, f\}.$$

If the system is in the state (q, p), the $\mathbf{t}$ in $\frac{df}{dt}$ represents a parameterization of the unique trajectory of X_H that passes through (q, p), so from this point of view the time evolution is formulated infinitesimally as a derivation. For this reason it should be emphasized that in order to be able to write this dynamical equation, the Jacobi identity (which can be understood as expressing a derivation property of the bracket $\{.,.\}$ with respect to itself) has played a pivotal role.

Mathematical formulations of quantum mechanics, on the other hand, have traditionally been based on a Hilbert space approach. In the Hilbert space approach the states of a system are represented by rays[8] in a Hilbert space $\mathcal{H}$, the observables are given by self adjoint operators on $\mathcal{H}$. The theory of measurement in quantum mechanics is markedly different than it is in classical mechanics. For an observable $\hat{O} \in End(\mathcal{H})$ which has no degenerate eigenvalues[9], the usual description of measurement theory (based on the Copenhagen interpretation of quantum mechanics) goes as follows. Before we make a measurement of the observable $\hat{O}$, the state ψ may be any superposition of eigenstates of $\hat{O}$ (for this we appeal to the appropriate spectral theorem). The result of a measurement of $\hat{O}$ will be an eigenvalue of $\hat{O}$ and the state ψ, will immediately collapse to the corresponding eigenstate.

One example of a mathematical formulation that falls into the Hilbert space paradigm is Matrix mechanics. The matrix formulation of quantum mechanics was the earliest formulation of quantum mechanics. Matrix mechanics was formulated by Werner Heisenberg in June of 1925[SBB], six months before Erwin Schrödinger published his formulation based on wavefunctions. In matrix mechanics observables are represented by $N \times N$ matrices. For a system with N basis[10] states each observ-

[8]By a ray we mean an equivalence class of vectors in $\mathcal{H}$ under the equivalence relation $v \sim w$ iff $v = \lambda w$ for some nonzero complex number λ. The reason for this is that the information that can be extracted from the theory (for example, the dynamics or the results of measurements) does not depend on the complex number λ (known as a phase factor) multiplying v.

[9]It is a postulate of quantum mechanics that those endomorphisms that correspond to observables should be self-adjoint so that their eigenvalues will be real.

[10]The number of basis states will normally be infinite, and often it will be uncountable as it is for example for position and momentum.

able is represented by a $N \times N$ Hermitian matrix. States are represented by $N \times 1$ column matrices. Suppose an observable A is represented by the martix $\hat{A}$, then the expectation value (the value that the results will tend toward after repeated measurements) for a measurement of A in the state ψ is given by the inner product

$$\langle \psi, \hat{A}\psi \rangle. \tag{1.3}$$

The dynamics are determined by a prefered self-adjoint matrix $\hat{H}$ called the Hamiltonian, which represents energy. The dynamical equation for any observable $\hat{A}$ is

$$\frac{d\hat{A}}{dt} = -\frac{i}{\hbar}[\hat{A}(t), \hat{H}].$$

In section 1.3, which is about (operator[11]) quantization we will discuss another Hilbert space formulation of quantum mechanics (wave mechanics).

1.2.1 Geometric formulation of quantum mechanics

It is possible to formulate quantum mechanics geometrically[AS] as a Hamiltonian system. In order to reach a geometric formulation the first major observation that is needed is the that every complex Hilbert space is naturally a Kähler manifold. We will now follow the chain of ideas that leads to a full fledged geometric formulation as far as we will need to for the discussions in chapters 2 and 3.

The complex Hilbert space of quantum mechanics may be viewed as a real vector space with a complex structure $J \in Aut(\mathcal{H})$, $J^2 = -I$. The complex structure J

[11] Quantization proceedures were initially proposed in the Hilbert space framework. This is in contrast to quantization procedures that are based on deformations of algebras, these were introduced in polished form for the first time in [BFFLS].

represents multiplication by i. Since we are viewing $\mathcal{H}$ as a real vector space it is natural to decompose the Hermitian inner product into it's real and imaginary parts,

$$\langle \psi, \phi \rangle = \frac{1}{2\hbar} G(\psi, \phi) + \frac{i}{2\hbar} \Omega(\psi, \phi).$$

From the properties of the Hermitian inner product one can show that Ω is a symplectic form and G is a positive definite real inner product, both Ω and G are non-degenerate, this means that G is a metric. From the identity $\langle \psi, J\phi \rangle = i\langle \psi, \phi \rangle$, we can conclude that the metric G and the symplectic form Ω are related as

$$G(\psi, \phi) = \Omega(\psi, J\phi).$$

This makes the triple $(\mathcal{H}, G, \Omega)$, into a (linear) Kähler space, this holds true for any complex Hilbert space. To make $\mathcal{H}$ into a Kähler manifold we can extend Ω naturally to a closed, non-degenerate 2-form (also called Ω) on $\mathcal{H}$ by using the canonical identification of the tangent space at any point of $\mathcal{H}$ with $\mathcal{H}$ itself. In this way we may view any complex Hilbert space as a cotangent bundle[12] (the simplest classical phase space). In terms of the complex structure J, the Schrödinger equation may be written

$$\dot{\psi} = -\frac{1}{\hbar} J \hat{H} \psi.$$

This motivates the assignment of a vector field

$$Y_{\hat{F}}(\psi) := -\frac{1}{\hbar} J \hat{F} \psi,$$

[12] Although admittedly not in a natural way as we are considering $\mathbb{R}^{2n}$ and there is no natural half dimensional subspace.

to each observable $\hat{F}$. $Y_{\hat{F}}$ is called the **Schrödinger vector field** associated to $\hat{F}$, it is defined so that the time evolution is given by the Schrödinger vector field associated to the Hamiltonian $\hat{H}$. By definition $Y_{\hat{F}}$ is the generator of a one-parameter family of unitary mappings of $\mathcal{H}$, this flow is the representation of the evolution of the system under study. Since the flow of $Y_{\hat{F}}$ preserves the Hermitian inner product, it must be simultaneously an isometry with respect to the metric G and a symplectomorphism with respect to the symplectic form Ω. This brings us to the second major point in the development of a geometric formulation, the Schrödinger vector field $Y_{\hat{F}}$ is Hamiltonian for the expectation function $F := \langle\psi, \hat{F}\psi\rangle : \mathcal{H} \to \mathbb{R}$, of $\hat{F}$. The calculation that demonstrates this fact is illustrative of the role of the (multiplication by i) complex structure J, which acts on the rescaled gradient vector $-\frac{1}{\hbar}\hat{F}\psi$, to turn it into the symplectic gradient vector $-\frac{1}{\hbar}J\hat{F}\psi$. The calculation also relies on the fact that $\hat{F}$ is self-adjoint. Let ξ be a tangent vector based at ψ, then

$$\begin{aligned} dF(\xi) = \frac{d}{dt}\langle\psi + t\xi\hat{F}(\psi + t\xi)\rangle|_{t=0} = \langle\psi, \hat{F}\xi\rangle + \langle\xi, \hat{F}\psi\rangle \\ = \frac{1}{\hbar}G(\hat{F}\xi, \psi, \xi) = \Omega(Y_{\hat{F}}, \xi) = i_{Y_{\hat{F}}}(\Omega)(\xi). \end{aligned}$$

This description of the the Schrödinger vector field (as the Hamiltonian vector field for F) already allows us to view each quantum mechanical system as a Hamiltonian system. The Hamiltonian systems that are obtained in this way have very different Hamiltonians than the usual classical systems, there will be terms in the Hamiltonian that represent quantum mechanical effects. In addition the subset of $C^\infty(\mathcal{H})$ that is comprised of expectation value functions of quantum observables is much smaller than $C^\infty(\mathcal{H})$ itself, where as in a classical system any $f \in C^\infty(M)$ may be regarded as an observavble. Let $\hat{F}$ and $\hat{K}$ be quantum observables and let F and K be their respective expectation values, there is a nice relation between Poisson bracket $\{F, K\}_\Omega$ and the expectation value of the bracket $[\hat{F}, \hat{K}]$ of operators that will be useful for the discussion in chapter 3,

$$\{F, K\}_\Omega = \Omega(X_F, X_K) = \frac{1}{i\hbar}\langle[\hat{F}, \hat{K}]\rangle.$$

Notice that the Poisson bracket that appears in the relation is the quantum Poisson bracket, so that this is not Dirac's principle[13]. So far we have been been working with the full Hilbert space $\mathcal{H}$ with the assumptions that the quantum states are normalized and that the states ψ and $e^{i\theta}\psi$ (we say that these states are related by a phase factor) are equivalent. These assumptions come out of the measurement theory, we take normalized states in so that the transition probabilities (the probability that the quantum system will end up in a particular state after a measurement) will be between 0 and 1. We regard two states that are related by a phase factor as the same because the expectation value of states that are related by a phase factor are the same. The upshot of all this is that it reveals that the apparent linearity[14] of quantum mechanics is an artifact of the Hilbert space formulations. The true physical phase space of quantum mechanics is the projectivization $\mathcal{P} := \mathcal{PH}$ of the Hilbert space we have been discussing so far. This observation, when carried to it's conclusion[AS], leads to a formulation of quantum mechanics that makes no reference to the Hilbert space or it's linear structure. We will not be needing the full formalism in this thesis, so we will conclude this section with this comment.

1.3 Quantization

The aim of quantization is to find a recipe that allows one to cook up a quantum description of a classical system with only the classical information and some as-

[13] If this comment does not make sense now, it will make sense after reading the next section.

[14] In physics the usual situation is that a linear theory is the linearization of a more general theory, this is the relation of special relativty to general relativity for example.

sumptions about how the quantum system should be related to the classical one as the ingredients, there are a variety of ways of doing this. Initially quantization was developed with reference to the Hilbert space formulations of quantum mechanics. Within this framework quantization can be described as a mapping

$$\text{classical observables} \xrightarrow{\mathcal{Q}} \text{quantum observables.}$$

This mapping is required to satisfy some properties (known as the quantization axioms), the exact list of axioms depends on the classical and quantum viewpoints that are being considered. This approach to quantization is known as operator quantization and it is the quantization mapping (along with its target Hilbert space) that we hope will provide the extra information and allow us to pass from a the classical description to a more detailed quantum description. We will present the axioms with a view toward the Kähler quantization of Kähler manifolds, in this context we will adopt the convention $C^\infty(M) := C^\infty(M;\mathbb{C})$, the Poisson structure is extended to the complex valued functions by complex linearity. We will take $C^\infty(M;\mathbb{C})$ as the algebra of observables, rather than just $C^\infty(M;\mathbb{R})$, however only the real valued functions have any physical meaning. Let $(M, \omega, \mathcal{H})$ be a Hamiltonian system, then the axioms can be expressed as follows:

1. $\mathcal{Q}$ should be $\mathbb{C}$-linear.

$$\mathcal{Q}(cf + g) = c\mathcal{Q}(f) + \mathcal{Q}(g)$$

2. The constant function 1 should be sent to identity operator I.

3. Real valued functions should be sent to Hermitian operators so that their eigenvalues will be real.

$$\mathcal{Q}(f) = \mathcal{Q}^*(f)$$

4. We will require that,

$$\frac{i}{\hbar}[\mathcal{Q}(f), \mathcal{Q}(g)] = \mathcal{Q}(\{f, g\}) + O(\hbar).$$

5. The final axiom can be stated in terms of the concept of a complete set of observables. If $\{f_1, \ldots, f_k\}$ is a complete set of observables, than we require that $\{\mathcal{Q}(f_1), \ldots, \mathcal{Q}(f_k)\}$ is a complete set of operators.

The first two conditions say that $\mathcal{Q}$ is a representation of (unital) associative algebras. The third condition is required to have a consistent interpretation for the measurement theory, since the eigenvalues of the operator $\mathcal{Q}(f)$ are the possible results of measurements. The fourth condition is a variation of what is often called Dirac's quantum condition, although Niels Bohr was the first to suggest it. Dirac referred to $\frac{i}{\hbar}[\hat{A}, \hat{B}] = \frac{i}{\hbar}(\hat{A}\hat{B} - \hat{B}\hat{A})$ as the "quantum Poisson bracket[15] ". According to Dirac [., .] should be analogous to the classical Poisson bracket, he showed that the classical Poisson bracket and the "quantum Poisson bracket" share the same algebraic structure, the JI. This 4th axiom is related to the concept of the classical limit. The big O notation means that in the limit $\hbar \to 0$ there should be equality. We will see shortly that the 5th axiom can be understood as an irreducibility condition, for now let us just say that a set of operators is called a **complete set of operators** if any operator that commutes with all of them is a multiple of the identity operator.

As another example of the Hilbert space approach to quantum theory take Schrödingers wave mechanics, it will be instructive to examine the quantization axioms in this context. We will revisit the example of a single particle moving in $\mathbb{R}^n$. Where in the classical theory the state of the particle was given by a point $(q, p) \in T^*\mathbb{R}^n$, the state of the quantum particle is given by a complex valued function of position $\psi(q)$, such that $|\psi(q)|^2$ is the probability density for the particles

[15]Although it is really only a Lie bracket because it does not satisfy the Leibniz rule.

position. The function $\psi(q)$ is called the wavefunction for the system. Where in the classical theory observables were functions $f \in C^\infty(T^*\mathbb{R}^n)$, in the quantum theory an observable is expressed as an operator on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. For an observable $\hat{f} \in L^2(\mathbb{R}^n)$ its possible values (the values that can be observed upon a measurement) are its eigenvalues, and the probability that that a particular eigenvalue will be observed depends on the eigenbasis decomposition of the state being observed. The expected value of the observable $\hat{f}$ for the state ψ is given by

$$\langle \psi, \hat{f}\psi \rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\hat{f}\psi(x,t)dx.$$

There are two ways of writing the time evolution. Recall that the equation for the Hamiltonian flow can be written in terms of the Lie bracket as

$$\frac{dq_i}{dt} = \{H, q_i\}$$
$$\frac{dp_i}{dt} = \{H, p_i\}$$

Then the time evolution of an arbitrary observable can be written

$$\frac{df}{dt} = \{H, f\}.$$

It is a basic principle of quantum mechanics (axiom 4) that the role played by the Poisson bracket of functions $\{.,.\}$ in the classical theory should be played by $\frac{i}{\hbar}$ times the commutator of operators in the quantum theory. This suggests the equation

$$\frac{d\hat{f}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{f}]$$

for the time evolution of a quantum observable, which has as its formal solution

$$\hat{A}(t) = \mathrm{e}^{\frac{-i}{\hbar}t\hat{H}}\hat{f}\mathrm{e}^{\frac{i}{\hbar}t\hat{H}}.$$

We are regarding the state of the system as stationary and the observable as dependant on time, this viewpoint would be natural for an observer moving along with the particle, this was Heisenberg's approach. Alternatively we could proceed according to Schrödinger's approach and put the time dependance into the state ψ while letting the observable remain constant. Then the equation for time evolution is Schrödinger's equation

$$\frac{\partial \psi}{\partial t} = \frac{-i}{\hbar} H\psi.$$

There are many other formulations of quantum mechanics, for a selection of nine, which includes the two we have mentioned so far see [SBB]. If we try to carry out quantization in the simplest case $N = \mathbb{R}^n$, $M = T^*N$ imposing a strict version of the 4th axiom, namely the equality $\frac{i}{\hbar}[\mathcal{Q}(f), \mathcal{Q}(g)] = \mathcal{Q}(\{f, g\})$, then for the coordinate functions q^i and p_j we will have

$$[\mathcal{Q}(q^i), \mathcal{Q}(q^j)] = \{q^i, q^j\} = 0$$

$$[\mathcal{Q}(p_i), \mathcal{Q}(p_j)] = \{p_i, p_j\} = 0$$

$$[\mathcal{Q}(q^i), \mathcal{Q}(p_j)] = i\hbar\{q^i, p_j\} = i\hbar\delta^i_j.$$

This is known as the Heisenberg algebra. Applying Schur's lemma we see that the 5th axiom asks us to find an irreducible representation of the Heisenberg algebra. This special case is the reason that the 5th axiom is regarded as an irreducibility condition. The Stone-von Neumann theorem says that any irreducible representation of the Heisenberg algebra[16] is unitarily equivalent to the one[17] on $L^2(N) = L^2(\mathbb{R}^n)$

[16] That exponentiates to a representation of the Heisenberg group.

[17] Notice that the correct Hilbert space is not functions on $L^2(T^*N)$ but rather $L^2(N)$, functions on a half dimensional subspace of the phase space T^*N.

given by

$$\mathcal{Q}(q^i)\psi(x) = x^i\psi(x),$$
$$\mathcal{Q}(p_j)\psi(x) = -i\hbar\frac{\partial\psi}{\partial x^j}(x)$$

This result was actually one of the motivating factors for the formulation of these quantization axioms. It turns out that up to a choice of ordering for the $\mathcal{Q}(q_j)'s$ and $\mathcal{Q}(p_j)'s$ (because on the quantum side the algebra of observables is not commutative) any observables quadratic in the $q_j's$ and $p_j's$ can be quantized while conforming to the strict version of the 4th axiom $\frac{i}{\hbar}[\mathcal{Q}(f), \mathcal{Q}(g)] = \mathcal{Q}(\{f, g\})$. If we try to quantize the cubic or higher observables by extending the above quantization, we will see that we run into problems with the 4th axiom. This can be understood as a shadow of the Groenewald-van Hove theorem. The Groenewald-van Hove theorem is a no-go theorem that essentially says that quantization of observables which are cubic or higer in the $q_j's$ and $p_j's$ is not possible if we insist on the strict form of the 4th axiom and on the 5th axiom[18].

Despite the efforts of many researchers working in mathematics and physics, the body of work collectively known as quantization has not yet been able to lead us to a theory of quantum mechanics that we can write down from scratch, avoiding the process of quantization altogether, and maybe it never will. It is clear that the classical theory does not contain all of the information, if it did there would be no need for a quantum theory or for quantization. When we try to get a quantum theory from a classical one we must somehow add some information, in operator quantization this happens when you introduce the quantization mapping (which is required to satisfy certain axioms) and its target Hilbert space, this gives an image of the classical theory inside what people understand as the mathematical

[18] For a review of the Groenewald-van Hove theorem see[Go]

framework for quantum mechanics. The reason why the Hilbert space along with its endomorphisms and their noncommutative structure are understood as "quantum" is that this setting works well in some particular situations that are of interest, like the free particle moving in $\mathbb{R}^n$. The free particle in $\mathbb{R}^n$ is quantized by Diracs' axioms of canonical quantization[19]. As we have seen, even in this (the simplest possible scenario) only observables that are quadratic (or of lower order) in the positions and momenta can be quantized. When one tries to quantized higher order observables in keeping with Diracs axioms, there will be algebraic inconsistencies [MJG]. Canonical quantization for the flat space $\mathbb{R}^n$ is really the best that we are able to do in that case. This means that whenever we write down quantization axioms for more general spaces (for example geometric quantization tries to quantize symplectic manifolds, Kähler quantization quantizes Kähler manifolds) if we try to do it using the strict version of axiom 4 and ask for a condition that reduces to irreducibility for a representation of the Heisenberg algebra we cannot quantize a general observable that is cubic or higher in the positions and momenta, if we would like to be able to quantize more observables we must somehow modify the axioms so that we may avoid the no go theorems. In deformation quantization and in Kähler quantization this is acomplished by adding the $O(\hbar)$ term to axiom 4. Therefore we see that there are challenges in building bridges between these two areas (classical mechanics and quantum mechanics), in a satisfying way, and people keep looking for new ideas.

1.4 Toeplitz operators

The Toeplitz operators that appear in chapter 2 are the "global" Toeplitz operators of Boutet de Monvel and Guillemin [BG]. The "global" Toeplitz operators can

[19] That is in keeping with axioms 1,2,3,4 (the strict version without the $O(\hbar)$ term).

be defined over ∂W, where ∂W is the smooth boundary of a strictly pseudoconvex domain W in a complex n-dimensional manifold, $n \geq 1$. We will introduce the "global" Toeplitz operators in 1.4.2. However Toeplitz operators were first defined on the circle S^1, so we will give this definition first.

1.4.1 Toeplitz operators on the circle

Toeplitz operators on the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ are defined as follows[BH]. Let μ denote the standard Lebesgue measure and let $e_n = e_n(z) = z^n$, $z \in S^1$, $n \in \mathbb{Z}$. The e_n are bounded measureable functions and they form a basis for $L^2 = L^2(S^1, \mu)$. The Hardy space H^2 is defined as the space of all functions in L^2 that are analytic, a function $f \in L^2$ is called analytic if $\int_{S^1} f\bar{e_n} d\mu = 0$ for all $n < 0$. Denote by $\Pi : L^2 \to H^2$ the orthogonal projector. For any bounded measurable function f there is a corresponding Toeplitz operator $T_f : H^2 \to H^2$ defined by $T_f := \Pi \circ M_f$, where M_f is the operator that multiplies by the function f. The Toeplitz operator corresponding to the constant function 1 is the identity operator $\mathcal{I}$ and for $f, g \in C^\infty(S^1)$ we have $T_{\alpha f + \beta g} = \alpha T_f + \beta T_g$.

Example 1.4.1.

1. The function $z \in C^\infty(S^1)$ has a Toeplitz operator that acts in the standard basis as multiplication by z.

$$c_1 + c_2 z + c_3 z^2 + \ldots \mapsto c_1 z + c_2 z^2 + c_3 z^3 + \ldots$$

2. The function $\frac{1}{z} \in C^\infty(S^1)$ has a Toeplitz operator that acts in the standard basis as multiplication by $\frac{1}{z}$ followed by a truncation.

$$c_1 + c_2 z + c_3 z^2 + \ldots \mapsto c_2 + c_3 z + c_4 z^2 + \ldots$$

1.4.2 Toeplitz operators on smooth boundaries of strictly pseudoconvex domains

Let W be a strictly pseudoconvex domain in a complex n-dimensional manifold M, assume that the boundary ∂W is smooth and that $\bar{W} = W \cup \partial W$ is compact, $n \geq 1$. Choose a defining function, $r \in C^\infty(M)$, with the following properties: $r|_W < 0$, $r|_{\partial W} = 0$, $dr \neq 0$ near ∂W. Let $j : \partial W \hookrightarrow \bar{W}$ be the inclusion map. The 1-form $\alpha = j^* Im(\bar{\partial} r)$ is a contact form on ∂W, the upshot of this is that the $(2n-1)$-form $\Omega = \alpha \wedge (d\alpha)^{n-1}$ is a volume form for ∂W. Denote by μ the measure associated to the volume form Ω and let $L^2 = L^2(\partial W, \mu)$. Let A(W) denote the space of functions defined on $\bar{W}$ which are continuous on $\bar{W}$, smooth on ∂W, and holomorphic on W. Define the *Hardy space* $H^2 = H^2(\partial W)$ to be the closure in $\{f|_{\partial W} \mid f \in A(W)\}$. Denote by $\Pi : L^2 \to H^2$ the orthogonal projector.

Definition 1.4.2. [Toeplitz operator]

An operator $T : C^\infty(\partial W) \to C^\infty(\partial W)$ is called a **Toeplitz operator of order k** if it is of the form $\Pi Q \Pi$, where Q is a pseudodifferential operator of order k. The *principal symbol* of T is defined by $\sigma(T) := \sigma(Q)|_\Sigma$, where $\sigma(Q)$ is the prinipal symbol of Q and

$$\Sigma = \{(x, \xi) \,|\, x \in \partial W,\ \xi = r\alpha_x,\ r > 0\}$$

is a symplectic submanifold of $T^*\partial W$.

In [BG] it is shown that the principal symbol of Toeplitz operators is well defined and that they form a ring. It is also shown that the principal symbol of Toeplitz operators obeys the same rules as the pricipal symbol of pseudo-differential operators:

$$\sigma(T_1 T_2) = \sigma(T_1)\sigma(T_2),$$

$$\sigma([T_1, T_2]) = i\{\sigma(T_1), \sigma(T_2)\},$$

where the Poisson bracket is the one coming from the symplectic structure of Σ. For a more detailed explaination of this theory in the context that is needed to understand section 2.3.1 of the thesis see [Bo] section 11.

1.4.3 Quantum line bundles

For a symplectic manifold (M, ω) a quantum line bundle is a triple (L, h, ∇), where L is a complex line bundle, h is a Hermitian metric on L, and ∇ is a connection compatible with the metric h and such that the following (pre)quantum condition is satisfied

$$\mathbf{curv}_{L,\nabla}(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = -i\omega(X, Y).$$

The symplectic manifold is called **quantizable** if there exists a quantum line bundle. Line bundles are topologically classified by their first Chern class $c_1(L) \in H^2(M, \mathbb{Z})$. In de Rham cohomology, $c_1(L)$ is represented by the curvature form of any connection on L, although the curvature form will depend on the connection, its cohomology class does not. This means that a necessary condition (it turns out that it is a sufficient condition also) for having a quantizable symplectic manifold is that $(\frac{1}{2\pi})\omega$ represent an integral cohomology class, in this case $\frac{\omega}{2\pi}$ is called **integral**.

When the manifold is Kähler, the quantum line bundle must be holomorphic and the connection must be compatible with the metric as well as the complex structure of the bundle. For compact Kähler manifolds the (pre)quantum condition leads to an embedding into projective space.

Theorem 1.4.3. *Kodaira's embedding theorem*

Let M be a compact complex manifold, M can be embedded into projective space if and only if there exists a positive line bundle L on M.

A line bundle $L \to M$ is called **positive** if$c_1(L)$ can be represented by a real positive closed (1,1) form. Kähler forms are positive, so the quantum condition implies that the quantum line bundle is positive, so that the embedding theorem applies to any quantizable Kähler manifold. The embedding is achieved by using the global holomorphic sections of some tensor power L^m of the quantizing line bundle L, in this case L is called an **ample** line bundle and L^m is called **very ample**. Kodaira's embedding theorem characterizes projective manifolds among compact Kähler manifolds.

1.5 Berezin-Toeplitz operator quantization

In this section we will introduce the Berezin-Toeplitz[20] operator quantization of a Kähler manifold.

Let (M, ω) be a compact Kähler manifold. Let (L, h, ∇) be a quantum line bundle over M and let $(L^{\otimes k}, h^{\otimes k}, \nabla^{\otimes k})$, $k \in \mathbb{N}_0$, be its tensor powers[21]. L is a holomorphic hermitian line bundle such that the curvature of the hermitian connection is $-i\omega$. We introduce a scalar product on $\Gamma_\infty(M, L^{\otimes k})$ (the space of smooth sections of $L^{\otimes k}$) and let $L^2(M, L^{\otimes k})$ be its L^2-completion under this inner product.

$$\langle \phi, \psi \rangle := \int_M h^k(\phi, \psi)\Omega,$$

where $\Omega := \frac{1}{n!}\omega^n$. Let $H^0(M, L^{\otimes k})$ be its closed subspace of holomorphic sections, this is a finite dimensional vector space due to the compactness of M. By Π^k we will

[20]This is also known as the Kähler quantization, we will see that although the Hilbert space for this quantization is the same as in geometric quantization, the choice of polarization and the definition of the mapping $C^\infty(M) \to End(\mathcal{H})$ make use of the Kähler structure.

[21]Sometimes we will simply write h^k for $h^{\otimes k}$ etc..

denote the natural projection

$$\Pi^k : L^2(M, L^k) \to H^0(M, L^k).$$

Smooth symbol

The reference used throughout this subsection is [BMS], where the method is based on the analysis of Toeplitz structures from [BG]. Results mentioned here and more extensive discussion can be found in surveys on Berezin-Toeplitz quantization, - for example in [S2].

The general theory developed by Boutet de Monvel and Guillemin (see 1.4.2), for W being the disk bundle in L^* and pseudo differential operators of order zero, leads to the following definition. For $f \in C^\infty(M)$, the Toeplitz operator (of level k) is defined by

$$T_f{}^{(k)} := \Pi^k \circ (M_f) : H^0(M, L^k) \to H^0(M, L^k).$$

Given a holomorphic section s we multiply it by f to obtain the section fs, which will not be holomorphic in general, to end up with a holomorphic section we keep only the holomorphic part of fs by acting with the natural projection. For a survey on Berezin-Toeplitz quantization see for example, [S2]. For $\alpha, \beta \in \mathbb{C}$ and $f, g \in C^\infty(M)$ we have that

$$T^{(k)}_{\alpha f + \beta g} = \alpha T^{(k)}_f + \beta T^{(k)}_g.$$

In this way we obtain for each $k \in \mathbb{N}_0$ a linear mapping

$$T^k : C^\infty(M) \to End(H^0(M, L^k))$$

$$f \to T_f{}^k$$

In general we have that

$$T_f{}^{(k)}T_g{}^{(k)} = \Pi^k(M_f)\Pi^k(M_g) \neq \Pi^k(M_{fg}) = T_{fg}{}^{(k)}$$

so that $T^{(k)}$ is not an associative algebra homomorpism, nor is it a Lie algebra homomorphism. Furthermore because $\Gamma_{hol}(M, L^k)$ is finite dimensional, $T^{(k)}$ is not even injective, it is however surjective[BMS].

Remark 1.5.1. 1. For $\alpha, \beta \in \mathbb{C}$ and $f, g \in C^\infty(M)$ we have that

$$T^{(k)}_{\alpha f+\beta g} = \alpha T^{(k)}_f + \beta T^{(k)}_g.$$

2. Any constant function c is clearly mapped to cI

3. Toeplitz operators satisfy the equality $T^{(k)*}_f = T^{(k)}_{\bar{f}}$. In particular for real valued functions $T^{(k)*}_f = T^{(k)}_f$.

These are the first three axioms of quantization, that these Toeplitz operators also satisfy the fourth axiom of quantization is the content of theorem 1.5.3 below.

Definition 1.5.2. (Berezin-Toeplitz quantization map)

$$T = \bigoplus_k T^{(k)} : C^\infty(M) \to \prod_k End(H^0(M, L^k))$$
$$f \mapsto \bigoplus_k (T^{(k)}_f)$$

In BT-quantization the index k plays the role of $\frac{1}{\hbar}$. In order to be a quantization of (M, ω), we should be able to approximate $C^\infty(M)$ with $End(H^0(M, L^k))$ by adjusting k, this is the content of the next theorem.

Theorem 1.5.3 ([BMS] Th. 4.1, 4.2, [S2] Th. 3.3)**.** *For $f, g \in C^\infty(M)$, as $k \to \infty$,*

(i)

$$||ik[T^{(k)}_f, T^{(k)}_g] - T^{(k)}_{\{f,g\}}|| = O(\frac{1}{k}),$$

(ii) there is a constant $C = C(f) > 0$ *such that*

$$|f|_\infty - \frac{C}{k} \leq ||T_f^{(k)}|| \leq |f|_\infty.$$

Proposition 1.5.4 ([BMS] p. 289, [S2] Prop. 3.5)**.** *For* $f, g \in C^\infty(M)$

$$\lim_{k\to\infty} ||[T_f^{(k)}, T_g^{(k)}]|| = 0.$$

Remark 1.5.5. Proof of this Proposition actually implies that

$$||[T_f^{(k)}, T_g^{(k)}]|| = O(\frac{1}{k})$$

as $k \to \infty$.

Part (i) of the theorem expresses the fact that although Dirac'c conditon is not satisfied at any fixed level k, asymptotically Dirac's condition is satisfied.

Proposition 1.5.6 ([BMS] p. 291, [S2] Prop. 3.4)**.** *For* $f_1, ..., f_p \in C^\infty(M)$

$$||T_{f_1}^{(k)} ... T_{f_p}^{(k)} - T_{f_1 ... f_p}^{(k)}|| = O(\frac{1}{k})$$

as $k \to \infty$.

At first glance these Berezin-Toeplitz operators do not appear to be related to the operator of geometric quantization (with Kähler polarization). They are however related by the well known Tuynman relation[Tu],

$$Q_f^{(k)} = iT_{f - \frac{1}{2k}\Delta f}^{(k)}. \tag{1.4}$$

It is a consequence of the Tuynman relation that the theorems and propositions of this section hold also for the operator of geometric quantization Q_f^k.

C^l symbol

The reference for theorems analogous to those above in subsection 1.5, with $f \in C^l(M)$, is [BMMP]. In [BMMP] the method is different from [BMS]. It relies on techniques developed in [MM1], [MM2], see also [MM3]. For $l = 4$ statements similar to Theorem 1.5.3, Prop. 1.5.6 follow from Cor. 4.5, Remark 5.7(b), Cor. 4.4 of [BMMP]. The fact that for $f, g \in C^4(M)$ $||[T_f^{(k)}, T_g^{(k)}]|| = O(\frac{1}{k})$ as $k \to \infty$ easily follows too, from Cor. 4.5 and Remark 5.7(b) [BMMP].

This concludes the first chapter of the thesis. In the second chapter we will discuss quantization in a more general context than we have been doing in this first chapter. Our goal will be to prove an analog of theorem 1.5.3 in this more general setting.

Chapter 2

m-plectic quantization via Toeplitz operators

This chapter contains the main results of the thesis. In section 2.3, by an application of Berezin-Toeplitz quantization we suggest a way to quantize a compact quantizable Kähler manifold regarded as a $(2n-1)$-plectic manifold. In section 2.4.3 we make another application of Berezin-Toeplitz quantization, this time to quantize a compact hyperkähler manifold equipped with a natural 4-form, we show that in both of these cases, the quantization has reasonable semiclassical properties.

2.1 Preliminaries

Here we will lay out some notations and definitions, as well as some lemmas concerning them, that we will use in the last two chapters of the thesis. S_n, for a positive integer n, will denote the symmetric group (the group of permutations of $1, \ldots, n$). For $\sigma \in S_n$, $\epsilon(\sigma)$ will denote the sign of the permutation (+1 for an even permutation and -1 for an odd permutation). For a complex vector space V and

$A, B \in End(V)$, $[A, B] = AB - BA$. I will denote the identity operator on V. Our vector spaces will be equipped with an inner product and as a consequence they will carry a norm, then $||A||$ will denote the operator norm of A defined for bounded[1] linear operators by

$$||A|| = inf\{c \geq 0 : ||Av|| \leq c||v|| \text{for all} v \in V\}.$$

A^* will denote the adjoint of the operator A. We will continue to denote the algebra of smooth complex-valued functions on a smooth manifold M (which will be a Kähler manifold in this section) by $C^\infty(M)$, for $f \in C^\infty(M)$ we will write $|f|_\infty = sup_{x \in M}|f(x)|$.

Definition 2.1.1. An $(m+1)$-form Ω on a smooth manifold M is called an **m-plectic form** or a **multisymplectic form** if it is closed ($d\Omega = 0$) and non-degenerate ($v \in T_xM, v \lrcorner \Omega_x = 0 \Rightarrow v = 0$). If Ω is a multisymplectic form on M, (M,Ω) is called a **multisymplectic** or **m-plectic**, manifold.

Example 2.1.2. The canonical example of an m-plectic manifold is $\bigwedge^m T^*M$, where M is any smooth manifold, this generalizes the 1-plectic (symplectic structure on) T^*M, which has already been introduced as the canonical example of a symplectic manifold. There is a canonical m-form α on $\bigwedge^m T^*M$ defined as:

$$\alpha(v_1, \ldots, v_m) = x(d\pi(v_1), \ldots, d\pi(v_m))$$

where each v_i is a tangent vector at a point $x \in \bigwedge^m T^*M$ and $\pi : \bigwedge^m T^*M \to M$ is the natural projection. The $(m+1)$-form

$$\omega = d\alpha$$

[1]If V is finite dimensional than every linear operator $V \to V$ is bounded, the vector spaces for Kähler quantization that we will be considering in this chapter are finite dimensional.

is m-plectic. To see this introduce coordinates $q^1, \ldots, q^d$ on an open set $U \subset M$. Then the vectors $dq^I = dq^{i_1} \wedge \cdots \wedge dq^{i_m}$ where $I = (i_1, \ldots, i_m)$ ranges over multi-indices of length m provide a basis for m-forms on U. To these m-forms there are corresponding fibre coordinates p_I which when combined with the coordinates q^i pulled back from the base give a coordinate system on $\bigwedge^m T^*U$. In this coordinate system we may write

$$\alpha = \sum_I p_I dq^I.$$

So that we have

$$d\alpha = \omega = \sum_I dp_I \wedge dq^I.$$

From this formula we can see that ω is closed (since it is exact) and non-degenerate (this follows from the linear independence of the dq^I).

Remark 2.1.3. The Jacobi identity may be written in any of a number of ways, we give some of the most common here. Let $X_i \in (A, \bullet, [.,.])$ where $(A, \bullet)$ is an algebra (which may or may not be associative) and $[.,.] : \bigwedge^2 A \to A$, $(A, [.,.])$ is a bracket that satisfies the Jacobi identity (JI), giving $(A, [.,.])$ the structure of a Lie algebra. The Jacobi identity may be a expressed in any of the following forms:

$$[X_1, [X_2, X_3]] = [[X_1, X_2], X_3] + [X_2, [X_1, X_3], \tag{2.1a}$$

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1] + [X_3, [X_1, X_2]] = 0, \tag{2.1b}$$

$$\sum_{\sigma \in \text{cyclic permutations}} [X_{\sigma(1)}, [X_{\sigma(2)}, X_{\sigma(3)}]] = 0, \tag{2.1c}$$

$$\sum_{\sigma \in S_3} \epsilon(\sigma)[X_{\sigma(1)}, [X_{\sigma(2)}, X_{\sigma(3)}]] = 0, \tag{2.1d}$$

(1a) emphasizes that the JI is a derivation property for any operator $[X, .]$ (where $X \in A$) with respect to the bracket [.,.], which may be viewed as a product. (1b) suggests another perspective on the JI, suppose the algebra $(A, \bullet)$ is associative, than we have:

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]$$
$$= X[Y,Z]-[Y,Z]X+Y[Z,X]-[Z,X]Y+Z[X,Y]-[X,Y]Z$$
$$= X(YZ)-X(ZY)-(YZ)X+(ZY)X+Y(ZX)-Y(XZ)+(ZX)Y-(XZ)Y+Z(XY)$$
$$-Z(YX)-(XY)Z+(YX)Z$$
$$= XYZ-XZY-YZX+ZYX+YZX-YXZ-ZXY+XZY+ZXY-ZYX-XYZ+YXZ=0$$

where in the last line associativiy permits us to drop the brackets and we have written $X \bullet Y = XY$ throughout. (1b),(1c) are precisely the same identity written in different notations. To see that (1a) is equivalent to (1b) simply apply the antisymmetry property of the bracket. The l.h.s of (1d) is proportional to the l.h.s of (1c), so these identities are equivalent as well.

Definition 2.1.4. ([T], [G]) Let M be a smooth manifold. A multilinear map

$$\{.,...,.\} : (C^\infty(M))^{\otimes j} \to C^\infty(M)$$

is called a **Nambu-Poisson bracket** or **(generalized) Nambu bracket of order** j if it satisfies the following properties:

- (skew-symmetry) $\{f_1,...,f_j\} = \epsilon(\sigma)\{f_{\sigma(1)},...,f_{\sigma(j)}\}$ for any $f_1,...,f_j \in C^\infty(M)$ and for any $\sigma \in S_j$,

- (Leibniz rule) $\{f_1,...,f_{j-1},g_1g_2\} = \{f_1,...,f_{j-1},g_1\}g_2 + g_1\{f_1,...,f_{j-1},g_2\}$ for any $f_1,...,f_{j-1},g_1,g_2 \in C^\infty(M)$,

- (fundamental identity)

$$\{f_1, ..., f_{j-1}, \{g_1, ..., g_j\}\} = \sum_{i=1}^{j} \{g_1, ..., \{f_1, ..., f_{j-1}, g_i\}, ..., g_j\},$$

  for all $f_1, ..., f_{j-1}, g_1, ..., g_j \in C^\infty(M)$.

The fundamental identity generalizes the Jacobi identity along the lines of (1a) above, that is, the fundamental identity generalizes the derivation property of the JI. It is natural to ask how to generalize the Hamiltonian formalism of symplectic geometry to the multisymplectic setting. This generalization relies on a bracket that satisfies the fundamental identity for the formulation of it's dynamics and is called Nambu mechanics, it first appeared in [N]. We don't need the full multisymplectic formalism for the purposes of this thesis, and we refer the reader to [T], [He], [R1].

Definition 2.1.5. Let M be a smooth manifold and let $\{\ldots, .\}$ be a Nambu-Poisson bracket defined on $C^\infty(M)$, than the pair $(M, \{., \ldots, .\})$ is called a **Nambu-Poisson manifold**.

Example 2.1.6. Let M be an oriented n-dimensional manifold with volume form v_M, we may define an n-ary bracket on $C^\infty(M)$ by the formula

$$df_1 \wedge df_2 \wedge \cdots \wedge df_n = \{f_1, f_2, \ldots, f_n\} v_M$$

This defines a Nambu-Poisson bracket[G].

Example 2.1.7. A smooth manifold M is called a **hyperkähler** manifold if it comes with a Riemannian metric g and three complex structures J_1, J_2, J_3 compatible with g in the sense that each $i \in \{1, 2, 3\}$, $\omega_i(X, Y) = g(X, J_i Y)$ is a Kähler form, where X, Y are vector fields on M and such that these complex structures satisfy the quaternionic relations, that is, $J_3 = J_1 J_2 = -J_2 J_1$.

Definition 2.1.8. ([APP], [AI]) Let M be a smooth manifold and suppose j is an even positive integer. A multilinear map

$$\{., ..., .\} : (C^\infty(M))^{\otimes j} \to C^\infty(M)$$

is called a **generalized Poisson bracket** if it satisfies the following properties:

- (skew-symmetry) $\{f_1, ..., f_j\} = \epsilon(\sigma)\{f_{\sigma(1)}, ..., f_{\sigma(j)}\}$ for any $f_1, ..., f_j \in C^\infty(M)$ and for any $\sigma \in S_j$,

- (Leibniz rule) $\{f_1, ..., f_{j-1}, g_1 g_2\} = \{f_1, ..., f_{j-1}, g_1\}g_2 + g_1\{f_1, ..., f_{j-1}, g_2\}$ for any $f_1, ..., f_{j-1}, g_1, g_2 \in C^\infty(M)$,

- (generalized Jacobi identity)

$$\mathrm{Alt}\{f_1, ..., f_{j-1}, \{f_j, ..., f_{2j-1}\}\} :=$$

$$\sum_{\sigma \in S_{2j-1}} \epsilon(\sigma)\{f_{\sigma(1)}, ..., f_{\sigma(j-1)}, \{f_{\sigma(j)}, ..., f_{\sigma(2j-1)}\}\} = 0$$

  for any $f_1, ..., f_{2j-1} \in C^\infty(M)$.

The generalized Jacobi identity, generalizes (1d) and for an associative algebra it is a consequence of the associativity[AI].

Definition 2.1.9. ([ILMM]) A bracket as in definition 2.1.8 satisfying only the first two conditions (skew-symmetry and Leibniz rule) is called an **almost Poisson bracket of order j**.

Remark 2.1.10. A Nambu-Poisson bracket of even order is a generalized Poisson bracket [ILMM].

2.1.1 Generalized commutator

Let $[.,.,.,.]$ denote the Nambu **generalized commutator** ([N], [T], [CT]): for a finite-dimensional complex vector space V and $A_1, ..., A_{2n} \in End(V)$

$$[A_1, ..., A_{2n}] = \sum_{\sigma \in S_{2n}} \epsilon(\sigma) A_{\sigma(1)} ... A_{\sigma(2n)}.$$

For example, for $n = 2$

$$[A_1, A_2, A_3, A_4] = \sum_{\sigma \in S_4} \epsilon(\sigma) A_{\sigma(1)} A_{\sigma(2)} A_{\sigma(3)} A_{\sigma(4)} =$$

$$\begin{aligned} &[A_1, A_2][A_3, A_4] - [A_1, A_3][A_2, A_4] + [A_1, A_4][A_2, A_3] + \\ &[A_3, A_4][A_1, A_2] - [A_2, A_4][A_1, A_3] + [A_2, A_3][A_1, A_4]. \end{aligned} \tag{2.2}$$

The bracket $[.,.,.,.]$ defines a map $\bigwedge^4 End(V) \to End(V)$ which does not satisfy the Leibniz rule and does not satisfy the fundamental identity. There has been some discussion of this in physics literature (e.g. [CZ1]) and they seem to think that requiring these two conditions is not necessary. There has been investigation into algebraic properties of this bracket - see e.g. [CJM] and [AI], where some ideas go back to [Br], [F], and earlier work by Kurosh and his school.

Remark 2.1.11. Because the composition of linear operators is associative, the generalized commutator satisfies the generalized Jacobi identity[AI].

Lemma 2.1.12.

$$[A_1, ..., A_{2n}] = \frac{1}{2^n} \sum_{\sigma \in S_{2n}} \epsilon(\sigma) [A_{\sigma(1)}, A_{\sigma(2)}][A_{\sigma(3)}, A_{\sigma(4)}] ... [A_{\sigma(2n-1)}, A_{\sigma(2n)}].$$

Proof. By straightforward comparison of the polynomials. Observe that each monomial from the left-hand side appears on the right-hand side exactly 2^n times, with appropriate sign, and this accounts for all the terms on the right hand side. □

Remark 2.1.13. Equality (2.2) is (93) [CZ1]. It is not hard to see that Lemma 2.1.12 is equivalent to (94) [CZ1].

2.2 n-plectic quantization

Where in classical mechanics the algebra of observables was $(C^\infty(M), \{.,.\})$, a Poisson algebra, the goal of the usual (1-plectic) quantization was to find a mapping $Q : C^\infty(M) \to End(\mathcal{H})$ taking classical observables to quantum observables satisfying some desirable properties which included a correspondence $\{.,.\} \to [.,.]$, between the classical and quantum Lie algebra structures. In both of the situations that we will be considering in the rest of this chapter we will have an almost Poisson algebra $(C^\infty(M), \{., \ldots, .\})$ where $\{., \ldots, .\}$ is an almost Poisson structure. We will be interested in a finding a mapping $Q : C^\infty(M) \to End(\mathcal{H})$ where the algebraic structure on the Hilbert space side is provided by the generalized commutator $[., \ldots, .]$. This mapping should satisfy some properties which mirror the axioms of the usual quantization (they will be outlined below) and which will include a correspondence $\{., \ldots, .\} \to [., \ldots, .]$ between the Nambu-Poisson structure and the generalized commutator. By an n-plectic quantization of $(C^\infty(M), \{., \ldots, .\})$ we will mean a mapping $Q : C^\infty(M) \to End(\mathcal{H})$ satisfying the following axioms:

1. $\mathcal{Q}$ should be $\mathbb{C}$-linear

$$\mathcal{Q}(cf + g) = c\mathcal{Q}(f) + \mathcal{Q}(g)$$

2. The constant function 1 should be sent to identity operator I.

3. Real valued functions should be sent to self-adjoint operators so that their eigenvalues will be real.

$$\mathcal{Q}(f) = \mathcal{Q}^*(f)$$

4. We will require that

$$(\frac{i}{\hbar})^n[\mathcal{Q}(f_1), \ldots, \mathcal{Q}(f_{2n})] = \mathcal{Q}(\{f_1, \ldots, f_{2n}\}) + O(\hbar).$$

Axioms 1,2, and 3 are the same ones that are desired for 1-plectic quantization (of which Berezin-Toeplitz quantization is an example). Axiom 4 is a straightforward generalization that we need in order to extend Berezin-Toeplitz quantization to the n-plectic case. Our focus in this chapter will be to demonstrate that this 4th axiom is satisfied in the two situations to which we will be applying Berezin-Toeplitz quantization. It should be noted that in the 1-plectic case both the classical algebra of observables and the quantum algebra of observables are Lie algebras and that although the Berezin-Toeplitz quantization map is not a Lie algebra morphism at any particular level k, the fact that it satifies axiom 4 means that asymptotically it becomes a Lie algebra homomorpism. The situation in the Nambu-Poisson case is a little different, the algebraic structure on the Hilbert space side is provided by the generalized commutator $[., \dots, .]$ which does not satisfy the same identity (fundamental identity) as the algebraic structure on the classical side. We will however show that when we extend Berezin-Toeplitz quantization to the n-plectic case the generalized version of axiom 4 holds, so that we will also have in this case that the mapping becomes a homomorphism asymptotically.

2.3 Quantization of the $(2n-1)$-plectic structure on an n-dimensional Kähler manifold

In this section we will use the Berezin-Toeplitz operator quantization to quantize a compact quantizable Kähler manifold of complex dimension n regarded as a $(2n-1)$-plectic manifold. We will show that the quantization has reasonable semiclassical properties.

Let (M, ω) be a compact n-dimensional Kähler manifold $(n \geq 1)$. Denote by $\{.,.\}$ the Poisson bracket on M coming from ω. Assume there is a hermitian holomorphic line bundle L on M such that the curvature of the hermitian connection is equal to $-i\omega$.

Every volume form is closed and non-degenerate so the volume form $\Omega = \frac{\omega^n}{n!}$ is a $(2n-1)$-plectic form. The bracket $\{.,...,.\} : \bigwedge^{2n} C^\infty(M) \to C^\infty(M)$ defined by

$$df_1 \wedge ... \wedge df_{2n} = \{f_1, ..., f_{2n}\}\Omega$$

is a Nambu-Poisson bracket (example 2.1.6) [G, Cor. 1 p. 106] .

Lemma 2.3.1. *For* $f_1, ..., f_{2n} \in C^\infty(M)$

$$\{f_1, ..., f_{2n}\} = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \epsilon(\sigma) \prod_{j=1}^{n} \{f_{\sigma(2j-1)}, f_{\sigma(2j)}\} \tag{2.3}$$

Remark 2.3.2. In particular, for $n = 2$

$$\{f_1, f_2, f_3, f_4\} = \{f_1, f_2\}\{f_3, f_4\} - \{f_1, f_3\}\{f_2, f_4\} + \{f_1, f_4\}\{f_2, f_3\}.$$

Proof of Lemma 2.3.1. Let's use the Darboux theorem and compare the left-hand side and the right-hand side of (2.3) in a local chart with coordinates x_1,...,x_{2n} such that in this chart $\omega = \sum_{j=1}^n dx_{2j-1} \wedge dx_{2j}$. Locally, in this chart, the Poisson bracket is

$$\{f_i, f_l\} = \sum_{j=1}^{n} (\frac{\partial f_i}{\partial x_{2j-1}} \frac{\partial f_l}{\partial x_{2j}} - \frac{\partial f_i}{\partial x_{2j}} \frac{\partial f_l}{\partial x_{2j-1}})$$

and $\{f_1, ..., f_{2n}\} = \det J$, where $J = (\frac{\partial f_i}{\partial x_l})$. det is the only function on $(2n) \times (2n)$ complex matrices which takes value 1 on the identity matrix, linear in the rows, and takes value zero on a matrix whose two adjacent rows are equal (axiomatic characterization of the determinant, see e.g. Theorem 3.14 [A]). The right-hand side of (2.3) is a polynomial in the entries of J that satisfies these three conditions, therefore it must be equal to $\det J$. □

Remark 2.3.3. The previous lemma holds true for any almost Poisson bracket of order j [AI], our proof makes use of the fact the bracket we are investigating is defined in terms of non-degenerate Poisson brackets.

The following theorem shows that, informally speaking, "$\{.,...,.\} \to [.,...,.]$ as $k \to \infty$".

Theorem 2.3.4. *For $f_1, ..., f_{2n} \in C^\infty(M)$*

$$||\frac{(ik)^n}{n!}[T^{(k)}_{f_1}, ..., T^{(k)}_{f_{2n}}] - T^{(k)}_{\{f_1,...,f_{2n}\}}|| = O(\frac{1}{k})$$

as $k \to \infty$.

Proof. By Theorem 1.5.3 (i)

$$||ik[T^{(k)}_{f_{2j-1}}, T^{(k)}_{f_{2j}}] - T^{(k)}_{\{f_{2j-1},f_{2j}\}}|| = O(\frac{1}{k}) \tag{2.4}$$

for $j = 1, ..., n$. Using Prop. 1.5.6 and the triangle inequality, we get:

$$||(ik)^n[T^{(k)}_{f_1}, T^{(k)}_{f_2}]...[T^{(k)}_{f_{2n-1}}, T^{(k)}_{f_{2n}}] - T^{(k)}_{\{f_1,f_2\}...\{f_{2n-1},f_{2n}\}}|| \leq$$

$$||(ik)^n[T^{(k)}_{f_1}, T^{(k)}_{f_2}]...[T^{(k)}_{f_{2n-1}}, T^{(k)}_{f_{2n}}] - T^{(k)}_{\{f_1,f_2\}}...T^{(k)}_{\{f_{2n-1},f_{2n}\}}||+$$

$$||T^{(k)}_{\{f_1,f_2\}...\{f_{2n-1},f_{2n}\}} - T^{(k)}_{\{f_1,f_2\}}...T^{(k)}_{\{f_{2n-1},f_{2n}\}}|| =$$

$$||\big((ik[T^{(k)}_{f_1}, T^{(k)}_{f_2}] - T^{(k)}_{\{f_1,f_2\}}) + T^{(k)}_{\{f_1,f_2\}}\big)...\big((ik[T^{(k)}_{f_{2n-1}}, T^{(k)}_{f_{2n}}] - T^{(k)}_{\{f_{2n-1},f_{2n}\}}$$

$$+T^{(k)}_{\{f_{2n-1},f_{2n}\}}\big) - T^{(k)}_{\{f_1,f_2\}}...T^{(k)}_{\{f_{2n-1},f_{2n}\}}|| + O(\frac{1}{k}).$$

This is $O(\frac{1}{k})$. Indeed, within $||.||$ the term $T^{(k)}_{\{f_1,f_2\}}...T^{(k)}_{\{f_{2n-1},f_{2n}\}}$ cancels and all the other terms are products of factors of the form $(ik[T^{(k)}_{f_{2j-1}}, T^{(k)}_{f_{2j}}] - T^{(k)}_{\{f_{2j-1},f_{2j}\}})$ (at least one of these appears) and of the form $T^{(k)}_{\{f_{2j-1},f_{2j}\}}$. Using the triangle inequality, (2.4) and Theorem 1.5.3 (ii), we get $O(\frac{1}{k})$. Thus, as $k \to \infty$,

$$||(ik)^n[T^{(k)}_{f_1}, T^{(k)}_{f_2}]...[T^{(k)}_{f_{2n-1}}, T^{(k)}_{f_{2n}}] - T^{(k)}_{\{f_1,f_2\}...\{f_{2n-1},f_{2n}\}}|| = O(\frac{1}{k}).$$

The exact same proof shows that

$$||(ik)^n[T^{(k)}_{f_{\sigma(1)}}, T^{(k)}_{f_{\sigma(2)}}]...[T^{(k)}_{f_{\sigma(2n-1)}}, T^{(k)}_{f_{\sigma(2n)}}] - T^{(k)}_{\{f_{\sigma(1)},f_{\sigma(2)}\}...\{f_{\sigma(2n-1)},f_{\sigma(2n)}\}}|| = O(\frac{1}{k}).$$

We note that

$$T^{(k)}_{\{f_1,...,f_{2n}\}} = \frac{1}{2^n n!}\sum_{\sigma\in S_{2n}} \epsilon(\sigma) T^{(k)}_{\prod_{j=1}^{n}\{f_{\sigma(2j-1},f_{\sigma(2j)}\}}$$

(by Lemma 2.3.1). The desired statement now follows from Lemma 2.1.12 and the triangle inequality. □

The following proposition is similar to Prop. 1.5.4. It implies that $\lim_{k\to\infty}|||[T^{(k)}_{f_1},...,T^{(k)}_{f_{2n}}]||| = 0$ (i.e. $T^{(k)}_{f_1}$, ..., $T^{(k)}_{f_{2n}}$ "Nambu-commute" as $k\to\infty'$.

Proposition 2.3.5. *For $f_1,...,f_{2n}\in C^\infty(M)$*

$$|||[T^{(k)}_{f_1},...,T^{(k)}_{f_{2n}}]||| = O(\frac{1}{k^n})$$

as $k\to\infty$.

Proof. Let us denote, for convenience,

$$\sum_{\sigma\in S_{2n}}{}' = \sum_{\substack{\sigma\in S_{2n}\\ \sigma(1)<\sigma(2),...,\\ \sigma(2n-1)<\sigma(2n)}} .$$

$$|||[T^{(k)}_{f_1},...,T^{(k)}_{f_{2n}}]||| =$$

$$||\sum_{\sigma\in S_{2n}}{}' \epsilon(\sigma)[T^{(k)}_{f_{\sigma(1)}},T^{(k)}_{f_{\sigma(2)}}][T^{(k)}_{f_{\sigma(3)}},T^{(k)}_{f_{\sigma(4)}}]...[T^{(k)}_{f_{\sigma(2n-1)}},T^{(k)}_{f_{\sigma(2n)}}]||| \leq$$

$$\sum_{\sigma\in S_{2n}}{}' |||[T^{(k)}_{f_{\sigma(1)}},T^{(k)}_{f_{\sigma(2)}}]|||...|||[T^{(k)}_{f_{\sigma(2n-1)}},T^{(k)}_{f_{\sigma(2n)}}]|||$$

which is $O(\frac{1}{k^n})$ by Remark 1.5.5. □

2.3.1 Alternative proof of theorem 2.3.4

We are also able to give a proof of theorem 2.3.4 by following the method that yeilded the result 1.5.3(i) in the first place.For a discussion of the background material needed in order to understand this alternative proof see [Bo] , along with further references, we write down the proof here.

Proof. Theorem 2.3.4

Let $f_i \in C^\infty(M)$, the commutator $[T_{f_i}, T_{f_j}]$ is a Toeplitz operator of order -1 with principal symbol $i\{\tau_\Sigma^\star f_i, \tau_\Sigma^\star f_j\}_\Sigma(t\alpha(\lambda)) = it^{-1}\{f_i, f_j\}_M(\tau(\lambda))$ [BMS]. It follows from lemma 2.3.1 that $[T_{f_1}, \ldots, T_{f_{2n}}]$ is a Toeplitz operator of order $-n$, its principal symbol can be obtained by applying lemma 2.1.12, lemma 2.3.1 and the multiplicative property of the principal symbol:

$$\sigma_p([T_{f_1}, \ldots, T_{f_{2n}}]) = \sigma_p(\frac{1}{2^n}\sum_{\sigma\in S_{2n}} \epsilon(\sigma)[A_{\sigma(1)}, A_{\sigma(2)}][A_{\sigma(3)}, A_{\sigma(4)}]...[A_{\sigma(2n-1)}, A_{\sigma(2n)}])$$

$$= \frac{1}{2^n}\sum_{\sigma\in S_{2n}} \epsilon(\sigma)\sigma_p([A_{\sigma(1)}, A_{\sigma(2)}][A_{\sigma(3)}, A_{\sigma(4)}]...[A_{\sigma(2n-1)}, A_{\sigma(2n)}])$$

$$= \frac{1}{2^n}\sum_{\sigma\in S_{2n}} \epsilon(\sigma)\sigma_p([A_{\sigma(1)}, A_{\sigma(2)}])\sigma_p([A_{\sigma(3)}, A_{\sigma(4)}])...\sigma_p([A_{\sigma(2n-1)}, A_{\sigma(2n)}])$$

$$= \frac{i^n t^{-n}}{2^n}\sum_{\sigma\in S_{2n}} \epsilon(\sigma)\{f_{\sigma(1)}, f_{\sigma(2)}\}\{f_{\sigma(3)}, f_{\sigma(4)}\}...\{f_{\sigma(2n-1)}, f_{\sigma(2n)}\}_M(\tau(\lambda))$$

$$= \frac{i^n t^{-n} n!}{1}\{f_{\sigma(1)}, \ldots, f_{\sigma(2n)}\}_M(\tau(\lambda))$$

Now consider the Toeplitz operator

$$A := D_\phi^{n+1}[T_{f_1}, \ldots, T_{f_{2n}}] - i^n n! D_\phi T_{\{f_{\sigma(1)}, \ldots, f_{\sigma(2n)}\}}$$

Apriori this operator is of order 1, but after a calculation we see that its principal symbol vanishes, so that it is infact of order 0:

$$\sigma_p(A) = t^{n+1}\sigma_p([T_{f_1}, \ldots, T_{f_{2n}}]) - i^n n! t\sigma_p(T_{\{f_1, \ldots, f_{2n}\}})$$

$$= i^n n! t\{f_1, \ldots, f_{2n}\}_M(\tau(\lambda)) - i^n n! t\{f_1, \ldots, f_{2n}\}_M(\tau(\lambda)) = 0$$

Zeroth order pseudo-differential operators on compact manifolds are bounded and since $\prod$ is bounded as an operator on $L^2(Q, \nu)$ (here Q is the unit cirlce bundle inside L^* and ν is a measure coming from the contact form described in section 1.4.2), it follows that A is bounded. Since $||A^{(m)}|| \leq ||A||$ we have

$$||A^{(m)}|| = ||A|H^{(m)}|| = ||m^{n+1}[T^{(m)}_{f_1}, \ldots, T^{(m)}_{f_{2n}}] - i^n n! m T^{(m)}_{\{f_1,\ldots,f_{2n}\}}|| \leq K$$

after dividing through by $i^n n! m$ we obtain

$$||\frac{m^n}{i^n n!}[T^{(m)}_{f_1}, ..., T^{(m)}_{f_{2n}}] - T^{(m)}_{\{f_1,\ldots,f_{2n}\}}|| = O(\frac{1}{m})$$

Like theorem 1.5.8, this applies to the Nambu-Poisson algebra (that comes from the bracket of) $\hat{A}$. $\hat{A}$ is obtained from A, after applying the Poisson algebra isomorphism given by multiplication by i. If we apply the inverse isomorphism (if we multiply by $-i$), we get the desired statement (theorem 2.3.4) which applies to the Nambu-Poisson algebra (that comes from the bracket of) A.

$$||\frac{(im)^n}{n!}[T^{(m)}_{f_1}, ..., T^{(m)}_{f_{2n}}] - T^{(m)}_{\{f_1,\ldots,f_{2n}\}}|| = O(\frac{1}{m})$$

as $m \to \infty$

and we are done.

$\square$

2.4 Quantization of a hyperkähler manifold

Let (M, g, J_1, J_2, J_3) be a compact connected hyperkähler manifold. Let $4q$ denote the real dimension of M. Denote $\omega_r = g(., J_r.)$ for $r = 1, 2, 3$. The 4-form

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$$

is 3-plectic (example 2.1.7)[CIL]. Define the brackets $\{., ., ., .\}_r$, $\{., ., ., .\}_{hyp}$ (maps $\bigwedge^4 C^\infty(M) \to C^\infty(M)$) as follows:

$$\{f_1, f_2, f_3, f_4\}_r = \{f_1, f_2\}_r\{f_3, f_4\}_r - \{f_1, f_3\}_r\{f_2, f_4\}_r + \{f_1, f_4\}_r\{f_2, f_3\}_r,$$

where $\{.,.\}_r$ is the Poisson bracket on (M, ω_r), $r = 1, 2, 3$,

$$\{f_1, f_2, f_3, f_4\}_{hyp} = \sum_{r=1}^{3} \{f_1, f_2, f_3, f_4\}_r. \tag{2.5}$$

In lemma 2.3.1 we proved that a bracket of order $2n$ where $2n = \dim M$ may be written in terms of Poisson brackets, for a bracket of order $m < 2n$ we are simply taking the formula in terms of Poisson brackets as the definition of the bracket. The map $C^\infty(M)^{\otimes 4} \to C^\infty(M)$ defined by (2.5) is multilinear and anti-symmetric. From the properties of the Poisson bracket it immediately follows that the Leibniz rule is satisfied:

$$\{f_1, f_2, f_3, f_4 f_5\}_{hyp} = f_4\{f_1, f_2, f_3, f_5\}_{hyp} + \{f_1, f_2, f_3, f_4\}_{hyp} f_5.$$

Therefore $\{., ., ., .\}_{hyp}$ is an almost Poisson bracket of order 4.

For $q = 1$ $\omega_r \wedge \omega_r$ $(r = 1, 2, 3)$ and Ω are volume forms. The standard bracket $\{., ., ., .\}^{(r)}$ is defined by

$$df_1 \wedge df_2 \wedge df_3 \wedge df_4 = \{f_1, f_2, f_3, f_4\}^{(r)} \frac{1}{2} \omega_r \wedge \omega_r.$$

From Lemma 2.3.1, or by a direct calculation (using the Darboux theorem, in local coordinates), we get:

Lemma 2.4.1. *For $q = 1$ $\{., ., ., .\}_r$ coincides with $\{., ., ., .\}^{(r)}$.*

From [G, Cor. 1 p.106] it immediately follows that for $q = 1$ (M is 4-dimensional) the fundamental identity

$$\{f_1, f_2, f_3, \{g_1, g_2, g_3, g_4\}_{hyp}\}_{hyp} = \{\{f_1, f_2, f_3, g_1\}_{hyp}, g_2, g_3, g_4\}_{hyp} +$$

$$\{g_1, \{f_1, f_2, f_3, g_2\}_{hyp}, g_3, g_4\}_{hyp} + \{g_1, g_2, \{f_1, f_2, f_3, g_3\}_{hyp}, g_4\}_{hyp} +$$

$$\{g_1, g_2, g_3, \{f_1, f_2, f_3, g_4\}_{hyp}\}_{hyp}$$

is satisfied. When $q > 1$ it is not necessarily a Nambu-Poisson bracket (it may not satisfy the fundamental identity if $q > 1$).

Assume that the Kähler forms ω_1, ω_2, ω_3 are integral. Let L_r be a holomorphic Hermitian line bundle with curvature of the Hermitian connection equal to $-i\omega_r$, for $r = 1, 2, 3$. For a positive integer k and $f \in C^\infty(M)$ denote by $T^{(k)}_{f;r} \in End(H^0(M, L_r^{\otimes k}))$ the Berezin-Toeplitz operator for f. There are two obvious ways to form a Hilbert space out of three Hilbert spaces $H^0(M, L_r^{\otimes k})$ ($r = 1, 2, 3$): by taking direct sum or tensor product. Another way to approach this is to say that the vector space of quantization is $H^0(M, (L_1 \otimes L_2 \otimes L_3)^{\otimes k})$, - this would be just the usual Berezin-Toeplitz quantization, with the line bundle $L_1 \otimes L_2 \otimes L_3$. Note: in general $H^0(M, (L_1 \otimes L_2 \otimes L_3)^{\otimes k})$ is not isomorphic to $H^0(M, L_1^{\otimes k}) \otimes H^0(M, L_2^{\otimes k}) \otimes H^0(M, L_3^{\otimes k})$.

Of course, the hyperkähler structure defines a whole S^2 of complex structures (and of Kähler forms) on M, not just three. A. Uribe pointed out to us that maybe an appropriate notion of quantization on a hyperkähler manifold should take into account all $J \in S^2$, and should involve an appropriate vector bundle over the twistor space, with fibers $H^0(M, L_J^{\otimes k})$. We look forward to seeing his work on this.

Note that the twistor space of a hyperkähler manifold is not Kähler (it is generally well-known, see for example [VK] p. 37, or [Hu]), so it's not possible to construct a Berezin-Toeplitz quantization on the twistor space.

2.4.1 Direct sum

Denote

$$\mathcal{H}^k = H^0(M, L_1^{\otimes k}) \oplus H^0(M, L_2^{\otimes k}) \oplus H^0(M, L_3^{\otimes k})$$

(direct sum of Hilbert spaces) and

$$\mathbf{T}^{(k)}_f = T^{(k)}_{f;1} \oplus T^{(k)}_{f;2} \oplus T^{(k)}_{f;3},$$

($\mathbf{T}^{(k)}_f$ acts on $\mathcal{H}^k$ by $\mathbf{T}^{(k)}_f(s_1, s_2, s_3) = (T^{(k)}_{f;1}s_1, T^{(k)}_{f;2}s_2, T^{(k)}_{f;3}s_3)$).

Remark 2.4.2. Since $||\mathbf{T}^{(k)}_f|| = \max\{||T^{(k)}_{f;1}||, ||T^{(k)}_{f;2}||, ||T^{(k)}_{f;3}||\}$, we immediately have:

- For $f, g \in C^\infty(M)$, as $k \to \infty$,
$$||ik[\mathbf{T}^{(k)}_f, \mathbf{T}^{(k)}_g] - \mathbf{T}^{(k)}_{\{f,g\}}|| = O(\frac{1}{k}),\ ||[\mathbf{T}^{(k)}_f, \mathbf{T}^{(k)}_g]|| = O(\frac{1}{k})$$
- For $f \in C^\infty(M)$, as $k \to \infty$, there is a constant $C = C(f) > 0$ such that
$$|f|_\infty - \frac{C}{k} \leq ||\mathbf{T}^{(k)}_f|| \leq |f|_\infty.$$
- For $f_1, ..., f_p \in C^\infty(M)$
$$||\mathbf{T}^{(k)}_{f_1}...\mathbf{T}^{(k)}_{f_p} - \mathbf{T}^{(k)}_{f_1...f_p}|| = O(\frac{1}{k})$$
as $k \to \infty$.

For $f, g, h, t \in C^\infty(M)$ we have:

$$[\mathbf{T}^{(k)}_f, \mathbf{T}^{(k)}_g, \mathbf{T}^{(k)}_h, \mathbf{T}^{(k)}_t] = \oplus^3_{r=1}[T^{(k)}_{f;r}, T^{(k)}_{g;r}, T^{(k)}_{h;r}, T^{(k)}_{t;r}].$$

Theorem 2.4.3. *For $f, g, h, t \in C^\infty(M)$*

$$|| - \frac{k^2}{2}[\mathbf{T}^{(k)}_f, \mathbf{T}^{(k)}_g, \mathbf{T}^{(k)}_h, \mathbf{T}^{(k)}_t] - \oplus^3_{r=1}T^{(k)}_{\{f,g,h,t\}_r;r}|| = O(\frac{1}{k})$$

as $k \to \infty$.

Proof. As $k \to \infty$, for $r = 1, 2, 3$, by Theorem 1.5.3 (i) for $f, g \in C^\infty(M)$

$$||ik[T^{(k)}_{f;r}, T^{(k)}_{g;r}] - T^{(k)}_{\{f,g\}_r;r}|| = O(\frac{1}{k}), \tag{2.6}$$

$$||ik[T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{h,t\}_r;r}|| = O(\frac{1}{k}). \tag{2.7}$$

Using Prop. 1.5.6, we get:

$$||(ik)^2[T^{(k)}_{f;r}, T^{(k)}_{g;r}][T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{f,g\}_r\{h,t\}_r;r}|| \leq$$

$$||(ik)^2[T^{(k)}_{f;r},T^{(k)}_{g;r}][T^{(k)}_{h;r},T^{(k)}_{t;r}]-T^{(k)}_{\{f,g\}_r;r}T^{(k)}_{\{h,t\}_r;r}||+||T^{(k)}_{\{f,g\}_r;r}T^{(k)}_{\{h,t\}_r;r}-T^{(k)}_{\{f,g\}_r\{h,t\}_r;r}||=$$

$$||(ik[T^{(k)}_{f;r},T^{(k)}_{g;r}]-T^{(k)}_{\{f,g\}_r;r}+T^{(k)}_{\{f,g\}_r;r})(ik[T^{(k)}_{h;r},T^{(k)}_{t;r}]-T^{(k)}_{\{h,t\}_r;r}+T^{(k)}_{\{h,t\}_r;r})$$

$$-T^{(k)}_{\{f,g\}_r;r}T^{(k)}_{\{h,t\}_r;r}||+O(\frac{1}{k})=$$

$$||(ik[T^{(k)}_{f;r},T^{(k)}_{g;r}]-T^{(k)}_{\{f,g\}_r;r})(ik[T^{(k)}_{h;r},T^{(k)}_{t;r}]-T^{(k)}_{\{h,t\}_r;r})+$$

$$(ik[T^{(k)}_{f;r},T^{(k)}_{g;r}]-T^{(k)}_{\{f,g\}_r;r})T^{(k)}_{\{h,t\}_r;r}+T^{(k)}_{\{f,g\}_r;r}(ik[T^{(k)}_{h;r},T^{(k)}_{t;r}]-T^{(k)}_{\{h,t\}_r;r})||+O(\frac{1}{k})$$

$$\leq||ik([T^{(k)}_{f;r},T^{(k)}_{g;r}]-T^{(k)}_{\{f,g\}_r;r})||\;||ik[T^{(k)}_{h;r},T^{(k)}_{t;r}]-T^{(k)}_{\{h,t\}_r;r})||+$$

$$||ik[T^{(k)}_{f;r},T^{(k)}_{g;r}]-T^{(k)}_{\{f,g\}_r;r}||\;||T^{(k)}_{\{h,t\}_r;r}||+||T^{(k)}_{\{f,g\}_r;r}||\;||ik[T^{(k)}_{h;r},T^{(k)}_{t;r}]-T^{(k)}_{\{h,t\}_r;r}||+O(\frac{1}{k})=$$

$$O(\frac{1}{k})O(\frac{1}{k})+|\{h,t\}_r|_\infty O(\frac{1}{k})+|\{f,g\}_r|_\infty O(\frac{1}{k})+O(\frac{1}{k})=O(\frac{1}{k}).$$

In the last line we used (2.6), (2.7), and applied Theorem 1.5.3 (ii) twice. Similarly we conclude, for f,h and g,t:

$$||(ik)^2[T^{(k)}_{f;r},T^{(k)}_{h;r}][T^{(k)}_{g;r},T^{(k)}_{t;r}]-T^{(k)}_{\{f,h\}_r\{g,t\}_r;r}||=O(\frac{1}{k}),$$

etc. (i.e. we get similar asymptotics for f,t and g,h, for h,t and f,g, for g,t and f,h, for g,h and f,t). Note:

$$T^{(k)}_{\{f,g,h,t\}_r;r}=T^{(k)}_{\{f,g\}_r\{h,t\}_r;r}-T^{(k)}_{\{f,h\}_r\{g,t\}_r;r}+T^{(k)}_{\{f,t\}_r\{g,h\}_r;r}.$$

Therefore, by (2.2) and the triangle inequality,

$$||-\frac{k^2}{2}[T^{(k)}_{f;r},T^{(k)}_{g;r},T^{(k)}_{h;r},T^{(k)}_{t;r}]-T^{(k)}_{\{f,g,h,t\}_r;r}||=O(\frac{1}{k}).$$

We get:

$$||-\frac{k^2}{2}[\mathbf{T}^{(k)}_f,\mathbf{T}^{(k)}_g,\mathbf{T}^{(k)}_h,\mathbf{T}^{(k)}_t]-\oplus^3_{r=1}T^{(k)}_{\{f,g,h,t\}_r;r}||=$$

$$\max_{1\leq r\leq 3}||-\frac{k^2}{2}[T^{(k)}_{f;r},T^{(k)}_{g;r},T^{(k)}_{h;r},T^{(k)}_{t;r}]-T^{(k)}_{\{f,g,h,t\}_r;r}||=O(\frac{1}{k}).$$

□

The following proposition is similar to Prop. 1.5.4. It implies that $\mathbf{T}^{(k)}_f$, $\mathbf{T}^{(k)}_g$, $\mathbf{T}^{(k)}_h$, $\mathbf{T}^{(k)}_t$ "Nambu-commute as $k\to\infty$".

Proposition 2.4.4. *For* $f_1, f_2, f_3, f_4 \in C^\infty(M)$

$$|||[\mathbf{T}^{(k)}_{f_1}, \mathbf{T}^{(k)}_{f_2}, \mathbf{T}^{(k)}_{f_3}, \mathbf{T}^{(k)}_{f_4}]||| = O(\frac{1}{k^2})$$

as $k \to \infty$.

Proof.

$$|||[\mathbf{T}^{(k)}_{f_1}, \mathbf{T}^{(k)}_{f_2}, \mathbf{T}^{(k)}_{f_3}, \mathbf{T}^{(k)}_{f_4}]||| = \max_{1\leq r\leq 3} |||[T^{(k)}_{f_1;r}, T^{(k)}_{f_2;r}, T^{(k)}_{f_3;r}, T^{(k)}_{f_4;r}]||| =$$

$$\max_{1\leq r\leq 3} ||\sum_{\sigma\in S_4}{}' \operatorname{sign}(\sigma)[T^{(k)}_{f_{\sigma(1)};r}, T^{(k)}_{f_{\sigma(2)};r}][T^{(k)}_{f_{\sigma(3)};r}, T^{(k)}_{f_{\sigma(4)};r}]|| \leq$$

$$\max_{1\leq r\leq 3} \sum_{\sigma\in S_4}{}' |||[T^{(k)}_{f_{\sigma(1)};r}, T^{(k)}_{f_{\sigma(2)};r}]||| \; |||[T^{(k)}_{f_{\sigma(3)};r}, T^{(k)}_{f_{\sigma(4)};r}]|||.$$

By Remark 1.5.5 it is $O(\frac{1}{k^2})$. $\square$

2.4.2 Direct sum: dimension 4

To discuss the correspondence between the the bracket on functions and the generalized commutator (as $k \to \infty$) in the hyperkähler case: we showed (Theorem 2.4.3) that for a hyperkähler manifold M of arbitrary dimension and smooth functions f, g, h, t on M $[\mathbf{T}^{(k)}_f, \mathbf{T}^{(k)}_g, \mathbf{T}^{(k)}_h, \mathbf{T}^{(k)}_t]$ is asymptotic to

$$\begin{pmatrix} T^{(k)}_{\{f,g,h,t\}_1;1} & & \\ & T^{(k)}_{\{f,g,h,t\}_2;2} & \\ & & T^{(k)}_{\{f,g,h,t\}_3;3} \end{pmatrix},$$

not to

$$\mathbf{T}^{(k)}_{\{f,g,h,t\}_{hyp}} = \begin{pmatrix} T^{(k)}_{\{f,g,h,t\}_{hyp};1} & & \\ & T^{(k)}_{\{f,g,h,t\}_{hyp};2} & \\ & & T^{(k)}_{\{f,g,h,t\}_{hyp};3} \end{pmatrix}.$$

To clarify, we have obtained an asymptotic relation between a map

$$\bigwedge^4 C^\infty(M) \to C^\infty(M) \times C^\infty(M) \times C^\infty(M)$$

$$f, g, h, t \mapsto (\{f, g, h, t\}_1, \{f, g, h, t\}_2, \{f, g, h, t\}_3)$$

and the Nambu generalized commutator $[., ., ., .]$. It is not the same as a correspondence between $\{., ., ., .\}_{hyp} : \bigwedge^4 C^\infty(M) \to C^\infty(M)$ and $[., ., ., .]$.

From now on M will be of real dimension 4 (so M is a compact hyperkähler 4-manifold, hence M is isomorphic to a K3-surface or a torus [Be] 14.22). In this case we get Theorem 2.4.5 below, and in the case when M is a 4-torus with three standard linear complex structures (Example 2.4.6 below) - we get that $[\mathbf{T}_f^{(k)}, \mathbf{T}_g^{(k)}, \mathbf{T}_h^{(k)}, \mathbf{T}_t^{(k)}]$ is asymptotic to $\mathbf{T}^{(k)}_{\{f,g,h,t\}_{hyp}}$.

We have: for $r = 1, 2, 3$

$$\Omega = \frac{\mu_r}{2}\omega_r \wedge \omega_r,$$

where μ_r is a smooth non-vanishing function on M. Denote by $\{., ., ., .\}$ the Nambu-Poisson bracket defined by

$$df_1 \wedge df_2 \wedge df_3 \wedge df_4 = \{f_1, f_2, f_3, f_4\}\Omega.$$

Therefore

$$\{f_1, f_2, f_3, f_4\}_r = \{f_1, f_2, f_3, f_4\}^{(r)} = \mu_r\{f_1, f_2, f_3, f_4\}.$$

Denote

$$\mathbf{T}_\mu^{(k)} = \begin{pmatrix} T^{(k)}_{\mu_1;1} & & \\ & T^{(k)}_{\mu_2;2} & \\ & & T^{(k)}_{\mu_3;3} \end{pmatrix}.$$

The following theorem shows that $[\mathbf{T}_{f_1}^{(k)}, \mathbf{T}_{f_2}^{(k)}, \mathbf{T}_{f_3}^{(k)}, \mathbf{T}_{f_4}^{(k)}]$ is asymptotic to $\mathbf{T}^{(k)}_{\{f_1,f_2,f_3,f_4\}}\mathbf{T}_\mu^{(k)}$.

Theorem 2.4.5. *For $f, g, h, t \in C^\infty(M)$*

$$|| - \frac{k^2}{2}[\mathbf{T}_f^{(k)}, \mathbf{T}_g^{(k)}, \mathbf{T}_h^{(k)}, \mathbf{T}_t^{(k)}] - \mathbf{T}^{(k)}_{\{f,g,h,t\}}\mathbf{T}_\mu^{(k)}|| = O(\frac{1}{k})$$

as $k \to \infty$.

Proof. For $r = 1, 2, 3$ the same argument as in the proof of Theorem 2.4.3 gives:

$$|| - \frac{k^2}{2}[T^{(k)}_{f;r}, T^{(k)}_{g;r}, T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{f,g,h,t\}_r;r}|| = O(\frac{1}{k}) \tag{2.8}$$

We have:

$$|| - \frac{k^2}{2}[T^{(k)}_{f;r}, T^{(k)}_{g;r}, T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{f,g,h,t\};r} T^{(k)}_{\mu_r;r}|| \leq$$

$$|| - \frac{k^2}{2}[T^{(k)}_{f;r}, T^{(k)}_{g;r}, T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{f,g,h,t\}\mu_r;r}|| + ||T^{(k)}_{\{f,g,h,t\}\mu_r;r} - T^{(k)}_{\{f,g,h,t\};r} T^{(k)}_{\mu_r;r}||.$$

This is $O(\frac{1}{k})$ by (2.8) and Prop. 1.5.6. Hence

$$|| - \frac{k^2}{2}[\mathbf{T}^{(k)}_{f}, \mathbf{T}^{(k)}_{g}, \mathbf{T}^{(k)}_{h}, \mathbf{T}^{(k)}_{t}] - \mathbf{T}^{(k)}_{\{f,g,h,t\}} \mathbf{T}^{(k)}_{\mu}||$$

$$= \max_{1 \leq r \leq 3} || - \frac{k^2}{2}[T^{(k)}_{f;r}, T^{(k)}_{g;r}, T^{(k)}_{h;r}, T^{(k)}_{t;r}] - T^{(k)}_{\{f,g,h,t\};r} T^{(k)}_{\mu_r;r}|| = O(\frac{1}{k}).$$

□

Example 2.4.6. Denote $\tilde{M} = \mathbb{R}^4$, with coordinates x_1, x_2, x_3, x_4, and equipped with three (linear) complex structures

$$J_1 = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} & & 1 & 0 \\ & & 0 & 1 \\ -1 & 0 & & \\ 0 & -1 & & \end{pmatrix}, \quad J_3 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$$

We have: $J_1 J_2 = J_3$ and, of course, $J_1^2 = J_2^2 = J_3^2 = -I$.

Note: if we regard $\tilde{M}$ as the one-dimensional quaternionic vector space, with basis 1, **i, j, k** ($\mathbf{i}^2$=$\mathbf{j}^2$=$\mathbf{k}^2$=-1, **ij**=**k**), then J_1, J_2, J_3 correspond to left multiplication by **i, j, k** respectively.

For the standard Riemannian metric on $\tilde{M}$, with the metric tensor $g = I$, the symplectic forms are as follows:

$$\omega_1 = -dx_1 \wedge dx_2 - dx_3 \wedge dx_4,$$

$$\omega_2 = -dx_1 \wedge dx_3 + dx_2 \wedge dx_4,$$

$$\omega_3 = -dx_1 \wedge dx_4 - dx_2 \wedge dx_3.$$

For $r = 1, 2, 3$

$$\frac{1}{2}\omega_r \wedge \omega_r = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4,$$

$$\Omega = \sum_{r=1}^{3} \omega_r \wedge \omega_r = 6dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

Everything is $\mathbb{Z}^4$-invariant and $g,\ J_1,\ J_2,\ J_3,\ \omega_1,\ \omega_2,\ \omega_3,\ \Omega$ descend to $M = \tilde{M}/\mathbb{Z}^4$. We get: $\mu_1 = \mu_2 = \mu_3 = 6$ and

$$6\{.,.,.,.\} = \{.,.,.,.\}_r = \{.,.,.,.\}^{(r)} = \frac{1}{3}\{.,.,.,.\}_{hyp}.$$

Theorem 2.4.5 gives: for $f, g, h, t \in C^\infty(M)$

$$|| - \frac{3}{2}k^2[\mathbf{T}^{(k)}_f, \mathbf{T}^{(k)}_g \mathbf{T}^{(k)}_h \mathbf{T}^{(k)}_t] - \mathbf{T}^{(k)}_{\{f,g,h,t\}_{hyp}}|| = O(\frac{1}{k}) \tag{2.9}$$

as $k \to \infty$.

2.4.3 Tensor product

Denote

$$\mathcal{H}_k = H^0(M, L_1^{\otimes k}) \otimes H^0(M, L_2^{\otimes k}) \otimes H^0(M, L_3^{\otimes k})$$

(tensor product of Hilbert spaces) and

$$\mathbb{T}^{(k)}_f = T^{(k)}_{f;1} \otimes T^{(k)}_{f;2} \otimes T^{(k)}_{f;3},$$

($\mathbb{T}^{(k)}_f(s_1 \otimes s_2 \otimes s_3) = T^{(k)}_{f;1}s_1 \otimes T^{(k)}_{f;2}s_2 \otimes T^{(k)}_{f;3}s_3$ and the action extends to $\mathcal{H}_k$ by linearity, also note: $||\mathbb{T}^{(k)}_f|| = ||T^{(k)}_{f;1}||\ ||T^{(k)}_{f;2}||\ ||T^{(k)}_{f;3}||$).

In the proofs below we shall need the following elementary statement.

Lemma 2.4.7. *If M_j, N_j are linear operators on a finite dimensional Hilbert space V_j $(j = 1, 2, 3)$, then*

$$||M_1 \otimes M_2 \otimes M_3 - N_1 \otimes N_2 \otimes N_3|| \leq ||M_1 - N_1||\ ||M_2 - N_2||\ ||M_3 - N_3|| +$$

$$||M_1 - N_1||\ ||M_2||\ ||N_3|| + ||M_1||\ ||N_2||\ ||M_3 - N_3|| + ||N_1||\ ||M_2 - N_2||\ ||M_3||$$

Proof. This immediately follows from the equality

$$(M_1 - N_1) \otimes (M_2 - N_2) \otimes (M_3 - N_3) = M_1 \otimes M_2 \otimes M_3 - N_1 \otimes N_2 \otimes N_3 -$$

$$(M_1 - N_1) \otimes M_2 \otimes N_3 - M_1 \otimes N_2 \otimes (M_3 - N_3) - N_1 \otimes (M_2 - N_2) \otimes M_3$$

□

We also note the following identity for tensor products of operators:

$$[A_1 \otimes A_2 \otimes A_3, B_1 \otimes B_2 \otimes B_3] = [A_1, B_1] \otimes [A_2, B_2] \otimes [A_3, B_3] + \tag{2.10}$$

$$[A_1, B_1] \otimes B_2 A_2 \otimes A_3 B_3 + A_1 B_1 \otimes [A_2, B_2] \otimes B_3 A_3 + B_1 A_1 \otimes A_2 B_2 \otimes [A_3, B_3].$$

Remark 2.4.8.

- For $f \in C^\infty(M)$, there is a constant $C = C(f) > 0$ such that, as $k \to \infty$,

$$(|f|_\infty - \frac{C}{k})^3 \leq ||\mathbb{T}_f^{(k)}|| \leq (|f|_\infty)^3.$$

- For $f_1, ..., f_p \in C^\infty(M)$

$$||\mathbb{T}_{f_1}^{(k)} ... \mathbb{T}_{f_p}^{(k)} - \mathbb{T}_{f_1 ... f_p}^{(k)}|| = O(\frac{1}{k})$$

  as $k \to \infty$.

The last statement holds for $p = 2$ by Lemma 2.4.7, Theorem 1.5.3 and Prop. 1.5.6. It follows for arbitrary p by induction.

Proposition 2.4.9. *For $f, g \in C^\infty(M)$*

$$||(ik)^3[T^{(k)}_{f;1}, T^{(k)}_{g;1}] \otimes [T^{(k)}_{f;2}, T^{(k)}_{g;2}] \otimes [T^{(k)}_{f;3}, T^{(k)}_{g;3}] - T^{(k)}_{\{f,g\}_1;1} \otimes T^{(k)}_{\{f,g\}_2;2} \otimes T^{(k)}_{\{f,g\}_3;3}|| = O(\frac{1}{k})$$

as $k \to \infty$.

Proof. This follows from Lemma 2.4.7, Theorem 1.5.3 and Remark 1.5.5. $\square$

Proposition 2.4.10. *For $f, g \in C^\infty(M)$*

$$||ik[\mathbb{T}^{(k)}_f, \mathbb{T}^{(k)}_g] - (T^{(k)}_{\{f,g\}_1;1} \otimes T^{(k)}_{fg;2} \otimes T^{(k)}_{fg;3} + T^{(k)}_{fg;1} \otimes T^{(k)}_{\{f,g\}_2;2} \otimes T^{(k)}_{fg;3} +$$
$$T^{(k)}_{fg;1} \otimes T^{(k)}_{fg;2} \otimes T^{(k)}_{\{f,g\}_3;3})|| = O(\frac{1}{k})$$

as $k \to \infty$.

Proof. Using (2.10), we get:

$$||ik[\mathbb{T}^{(k)}_f, \mathbb{T}^{(k)}_g] - (T^{(k)}_{\{f,g\}_1;1} \otimes T^{(k)}_{fg;2} \otimes T^{(k)}_{fg;3} + T^{(k)}_{fg;1} \otimes T^{(k)}_{\{f,g\}_2;2} \otimes T^{(k)}_{fg;3} +$$
$$T^{(k)}_{fg;1} \otimes T^{(k)}_{fg;2} \otimes T^{(k)}_{\{f,g\}_3;3})|| \leq$$
$$||ik[T^{(k)}_{f;1}, T^{(k)}_{g;1}] \otimes T^{(k)}_{g;2} T^{(k)}_{f;2} \otimes T^{(k)}_{f;3} T^{(k)}_{g;3} - T^{(k)}_{\{f,g\}_1;1} \otimes T^{(k)}_{fg;2} \otimes T^{(k)}_{fg;3}|| +$$
$$||T^{(k)}_{f;1} T^{(k)}_{g;1} \otimes ik[T^{(k)}_{f;2}, T^{(k)}_{g;2}] \otimes T^{(k)}_{g;3} T^{(k)}_{f;3} - T^{(k)}_{fg;1} \otimes T^{(k)}_{\{f,g\}_2;2} \otimes T^{(k)}_{fg;3}|| +$$
$$||T^{(k)}_{g;1} T^{(k)}_{f;1} \otimes T^{(k)}_{f;2} T^{(k)}_{g;2} \otimes ik[T^{(k)}_{f;3}, T^{(k)}_{g;3}] - T^{(k)}_{fg;1} \otimes T^{(k)}_{fg;2} \otimes T^{(k)}_{\{f,g\}_3;3}|| +$$
$$k||[T^{(k)}_{f;1}, T^{(k)}_{g;1}] \otimes [T^{(k)}_{f;2}, T^{(k)}_{g;2}] \otimes [T^{(k)}_{f;3}, T^{(k)}_{g;3}]||.$$

Each of the first three terms is $O(\frac{1}{k})$ by Lemma 2.4.7, Theorem 1.5.3, Prop. 1.5.6 and Remark 1.5.5. The last term is $O(\frac{1}{k^2})$ by Remark 1.5.5. $\square$

Corollary 2.4.11. *For $f, g \in C^\infty(M)$*

$$||[\mathbb{T}_f^{(k)}, \mathbb{T}_g^{(k)}]|| = O(\frac{1}{k})$$

as $k \to \infty$.

Proof. Follows from Proposition 2.4.10 and Theorem 1.5.3(ii) by triangle inequality. $\square$

Corollary 2.4.12. *For $f, g, h, t \in C^\infty(M)$*

$$||[\mathbb{T}_f^{(k)}, \mathbb{T}_g^{(k)}, \mathbb{T}_h^{(k)}, \mathbb{T}_t^{(k)}]|| = O(\frac{1}{k^2})$$

as $k \to \infty$.

Proof. Follows from equality (2.2) and Corollary 2.4.11 by triangle inequality. $\square$

Proposition 2.4.13. *For $f, g, h, t \in C^\infty(M)$*

$$|| - \frac{k^6}{8}[T_{f;1}^{(k)}, T_{g;1}^{(k)}, T_{h;1}^{(k)}, T_{t;1}^{(k)}] \otimes [T_{f;2}^{(k)}, T_{g;2}^{(k)}, T_{h;2}^{(k)}, T_{t;2}^{(k)}] \otimes [T_{f;3}^{(k)}, T_{g;3}^{(k)}, T_{h;3}^{(k)}, T_{t;3}^{(k)}] -$$

$$T_{\{f,g,h,t\}_1;1}^{(k)} \otimes T_{\{f,g,h,t\}_2;2}^{(k)} \otimes T_{\{f,g,h,t\}_3;3}^{(k)}|| = O(\frac{1}{k^2})$$

as $k \to \infty$.

Proof. For $r = 1, 2, 3$

$$||[T_{f;r}^{(k)}, T_{g;r}^{(k)}, T_{h;r}^{(k)}, T_{t;r}^{(k)}]|| = O(\frac{1}{k^2})$$

as $k \to \infty$ (this follows by triangle inequality from (2.2) and Remark 1.5.5). The statement now follows from Lemma 2.4.7, Theorem 2.4.3 and Theorem 1.5.3 (ii). $\square$

It is natural to ask about asymptotics of $[\mathbb{T}_f^{(k)}, \mathbb{T}_g^{(k)}, \mathbb{T}_h^{(k)}, \mathbb{T}_t^{(k)}]$ for given $f, g, h, t \in C^\infty(M)$. Proposition 2.4.10 dictates the following very technical statement.

Theorem 2.4.14. *For $f_1, f_2, f_3, f_4 \in C^\infty(M)$*

$$||-\frac{k^2}{2}[\mathbb{T}^{(k)}_{f_1}, \mathbb{T}^{(k)}_{f_2}, \mathbb{T}^{(k)}_{f_3}, \mathbb{T}^{(k)}_{f_4}] - \mathbb{W}^{(k)}_{f_1,f_2,f_3,f_4}|| = O(\frac{1}{k})$$

as $k \to \infty$, where

$$\mathbb{W}^{(k)}_{f_1,f_2,f_3,f_4} = T^{(k)}_{\{f_1,f_2,f_3,f_4\}_1;1} \otimes T^{(k)}_{f_1f_2f_3f_4;2} \otimes T^{(k)}_{f_1f_2f_3f_4;3} +$$
$$T^{(k)}_{f_1f_2f_3f_4;1} \otimes T^{(k)}_{\{f_1,f_2,f_3,f_4\}_2;2} \otimes T^{(k)}_{f_1f_2f_3f_4;3} + T^{(k)}_{f_1f_2f_3f_4;1} \otimes T^{(k)}_{f_1f_2f_3f_4;2} \otimes T^{(k)}_{\{f_1,f_2,f_3,f_4\}_3;3} +$$
$$\sum_{\substack{(i,j,m,l)=(1,2,3,4),\\(1,3,2,4),(1,4,2,3)}} \mathrm{sign}(i,j,m,l)\Big[T^{(k)}_{f_if_j\{f_m,f_l\}_1;1} \otimes (T^{(k)}_{f_mf_l\{f_i,f_j\}_2;2} \otimes T^{(k)}_{f_if_jf_mf_l;3} +$$
$$T^{(k)}_{f_if_jf_mf_l;2} \otimes T^{(k)}_{f_mf_l\{f_i,f_j\}_3;3}) + T^{(k)}_{f_mf_l\{f_i,f_j\}_1;1} \otimes$$
$$(T^{(k)}_{f_if_j\{f_m,f_l\}_2;2} \otimes T^{(k)}_{f_if_jf_mf_l;3} + T^{(k)}_{f_if_jf_mf_l;2} \otimes T^{(k)}_{f_if_j\{f_m,f_l\}_3;3}) +$$
$$T^{(k)}_{f_if_jf_mf_l;1} \otimes (T^{(k)}_{f_if_j\{f_m,f_l\}_2;2} \otimes T^{(k)}_{f_mf_l\{f_i,f_j\}_3;3} + T^{(k)}_{f_mf_l\{f_i,f_j\}_2;2} \otimes T^{(k)}_{f_if_j\{f_m,f_l\}_3;3})\Big].$$

Proof. First, we observe: as $k \to \infty$

$$||(ik)^2[\mathbb{T}^{(k)}_{f_i}, \mathbb{T}^{(k)}_{f_j}][\mathbb{T}^{(k)}_{f_m}, \mathbb{T}^{(k)}_{f_l}] - (T^{(k)}_{\{f_i,f_j\}_1;1} \otimes T^{(k)}_{f_if_j;2} \otimes T^{(k)}_{f_if_j;3} +$$
$$T^{(k)}_{f_if_j;1} \otimes T^{(k)}_{\{f_i,f_j\}_2;2} \otimes T^{(k)}_{f_if_j;3} + T^{(k)}_{f_if_j;1} \otimes T^{(k)}_{f_if_j;2} \otimes T^{(k)}_{\{f_i,f_j\}_3;3})$$
$$(T^{(k)}_{\{f_m,f_l\}_1;1} \otimes T^{(k)}_{f_mf_l;2} \otimes T^{(k)}_{f_mf_l;3} + T^{(k)}_{f_mf_l;1} \otimes T^{(k)}_{\{f_m,f_l\}_2;2} \otimes T^{(k)}_{f_mf_l;3} +$$
$$T^{(k)}_{f_mf_l;1} \otimes T^{(k)}_{f_mf_l;2} \otimes T^{(k)}_{\{f_m,f_l\}_3;3})|| = O(\frac{1}{k}).$$

This follows from the elementary inequality

$$||M_1M_2 - N_1N_2|| = ||M_1M_2 - M_2N_1 + M_2N_1 - N_1N_2|| \leq$$
$$||M_2||||M_1 - N_1|| + ||N_1||||M_2 - N_2||$$

by setting

$$M_1 = ik[\mathbb{T}^{(k)}_{f_i}, \mathbb{T}^{(k)}_{f_j}],\ M_2 = ik[\mathbb{T}^{(k)}_{f_m}, \mathbb{T}^{(k)}_{f_l}]$$

$$N_1 = T^{(k)}_{\{f_i,f_j\}_1;1} \otimes T^{(k)}_{f_if_j;2} \otimes T^{(k)}_{f_if_j;3} + T^{(k)}_{f_if_j;1} \otimes T^{(k)}_{\{f_i,f_j\}_2;2} \otimes T^{(k)}_{f_if_j;3} +$$
$$T^{(k)}_{f_if_j;1} \otimes T^{(k)}_{f_if_j;2} \otimes T^{(k)}_{\{f_i,f_j\}_3;3},$$
$$N_2 = T^{(k)}_{\{f_m,f_l\}_1;1} \otimes T^{(k)}_{f_mf_l;2} \otimes T^{(k)}_{f_mf_l;3} + T^{(k)}_{f_mf_l;1} \otimes T^{(k)}_{\{f_m,f_l\}_2;2} \otimes T^{(k)}_{f_mf_l;3} +$$
$$T^{(k)}_{f_mf_l;1} \otimes T^{(k)}_{f_mf_l;2} \otimes T^{(k)}_{\{f_m,f_l\}_3;3},$$

with the use of Theorem 1.5.3(ii), Prop. 2.4.10 and Cor. 2.4.11. Next, using Lemma 2.4.7, Theorem 1.5.3(ii) and Prop. 1.5.6, we get:

$$|| - k^2[\mathbb{T}^{(k)}_{f_i}, \mathbb{T}^{(k)}_{f_j}][\mathbb{T}^{(k)}_{f_m}, \mathbb{T}^{(k)}_{f_l}] - \Big[T^{(k)}_{\{f_i,f_j\}_1\{f_m,f_l\}_1;1} \otimes T^{(k)}_{f_if_jf_mf_l;2} \otimes T^{(k)}_{f_if_jf_mf_l;3} +$$
$$T^{(k)}_{f_if_jf_mf_l;1} \otimes T^{(k)}_{\{f_i,f_j\}_2\{f_m,f_l\}_2;2} \otimes T^{(k)}_{f_if_jf_mf_l;3} + T^{(k)}_{f_if_jf_mf_l;1} \otimes T^{(k)}_{f_if_jf_mf_l;2} \otimes T^{(k)}_{\{f_i,f_j\}_3\{f_m,f_l\}_3;3} +$$
$$T^{(k)}_{f_if_j\{f_m,f_l\}_1;1} \otimes (T^{(k)}_{f_mf_l\{f_i,f_j\}_2;2} \otimes T^{(k)}_{f_if_jf_mf_l;3} + T^{(k)}_{f_if_jf_mf_l;2} \otimes T^{(k)}_{f_mf_l\{f_i,f_j\}_3;3}) +$$
$$T^{(k)}_{f_mf_l\{f_i,f_j\}_1;1} \otimes (T^{(k)}_{f_if_j\{f_m,f_l\}_2;2} \otimes T^{(k)}_{f_if_jf_mf_l;3} + T^{(k)}_{f_if_jf_mf_l;2} \otimes T^{(k)}_{f_if_j\{f_m,f_l\}_3;3}) +$$
$$T^{(k)}_{f_if_jf_mf_l;1} \otimes (T^{(k)}_{f_if_j\{f_m,f_l\}_2;2} \otimes T^{(k)}_{f_mf_l\{f_i,f_j\}_3;3} + T^{(k)}_{f_mf_l\{f_i,f_j\}_2;2} \otimes T^{(k)}_{f_if_j\{f_m,f_l\}_3;3})\Big]|| = O(\frac{1}{k}).$$

After that we note:

$$[\mathbb{T}^{(k)}_{f_1}, \mathbb{T}^{(k)}_{f_2}, \mathbb{T}^{(k)}_{f_3}, \mathbb{T}^{(k)}_{f_4}] = \sum_{\substack{(i,j,m,l)= \\ (1,2,3,4),(1,3,2,4),(1,4,2,3) \\ (3,4,1,2),(2,4,1,3),(2,3,1,4)}} \mathrm{sign}(i,j,m,l)[\mathbb{T}^{(k)}_{f_i}, \mathbb{T}^{(k)}_{f_j}][\mathbb{T}^{(k)}_{f_m}, \mathbb{T}^{(k)}_{f_l}]$$

(see (2.2)). Taking the sum, we get:

$$|| - k^2[\mathbb{T}^{(k)}_{f_1}, \mathbb{T}^{(k)}_{f_2}, \mathbb{T}^{(k)}_{f_3}, \mathbb{T}^{(k)}_{f_4}] - 2\mathbb{W}^{(k)}_{f_1,f_2,f_3,f_4}|| = O(\frac{1}{k}).$$

□

2.5 Conclusions

In section 2.3 we were able to use the Berezin-Toeplitz quantization of the Poisson algebra $(C^\infty(M), \{\cdot,\cdot\})$ (where the bracket is defined using the 1-plectic form ω defined on M) to quantize the Nambu-Poisson algebra $(C^\infty(M), \{\cdot,\cdots,\cdot\})$ (where the bracket is defined using the (2n-1)-plectic[2] structure $\frac{\omega^n}{n!}$). We kept the same Hilbert space $\mathcal{H}$ as is used in Berezin-Toeplitz quantization, but we use the generalized commutator $[\cdot,\cdots,\cdot]$ as the algebraic structure on $End(\mathcal{H})$ rather than the usual commutator $[\cdot,\cdot]$. We prove theorem 2.3.4, which generalizes the well know theorem 1.5.3(i) [BMS].

In section 2.4 we worked on a hyperkähler manifold $(M, \omega_1, \omega_2, \omega_3)$. The manifold M comes with three integral symplectic forms and thus is quantized by Berezin-Toeplitz in three different ways. We combine the three Hilbert spaces of Berezin-Toeplitz quantization in two different ways, by taking their direct sum and by taking their tensor product. We prove theorem 2.4.5 and show that when $dim_{\mathbb{R}}(M) = 4$, the direct sum Hilbert space can be used to quantize M. In theorem 2.1.14, where the three Hilbert spaces are combined by taking their tensor product we extend the Berezin-Toeplitz quantization map to the tensor product and in analogy with theorem 1.5.3(i) [BMS] we compute the $k \to \infty$ asymptotics of the extended mapping. We note that in all cases the map functions→ operators is linear, except for the map $f \to \mathbb{T}_f^{(k)}$.

[2]recall that $dim(M) = 2n$.

Chapter 3

Two additional results

3.1 Introduction

In this chapter we have collected together two additional results. In section 3.3 we discuss the deformation quantization of the volume form Nambu-Poisson structure from chapter 2. In section 3.4 we write down a set of conditions on operators $\hat{F}_1, \ldots, \hat{F}_{2m-1}, \hat{G}_1, \ldots, \hat{G}_{2m} \in End(\mathcal{H})$ (where $\mathcal{H}$ is any complex Hilbert space) that ensures that the generalized commutator $[., \ldots, .]$ will satisfy the fundamental identity of a Nambu-Poisson bracket with respect to these operators.

3.2 Deformation quantization

Let $A = C^\infty(M)[[t]]$, the space of formal power series with coefficients in $C^\infty(M)$. A product $\star$ on A is called a **(formal) star product** if it is an associative $\mathbb{C}$-linear product such that

1. $A/tA \cong C^\infty(M)$, in particular $f \star g \mod t = fg$, for $f, g \in C^\infty(M) \subset C^\infty(M)[[t]]$.

2. $\frac{1}{t}(f \star g - g \star f) \bmod t = -i\{f, g\}$,

where $f, g \in C^\infty(M)$. We can also write

$$f \star g = \sum_{j=0}^{\infty} C_j(f,g)t^j, \tag{3.1}$$

with $C_j(f,g) \in C^\infty(M)$. The C_j should be $\mathbb{C}$-bilinear in f and g. The conditions 1. and 2. can be reformulated as

$$C_0(f,g) = fg, \tag{3.2}$$

and

$$C_1(f,g) - C_1(g,f) = -i\{f,g\} \tag{3.3}$$

The billinearity of the C_j guarantees that the star product will be bilinear. The condition 1. says that the star product is in fact a deformation of the associative algebra $(C^\infty(M), \bullet)$, where $f \bullet g$ is the usual pointwise multiplication of functions [Ger]. Every star product defines a skew symmetric bracket of functions by the formula

$$[f,g]_Q := \frac{1}{t}(f \star g - g \star f). \tag{3.4}$$

Condition 2. is equivalent to the correspondence principle[1] (of quantum mechanics) for the quantum bracket defined by 3.4. That is, 2. says that $[f,g]_Q$ is a deformation of the Poisson algebra $(C^\infty(M), \{,\})$ [Ger].

In the article [MS], Schlichenmaier gives a proof of the following theorem.

Theorem 3.2.1. *[MS][thm 2.2] There exists a unique (formal) star product on* $C^\infty(M)$

$$f \star g \equiv \sum_{j=0}^{\infty} \nu^j C_j(f,g), \tag{3.5}$$

[1] This is the quantization axiom 4.

in such a way that for f,g$\in C^\infty(M)$ *and for every* $N \in \mathbb{N}$ *we have with suitable constants* $K_N(f,g)$ *for all* m

$$||T_f^{(m)}T_g^{(m)} - \sum_{0\le j<N}(\frac{1}{m})^j T_{C_j(f,g)}^{(m)}|| \le K_N(f,g)(\frac{1}{m})^N \tag{3.6}$$

For $N = 1$ 3.6 reads

$$|||T_f^{(m)}T_g^{(m)} - T_{fg}^{(m)}|| = O(\frac{1}{m}) \tag{3.7}$$

For N=2 it reads

$$||T_f^{(m)}T_g^{(m)} - T_{fg}^{(m)} - (\frac{1}{m})T_{C_1(f,g)}^{(m)}|| = O(\frac{1}{m})^N \tag{3.8}$$

from which we can obtain

$$||(T_f^{(m)}T_g^{(m)} - T_{fg}^{(m)} - (\frac{1}{m})T_{C_1(f,g)}^{(m)}) - (T_g^{(m)}T_f^{(m)} - T_{gf}^{(m)} - (\frac{1}{m})T_{C_1(g,f)}^{(m)})|| =$$

$$||[T_f^{(m)}, T_g^{(m)}] - T_{fg}^{(m)} + T_{gf}^{(m)} - \frac{1}{m}(T_{C_1(f,g)}^{(m)} - T_{C_1(g,f)}^{(m)})|| =$$

$$||[T_f^{(m)}, T_g^{(m)}] + \frac{1}{m}T_{i\{f,g\}}^{(m)}|| =$$

$$||[T_f^{(m)}, T_g^{(m)}] + \frac{i}{m}T_{\{f,g\}}|| =$$

$$\frac{||im[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}||}{m} \le$$

$$||T_f^{(m)}T_g^{(m)} - T_{fg}^{(m)} - \frac{1}{m}T_{C_1(f,g)}^{(m)}|| + ||T_g^{(m)}T_f^{(m)} - T_{gf}^{(m)} - \frac{1}{m}T_{C_1(g,f)}^{(m)}|| = O(\frac{1}{m^2})$$

That is

$$||im[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}|| = O(\frac{1}{m}) \tag{3.9}$$

Together 3.7 and 3.9 comprise one part of Theorem 1.5.3 [BMS].

3.3 A deformation of the Nambu bracket

In chapter 2, section 2.3 we showed how to use Berezin-Toeplitz quantization in one kind of n-plectic setting to generalize theorem 1.5.3 from the symplectic setting. As has been explained already, Berezin-Toeplitz quantization leads to a deformation quantization and the Berezin-Toeplitz star product[**?**]. In this section we start to perform this step in the multisymplectic setting of section 2.3. We propose to define a higher order analogue of the Berezin-Toeplitz star product.

Consider the (2n-1)-plectic manifold (M, Ω) obtained from the symplectic manifold (M, ω), where $\Omega = \frac{1}{n!}\omega^n$. Define a star product (the terminology is justified by the next lemma) of 2n functions in $C^\infty(M)$ by the formula

$$\star(f_1, f_2, \ldots, f_{2n-1}, f_{2n}) = \sum_{j=0}^{\infty} t^j D_j(f_1, f_2, \ldots, f_{2n-1}, f_{2n}) \tag{3.10}$$

where

$$D_j(f_1, f_2, \ldots, f_{2n-1}, f_{2n}) := C_j(f_1, f_2) \ldots C_j(f_{2n-1}, f_{2n}) \tag{3.11}$$

and the C_j are the coefficients in the Berezin-Toeplitz star product 3.2.1, then we have

Proposition 3.3.1. *The 2n-ary product defined above has as its classical limit* [2] *the volume form Nambu-Poisson structure.*

1. $D_0(f_1, f_2, \ldots, f_{2n-1}, f_{2n}) = f_1 f_2 \ldots f_{2n-1} f_{2n}$

2. $\sum_{\sigma \in S_{2n}} \epsilon(\sigma) D_1(f_{\sigma(1)}, f_{\sigma(2)}, \ldots, f_{\sigma(2n-1)}, f_{\sigma(2n)}) = n!(-i)^n \{f_1, f_2, \ldots, f_{2n-1}, f_{2n}\}$

Where the bracket $\{., ..., .\} : \bigwedge^{2n} C^\infty(M) \to C^\infty(M)$ is the Nambu-Poisson bracket defined by

[2] The limit $t \to 0$.

$$df_1 \wedge ... \wedge df_{2n} = \{f_1, ..., f_{2n}\}\Omega \tag{3.12}$$

In order to prove the proposition we will use a formula for the 2n-ary bracket defined by 3.12, in terms of the usual Poisson bracket and a (2n-2)-ary bracket. One may define a 2j-ary bracket ($j \leq n$) for any Poisson manifold $(M, \{.,.\})$ of dimension 2n by the formula,

$$\{f_1, \ldots, f_{2j}\} = \frac{1}{2^j j!} \sum_{\sigma \in S_{2j}} \epsilon(\sigma) \prod_{i=1}^{j} \{f_{\sigma(2i-1)}, f_{\sigma(2i)}\} \tag{3.13}$$

In chapter 2 we demonstrated that the (2n)-ary bracket defined by 3.12 agrees with the one defined by 3.13 when $j = n$, so that 3.13 may be regarded as a generalization of the Nambu-Poisson bracket 3.12. It should be noted that for $j < n$ the bracket defined by the formula 3.13 does not satisfy the fundamental identity of the Nambu-Poisson bracket. Those identities are not needed to establish the relation 3.14 that we will need.

$$\begin{aligned}\{f_1, \ldots, f_{2n}\} = \{f_1, f_2, \}\{f_3, \ldots, f_{2n}\} \\ + \sum_{i=3}^{2n-1} (-1)^i \{f_1, f_j\}\{f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{2n}\} \\ + \{f_1, f_{2n}\}\{f_2, \ldots, f_{2n-1}\}\end{aligned} \tag{3.14}$$

For j=3 the formula reads

$$\begin{aligned}\{f_1, f_2, f_3, f_4, f_5, f_6\} = \{f_1, f_2\}\{f_3, f_4, f_5, f_6\} - \{f_1, f_3\}\{f_2, f_4, f_5, f_6\} \\ + \{f_1, f_4\}\{f_2, f_3, f_5, f_6\} - \{f_1, f_5\}\{f_2, f_3, f_4, f_6\} \\ + \{f_1, f_6, \}\{f_2, f_3, f_4, f_5\}\end{aligned} \tag{3.15}$$

3.14 can be established by simply substituting the definition (13) for all of the brackets on both sides of the relation 3.14.

Proof. Proof of proposition 3.3.1

The proof is an induction with base case n=1 provided by 3.3 and involving 3.14 in the induction step.

$$\sum_{\sigma\in S_{2n}} \epsilon(\sigma)C_1(f_{\sigma(1)}, f_{\sigma(2)})\dots C_1(f_{\sigma(2n-1)}, f_{\sigma(2n)})$$

$$= n(C_1(f_1, f_2) - C_1(f_2, f_1))(\sum_{\sigma\in S_{2n-2}} \epsilon(\sigma)C_1(f_{\sigma(3)}, f_{\sigma(4)})\dots C_1(f_{\sigma(2n-1)}, f_{\sigma(2n)}))$$

$$+\sum_{i=3}^{2n-1} n(C_1(f_1, f_i) - C_1(f_i, f_1))(\sum_{\sigma\in S_{2n-2}} \epsilon(\sigma)C_1(f_{\sigma(1)}, f_{\sigma(2)})\dots C_1(f_{\sigma(i-1)}, f_{\sigma(i+1)})\dots C_1(f_{\sigma(2n-1)}, f_{\sigma(2n)}))$$

$$+n(C_1(f_1, f_{2n}) - C_1(f_{2n}, f_1))(\sum_{\sigma\in S_{2n-2}} \epsilon(\sigma)C_1(f_{\sigma(3)}, f_{\sigma(4)})\dots C_1(f_{\sigma(2n-2)}, f_{\sigma(2n-1)}))$$

$$= n!(-i)^n(\{f_1, f_2,\}\{f_3, \dots, f_{2n}\}$$

$$+\sum_{i=3}^{2n-1}(-1)^i\{f_1, f_j\}\{f_2, \dots, f_{i-1}, f_{i+1}, \dots, f_{2n}\}$$

$$+\{f_1, f_{2n}\}\{f_2, \dots, f_{2n-1}\})$$

$$= n!(-i)^n\{f_1, \dots, f_{2n}\}$$

This proves 2., 1. follows from 3.2 and 3.11. □

Remark 3.3.2. The requirement $\sum_{\sigma\in S_4}\epsilon(\sigma)D_1(f_{\sigma(1)}, f_{\sigma(2)}, \dots, f_{\sigma(2n-1)}, f_{\sigma(2n)}) = n!(-i)^n\{f_1, f_2, \dots, f_{2n-1}, f_{2n}\}$ is a generalization of the requirement $C_1(f, g) - C_1(g, f) = -i\{f, g\}$ which we make for binary star products. As we will see by the next definition, this constraint ensures that our star products (which are deformations, of the associative product of two or more functions, vis-a-vis condition 1.) lead to infinitesimal deformations of the 2n-bracket 3.12 (we will denote this family of deformations by $\star[f_1, \dots, f_{2n}]_t$) .

$$\star[f_1,\ldots,f_{2n}]_t := \frac{1}{t}(\sum_{\sigma\in S_{2n}}\epsilon(\sigma)\star(f_{\sigma(1)},\ldots,f_{\sigma(2n)})$$

$$= \sum_{\sigma\in S_{2n}}\epsilon(\sigma)D_1(f_{\sigma(1)},\ldots,f_{\sigma 2n}) + t\sum_{\sigma\in S_{2n}}\epsilon(\sigma)D_2(f_{\sigma(1)}\ldots,f_{\sigma(2n)}) + O(t^2)$$

$$= n!i^n\{f_1,\ldots,f_{2n}\} + t\sum_{\sigma\in S_{2n}}\epsilon(\sigma)D_2(f_{\sigma(1)},\ldots,f_{\sigma(2n)}) + O(t^2)$$

$$= n!i^n\{f_1,\ldots,f_{2n}\} + t\alpha(f_1,\ldots,f_{2n}) + O(t^2)$$

where

$$\alpha : \wedge^{2n}C^\infty(M) \to C^\infty(M)$$

Remark 3.3.3. The star product $\star(.,\ldots,.)$ satisfies some additional properties that are worth mentioning.

1. $\star(1,f_2,\ldots,f_{2n}) = \star(f_2,1,\ldots,f_{2n}) = \cdots = \star(f_2,f_3,\ldots,f_{2n-1},1) = f_2f_3\ldots f_{2n-1}f_{2n}$ This means, in other words, that multiplication by a constant c in classical mechanics corresponds throughout the deformation to multiplication by the constant power series c. . This is one of the usual axioms of quantization. This property is equivalent to $D_k(1,f_2,\ldots,f_{2n}) = D_k(f_2,1,\ldots,f_{2n}) = \cdots = D_k(f_2,f_3,\ldots,f_{2n-1},1) = 0$ for $k \geq 1$, it follows from the corresponding property of 3.5 and the definition of the D_k.

2. A star product of 2n functions should be called **local** if for all $f,g \in C^\infty(M)$, the support $suppD_j(f_1,\ldots,f_{2n})$ is contained in $suppf_1\bigcap\cdots\bigcap suppf_{2n}$ for all $j \in \mathbb{N}_0$. This is the obvious generalization of locality of a star product for n=1. The locality of our star product depends on the locality of the star product defined by Schlichenmaier, see [MS] and the comments therein.

Remark 3.3.4. To establish analogy with deformations of associative algebras in the sense of Gerstenhaber[Ger] we would need a (generalized) associativity property.

3.4 Generalized commutator

In this section we will write down a set of conditions on operators $\hat{F}_1, \ldots, \hat{F}_{2n-1}$, $\hat{G}_1, \ldots, \hat{G}_{2n} \in End(\mathcal{H})$ (where $\mathcal{H}$ is any complex Hilbert space) that ensures that the generalized commutator $[., \ldots, .]$ will satisfy the fundamental identity of a Nambu-Poisson bracket with respect to these operators.. Throughout this section our Hilbert space $\mathcal{H}$ will be finite dimensional with $dim_{\mathbb{R}}(\mathcal{H}) = 2n$ and $m \leq n$.

Recall from our discussion of the geometric formulation of quantum mechanics in chapter one that for any $\hat{F}, \hat{K} \in End(\mathcal{H})$ we have the relation

$$\{F, K\}_\Omega = \frac{1}{i\hbar}\langle[\hat{F}, \hat{K}]\rangle, \tag{3.16}$$

where F, K are the expectation values of $\hat{F}$ and $\hat{K}$ respectively (these are functions $F, K \in C^\infty(\mathcal{H})$) and $\{., .\}_\Omega$ is the natural Poisson structure that can be defined on Hilbert space[AS]. The expectation value of an operator $\hat{F}$ is defined as

$$F(\psi) = \langle\hat{F}\rangle_\psi = \langle\psi, \hat{F}\psi\rangle.$$

Lemma 3.4.1. *For any self-adjoint operator $\hat{G} \in End(\mathcal{H})$, if $\langle\hat{G}\rangle_\psi = 0$ for every $\psi \in \mathcal{H}$ then $\hat{G} = 0$. In other words, if $G = 0$ than $\hat{G} = 0$ (the converse statement is of course true as well).*

Proof. Assume that $\hat{G}$ is not the zero operator. $\hat{G}$ is self-adjoint so it will have at least one non-zero eigenvalue c (which will be real) with non-zero eigenvector ψ, then

$$\langle \psi, \hat{G}\psi \rangle = \langle \psi, c\psi \rangle = c\langle \psi, \psi \rangle = c||\psi||^2 = 0.$$

This last equality contradicts the assumption that both c and ψ are non-zero.

□

Remark 3.4.2. Recall that an operator corresponding to a real valued function is always self-adjoint. From a physical point of view restricting to self-adjoint operators is natural.

Lemma 3.4.3. *For even n the generalized commutator of $2n$ Hermitian (self-adjoint[3]) operators is again Hermitian.*

Proof. In order to prove the lemma first of all we will use the following calculation which demonstrates that when $n = 1$ the commutator of Hermitian operators is skew-Hermitian. Let $A^* = A$ and $B^* = B$ be two Hermitian operators and consider their commutator,

$$[A, B]^* = (AB - BA)^* = (AB)^* - (BA)^* = B^*A^* - A^*B^* = BA - AB = -[A, B]. \tag{3.17}$$

For the general case the calculation goes like this, where $A_i^* = A_i$,

$$\begin{aligned} [A_1, \ldots, A_{2n}]^* &= \frac{1}{2^n} \sum_{\sigma \in S_n} \epsilon(\sigma)([A_{\sigma(1)}, A_{\sigma(2)}] \ldots [A_{\sigma(2n-1)}, A_{\sigma(2n)}])^* \\ &= \frac{1}{2^n} \sum_{\sigma \in S_n} \epsilon(\sigma)[A_{\sigma(2n-1)}, A_{\sigma(2n)}]^* \ldots [A_{\sigma(1)}, A_{\sigma(2)}]^* \end{aligned}$$

[3]For the finite dimensional vector spaces that we are considering these notions coincide. When the vector spaces are infinite dimensional the domain of definition of a operator need not be the entire space and these notions diverge.

$$= \frac{1}{2^n} \sum_{\sigma \in S_n} \epsilon(\sigma)(-1)^n [A_{\sigma(2n-1)}, A_{\sigma(2n)}] \dots [A_{\sigma(1)}, A_{\sigma(2)}]$$
$$= \frac{1}{2^n} \sum_{\sigma \in S_n} \epsilon(\sigma)(-1)^n [A_{\sigma(1)}, A_{\sigma(2)}] \dots [A_{\sigma(2n-1)}, A_{\sigma(2n)}]$$
$$= (-1)^n [A_1, \dots, A_{2n}]$$

This proves the lemma.

$\square$

We would like to compare the generalized commutator $[., \dots, .]$ to the Nambu-Poisson bracket $\{., \dots, .\}_\Omega$, these cannot be compared directly because one of them is an operator on Hilbert space and the other is a function on Hilbert space. We can instead compare the expectation value $\langle [., \dots, .] \rangle$ which is a function on Hilbert space to the Nambu Poisson bracket. Consider the difference

$$\mathcal{D}(F_1, \dots, F_{2n}) \equiv \{F_1, \dots, F_{2n}\}_\Omega - \frac{1}{(i\hbar)^n} \langle [\hat{F}_1, \dots, \hat{F}_{2n}] \rangle \tag{3.18}$$

Now, the fundamental identity for $\{., \dots, .\}_\Omega$ with respect to the operators $\hat{F}_1, \dots, \hat{F}_{2n-1}$, $\hat{G}_1, \dots, \hat{G}_{2n}$ reads,

$$\{F_1, \dots, F_{2n-1}, \{G_1, \dots, G_{2n}\}\} - \{\{F_1, \dots, F_{2n-1}, G_1\}, G_2, \dots, G_{2n}\}$$

$$-\{G_1, \{F_1, \dots, F_{2n-1}, G_2\}, G_3, \dots, G_{2n}\} - \dots - \{G_1, \dots, G_{2n-1}, \{F_1, \dots, F_{2n-1}, G_{2n}\}\} = 0$$

Where the F_i and the G_i are expectation values of self-adjoint operators $\hat{F}_i$ and $\hat{G}_i$ respectively. When $\mathcal{D}(G_1, \dots, G_{2n}) = 0$ and $\mathcal{D}(F_1, \dots, F_{2n-1}, G_i) = 0$ for $i \in \{1, \dots, 2n\}$, we may substitute from 3.18 to get,

$$\frac{1}{(i\hbar)^n} (\{F_1, \dots, F_{2n-1}, \langle [\hat{G}_1, \dots, \hat{G}_{2n}] \rangle\} - \{\langle [\hat{F}_1, \dots, \hat{F}_{2n-1}, \hat{G}_1] \rangle, G_2, \dots, G_{2n}\}$$

$$-\{G_1, \langle[\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_2]\rangle, G_3, \ldots, G_{2n}\} - \cdots - \{G_1, \ldots, G_{2n-1}, \langle[\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_{2n}]\rangle\}) = 0.$$

If furthermore $\mathcal{D}(F_1, \ldots, F_{2n-1}, \langle[\hat{G}_1, \ldots, \hat{G}_{2n}]\rangle) = 0$, $\mathcal{D}(\langle[\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_1]\rangle, G_2, \ldots, G_{2n}) = 0$, $\mathcal{D}(G_1, \langle[\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_2]\rangle, G_3, \ldots, G_{2n}) = 0, \ldots, \mathcal{D}(G_1, \ldots, G_{2n-1}, \langle[\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_{2n}]\rangle) = 0$, we may further substitute from 3.18 to get,

$$\frac{1}{(i\hbar)^{2n}}(\langle[\hat{F}_1, \ldots, \hat{F}_{2n-1}, [\hat{G}_1, \ldots, \hat{G}_{2n}]]\rangle - \langle[[\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_1], \hat{G}_2, \ldots, \hat{G}_{2n}]\rangle$$

$$-\langle[\hat{G}_1, [\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_2], \hat{G}_3, \ldots, \hat{G}_{2n}]\rangle - \cdots - \langle[\hat{G}_1, \ldots, \hat{G}_{2n-1}, [\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_{2n}]]\rangle) = 0.$$

We may drop the overall constant and because expectation is linear we may bring the angled brackets to the outside to get,

$$\langle[\hat{F}_1, \ldots, \hat{F}_{2n-1}, [\hat{G}_1, \ldots, \hat{G}_{2n}]] - [[\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_1], \hat{G}_2, \ldots, \hat{G}_{2n}]$$

$$-[\hat{G}_1, [\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_2], \hat{G}_3, \ldots, \hat{G}_{2n}] - \cdots - [\hat{G}_1, \ldots, \hat{G}_{2n-1}, [\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_{2n}]]\rangle = 0.$$

The expression inside the angled brackets is a Hermitian operator whenever n is even by lemma 3.4.3, so we may apply lemma 3.4.1 to drop the angled brackets and obtain the fundamental identity for the generalized commutator.

$$[\hat{F}_1, \ldots, \hat{F}_{2n-1}, [\hat{G}_1, \ldots, \hat{G}_{2n}]] - [[\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_1], \hat{G}_2, \ldots, \hat{G}_{2n}]$$

$$-[\hat{G}_1, [\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_2], \hat{G}_3, \ldots, \hat{G}_{2n}] - \cdots - [\hat{G}_1, \ldots, \hat{G}_{2n-1}, [\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_{2n}]] = 0.$$

We have proven the following proposition.

Proposition 3.4.4. *Let $\mathcal{H}$ be a complex Hilbert space with $dim_{\mathbb{R}}(\mathcal{H}) = 2n$ and let $\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_1, \ldots, \hat{G}_{2n} \in End(\mathcal{H})$ be self-adjoint operators. Assume also that m is even and that the difference function 3.18 is zero for all of the following combinations:*

1. $\mathcal{D}(G_1, \ldots, G_{2n}) = 0$ *and* $\mathcal{D}(F_1, \ldots, F_{2n-1}, G_i) = 0$ *for* $i \in \{1, \ldots, 2n\}$.

2. $\mathcal{D}(F_1, \ldots, F_{2n-1}, \langle[\hat{G}_1, \ldots, \hat{G}_{2n}]\rangle) = 0,$
$\mathcal{D}(\langle[\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_1]\rangle, G_2, \ldots, G_{2n}) = 0,$
$\mathcal{D}(G_1, \langle[\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_2]\rangle, G_3, \ldots, G_{2n}) = 0, \ \ldots,$
$\mathcal{D}(G_1, \ldots, G_{2n-1}, \langle[\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_{2n}]\rangle) = 0$

Than the generalized commutator $[., \ldots, .]$ *satisfies the fundamental identity with respect to the operators* $\hat{F}_1, \ldots, \hat{F}_{2n-1}, \hat{G}_1, \ldots, \hat{G}_{2n} \in End(\mathcal{H})$.

Remark 3.4.5.

In order to be able to say something about when we will expect $\mathcal{D} = 0$ we can write both of $\{., \ldots, .\}_\Omega$ and $[., \ldots, .]$ in terms of the brackets $\{., .\}_\Omega$ and $[., .]$ respectively and apply the relation 3.16.

$$\begin{aligned}\mathcal{D} &\equiv \{F_1, \ldots, F_{2n}\}_\Omega - \frac{1}{(i\hbar)^n}\langle[\hat{F}_1, \ldots, \hat{F}_{2n}]\rangle \\ &= \frac{1}{2^n n!}\sum_{\sigma\in S_n}\epsilon(\sigma)\{F_{\sigma(1)}, F_{\sigma(2)}\}_\Omega \ldots \{F_{\sigma(2n-1)}, F_{\sigma(2n)}\}_\Omega \\ &-\frac{1}{(i\hbar)^n}\langle\frac{1}{2^n}\sum_{\sigma\in S_n}\epsilon(\sigma)[\hat{F}_{\sigma(1)}, \hat{F}_{\sigma(2)}] \ldots [\hat{F}_{\sigma(2n-1)}, \hat{F}_{\sigma(2n)}]\rangle\end{aligned}$$

After substituting from 3.16 and bringing the angled brackets to the inside of the sum using linearity of the expectation we get,

$$\begin{aligned}\mathcal{D} = &\frac{1}{2^n n!(i\hbar)^n}\sum_{\sigma\in S_n}\epsilon(\sigma)\langle[\hat{F}_{\sigma(1)}\hat{F}_{\sigma(2)}]\rangle \ldots \langle[\hat{F}_{\sigma(2n-1)}, \hat{F}_{\sigma(2n)}]\rangle \\ &-\frac{1}{2^n(i\hbar)^n}\sum_{\sigma\in S_n}\epsilon(\sigma)\langle[\hat{F}_{\sigma(1)}, \hat{F}_{\sigma(2)}] \ldots [\hat{F}_{\sigma(2n-1)}, \hat{F}_{\sigma(2n)}]\rangle.\end{aligned}$$

From this we see that the difference $\mathcal{D}$ is a sum of differences of the form

$$\epsilon(\sigma)(\langle[\hat{F}_{\sigma(1)}, \hat{F}_{\sigma(2)}]\rangle \dots \langle[\hat{F}_{\sigma(2n-1)}, \hat{F}_{\sigma(2n)}]\rangle - \langle[\hat{F}_{\sigma(1)}, \hat{F}_{\sigma(2)}] \dots [\hat{F}_{\sigma(2n-1)}, \hat{F}_{\sigma(2n)}]\rangle)$$

When n=2 this is exactly the covariance of the operators $[\hat{F}_{\sigma(1)}, \hat{F}_{\sigma(2)}]$ and $[\hat{F}_{\sigma(3)}, \hat{F}_{\sigma(4)}]$. For $n \geq 3$ this suggests a generalization of covariance to more than two operators.

3.5 Conclusions

The Berezin Toeplitz operator quantization leads to a deformation quantization, the Berezin Toeplitz star product, theorem 3.2.1. In equation (3.10) we define a higher order analog of the Berezin Toeplitz star product. We prove in proposition 3.3.1 that this higer order star product reduces to the Nambu-Poisson algebra $(C^\infty(M), \{\cdot, \cdots, \cdot\})$ in the limit $t \to \infty$.

In section 3.4 we investigate the generalized commutator $[\cdot, \cdots, \cdots]$, which satisfies the generalized Jacobi identity. In proposition 3.4.4 we determine conditions on operators $\hat{F}_1, \dots, \hat{F}_{2n-1}$, $\hat{G}_1, \dots, \hat{G}_{2n} \in End(\mathcal{H})$ (where $\mathcal{H}$ is any complex Hilbert space) that ensures that the generalized commutator will satisfy the fundamental identity.

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