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Combinatorial Polynomial Identity Theory

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by

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Abstract

This dissertation consists of two parts. Part I examines certain Burnside-type conditions on the multiplicative semigroup of an (associative unital) algebra A.

A semigroup S is called n-collapsing if, for every $a_1, \ldots, a_n \in S$, there exist functions $f \neq g$ on the set $\{1, 2, \ldots, n\}$ such that

$$s_{f(1)} \cdots s_{f(n)} = s_{g(1)} \cdots s_{g(n)}.$$

If f and g can be chosen independently of the choice of $s_1, \ldots, s_n$, then S satisfies a semigroup identity. A semigroup S is called n-rewritable if f and g can be taken to be permutations. Semple and Shalev extended Zelmanov's Fields Medal writing solution of the Restricted Burnside Problem by proving that every finitely generated residually finite collapsing group is virtually nilpotent.

The primary result of Part I is that the following conditions are equivalent for every algebra A over an infinite field: the multiplicative semigroup of A is collapsing, A satisfies a multiplicative semigroup identity, and A satisfies an Engel identity: $[x,_m y] = 0$. Furthermore, in this case, A is locally (upper) Lie nilpotent. It is also shown that, if the multiplicative semigroup of A is rewritable, then A must be commutative.

In Part II of this dissertation, we study algebraic analogues to well-known problems of Philip Hall on the verbal and marginal subgroups of a group. We begin by proving two algebraic analogues of the Schur-Baer-Hall Theorem: if G is a group such that $G/\mathcal{Z}_n(G)$ is finite, where $\mathcal{Z}_n(G)$ is the n^{th} higher centre of G, then the $(n+1)^{\text{st}}$ term, $\gamma_{n+1}(G)$, of the lower central series of G is also finite; conversely, if $\gamma_{n+1}(G)$ is finite, then so is $G/\mathcal{Z}_{2n}(G)$. Next, we prove results of a more general type.

Given an algebra A and a polynomial f, we define the verbal subspace, $\mathcal{S}_A(f)$, of A to be spanned by the set of f-values in A, the verbal subalgebra, $\mathcal{A}_A(f)$, and the verbal ideal, $\mathcal{I}_A(f)$, of A to be generated by the set of f-values in A. We also define the marginal subspace $\widehat{\mathcal{S}_A}(f)$ of A to be the set of all elements $z \in A$ such that

$$f(b_1, \ldots, b_{i-1}, b_i + \alpha z, b_{i+1}, \ldots, b_n) = f(b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_n),$$

for all $i = 1, 2, \ldots, n$, $b_1, \ldots, b_n \in A$, and $\alpha \in K$. Furthermore, we define the marginal subalgebra, $\widehat{\mathcal{A}}_A(f)$, and the marginal ideal, $\widehat{\mathcal{I}}_A(f)$, to be the largest subalgebra, respectively, largest ideal, of A contained in $\widehat{\mathcal{S}}_A(f)$. We consider the following problems:

1. If $\widehat{\mathcal{S}}_A(f)$ is of finite codimension in A, is $\mathcal{S}_A(f)$ finite-dimensional?

2. If $\mathcal{S}_A(f)$ is finite-dimensional, is $\widehat{\mathcal{S}}_A(f)$ of finite codimension in A?

3. If $\mathcal{S}_A(f)$ is finite-dimensional, is $\mathcal{A}_A(f)$ or $\mathcal{I}_A(f)$ finite-dimensional?

4. If $A/\widehat{\mathcal{S}}_A(f)$ is finite-dimensional, is $A/\widehat{\mathcal{A}}_A(f)$ or $A/\widehat{\mathcal{I}}_A(f)$ finite-dimensional?

Keywords: Multiplicative semigroup, collapsing, rewritable, Engel identity, upper Lie nilpotent, verbal and marginal subspaces.

Co-Authorship Statement

This dissertation consists of two parts. Part I is based in part on a paper entitled: 'Multiplicatively collapsing and rewritable algebras', to appear in *Proc. Amer. Math. Soc.* My co-authors are my supervisor, Professor David Riley, and Professor Eric Jespers from Vrije Universiteit Brussel, Belgium. The project was initiated by Professor Riley and myself, and we proved the key lemma together. Later, Professor Jespers added some ideas using idempotents to help bring the paper into its final form. All three authors made important intellectual contributions.

Part II is based in part on a preprint entitled: 'Relationships between the canonical descending and ascending central series of an associative algebra'. My co-author is Professor Riley. While the idea for this project was Professor Riley's, the intellectual contributions made for the paper are roughly equal. I was the principal author of the paper.

Acknowlegements

As my graduate education nears its end, it is a pleasure to think back at these past four years and remember all those people who have made this part of my life so enjoyable.

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I would like also to thank my father, Khalil Shahada, and all of my brothers and sisters. They were always supportive and encouraged me with their best wishes.

Finally, I would like to thank my husband, Ahmed. He was always there cheering me up and stood by my side through the good, as well as, the bad times.

Dedication

This dissertation is dedicated to my mother

”Sohayla El-Mugari”.

I wish you were here with me to witness what I have done just becouse of you.

Contents

List of Abbreviations and Symbols

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$	Set of integers, positive integers, rational numbers
$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$	The binomial coefficient ($n \geq k$ are positive integers)
G	Used to represent a group
$H \leq G$	H is a subgroup of G
$N \lhd G$	N is a normal subgroup of G
$\mathbb{Z}_n$	$\mathbb{Z}/n\mathbb{Z}$
$\lvert S \rvert$	Cardinality of a set S
$\lvert G : H \rvert$	Index of the subgroup H in the group G
$\mathrm{Aut}(G)$, $\mathrm{Inn}(G)$	Automorphism group, inner automorphism group of G
K	Used to represent a field
$\mathrm{char}(K)$	The characteristic of K
F	Used to represent a field extension of K
$K(\alpha)$	Simple field extension of K
$\mathbb{F}_{p^k}$	A field of cardinality p^k
$\mathbb{F}_{p^\infty}$	$\bigcup_{k\geq 1} \mathbb{F}_{p^k}$
$\mathcal{V}$	Used to represent a variety of (associative) algebras
A	Used to represent an (associative) algebra
A_1	The unital hull of a (nonunital) algebra A
$A^\times$	group of units of an algebra A
$[A]$	The Lie algebra associated with an algebra A
$(A, \cdot)$	The multiplicative semigroup of an algebra A
$\mathcal{Z}(G)$, $\mathcal{Z}(A)$	The centre of a group G, the centre of an algebra A
$\mathcal{J}(A)$	The Jacobson radical of an algebra A
L	Used to represent a Lie algebra

$M_n(K)$	The algebra of all $n \times n$ matrices over a field K
$U_n(K)$	The subalgebra of all $n \times n$ upper triangular matrices over a field K
e_{ij}	An $n \times n$ matrix with 1 in the position (i, j) and zero elsewhere
$R_2(K) = Ke_{11} + Ke_{12}$	Subalgebra of $M_2(K)$
$C_2(K) = Ke_{11} + Ke_{21}$	Subalgebra of $M_2(K)$
$B(K, F, \sigma)$	Subalgebra of $M_2(F)$; $K \subseteq F$ is a field extension
X	Used to represent a set of indeterminates
$\mathcal{A}\langle X\rangle$	The free (associative) algebra on the set X
$\mathcal{L}\langle X\rangle$	The free Lie algebra on the set X
$K[x]$	The ring of polynomials in x over a field K
$K(x)$	The field of fractions of $K[x]$
PI-algebra	Polynomial identity algebra
S_n, A_n	Symmetric, alternating groups of degree n
(x, y)	$x^{-1}y^{-1}xy$
x^y	$y^{-1}xy$
$G' = (G, G)$	Derived subgroup of a group G
$[a, b]$	The Lie bracket of a and b
$[\![a, b, c]\!] = [a, b]c$	The Jennings triple product of a, b and c
$g_n(x, y_1, z_1, \ldots, y_n, z_n)$	The polynomial $[\![x, y_1, z_1, \ldots, y_n, z_n]\!]$
$\mu_n(B, A) = [\![\mu_{n-1}(B, A), A, A]\!]$	Defined for a subspace B of an algebra A, $n \geq 2$ and $\mu_1(B, A) = [\![B, A, A]\!]$
$a \circ b = a + b + ab$	The adjoint operation
$(A, \circ)$	The adjoint semigroup of an algebra A
(X_1, X_2)	The commutator subgroup of the sets X_1 and X_2
$[X_1, X_2]$	The additive subgroup generated by $\{[x_1, x_2] \mid x_1 \in X_1, x_2 \in X_2\}$

$f_n(x_1, \ldots, x_{n+1})$	The polynomial $x_1 \cdots x_{n+1}$
G^n, A^n	The n^{th} power of a group G, the associative n^{th} power of an algebra A
$\gamma_n(G)$, $\gamma_n(A)$	The n^{th} term of the lower central series of a group G, the n^{th} term of the lower central series of an algebra A
$A^{[n]}$	The associative ideal generated by $\gamma_n(A)$
$A^{(n)}$	The n^{th} Lie power of an algebra A
$\mathcal{Z}_n(G)$, $\mathcal{Z}_n(A)$	The n^{th} higher centre of a group G, the n^{th} higher centre of an algebra A
$\mathrm{Ann}^n(A)$	The ideal of an algebra A given by $\mathrm{Ann}^n(A)/\mathrm{Ann}^{n-1}(A) = \mathrm{Ann}(A/\mathrm{Ann}^{n-1}(A))$
$\mathcal{Z}_{[n]}(A)$	The largest associative ideal of an algebra A contained in $\mathcal{Z}_n(A)$
$F^{(n)}(A)$	The n^{th} strong centre of an algebra A
$\theta(G)$	The verbal subgroup of G determined by a word θ
$\theta^*(G)$	The marginal subgroup of G determined by a word θ
$f(A)$	Set of values of a polynomial f in an algebra A
$\mathcal{S}_A(f)$, $\mathcal{A}_A(f)$, $\mathcal{I}_A(f)$	verbal subspace, verbal subalgebra, verbal ideal of A determined by f
$\widehat{\mathcal{S}}_A(f)$, $\widehat{\mathcal{A}}_A(f)$, $\widehat{\mathcal{I}}_A(f)$	marginal subspace, marginal subalgebra, marginal ideal of A determined by f

Part I

Multiplicatively Collapsing and Rewritable Algebras

Chapter 1

Burnside-Type Problems

1.1 The Burnside Problems for Groups

Recall that a *periodic group* G is a group in which each element has finite order. Notice that, if there is an $n \in \mathbb{N}$ such that $g^n = 1$, for all $g \in G$, then G is a periodic group of bounded exponent. The minimal such n is called the *exponent* of G.

In 1902, Burnside ([7]) formulated his famous problems for periodic groups which has been one of the main driving forces of the theory of infinite groups for a long time.

The General Burnside Problem 1.1.1 *Is it true that every finitely generated periodic group is finite?*

The Burnside Problem 1.1.2 *Is it true that every finitely generated group of bounded exponent is finite?*

After many unsuccessful attempts to obtain a proof in the late 30s-early 40s, the following weaker version of the Burnside Problem was studied: is it true that there are only finitely many m-generated finite groups of exponent n? In other words, the question is whether there exists a universal finite m-generated group of exponent n having all other finite m-generated groups of exponent n as homomorphic images. Later (thanks to Magnus ([38])) this question became known as the *Restricted Burnside Problem*.

The Restricted Burnside Problem 1.1.3 *If G is a finite, m-generated group of exponent n, does there exist a bound for the order of G in terms of m and n?*

In 1964, Golod and Shafarevich gave a negative answer to the General Burnside Problem ([12]). Since then, a considerable array of infinitely generated periodic groups was constructed by other authors (*cf.* Alyoshin [2], Suschansky [66], Grigorchuk [13], Gupta-Sidki [14]).

In 1968, Novikov and Adian ([45]) constructed counterexamples to the Burnside Problem for groups of odd exponents $n > 4381$ (now for odd exponents $n > 115$, *cf.* Lysenok [37]). Olshansky ([46]) showed how wildly periodic groups may behave.

At the same time there were two major reasons to believe that the Restricted Burnside Problem would have a positive solution. One of these reasons was the following reduction theorem obtained by Hall and Higman ([21]).

Theorem 1.1.4 *Let $n = p_1^{k_1} \cdots p_r^{k_r}$, where p_i are distinct prime numbers, $k_i \geq 1$, and assume that the following statements hold.*

1. *The Restricted Burnside Problem for groups of exponents $p_i^{k_i}$ has a positive solution.*

2. *There are only a finite number of finite simple groups of exponent n.*

3. *The factor group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is solvable for any finite simple group of exponent n.*

Then the Restricted Burnside Problem for groups of exponent n also has positive solution.

Another reason was the close relation of the problem to *Lie algebras* satisfying the *Engel identity*. Using this approach, Kostrikin ([31]) proved that for any prime p, the order of an m-generated finite group G of exponent p is bounded by a function $f(m, p)$. Later, using also a ring theory approach, Zelmanov ([72],[73]) completed Kostrikin's proof by finding such a function for any group G of exponent p^{α}.

Zelmanov's Solution to the Resrtricted Burnside Problem 1.1.5 *The Restricted Burnside Problem has a positive solution.*

1.2 The Kurosh-Levitzki Problems for Algebras

Definition 1.2.1 *An associative algebra A over a field K is called* algebraic*, if for every $a \in A$, there exists nontrivial $f_a(x) \in K[x]$ such that $f_a(a) = 0$.*

Recall that a (nonunital) algebra A is *nil of bounded index* if there exists an $n \in \mathbb{N}$ such that $a^n = 0$, for all $a \in A$; the algebra A is *nilpotent of class $\leq n$* if, for all $a_1, \ldots, a_n \in A$, $a_1 \cdots a_n = 0$.

Kurosh ([33]) and, independently, Levitzki ([36]), formulated two problems for algebras, which were similar to Burnside Problems.

The Kurosh-Levitzki Problems 1.2.2 *Let A be a finitely generated associative algebraic algebra.*

1. General version: *Is A necessarily finite-dimensional? In particular, if A is nil, is it nilpotent?*

2. Bounded version: *If every element of A is algebraic of bounded degree, is A finite-dimensional? If A is nil of bounded index, is it nilpotent?*

Let $f(x_1, \ldots, x_n)$ be a polynomial in the free algebra on the set $\{x_1, x_2, \ldots\}$ over a field K. Recall that f is called *homogeneous* if each f-monomial is of the same degree in each indeterminate (where this degree may depend upon the indeterminate). In particular, each f-monomial has the same degree. By collecting together the f-monomials of given degree in each indeterminate, we can express a given polynomial f in a natural way as a sum of homogeneous polynomials; these are the *homogeneous components* of f. Moreover, f is called *multilinear* if it is linear in each of its indeterminates. In other words,

$$f(x_1, \ldots, x_n) = \textstyle\sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)},$$

for some $\alpha_\sigma \in K$, where S_n is the symmetric group of degree n.

Golod and Shafarevich ([12]) gave a counterexample to the General Kurosh-Levitzki Problem using *class field tower theory*. Actually, this counterexample was used to construct the first counterexample to the General Burnside Problem.

Golod-Shafarevich Theorem 1.2.3 *Let A be the free algebra over a field K in $n = d + 1$ non-commuting variables x_i. Let J be the 2-sided ideal of A generated by homogeneous elements f_j of A of degree d_j with*

$$2 \leq d_1 \leq d_2 \leq \cdots,$$

where d_j tends to infinity. Let r_i be the number of d_j equal to i. Let $B = A/J$, a graded algebra. Let $b_j = \dim B_j$. The fundamental inequality of Golod and Shafarevich states that

$$b_j \geq nb_{j-1} - \textstyle\sum_{i=2}^{j} b_{j-i} r_i.$$

As a consequence:

1. *B is infinite-dimensional if $r_i \leq d^2/4$, for all i.*

2. *If B is finite-dimensional, then $r_i > d^2/4$, for some i.*

Corollary 1.2.4 *For each prime p, there is an infinite group G generated by three elements in which each element has order a power of p.*

Concerning the Bounded Kurosh-Levitzki Problem, if the characteristic of the field is 0 or sufficiently large, the Dubnov-Ivanov-Nagata-Higman Theorem (see [8]) gives that the algebra is nilpotent even without the assumption of finite generation.

Dubnov-Ivanov-Nagata-Higman Theorem 1.2.5 *Let A be a nonunital associative algebra over a field K such that $\mathrm{char}(K) = 0$. If A satisfy the polynomial identity $x^k = 0$, then there exists an integer $d = d(k)$ such that A is nilpotent of class d.*

1.3 The Solution of the Bounded Kurosh-Levitzki Problem for Associative PI-algebras

Definition 1.3.1 *Let $X = \{x_1, x_2, \ldots\}$ be a countably infinite set of noncommuting indeterminates. Let $f = f(x_1, \ldots, x_n)$ be a polynomial in the free associative algebra on X and let A be any associative algebra. We say that $f = 0$ is a* polynomial identity *for A if*

$$f(a_1, \ldots, a_n) = 0,$$

for arbitrary $a_1, \ldots, a_n \in A$. A is called a PI-algebra if A has a nontrivial polynomial identity.

Algebras with polynomial identities generalize commutative and finite-dimensional algebras. This generalization is not only formal. PI-algebras share many structural properties with commutative and finite-dimensional algebras (see chapter 8 in [8], for example).

Nil algebras of bounded index satisfy an identity of the form $x^n = 0$, for some positive integer n. Similarly, if all elements of the algebra A are algebraic of bounded degree n, then $1, a, a^2, \ldots, a^n$ are linearly dependent for any $a \in A$, and this implies that A satisfies the *identity of algebraicity*

$$\textstyle\sum_{\sigma \in S_{n+1}} (-1)^\sigma x^{\sigma(0)} y_1 x^{\sigma(1)} y_2 \cdots y_n x^{\sigma(n)} = 0,$$

where S_{n+1} acts on the set $\{0, 1, \ldots, n\}$. Hence, every algebra A that satisfies the hypotheses of the Bounded Kurosh-Levitzki Problem is, in fact, a PI-algebra. Therefore, it makes sense to ask whether the Kurosh-Levitzki Problem has a positive solutions for PI-algebras.

The General Kurosh-Levitzki Problem for PI-algebras 1.3.2 *If A is a finitely generated algebraic (respectively, nil) PI-algebra, is A finite dimensional (respectively, nilpotent)?*

For nil algebras of bounded index the problem was answered positively by Levitzki ([36]), and the general problem for algebraic PI-algebras by Kaplansky ([29]). Both proofs involved structure theory of rings and can be found in Herstein's book ([24]).

Levitzki's Theorem 1.3.3 *Let A be a finitely generated nil algebra of bounded index. Then A is nilpotent.*

Kaplansky's Theorem 1.3.4 *Let A be a finitely generated algebraic PI-algebra. Then A is finite-dimensional.*

In 1957, Shirshov ([64]) suggested another, purely combinatorial, direct approach to the Kurosh-Levitzki Problem for PI-algebras. In fact, he showed, using the concept of the *height of a finitely generated algebra*, that the theorems of Levitzki and Kaplansky follow from his theorem.

Definition 1.3.5 *Let A be an algebra generated by $a_1, \ldots, a_d$, and let H be a finite set of monomials in $a_1, \ldots, a_d$. One says that A is of* height h with respect to H *if h is the minimal integer with the property that, as a vector space, A is spanned by the monomials*

$$u_{i_1}^{j_1} \cdots u_{i_k}^{j_k},$$

such that $u_{i_1}, \ldots, u_{i_k} \in H$ and $k \leq h$.

Shirshov's Height Theorem 1.3.6 *Let A be a PI-algebra satisfying a polynomial identity of degree n and generated by elements $a_1, \ldots, a_d$. Then A is of finite height with respect to the set of all monomials $a_{i_1} \cdots a_{i_k}$ of length $k < n$.*

As we mentioned above, we can apply Shirshov's Height Theorem to obtain the results of Levitzki and Kaplansky on the General Kurosh-Levitzki Problem for PI-algebras. Note that the statements below do not require the nil or the algebraic assumption of all elements of the algebra as in the original results of Levitzki and Kaplansky.

Corollary 1.3.7 *Let A be a PI-algebra satisfying a polynomial identity of degree n and let A be generated by the elements $a_1, \ldots, a_d$. Let H be the (finite) set of all products $a_{i_1} \cdots a_{i_k}$, $k < n$. Then the following statements holds.*

1. *If every element of H is nil, then A is nilpotent (with bounded index with respect to n).*

2. *If every element of H is algebraic, then A finite-dimensional (of bounded dimension with respect to n).*

1.4 Lie Algebras Satisfying an Engel Identity

In this section we will consider an analogue of Levitzki's Theorem for *Lie algebras* satisfying an *Engel identity*.

Definition 1.4.1

1. *A* Lie algebra *is an algebra L together with a bilinear binary operation $[\cdot,\cdot] : L \times L \to L$ which satisfies the following axioms: for every $a, b, c \in L$,*

$$[a, a] = 0, \quad \textit{(the anticommutative law)}$$

and

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0, \quad \textit{(the Jacobi identity).}$$

2. *A Lie algebra L is said to be* Engelian *or simply* Engel *if there exists $m \in \mathbb{N}$ such that it satisfies the polynomial identity*

$$[x,_m y] := [x, \underbrace{y, y, \ldots, y}_{m}] = 0.$$

3. *Let L be a Lie algebra. Set $\gamma_1(L) = L$ and recursively define $\gamma_n(L) = [\gamma_{n-1}(L), L]$, for each $n \geq 2$. Then L is called* nilpotent of class c, *if $\gamma_{c+1}(L) = 0$ for some $c \in \mathbb{N}$ and c is minimal.*

Notice that, throughout Part I, all Lie commutators $[x_1, x_2, \ldots, x_n]$ are assumed to be *left normed* Lie commutators.

In Lie algebras over a field of characteristic zero, the Engel condition implies nilpotence (not just local nilpotence). This remarkable result was proved by Zelmanov in [71]. Although not all m-Engel Lie algebras over a field of positive characteristic are nilpotent, Zelmanov proved this to be the case whenever the Lie algebra is finitely generated ([72, 73]).

Zelmanov's Theorem for Engelian Lie Algebras 1.4.2

1. *Let L be a Lie algebra over a field K of characteristic 0. If L is Engel, then L is nilpotent.*
2. *Let L be a finitely generated Lie algebra over a field K of positive characteristic. If L is Engel, then L is nilpotent.*

As we mentioned before, Zelmanov used (a stronger version of) part (2) of the above result to solve the Restricted Burnside Problem for groups.

1.5 Associative Algebras Satisfying an Engel Identity

Recall that an associative algebra A has a natural Lie structure given by the Lie bracket

$$[a, b] = ab - ba.$$

To avoid ambiguity, an associative algebra A is said to be *Lie nilpotent* if it is nilpotent when viewed as a Lie algebra. It is known that there exist Engelian Lie algebras arising from associative algebras over fields of positive characteristic that are not Lie nilpotent ([56]). Perhaps the simplest example is the tensor square of the (nonunital) Grassmann algebra of a countably-infinite-dimensional vector space over a field of characteristic $p > 2$. In fact, it is nil of bounded index $2p$ but not Lie nilpotent (see [48, 52]).

Definition 1.5.1

1. *For an associative algebra A, we set* $A^{(1)} = A$ *and recursively define* $A^{(n+1)}$ *to be the associative ideal in A generated by the Lie ideal* $[A^{(n)}, A]$*, for each* $n \geq 1$. *The ideal* $A^{(n)}$ *is sometimes called the* n^{th} Lie power *of A.*

2. *An algebra A is upper (or strongly) Lie nilpotent of class at most c if $A^{(c+1)} = 0$ and such c is minimal.*

Although upper Lie nilpotence clearly implies Lie nilpotence, the converse need not be true (as was conjectured by S.A. Jennings). Indeed, the Grassmann algebra over a field of characteristic not 2 is Lie nilpotent of class 2 but not upper Lie nilpotent, while Gupta and Levin constructed a similar example in characteristic 2 ([15]).

Riley and Wilson, in [54], were able to construct the associative analogue of Zelmanov's Theorem. In fact, something stronger holds.

Theorem 1.5.2 *Let C be a commutative ring, and let R be a finitely generated associative C-algebra that is generated by d elements and satisfies the Engel identity of degree n. Then R is upper Lie nilpotent of class bounded by a function depending on d and n only.*

1.6 Mal′cev Semigroup Identities

The usual multiplicative semigroup of an associative algebra A will be presented by $(A, \cdot)$. If A is unital, then the elements of $(A, \cdot)$ possessing an inverse form the group of units, $A^{\times}$, of A. Another important semigroup structure on A is given by the adjoint operation

$$a \circ b = a + b + ab.$$

By embedding A into its unital hull, A_1, we can express the adjoint operation on A by

$$a \circ b = (1 + a)(1 + b) - 1.$$

In the case when A is itself unital, $(A, \circ) \to (A, \cdot) : a \mapsto 1 + a$ is an isomorphism of monoids. We recall that the *Jacobson radical*, $\mathcal{J}(A)$, of A is the largest ideal of A that forms a group under the adjoint operation.

Definition 1.6.1 *A semigroup S is said to* satisfy a (nontrivial) identity

$$\omega_1(x_1,\ldots,x_n) = \omega_2(x_1,\ldots,x_n),$$

where ω_1 *and* ω_2 *are distinct monomials in the free monoid on* $\{x_1, x_2, \ldots\}$*, if, for arbitrary* $s_1,\ldots,s_n \in S$*,*

$$\omega_1(s_1,\ldots,s_n) = \omega_2(s_1,\ldots,s_n).$$

The identity $\omega_1 = \omega_2$ *is called* reduced *if* ω_1 *and* ω_2 *begin and end with different letters.*

Definition 1.6.2 *Define the sequences* λ_n *and* ρ_n *in the free semigroup on* $\{x, y, z_0, z_1, \ldots\}$ *by*

$$\lambda_0 = x, \quad \rho_0 = y,$$

and

$$\lambda_{n+1} = \lambda_n z_n \rho_n, \quad \rho_{n+1} = \rho_n z_n \lambda_n,$$

for each $n \geq 0$*. The* n^{th} Mal′cev identity *is the reduced identity*

$$\lambda_n(x, y, z_0, \ldots, z_{n-1}) = \rho_n(x, y, z_0, \ldots, z_{n-1}).$$

Mal′cev ([40]), and independently, Neumann and Taylor ([44]), gave the following characterization for groups satisfying a Mal′cev identity.

Theorem 1.6.3 *A group G is nilpotent of class at most n if and only if it satisfies the* n^{th} *Mal′cev identity.*

A similar result to the above theorem holds in the category of rings. Riley and Tasić ([53]) proved the following.

Theorem 1.6.4 *Let R be an associative ring. If R is Lie nilpotent of class at most n, then* $(R, \circ)$ *satisfies the* n^{th} *Mal′cev identity.*

The complete converse of the above result was later verified by Amberg and Sysak ([3]).

Theorem 1.6.5 *Let R be an associative ring. If* $(R, \circ)$ *satisfies the* n^{th} *Mal′cev identity, then R is Lie nilpotent..*

1.7 Thue-Morse Semigroup Identities

Definition 1.7.1 *The n^{th} Thue-Morse identity $\mu_n(x, y) = \nu_n(x, y)$ is the n^{th} Mal′cev identity with*

$$z_0 = z_1 = \cdots = z_{n-1} = 1.$$

For example, the 2^{nd} Thue-Morse identity is the identity $xy^2x = yx^2y$.

Riley and Wilson, in [55], proved that if A is an algebra over an infinite field satisfying an Engel identity, then $(A, \circ)$ satisfies a semigroup identity; moreover, in this case, $(A, \circ)$ satisfies a Thue-Morse identity. The converse is also true:

Theorem 1.7.2 *Let A be an associative algebra over an infinite field K. Then the following conditions are equivalent.*

1. *A satisfies an adjoint semigroup identity.*

2. *A satisfies a reduced (multiplicative) semigroup identity.*

3. *A satisfies a reduced binomial identity.*

4. *A satisfies an identity of the form*

   $$\textstyle\sum_{i=0}^{n} \alpha_i y^i x y^{n-i} = 0,$$

   where $\alpha_i \in K$, $\alpha_0 \neq 0$, and $\alpha_n \neq 0$.

5. *A satisfies an Engel identity.*

6. *$(A, \circ)$ satisfies a Thue-Morse identity.*

Chapter 2

Collapsing and Rewritable Groups

2.1 Collapsing Groups

The following definition was introduced by Semple and Shalev in [62, 63].

Definition 2.1.1 *Let S be a semigroup, and let n be a positive integer. Then S is said to be n-*collapsing *if, for every $s_1, \ldots, s_n$ in S, there exist distinct functions $f, g : \{1, \ldots, n\} \to \{1, \ldots, n\}$, such that*

$$s_{f(1)} \cdots s_{f(n)} = s_{g(1)} \cdots s_{g(n)}.$$

If S is n-collapsing, for some n, then we simply say that S is collapsing.

Notice that the semigroup S clearly satisfies an identity when the functions f and g in the above definition can be chosen independently of the particular elements $s_1, \ldots, s_n$; conversely, it is easy to see that every semigroup satisfying an identity is collapsing.

Recall that a group G is said to be *residually finite* if, given $g \neq 1$ in G, there is an $N \triangleleft G$ such that $g \notin N$ and G/N is finite (for more on residual properties, see [58] section 2.3).

In [62], Semple and Shalev extended Zelmanov's proof of the Restricted Burnside Problem by proving that every finitely generated residually finite group is nilpotent-by-finite (or virtually nilpotent) precisely when it is collapsing (since any group satisfying a Burnside identity $x^m = 1$ is clearly collapsing).

Theorem 2.1.2 *Let G be a finitely generated residually finite group. Then G is collapsing if and only if G is nilpotent-by-finite.*

Notice that every nilpotent-by-finite group satisfies a semigroup identity of the form

$$\mu_n(x^e, y^e) = \nu_n(x^e, y^e),$$

for some positive integers n and e. Therefore, every finitely generated residually finite collapsing group satisfies a semigroup identity. In fact, in [63], Shalev completed the characterization of residually finite collapsing groups by addressing the non-finitely generated case. In order to formulate his result, let us say that a group G is *strongly locally nilpotent* if it generates a locally nilpotent variety; this means that, for some function g (depending on G) and for all positive integers d, every d-generated subgroup of G is nilpotent of class at most $g(d)$.

Theorem 2.1.3 *A residually finite group is collapsing if and only if it is an extension of a strongly locally nilpotent group by a group of finite exponent.*

On the other hand, Shalev raised the following natural question in [63].

Does every collapsing group satisfy a semigroup identity?

We think that the following logical extension also remains open.

Does every collapsing semigroup satisfy an identity?

2.2 Rewritable Groups

Definition 2.2.1 *A* permutational semigroup identity *is one of the form*

$$x_1 \cdots x_n = x_{\alpha(1)} \cdots x_{\alpha(n)},$$

for some non-identity permutation α on the set $\{1, \ldots, n\}$.

Let $R_2(K) = Ke_{11} + Ke_{12}$ be the subalgebra contained in the algebra, $M_2(K)$, of all 2×2 matrices over a field K. Then the semigroup $(R_2(K), \cdot)$ satisfies the (unreduced) permutational identity

$$xyz = yxz,$$

of degree 3. A similar statement holds for the semigroup $(C_2(K), \cdot)$, where $C_2(K) = Ke_{11} + Ke_{21}$ (it satisfies the identity $zxy = zyx$).

We will denote the *group commutator* of x and y in a group G by

$$(x, y) = x^{-1}y^{-1}xy.$$

Furthermore, we will denote by G' the derived subgroup of G, the subgroup generated by all commutators (x, y).

In 1985, Maj, *etal* ([39]) gave the following characterization for groups satisfying a permutational identity.

Theorem 2.2.2 *A group G satisfies a permutational identity if and only if it is finite-by-abelian-by-finite; that is, there is a normal subgroup N of G, of finite index, such that its derived subgroup N' is finite.*

Definition 2.2.3 *Let S be a semigroup, and let n be a positive integer. Then S is said to be n-*rewritable *if, for every $s_1, \ldots, s_n$ in S, there exist distinct permutations $\alpha, \beta : \{1, \ldots, n\} \to \{1, \ldots, n\}$, such that*

$$s_{\alpha(1)} \cdots s_{\alpha(n)} = s_{\beta(1)} \cdots s_{\beta(n)}.$$

If S is n-rewitable, for some n, then we simply say that S is rewritable.

Clearly, every semigroup satisfying a permutational identity is rewritable, and every rewritable semigroup is collapsing.

In [6], Blyth extended Theorem 2.2.2 to rewritable groups:

Theorem 2.2.4 *A group G is rewritable if and only if it is finite-by-abelian-by-finite.*

More recently, Elashiry and Passman gave a quantitative version of Blyth's theorem in [9] by describing explicit bounds. Recall that a *characteristic* subgroup is a subgroup that is invariant under all automorphisms of the parent group. Because conjugation is an automorphism, every characteristic subgroup is normal.

Theorem 2.2.5 *Let G be an n-rewritable group. Then G has a characteristic subgroup N such that $|G : N|$ and $|N'|$ are finite and have sizes bounded by functions depending only on n.*

Chapter 3

Collapsing and Rewritable Algebras

Henceforth, we shall use the term 'algebra' (without modification) to indicate an associative algebra over a field K of characteristic $p \geq 0$; we do not assume that algebras are unital unless specified. Let A be an algebra. We will study certain Burnside-type conditions on the adjoint semigroup $(A, \circ)$ of A in terms of its natural Lie structure via the Lie bracket

$$[a, b] = ab - ba,$$

for all $a, b \in A$.

3.1 Collapsing Algebras

We will prove the following result in Chapter 5. It is the first of two main results in Part I of this dissertation.

Theorem 3.1.1 *Let A be an algebra over a field K. If K is infinite or A is nil, then the following conditions are equivalent.*

1. *The adjoint semigroup $(A, \circ)$ is collapsing.*

2. *A satisfies an Engel identity.*

3. *$(A, \circ)$ satisfies a Thue-Morse identity.*

4. *$(A, \circ)$ satisfies an identity.*

First notice that it is not possible to add the statement 'The multiplicative semigroup $(A, \cdot)$ is collapsing' to the list of the above statements (at least in the case when A is without unity). Indeed, take A to the subalgebra $R_2(K)$ (or $C_2(K)$), contained in the algebra of 2×2 matrices over the field K. Then, as we saw in Chapter 2, $(A, \cdot)$ satisfies the unreduced semigroup identity $xyz = yxz$ (or $zxy = zyx$), and yet A does not satisfy any Engel identity; clearly $[e_{12},_m e_{11}] = \pm e_{12} \neq 0$ for all $m \geq 1$. Also, Theorem 3.1.1 does not extend in a natural way to all algebras over all finite base fields: it is clear that every finite algebra is collapsing, but not every finite algebra satisfies an Engel identity.

3.2 Rewritable Algebras

We will also obtain the following result. It is the second primary result in Part I.

Theorem 3.2.1 *Let A be an algebra over an infinite field. Then the following conditions are equivalent.*

1. *The adjoint semigroup $(A, \circ)$ is rewritable.*

2. *The adjoint semigroup of every 2-generated subalgebra of A is rewritable.*

3. *A is commutative.*

4. *The adjoint semigroup $(A, \circ)$ satisfies a permutational identity.*

Again, as the case for Theorem 3.1.1, there is no hope to extend Theorem 3.2.1 to all algebras over all base fields, since every finite algebra is rewritable but not necessarily commutative. However, we will prove that Theorem 3.2.1 does extend to all nil algebras over 'sufficiently large' finite base fields.

Theorem 3.2.2 *Let A be an algebra over a field K such that either K is infinite or A is nil and $|K| \geq \kappa(n)$. If the adjoint semigroup of every 2-generated subalgebra of A is n-rewritable, then A is commutative.*

An explicit description of the function $\kappa(n)$ will be given in the proof of Theorem 3.2.2 in Chapter 6.

Chapter 4

Minimal Non-Engel Varieties

4.1 The Characterization of Minimal Non-Engel Varieties

Recall that a variety $\mathcal{V}$ of associative algebras is said to be *Engelian* or simply *Engel* if there exists an integer $m \geq 1$ such that, for every $A \in \mathcal{V}$, A satisfies an Engel identity $[x,_m y] = 0$.

Zorn's Lemma implies that every non-Engel variety contains, as a subvariety, at least one such 'boundary' variety. Therefore, a check on being Engel would consist in verifying whether a given variety contains a boundary variety falling in a prescribed list. Such a description for just non-Engel varieties over a field of characteristic 0 was obtained by Mal′tsev in [41].

Theorem 4.1.1 *A variety $\mathcal{V}$ of associative algebras over a field of characteristic 0 is Engel if and only if $\mathcal{V}$ contains neither $R_2(K)$ nor $C_2(K)$.*

O.B. Finogenova (formerly known as O.B. Paison) proved the same result for varieties of associative algebras over infinite fields of positive characteristic ([10]). The analogous characterization when the base field is finite is more complicated:

Definition 4.1.2 *Let $|K| = r < \infty$, and consider any extension field F of K with r^{s^t} elements, where s is a prime and t is a positive integer, and let σ be any one of the automorphisms of F defined by $a^\sigma = a^{r^{ks^{t-1}}}$, where $k \in \{1, \ldots, s-1\}$. Because s is a prime, F^σ is the unique maximal subfield of F containing K.*

The associative (unital) K-algebra $B(K, F, \sigma)$ *consists of all matrices of the form*

$$\begin{pmatrix} a & b \\ 0 & a^{\sigma} \end{pmatrix} = ae_{11} + be_{12} + a^{\sigma}e_{22},$$

where a and b belong to F.

In [10], Finogenova completed the characterization for non-Engel varieties over a finite field and proved the following result.

Theorem 4.1.3 *A variety $\mathcal{V}$ over a finite field K is Engel precisely when $\mathcal{V}$ does not contain either of the algebras $R_2(K)$ or $C_2(K)$, nor any algebra of the form $B(K, F, \sigma)$.*

Chapter 5

Proof of Collapsing Algebras Results

In this chapter, we will prove Theorem 3.1.1, which we repeat here for convenience: *Let A be an algebra over a field K. If K is infinite or A is nil, then the following conditions are equivalent.*

1. *The adjoint semigroup $(A, \circ)$ is collapsing.*

2. *A satisfies an Engel identity.*

3. *$(A, \circ)$ satisfies a Thue-Morse identity.*

4. *$(A, \circ)$ satisfies an identity.*

Note first that the implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are trivial. The implication (2) $\Rightarrow$ (3) follows from Theorem 1.7.2, proved by Riley and Wilson in [54].

For the remaining implication, (1) $\Rightarrow$ (2) of Theorem 3.1.1, we will consider the case of collapsing nil algebras and the case of collapsing algebras over infinite fields separately in the following two sections.

5.1 Collapsing Nil Algebras

The main purpose of this section is to prove the following result. Note that the theorem below was proved by Riley, in [49], for all fields of characteristic zero and, in [50], for all infinite fields of characteristic $p > 0$.

Theorem 5.1.1 *Let A be a nil algebra over any field. If the adjoint group $(A, \circ)$ is n-collapsing, then A is m-Engel for some m depending only on n.*

It remains to consider only the case of nil algebras over a finite field K. Therefore, we assume that A is an algebra over a finite field K.

Definition 5.1.2

1. *Define the words ϵ_m and η_m in the free monoid on $\{x, y, z\}$ by*

$$\epsilon_m(x, y, z) = \lambda_m(x, y, 1, z, z^2, \ldots, z^{m-1}),$$

$$\eta_m(x, y, z) = \rho_m(x, y, 1, z, z^2, \ldots, z^{m-1});$$

   in other words, set $z_i = z^i$, for each i in the words λ_m and ρ_m.

2. *Let S be a semigroup, and let m be a positive integer. Then S is said to be* positively m-Engel *whenever S satisfies the identity*

$$\epsilon_m(x, y, z) = \eta_m(x, y, z).$$

   If S is positively m-Engel, for some m, then S is called positively Engel.

Riley, in [51], was the first to study positively Engel rings. In fact, using Lemma 5.1.4 below, he proved the following result.

Theorem 5.1.3 *Let R be an associative ring. Then the following conditions are equivalent.*

1. *The associated Lie ring,* $[R]$*, of* R *is Engel.*

2. *The adjoint semigroup* $(R, \circ)$ *of* R *is positively Engel.*

3. *The multiplicative semigroup* $(R, \cdot)$ *of* R *is positively Engel.*

Lemma 5.1.4 *Consider elements* x, y, z *in* $B(K, F, \sigma)$ *of the form*

$$x = a_0 e_{11} + b_0 e_{12} + a_0^\sigma e_{22},$$
$$y = a_0 e_{11} + c_0 e_{12} + a_0^\sigma e_{22}, \quad \text{and}$$
$$z_m = d_m e_{11} + d_m^\sigma e_{22} \quad (m \geq 0).$$

Then, working in the multiplicative semigroup of $B(K, F, \sigma)$*, the following recursive formulas hold:*

$$\lambda_m(x, y, z_0, \ldots, z_{m-1}) = a_m e_{11} + b_m e_{12} + a_m^\sigma e_{22} \quad \text{and}$$
$$\rho_m(x, y, z_0, \ldots, z_{m-1}) = a_m e_{11} + c_m e_{12} + a_m^\sigma e_{22},$$

where

$$a_m = a_{m-1}^2 d_{m-1},$$
$$b_m = a_{m-1} c_{m-1} d_{m-1} + a_{m-1}^\sigma b_{m-1} d_{m-1}^\sigma, \quad \text{and}$$
$$c_m = a_{m-1} b_{m-1} d_{m-1} + a_{m-1}^\sigma c_{m-1} d_{m-1}^\sigma,$$

for each $m \geq 1$.

Recall from Theorem 4.1.3 that a variety $\mathcal{V}$ of associative algebras over a finite field K is *Engel* if and only if $\mathcal{V}$ does not contain either of the algebras $R_2(K)$ or $C_2(K)$, nor any algebra of the form $B(K, F, \sigma)$ (*cf.* [10]). Based on this characterization, we have the following result.

Lemma 5.1.5 *Let A be an algebra over any field K of characteristic $p > 0$, and suppose that, for some integers $e, m \geq 0$, $(A, \circ)$ satisfies the identity*

$$\epsilon_m(x^{p^e}, y^{p^e}, z^{p^e}) = \eta_m(x^{p^e}, y^{p^e}, z^{p^e}).$$

Then A satisfies an Engel identity.

Proof. Without loss of generality, let $K = \mathbb{F}_p$. Clearly the condition on A can be framed in terms of polynomial identities, which uniquely determine a variety $\mathcal{V}$ over K. Thus, as discussed above, it suffices to show that $\mathcal{V}$ does not contain $R_2(K)$ or $C_2(K)$, nor any algebra of the form $B(K, F, \sigma)$.

First observe that p-powers in $(A, \circ)$ coincide with p-powers in $(A, \cdot)$. Indeed, simple inductive argument shows that

$$\underbrace{x \circ \cdots \circ x}_{p^\alpha} = \begin{pmatrix} p^\alpha \\ 1 \end{pmatrix} x + \begin{pmatrix} p^\alpha \\ 2 \end{pmatrix} x^2 + \cdots + \begin{pmatrix} p^\alpha \\ p^\alpha - 1 \end{pmatrix} x^{p^\alpha - 1} + x^{p^\alpha} = x^{p^\alpha}.$$

To prove that $A = R_2(K) \notin \mathcal{V}$, set $x = -e_{11}$, $y = -e_{11} + e_{12}$, and $z = 0$. Then $x^{p^e} = x$, $y^{p^e} = y$, and $z^{p^e} = 0$ in $(A, \circ)$. Hence, by induction on m,

$$\epsilon_m(x^{p^e}, y^{p^e}, z^{p^e}) - \eta_m(x^{p^e}, y^{p^e}, z^{p^e}) = \mu_m(x, y) - \nu_m(x, y) = x - y = -e_{12} \neq 0,$$

for every $m \geq 0$. The proof that $C_2(K) \notin \mathcal{V}$ is analogous.

It remains to prove that none of the algebras $B(K, F, \sigma)$ lies in $\mathcal{V}$. Since $B(K, F, \sigma)$ is unital, it suffices to show that its multiplicative semigroup does not satisfy the identity $\epsilon_m(x^{p^e}, y^{p^e}, z^{p^e}) = \eta_m(x^{p^e}, y^{p^e}, z^{p^e})$. So, let n be a positive integer, and let $x = \alpha e_{11} + \beta e_{12} + \alpha^\sigma e_{22} \in B(K, F, \sigma)$ be an arbitrary element in the multiplicative semigroup. A simple induction argument on n yields the expression

$$x^n = \alpha^n e_{11} + (\textstyle\sum_{i+j=n-1} \alpha^{i+j\sigma})\beta e_{12} + \alpha^{n\sigma} e_{22}.$$

Now choose $\alpha \in F$ such that $\alpha^\sigma \neq \alpha$, and put $x = \alpha e_{11} + e_{12} + \alpha^\sigma e_{22}$, $y = \alpha e_{11} + \alpha^\sigma e_{22}$, and $z = \alpha^{-1} e_{11} + \alpha^{-\sigma} e_{22}$. Then

$$\begin{aligned} x^{p^e} &= \alpha^{p^e} e_{11} + \Big(\sum_{i+j=p^e-1} \alpha^{i+j\sigma}\Big) e_{12} + \alpha^{p^e\sigma} e_{22}, \\ y^{p^e} &= \alpha^{p^e} e_{11} + \alpha^{p^e\sigma} e_{22}, \quad \text{and} \\ z^{p^e} &= \alpha^{-p^e} e_{11} + \alpha^{-p^e\sigma} e_{22}. \end{aligned}$$

Setting $a_0 = \alpha^{p^e}$, $b_0 = \sum_{i+j=p^e-1} \alpha^{i+j\sigma}$, $c_0 = 0$, and $d_m = \alpha^{-mp^e}$, for each $m \geq 0$, we discover from Lemma 5.1.4 that

$$\begin{aligned} \epsilon_m(x^{p^e}, y^{p^e}, z^{p^e}) &= a_m e_{11} + b_m e_{12} + a_m^\sigma e_{22} \quad \text{and} \\ \eta_m(x^{p^e}, y^{p^e}, z^{p^e}) &= a_m e_{11} + c_m e_{12} + a_m^\sigma e_{22}, \end{aligned}$$

where

$$\begin{aligned} a_m &= a_{m-1}^2 d_{m-1}, \\ b_m &= a_{m-1} c_{m-1} d_{m-1} + a_{m-1}^\sigma b_{m-1} d_{m-1}^\sigma, \quad \text{and} \\ c_m &= a_{m-1} b_{m-1} d_{m-1} + a_{m-1}^\sigma c_{m-1} d_{m-1}^\sigma, \end{aligned}$$

for each $m \geq 1$. Induction shows that $a_m = a_0^{m+1}$ and $a_m d_m = a_0$, for each m. Consequently, $b_m - c_m = (a_0^\sigma - a_0)(b_{m-1} - c_{m-1})$, and hence $b_m - c_m = (a_0^\sigma - a_0)^m b_0$, for each m. This yields

$$\epsilon_m(x^{p^e}, y^{p^e}, z^{p^e}) - \eta_m(x^{p^e}, y^{p^e}, z^{p^e}) = (a_0^\sigma - a_0)^m b_0 e_{12},$$

for each $m \geq 0$. Thus, it suffices to show that neither $a_0^\sigma = a_0$ nor $b_0 = 0$. So, suppose to the contrary that $a_0 = \alpha^{p^e}$ is fixed by σ. Then

$$(\alpha^\sigma - \alpha)^{p^e} = (\alpha^{p^e})^\sigma - \alpha^{p^e} = 0,$$

contrary to our choice of α. Now suppose $b_0 = 0$; in other words,

$$\sum_{i+j=p^e-1} \alpha^{i+j\sigma} = \alpha^{p^e-1} \sum_{i=0}^{p^e-1} (\alpha^{\sigma-1})^i = 0.$$

Then

$$(\alpha^{p^e})^{\sigma-1} - 1 = (\alpha^{\sigma-1})^{p^e} - 1 = (\alpha^{\sigma-1} - 1) \sum_{i=0}^{p^e-1} (\alpha^{\sigma-1})^i = 0,$$

so that σ fixes α^{p^e}, contrary to what we have seen. □

Definition 5.1.6 *For each positive integer n, define a polynomial p_n in the free algebra on indeterminates $x_1, x_2, \ldots, x_n, y$ by*

$$p_n = \prod (x_{f(1)} \circ \cdots \circ x_{f(n)} - x_{g(1)} \circ \cdots \circ x_{g(n)}) y,$$

where the product runs over all pairs of distinct functions f, g on $\{1, \ldots, n\}$ (in some fixed order).

It is clear to see that, if $(A, \circ)$ is n-collapsing, then A satisfies $p_n = 0$.

Theorem 2.1.3 has the following consequence (see [63]):

Theorem 5.1.7 *Let G be a d-generated n-collapsing residually finite group. Then G has a nilpotent normal subgroup N whose index and nilpotency class are n, d-bounded.*

Proof of Theorem 5.1.1: As remarked before, we may assume that the base field K is finite with prime characteristic p. It is also safe to assume that A is 2-generated. Since A satisfies the polynomial identity $p_n = 0$, A is a 'finite' nilpotent algebra by Kaplansky's celebrated solution to the Kurosh-Levitzki problem (see Theorems 1.3.3 and 1.3.4). It follows that $G = (A, \circ)$ is an n-collapsing finite group. Consequently, by Theorem 5.1.7, G contains a normal subgroup N such that the exponent e of G/N and the nilpotence class c of N are bounded by functions of n only. Thus, since G is a p-group, G satisfies the semigroup identity

$$\lambda_c(x^{p^t}, y^{p^t}, z_0^{p^t}, \ldots, z_{c-1}^{p^t}) = \rho_c(x^{p^t}, y^{p^t}, z_0^{p^t}, \ldots, z_{c-1}^{p^t}),$$

where p^t is the largest power of p dividing e. Therefore, by Lemma 5.1.5, A is m-Engel, for some m, which may depend on p. However, if $p > e$, then G is nilpotent of class at most c, which is tantamount to A being Lie nilpotent of class at most c (see Theorem 1.6.5). Since there are only finitely many prime-power divisors of e, clearly m can be chosen independent of K and p. □

5.2 Collapsing Algebras over Infinite Fields

In this section, we will prove the following result, which is implication (1) ⇒ (2) of Theorem 3.1.1 for an algebra A over an infinite field K.

Theorem 5.2.1 *Let A be an algebra over an infinite field. If $(A, \circ)$ is n-collapsing, then A is m-Engel, for some m depending only on n.*

Definition 5.2.2 *We say that a ring B is* involved *in another ring A if B is a direct limit of rings that are each a homomorphic image of a subring of A.*

Observe that, if a ring A has the property that $(A, \circ)$ is n-collapsing, for some positive integer n, then so does every ring B involved in A.

Definition 5.2.3 *For each prime $p > 0$ and integer $k \geq 1$, let $\mathbb{F}_{p^k}$ denote the field of cardinality p^k. We embed each $\mathbb{F}_{p^k}$ into $\mathbb{F}_{p^{k+1}}$ and set*

$$\mathbb{F}_{p^\infty} = \textstyle\bigcup_{k\geq 1} \mathbb{F}_{p^k}.$$

Let K_0 denote the field

$$K_0 = \begin{cases} \mathbb{Q}, & \text{if} \quad p = 0 \\ \mathbb{F}_{p^\infty}, & \text{if} \quad p > 0. \end{cases}$$

Lemma 5.2.4 *Let K be an infinite field. Then the following statements hold.*

1. *K involves K_0.*

2. *$R_2(K)$, respectively $C_2(K)$, involves $R_2(K_0)$, respectively $C_2(K_0)$.*

Proof. If $\mathbb{Q}$ or $\mathbb{F}_{p^\infty}$ is a subfield of K, then both assertions are obvious. If not, $\mathbb{F}_p(t)$ is a subfield of K, for some transcendental element t over $\mathbb{F}_p$, and thus $\mathbb{F}_p[t]$ is a subring of K. Since each field $\mathbb{F}_{p^k}$ is a (ring) homomorphic image of $\mathbb{F}_p[t]$, this proves part (1). The proof of part (2) is analogous. □

Lemma 5.2.5 *Let A be an algebra over an infinite field K, and let e be any idempotent in A.*

1. *If $eA(1-e) = \{ea - eae | a \in A\} \neq 0$, then A involves $R_2(K_0)$.*

2. *If $(1-e)Ae = \{ae - eae | a \in A\} \neq 0$, then A involves $C_2(K_0)$.*

3. *If A involves neither $R_2(K_0)$ nor $C_2(K_0)$, then e is central in A.*

Proof. Suppose that $x \in eA(1-e)$ is nonzero. Then $ex = x$, $xe = 0$, and $x^2 = 0$, so that $Ke + Kx \cong R_2(K)$ is a subalgebra of A. This proves part (1) by Lemma 5.2.4. Because the statement of part (2) is left-right symmetric to the statement of part (1), it remains to deduce part (3): if A involves neither $R_2(K_0)$ nor $C_2(K_0)$, then by parts (1) and (2),

$$eA(1-e) = 0 = (1-e)Ae,$$

so that e is central. □

Lemma 5.2.6 *Neither $R_2(K_0)$ nor $C_2(K_0)$ is collapsing.*

Proof. By symmetry, it suffices to address the claim for $R_2(K_0)$. We consider the characteristic zero case first. Let $n \geq 1$ be given, and put $a_i = e_{11} + b^i e_{12}$, where $b = 2^n$, for each i. Let f be any function on $\{1, \ldots, n\}$. Then, by induction on n,

$$a_{f(1)} \circ a_{f(2)} \circ \cdots \circ a_{f(n)} = (b-1)e_{11} + (b^{f(1)} + 2b^{f(2)} + \cdots + 2^{n-1}b^{f(n)})e_{12}.$$

It remains to prove that the integer $a_f = b^{f(1)} + 2b^{f(2)} + \cdots + 2^{n-1}b^{f(n)}$ is uniquely determined by f. Indeed, by our choice of b, collecting like powers of b yields the unique b-adic expansion of the integer a_f. Because there is a one-to-one correspondence between the nonempty subsets of $\{1, 2, \ldots, 2^{n-1}\}$ and the sums of those subsets, the claim follows.

Now consider the case when $K_0 = \mathbb{F}_{p^\infty}$. Choose k minimally such that $\phi(p^k - 1) > (n-1)n^n$, where ϕ is Euler's function, and choose $\beta_1, \ldots, \beta_n \in K_0$ linearly independent over $F := \mathbb{F}_{p^k}$. Let α be a primitive $(p^k - 1)$th root of unity in F, and set $\alpha_i = \alpha^{n^i} - 1 \in F$ and $a_i = \alpha_i e_{11} + \beta_i e_{12}$, for each i. Let f be any function on $\{1, \ldots, n\}$. Then, by induction on n,

$$a_{f(1)} \circ \cdots \circ a_{f(n)} = (\alpha_{f(1)} \circ \cdots \circ \alpha_{f(n)})e_{11} + \textstyle\sum_{i=1}^{n}[(\alpha_{f(1)} \circ \cdots \circ \alpha_{f(i-1)}) + 1]\beta_{f(i)}e_{12},$$

where the empty circle product is zero. It remains to show that the scalar

$$a_f = \textstyle\sum_{i=1}^{n}[(\alpha_{f(1)} \circ \cdots \circ \alpha_{f(i-1)}) + 1]\beta_{f(i)} = \sum_{i=1}^{n} \alpha^{n^{f(1)}+\cdots+n^{f(i-1)}}\beta_{f(i)}$$

uniquely determines f. Indeed, observe first that the subset

$$S_f = \{1, \alpha^{n^{f(1)}}, \ldots, \alpha^{n^{f(1)}+\cdots+n^{f(n-1)}}\}$$

of F is linearly independent over $\mathbb{F}_p$ since the degree of the minimal polynomial over $\mathbb{F}_p$ satisfied by α is $\phi(p^k - 1) > (n-1)n^n$, by construction. In particular, the nonempty subsets of S_f are uniquely determined by the sum of their elements. Moreover, the coefficients of the unique F-linear expansion of a_f in terms of the β_i determines a partition of S_f. Suppose now that $a_f = a_g$ for some function g on $\{1, \ldots, n\}$. Then, for every i, there exists a j such that the coefficient of $\beta_{f(i)} = \beta_{g(j)}$ in $a_f = a_g$ corresponds to both a subset X_i of S_f containing the element $\alpha^{n^{f(1)}+\cdots+n^{f(i-1)}}$ and to a subset Y_j of S_g containing the element $\alpha^{n^{g(1)}+\cdots+n^{g(j-1)}}$. Again, since the powers of α contained in $S_f \cup S_g$ are all less than the degree of its minimal polynomial, it follows that

$$\alpha^{n^{f(1)}+\cdots+n^{f(i-1)}} \in Y_j.$$

In fact, the only possibility is

$$\alpha^{n^{f(1)}+\cdots+n^{f(i-1)}} = \alpha^{n^{g(1)}+\cdots+n^{g(i-1)}},$$

since

$$n^{f(1)} + \cdots + n^{f(i-1)} \equiv i - 1 \mod n - 1.$$

Therefore, $i = j$, and so $g(i) = g(j) = f(i)$, as required. □

Let $\mathcal{J}(A)$ denote the Jacobson radical of A, and, for simplicity, we will let $C(A) = A^{(2)}$; the associative ideal in A generated by the Lie ideal $\gamma_2(A)$.

Proposition 5.2.7 *Let n be a positive integer, and A be an algebra over an infinite field K. Then the following statements hold.*

1. *If $A = M_2(K)$, then A does not satisfy $p_n = 0$.*
2. *If $(A, \circ)$ is n-collapsing, then every idempotent e of A is central and $A/\mathcal{J}(A)$ is commutative. Furthermore, if A is generated by $m < \infty$ elements, then $C(A)$ is nilpotent of index bounded by a function determined by m and n only.*

Proof. From Lemma 5.2.4, $M_2(K_0)$ is involved in $M_2(K)$. Thus, if $M_2(K)$ satisfies $p_n = 0$, then so would $M_2(K_0)$. However, the latter does not: evaluate each $x_i = a_i$ and $y = e_{21}$ with a_i as in the proof of Lemma 5.2.6. Now suppose that $(A, \circ)$ is n-collapsing. Then A involves neither $R_2(K_0)$ nor $C_2(K_0)$ by Lemma 5.2.6; hence, by Lemma 5.2.5, all idempotents are central in A. By part (1), A satisfies a non-matrix identity; hence, $A/\mathcal{J}(A)$ is commutative and $C(A)$ is nilpotent when A is finitely generated (see [5] or [43], for example). Because the last statement depends only on the K-variety satisfied by $p_n = 0$, the nilpotence index of $C(A)$ is bounded by a function depending only on m, n, and possibly K. To see there is a bound independent of K, let t_0 be nilpotence index of $C(A)$ when A is the relatively-free (countably generated) algebra $\mathbb{Q}$-algebra satisfying $p_n = 0$. Then, for all sufficiently large primes p, if an $\mathbb{F}_p$-algebra A satisfies $p_n = 0$, then $C(A)$ is nilpotent of index at most t_0. Indeed, otherwise, for each $k \geq 1$, there exists a prime $q_k \geq k$ and an $\mathbb{F}_{q_k}$-algebra A_k satisfying $p_n = 0$ such that $C(A_k)$ is not nilpotent of index at most t_0. It follows that the ultraproduct

$$A = \prod_{k\geq 1} A_k/\mathcal{F},$$

with respect to the Fréchet ultrafilter $\mathcal{F}$, is an algebra over the field

$$K = \prod_{k\geq 1} \mathbb{F}_{q_k}/\mathcal{F},$$

that satisfies $p_n = 0$ but $C(A)$ is not nilpotent of index at most t_0. However, this cannot happen because K has characteristic zero. Since every K-algebra is also an algebra over one of $\mathbb{Q}$, $\mathbb{F}_{p^\infty}$ or $\mathbb{F}_p(t)$, the claim follows. □

Recall that a unital algebra A is called local if $A = A^\times \cup \mathcal{J}(A)$.

Lemma 5.2.8 *Let A be a finite-dimensional unital algebra over an infinite field K such that A is local and $C(A)^2 = 0$. If $(A, \cdot)$ is n-collapsing, then A is Lie nilpotent.*

Proof. Suppose that $(A, \cdot)$ is n-collapsing. Using the identity

$$u^{-1}v^{-1}uv = 1 + u^{-1}v^{-1}[u, v],$$

twice, it is easy to check that the condition $C(A)^2 = 0$ forces $A^\times$ to be metabelian. Thus, every 2-generated subgroup G of $A^\times$ is residually finite by a theorem of P. Hall in [19]. Since G is a subsemigroup of $(A, \cdot)$, it is n-collapsing, too. It now follows from Theorem 5.1.7 that G contains a normal subgroup N such that the exponent e of G/N and the nilpotence class c of N are bounded functions of n only. Thus, every such G, and hence $A^\times$, satisfies a semigroup identity of the form $\mu_c(x^e, y^e) = \nu_c(x^e, y^e)$. Moreover, since $A = A^\times \cup \mathcal{J}(A)$, $(A, \cdot)$ satisfies the reduced semigroup law $\mu_c(x^t, y^t) = \nu_c(x^t, y^t)$, where t is the greater of e and the nilpotence index of $\mathcal{J}(A)$. It now follows from Theorem 1.7.2 that A must be Lie nilpotent. □

The following result by Kublanovskii ([32]) is used to prove Lemma 5.2.10 below.

Theorem 5.2.9 *Let $\mathcal{V}$ be a variety of associative algebra satisfying a polynomial identity of the form*

$$xy^m z = \sum_{m_i<m} u_i x y^{m_i} z v_i,$$

for some u_i, v_i *in the free algebra. Then, for every finitely generated algebra* A *in* $\mathcal{V}$*,* A *is residually finite-dimensional algebra.*

Lemma 5.2.10 *Let* A *be any algebra over an infinite field such that* $C(A)^2 = 0$*. If* $(A, \circ)$ *is* n*-collapsing, then* A *satisfies an Engel identity of degree bounded by a function of* n *only.*

Proof. It suffices to assume that A is 2-generated. Since $C(A)^2 = 0$ by hypothesis, A satisfies the polynomial identity

$$xy^2z - (xyz)y - y(xyz) + y(xz)y = [x, y][y, z] = 0;$$

hence, by Theorem 5.2.9, A is residually finite-dimensional. Thus, because our hypotheses are preserved by homomorphic images, we may assume that A is finite-dimensional. In this case, $A/\mathcal{J}(A)$ is finite-dimensional and semiprimitive. By embedding $A/\mathcal{J}(A)$ into its unital hull, standard arguments force $A/\mathcal{J}(A)$ to be semisimple, and hence unital. Lift this unity to an idempotent e in A. Then, by Proposition 5.2.7, e is central in A. In particular, A decomposes into a direct sum of ideals:

$$A = eA \oplus (1 - e)A.$$

By construction, $(1 - e)A \subseteq \mathcal{J}(A)$, which satisfies an Engel identity of degree bounded by a function of n only by Theorem 5.1.1. Since the ideal eA inherits our hypotheses, it suffices to proceed under that assumption that A is unital.

Since $\mathcal{J}(A)$ is nilpotent, the primitive central idempotents $f_1, \ldots, f_r$ in $A/\mathcal{J}(A)$ lift to orthogonal idempotents $e_1, \ldots, e_r$ in A. These idempotents are central in A by Proposition 3.4. Furthermore, because $1 - \sum e_i$ is an idempotent contained in $\mathcal{J}(A)$, we have $\sum e_i = 1$. Hence,

$$A = (\textstyle\sum e_i)A = e_1A \oplus \cdots \oplus e_rA,$$

where $\mathcal{J}(e_iA) = e_i\mathcal{J}(A)$, for each i. Because $A/\mathcal{J}(A)$ is commutative (by Proposition 5.2.7), each Wedderburn factor $e_iA/e_i\mathcal{J}(A) = f_i(A/\mathcal{J}(A))$ is a field extension of K. Thus, Lemma 5.2.8 applies to each local algebra A_i, so that A is Lie nilpotent. To see that A satisfies an Engel identity of bounded degree, consider the unital subalgebra B generated by arbitrary elements $x, y \in A$ over the prime subfield $\mathbb{F}$ of K. Then, as above, B is residually finite-dimensional as an $\mathbb{F}$-algebra. Thus, we may assume that B is a finite-dimensional Lie nilpotent algebra over the perfect field $\mathbb{F}$. Consequently,

$$B = \mathcal{Z}(B) + \mathcal{J}(B),$$

by a theorem of Sweedler ([67]), where $\mathcal{Z}(B)$ denotes the centre of B. Since $(\mathcal{J}(B), \circ)$ is n-collapsing, Theorem 5.1.1 informs us that $\mathcal{J}(B)$, and hence B, satisfies an Engel identity of bounded degree, as required. □

Proposition 5.2.11 *Let R be any associative ring such that $R/C(R)^2$ is Lie nilpotent of class c and $C(R)$ is (associatively) nilpotent of index t. If $c = 1$, then R is commutative. If $c \geq 2$, then R is Lie nilpotent of class at most*

$$t + \tfrac{1}{2}(c-2)t(t-1).$$

Proof. It suffices to assume that R is unital, so that $C(R) = [R, R]R$. Denote the ith term of the lower central series of R by $\gamma_i(R)$, so that $\gamma_{c+1}(R) \subseteq C(R)^2$, by hypothesis. If $c = 1$, then $C(R) \subseteq C(R)^2 = 0$ since $C(R)$ is nilpotent. Now suppose $c \geq 2$. Using the identity $[ab, c] = a[b, c] + [a, b]c$ repeatedly yields

$$\gamma_{c+1+k}(R) \subseteq \textstyle\sum_{i+j=k}[C(R),_i R][C(R),_j R],$$

for every integer $k \geq 0$. Moreover,

$$[C(R),_i R] = [[R, R]R,_i R] \subseteq [R, R]\gamma_{i+1}(R) + \gamma_{i+2}(R)R \subseteq C(R)^2,$$

provided $i \geq c - 1$. But, if $i + j = k = 2(c-2) + 1$, then either $i \geq c - 1$ or $j \geq c - 1$; hence, in either case,

$$\gamma_{(c+1)+[2(c-2)+1]}(R) \subseteq C(R)^3.$$

The above pigeon-hole argument extends naturally to longer products proving that

$$\gamma_{(c+1)+[2(c-2)+1]+[3(c-2)+1]+\cdots+[(t-1)(c-2)+1)]}(R) \subseteq C(R)^t = 0.$$

□

Proof of Theorem 5.2.1: It suffices to assume that A is 2-generated. Observe that $A/C(A)^2$ satisfies an Engel identity of bounded index by Lemma 5.2.10, so that $A/C(A)^2$ is Lie nilpotent of bounded class by Theorem 1.7.2. It now follows from Propositions 5.2.7 and 5.2.11 that A is Lie nilpotent of bounded class. □

Using Theorems 5.1.1 and 5.2.1 (Theorem 3.1.1, implication (1) $\Rightarrow$ (2)), we can add one more result which shows that being collapsing is a 'local' condition.

Corollary 5.2.12 *Let A be an algebra over a field K such that either K is infinite or A is nil. If $(B, \circ)$ is n-collapsing, for every 2-generated subalgebra B of A, then $(A, \circ)$ is N-collapsing, for some N depending only on n.*

Proof. By Theorems 5.1.1 and 5.2.1, each B is m-Engel, for some m depending only on n. Hence, by Theorem 1.7.2, each $(B, \circ)$ satisfies the Thue-Morse identity of index $N = c(m, 2)$, and, consequently, $(A, \circ)$ satisfies the Thue-Morse identity of index N, as well. □

Chapter 6

Proof of Rewritable Algebras Results

Recall from Theorem 2.2.4 that a n-rewritable group G contains a normal subgroup N of finite index such its derived subgroup, N', is finite. Elashiry and Passman gave a quantitative version of this result (see Theorem 2.2.5) by describing explicit integer-valued functions $r(n)$ and $s(n)$, depending only on n, such that $|G : N| \leq r(n)$ and $|N'| \leq s(n)$.

Recall that the group commutator of units x and y in an algebra A is denoted by

$$(x, y) = x^{-1}y^{-1}xy,$$

so that $(x, y)^{-1} = x^{-1}y^{-1}[x, y]$.

6.1 Proof of Theorems 3.2.1 and 3.2.2

Lemma 6.1.1 *Let A be a nilpotent algebra over a field K such that $|K| \geq r(n)(s(n) + 1)$. If $G = (A, \circ)$ is n-rewritable, then A is commutative.*

Proof. Suppose G is n-rewritable. Then, as discussed above, G contains a normal subgroup N such that $|G : N| \leq r(n)$ and $|N'| \leq s(n)$. We claim that N must be abelian, for otherwise we would have $|N'| > s(n)$. Indeed, if $N' \neq 1$, then there exist elements $x, y \in N$ such that

$$x \circ y - y \circ x = xy - yx = [x, y] \neq 0.$$

Because A is nilpotent, there exists a maximal integer $k \geq 1$ such that $[x, y] \in A^k$ but $[x, y] \notin A^{k+1}$. It suffices to prove that $|N' + A^{k+1}/A^{k+1}| > s(n)$, and so we may assume that $A^{k+1} = 0$. Since $|K| \geq r(n)(s(n) + 1)$ and $|G/N| \leq r(n)$, there exists some $\beta \in K$ such that, for at least $(s(n) + 1)$-many elements $\alpha \in K$, we have $(\alpha x) \circ N = (\beta x) \circ N$; in other words,

$$x_\alpha := (\beta - \alpha)x(1 - \alpha x + \alpha^2 x^2 - \cdots) = (1 - \alpha x + \alpha^2 x^2 - \cdots)(1 + \beta x) - 1 \in N.$$

Hence, because $[x, y] \in A^k$ and $A^{k+1} = 0$, we have

$$(1 + x_\alpha, 1 + y) - 1 = (1 - x_\alpha + x_\alpha^2 - \cdots)(1 - y + y^2 - \cdots)[x_\alpha, y] = (\beta - \alpha)[x, y],$$

and so $|N'| \geq s(n) + 1$, as claimed. Now notice that the subalgebra of A generated by N is also an abelian normal subgroup of G of index at most $r(n)$. Therefore, in order to prove that G itself is abelian, it suffices for us to assume that N is a subalgebra of A. We claim that, in this case, N must coincide with G. So, let us suppose, to the contrary, that there is some x in G that does not lie in N, and let $\alpha \in K^\times$. Then, since N is both a subgroup of G and a subalgebra of A, neither αx nor its inverse $-\alpha x + \alpha^2 x^2 - \alpha^3 x^3 + \cdots$ can lie in N. Now, for every $\alpha \neq \beta$ in $K^\times$, we have distinct cosets $(\alpha x) \circ N \neq (\beta x) \circ N$. Indeed, otherwise

$$(\beta - \alpha)x(1 - \alpha x + \alpha^2 x^2 - \cdots) = (1 - \alpha x + \alpha^2 x^2 - \cdots)(1 + \beta x) - 1 \in N,$$

so that

$$-\alpha x + \alpha^2 x^2 - \alpha^3 x^3 + \cdots = -\alpha(\beta - \alpha)^{-1}(\beta - \alpha)x(1 - \alpha x + \alpha^2 x^2 - \cdots) \in N,$$

contrary to what we have just observed. Since

$$|K^\times| = |K| - 1 \geq r(n)(s(n) + 1) - 1 > r(n),$$

this contradicts the fact that $|G : N| \leq r(n)$, completing the proof. □

To prove Theorem 3.2.1, it suffices to prove the following stronger result.

Theorem 6.1.2 *Let A be an algebra over a field K such that either K is infinite or A is nil and $|K| \geq r(n)(s(n) + 1)$. If the adjoint semigroup of every 2-generated subalgebra of A is n-rewritable, then A is commutative.*

Proof. We may assume that A itself is 2-generated and n-rewritable. Thus, by Theorem 3.1.1, A is m-Engel, for some m, so that A satisfies the identity

$$xy^m z + \sum_{i=1}^{m}(-1)^i\binom{m}{i}y^i xy^{m-i}z = [x,_m y]z = 0.$$

Hence, by Theorem 5.2.9, we may assume that A is finite-dimensional and Lie nilpotent (by Engel's theorem). Thus, in the case when the base field K is perfect,

$$A_1 = \mathcal{Z}(A_1) + \mathcal{J}(A_1) = \mathcal{Z}(A_1) + \mathcal{J}(A),$$

by a theorem of Sweedler [67], and so A is commutative by Lemma 6.1.1. When K is not perfect, either K contains the perfect field $\mathbb{F}_{p^\infty}$, in which case we are done by replacing K with $\mathbb{F}_{p^\infty}$, or K contains a transcendental element over $\mathbb{F}_p$. In the latter case, the structural constants of A are contained in a purely transcendental field extension $F(t_1, \ldots, t_m)$ of a finite field F with $m \geq 1$. By clearing denominators, we may assume that the structural constants lie in $R = F[t_1, \ldots, t_m]$, and hence that A is a finitely generated free R-algebra. Since R is the subdirect product of finite fields, all of arbitrarily large cardinality, it follows that A is commutative by the perfect case. □

Part II

Verbal and Marginal Properties of an Associative Algebra

Chapter 1

Verbal and Marginal Subgroups

1.1 Commutators

Let G be a group and let $g_1, g_2 \dots$ be elements of G. Recall that the *commutator* of g_1 and g_2 is

$$(g_1, g_2) = g_1^{-1} g_2^{-1} g_1 g_2.$$

More generally, a *simple commutator of weight* $n \geq 2$ is defined recursively by the rule

$$(g_1, \dots, g_n) = ((g_1, \dots, g_{n-1}), g_n),$$

where by convention $(g_1) = g_1$. A useful shorthand notation is

$$(g,_m h) = (g, \underbrace{h, \dots, h}_{m}).$$

There are many group commutators' useful identities, even though their proof is quite elementary.

Lemma 1.1.1 *The following identities hold for any group G.*

1. $(x, y) = (y, x)^{-1}$.

2. *If we denote* $y^{-1}xy$ *by* x^y*, then*

$$[x, yz] = [x, z][x, y]^z \quad \textit{and} \quad [xy, z] = [x, z]^y[y, z].$$

3. *The Hall-Witt identity (or the Jacobi identity) holds,*

$$(x, y^{-1}, z)^y (y, z^{-1}, x)^z (z, x^{-1}, y)^x = 1.$$

1.2 Verbal Subgroups

Definition 1.2.1 *Let $\theta(x_1, \ldots, x_n) = x_{i_1}^{r_1} \cdots x_{i_t}^{r_t}$ be a word in the free group on a countably infinite set $\{x_1, x_2, \ldots,\}$. If $g_1, \ldots, g_n$ are elements of a group G, we define the value of the word θ at the elements $g_1, \ldots, g_n$ to be*

$$\theta(g_1, \ldots, g_n) = g_{i_1}^{r_1} \cdots g_{i_t}^{r_t}.$$

The subgroup of G generated by all values in G of the word θ is called the verbal subgroup of G determined by θ, *and will be denoted by $\theta(G)$.*

For example, if $\theta = (x_1, x_2)$, then $\theta(G) = (G, G) = G'$, the derived subgroup of G; if $\theta = x_1^n$, then $\theta(G) = G^n$, the subgroup generated by all the n^{th} *powers* in G. Note that if $\alpha : G \to H$ is a group homomorphism, then

$$\alpha(\theta(g_1, \ldots, g_n)) = \theta(\alpha(g_1), \ldots, \alpha(g_n)),$$

which shows at once that $\alpha(\theta(G)) \leq \theta(H)$. In particular, every verbal subgroup is full-invariant. The converse is false in general; for instance, in the group of all roots of unity, the subgroup of n^{th} roots for fixed n is fully-invariant but not verbal (see Exercise 2.3.3 in [58]).

1.3 Marginal Subgroups

Definition 1.3.1 *If θ is a word in $x_1, x_2, \ldots$ and G is any group, a normal subgroup N is said to be θ-*marginal *in G if*

$$\theta(g_1, \ldots, g_{i-1}, g_i a, g_{i+1}, \ldots, g_n) = \theta(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_n),$$

for all $i = 1.2. \ldots, n$, $g_1, \ldots, g_n \in G$, *and all* $a \in N$. *This is equivalent to the requirement:* $g_i \equiv h_i$ *mod* N, $(1 \leq i \leq n)$, *always implies that*

$$\theta(g_1, \ldots, g_n) = \theta(h_1, \ldots, h_n).$$

We see from the above definition that the θ-marginal subgroups of G generate a normal subgroup which is also θ-marginal. This is called *the θ-marginal subgroup of* G and is written

$$\widehat{\theta}(G).$$

For example, suppose that $\theta = (x_1, x_2)$: if $a \in \widehat{\theta}(G)$ and $g \in G$, then

$$(g, a) = (g, 1a) = (g, 1) = 1,$$

for all $g \in G$, that is, a belongs to the centre of G, $\mathcal{Z}(G)$. Conversely, if $a \in \mathcal{Z}(G)$, then it is easy to see that

$$(g_1, g_2 a) = (g_1, g_2),$$

so that $\widehat{\theta}(G) = \mathcal{Z}(G)$ in this case.

A marginal subgroup is always *characteristic*; invariant under all automorphisms of the parent group. but need not be fully-invariant; for example, the centre of the group $A_4 \times \mathbb{Z}_2$ is not fully-invariant (see Exercise 1.5.9 in [58]).

The following lemma indicates a connection between verbal and marginal subgroups. The proof is easy and can be found in [58].

Lemma 1.3.2 *Let θ be a nontrivial word in $x_1, x_2, \ldots$, and let G be any group. Then $\theta(G) = 1$ if and only if $\widehat{\theta}(G) = G$.*

1.4 Hall's Problems

Recall that a partially ordered set Λ with partial order $\leq$ satisfies the *maximal condition* if each nonempty subset Λ_0 contains at least one maximal element, that is, an element which does not precede any other element of Λ_0. We also say that Λ satisfies the *ascending chain condition* if there does not exist an infinite properly ascending chain

$$\lambda_1 < \lambda_2 < \cdots,$$

in Λ. In fact these properties are identical (see [58]).

Philip Hall posed some questions regarding the relationship between verbal and marginal subgroups (see [57]).

Hall's Problems 1.4.1 *Let θ be a nontrivial word in n variables, and let G be a group.*

1. *If π is a set of primes and $|G : \widehat{\theta}(G)|$ is a finite π-group, is $\theta(G)$ also a finite π-group?*
2. *If $\theta(G)$ is finite and G satisfies maximal condition on its subgroups, is $|G : \widehat{\theta}(G)|$ finite?*
3. *If the set $\{\theta(g_1, \ldots, g_n) | g_1, \ldots, g_n \in G\}$ is finite, does it follows that $\theta(G)$ is finite?*

In the coming chapters, we will see that all Hall's Problems have positive solutions for the case when $\theta = (x_1, x_2)$. When θ is an arbitrary word, none of these problems has been settled. However there has been a good deal of progress thanks to the work of Hall, Merzlyakov, Schur, Baer and Turner-Smith. In Chapters 2 and 3, we will give an account of some of the results obtained by these authors.

Definition 1.4.2

1. *A group word θ, for which Hall's First Problem holds for all groups G, is called* robust.
2. *If every word θ is robust in a particular group G, then we will say that G is* verbally-robust.

Chapter 2

Hall's First Problem and Schur-Baer Theorems for Groups

2.1 Schur's Theorem

Recall that if $\theta = (x_1, x_2)$ is the commutator word in 2 indeterminates, then, for any group G, $\theta(G) = G'$ and $\widehat{\theta}(G) = \mathcal{Z}(G)$. A basic theorem of Schur ([60]) reflects a relationship between these terms.

Schur's Theorem 2.1.1 *If G is a group with $G/Z(G)$ is finite, then G' is also finite.*

Roughly speaking, Schur's Theorem says that if the centre of a group is large, the derived subgroup is small. Notice that this result provides a partial solution to Hall's First Problem. In other words, the word $\theta = (x_1, x_2)$ is robust.

Some other authors have tried to find under which conditions, the converse of Schur's theorem would be true. For example, Isaacs ([25]) proved that if a group G is *capable* and $|G'| < \infty$ then $G/\mathcal{Z}(G)$ is finite. Recall that a group is said to be capable if it occurs as the inner automorphism group of some group. Also, Halasi and Podoski ([16]) proved that the converse of Schur's Theorem holds for any group G with trivial *Frattini subgroup*; the intersection of all maximal subgroups of G.

2.2 Commutators Subgroups

It is useful to be able to form commutators of subsets as well as elements. Let $X_1, X_2, \ldots$ be nonempty subsets of a group G. Define the *commutator subgroup* of X_1 and X_2 to be

$$(X_1, X_2) = \langle (x_1.x_2) | x_1 \in X_1, x_2 \in X_2 \rangle.$$

More generally, let

$$(X_1, \ldots, X_n) = ((X_1, \ldots, X_{n-1}), X_n),$$

where $n \geq 2$. Note that the set of all commutators need not be a subgroup (see Exercise 2.43 in [59]); in order for (X_1, X_2) to be a subgroup, therefore, we must take the subgroup generated by the indicated commutators. Observe that $(X_1, X_2) = (X_2, X_1)$ by part (1) of Lemma 1.1.1. It is sometimes convenient to write $(X,_m Y)$ for $(X, \underbrace{Y, \ldots, Y}_{m})$.

2.3 The Descending and Ascending Central Series of Groups

Recall that a *characteristic subgroup* is a subgroup that is invariant under all automorphisms of the parent group.

Definition 2.3.1 *Define the characteristic subgroups $\gamma_n(G)$ of a group G by induction:*

$$\gamma_1(G) = G; \quad \gamma_{n+1}(G) = (\gamma_n(G), G),$$

for all integers $n \geq 1$.

Notice that $\gamma_2(G) = (\gamma_1(G), G) = (G, G) = G'$. It is easy to check that $\gamma_{n+1}(G) \leq \gamma_n(G)$, for all $n \geq 1$.

Definition 2.3.2 *The* descending central series (or lower central series) *of a group G is the series*

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots.$$

Notice that $\gamma_n(G)/\gamma_{n+1}(G)$ lies in the centre of $G/\gamma_{n+1}(G)$ and that each $\gamma_n(G)$ is fully-invariant in G. Also the descending central series does not in general reach 1. There is another series of interest; an ascending series of subgroups that is dual to the descending central series in the same sense that the centre is dual to the commutator subgroup.

Definition 2.3.3 *The* higher centres $\mathcal{Z}_n(G)$ *are the following characteristic subgroups of G recursively defined by:*

$$\mathcal{Z}_0(G) = 1; \quad \mathcal{Z}_{n+1}(G)/\mathcal{Z}_n(G) = \mathcal{Z}(G/\mathcal{Z}_n(G)),$$

for all integers $n \geq 1$.

Of course, $\mathcal{Z}_1(G) = \mathcal{Z}(G)$; the centre of G.

Definition 2.3.4 *The* ascending central series (or upper central series) *of G is the series*

$$1 = \mathcal{Z}_0(G) \leq \mathcal{Z}_1(G) \leq \mathcal{Z}_2(G) \leq \cdots.$$

It is known that $\mathcal{Z}_n(G)$ is not necessarily fully-invariant in G. The ascending central series need not reach G, but if G is finite, the series terminates at a subgroup called the *hypercentre*.

The crucial properties of these central series are displayed in the next result (the proof can be found in [59]).

Theorem 2.3.5 *If G is a group, then there is an integer c with* $\mathcal{Z}_c(G) = G$ *if and only if* $\gamma_{c+1}(G) = 1$. *Moreover, in this case,*

$$\gamma_{i+1}(G) \leq \mathcal{Z}_{c-i}(G) \quad \textit{for all } i.$$

Recall that a group G is *nilpotent* if there is an integer c such that $\gamma_{c+1}(G) = 1$; the least such c is called the *class* of the nilpotent group G. In particular, a group is nilpotent of class 1 if and only if it is abelian. Note that Theorem 2.3.5 shows, for nilpotent groups, that the descending and ascending central series are of the same length.

The following Lemma, due to Kaluznin and Hall (see [57]), is called the Three Subgroup Lemma.

Lemma 2.3.6 *Let H, K, L be subgroups of a group G. If two of the commutator subgroups (H, K, L), (K, L, H), (L, H, K) are contained in a normal subgroup of G, then so is the third.*

This result, which follows easily from the Hall-Witt identity, finds immediate application in the simple induction arguments which establish a number of useful properties of the descending and ascending central series of a group G.

Theorem 2.3.7 *Let G be any group and let m and n be positive integers. Then the following statements holds.*

1. $(\gamma_m(G), \gamma_n(G)) \leq \gamma_{m+n}(G)$.

2. $\gamma_m(\gamma_n(G)) \leq \gamma_{mn}(G)$.

3. *If $m \leq n$, then* $(\gamma_m(G), \gamma_n(G)) \leq \mathcal{Z}_{n-m}(G)$.

4. $\mathcal{Z}_m(G/\mathcal{Z}_n(G)) = \mathcal{Z}_{m+n}(G)/\mathcal{Z}_n(G)$.

Proof. See 5.1.11 in [57].

2.4 Baer's Theorem

Schur's Theorem raises the following natural question:

Is there a generalization to higher terms of the descending and ascending central series?

A theorem of Baer ([4]) provides positive answer to this question.

Baer's Theorem 2.4.1 *If G is a group such that $G/\mathcal{Z}_n(G)$ is finite, then $\gamma_{n+1}(G)$ is finite.*

The case $n = 1$ is, of course, Schur's Theorem. Observe that if $\theta = (x_1, \ldots, x_n)$ then, for any group G, $\theta(G) = \gamma_{n+1}(G)$ and $\widehat{\theta}(G) = \mathcal{Z}_n(G)$. Therefore, Baer's Theorem also provides a partial solution to Hall's First Problem. Furthermore, in this case, the word $\theta = (x_1, \ldots, x_n)$ is robust.

Regarding the converse of Baer's Theorem, Hekster ([23]) proved that the converse holds for finitely generated groups.

Theorem 2.4.2 *If G is a finitely generated group such that $\gamma_{n+1}(G)$ is finite, then so is $G/\mathcal{Z}_n(G)$.*

Moreover, Hatamian and others ([22]) strengthened the above result and proved it under the weaker condition that $G/\mathcal{Z}_n(G)$ is finitely generated.

Chapter 3

Hall's Second and Third Problems for Groups

3.1 Hall's Second Problem and Hall's Theorem

Recall Hall's Second Problem: *if $\theta(G)$ is finite and G satisfies maximal condition on its subgroups, is $|G:\widehat{\theta}(G)|$ finite?*

Definition 3.1.1 *Define Γ to be the set of all commutator subgroups functions obtainable from the identity function γ (defined by $\gamma(G) = G$ for all groups G) by a finite succession of commutator operations. For $\phi, \psi \in \Gamma$, define*

$$(\phi\psi)(G) = (\phi(G), \psi(G)),$$

so that Γ is a commutative groupoid generated by the single element γ. For each $\phi \in \Gamma$, define the length $l(\phi)$, by taking $l(\gamma) = 1$, and $l(\alpha\beta) = l(\alpha) + l(\beta)$ for $\alpha, \beta \in \Gamma$. We now associate with each element of Γ a word as follows:

1. *γ is associated with the word x_1.*

2. *If the words $u(x_1, \ldots, x_r)$ and $v(x_1, \ldots, x_s)$ are associated with ϕ and ψ from Γ, respectively, then*

$$(u(x_1, \dots, x_r), v(x_{r+1}, \dots, x_{r+s})),$$

is associated with $\phi\psi$.

The collection of all words associated with elements of Γ *are called* outer-commutator words.

In other words, outer-commutator words are those obtained by 'nesting' commutators, but using always different indeterminates. For example, $((x_1, x_2), (x_3, x_4, x_5), x_6)$ is an outer-commutator word, while (x_1, x_2, x_2) is not. Hall's Second Problem has a positive solution by Turner-Smith for outer-commutator words (see [68]):

Theorem 3.1.2 *Let* θ *be an outer commutator word and let* G *be group which satisfies maximal condition on its subgroups. If* $\theta(G)$ *is finite, then* $|G : \widehat{\theta}(G)|$ *is finite.*

It should be observed that if we omit the requirement that G satisfy maximal condition from its statement, Hall's Second Problem has a negative solution, even when $\theta = (x_1, x_2)$ (see Chapter 4 in [57], vol. I).

Along this same line, P. Hall, in [18], has proved the following partial converse of Baer's Theorem.

Hall's Theorem 3.1.3 *If* G *is a group such that* $\gamma_{n+1}(G)$ *is finite, then* $G/\mathcal{Z}_{2n}(G)$ *is finite.*

Combining Baer and Hall theorems, we can state that:

Some term of the ascending central series of a group G has finite index if and only if some of the descending central series of G is finite.

In the sequel, we shall refer to the collection of Schur-Baer-Hall Theorems (Theorems 2.1.1, 2.4.1 and 3.1.3, respectively) simply as the *Schur-Baer-Hall Theorem*.

3.2 Hall's Third Problem

Recall Hall's Third Problem:

If the set $\{\theta(g_1, \ldots, g_n) | \, g_1, \ldots, g_n \in G\}$ is finite, does it follows that $\theta(G)$ is finite?

Definition 3.2.1

1. *A group word θ, for which Hall's Third Problem holds for all groups G, is called* concise. *A word which is not concise is called* verbose.

2. *If every word θ is concise in a particular group G, then we will say that G is* verbally-concise.

According to the above definition, Hall's Third Problem asks whether every word is concise. Ivanov ([26]) proved that this problem has a negative solution in its form.

Theorem 3.2.2 *There exists a* 2*-generated torsion-free group G with nontrivial cyclic centre whose quotient group is an infinite periodic group of period p^2n (n odd, $n < 10^{10}$, p prime, $p >$ 5000); the word $\theta(x, y) = ((x^{pn}, y^{pn})^n, y^{pn})^n$ takes exactly two values on the group G, and the value of the word $\theta(x, y)$, that is not the unit is exactly the generating element of the centre of G.*

On the other hand, many relevant words are known to be concise. For instance, the outer commutator words by a result of John Wilson ([70]):

Theorem 3.2.3 *All outer-commutator words are concise.*

Regarding the verbally-concise groups, both Merzlyakov ([42]) and Turner-Smith ([69]) provided positive support.

Theorem 3.2.4

1. <u>Merzlyakov [42]</u>: *All linear groups are verbally-concise.*

2. Turner-Smith [69]: *All residually finite groups whose quotients are again residually finite are verbally-concise.*

In fact, all three of Hall's Problems have a positive solution in the above two classes of groups (see [42, 69]). Both of these classes contain, for example. all polycyclic groups, while the second class contains all finitely generated abelian-by-nilpotent groups. Notice that Hall's Third Problem is still open for all residually finite groups (see section 1.4 in [61]).

Chapter 4

Algebra Analogues to Hall's Problems

4.1 Verbal and Marginal Subspaces

Stewart was the first to consider (nonassociative) algebraic analogues to the concepts of verbal and marginal subgroups ([65]). From now on, we will restrict our attention only to associative algebras. Henceforth, we will reserve the term 'algebra' for associative algebras A over an arbitrary but fixed base field K. We do not assume that algebras are necessarily unital.

Definition 4.1.1 *Let $f(x_1, \ldots, x_n)$ be a polynomial in the free algebra on the set of indeterminates $\{x_1, x_2, \ldots\}$ over a field K, and let A be any algebra. We will denote by*

$$f(A) = \{f(a_1, \ldots, a_n) | \, a_1, \ldots, a_n \in A\},$$

*the set of f-*values *in A.*

1. *The* verbal subspace $\mathcal{S}_A(f)$ *of A is the subspace spanned by the set $f(A)$.*

2. *The* verbal subalgebra $\mathcal{A}_A(f)$ *of A is the subalgebra generated by the set $f(A)$.*

3. *The* verbal ideal $\mathcal{I}_A(f)$ *of A is generated, as an ideal, by the set $f(A)$.*

Definition 4.1.2 *The* marginal subspace, *$\widehat{\mathcal{S}}_A(f)$, of A is defined to be the set of all elements $z \in A$ with the property that*

$$f(b_1, \ldots, b_{i-1}, b_i + \alpha z, b_{i+1}, \ldots, b_n) = f(b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_n),$$

for each $i = 1, 2, \ldots, n$, for all choices of $b_1, \ldots, b_n$ in A and $\alpha \in K$. As with verbal subspaces, but dually, we define the marginal subalgebra $\widehat{\mathcal{A}}_A(f)$ *to be the largest subalgebra of A contained in $\widehat{\mathcal{S}}_A(f)$, and the* marginal ideal $\widehat{\mathcal{I}}_A(f)$ *to be the largest ideal of A contained in $\widehat{\mathcal{S}}_A(f)$.*

We repeat here the definition for homogeneous and multilinear polynomials for convenience.

Definition 4.1.3 *Let $f(x_1, \ldots, x_n)$ be a polynomial in the free algebra on the set $\{x_1, x_2, \ldots\}$ over a field K.*

1. *f is called* homogeneous *if each f-monomial is of the same degree in each indeterminate (where this degree may depend upon the indeterminate). By collecting together the f-monomials of given degree in each indeterminate, we can express a given polynomial f in a natural way as a sum of homogeneous polynomials; these are the* homogeneous components *of f.*

2. *f is called* multilinear *if it is linear in each of its indeterminates. In other words,*

$$f(x_1, \ldots, x_n) = \textstyle\sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)},$$

for some $\alpha_\sigma \in K$, where S_n is the symmetric group of degree n.

Notice that, when f is homogeneous, $z \in \widehat{\mathcal{S}}_A(f)$ if and only if

$$f(b_1, \ldots, b_{i-1}, z, b_{i+1}, \ldots, b_n) = 0,$$

for each $i = 1, 2, \ldots, n$ and all choices of $b_1, \ldots, b_n$ in A.

4.2 Analogues to Hall's Problems

The first three of the following problems may be regarded as analogues to Hall's Problems 1.4.1.

Problems 4.2.1 *Let f be a polynomial, and let A be any algebra.*

1. *If $\widehat{\mathcal{S}}_A(f)$ is of finite codimension in A, is $\mathcal{S}_A(f)$ finite-dimensional?*

2. *If $\mathcal{S}_A(f)$ is finite-dimensional, is $\widehat{\mathcal{S}}_A(f)$ of finite codimension in A? If this is not the case, under what extra hypothesis is $A/\widehat{\mathcal{S}}_A(f)$ finite-dimensional?*

3. *If $\mathcal{S}_A(f)$ finite-dimensional, is $\mathcal{A}_A(f)$ or $\mathcal{I}_A(f)$ finite-dimensional?*

4. *If $A/\widehat{\mathcal{S}}_A(f)$ is finite-dimensional, is $A/\widehat{\mathcal{A}}_A(f)$ or $A/\widehat{\mathcal{I}}_A(f)$ finite-dimensional?*

The following result of Stewart is a special case of Theorem 5.2 in [65]: it solves the above first problem.

Theorem 4.2.2 *Let A be an algebra, and let f be a polynomial with the property that $\widehat{\mathcal{S}}_A(f)$ is of finite codimension in A. Then $\mathcal{S}_A(f)$ is finite-dimensional.*

Recall that an algebra A can be viewed as a Lie algebra via the Lie bracket $[a, b] = ab - ba$, for all $a, b \in A$. As in Part I, throughout this part also, Lie commutators $[x_1, x_2, \ldots, x_n]$ are assumed to be *left normed* Lie commutators.

The following Lie bracket identities will be frequently used. Each identity follows easily by expansion of the Lie products.

Lemma 4.2.3 *The following identities hold for all algebras A.*

1. *Adjoint maps are derivations; in other words, for all $a, b, c \in A$,*

$$[ab, c] = a[b, c] + [a, c]b.$$

2. *The semi-Jacobi identity holds; namely, for all $a, b, c \in A$,*

$$[ab, c] = [a, bc] + [b, ca].$$

Stewart also proved that, whenever f is multilinear, $\widehat{\mathcal{S}}_A(f)$ is closed under derivations (see Proposition 5.1 in [65]). Consequently, we have the following result.

Theorem 4.2.4 *Let $f(x_1, \ldots, x_n)$ be any multilinear polynomial. Then $\widehat{S}_A(f)$ is a Lie ideal in the associative algebra A.*

Proof. The result follows from part (1) of Lemma 4.2.3 and Proposition 5.1 in [65]. $\square$

Chapter 5

An Analogue of the Schur-Baer-Hall Theorem for Lie Algebras

5.1 Lie Algebra Analogue of the Schur-Baer-Hall Theorem

Definition 5.1.1 *We shall denote the* descending central series *of a Lie algebra L by*

$$L = \gamma_1(L) \supseteq \gamma_2(L) \supseteq \gamma_3(L) \supseteq \cdots,$$

where $\gamma_{n+1}(L) = [\gamma_n(L), L] = [L,_n L]$, for all positive integers n, and its ascending central series *by*

$$0 = \mathcal{Z}_0(L) \subseteq \mathcal{Z}_1(L) \subseteq \mathcal{Z}_2(L) \subseteq \cdots,$$

where $\mathcal{Z}_{n+1}(L)/\mathcal{Z}_n(L)$ is the centre of $L/\mathcal{Z}_n(L)$, for each $n \geq 0$.

Recall that $\gamma_{c+1}(L) = 0$ precisely when $\mathcal{Z}_c(L) = L$; if such c is minimal, then L is said to be *nilpotent of class c*. In the case when L is an associative algebra being viewed as a Lie algebra, L is said to be *Lie nilpotent of class c*.

Using a similar approach used to prove the Schur-Baer-Hall Theorem, Stewart proved in [65] the following precise analogues in the category of Lie algebras:

Theorem 5.1.2 *Let L be a Lie algebra, and let n be a positive integer.*

1. *If $L/\mathcal{Z}_n(L)$ is finite-dimensional, then so is $\gamma_{n+1}(L)$.*

2. *If $\gamma_{n+1}(L)$ is finite-dimensional, then so is $L/\mathcal{Z}_{2n}(L)$.*

Notice that, since every associative algebra can be viewed as a Lie algebra, Theorem 5.1.2 holds, in particular, for associative algebras.

Chapter 6

The Canonical Central Series of Ideals of an Associative Algebra

6.1 Lie Powers and Higher Strong Centres

Notice that, if A is an associative algebra, then $\gamma_n(A)$ and $\mathcal{Z}_n(A)$ are very rarely ideals of A in the associative sense. Thus, within the category of associative algebras, neither $\{\gamma_n(A)\}$ nor $\{\mathcal{Z}_n(A)\}$ is a central series of ideals. Furthermore, if $A^{[n]}$ is taken to be the associative ideal in A generated by $\gamma_n(A)$, then the series of ideals

$$A = A^{[1]} \supseteq A^{[2]} \supseteq A^{[3]} \supseteq \cdots$$

is not normally a central series. For instance, consider the case when A is a Grassmann algebra of a vector space of dimension at least 4 over a field of characteristic not 2. Then it is easy to check that $[A^{[2]}, A] \neq 0$, while $A^{[3]} = 0$. Thus, we need to consider the following more subtle modifications. Notice first that Jennings, in [27], was the first to study central series in the category of associative algebras.

Definition 6.1.1 *Let A be an associative algebra.*

1. *Set $A^{(1)} = A$ and recursively define $A^{(n+1)}$ to be the associative ideal in A generated by the*

Lie ideal $[A^{(n)}, A]$*, for each* $n \geq 1$*. The ideal* $A^{(n)}$ *is sometimes called the* n^{th} Lie power *of* A*.*

2. *Let* $F(A)$ *denote the largest associative ideal contained in the centre of* A*. Set* $F^{(0)}(A) = 0$*. Then, for each* $n \geq 0$*,* $F^{(n+1)}(A)$ *is the ideal in* A *given by*

$$F^{(n+1)}(A)/F^{(n)}(A) = F(A/F^{(n)}(A)).$$

The ideal $F^{(n)}(A)$ *is sometimes referred to as the* n^{th} strong centre *of* A*.*

In particular, Jennings proved in [27] that the series $\{A^{(n)}\}$ and $\{F^{(n)}(A)\}$ are, respectively, the canonical fastest descending and ascending central series of A in the category of associative algebras. Moreover, he proved that $A^{(n+1)} = 0$ precisely when $F^{(n)}(A) = A$. As we saw in Part I, in the case when $A^{(c+1)} = 0$, and such c is minimal, A is called *upper Lie nilpotent of class* c.

Since $\gamma_n(A) \subseteq A^{(n)}$ and $F^{(n)}(A) \subseteq Z_n(A)$, for all positive integers n, an upper Lie nilpotent algebra is always Lie nilpotent. The converse, however, need not be true: an infinite-dimensional Grassmann algebra over a field of characteristic not 2 is Lie nilpotent of class 2 but not upper Lie nilpotent of any class. Gupta and Levin ([15]) constructed a similar example in characteristic 2. On the other hand, it follows from Theorem 1.5.2, due to Riley and Wilson, that upper Lie nilpotence does follow from Lie nilpotence whenever the algebra is finitely generated.

Chapter 7

Associative Algebra Analogues of the Schur-Baer-Hall Theorem

7.1 Associative Powers

Consider the *descending* series of an algebra A given by the associative powers of A,

$$A = A^1 \supseteq A^2 \supseteq A^3 \supseteq \cdots,$$

and the corresponding *ascending* series of A,

$$0 = \mathrm{Ann}^0(A) \subseteq \mathrm{Ann}^1(A) \subseteq \mathrm{Ann}^2(A) \subseteq \cdots,$$

where $\mathrm{Ann}^{n+1}(A)$, for all $n \geq 0$, is the ideal of A given by

$$\mathrm{Ann}^{n+1}(A)/\mathrm{Ann}^n(A) = \mathrm{Ann}(A/\mathrm{Ann}^n(A)),$$

the two-sided annihilator of the algebra $A/\mathrm{Ann}^n(A)$.

We will prove the following analogue of the Schur-Baer-Hall Theorem in this case.

Theorem 7.1.1 *Let A be an algebra, and let n be a positive integer. Then the following statements hold.*

1. *If $A/\mathrm{Ann}^n(A)$ is finite-dimensional, then so is A^{n+1}.*

2. *Conversely, if A is finitely generated and A^{n+1} is finite-dimensional, then $A/\mathrm{Ann}^n(A)$ is also finite-dimensional.*

3. *In general, if A^{n+1} is finite-dimensional, then so is $A/\mathrm{Ann}^{2n+1}(A)$.*

Later, we will see that part (2) of Theorem 7.1.1 support positively Problem (2) in 4.2.1 for the case when $f = x_1 \cdots x_{n+1}$. In Chapter 10, we will provide a counterexample shows that Problem (2) in 4.2.1 fails to hold in general.

7.2 Lie Powers

We also will prove the following more natural associative algebra analogue of the Schur-Baer-Hall Theorem.

Theorem 7.2.1 *Let A be an algebra. Then the following statements hold for every positive integer n.*

1. *If $A/F^{(n)}(A)$ is finite-dimensional, then so is $A^{(n+1)}$.*

2. *Conversely, if A is finitely generated and $A^{(n+1)}$ is finite-dimensional, then $A/F^{(n)}(A)$ is finite-dimensional.*

3. *For an arbitrary algebra A, if $A^{(n+1)}$ is finite-dimensional, then so is $A/F^{(3n-1)}(A)$.*

Part (2) of Theorem 7.2.1 provides a positive solution for Problem (2) in 4.2.1 for the polynomial g_n (which will be defined in Chapter 8). Moreover, in Chapter 10, we will give an example showing that Problem (2) need not hold for non-finitely generated algebras with respect to the polynomial g_n.

7.3 Some Related Results

We will use Theorem 7.2.1 to prove the following result.

Theorem 7.3.1 *Let A be a finitely generated algebra. Then the following statements hold for every positive integer n.*

1. *If $\gamma_{n+1}(A)$ is finite-dimensional, then there exists a positive integer m such that $A^{(m+1)}$ is also finite-dimensional.*

2. *If $A/\mathcal{Z}_n(A)$ is finite-dimensional, then there exists a positive integer m such that $A/F^{(m)}(A)$ is also finite-dimensional.*

Note that the examples, mentioned in Chapter 6 showing that Lie nilpotence does not, in general, implies upper Lie nilpotence, show that the finite generation hypothesis in Theorem 7.3.1 cannot be omitted.

In order to prove Theorem 7.3.1, we will also require the following result, which is of interest in its own right.

Theorem 7.3.2 *Let A be an algebra and let n be a positive integer. Then the following statements hold.*

1. *If $\gamma_{n+1}(A)$ is finite-dimensional, then so is the associative ideal, $A^{[n+1]}$, it generates.*

2. *Let $\mathcal{Z}_{[n]}(A)$ be the largest associative ideal of A contained in $\mathcal{Z}_n(A)$. If $A/\mathcal{Z}_n(A)$ is finite-dimensional, then so is $A/\mathcal{Z}_{[n]}(A)$. Thus, in particular, if $A/\mathcal{Z}(A)$ is finite-dimensional, then so is $A/F(A)$.*

Chapter 8

The Jennings Triple Product

Throughout this chapter, it will be convenient to assume that all algebras are unital. The principal reason is that this will allow us to write

$$A^{(n+1)} = A[A^{(n)}, A]A = [A^{(n)}, A]A,$$

for each positive integer n.

8.1 Jennings Triple Product

Definition 8.1.1

1. *Let $a_1, a_2, \ldots$ be elements in A. We define the* Jennings triple product *on A by*

$$[\![a_1, a_2, a_3]\!] = [a_1, a_2]a_3.$$

   Furthermore, for each $n \geq 2$, we write

$$[\![a_1, \ldots, a_{2n+1}]\!] = [\![[\![a_1, \ldots, a_{2n-1}]\!], a_{2n}, a_{2n+1}]\!].$$

2. *Given a subspace B of A, we set $\mu_1(B, A) = [\![B, A, A]\!]$ and*

$$\mu_n(B, A) = [\![\mu_{n-1}(B, A), A, A]\!],$$

for each $n \geq 2$.

Both parts of the following lemma follow from simple inductive arguments.

Lemma 8.1.2 *For every (unital) algebra* A *and integers* $m, n \geq 1$*, the following statements hold.*

1. *For every subspace* B *of* A*,* $\mu_m(\mu_n(B, A), A) = \mu_{m+n}(B, A)$.
2. $\mu_n(A, A) = A^{(n+1)}$.

Parts (1)-(4) of the following result are due to Jennings ([27]). Part (5) follows easily from part (4) by induction.

Theorem 8.1.3 *Let* A *be an algebra, and let* m *and* n *be positive integers. Then the following statements hold.*

1. $[A^{(m)}, A^{(n)}] \subseteq A^{(m+n)}$.
2. $A^{(m)}A^{(n)} \subseteq A^{(m+n-1)}$.
3. *If* $m \leq n$*, then* $A^{(m)}F^{(n)}(A) \subseteq F^{(n-m+1)}(A)$.
4. *If* $m \leq n$*, then* $[A^{(m)}, F^{(n)}(A)] \subseteq F^{(n-m)}(A)$.
5. *If* $m \leq n$*, then* $\mu_m(F^{(n)}(A), A) \subseteq F^{(n-m)}(A)$.

Lemma 8.1.4 *In the free algebra on the indeterminates* w, x, x', y, y', z*, the following identities hold.*

1. $[\![xx', y, z]\!] = [\![x, x'y, z]\!] + [\![x', yx, z]\!]$.
2. $$[\![wxx', y, z]\!] = [\![wx, x'y, z]\!] - [\![wx, x', yz]\!] + [\![wx, x', 1, y, z]\!] + [\![wx', y, xz]\!] - [\![w, x', 1, y, xz]\!] - [\![w, y, x'xz]\!] + [\![w, y, 1, x', xz]\!].$$
3. $[\![wx, yy', z]\!] = [\![wx, y, y'z]\!] + [\![wx, y', yz]\!] - [\![wx, y', 1, y, z]\!]$.

Proof. Using the semi-Jacobi identity (part (2) of Lemma 4.2.3), we have

$$\begin{aligned}
[\![xx', y, z]\!] &= [xx', y]z \\
&= [x, x'y]z + [x', yx]z \\
&= [\![x, x'y, z]\!] + [\![x', yx, z]\!].
\end{aligned}$$

This prove part (1). In what follows, we use the two identities given in Lemma 4.2.3 freely. To prove part (2), observe that

$$\begin{aligned}
[\![wxx', y, z]\!] &= [wxx', y]z \\
&= [wx, x'y]z + [x', ywx]z \\
&= [\![wx, x'y, z]\!] + y[x', wx]z + [x', y]wxz.
\end{aligned}$$

Clearly,

$$\begin{aligned}
y[x', wx]z &= [x', wx]yz + [y, [x', wx]]z \\
&= -[\![wx, x', yz]\!] + [\![wx, x', 1, y, z]\!].
\end{aligned}$$

It remains to observe that

$$\begin{aligned}
[x', y]wxz &= [x'w, y]xz - x'[w, y]xz \\
&= [wx', y]xz + [[x', w], y]xz - [w, y]x'xz - [x', [w, y]]xz \\
&= [\![wx', y, xz]\!] - [\![w, x', 1, y, xz]\!] - [\![w, y, x'xz]\!] + [\![w, y, 1, x', xz]\!].
\end{aligned}$$

To prove part (3), notice that

$$\begin{aligned}
[\![wx, yy', z]\!] &= [wx, yy']z \\
&= [wx, y]y'z + y[wx, y']z \\
&= [\![wx, y, y'z]\!] + [wx, y']yz + [y, [wx, y']]z \\
&= [\![wx, y, y'z]\!] + [\![wx, y', yz]\!] - [\![wx, y', 1, y, z]\!].
\end{aligned}$$

□

Definition 8.1.5 *For each positive integer n, we define the polynomial g_n by*

$$g_n(x, y_1, z_1, \dots, y_n, z_n) = [\![x, y_1, z_1, y_2, z_2, \dots, y_n, z_n]\!].$$

Notice that, for every (unital) algebra A, we have $\mathcal{S}_A(g_n) = A^{(n+1)}$. Moreover, A is upper Lie nilpotent of class at most n precisely when A satisfies the polynomial identity $g_n = 0$.

Lemma 8.1.6 *In the free algebra on $X = \{w, x, y_1, z_1, \dots, y_n, z_n\}$, the following identities hold.*

1. *For each positive integer n, we have*

$$g_n(wx, y_1, z_1, \dots, y_n, z_n) = g_n(w, xy_1, z_1, \dots, y_n, z_n) + g_n(x, y_1w, z_1, \dots, y_n, z_n).$$

2. *For each $1 \le i \le n$, if $v_{i-1} = [\![x, y_1, z_1, \dots, y_{i-1}, z_{i-1}]\!]$, then*

$$\begin{aligned}
g_n(v_{i-1}, y_iw, z_i, \dots, y_n, z_n) &= g_n(v_{i-1}, y_i, wz_i, \dots, y_n, z_n) + g_n(v_{i-1}, w, y_iz_i, \dots, y_n, z_n) \\
&\quad - [\![g_n(v_{i-1}, w, 1, y_i, z_i, \dots, y_{n-1}, z_{n-1}), y_n, z_n]\!].
\end{aligned}$$

3. *For each* $1 \leq i \leq n-1$, *if* $v_i = [\![x, y_1, z_1, \ldots, y_i, 1]\!]$, *then*

$$\begin{aligned} g_n(v_i w z_i, y_{i+1}, z_{i+1}, \ldots, y_n, z_n) &= g_n(v_i, w z_i y_{i+1}, z_{i+1}, \ldots, y_n, z_n) \\ &\quad - g_n(v_i y_{i+1}, w z_i, z_{i+1}, \ldots, y_n, z_n) \\ &\quad + [\![g_n(v_i, y_{i+1}, 1, w z_i, z_{i+1}, \ldots, y_{n-1}, z_{n-1}), y_n, z_n]\!]. \end{aligned}$$

Proof. Part (1) follows by part (1) of Lemma 8.1.4, while part (2) follows directly from part (3) of Lemma 8.1.4.

To prove part (3), we use part (1) of Lemma 8.1.4 to observe that

$$\begin{aligned} [\![v_i w z_i, y_{i+1}, z_{i+1}, \ldots, y_n, z_n]\!] &= [\![v_i, w z_i y_{i+1}, z_{i+1}, \ldots, y_n, z_n]\!] + [\![w z_i, y_{i+1} v_i, z_{i+1}, \ldots, y_n, z_n]\!] \\ &= g_n(v_i, w z_i y_{i+1}, z_{i+1}, \ldots, y_n, z_n) + [\![w z_i, v_i y_{i+1}, z_{i+1}, \ldots, y_n, z_n]\!] \\ &\quad + [\![w z_i, [y_{i+1}, v_i], z_{i+1}, \ldots, y_n, z_n]\!]. \end{aligned}$$

On the other hand,

$$[\![w z_i, v_i y_{i+1}, z_{i+1}, \ldots, y_n, z_n]\!] = -g_n(v_i y_{i+1}, w z_i, z_{i+1}, \ldots, y_n, z_n),$$

and

$$\begin{aligned} [\![w z_i, [y_{i+1}, v_i], z_{i+1}, \ldots, y_n, z_n]\!] &= [\![v_i, y_{i+1}, 1, w z_i, z_{i+1}, \ldots, y_n, z_n]\!] \\ &= [\![g_n(v_i, y_{i+1}, 1, w z_i, z_{i+1}, \ldots, y_{n-1}, z_{n-1}), y_n, z_n]\!], \end{aligned}$$

as required. □

Remark 8.1.7 *Observe that, since A was assumed to be unital, we have*

$$F^{(n)}(A) = \{z \in A \mid [zA, A]A \subseteq F^{(n-1)}(A)\},$$

for all $n \geq 1$.

We can further characterize the ideal $F^{(n)}(A)$ as follows:

Theorem 8.1.8 *Let A be an algebra, and let $n \geq 1$ be an integer. Then the following statements hold.*

1. $F^{(n)}(A) = \{z \in A \mid \mu_n(zA, A) = 0\}$*; in other words,*

$$F^{(n)}(A) = \{z \in A \mid g_n(za, b_1, c_1, \ldots, b_n, c_n) = 0, \textit{ for all } a, b_i, c_i \in A\}.$$

2. $F^{(n)}(A) = \widehat{\mathcal{S}}_A(g_n)$.

Proof. We use induction on n to prove part (1). If $n = 1$, then

$$F^{(1)}(A) = \{z \in A \mid [zA, A]A = 0\} = \{z \in A \mid \mu_1(zA, A) = 0\}.$$

Notice that, by part (1) of Lemma 8.1.2,

$$\mu_{n-1}([zA, A]A, A) = \mu_{n-1}(\mu_1(zA, A), A) = \mu_n(zA, A).$$

Thus, if $F^{(n-1)}(A) = \{z \in A \mid \mu_{n-1}(zA, A) = 0\}$, then

$$\begin{aligned} F^{(n)}(A) &= \{z \in A \mid [zA, A]A \subseteq F^{(n-1)}(A)\} \\ &= \{z \in A \mid \mu_{n-1}([zA, A]A, A) = 0\} \\ &= \{z \in A \mid \mu_n(zA, A) = 0\}, \end{aligned}$$

as required.

To prove the inclusion $F^{(n)}(A) \subseteq \widehat{\mathcal{S}}_A(g_n)$ in part (2), it suffices to show that $g_n = 0$ whenever one of the indeterminates in

$$g_n(x, y_1, z_1, \ldots, y_n, z_n) = [\![x, y_1, z_1, y_2, z_2, \ldots, y_n, z_n]\!],$$

is evaluated in $F^{(n)}(A)$. First suppose that x is evaluated in $F^{(n)}(A)$. Then, using part (5) of Theorem 8.1.3,

$$g_n \in \mu_n(F^{(n)}(A), A) \subseteq F^{(n-n)}(A) = 0.$$

Similarly, if y_1 is evaluated in $F^{(n)}(A)$, $g_n = 0$. Now suppose that $2 \le i \le n$ and y_i is evaluated in $F^{(n)}(A)$. Then, by parts (4) and (5) of Theorem 8.1.3,

$$g_n \in \mu_{n-i}([A^{(i)}, F^{(n)}(A)]A, A) \subseteq \mu_{n-i}(F^{(n-i)}(A), A) = 0.$$

Finally suppose that $1 \le i \le n$ and z_i is evaluated in $F^{(n)}(A)$. Then, by parts (3) and (5) of Theorem 8.1.3,

$$g_n \in \mu_{n-i}(A^{(i+1)}F^{(n)}(A), A) \subseteq \mu_{n-i}(F^{(n-i)}(A), A) = 0.$$

This proves that $F^{(n)}(A) \subseteq \widehat{\mathcal{S}}_A(g_n)$. To prove the reverse inclusion, we use the characterization of $F^{(n)}(A)$ given in part (1). So, let $z \in \widehat{\mathcal{S}}_A(g_n)$, and let $a, b_1, c_1, \ldots, b_n, c_n$ be arbitrary elements in A. Then, by part (1) of Lemma 8.1.6, we have

$$\begin{aligned} g_n(za, b_1, c_1, \ldots, b_n, c_n) &= g_n(z, ab_1, c_1, \ldots, b_n, c_n) + g_n(a, b_1z, c_1, \ldots, b_n, c_n) \\ &= g_n(a, b_1z, c_1, \ldots, b_n, c_n). \end{aligned}$$

Now, an inductive argument based on the other parts of Lemma 8.1.6 and the fact that $\widehat{\mathcal{S}}_A(g_n)$ is a Lie ideal (by Theorem 4.2.4) allows us to continue 'shifting z towards the right', so that ultimately

$$g_n(za, b_1, c_1, \ldots, b_n, c_n) \in \mathcal{S}_A(g_n)zA = 0,$$

as required. □

Notice that it is unusual to be able to conclude that a polynomial identity holds on an entire algebra knowing only that it holds on a given set of generators; however, this is indeed the case for the polynomials g_n:

Theorem 8.1.9 *Let A be an algebra generated by a set Y. Then the following statements hold.*

1. *A is an upper Lie nilpotent of class at most n if and only if*

$$g_n(a, b_1, c_1, \ldots, b_n, c_n) = [\![a, b_1, c_1, \ldots, b_n, c_n]\!] = 0,$$

for all $a, b_1, \ldots, b_n, c_1, \ldots, c_n \in Y$.

2. *An element* z *in* A *lies in* $F^{(n)}(A)$ *if and only if*

$$g_n(za, b_1, c_1, \ldots, b_n, c_n) = [\![za, b_1, c_1, \ldots, b_n, c_n]\!] = 0,$$

for all $a, b_1, \ldots, b_n, c_1, \ldots, c_n \in Y$.

Proof. Necessity in part (1) is trivial. For sufficiency, let $a, b_1, c_1, \ldots, b_n, c_n$ be arbitrary products of elements from Y. Then, by Lemma 8.1.6 and induction, it is clear that the element $g_n(a, b_1, c_1, \ldots, b_n, c_n)$ lies in $g_n(Y)A$. Thus, if $g_n(Y) = 0$, then A is an upper Lie nilpotent of class at most n. The proof of part (2) is similar to part (1) by first using part (1) of Theorem 8.1.8. □

Chapter 9

Proofs of the Algebra Analogues of Schur-Baer-Hall Theorem

9.1 Proof of Theorem 7.1.1

Recall Theorem 7.1.1: *Let A be an algebra, and let n be a positive integer. Then the following statements hold.*

1. *If $A/\mathrm{Ann}^n(A)$ is finite-dimensional, then so is A^{n+1}.*

2. *Conversely, if A is finitely generated and A^{n+1} is finite-dimensional, then $A/\mathrm{Ann}^n(A)$ is also finite-dimensional.*

3. *In general, if A^{n+1} is finite-dimensional, then so is $A/\mathrm{Ann}^{2n+1}(A)$.*

Definition 9.1.1 *For each integer $n \geq 1$, we define the polynomial f_n by*

$$f_n(x_1, \ldots, x_{n+1}) = x_1 \cdots x_{n+1}.$$

Therefore, it follows from the above definition that, for every algebra A, $\mathcal{S}_A(f_n) = A^{n+1}$.

Lemma 9.1.2 *Let A be an algebra, and let n be a positive integer. Then the following statements hold.*

1. $\mathrm{Ann}^n(A) = \{z \in A \mid \sum_{i=0}^{n} A^{n-i}zA^i = 0\} = \widehat{\mathcal{S}}_A(f_n)$, *where* $A^0 = K$.

2. *If A is generated by a set Y, then* $\mathrm{Ann}^n(A)$ *consists of all the elements z in A such that, for all* $a_1, \ldots, a_n \in Y$ *and* $0 \le i \le n$*, we have*

$$a_1 \cdots a_i z a_{i+1} \cdots a_n = 0.$$

Proof. We use induction on $n \ge 0$ to prove part (1). The statement is trivial for $n = 0$. Let $z \in \mathrm{Ann}^{n+1}(A)$. Then $zA, Az \subseteq \mathrm{Ann}^n(A)$. Hence

$$\textstyle\sum_{i=0}^{n} A^{n-i}zA^i = 0.$$

by the induction hypotheses. The reverse inclusion is trivial. Part (2) follows easily from part (1). □

Proof of part (1) of Theorem 7.1.1: Suppose that the dimension of $A/\mathrm{Ann}^n(A)$ is finite. Then, since $\mathrm{Ann}^n(A) = \widehat{\mathcal{S}}_A(f_n)$, by part (1) of Lemma 9.1.2, it follows from Theorem 4.2.2 that $A^{n+1} = \mathcal{S}_A(f_n)$ is also finite-dimensional, as required. □

Proof of part (2) of Theorem 7.1.1: Let A be an algebra generated by a finite set Y, and suppose that $\dim A^{n+1} < \infty$. Fix $a_1, \ldots, a_n \in Y$ and $0 \le i \le n$, and consider the linear map

$$\lambda : A \to A^{n+1} : z \mapsto a_1 \cdots a_i z a_{i+1} \cdots a_n.$$

Then, since $\dim A^{n+1} < \infty$, $\ker(\lambda)$ is of finite codimension in A. Because there are only finitely many maps of this form, $\bigcap_\lambda \ker(\lambda)$ is also of finite codimension in A. But, by part (2) of Lemma 9.1.2, $\mathrm{Ann}^n(A) = \bigcap_\lambda \ker(\lambda)$, and hence

$$\dim A/\mathrm{Ann}^n(A) < \infty,$$

as required. □

Proof of part (3) of Theorem 7.1.1: Let A be any algebra such that $\dim A^{n+1} < \infty$, for some integer $n \ge 1$. For each $a \in A^{n+1}$, define the linear maps

$$\lambda_a : A \to A^{n+2} : z \mapsto az \quad \text{and} \quad \rho_a : A \to A^{n+2} : z \mapsto za.$$

It is clear that $\ker(\lambda_a)$ is the right annihilator of a in A, while $\ker(\rho_a)$ is the left annihilator of a in A. Thus, since $\dim A^{n+2} < \infty$, the subspaces $\ker(\lambda_a)$ and $\ker(\rho_a)$ are both of finite codimension in A, for all $a \in A^{n+1}$. Therefore,

$$\mathrm{Ann}(A^{n+1}) = \bigcap_{a \in A^{n+1}} \ker(\lambda_a) \cap \ker(\rho_a),$$

is of finite codimension in A. Since

$$\mathrm{Ann}^{2n+1}(A) = \{z \in A \mid \sum_{i+j=2n+1} A^i z A^j = 0\},$$

by part (1) of Lemma 9.1.2, we have $\mathrm{Ann}(A^{n+1}) \subseteq \mathrm{Ann}^{2n+1}(A)$. Consequently, $A/\mathrm{Ann}^{2n+1}(A)$ is finite-dimensional, as required. □

9.2 Proof of Theorem 7.2.1

Recall Theorem 7.2.1: *Let A be an algebra. Then the following statements hold for every positive integer n.*

1. *If $A/F^{(n)}(A)$ is finite-dimensional, then so is $A^{(n+1)}$.*
2. *Conversely, if A is finitely generated and $A^{(n+1)}$ is finite-dimensional, then $A/F^{(n)}(A)$ is finite-dimensional.*
3. *For an arbitrary algebra A, if $A^{(n+1)}$ is finite-dimensional, then so is $A/F^{(3n-1)}(A)$.*

If A happens to be non-unital, let A_1 be its unital hull. Then it is easy to see that, for every integer $n \geq 1$, we have

$$A_1^{(n)} = A^{(n)} \qquad \text{and} \qquad F^{(n)}(A_1) = F^{(n)}(A).$$

Therefore, to prove Theorem 7.2.1, it is fair to assume that A is unital.

Proof of part (1) of Theorem 7.2.1: Suppose that $A/F^{(n)}(A)$ is finite-dimensional. Then, since $F^{(n)}(A) = \widehat{\mathcal{S}}_A(g_n)$ by part (2) of Theorem 8.1.8, it follows from Theorem 4.2.2 that $A^{(n+1)} = \mathcal{S}_A(g_n)$ is also finite-dimensional, as claimed. □

Our proof of the remaining parts of Theorem 7.2.1 requires the following two propositions.

Proposition 9.2.1 *For every algebra A, the following statements hold.*

1. *For each positive integer n,*

$$A^{(n+1)} = \sum \gamma_{n_1}(A)\cdots\gamma_{n_t}(A)A,$$

where the sum is over all integers $t \geq 1$ *and* $n_1, \ldots, n_t \geq 2$ *with the property that*

$$\textstyle\sum_{i=1}^{t}(n_i - 1) = n.$$

2. *If* $A^{(n+1)} = 0$*, then* $\gamma_2(A)^n = 0$ *and* $\gamma_{n+1}(A) = 0$*. Conversely, if, for some positive integers m and c,* $\gamma_2(A)^m = 0$ *and* $\gamma_{c+1}(A) = 0$*, then* $A^{(n+1)} = 0$*, where* $n = (m-1)(c-2)+1$*.*

Proof. Using part (2) of Theorem 8.1.3, we have

$$\sum \gamma_{n_1}(A)\cdots\gamma_{n_t}(A)A \subseteq A^{(n+1)}.$$

We will prove the reverse inclusion using induction on $n \geq 0$. Clearly the inclusion is true for $n = 0$. Assume that

$$A^{(n+1)} \subseteq \sum \gamma_{n_1}(A)\cdots\gamma_{n_t}(A)A.$$

Then, by part (1) of Lemma 4.2.3, we have

$$\begin{aligned} A^{(n+2)} &= [A^{(n+1)}, A]A \\ &\subseteq \sum_{\sum_{i=1}^{t}(n_i-1)=n} [\gamma_{n_1}(A)\cdots\gamma_{n_t}(A)A, A]A \\ &\subseteq \sum_{\sum_{i=1}^{t}(n_i-1)=n} \sum_{j=1}^{t} \gamma_{n_1}(A)\cdots\gamma_{n_j}(A)\gamma_{n_j+1}(A)\gamma_{n_{j+1}}(A)\cdots\gamma_{n_t}(A)A, \end{aligned}$$

as required. The first claim in part (2) follows from part (1). Now suppose that $\gamma_2(A)^m = 0$ and $\gamma_{c+1}(A) = 0$, for some positive integers m and c, and set $n = (m-1)(c-2)+1$. Then $A^{(n+1)} = 0$ because, if, in the expression for $A^{(n+1)}$ given in part (1) we have

$$\sum_{i=1}^{t}(n_i - 1) = n = (m-1)(c-2) + 1,$$

then either $t \geq m$ or some $n_i \geq c + 1$ (by a pigeonhole type argument). □

Proposition 9.2.2 *In the free algebra $\mathcal{A}\langle X\rangle$ on $X = \{w, x, y_1, z_1, y_2, z_2, \ldots\}$, the following statements hold.*

1. *Let $1 \leq i \leq m$ be integers. If $v_{i-1} = [y_1, \ldots, y_{i-1}]$, then*

   $$[v_{i-1}, y_i x, y_{i+1}, \ldots, y_m]$$

   is a linear combination of products $v_1 \cdots v_t$ of left-normed Lie monomials $v_1, \ldots, v_t$ in the free Lie algebra $\mathcal{L}\langle X\rangle$ on X, where $t \geq 1$, each $v_j \in \gamma_{m_j}(\mathcal{L}\langle X\rangle)$, v_1 starts with y_1, and $\sum_{j=1}^{t}(m_j - 1) = m$. The corresponding statement also holds for $[v_{i-1}, xy_i, y_{i+1}, \ldots, y_m]$.

2. *The polynomial $g_n(wx, y_1, z_1, \ldots, y_n, z_n)$ is a linear combination of multilinear products of degree $2n+2$ of the form $v_1 \cdots v_t$, where $t \geq 1$ and each v_j is a left-normed Lie monomial of length n_j in $\mathcal{L}\langle X\rangle$ such that $\sum_{j=1}^{t}(n_j - 1) = n$. Furthermore, we may assume that each v_1 starts with any choice of the indeterminates $w, x, y_1, z_1, \ldots, y_n, z_n$.*

Proof. To prove part (1), we use part (1) of Lemma 4.2.3 to first write

$$\begin{aligned}
[[v_{i-1}, y_i x], y_{i+1}, \ldots, y_m] &= [y_i[v_{i-1}, x] + [v_{i-1}, y_i]x, y_{i+1}, \ldots, y_m] \\
&= y_i[v_{i-1}, x, y_{i+1}, \ldots, y_m] + [y_i, y_{i+1}, \ldots, y_m][v_{i-1}, x] \\
&\quad + [v_{i-1}, y_i][x, y_{i+1}, \ldots, y_m] + [v_{i-1}, y_i, y_{i+1}, \ldots, y_m]x.
\end{aligned}$$

The last two terms of the last sum are already in the correct form, while the first term is easily rewritten as a linear combination of two terms, each in correct form. The obstruction to rewriting the second term is

$$[[v_{i-1}, x], [y_i, y_{i+1}, \ldots, y_m]].$$

But this can be rewritten as a linear combination of terms in correct form using induction and the Jacobi identity. This proves part (1).

To prove (2), first observe that, if $v_1 = [v_k, x, z_1, \ldots, z_l]$, say, where $k + l + 1 = n_1$, then, by another inductive argument using the Jacobi identity,

$$v_1 = -[[x, v_k], z_1, \ldots, z_l] \in [x, \gamma_k(\mathcal{L}\langle X\rangle),_l \mathcal{L}\langle X\rangle] \subseteq [x,_{k+l} \mathcal{L}\langle X\rangle].$$

Similarly, we have

$$v_j v_1 \subseteq v_1 v_j + [v_1,_{n_j} \mathcal{L}\langle X\rangle].$$

With these two facts in hand, it is now easy to see that part (2) follows from part (1) of Proposition 9.2.1 together with part (1) above. □

Proof of part (2) of Theorem 7.2.1: Let A be an algebra generated by a finite set Y, and suppose that $\dim A^{(n+1)} < \infty$. We need to prove that $\dim A/F^{(n)}(A) < \infty$, as well. To this end, let $t \geq 1$ and let $v_1, \ldots, v_t$ be Lie monomials of lengths $n_1, \ldots, n_t$ in $\mathcal{L}\langle X\rangle$ such that $v_1 \cdots v_t$ is multilinear of degree $2n + 2$ and $\sum_{i=1}^{t}(n_i - 1) = n$. Furthermore, suppose that v_1 starts with w. Then, by part (2) of Theorem 8.1.8 and part (1) of Proposition 9.2.1, we have

$$v_1 \cdots v_t \in \mathcal{A}\langle X\rangle^{(n+1)} = \widehat{\mathcal{S}}_{\mathcal{A}\langle X\rangle}(g_n).$$

Evaluating w by z in A and the remaining indeterminates $x, y_1, z_1, \ldots, y_n, z_n$ with elements in Y induces a linear map

$$\lambda : A \to A^{(n+1)} : z \mapsto v_1 \cdots v_t.$$

Since $\dim A^{(n+1)} < \infty$, $\ker(\lambda)$ is of finite codimension in A. Because there are only finitely many maps of this form, $\bigcap_\lambda \ker(\lambda)$ is also of finite codimension in A. Consequently, it suffices to show that

$$F^{(n)}(A) \supseteq \bigcap_\lambda \ker(\lambda).$$

Let $z \in \bigcap_\lambda \ker(\lambda)$. Then, by part (2) of Theorem 8.1.9, $z \in F^{(n)}(A)$ precisely when, for all $a, b_1, \ldots, b_n, c_1, \ldots, c_n \in Y$, we have

$$g_n = g_n(za, b_1, c_1, \ldots, b_n, c_n) = 0.$$

However, by part (2) Proposition 9.2.2, g_n is a linear combination of products $v_1 \cdots v_t$ of left-normed Lie monomials $v_1, \ldots, v_t$, where $t \geq 1$, v_1 starts with z, each $v_i \in \gamma_{n_i}(A)$, and

$$\textstyle\sum_{i=1}^{t}(n_i - 1) = n.$$

But each of these products $v_1 \cdots v_t$ is trivial since $z \in \bigcap_\lambda \ker(\lambda)$; hence, $g_n = 0$, as required. □

Remark 9.2.3 *Observe that in the proof of part (2) of Theorem 7.2.1,*

$$F^{(n)}(A) = \textstyle\bigcap_\lambda \ker(\lambda),$$

for any set of generators Y of A, yielding yet another characterization of $F^{(n)}(A)$.

To see why the reverse inclusion $F^{(n)}(A) \subseteq \bigcap_\lambda \ker(\lambda)$ also holds, let $z \in F^{(n)}(A)$. Then, for each map λ, $\lambda(z) = v_1 \cdots v_t \in A^{(n+1)} = \mathcal{S}_A(g_n)$, and so $\lambda(z) = 0$ since z is an element of the ideal $F^{(n)}(A) = \widehat{\mathcal{S}}_A(g_n)$.

Proof of part (3) of Theorem 7.2.1: As we mentioned before, we may assume that the algebra A is unital. Suppose now that $\dim A^{(n+1)} < \infty$. Then, by part (2) of Theorem 5.1.2, $\dim A/Z_{2n}(A) < \infty$. Let $a \in A^{(n+1)}$ and consider the linear maps

$$\lambda_a : A \to A^{(n+1)} : z \mapsto az \quad \text{and} \quad \rho_a : A \to A^{(n+1)} : z \mapsto za.$$

It is clear that $\ker(\lambda_a)$ and $\ker(\rho_a)$ are, respectively, the right and left annihilators of a in A. Furthermore, since $\dim A^{(n+1)} < \infty$, $\ker(\lambda_a)$ and $\ker(\rho_a)$ are each of finite codimension in A, for all $a \in A^{(n+1)}$. Therefore, $\mathrm{Ann}(A^{(n+1)})$ is of finite codimension in A. Let $J = \mathrm{Ann}(A^{(n+1)}) \cap Z_{2n}(A)$. Then, from what we have just seen, J is of finite codimension in A. We claim that $J \subseteq F^{(m)}(A)$ for $m = 3n - 1$. Indeed, let $z \in J$. In order to show that $z \in F^{(m)}(A) = \widehat{\mathcal{S}}_A(g_m)$, we need to prove that $g_m = 0$ whenever any of the indeterminates x, y_i, z_i (for some $i = 1, \ldots, n$) is evaluated to

z. By part (2) of Proposition 9.2.1, we can expand the polynomial g_m into a linear combination of products $v_1 \cdots v_t$, where each $v_i \in \gamma_{m_i}(\mathcal{L}\langle X \rangle)$, $\sum_{i=1}^{t}(m_i - 1) = m$, and v_1 starts with the indeterminate to be evaluated to z. It suffices to show that each of these products $v_1 \cdots v_t$ in the expansion of g_m evaluates to zero. Indeed, after evaluation, we have

$$v_1 \in [Z_{2n}(A),_{m_1-1} A] \subseteq Z_{2n-(m_1-1)}(A).$$

Thus, if $m_1 \geq 2n + 1$, then $g_m = 0$, as required. On the other hand, if $m_1 \leq 2n$, then

$$(m_2 - 1) + \cdots + (m_t - 1) = m - (m_1 - 1) \geq m - 2n + 1 = n,$$

so that $v_2 \cdots v_t \in A^{(n+1)}$. Therefore, since $v_1 \in \mathrm{Ann}(A^{(n+1)})$, it follows that $v_1 \cdots v_t = 0$, as required. □

9.3 Proof of Theorems 7.3.1 and 7.3.2

Recall Theorem 7.3.2: *Let A be an algebra and let n be a positive integer. Then the following statements hold.*

1. *If $\gamma_{n+1}(A)$ is finite-dimensional, then so is the associative ideal, $A^{[n+1]}$, it generates.*

2. *Let $\mathcal{Z}_{[n]}(A)$ be the largest associative ideal of A contained in $\mathcal{Z}_n(A)$. If $A/\mathcal{Z}_n(A)$ is finite-dimensional, then so is $A/\mathcal{Z}_{[n]}(A)$. Thus, in particular, if $A/\mathcal{Z}(A)$ is finite-dimensional, then so is $A/F(A)$.*

Proof of part (1) of Theorem 7.3.2: First, observe that it suffices to assume that A is unital. Suppose now that $\dim\gamma_{n+1}(A) = m < \infty$. It follows that $\dim A/C \leq m^2$, where C is the centralizer of $\gamma_{n+1}(A)$ in A. Moreover, $\gamma_{n+1}(A)C = C\gamma_{n+1}(A) \subseteq \gamma_{n+1}(A)$ since, by part(1) of Lemma 4.2.3, if $a_1, \ldots, a_{n+1} \in A$ and $c \in C$, we have

$$\begin{aligned}[[a_1, \ldots, a_n], a_{n+1}]c &= [[a_1, \ldots, a_n], a_{n+1}c] - a_{n+1}[[a_1, \ldots, a_n], c] \\ &= [[a_1, \ldots, a_n], a_{n+1}c] \in \gamma_{n+1}(A).\end{aligned}$$

Consequently,

$$A^{[n+1]} = \gamma_{n+1}(A)A = \gamma_{n+1}(A)b_1 + \cdots + \gamma_{n+1}(A)b_{m^2},$$

for some $b_1 = 1, b_2, \ldots, b_{m^2} \in A$. It follows that $\dim A^{[n+1]} \leq m^3 < \infty$, as required. □

The proof of part (2) of Theorem 7.3.2 requires the following lemma (see [34]).

Lemma 9.3.1 *For every algebra A and integer $n \geq 0$, the Lie ideal $\mathcal{Z}_n(A)$ is an associative subalgebra of A.*

Proof. Observe first that, for every integer $n \geq 1$,

$$\mathcal{Z}_n(A) = \{z \in A \mid [z, A] \in \mathcal{Z}_{n-1}(A)\}.$$

Let $z_1, z_2 \in \mathcal{Z}_n(A)$ and $a \in A$. Then, by the semi-Jacobi identity (part (2) of Lemma 4.2.3), we have

$$[z_1 z_2, a] = [z_1, z_2 a] + [z_2, a z_1] \in \mathcal{Z}_{n-1}(A).$$

Hence $z_1 z_2 \in \mathcal{Z}_n(A)$, as required. □

The following useful result, due to Lee and Liu ([35]), will be used to prove the remaining part of Theorem 7.3.2.

Proposition 9.3.2 *Let R be an algebra over a field K, and let A be a subalgebra of R such that* $\dim_K R/A < \infty$. *Then there exists an ideal I of R contained in A such that* $\dim_K R/I < \infty$.

Proof of part (2) of Theorem 7.3.2: Suppose that $\dim A/\mathcal{Z}_n(A) < \infty$, for some positive integer n. By Lemma 9.3.1, $\mathcal{Z}_n(A)$ is an associative subalgebra of A. However, by Proposition 9.3.2, $\mathcal{Z}_n(A)$ contains an ideal that is also of finite codimension. It follows that $A/\mathcal{Z}_{[n]}(A)$ is finite-dimensional, as required. □

We will use Theorem 7.3.2 to prove Theorem 7.3.1, which we recall here for convenience: *Let A be a finitely generated algebra. Then the following statements hold for every positive integer n.*

1. *If $\gamma_{n+1}(A)$ is finite-dimensional, then there exists a positive integer m such that $A^{(m+1)}$ is also finite-dimensional.*

2. *If $A/\mathcal{Z}_n(A)$ is finite-dimensional, then there exists a positive integer m such that $A/F^{(m)}(A)$ is also finite-dimensional.*

Proof of Theorem 7.3.1: Let A be a finitely generated algebra, and suppose that $\gamma_{n+1}(A)$ is finite-dimensional. Then, by Theorem 7.3.2, $A^{[n+1]}$ is also finite-dimensional. Hence,

$$\bar{A} := A/A^{[n+1]},$$

is a finitely generated Lie nilpotent (associative) algebra. Consequently, $\bar{A}$ is upper Lie nilpotent. This is a special case of the fact that every finitely generated associative algebra satisfying an Engel identity is upper Lie nilpotent, as proved by Riley and Wilson (see [54]). Therefore, $A^{(m+1)} \subseteq A^{[n+1]}$, for some positive integer m. Since $A^{[n+1]}$ is finite-dimensional, so is $A^{(m+1)}$. This proves part (1). To prove part (2), suppose that $\dim A/\mathcal{Z}_n(A) < \infty$, for some positive integer n. Then, by part (1) of Theorem 5.1.2, $\dim \gamma_{n+1}(A) < \infty$. Thus, as shown above, $\dim A^{(m+1)} < \infty$, for some integer m. Hence, by part (2) of Theorem 7.2.1, $\dim A/F^{(m)}(A) < \infty$, as required. □

Chapter 10

Counterexamples

Let $K(\alpha)$ be any simple field extension of our base field K, and let V be a vector space with basis $\{v_1, \ldots, v_{n+1}\}$ over $K(\alpha)$, for some fixed positive integer n. Now let E denote the (non-unital) Grassmann-like $K(\alpha)$-algebra generated by V subject to the relations

$$v_j v_i = \alpha v_i v_j,$$

for all $1 \leq i \leq j \leq n+1$. Notice that these relations imply that $v_i^2 = 0$ except when $\alpha = 1$; thus, in the case when $\alpha = 1$, we impose the additional relations $v_i^2 = 0$, for each i. It is easy to see that E has a $K(\alpha)$-basis consisting of all the monomials of the form

$$v_{i_1} \cdots v_{i_k},$$

where $1 \leq i_1 < \cdots < i_k \leq n+1$ and $1 \leq k \leq n+1$. Clearly $E^{n+2} = 0$ and $\mathrm{Ann}(E) = E^{n+1} = K(\alpha)v_1 \cdots v_{n+1}$. Simple induction argument shows also that

$$\mathrm{Ann}^m(E) = E^{n-m+2},$$

for each $0 \leq m \leq n+1$.

Let A be the algebra formed by identifying the elements corresponding to $v_1 \cdots v_{n+1}$ in each copy of a direct sum of countably many copies of E.

10.1 Associative Powers

The following example shows that part (2) of Theorem 7.1.1 does not extend to non-finitely generated algebras.

Example 10.1.1 *Let $\alpha = 1$ and define the algebra A as before. Then A is a commutative K-algebra such that* $\dim_K(A^{n+1}) = 1$ *while* $A/\mathrm{Ann}^n(A) = A/A^2$ *is infinite-dimensional.*

10.2 Lie Powers

Let α be a primitive root of unity whose order exceeds n. Notice if $1 \leq i_1 < \cdots < i_k < i_{k+1} \leq n+1$, then

$$[v_{i_1} \cdots v_{i_k}, v_{i_{k+1}}] = (1 - \alpha^k) v_{i_1} v_{i_2} \cdots v_{i_{k+1}}.$$

It follows that

$$\gamma_m(E) = E^{[m]} = E^{(m)} = E^m,$$

for all m. We claim that

$$\mathcal{Z}(E) = \begin{cases} E^{n+1}, & \text{if } \; n \text{ is odd} \\ K(\alpha) v_1 \cdots v_k v_{k+2} \cdots v_{n+1} + E^{n+1}, & \text{if } \; n = 2k \text{ is even.} \end{cases}$$

Indeed, notice that, if $z = z_1 + \cdots + z_{n+1} \in \mathcal{Z}(E)$, where each component z_m lies in E^m, linear independence forces each monomial $v_{i_1} \cdots v_{i_m}$ $(1 \leq i_1 < \cdots < i_m \leq n+1)$ in the support of each z_m to lie in $\mathcal{Z}(E)$, too. Suppose now that $z_m \neq 0$, for some $1 \leq m \leq n$. Let $v_{i_1} \cdots v_{i_m}$ $(1 \leq i_1 < \cdots < i_m \leq n+1)$ be in the support of z_m, and let $1 \leq j \leq n+1$ be such that

$$i_1 < \cdots < i_r < j < i_{r+1} < \cdots < i_m,$$

for some $0 \leq r \leq m$. Then

$$0 = [v_{i_1} \cdots v_{i_m}, v_j] = (\alpha^{m-r} - \alpha^r) v_{i_1} \cdots v_{i_r} v_j v_{i_{r+1}} \cdots v_{i_m},$$

so that $\alpha^{m-2r} = 1$. Since the order of α exceeds n and $-n \leq m - 2r \leq n$, we have $r = \frac{m}{2}$. Thus, for all choices of j, $i_r < j < i_{r+1}$. So, in particular, $i_1 = 1$ and $i_m = n + 1$. It now follows that $m = n = 2k$ is even, and

$$v_{i_1} \cdots v_{i_m} = v_1 \cdots v_k v_{k+2} \cdots v_{n+1}.$$

Thus, the claim has been proved (the reverse inclusion being obvious). Consequently,

$$F(E) = \mathcal{Z}(E) = \begin{cases} E^{n+1}, & \text{if} \quad n \text{ is odd} \\ K(\alpha)v_1 \cdots v_k v_{k+2} \cdots v_{n+1} + E^{n+1}, & \text{if} \quad n = 2k \text{ is even.} \end{cases}$$

Next we claim

$$F^{(m)}(E) = \mathcal{Z}_m(E) = E^{n-m+2},$$

for each $2 \leq m \leq n+1$. This is clear when n is odd. So, suppose that $n = 2k$ is even. The claim follows easily from the base step: $F^{(2)}(E) = \mathcal{Z}_2(E) = E^n$. So, let

$$z + \mathcal{Z}(E) = z_1 + \cdots + z_n + \mathcal{Z}(E) \in \mathcal{Z}(E/\mathcal{Z}(E)),$$

where each component $z_m \in E^m$. Thus, for each $1 \leq i \leq n + 1$, we have

$$[z_1, v_i] + \cdots + [z_n, v_i] \in \mathcal{Z}(E) = K(\alpha)v_1 \cdots v_k v_{k+1} \cdots v_{n+1} + E^{n+1}.$$

Thus, by linear independence,

$$z_1 = \cdots = z_{n-2} = 0.$$

Therefore,

$$z + \mathcal{Z}(E) = z_{n-1} + z_n + \mathcal{Z}(E).$$

Suppose that $z_{n-1} \neq 0$. Then there exists a monomial of the form

$$v_1 \cdots v_{i-1} v_{i+1} \cdots v_{j-1} v_{j+1} \cdots v_{n+1},$$

with $1 \leq i < j \leq n + 1$ in the support of z_{n-1}. So, by our choice of α, neither $[z_{n-1}, v_i]$ nor $[z_{n-1}, v_j]$ is zero in E. From above, however,

$$[z_{n-1}, v_i], [z_{n-1}, v_j] \in \mathcal{Z}(E) = K(\alpha)v_1 \cdots v_k v_{k+1} \cdots v_{n+1} + E^{n+1},$$

forcing $i = j = k + 1$, a contradiction. Hence, we have shown that $\mathcal{Z}_2(E) = E^n$. Thus, since $F(E) = \mathcal{Z}(E)$, it now follows that $F^{(2)}(E) = \mathcal{Z}_2(E) = E^n$, as required.

Example 10.2.1 *If α is a primitive root of unity whose order exceeds n, then A, when viewed as a K-algebra, has the property that* $\dim_K(A^{(n+1)}) = \dim_K(K(\alpha)) < \infty$*, and yet* $A/F^{(n)}(A) = A/A^2$ *is infinite-dimensional.*

This shows that part (2) of Theorem 7.2.1 cannot be extended to non-finitely generated algebras. Furthermore, using the same algebra A viewed as a Lie K-algebra, we find that the converse of part (1) of Stewart's result, Theorem 5.1.2, does not hold in general either.

Example 10.2.2 *If α is a primitive root of unity with order exceeding n, then A, when viewed as a Lie K-algebra, has the property that* $\dim_K(\gamma_{n+1}(A)) = \dim_K(K(\alpha)) < \infty$*, and yet* $A/\mathcal{Z}_n(A) = A/A^2$ *is infinite-dimensional.*

Chapter 11

Conciseness

11.1 Concise and Marginally Concise Polynomials

Recall that a polynomial f is called homogeneous if each f-monomial is of the same degree in each indeterminate. Also f is called multilinear if it is linear in each of its indeterminates. Let A be a K-algebra. In [65], Stewart proved that, whenever K is infinite or $f(x_1, \ldots, x_n)$ is a homogeneous polynomial of degree m_i in x_i and $|K| \geq m_i$ $(1 \leq i \leq n)$, $\mathcal{S}_A(f)$ is invariant under all derivations of A. Consequently, we have the following result.

Theorem 11.1.1 *Let A be an algebra over a field K, and let $f(x_1, \ldots, x_n)$ be a polynomial. If K is infinite or f is homogeneous of degree m_i in x_i and $|K| \geq m_i$ $(1 \leq i \leq n)$, then $\mathcal{S}_A(f)$ is a Lie ideal.*

Proof. Part (1) of Lemma 4.2.3 say that ad maps are associative derivations. Thus, the result follows from Theorem 3.1 in [65]. □

For an algebra A, $C_A(a)$, for $a \in A$, will denote the *centralizer* of a in A.

Lemma 11.1.2 *Let $\mathcal{U}$ be a Lie ideal of an algebra A such that* $\dim \mathcal{U} = m < \infty$. *If*

$$C_A(\mathcal{U}) := \bigcap_{u \in \mathcal{U}} C_A(u),$$

then $\dim(A/C_A(\mathcal{U})) \leq m^2$.

Proof. For every $u \in \mathcal{U}$, $\dim(A/C_A(u)) \leq m$. Clearly, it follows that $\dim(A/C_A(\mathcal{U})) \leq m^2$. □

Proposition 11.1.3 *Let $f(x_1, \ldots, x_n)$ be a polynomial in the free algebra on the set $\{y, x_1, x_2 \ldots\}$ over a field K, and define*

$$g(x_1, \ldots, x_n, y) = [f(x_1, \ldots, x_n), y].$$

Let A be a K-algebra, and suppose that $\dim \mathcal{S}_A(g) < \infty$. *If K is infinite or f is homogeneous of degree m_i in x_i and $|K| \geq m_i$ $(1 \leq i \leq n)$, then* $\dim \mathcal{I}_A(g) < \infty$.

Proof. Suppose that $\dim \mathcal{S}_A(g) = m < \infty$. Let $g_i = [f_i, z_i]$, for $i = 1, \ldots, m$, be a basis for $\mathcal{S}_A(g)$. Then, for each $1 \leq i \leq m$, $\dim(A/C_A(f_i)) \leq m$ (consider the map $A \to \mathcal{S}_A(g) : a \mapsto [f_i, a]$). Hence $\dim(A/C) \leq m^2$ where

$$C = \textstyle\bigcap_{i=1}^m C_A(f_i).$$

Moreover, $\mathcal{S}_A(g)C = C\mathcal{S}_A(g) \subseteq \mathcal{S}_A(g)$ since, by part (1) of Lemma 4.2.3, for every $i = 1, \ldots, m$ and $c \in C$, we have

$$\begin{aligned} [f_i, z_i]c &= [f_i, z_i c] - z_i[f_i, c] \\ &= [f_i, z_i c] \in \mathcal{S}_A(g). \end{aligned}$$

Consequently, by embedding A into its unital hull, A_1,

$$\begin{aligned} \mathcal{I}_A(g) &= \mathcal{S}_A(g)A_1 \\ &= \mathcal{S}_A(g)[K1 + Kb_1 + \cdots + Kb_{m^2} + C], \end{aligned}$$

for some $b_1, b_2, \ldots, b_{m^2} \in A$. It follows that $\dim \mathcal{I}_A(g) \leq m(m^2 + 1) < \infty$, as required. □

Corollary 11.1.4 *Let A be a K-algebra, and let $f(x_1, \ldots, x_n)$ be a homogeneous polynomial of degree m_i in x_i and $|K| \geq m_i$ $(1 \leq i \leq n)$. Set*

$$g(x_1, \ldots, x_n, y) = [f(x_1, \ldots, x_n), y].$$

If $\dim \mathcal{S}_A(f) < \infty$*, then* $\dim \mathcal{I}_A(g) < \infty$*.*

Proof. By Theorem 11.1.1, $\mathcal{S}_A(f)$ is Lie ideal. It follows that $\mathcal{S}_A(g) \subseteq \mathcal{S}_A(f)$. Hence $\dim \mathcal{S}_A(g) < \infty$, and, by Proposition 11.1.3, $\dim \mathcal{I}_A(g) < \infty$, as required. □

Lemma 11.1.5 *Let $f(x_1, \ldots, x_n)$ be a multilinear polynomial in the free algebra on $\{y, x_1, x_2, \ldots\}$ over a field K, and set*

$$g(x_1, \ldots, x_n, y) = [f(x_1, \ldots, x_n), y].$$

If $\bar{A} = A/\mathcal{I}_A(g)$, then $\mathcal{S}_{\bar{A}}(f)$ is a subalgebra of $\bar{A}$. In other words, $\mathcal{A}_A(f) \subseteq \mathcal{S}_A(f) + \mathcal{I}_A(g)$.

Proof. The polynomial f has the form

$$f(x_1, \ldots, x_n) = \textstyle\sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)},$$

for some $\alpha_\sigma \in K$. Let $a_1, \ldots, a_n \in A$ and $z \in \mathcal{S}_{\bar{A}}(f) \subseteq \mathcal{Z}(\bar{A})$. Then

$$\begin{aligned} f(a_1, \ldots, a_n)z &= \sum_{\sigma \in S_n} \alpha_\sigma a_{\sigma(1)} \cdots a_{\sigma(n)} z \\ &= f(a_1 z, a_2, \ldots, a_n) \in \mathcal{S}_{\bar{A}}(f). \end{aligned}$$

Therefore, $\mathcal{S}_{\bar{A}}(f)$ is a subalgebra of $\bar{A}$, as required. □

Theorem 11.1.6 *Let f be a multilinear polynomial, and let A be an algebra. If $\mathcal{S}_A(f)$ is finite-dimensional, then so is the subalgebra, $\mathcal{A}_A(f)$, it generates.*

Proof. The result follows from combining Corollary 11.1.4 and Lemma 11.1.5 □

It now makes sense to propose the following algebraic-analogue definition of concise polynomials. Also, recall from Proposition 9.3.2 that, if B is a subalgebra of finite codimension in an associative algebra A, then B contains an ideal that is, also, of finite codimension in A. Thus, we may also study 'marginal' concise polynomials defined as follows.

Definition 11.1.7 *Let f be a polynomial, and let A be any algebra.*

1. *f is called* concise *in A if* $\dim\mathcal{S}_A(f) < \infty$ *implies* $\dim\mathcal{I}_A(f) < \infty$.
2. *f is called* marginally concise *in A if* $\dim(A/\widehat{\mathcal{S}}_A(f)) < \infty$ *implies* $\dim(A/\widehat{\mathcal{I}}_A(f)) < \infty$.

Thus, Problem (3) in 4.2.1 becomes:

is every polynomial f concise?

While Problem (4) becomes:

is every polynomial f marginally concise?

Definition 11.1.8 *Let f be a polynomial, and let A be any algebra. We say that $f = 0$ is* virtually a polynomial identity *of A if there exists $I \lhd A$ such that* $\dim(A/I) < \infty$ *and I satisfies* $f = 0$.

Recall that, if $f(x_1, \ldots, x_n)$ is homogeneous,

$$\widehat{\mathcal{S}}_A(f) = \{z \in A |\, f(a_1, \ldots, a_{i-1}, z, a_{i+1}, \ldots, a_n) = 0, a_1, \ldots, a_n \in A, 1 \le i \le n\}.$$

Thus, for all $z_1, \ldots, z_n \in \widehat{\mathcal{S}}_A(f)$, $f(z_1, \ldots, z_n) = 0$. Hence, $\widehat{\mathcal{I}}_A(f) \subseteq \widehat{\mathcal{S}}_A(f)$ satisfies $f = 0$. Therefore, if f is marginally concise and $\dim(A/\widehat{\mathcal{S}}_A(f)) < \infty$, then $f = 0$ is virtually a PI for A.

Following Drensky (see Chapter 4 in [8]), we recall the definition of *proper polynomials*:

Definition 11.1.9 *A polynomial f in the free algebra over a field K is called* proper (or commutator of length of least 2) *polynomial if it is a linear combination of products of commutators:*

$$f(x_1,\dots,x_n)=\sum \alpha_{i,\dots,j}[x_{i_1},\dots,x_{i_p}]\cdots[x_{j_1},\dots,x_{j_q}],$$

for some $\alpha_{i,\dots,j}\in K$

Definition 11.1.10 *A multilinear polynomial is called* distinctly *proper if it can be written in the form*

$$f([x_1,\dots,x_m],\dots,[z_1,\dots,z_n]),$$

where f *is a multilinear polynomial and each of the commutators of length at least* 2.

Many PI-algebras are known to satisfy distinctly proper polynomial identities. For example, the subalgebra of all $n\times n$ upper triangular matrices, $U_n(K)$, over a field K satisfies the distinctly proper identity $[x_1,y_1][x_2,y_2]\cdots[x_n,y_n]=0$ (see Chapter 5 in [8]). Furthermore, a Lie algebra L is called metabelian if it satisfies the metabelian identity $[[x_1,x_2],[x_3,x_4]]=0$, which is clearly a distinctly proper identity. More generally, Lie solubility of derived length d corresponds, as well, to a distinctly proper polynomial.

Theorem 11.1.11 *Let* f *be a distinctly proper polynomial, and let* A *be any algebra. Then* $\widehat{\mathcal{S}}_A(f)$ *is a subalgebra of* A.

Proof. First, we prove the statement for Lie commutators $[x_1,\dots,x_n]$ with $n\geq 2$. Consider the free algebra $\mathcal{A}\langle X\rangle$ on the set $\{w_1,w_2,x_1,\dots,x_n\}$. Then, using inductive argument with the Jacobi identity, we have

$$\begin{aligned}[][[x_1,\dots,x_m],w_1w_2,x_{m+2},\dots,a_n]&=-[[w_1w_2,[x_1,\dots,x_m]],x_{m+2},\dots,x_n]\\&\in\sum[w_1w_{2,n-1}\,\mathcal{A}\langle X\rangle].\end{aligned}$$

However, by the semi-Jacobi identity (part (2) of Lemma 4.2.3), if $a_1,\dots,a_{n-1}\in A$ and $z_1,z_2\in\widehat{\mathcal{S}}_A([x_1,\dots,x_n])$, then

$$[z_1z_2,a_1.\dots,a_{n-1}]=[z_1,z_2a_1,\dots,a_{n-1}]+[z_2,a_1z_1,\dots,a_{n-1}]=0.$$

It follows that $[b_1, \ldots, b_{i-1}, z_1z_2, b_{i+1}, \ldots, b_n] = 0$ for each $i = 1, 2, \ldots, n$ and for all choices of $b_1, \ldots, b_n$ in A.Thus, $z_1z_2 \in \widehat{\mathcal{S}}_A([x_1, \ldots, x_n])$, as required. Next, let $f(x_1, \ldots, x_n)$ be any distinctly proper polynomial. Then f can be written in the form

$$f(x_1, \ldots, x_n) = h(f_1, \ldots, f_m),$$

where h is a multilinear polynomial and $f_1, \ldots, f_m$ are Lie commutators (of distinct indeterminates) each has length at least 2. Let $z_1, z_2 \in \widehat{\mathcal{S}}_A(f)$. Now, evaluate x_i, for some $1 \leq i \leq n$, by z_1z_2 and the remaining indeterminates in f with elements $a_j \in A$. Observe that x_i must fall in exactly one Lie commutator, say, without loss of generality, f_1. Applying the above proof to the Lie commutator f_1, it follows that $f(a_1, \ldots, a_{i-1}, z_1z_2, a_{i+1}, \ldots, a_n)$ is a sum of two evaluations of f with the first indeterminate of the Lie commutator f_1 is evaluated with z_1, in one of these evaluations, while evaluated by z_2 in the other. This implies that $f(a_1, \ldots, a_{i-1}, z_1z_2, a_{i+1}, \ldots, a_n) = 0$, and hence $z_1z_2 \in \widehat{\mathcal{S}}_A(f)$, as required. □

In particular, we have the following result:

Corollary 11.1.12 *Every distinctly proper polynomial is marginally concise.*

Proof. Let f be a distinctly proper polynomial, and let A is an algebra such that $\dim(A/\widehat{\mathcal{S}}_A(f)) < \infty$. By Theorem 11.1.11, $\widehat{\mathcal{S}}_A(f)$ is a subalgebra of A. However, by Proposition 9.3.2, $\widehat{\mathcal{S}}_A(f)$ contains an ideal I of finite codimension. But $I \subseteq \widehat{\mathcal{I}}_A(f)$. Thus, $\dim(A/\widehat{\mathcal{I}}_A(f)) < \infty$, and hence f is marginally concise. □

Examples 11.1.13

1. *Define* $g(x_1, \ldots, x_n, y) = f(x_1, \ldots, x_n)y$ *for any polynomial* $f(x_1, \ldots, x_n)$*. Then, clearly for any algebra A,* $\mathcal{S}_A(g) = \mathcal{I}_A(g)$*. Thus, g is concise. In particular,*

$$g_n = [[\![x, y_1, z_1, \ldots, y_{n-1}, z_{n-1}]\!], y_n]z_n$$

is concise.

2. *By Proposition 11.1.3, if K is infinite or $f(x_1, \ldots, x_n)$ is a homogeneous polynomial of degree m_i in x_i and $|K| \geq m_i$ $(1 \leq i \leq n)$ (in particular if f is a multilinear polynomial), then the polynomial $g = [f(x_1, \ldots, x_n), y]$ is concise. Thus, by induction, all Lie commutators $[x_1, \ldots, x_n]$ are concise.*

3. *By part (2) of Theorem 8.1.8, $\widehat{\mathcal{S}}_A(g_n) = F^{(n)}(A)$ is an ideal, and hence*

$$g_n = [\![x, y_1, z_1, \ldots, y_n, z_n]\!],$$

is marginally concise.

4. *By Corollary 11.1.12, all distinctly proper polynomials are marginally concise. In particular, the Lie metabelian polynomial $[[x_1, x_2], [x_3, x_4]]$, and the Lie soluble polynomial $f_n = f_n(x_1, \ldots, x_{2^n})$, which is defined inductively by:*

$$f_1(x_1, x_2) = [x_1, x_2],$$

and, for all $n > 1$,

$$f_n = [f_{n-1}(x_1, \ldots, x_{2^{n-1}}), f_{n-1}(x_{2^{n-1}+1}, \ldots, x_{2^n}].$$

Definition 11.1.14 *Let A be an algebra. If every polynomial f is concise in A, then A is called* verbally-concise.

Example 11.1.15 *Let $A = KG$ be a group algebra of an infinite group G over a field K. Clearly then A cannot have a nontrivial finite-dimensional ideal. Therefore, A is verbally-concise only if, for all polynomials f with the property that $\mathcal{S}_A(f)$ is finite-dimensional, f is a polynomial identity of A.*

Open Problems 11.1.16 *The following questions are still open problems:*

1. *Is it true that all group algebras are verbally-concise?*

2. *Are all polynomials concise?*

3. *Are all polynomials marginally concise?*

Bibliography

[1] S.I. Adyan, ‘The Burnside Problem and Identities in Groups’, (Russian), *Nauka*, Moscow (1975). Translation: *Ergebnisse der Math. und ihrer Grenzgebiete* 95, Springer-Verlag, Berlin-New York (1979).

[2] S.V. Alyoshin, ‘Finite automata and the Burnside problem on periodic groups’, *Matem. Zametky* 11 (1972) 319-328.

[3] B. Amberg and Y. Sysak, ‘Associative rings whose adjoint group semigroup is locally nilpotent’, *Arch. Math.* 76 (2001) 426-435.

[4] R. Baer, ‘Endlichkeitskriterien fur Kommutator-gruppen’, *Math. Ann.* 124 (1952) 161-177.

[5] Y. Billig, D.M. Riley, and V. Tasić, ‘Nonmatrix varieties and nil-generated algebras whose units satisfy a group identity’ *J. Algebra* 190 (1997) 241-252.

[6] R. Blyth, ‘Rewriting products of group elements, II’, *J. Algebra* 261 (2003) 19-30.

[7] W. Burnside, ‘On an unsettled question in the theory of discontinuous groups’, *Quart. J. Math.* 33 (1902) 230-238.

[8] V. Drensky, ‘Free Algebras and PI-Algebras’, *Springer* (1999).

[9] M.I. Elashiry and D.S. Passman, ‘Rewritable groups’, *J. Algebra* 345 (2011) 190-201.

[10] O.B. Finogenova, 'Varieties of associative algebras satisfying Engel identities', *Algebra Logic* 43 (2004) 271-284.

[11] E.S. Golod, 'On nil-algebras and finitely approximable p-groups', (Russian), *Izv. Akad. Nauk SSSR, Ser. Mat.* 28 (1964) 273-276.

[12] E.S. Golod and I.R. Shafarevich, 'On the class field tower', (Russian), *Izv. Akad. Nauk SSSR, Ser. Mat.* 28 (1964) 261-272.

[13] R.I. Grigorchuk, 'On Burnside's problem on periodic groups', (Russian), *Funk. Analiz i ego Prilozh.* 14 (1980) 53-54. Translation: *Functional Anal. Appl.* 14 (1980) 41-43.

[14] N.D. Gupta and S. Sidki, 'On the Burnside problem on periodic groups', *Math. Z.* 182 (1983) 385-388.

[15] N.D. Gupta and F. Levin, 'On the Lie ideals of a ring', *J. Algebra* 81 (1983) 225-231.

[16] Z. Halasi and K. Podoski, 'Bounds in groups with trivial Frattini subgroup', *J. Algebra* 3 (2008) 893-896.

[17] P. Hall, 'Finiteness conditions for soluble groups', *Proc. London Math. Soc.* 4 (1954) 419-436.

[18] P. Hall, 'Finite-by-nilpotent groups', *Proc. Cambridge Philos. Soc.* 52 (1956) 611-616.

[19] P. Hall, 'On finiteness of certain soluble groups', *Proc. London Math. Soc.* 9 (1959) 595-622.

[20] P. Hall, 'The Edmonton notes on nilpotent groups', *Queen Mary College Mathematics Notes* (1969) (originally published in 1957 by the Can. Math. Congress).

[21] P. Hall and G. Higman, 'On the p-length of p-soluble groups and reduction theorems for Burnside's problem', *Proc. London Math. Soc. 9* 3 (1956) 1-42.

[22] R. Hatamian, M. Hassanzadeh and S. Kayvanfar, 'A converse of Baer's theorem', *Ricerche mat.*, Springer 63 (2014) 183-187.

[23] N. S. Hekster, 'On the structure of n-isoclinism classes of groups', *J. Pure Appl. Algebra* 40(1) (1986) 63-85.

[24] N.I. Herstein, 'Noncommutative Rings', *Carus Math. Monographs* 15, Wiley and Sons, Inc., New York (1968).

[25] I.M. Isaacs, 'Derived subgroups and centers of capable groups', *Proc. Amer. Math. Soc.* 129(10) (2001) 2853-2859.

[26] S.V. Ivanov, 'P. Hall's conjecture on the finiteness of verbal subgroups', *Izv. Vyssh. Uchebn. Zaved.* 325 (1989) 60-70.

[27] S.A. Jennings, 'Central chains of ideals in an associative ring', *Duke Math. J.* 9 (1942) 341-355.

[28] I. Kaplansky, 'Rings with polynomial identity', *Bull. Amer. Math. Soc.* 54 (1948) 575-580.

[29] I. Kaplansky, 'Topological representations of algebras', I, *Trans. Amer. Math. Soc.* 68 (1950) 62-75.

[30] A. Kemer, 'Nonmatrix varieties', *Algebra and Logic* 19 (1981) 157-178.

[31] A.I. Kostrikin, 'Around Burnside ', (Russian), *Nanka*, Moscow (1986). Translation: *Ergebnisse der Math. und ihrer Grenzgebiete* 20, Springer-Verlag, Berlin (1990).

[32] S.I. Kublanovskii, 'On varieties of associative algebras with local finiteness conditions', *St. Petersburg Math. J.* 9 (1998) 763-813.

[33] A.G. Kurosch, 'Ringtheoretische Probleme, die mit dem Burnsideschen Problem uber periodische Gruppen in Zusammenhang stehen', (Russian, German summary), *Bull. Acad. Sci. URSS Ser. Math. (Izv. Akad. Nauk SSSR)* 5 (1941) 233-240.

[34] H. Laue, 'On the associated Lie ring and the adjoint group of a radical ring', *Canad. Math. Bull.* 27 (1984) 215-222.

[35] T.K. Lee and K.S. Liu, 'Algebras with a finite-dimensional maximal subalgebra', *Comm. in Algebra* 33 (2005) 339-342.

[36] I. Levitzki, 'On a problem of A. Kurosch', *Bull. Amer. Math. Soc.* 52 (1946) 1033-1035.

[37] L. Lysenok, 'On the Burnside problem for odd exponents $n > 115$', Materials of the International Conference in Algebra, Novosibirsk (1989).

[38] W. Magnus 'A connection between the Baker-Hausdorff formula and a problem of Burnside', *Ann. Math.* 52 (1950) 11-26.

[39] M. Maj, M. Curzio, P. Longobardi and D.J.S. Robinson, 'A permutational property of groups', *Archiv der Mathematik* 44 (1985) 385-389.

[40] A.I. Mal'cev, 'Nilpotent semigroups', (Russian), *Ivanov. Gos. Ped. Inst. Uchen. Zap. Fiz.-Mat. Nauki* 4 (1953) 107-111.

[41] Yu.N. Mal'tsev, 'Varieties of associative algebras', *Algebra Logika* 15 (1976) 579-584 (in Russian).

[42] Ju.I. Merzlyakov, 'Verbal and marginal subgroups of linear groups', *Doki. Akad. Nauk SSSR* 177 (1967) 1008-1011.

[43] S.P. Mishchenko, V.M. Petrogradsky, and A. Regev, 'Characterization of non-matrix varieties of associative algebras', *Israel J. Math.* 182 (2011) 337- 348.

[44] B.H. Neumann and T. Taylor, ‘Subsemigroups of nilpotent groups’, *Proc. Royal Soc. Ser. A* 274 (1963) 1-4.

[45] P.S. Novikov and S.I. Adyan, ‘Infinite periodic groups’,I, II, III (Russian), *Izv. Akad. Nank SSSR, Ser. Mat.* 32 (1968) 212-244, 251-524, 709-73. Translation: *Math. USSR, Izv.* 2 (1968) 209-236, 241-480, 665-685.

[46] A.Yu. Olshanskii, ‘Some infinite systems of identities’, (Russian), *Trudy Sem. Petrovskogo* 3 (1978) 139-146. Translation: *AMS, II Ser.* 119 (1983) 81-88.

[47] Yu.P. Razmyslov, ‘On Lie algebras satisfying the Engel condition’, *Algebra and Logic* 10 (1971) 21-29.

[48] D.M. Riley, ‘The tensor product of a Lie soluble algebras’, *Arch. Math. (Basel)* 66 (1996) 372-377.

[49] D.M. Riley, ‘Algebras with collapsing monomials’, *Bull. London Math. Soc.* 30 (1998) 521-528.

[50] D.M. Riley, ‘Algebras with collapsing monomials, II’, *Comm. Algebra* 29 (2001) 2745-2756.

[51] D.M. Riley, ‘Engel varieties of associative rings and the number of Mersenne primes’, *J. Algebra* 261 (2003) 19-30.

[52] D.M. Riley and V. Tasić, ‘The transfer of a commutator law from a nil-ring to its adjoint group’, *Canad. Math. Bull.* 40 (1997) 103-107.

[53] D.M. Riley and V. Tasić, ‘Malćev nilpotent algebras’, *Arch. Math. (Basel)* 72 (1999) 22-27.

[54] D.M. Riley and M.C. Wilson, ‘Associative rings satisfying the Engel condition’, *Proc. Amer. Math. Soc.* 127 (1999) 973-976.

[55] D.M. Riley and M.C. Wilson, 'Associative algebras satisfying a semigroup identity', *Glasgow Math J.* 41 (1999) 453-462.

[56] E. Rips and A. Shalev, 'The Baer condition for group algebras', *J. Algebra* 140 (1991) 83-100. MR 92g:17008

[57] D.J.S. Robinson, 'Finitess conditions and generalized soluble groups', vol. I, II, Springer, Berlin, 1972.

[58] D.J.S. Robinson, 'A course in the theory of groups', Springer-Verlag New York, Inc., 1996.

[59] J.J. Rotman, 'An introduction to the theory of groups ', Springer-Verlag New York, Inc., 1995.

[60] I. Schur, 'Uber die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutioen', *J. Reine Angew. Math.* 127 (1904) 20-50.

[61] D. Segal, 'Words, notes on verbal width in groups', Cambridge University Press, 2009.

[62] J.F. Semple and A. Shalev, 'Combinatorial conditions in residually finite groups, I', *J. Algebra* 157 (1993) 43-50.

[63] A. Shalev, 'Combinatorial conditions in residually finite groups, II', *J. Algebra* 157 (1993) 51-62.

[64] A.I. Shirshov, 'On rings with identity relations', *Mat. Sb.* 41 (1957) 277-283.

[65] I. Stewart, 'Verbal and marginal properties of non-associative algebras', *Proc. London Math. Soc.* (3) 28 (1974) 129-140.

[66] V.I. Suschansky, 'Periodic p-groups of permutations and the General Burnside Problem', *Dokl. Akad. Nauk SSSR* 247 (1979) 447-461.

[67] M.E. Sweedler, 'Lie nilpotent and solvable associative algebras', *J. Algebra* 26 (1973) 422-430.

[68] R.F. Turner-Smith, 'Marginal subgroup properties for outer commutator words', *Proc. London Math. Soc.* 14 (1964) 321-341.

[69] R.F. Turner-Smith, 'Finiteness conditions for verbal subgroups', *J. London Math. Soc.* 41 (1966) 166-176.

[70] J. Wilson, 'On outer-commutator words', *Can. J. Math.* 3 (1974) 608-620.

[71] E.I. Zelmanov, 'Engelian Lie algebras', *Siberian Math. J.* 29 (1988) 777-781.

[72] E.I. Zelmanov, 'Solution of the Restricted Burnside Problem for group of odd exponent', *Math. USSR-Izv.* 36 (1991) 41-60.

[73] E.I. Zelmanov, 'Solution of the Restricted Burnside Problem for 2-groups', *Math. USSR-Sb.* 72 (1992) 543-565.

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