4-20-2015 12:00 AM

On Spectral Invariants of Dirac Operators on Noncommutative Tori and Curvature of the Determinant Line Bundle for the Noncommutative Two Torus

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A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Mathematics

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(Thesis format: Integrated-Article)

by

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A thesis submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

The School of Graduate and Postdoctoral Studies
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London, Ontario, Canada

April 2015

Abstract

We extend the canonical trace of Kontsevich and Vishik to the algebra of non-integer order classical pseudodifferntial operators on noncommutative tori. We consider the spin spectral triple on noncommutative tori and prove the regularity of eta function at zero for the family of operators $e^{th/2}De^{th/2}$ and the coupled Dirac operator $D+A$ on noncommutative 3-torus. Next, we consider the conformal variations of $\eta_D(0)$ and we show that the spectral value $\eta_D(0)$ is a conformal invariant of noncommutative 3-torus. Next, we study the conformal variation of $\zeta'_{|D|}(0)$ and show that this quantity is also a conformal invariant of odd noncommutative tori. This the analogue of the vanishing of the conformal anomaly of LogDet in odd dimensions in commutative case. We also consider $\eta_{D+A}(0)$ for the coupled Dirac operator $D+A$ on noncommutative 3-torus and compute a local formula for the variation of $\eta_{D+A}(0)$ with respect to the vector potential A.

In the second part, we consider a family of elliptic first order differential operators $\bar{\partial}_A$ on noncommutative two torus which are the noncommutative analogues of Cauchy-Riemann operators on a closed Riemann surface. We consider the Quillen determinant line bundle associated to this family and by using the machinery of the canonical trace, we compute the second variation of the $\zeta'_\Delta(0)$ where $\Delta = \bar{\partial}_A^2$ is the Dolbeault Laplacian. This gives the analogue of the Quillen's computations for the curvature form of the determinant line bundle in commutative case.

Keywords: Noncommutative tori, Kontsevich-Vishik canonical trace, eta invariant, Cauchy-Riemann operators, Quillen determinant line bundle.

Declaration of Authorship

I, Ali Fathi baghbadorani, declare that this thesis titled, 'On Spectral Invariants of Dirac Operators on Noncommutative Tori and Curvature of the Determinant Line Bundle for the Noncommutative Two Torus' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

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Date: ______________________________

Co-Authorship

This thesis incorporates material that is result of joint research, as follows:

- Chapter [3] is based on the paper
  On certain spectral invariants of Dirac operators on noncommutative tori, *arXiv:1504.01174v1 [math.QA], Apr 2015,*
  which is the outcome of my research conducted under the supervision of Professor Masoud Khalkhali.

- Chapter [4] is based on the paper
  Curvature of the Determinant Line Bundle for the Noncommutative Two Torus, *arXiv:1410.0475 [math.QA]*, Oct 2014,
  which is the outcome of a joint research undertaken in collaboration with Asghar Ghorbanpour under the supervision of Professor Masoud Khalkhali.

Acknowledgements

First and foremost, I wish to thank my supervisor Maosud Khalkhali whose intellectual achievements have inspired me for more than ten years of my life. I also feel indebted to my old friend Hooman Safizadeh whose passion towards mathematics had a deep impact on my personal mathematical aesthetics.

I am deeply grateful to my parents from whom I learned generosity and dedication. And at last, non of this was possible without the love and support of my small family, dearest Nazi and Annie. To both of you I am forever indebted.

Contents

Chapter 1

Introduction

As the title suggests, the material covered in this thesis grow up from the soil of Alain Connes' noncommutative Geometry. In the past thirty five years, Alain Connes has developed a framework where the concepts of differential geometry, algebraic topology and index theory have been extended to a more general algebraic setting [2, 3]. At the heart of the program of noncommutative geometry lies the fundamental duality between a point space X and $\mathcal{F}(X)$, the algebra of functions on it. This duality is perhaps one of the oldest ideas in mathematics. For instance, from early on it was known that one can study a Riemann surface X by studying $\mathfrak{M}(X)$, the field of meromorphic functions on it. Along these lines, in 1943 Gelfand and Naimark published a fundamental result where they characterized a compact Hausdorff topological space X by $C(X)$, the C^*-algebra of complex valued continuous functions on X. The noncommutative geometry starts where we consider a *noncommutative* C^*-algebra as the algebra of functions on a *noncommutative space* and try to extend the concepts of topology, analysis and geometry to this context.

The concept of a Riemannian geometry is extended to the noncommutative setting through the notion of a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ [3]. Here $\mathcal{A}$ is a $*$-algebra represented on a Hilbert space $\mathcal{H}$ and D is a self adjoint densely defined operator on $\mathcal{H}$ satisfying certain conditions. This data encapsulates and generalizes the data of a classical Riemannian spin manifold with the algebra $\mathcal{A}$ of complex valued smooth functions, the Hilbert space $\mathcal{H}$ of L^2-spinors, and the *Dirac operator* $\not{D}$. One should perceive the spectral approach of noncommutative geometry to Riemannian geometry by observing that the local description of Riemann metric tensor $g_{\mu\nu}$ is not accessible in the noncommutative setting anymore. Here, the *spectral geometry* offers a solution, namely, to look at the Riemannian metric through the window of spectral invariants of *elliptic (pseudo)differential operators* on Riemannian manifolds. Therefore within this paradigm, those properties and invariants of Riemannaian geometry which can be formulated in terms of spectral data of elliptic operators, stand a chance of being translated to noncommutaive language. For example in the case of a Riemannian spin manifold, one can read off the dimension and volume of the manifold from the spectrum of the Dirac operator (*Wayl's law*). Also *Connes' distance formula* [3] gives the geodesic distance between two points using the Dirac operator..

One can go further and ask for more refined concepts of Riemannian geometry such as *scalar curvature*. Again, the problem here is that the scalar curvature is defined by the local description of the Riemann metric tensor. For this, spectral geometry offers a solution as follows. Consider a closed Riemannian manifold (M^n, g) and the *Laplace-Beltrami* operator Δ_g associated to the metric g acting on smooth functions on M. One has the following assymptotic expansion for the *heat trace* around zero,

$$\mathrm{Tr}(e^{-t\Delta}) \sim (4\pi t)^{-n/2} \sum_{i=0}^{\infty} a_i t^{i/2}, \quad t \to 0 \tag{1.1}$$

and it can be shown that

$$a_2 = \frac{1}{6} \int_M s(x) \mathrm{dvol}_g,$$

where $s(x)$ is the scalar curvature. Therefore if one can write the asymptotic expansion (1.1) for the Dirac Laplacian $\Delta = D^2$ of an abstract spectral triple $(\mathcal{A}, \mathcal{H}, D)$, then there is a chance for defining the density involved in a_2 as the noncommutative scalar curvature. The only known example of such spectral triple is built on noncommutative tori [3] where there exists a powerful notion of pseudodifferential calculus [4] and therefore the asymptotic expansion (1.1) can be established and one can compute the noncommutative scalar curvature [4], [5].

There exists another approach for studying the spectral properties of elliptic (classical) pseudodifferential operators, namely the *zeta function* approach. It can be thought as the Mellin transformed counterpart of the heat kernel method. For the special case of the Laplacian Δ_g, the spectral zeta function is defined by

$$\zeta_\Delta(z) = \mathrm{TR}\left(\Delta^{-z}\right),$$

where the TR on the right side is the *Kontsevich-Vishik canonical trace* [6] on classical pseudodifferential operators of non-integer order. This trace is the analytic continuation of the ordinary trace on trace-class pseudodifferential operators. In [6] it was shown that for general non-integer order holomorphic families of elliptic classical pseudodifferential operators P_z, the map $\mathrm{TR}\,(P_z)$ is a meromorphic function with simple poles. In the special case of the families of the form AQ^{-z} it turns out that $z = 0$ is a simple pole with residue given by,

$$\mathrm{Res}_{z=0}\mathrm{TR}(AQ^{-z}) = \frac{1}{\mathrm{ord}(Q)}\mathrm{Wres}\,(A)\,,$$

where $\mathrm{Wres}(A)$ is the Wodzicki residue [9] on the integer order classical pseudodifferential operators. Furthermore, in [7] a full description of the Laurent expansion of the map $\mathrm{TR}(AQ^{-z})$ at the poles was given.

One advantage of the zeta function method is that by using the canonical trace, one can express various spectral invariants of elliptic operators in terms of integrals of local densities computed from the homogeneous terms in the symbol of the operator. For instance, considering the Laplacian operator Δ_g on a closed Riemannian manifold (M, g), from the pole structure of $\mathrm{TR}(\Delta^{-z})$ we see that $z = 0$ is a regular point and the value $\zeta_\Delta(0)$ is given by a local formula

$$\zeta_\Delta(0) = -\frac{1}{2}\int_M \mathrm{res}_x\,(\log\Delta)\,\mathrm{dvol}_g - \mathrm{Tr}(\Pi_\Delta), \tag{1.2}$$

where $\mathrm{res}_x\,(\log\Delta)$ is a certain density computed from the operator $\log\Delta$ and Π_Δ is the orthogonal projection onto $\ker(\Delta)$.

The central idea of this thesis is to study the spectral invariants of elliptic operators on noncommutative tori by using the machinery of canonical trace. The backbone of the results in both chapter [3] and chapter [4] is the generalization of the canonical trace to Connes' algebra of classical pseudodifferential operators on noncommutative tori. Here we briefly review the contents of the chapters in this thesis. In chapter [2] we review the basics of pseudodifferential operators on a closed manifold. We also review the construction of the the canonical trace on non-integer order classical pseudodifferential operators and express some important spectral invariants of pseudodifferential operators within this picture. We also review some of the results on conformal invariants of elliptic operators obtained by the method of variations of the canonical trace. At the end we give a quick introduction to the framework of noncommutative geometry through Dirac operators and spectral triples.

In chapter [3] we consider the spin spectral triple on noncommutative tori and by studying the meromorphic structure of the canonical trace of holomorphic families of pseudodifferntial operators on noncommutative tori, we prove the regularity of eta function at zero for the family of conformally perturbed Dirac operators $e^{th/2}De^{th/2}$ and the coupled Dirac operator $D+A$ on noncommutative 3-torus. The spectral eta function was first introduced in [1] where the value $\eta_P(0)$ of a self adjoint pseudodifferntial operator P appeared as a boundry correction term in Atiyah-Patodi-Singer index theorem. Using the method of variations of canonical trace we consider the conformal variations of $\eta_D(0)$ and we show that the spectral value $\eta_D(0)$ is a conformal invariant of noncommutative 3-torus. Next, we study the conformal variation of $\zeta'_{|D|}(0)$ and show that this quantity is also a conformal invariant of odd noncommutative tori. This is the analogue of the vanishing of the conformal anomaly of LogDet in odd dimensions in the commutative case. We also consider $\eta_{D+A}(0)$ for the coupled Dirac operator $D+A$ on noncommutative 3-torus and compute a local formula for the variation of $\eta_{D+A}(0)$ with respect to the vector potential A.

In chapter [4] we consider a family of twisted Dolbeault spectral triples on noncommutative two-torus. This gives rise to a family of elliptic first order differential operators $\bar{\partial}_A$ which are the noncommutative analogues of Cauchy-Riemann operators on a closed Riemann surface. In [8], Quillen considered a line bundle associated to a family of Cauchy-Riemann operators on a closed Riemann surface and equiped the line bundle with a Hemitian metric. Then by using the variations of zeta functions he was able to compute the first Chern form of the line bundle. We consider the Quillen determinant line bundle associated to the family of Cauchy-Riemann operators on noncommutative two-torus and by using the machinery of the canonical trace, we compute the second variation of the $\zeta'_{\Delta}(0)$ where $\Delta = \bar{\partial}_A^2$ is the Dolbeault Laplacian and we obtain the analogue of the Quillen's computations for the curvature form of the determinant line bundle in commutative case.

Bibliography

[1] M.F. Atiyah, V.K. Patodi, I.M. Singer. Spectral asymmetry and Riemannian geometry.I. *Math. Proc. Cambridge Philos. Soc.*, Volume77:43–69, 1975.

[2] A. Connes. C^* algèbres et géométrie différentielle. *C. R. Acad. Sci. Paris Sér. A-B*, 290(13):A599–A604, 1980.

[3] A. Connes. *Noncommutative geometry.* Academic Press, Inc., San Diego, CA, 1994.

[4] A. Connes, H. Moscovici. Modular curvature for noncommutative two-tori. *J. Amer. Math. Soc.*, 27(3):639–684, 2014.

[5] F. Fathizadeh, M. Khalkhali. Scalar curvature for the noncommutative two torus. *J. Noncommut. Geom.*, 7(4):1145–1183, 2013.

[6] M. Kontsevich and S. Vishik. Geometry of determinants of elliptic operators. In *Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993)*, volume 131 of *Progr. Math.*, pages 173–197. Birkhäuser Boston, Boston, MA, 1995.

[7] S. Paycha and S. Scott. A Laurent expansion for regularized integrals of holomorphic symbols. *Geom. Funct. Anal.*, 17(2):491–536, 2007.

[8] D. Quillen. Determinants of Cauchy-Riemann operators on Riemann surfaces. *Funktsional. Anal. i Prilozhen.*, 19(1):37–41, 96, 1985.

[9] M. Wodzicki. Noncommutative residue. I. Fundamentals. K-theory, Arithmetic and Geometry (Moscow, 1984–1986), Lecture Notes in Math., 1289, Springer, Berlin, 1987, p. 320-399.

Chapter 2

Background Material

2.1 Pseudodifferntial operators

In this section we recall the basic properties of pseudodifferential operators on manifolds. For the proof of the results we refer to [17] and [27].

2.1.1 Pseudodifferential operators on Euclidean space

We begin by definition of differential operators on open sets of Euclidean space.

Let $U \subset \mathbb{R}^d$ be an open set. A differential operator with smooth coefficients acting on smooth functions on U is defined by

$$Pu(x) = \sum_{|\alpha| \leqslant a} c_\alpha(x) D_x^\alpha u(x), \quad u \in C^\infty(U), \quad c_\alpha \in C^\infty(U),$$

where $D_x^\alpha := (-i)^\alpha \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$ with $\alpha = (\alpha_1, \cdots \alpha_d)$, $\alpha_i \geqslant 0$. The non negative integer a is called the order of P.

The *symbol* of the operator P is defined by

$$\sigma(P)(x, \xi) = \sum_{|\alpha| \leqslant a} c_\alpha(x) \xi^\alpha.$$

By the basic properties of Fourier transform, one has the relation

$$Pu(x) = \int e^{i\langle x,\xi\rangle}\sigma(P)(x,\xi)\hat{u}(\xi)d\xi, \quad x \in U, u \in C_0^\infty(U), \tag{2.1}$$

where $d\xi$ is the Lebesgue measure on $\mathbb{R}^d$ with an additional normalizing factor of $(2\pi)^{-d/2}$. A pseud differential operator on U is defined by the above formula by using a generalized class of symbols.

Definition 2.1.1. *A scalar symbol σ of order $a \in \mathbb{R}$ on U is a smooth function in $C^\infty(U \times \mathbb{R}^d)$ with property that there exists a real number a such that for any multi-indices $\gamma, \delta \in \mathbb{Z}^d_{\geqslant 0}$ and for any compact subset $K \subset U$, there exists a constant $C_{\gamma,\delta,K} \in \mathbb{R}^+$ such that*

$$\left|\partial_x^\gamma \partial_\xi^\delta \sigma(x,\xi)\right| \leqslant C_{\gamma,\delta,K} \langle\xi\rangle^{a-|\delta|}, \ x \in K, \ \xi \in \mathbb{R}^d,$$

where $\langle\xi\rangle := \sqrt{1+|\xi|^2}$.

We denote the set of scalar symbols of order a by $\mathcal{S}^a(U)$ and the set of all symbols on U by $\mathcal{S}(U)$. The order of symbols naturally equips the set of scalar symbols with a filtration,

$$\mathcal{S}(U) = \bigcup_a \mathcal{S}^a(U).$$

If E is a trivial vector bundle on U with the complex vector space V as fibers, we define a symbols with matrix coefficients as an element in

$$\mathcal{S}^a(U) \otimes \mathrm{End}(V). \tag{2.2}$$

Also, the class of scalar *smoothing symbols* is defined by

$$\mathcal{S}^{-\infty}(U) = \bigcap_{a\in\mathbb{R}} \mathcal{S}^a(U). \tag{2.3}$$

The equivalence of symbols is defined by the relation

$$\sigma \sim \sigma' \longleftrightarrow \ \sigma - \sigma' \in \mathcal{S}^{-\infty}(U). \tag{2.4}$$

Next we define the class of *classical* symbols on U.

Definition 2.1.2. *A symbol $\sigma \in \mathcal{S}(U)$ is called a classical symbol of order $\alpha \in \mathbb{C}$ if for any N and each $0 \leqslant j \leqslant N$ there exist $\sigma_{\alpha-j}(x,\xi)$, positive homogeneous of degree $\alpha - j$*

in ξ variable, and a symbol $\sigma^N \in \mathcal{S}^{\mathrm{Re}(\alpha)-N-1}(U)$, such that

$$\sigma(x,\xi) = \sum_{j=0}^{N} \chi(\xi)\sigma_{\alpha-j}(x,\xi) + \sigma^N(x,\xi) \quad \xi \in \mathbb{R}^d. \tag{2.5}$$

Here χ is a smooth cut off function on $\mathbb{R}^d$ which is equal to zero on a small ball around the origin, and is equal to one outside the unit ball. It can be shown that the homogeneous terms in the expansion are uniquely determined by σ. The set of all classical symbols of order α over U is denoted by $\mathcal{S}^{\alpha}_{cl}(U)$.

The above definition can be easily adapted to symbols with matrix coefficient on U.

Definition 2.1.3. *The first term in homogeneous expansion 2.5 for the symbol σ is called the leading symbol and is denoted by $\sigma^L(x,\xi)$.*

For a fixed order α one can equip the the space of symbols $\mathcal{S}^{\alpha}_{cl}(U) \otimes \mathrm{End}(V)$ with a Fréchet structure using the following semi norms [27],

$$\begin{aligned}
&\sup_{x\in K_i,\xi\in\mathbb{R}^d} \langle\xi\rangle^{-\mathrm{Re}(\alpha)+|\beta|} ||\partial_x^{\alpha}\partial_\xi^{\beta}\sigma(x,\xi)||,\\
&\sup_{x\in K_i,\xi\in\mathbb{R}^d} \langle\xi\rangle^{-\mathrm{Re}(\alpha)+N+|\beta|} ||\partial_x^{\alpha}\partial_\xi^{\beta}\left(\sigma - \sum_{j=0}^{N-1}\chi(\xi)\sigma_{\alpha-j}\right)(x,\xi)||,\\
&\sup_{x\in K_i,|\xi|=1} ||\partial_x^{\alpha}\partial_\xi^{\beta}\sigma_{\alpha-j,l}(x,\xi)||,
\end{aligned}$$

where α,β are multi-indices, $j \geqslant 0$, $\{K_i,\ i\in\mathbb{N}\}$ is a countable covering of U with compact sets and χ is any smooth function vanishing around zero and equal to one outside of the unit ball.

On can define the star-product of two classical symbols $\sigma \in \mathcal{S}^{a}_{cl}(U)$ and $\tau \in \mathcal{S}^{b}_{cl}(U)$ by

$$\sigma \star \tau = \lim_{N\to\infty} \sum_{|\alpha|=0}^{N-1} \frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^{\alpha}\sigma(x,\xi)\partial_x^{\alpha}\tau(x,\xi), \tag{2.6}$$

where the limit is taken in the above Fréchet topology on the space of symbols of constant order $a+b$ and also the following multi-index notation is used: $\alpha = (\alpha_1,\cdots,\alpha_d)$, $\alpha_j \geqslant 0$, $\alpha! = \alpha_1!\cdots\alpha_d!$, $\partial_\xi^{\alpha} = \partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_d}^{\alpha_d}$ and $\partial_x^{\alpha} = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}$. In particular, the above limit means that for any $N\in\mathbb{N}$

$$\sigma \star \tau - \sum_{|\alpha|=0}^{N-1} \frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^{\alpha}\sigma(x,\xi)\partial_x^{\alpha}\tau(x,\xi) \in \mathcal{S}^{a+b-N}_{cl}(U).$$

Let $\sigma \in \mathcal{S}^a(U)$ and $\tau \in \mathcal{S}^b(U)$, then

$$\sigma \star \tau \in \mathcal{S}^{a+b}(U) \tag{2.7}$$

and the leading symbols are related by

$$(\sigma \star \tau)^L = \sigma^L.\tau^L.$$

Recall that for a function $u \in C_0^\infty(U)$ the Fourier transform of u is defined by

$$\mathcal{F}(u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-i\langle y,\xi\rangle} u(y)dy.$$

Definition 2.1.4. *Consider a symbol $\sigma \in \mathcal{S}(U)$, the pseudodifferential operator corresponding to σ is a linear operator*

$$\mathrm{Op}(\sigma) : C_0^\infty(U) \longrightarrow C_0^\infty(U) \tag{2.8}$$

defined by

$$\left(\mathrm{Op}(\sigma)u\right)(x) = \mathcal{F}^{-1}\left(\sigma(x,.)\hat{u}\right), \; u \in C_0^\infty(U). \tag{2.9}$$

We say that the pseudodifferential operator $\mathrm{Op}(\sigma)$ *is classical of order a if $\sigma \in \mathcal{S}^a_{cl}(U)$. Also* $\mathrm{Op}(\sigma)$ *is a matrix pseudodifferential operator if $\sigma \in \mathcal{S}(U) \otimes \mathrm{End}(V)$, and* $\mathrm{Op}(\sigma)$ *is smoothing if $\sigma \in \mathcal{S}^{-\infty}(U)$.*

We denote the collection of all matrix pseudodifferential operators on U by $\Psi^*(U,V)$ and classical matrix pseudodifferential operators by $\Psi^*_{cl}(U,V)$. If a linear operator $A : C_0^\infty(U) \to C_0^\infty(U)$ is given by $A = \mathrm{Op}(\rho)$ where $\rho \in \mathcal{S}^*(U)$, we write $\sigma(A) \sim \rho$. Note that $\Psi^*(U,V)$ is equipped with a natural filtration given by the order of symbols. The following proposition shows that the composition of pseudodifferential operators respects this filtration and hence, $\Psi^*(U,V)$ is a filtered algebra (see [17] for a proof).

Proposition 2.1.5. *Let $A, B : C_0^\infty(U,V) \longrightarrow C_0^\infty(U,V)$ be two matrix pseudodifferential operators with the symbols $\sigma(A)$ and $\sigma(B)$. The composition of A and B is a pseudodifferntial operator with the following symbol*

$$\sigma(AB) \sim \sigma(A) \star \sigma(B), \tag{2.10}$$

where the $\star$-product is defined in (2.6). Furthermore, the leading symbol of AB is the product of the leading symbols

$$\sigma^L(AB) = \sigma^L(A) \circ \sigma^L(B).$$

□

2.1.2 Pseudodifferential operators on manifolds

In this section we extend the notion of pseudodifferential operators on euclidean spaces to smooth manifolds. Let M be a $d-$dimensional smooth manifold with an atlas $\{U_i, \phi_i\}$ and a partition of unity $\{\chi_i\}$ subordinate to this atlas. Now consider a linear operator $A : C^\infty(M) \to C^\infty(M)$ on M. The localized operator on the coordinate patch (U, ϕ) around a point $x \in M$ is given by $A_U := \chi A \tilde{\chi}$ where χ and $\tilde{\chi}$ are smooth functions with compact support in U which are equal to one in a neighborhood of x.

Definition 2.1.6. *We call A a pseudodiffrential operator, if on each chart (U, ϕ), the following operator on the open set $\phi(U)$ of $\mathbb{R}^d$ is a pseudodifferential operator,*

$$\phi_* A_U := \phi^* \circ A_U \circ \phi_*, \tag{2.11}$$

where $\phi^* f := f \circ \phi$ and $\phi_* f := f \circ \phi^{-1}$. We define the symbol of A in the local chart (U, ϕ) to be the symbol of $\phi_* A_U$.

Now, given the partition of unity $\{\chi_i\}$ for M, one can write,

$$A = \sum_{\mathrm{Supp}(\chi_i) \cap \mathrm{Supp}(\chi_j) \neq \emptyset} \chi_i A \chi_j + R(A), \tag{2.12}$$

where $R(A) = \sum_{\mathrm{Supp}(\chi_i) \cap \mathrm{Supp}(\chi_j) = \emptyset} \chi_i A \chi_j$ is a smoothing operator. Those properties of pseudodifferential operators on euclidean space which are invariant under diffeomorphisms can be extended to smooth manifolds. Let (U, ϕ) and (V, ψ) be two coordinate patches and $\kappa := \psi \circ \phi^{-1}$ be the coordinate transformation. Then

$$\psi_* A_{U \cap V} = \kappa^* \circ \phi_* A_{U \cap V} \circ \kappa_* = \kappa_* \left(\phi_* A_{U \cap V} \right). \tag{2.13}$$

One can show that (see [17]) $\psi_* A_{U \cap V}$ and $\phi_* A_{U \cap V}$ have the same order and differ by an operator of strictly smaller order. Therefore it makes sense to define an operator A

to be **classical** if on any chart (U, ϕ) and any localization A_U, the localized operator $A_{\phi(U)}$ is classical. The operator A has the **order** α if on any chart (U, ϕ) and any localization A_U, the localized operator $\phi_* A$ has order α. We denote the set of classical pseudodifferential operators of order α by $\Psi^{\alpha}_{cl}(M)$. At the local chart (U, ϕ), the **leading symbol** $\sigma^L(A)(x)$ at the point $x \in U$ is defined by $\sigma^L(\phi_* A_U)(\phi(x))$. Based on above, we have the following two definitions.

Definition 2.1.7. *A pseudodifferential operator A is called* **elliptic** *if for $x \in M$ and $\xi \neq 0$, the leading symbol $\sigma^L(A)(x, \xi)$ is invertible.*

Definition 2.1.8. *The algebra of classical pseudodifferential operators on M, denoted by $\Psi^*_{cl}(M)$ is the algebra generated by $\bigcup_{\alpha \in \mathbb{C}} \Psi^{\alpha}_{cl}(M)$. It is endowed with the natural filtration resulting from the order and the product of symbols makes it into a filtered algebra. The two sided ideal of smoothing operators is given by*

$$\Psi^{-\infty}_{cl}(M) = \bigcap_{\alpha \in \mathbb{C}} \Psi^{\alpha}_{cl}(M).$$

The above definitions and properties extend to matrix pseudodifferential operators acting on sections $C^{\infty}_0(M, E)$ of a smooth vector bundle E on M. This leads to the algebra of matrix pseudodifferential operators $\Psi^*_{cl}(M, E)$.

2.1.3 Traces on pseudodifferential operators

In this section we review the different types of traces defined on algebra of classical pseudodifferential operators on a compact closed $d-$dimensional Riemannian manifold M, all of the results hold in the case of matrix pseudodifferential operators as well. We follow [27] and [31] in this section.

Consider an open subset $U \subset \mathbb{R}^d$. Recall that a pseudodifferential operator $A : C^{\infty}_0(U) \mapsto C^{\infty}_0(U)$ with the symbol σ can be written as follows,

$$Au(x) = \int_{\mathbb{R}^d} e^{i<x,\xi>} \sigma(x, \xi) \hat{u}(\xi) d\xi, \ u \in C^{\infty}_0(U),$$

and one can write

$$
\begin{aligned}
Au(x) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i<x-y,\xi>}\sigma(x,\xi)u(y)d\xi dy \\
&= \int_{\mathbb{R}^d} K_A(x,y)u(y)dy,
\end{aligned} \tag{2.14}
$$

where

$$K_A(x,y) = \int_{\mathbb{R}^d} e^{i<x-y,\xi>}\sigma(x,\xi)d\xi$$

is the *Schwartz Kernel* of the operator A. In general, this is a distribution kernel with the singularities located on the diagonal.

The notion of Schwartz Kernel can be extended to pseudodifferential operators on manifolds (see [31] for a detailed review). However, all we need here is that, when M is compact and $A \in \Psi_{cl}^{<-d}(M)$, the restriction of the Schwartz kernel $K_A(x,y)$ to the diagonal of $M \times M$ defines a smooth 1-density (therefore it can be integrated over M). Also the operator A belongs to the ideal of trace class operators and its trace is given by the integral over the diagonal of $M \times M$ of the kernel,

$$\mathrm{Tr}(A) = \int_M K_A(x,x)dx. \tag{2.15}$$

We are seeking for an extension of ordinary trace possibly to the algebra $\Psi_{cl}^{*}(M)$. It turns out that the *canonical trace* ([23]),

$$\mathrm{TR} : \Psi_{cl}^{\notin\mathbb{Z}}(M) \to \mathbb{C}, \tag{2.16}$$

on classical operators of non-integer order is the unique analytic continuation of the ordinary trace on $\Psi_{cl}^{<-d}(M)$. Roughly speaking, being analytic means that if A_z for $z \in \mathbb{C}$ is a holomorphic family of pseudodifferential operators in $\Psi_{cl}^{\mathbb{C}\setminus\mathbb{Z}}(M)$, then $\mathrm{TR}\,(A_z)$ is a holomorphic map. The basic idea behind the construction of the canonical trace is actually quite old and in mathematics literature it is known as the Hadamard finite part method. Also the idea was widely used by physicists when dealing with integrals of the form

$$\int_{\mathbb{R}^d} f(\xi)d\xi,$$

when $f \in L^1(B_R(0))$ for any $R > 0$ but $f \notin L^1(\mathbb{R}^d)$ (*ultra-violet divergence*) where it is known as momentum cut-off integral method.

We start by the following lemma on the cut-off integral of classical symbols on euclidean space [27],[29].

Lemma 2.1.9. *Let $\sigma \in \mathcal{S}^{\alpha}_{cl}(U)$ where $U \in \mathbb{R}^d$ is an open set and $B_R(0)$ be the ball of radius R around the origin. One has the following asymptotic expansion*

$$\int_{B(R)} \sigma(\xi)d\xi \sim_{R\to\infty} \sum_{j=0,\alpha-j+d\neq 0}^{\infty} \alpha_j(\sigma)R^{\alpha-j+d} + \beta(\sigma)\log R + c(\sigma), \tag{2.17}$$

where $\beta(\sigma) = \int_{|\xi|=1} \sigma_{-d}(\xi)d\xi$ and the constant term in the expansion, $c(\sigma)$, is given by

$$c(\sigma) = \int_{\mathbb{R}^n} \sigma^N(x,\xi)d\xi + \sum_{j=0}^{N} \int_{B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi - \sum_{j=0,\alpha-j+d\neq 0}^{N} \frac{1}{\alpha-j+d} \int_{S^{d-1}} \sigma_{\alpha-j}(\omega)d\omega, \tag{2.18}$$

where we have used the notation in definition (2.1.2).

Proof. In order to simplify the notations we drop the variable $x \in U$ and write $\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^N(\xi)$ with large enough N, so that σ^N is integrable. Then we have,

$$\int_{B(R)} \sigma(\xi)d\xi = \sum_{j=0}^{N} \int_{B(R)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi + \int_{B(R)} \sigma^N(\xi)d\xi. \tag{2.19}$$

For $N > \alpha+1$, $\sigma^N \in \mathcal{L}^1(\mathbb{R}^d)$, so

$$\int_{B(R)} \sigma^N(\xi)d\xi \to \int_{\mathbb{R}^d} \sigma^N(\xi)d\xi, \quad R \to \infty.$$

Now for each $j \leqslant N$ we have

$$\int_{B(R)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi = \int_{B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi + \int_{B(R)\backslash B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi.$$

Obviously $\int_{B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi < \infty$ and by using polar coordinates $\xi = r\omega$, and homogeneity of $\sigma_{\alpha-j}$, we have

$$\int_{B(R)\backslash B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi = \int_1^R r^{\alpha-j+d-1}dr \int_{|\xi|=1} \sigma_{\alpha-j}(\xi)d\xi. \tag{2.20}$$

Note that the cut-off function is equal to one on the set $\mathbb{R}^d \backslash B(1)$. For the term with $\alpha - j = -d$ one has

$$\int_{B(R)\backslash B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi = \log R \int_{|\xi|=1} \sigma_{\alpha-j}(\xi)d\xi.$$

The terms with $\alpha - j \neq -d$ will give us the following:

$$\begin{aligned} \int_{B(R)\backslash B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi &= \\ \frac{R^{\alpha-j+d}}{m-j+d}\int_{|\xi|=1} \sigma_{\alpha-j}(\xi)d\xi &- \frac{1}{\alpha-j+d}\int_{|\xi|=1} \sigma_{\alpha-j}(\xi)d\xi. \end{aligned} \tag{2.21}$$

Adding all the constant terms in (2.19)-(2.21), we get the constant term given in (2.18).

□

The cut-off integral of the classical symbol $\sigma \in \mathcal{S}^{\alpha}_{cl}(U)$ *regularizes* the possible divergence in $\int_{\mathbb{R}^d} \sigma(\xi)d\xi$, by only picking the constant term (finite part) in the asymptotic expansion of $\int_{B_R(0)} \sigma(\xi)d\xi$ as $R \to \infty$.

Definition 2.1.10. *Let $\sigma \in \mathcal{S}^{\alpha}_{cl}(U)$ be a classical symbol. The cut-off integral is given by the constant term in the equation (2.17),*

$$\begin{aligned} &⨍ \sigma(x,\xi)d\xi = c(\sigma)(x,\xi) \\ &\quad = \int_{\mathbb{R}^n} \sigma^N(x,\xi)d\xi + \sum_{j=0}^{N}\int_{B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi - \sum_{j=0,\alpha-j+d\neq 0}^{N} \frac{1}{\alpha-j+d}\int_{S^{d-1}} \sigma_{\alpha-j}(\omega)d\omega. \end{aligned}$$

It is immediate from the definition of the cut-off integral that for $\sigma \in \mathcal{S}^{|\alpha|<-d}_{cl}(U)$,

$$⨍ \sigma(\xi)d\xi = \int_{\mathbb{R}^d} \sigma(\xi)d\xi.$$

Also a quick computation (see [29]) shows that if $A \in \Psi^{*}_{cl}(U)$ is a *differential operator*, then

$$⨍ \sigma\left(A\right)(\xi)d\xi = 0.$$

In order to define the cut-off integral for symbols on manifolds we need to see how it changes under the coordinate transformations. Let U and U' be two open sets of $\mathbb{R}^d$ and

$\phi : U \to U'$ be a diffeomorphism. For a pseudodifferential operator A on U we define its push forward by

$$\phi_* A := \phi_* \circ A \circ \phi^*, \tag{2.22}$$

where $\phi^* f = f \circ \phi$ and $\phi_* g = g \circ \phi^{-1}$ for any $f \in C^\infty(U')$ and $g \in C^\infty(U)$. It can be shown (see [17]) that $\phi_* \mathrm{Op}(\sigma)$ and $\mathrm{Op}(\phi_* \sigma)$ are related as follows, at $(x', \xi') \in T^*U'$,

$$\phi_* \sigma(x', \xi') = \sigma\left(\phi^{-1}(x'), d\phi\left(\phi^{-1}(x')\right)^t \xi'\right) \tag{2.23}$$

where $d\phi(x) : T_x U \to T_{x'} U'$ and $(d\phi(x))^t : T^*_{\phi(x)} U' \to T^*_x U$.

Therefore, in order to lift the cut-off integral to a density on a manifold M, we need to know how the expression in Definition 2.1.10 transforms under the ξ-changes of variable of the form $A\xi$ where $A \in GL(d, \mathbb{R})$. It turns out that unlike the ordinary integrals which are invariant under changes of variable, the cut-off integral suffers from lack of covariance. The following lemma make this more precise, we refer the reader to [23] or [27] for the proof.

Lemma 2.1.11. *Let $A \in GL(d, \mathbb{R})$, $U \subset \mathbb{R}^d$ and open set and $\sigma \in \mathcal{S}^\alpha_{cl}(U)$ a classical symbol. Then,*

$$-\!\!\!\!\!\!\int \sigma(A\xi) d\xi = -\!\!\!\!\!\!\int \sigma(\xi) d\xi - \int_{S^{d-1}} \sigma_{-d}(\omega) \log\left(|A^{-1}\omega|\right) d\omega. \tag{2.24}$$

□

It follows from the above lemma that in order to have a well-defined trace density for the canonical trace on a manifold M we need to restrict its domain to non-integer order classical pseudodifferential operators.

Definition 2.1.12. *The* TR*−functional,* $\mathrm{TR} : \Psi^{\mathbb{C}\setminus\mathbb{Z}}_{cl}(M) \to \mathbb{C}$ *is defined by*

$$\mathrm{TR}(A) := \int_M -\!\!\!\!\!\!\int \sigma(A)(x, \xi) d\xi dx, \;\; A \in \Psi^{\mathbb{C}\setminus\mathbb{Z}}_{cl}(M). \tag{2.25}$$

The trace property of the functional $\mathrm{TR} : \Psi^{\mathbb{C}\setminus\mathbb{Z}}_{cl}(M) \to \mathbb{C}$ follows from the fact that it is the analytic continuation of the ordinary trace $\mathrm{Tr} : \Psi^{<-d}_{cl}(M) \to \mathbb{C}$ to $\Psi^{\mathbb{C}\setminus\mathbb{Z}}_{cl}(M)$. In the following we establish this important relationship in detail, we follow [29] and [31] in this part.

The notion of a *holomorphic family* of symbols was first used in [19] and then used by Kontsevich and Vishik [23] in studying the multiplicative anomaly for determinants in quantum field theory. Let $W \subset \mathbb{C}$ be a complex domain, a map $f : W \to E$ with values in a normed vector space $(E, ||.||)$ is called holomorphic if for any $w_0 \in W$ there exists a a vector $f'(w_0) \in E$ such that

$$\lim_{w \to w_0} \| \frac{f(w) - f(w_0)}{w - w_0} - f'(w_0) \| = 0.$$

The map is called holomorphic if the above property holds for any $w_0 \in W$. The known results for holomorphic maps with values in Banach spaces can be generailized to the maps with values in Fréchet spaces (see [27]). We now give the precise definition of holmorphic families of symbols and pseudodifferential operators. We restrict to the case of classical symbols on Euclidean spaces.

Definition 2.1.13. *A family $\sigma(z)$ of classical symbols on the open set $U \subset \mathbb{R}^d$ parametrized by a complex domain $W \subset \mathbb{C}$ is called holomorphic if*

- *$\sigma(z)$ is a holomorphic map of variable z with values in $C^\infty(U \times \mathbb{R}^d)$ and*

$$\sigma(z) \sim \sum_{j \geqslant 0} \sigma_{\alpha(z)-j}(z) \in \mathcal{S}_{cl}^{\alpha(z)}(U), \tag{2.26}$$

where the order $\alpha : W \to \mathbb{C}$ is holomorphic.

- *for $N \geqslant 1$, the remainder*

$$\sigma^N(z) := \sigma(z) - \sum_{j=0}^{N-1} \sigma_{\alpha(z)-j}(z)\chi$$

is holomorphic as a function with values in $C^\infty(\mathbb{R}^d)$, and the k-th $z-$derivative

$$\partial_z^k \left(\sigma^N(z) \right)$$

is a symbol on $\mathbb{R}^d$ of order $\alpha(z) - N + \epsilon$ for any $\epsilon > 0$ and locally uniformly in z, i.e. the k-th $z-$derivative $\partial_z^k \left(\sigma^N(z) \right)$ satisfies the estimate in Definition 2.1.1 and the estimate is uniform in z on compact sets K.

A family of operators A_z is called holomorphic if $A_z = \mathrm{Op}(\sigma(z))$ for a holomorphic family of symbols $\sigma(z)$.

The following important result of Kontsevich and Vishik [23] describes the pole structure of cut-off integrals of holomorphic families of classical symbols.

Theorem 2.1. *Given a holomorphic family $\sigma(z) \in \mathcal{S}^*_{cl}(\mathbb{R}^d)$, $z \in W \subset \mathbb{C}$, the map*

$$z \mapsto -\!\!\!\!\!\!\int \sigma(z)(\xi) d\xi,$$

is meromorphic with at most simple poles located in

$$P = \{z_0 \in W;\ \alpha(z_0) \in \mathbb{Z} \cap [-d, +\infty]\}.$$

The residues at poles are given by

$$\mathrm{Res}_{z=z_0} -\!\!\!\!\!\!\int \sigma(z)(\xi) d\xi = -\frac{1}{\alpha'(z_0)} \int_{|\xi|=1} \sigma(z_0)_{-d} d\xi.$$

Proof. By definition, one can write $\sigma(z) = \sum_{j=0}^{N} \chi(\xi)\sigma(z)_{\alpha(z)-i}(\xi) + \sigma(z)^N(\xi)$, and by Lamma 2.1.9 we have,

$$\begin{aligned} -\!\!\!\!\!\!\int \sigma(z)(\xi) d\xi = & \int_{\mathbb{R}^d} \sigma(z)^N(\xi) d\xi + \sum_{j=0}^{N} \int_{B(1)} \chi(\xi)\sigma(z)_{\alpha(z)-j}(\xi) \\ & - \sum_{j=0}^{N} \frac{1}{\alpha(z)+d-j} \int_{|\xi|=1} \sigma(z)_{\alpha(z)-j}(\xi) d\xi. \end{aligned}$$

Now suppose $\alpha(z_0) + d - j_0 = 0$. By holomorphicity of $\sigma(z)$, we have

$$\alpha(z) - \alpha(z_0) = \alpha'(z_0)(z - z_0) + o(z - z_0),$$

hence

$$\mathrm{Res}_{z=z_0} -\!\!\!\!\!\!\int \sigma(z) = \frac{-1}{\alpha'(z_0)} \int_{|\xi|=1} \sigma(z_0)_{-d}(\xi) d\xi.$$

□

One has the following immediate corollary.

Corollary 2.1.14. *The functional* TR $: \Psi^{\mathbb{C}\setminus\mathbb{Z}}_{cl} \to \mathbb{C}$ *is the analytic continuation of the ordinary trace on trace-class pseudodifferential operators.*

Proof. First observe that, by the above result, for a non-integer order holomorphic family of symbols $\sigma(z)$, the map $z \mapsto -\!\!\!\!\!\!\int \sigma(z)(\xi) d\xi$ is holomorphic. Hence, the map

$\sigma \mapsto -\!\!\!\!\!\!\int \sigma(\xi) d\xi$ is the unique analytic continuation of the map $\sigma \mapsto \int_{\mathbb{R}^d} \sigma(\xi) d\xi$ from $\mathcal{S}_{cl}^{<-d}(\mathbb{R}^d)$ to $\mathcal{S}_{cl}^{\notin\mathbb{Z}}(\mathbb{R}^d)$. Since the canonical trace involves an integration over $x-$variable of above densities, we obtain the result. □

Complex powers of operators form an important class of holomorphic families. We briefly explain their construction here, following [27].

Let $Q \in \Psi_{cl}^q(M)$ be a positive elliptic pseudodifferential operator of order $q > 0$. The complex power of such an operator, Q_ϕ^z, for $\mathrm{Re}(z) < 0$ can be defined by the following Cauchy integral formula.

$$Q_\phi^z = \frac{i}{2\pi} \int_{C_\phi} \lambda_\phi^z (Q - \lambda)^{-1} d\lambda. \tag{2.27}$$

Here λ_ϕ^z is the complex power with branch cut $L_\phi = \{re^{i\phi}, r \geqslant 0\}$ and C_ϕ is a contour given by

$$\begin{aligned} C_\phi &= C_\phi^1 \cup C_\phi^2 \cup C_\phi^3, \\ C_\phi^1 &= \{\rho e^{i\phi},\ \infty > \rho \geqslant r\}, \\ C_\phi^2 &= \{\rho e^{i(\phi - 2\pi)},\ \infty > \rho \geqslant r\}, \\ C_\phi^3 &= \{re^{it},\ \phi - 2\pi \leqslant t < \phi\} \end{aligned}$$

around the branch cut L_ϕ and the non-zero spectrum of A. the constant r is chosen small enough such that the contour does not intersect the spectrum of A.

In general an operator for which one can find a ray L_ϕ with the above property, is called an *admissible operator* with the spectral cut L_ϕ. Positive elliptic operators are admissible and we take the ray L_π as the spectral cut, and in this case we drop the index ϕ and write Q^z.

To extend (2.27) to $\mathrm{Re}(z) > 0$ we choose a positive integer such that $\mathrm{Re}(z) < k$ and define

$$Q_\phi^z := Q^k Q_\phi^{z-k}.$$

It can be proved (see [27]) that this definition is independent of the choice of k.

Now we can prove the trace property of TR-functional. The basic idea is to embed the symbol $\sigma(A)$ of the operator A into the holomorphic family of symbols $\sigma\left(AQ^{-z}\right)$.

Proposition 2.2. *We have* $\mathrm{TR}(AB) = \mathrm{TR}(BA)$ *for any* $A, B \in \Psi_{cl}(M)$, *provided that* $\mathrm{ord}(A) + \mathrm{ord}(B) \notin \mathbb{Z}$.

Proof. Consider the families A_z and B_z such that $A_0 \sim A$, $B_0 \sim B$, $\mathrm{ord}(A_z) = \mathrm{ord}(A)+z$ and $\mathrm{ord}(B_z) = \mathrm{ord}(B) + z$. For $z \in W = -(\mathrm{ord}(A) + \mathrm{ord}(B)) + \mathbb{Z}$ the families $\{A_zB_z\}$ and $\{B_zA_z\}$ have non-integer order . For $\mathrm{Re}(z) \ll 0$, the two famillIes are trace-class and $\mathrm{Tr}(A_zB_z) = \mathrm{Tr}(B_zA_z)$. now by analytic continuation we have $\mathrm{TR}(A_zB_z) = \mathrm{TR}(B_zA_z)$, for $z \in \mathbb{C} - W$. Putting $z = 0$ gives $\mathrm{TR}(AB) = \mathrm{TR}(BA)$. □

Another important functional on algebra of classical pseudodifferential operators is the *noncommutative (Wodzicki) residue*. It is a trace on the algebra of integer order classical pseudodifferential operators ([19], [20], [37]),

$$\mathrm{Wres} : \Psi^{\mathbb{Z}}_{cl}(M) \mapsto \mathbb{C}. \tag{2.28}$$

We begin by the following definition for pseudodifferential operators on $U \subset \mathbb{R}^d$.

Definition 2.1.15. *For* $\sigma \in \mathcal{S}^m_{cl}(U)$, *the residue density of* σ *is defined as follows.*

$$\mathrm{res}\,(\sigma)\,(x) = \int_{|\xi|=1} \sigma_{-d}(x,\xi) d\xi. \tag{2.29}$$

The Wodzicki residue of a classical pseudodifferential operator $A \in \Psi^{\mathbb{Z}}_{cl}(U)$ is defined as,

Definition 2.1.16.

$$\mathrm{Wres}(A) = \int_U \mathrm{res}\,(\sigma(A))\, dx = \int_U \int_{|\xi|=1} \sigma(A)_{-d}(x,\xi) d\xi dx. \tag{2.30}$$

In order to extend the above definition to operators on manifold we need to make sure that the residue density of $\sigma(A)$ is in fact a density. This can be established by making use of a deeper relation between the canonical trace on non-integer order operators and noncommutative residue on algebra of integer order operators.

Proposition 2.1.17. *Let* $A \in \Psi^{\alpha}_{cl}(M)$ *be of order* $\alpha \in \mathbb{Z}$ *and let* Q *be a positive elliptic classical pseudodifferential operator of positive order* q. *Then,*

- *For the holomorphic family* $\sigma(z) = \sigma(AQ^{-z})$, $z = 0$ *is a simple pole for the map*

$$z \mapsto \fint \sigma(z)(\xi) d\xi,$$

whose residue is given by

$$\mathrm{Res}_{z=0}\left(z \mapsto \fint \sigma(z)(\xi) d\xi\right) = -\frac{1}{\alpha'(0)} \int_{|\xi|=1} \sigma_{-d}(0) d\xi = -\frac{1}{\alpha'(0)} \mathrm{res}(A). \quad (2.31)$$

- $$\mathrm{Res}_{z=0}\mathrm{TR}(AQ^{-z}) = \frac{1}{q}\mathrm{Wres}(A). \quad (2.32)$$

Proof. This is simply a corollary of Theorem 2.1 applied to the holomorphic family $\sigma(z) = \sigma(AQ^{-z})$. Also , since $\sigma(z)$ has non-integer order, $\fint \sigma(z)(\xi) d\xi$ is a well defined density and therefore

$$\mathrm{res}(A)(x, \xi) = \mathrm{Res}_{z=0} \fint \sigma(z)(\xi) d\xi \quad (2.33)$$

is a well defined density on M. $\square$

The trace property of the noncommutative residue can be derived from the above result.

Proposition 2.1.18. *Consider the operators* $A, B \in \Psi^{\mathbb{Z}}_{cl}(M)$, *then*

$$\mathrm{Wres}([A, B]) = 0. \quad (2.34)$$

Proof. We write,

$$\mathrm{Wres}([A, B]) = \mathrm{Res}_{z=0}\mathrm{TR}([A, B]Q^{-z}) = \mathrm{Res}_{z=0}\mathrm{TR}(C_z) + \mathrm{Res}_{z=0}\mathrm{TR}([AQ^{-z}, B]),$$

where $C_z = ABQ^{-z} - AQ^{-z}B$. For $Re(z) \gg 0$, the operator AQ^{-z} is trace-class and $\mathrm{Tr}([AQ^{-z}, B]) = 0$, so by analytic continuation,$\mathrm{TR}([AQ^{-z}, B]) = 0$ and therefore,

$$\mathrm{Res}_{z=0}\mathrm{TR}([A, B]Q^{-z}) = \mathrm{Res}_{z=0}\mathrm{TR}(C_z). \quad (2.35)$$

Finally, $C_0 = ABQ^0 - AQ^0B \in \Psi^{-\infty}_{cl}(M)$, and since the noncommutative residue of a smoothing operator is zero, we obtain

$$\mathrm{Wres}([A, B]) = \mathrm{Res}_{z=0}TR(C_z) = \mathrm{Wres}(C_0) = 0. \quad (2.36)$$

□

The result of Theorem 2.1 has been generalized in [29] and the full Laurent expansion can be explicitly calculated.

Theorem 2.3. *Consider the holomorphic family $\sigma(z)$ of classical symbols on an open set $U \subset \mathbb{R}^d$. Also, assume that that the order of the family is linear, $\alpha(z) = \alpha'(0)z + \alpha(0)$, $\alpha'(0) \neq 0$. Then at $z = 0$ the map $z \mapsto ⨍ \sigma(z)(x,\xi)d\xi$ has the following Laurent expansion,*

$$⨍ \sigma(z)(x,\xi)d\xi = \left(\frac{-1}{\alpha'(0)}\mathrm{res}\sigma(0)\right)\frac{1}{z} + \sum_{k=0}^{K}\frac{z^k}{k!}\left(⨍ \sigma^{(k)}(0) - \frac{1}{\alpha'(0)(k+1)}\mathrm{res}\sigma^{(k+1)}(0)\right) + O(z^k). \quad \square$$

One has to note that the $z-$derivative $\sigma^(k)(0)$ need not be a classical symbol, however the $z-$derivatives of holomorphic families of the form $\sigma(z) = AQ^{-z}$ fall into a larger class of symbols called *log-polyhomogenous symbols*. Here we briefly review their definition and properties and refer the reader to [29] and [26] for details.

Definition 2.1.19. *Let $U \subset \mathbb{R}^d$ be an open set, $\alpha \in \mathbb{C}$ and $k \in \mathbb{Z}$. a symbol $\sigma \in \mathcal{S}^\alpha(U)$ is called log-polyhomogeneous of order α and $\log -$ type k if for any N and each $0 \leqslant j \leqslant N$ there exist a symbol $\sigma^N \in \mathcal{S}^{\mathrm{Re}(\alpha)-N-1}(U)$, such that*

$$\sigma(\xi) = \sum_{j=0}^{N}\chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^N(\xi) \quad \xi \in \mathbb{R}^d, \tag{2.37}$$

where

$$\sigma_{\alpha-j}(x,\xi) = \sum_{l=0}^{k}\sigma_{\alpha-j,l}(x,\xi)\log^l(|\xi|),\ \xi \neq 0, \tag{2.38}$$

and $\sigma_{\alpha-j,l}(x,\xi)$ are positively homogeneous of order $\alpha - j$ in ξ. Here χ is a smooth cut off function on $\mathbb{R}^d$ which is equal to zero on a small ball around the origin, and is equal to one outside the unit ball. We denote the collection of these symbols by $\mathcal{S}_{cl}^{\alpha,k}(U)$.

An important example of an operator with a log-polyhomogenous symbol is $\log Q$ where $Q \in \Psi_{cl}^q(M)$ is a positive elliptic differential operator of order $q > 0$. The logarithm of Q can be defined by

$$\log Q = Q \left.\frac{d}{dz}\right|_{z=0} Q^{z-1} = Q \left.\frac{d}{dz}\right|_{z=0} \frac{i}{2\pi}\int_C \lambda^{z-1}(Q-\lambda)^{-1}d\lambda.$$

It is a pseudodifferential operator with symbol

$$\sigma(\log Q) \sim \sigma(Q) \star \sigma\Big(\left.\frac{d}{dz}\right|_{z=0} Q^{z-1}\Big), \tag{2.39}$$

where $\star$ denotes the product of symbols. Using symbol calculus, it can be shown that (2.39) is a log-homogeneous symbol of the form

$$\sigma(\log Q)(\xi) = q\log|\xi| I + \sigma_{cl}(\log Q)(\xi),$$

where $\sigma_{cl}(\log Q)$ is a classical symbol of order zero. This symbol can be computed using the homogeneous parts of the classical symbol $\sigma(Q^z) = \sum_{j=0}^{\infty} b(z)_{qz-j}(\xi)$ and it is given by the following formula (see e.g. [27]).

$$\begin{aligned}\sigma_{cl}(\log Q)(\xi) = \\ \sum_{k=0}^{\infty}\sum_{i+j+|\alpha|=k}\frac{1}{\alpha!}\partial^\alpha\sigma_{q-i}(Q)\delta^\alpha\left[|\xi|^{-q-j}\left.\frac{d}{dz}\right|_{z=0} b(z-1)_{qz-q-j}\left(\xi/|\xi|\right)\right].\end{aligned} \tag{2.40}$$

The definition of the cut-off integral of symbols can be extended to log-polyhomogenous symbols [26] and for a non-integer order symbol $\sigma \in \mathcal{S}^{\alpha,k}_{cl}(M)$, $⨍ \sigma(x,\xi)d\xi$ is a well defined density, and hence one can define the canonical trace $\mathrm{TR}(A)$ by (2.25). Also it is shown in [26] that the noncommutative residue can be extended to log-polyhomogenous symbols of log type k by the following formula for the residue density,

$$\mathrm{res}\,(\sigma)\,(x,\xi) = \int_{|\xi|=1}\sigma_{-d,k}(x,\xi)d\xi,\ \sigma\in\mathcal{S}^{\alpha,k}_{cl},\ U\subset\mathbb{R}^d. \tag{2.41}$$

The Theorem 2.3 takes the following form when applied to the holomorphic family $A_z = AQ^{-z}$ where M^d is a closed manifold, $A \in \Psi^*_{cl}(M)$ and Q is a positive elliptic differential operator (more generally, an admissible operator) of order $q > 0$. We only state the result here and refer to [29] for the proof in a more general setting.

Theorem 2.4. *For the pseudodifferential operator $A \in \Psi^*_{cl}M$ and Q a positive elliptic differential operator of order $q > 0$, $\mathrm{TR}\,(AQ^{-z})$ has a pole of order at most 1 at $z = 0$*

with the following Laurent expansion,

$$\begin{aligned}
\mathrm{TR}(AQ^{-z}) &= \frac{1}{q}\mathrm{Wres}(A)\frac{1}{z} \\
&+ \int_M \left(⨍ \sigma(A)(\xi) - \frac{1}{q}\mathrm{res}(A \log Q) \right) - \mathrm{Tr}(A\Pi_Q) \\
&+ \sum_{k=1}^{K} (-1)^k \frac{(z)^k}{k!} \\
&\times \left(\int_M \left(⨍ \sigma(A(\log Q)^k) d\xi - \frac{1}{q(k+1)}\mathrm{res}(A(\log Q)^{k+1}) \right) - \mathrm{Tr}(A \log^k Q \Pi_Q) \right) \\
&+ o(z^K),
\end{aligned}$$

where Π_Q *is the projection over the generalized kernel of* Q*.* □

2.1.4 Spectral functions of elliptic operators

In this section we define and review the basic properties of two important spectral function associated to an elliptic operator on a closed manifold. For a thorough review of the subject, we refer the reader to [17] and to [5] for the applications in quantum field theory. We start by the definition of admissible operators.

Definition 2.1.20. *An operator* $A \in \Psi^*_{cl}(M)$ *is called admissible if for any* $(x, \xi) \in T^*M \backslash M \times \{0\}$ *the leading symbol* $\sigma^L(A)(x, \xi)$ *has no eigenvalue on the ray* $L_\theta = \{re^{i\theta},\ r \geqslant 0\}$ *and also the spectrum of* A *does not intersect* L_θ*.* θ *is called the principal angle and* L_θ *is the spectral cut.*

Remark 2.1.21. From the definition, it follows that an admissible operator is elliptic and invertible. In applications, many of the interesting elliptic operators have kernel, however one can obtain an invertible operator as follows. Let $A \in \Psi^*_{cl}(M)$ be a self adjoint elliptic operator. One has the splitting $L^2(M) = \ker(A) \oplus \mathrm{Range}(A)$ where $\ker(A)$ is the finite dimensional kernel of A and $\mathrm{Range}(A)$ is the closed range of A. By setting $A' = A \oplus \Pi_A$, where Π_A is the projection onto $\ker(A)$ one gets an invertible operator. Classical pseudodifferential operators with positive leading symbol such as generalized Laplacians, also formally self adjoint elliptic classical pseudodifferential operators such as Dirac operators in odd dimensions are all examples of admissible operators.

Definition 2.1.22. *Let $A \in \Psi^*_{cl}(M)$ and Q be an admissible operator, then the generalized ζ-function of Q is given by,*

$$\zeta_\phi(A, Q) := \mathrm{TR}\left(AQ_\phi^{-z}\right), \tag{2.42}$$

where TR *is the canonical trace and ϕ is a fixed principal angle for Q and Q_ϕ^{-z} is the complex power of Q using the spectral cut L_ϕ.*

In the special case of $A = I$ in above definition one gets the *spectral ζ-function* of Q,

$$\zeta_\phi(Q)(z) = \mathrm{TR}\left(Q_\phi^{-z}\right). \tag{2.43}$$

The meromorphic structure of the spectral $\zeta-$function easily follows from the pole structure of canonical trace of the family Q_ϕ^{-z} (see Theorem 2.4).

Proposition 2.1.23. *Let $Q \in \Psi^q_{cl}(M)$ be an admissible operator of order $q > 0$, the spectral $\zeta-$function $\zeta_\phi(Q)(z) = \mathrm{TR}\left(Q_\phi^{-z}\right)$ is a meromorphic function with at most simple poles located in the set $\Sigma = \{\frac{d-k}{q},\ k = 0, 1, \cdots\}$, where d is the dimension of M. Furthermore, $z = 0$ is always a regular point and residues at poles are given by*

$$\mathrm{Res}_{z=\sigma}\zeta_\phi(Q)(z) = q\mathrm{Wres}\left(Q_\phi^{-\sigma}\right),\ \sigma \in \Sigma. \tag{2.44}$$

Proof. The pole structure of $\zeta_\phi(Q)(z)$ follows from Theorem 2.1. Also by Theorem 2.4 we see that

$$\mathrm{Res}_{z=0}\zeta_\phi(Q)(z) = \frac{1}{q}\mathrm{Wres}(I) = 0,$$

therefore $z = 0$ is a regular point. For the residues at other poles, we set, $w = z - \sigma$ and therefore,

$$\mathrm{Res}_{z=\sigma}\zeta_\phi(Q)(z) = \mathrm{Res}_{w=0}\mathrm{TR}\left(Q_\phi^{-w-\sigma}\right) = \mathrm{Res}_{w=0}\mathrm{TR}\left(Q_\phi^{-\sigma}Q_\phi^{-w}\right) = \frac{1}{q}\mathrm{Wres}\left(Q_\phi^{-\sigma}\right).$$

$\square$

Remark 2.1.24. It follows from the above result that for an admissible operator $Q \in \Psi^q_{cl}(M)$, the values $\zeta_\theta(Q)(0)$ and $\zeta'_\theta(Q)(0)$ make sense. Also, from Theorem 2.4 we obtain the following formulas,

$$\zeta_\theta(Q)(0) = -\frac{1}{q}\int_M \mathrm{res}\left(A\log Q\right) - \mathrm{Tr}(A\Pi_Q). \tag{2.45}$$

and

$$\zeta'_\theta(Q)(0) = -\left(\int_M \left(⨍ \sigma(\log Q)d\xi - \frac{1}{2q}\mathrm{res}((\log Q)^2)\right) - \mathrm{Tr}(\log Q\Pi_Q)\right). \tag{2.46}$$

Note that unlike $\zeta_\theta(Q)(0)$, the value $\zeta'_\theta(Q)(0)$ is *non local* due to presence of the cut off integral which consists of infinitely many homogeneous terms of the symbol of $\log Q$.

Form the definition of the complex power of an elliptic operator it is obvious that the values of the spectral zeta function has a dependence on the choice of the spectral cut. Here we briefly review some of the results regarding this dependence and refer the reader to [30] for details.

Consider the elliptic operator $A \in \Psi^\alpha_{cl}(M)$ with $\alpha > 0$. Let L_θ and $L_{\theta'}$ be two different spectral cuts with $\theta < \theta' \leqslant \theta + 2\pi$ and let $\Lambda_{\theta,\theta'}$ be the sector of the plane $\theta < \arg\lambda < \theta'$. The corresponding *sectorial projection* of A is defined by

$$\Pi_{\theta,\theta'}(A) = \int_{\Gamma_{\theta,\theta'}} \lambda^{-1}(A-\lambda)^{-1}d\lambda, \tag{2.47}$$

where

$$\Gamma_{\theta,\theta'} = \{\rho e^{i\theta};\ \infty > \rho \geqslant r\} \cup \{re^{it};\ \theta < t < \theta'\} \cup \{\rho e^{i\theta'};\ r \leqslant \rho < \infty\},$$

and r is small enough such that no non-zero eigenvalue of A lies inside the disk $|\lambda| \leqslant r$. It can be shown (see [30]) that $\Pi_{\theta,\theta'}(A)$ is a psedodifferential operator of ord $\leqslant 0$ and hence is a bounded operator on $L^2(M)$. Furthermore, when the principal symbol of A has no eigenvalues in the angular sector $\Lambda_{\theta,\theta'}$, the operator $\Pi_{\theta,\theta'}(A)$ is smoothing.

The following result is due to M.Wodzicki [35] .

Proposition 2.1.25. *One has the following equality of meromorphic functions,*

$$\zeta_\theta(A)(z) - \zeta_{\theta'}(A)(z) = \left(1 - e^{-2\pi i z}\right)\mathrm{TR}\left(\Pi_{\theta,\theta'}A_\theta^{-z}\right),\ z \in \mathbb{C}. \quad \square \tag{2.48}$$

One also has the following result on the value of the spectral $\zeta-$function at zero [35].

Proposition 2.1.26. *Consider the elliptic operator $A \in \Psi^\alpha_{cl}(M)$ with $\alpha > 0$ and let L_θ be a spectral cut. then the value $\zeta_\theta(A)(0)$ is independent of θ.*

$\square$

Remark 2.1.27. The Wodzicki's proof of the above proposition is quite involved and uses very careful analysis of local spectral invariants. Note that the difference

$$\zeta_\theta(A)(0) - \zeta_{\theta'}(A)(0)$$

is a multiple of the Wodzicki residue of a pseudodifferntial projection. In fact he used the above result to prove that the Wodzicki residue of any pseudodifferential projection is zero [36].

Another important spectral function associated to a self adjoint elliptic pseudodifferential operator is the *spectral eta function*. It was introduced in [1] and it's value at origin appeared as a boundary correction term in Atiyah-Patodi-Singer index theorem.

Definition 2.1.28. *Let $A \in \Psi^*_{cl}(M)$ be a self adjoint elliptic pseudodifferential operator. The spectral eta function is defined by*

$$\eta(A)(z) = \mathrm{TR}\left(A|A|^{z-1}\right) = \mathrm{TR}\left(F|A|^{-z}\right), \tag{2.49}$$

where $F = A|A|^{-1}$.

Regarding the meromorphic structure of the spectral eta function one has the following result.

Proposition 2.1.29. *Let $A \in \Psi^\alpha_{cl}(M^d)$ be a self adjoint pseudodifferential operator. The spectral eta function $\eta(A)(z)$ is a meromorphic function with at most simple poles located in the set $\Sigma = \{\frac{d-k}{\alpha},\ k = 0, 1, \cdots\}$. Also the residue at $z = 0$ is given by*

$$\mathrm{Res}_{z=0}\eta(A)(z) = \frac{1}{\alpha}\mathrm{Wres}\,(F)\,. \tag{2.50}$$

Proof. The result follows easily from Theorem 2.4. □

It turns out that the eta function is in fact regular at $z = 0$ and the value $\eta(A)(0)$ should be considered as the infinite dimensional analogue of the signature of a self adjoint matrix. However, the proof of regularity of $\eta(A)(z)$ at $z = 0$ for a self adjoint pseudodifferential operator is much harder than regularity of zeta functions at zero. This was proved in [2] and [16]. The proof uses topological arguments to reduce the problem to the case of a twisted spin Dirac operator on an odd dimensional spin manifold and then uses invariant theory. Along a different line, one can give a different proof, using the result of Wodzicki on the Wodzicki residue of a pseudodifferential projection (see Remark 2.1.27 and [4]). First one obreves that for a self adjoint pseudodifferential operator $A \in \Psi^{\alpha}_{cl}(M^d)$,

$$\mathrm{Res}_{z=0}\eta(A)(z) = \frac{1}{\alpha}\mathrm{Wres}\,(F)\,,$$

where $F = \frac{A}{|A|}$. Now consider the operator $P = \frac{F+1}{2}$, it is a pseudodifferential projection (in fact it is the projection onto the $+1$ eigenspace of F). One has

$$F = 2P - 1 \tag{2.51}$$

and therefore by appealing to the result of Wodzicki, one has

$$\mathrm{Wres}(F) = 0. \tag{2.52}$$

2.1.5 Conformal invariants of elliptic operators

In this section we study those spectral quantities associated to geometric operators that remain invariant under the conformal changes of the metric. Let (M, g) be a closed Riemannian maifold and $E \to M$ be a complex vector bundle. By a geometric operator we mean a classical pseudodifferential operator $A_g \in \Psi_{cl}(M, E)$ acting on the smooth sections of E. Recall that a Riemannian metric $\tilde{g}$ is called *conformally equivalent* to the metric g if,

$$\tilde{g} = e^f g, \;\; f \in C^\infty(M). \tag{2.53}$$

Among the geometric operators we restrict our selves to the class of *conformally covariant* operators.

Definition 2.1.30. *An operator $A_g \in \Psi_{cl}(M, E)$ made from the Riemannian metric g is called conformally covariant, if there exists two numbers a and b such that,*

$$A_{\tilde{g}} = e^{-bf} A_g e^{af}, \tag{2.54}$$

where $\tilde{g} = e^{2f} g$.

In the following, we list some of the important examples of conformally covariant differential operators.

Example 2.1.31. Our first example of a conformally covariant operator is the *Dirac operator*. Consider the triple $(C^\infty(M), L^2(M, S)_g, D_g)$ encoding the data of a closed

n-dimensional spin Riemannian manifold with the spin Dirac operator on the space of spinors. By varying g within its conformal class, we consider $\tilde{g} = k^{-2}g$ for some $k = e^h > 0$ in $C^\infty(M)$. The volume form for the perturbed metric is given by $dvol_{\tilde{g}} = k^{-n}dvol_g$ and one has a unitary isomorphism

$$U : L^2(M, S)_g \longrightarrow L^2(M, S)_{\tilde{g}}$$

by

$$U(\psi) = k^{\frac{n}{2}}\psi,$$

It can be shown that (see [21]) $D_{\tilde{g}} = k^{\frac{n+1}{2}} D_g k^{\frac{-n+1}{2}}$ and hence

$$U^* D_{\tilde{g}} U = k^{\frac{-n}{2}} (k^{\frac{n+1}{2}} D_g k^{\frac{-n+1}{2}}) k^{\frac{n}{2}} = \sqrt{k} D_g \sqrt{k}.$$

Example 2.1.32. Let $D : L^2(M, S)_g \to L^2(M, S)_g$ be the Dirac operator on the space of spinors and $F_g = D|D|^{-1}$. By a conformal change of metric $\tilde{g} = k^{-2}g$ for some $k = e^h > 0$ in $C^\infty(M)$ and under the above unitary equivalence of Hilbert spaces one has (see [3]),

$$U^* \tilde{F} U = F \text{ mod } \mathcal{K},$$

where $\mathcal{K}$ is the ideal of compact operators on $L^2(M, S)_g$. Also consider another metric g', the corresponding Dirac operator $D' : L^2(M, S)_{g'} \to L^2(M, S)_{g'}$ and $F' = D'|D'|^{-1}$, then one can show that if

$$U^* \tilde{F} U = F' \text{ mod } \mathcal{K},$$

for an isomorphism U between the Hilbert spaces, then the metrics g and g' are conformally equivalent. This is the precise statement of the folklore fact that the $F = \frac{D}{|D|}$ encodes the conformal class of the Riemannian metric.

Example 2.1.33. The *Laplace-Beltrami opertor* on a closed surface defined by

$$\Delta_g = d^*d + dd^* : C^\infty(M) \to C^\infty(M)$$

is conformally covariant [8]. In higher dimension n, the *conformal Laplacian* defined by

$$L_g = \Delta_g + c_n R_g$$

is coformally covariant, where $c_n = \frac{n-2}{4(n-1)}$ and R_g is the scalar curvature.

In the rest of this section we study the following three spectral quantity associated to a conformally covariant operator A_g,

$$\zeta(A_g)(0), \quad \zeta'(A_g)(0), \quad \eta(A_g)(0)$$

and the way they change under the transformations

$$g \to e^{2f} g, \quad A_g \to e^{-bf} A_g e^{af}.$$

We follow the approach of [28] where the method of variations of the canonical trace for smooth families is used for studying the *conformal anomalies*. For a different approach, using the variations of heat kernels, we refer to [33].

First we introduce the notion of the *weighted trace.*

Definition 2.1.34. *Let $Q \in \Psi^q_{cl}(M)$ be an admissible operator of order $q > 0$ and $A \in \Psi^*_{cl}(M)$. The weighted trace of A is defined by*

$$\mathrm{tr}^Q(A) := \mathrm{f.p.}_{z=0}\mathrm{TR}\left(AQ^{-z}\right) + \mathrm{tr}\left(A\Pi_Q\right), \tag{2.55}$$

where $\mathrm{f.p.}_{z=0}\mathrm{TR}\left(AQ^{-z}\right)$ is the finite part (the constant term) in the Laurent expansion of $\mathrm{TR}(AQ^{-z})$ at $z = 0$ (see Theorem 2.4).

From the above definition it is clear that the three spectral quantities $\zeta(A)(0)$, $\zeta'(A)(0)$ and $\eta(A)$ (for A self adjoint) can be expressed as weighted traces,

$$\begin{aligned} \zeta(A)(0) &= \mathrm{tr}^A\left(I\right), \\ \zeta'(A)(0) &= \mathrm{tr}^A\left(\log A\right), \\ \eta(A)(0) &= \mathrm{tr}^{|A|}\left(A|A|^{-1}\right). \end{aligned}$$

Next, we need the following results on variation of the Wodzicki residue and the canonical trace of differentiable families of operators, we refer to [28] or [27] for the proof. A C^k-differentiable family of operators A_t satisfies the conditions of definition 2.1.13, with holomorphic replaced by C^k differentiable.

Proposition 2.1.35. *Consider a differentiable family $A_t \in \Psi^*_{cl}(M)$ of constant order α. Then,*

- *The Wodzicki residue commutes with the variation,*

$$\frac{d}{dt}\mathrm{Wres}(A_t) = \mathrm{Wres}(\dot{A}_t), \tag{2.56}$$

where $\dot{A}_t = \frac{d}{dt}A_t$.

- *If* α *is non-integer,*

$$\frac{d}{dt}\mathrm{TR}(A_t) = \mathrm{TR}(\dot{A}_t). \tag{2.57}$$

□

Also, consider a C^1 map $h : W \subset \mathbb{C} \to \mathbb{R}$ such that for any admissible operator $A \in \Psi^*_{cl}(M)$,one has $h(A), h'(A) \in \Psi^*_{cl}(M)$. We have the following proposition [28].

Proposition 2.1.36. *Let* A_t *be a differentiable family of admissible operators of constant non-integer order* α*. then,*

$$\frac{d}{dt}\mathrm{TR}\left(h(A_t)A_t^{-z}\right) = \mathrm{TR}\left(h'(A_t)\dot{A}_tA_t^{-z}\right) - z\mathrm{TR}\left(h(A_t)\dot{A}_tA_t^{-z-1}\right). \tag{2.58}$$

□

Corollary 2.1.37. *By equating the Laurent expansions on both sides of above equation we obtain,*

$$\frac{d}{dt}\mathrm{Wres}\left(h(A_t)\right) = \mathrm{Wres}\left(h'(A_t)\dot{A}_t\right), \tag{2.59}$$

$$\frac{d}{dt}\mathrm{tr}^{A_t}\left(h(A_t)\right) = \mathrm{tr}^{A_t}\left(h'(A_t)\dot{A}_t\right) - \frac{1}{\alpha}\mathrm{Wres}\left(h(A_t)\dot{A}_tA_t^{-1}\right), \tag{2.60}$$

$$\frac{d}{dt}\mathrm{tr}^{A_t}\left(h(A_t)\log^j A_t\right) = \mathrm{tr}^{A_t}\left(h'(A_t)\dot{A}_t\log^j A_t\right) + j\mathrm{tr}^{A_t}\left(h(A_t)\dot{A}_tA_t^{-1}\log^{j-1} A_t\right), \tag{2.61}$$

$j \in \mathbb{Z}^+$. □

Now consider a self adjoint conformally covariant operator A_g and let $A_t = A_{e^{tf}g}$ be the one-parameter family of operators obtained from the conformal perturbation of the metric, i.e. the one parameter family of the Riemannian metrics

$$g_t = e^{2tf}g,\ f \in C^\infty(M).$$

Remark 2.1.38. Note that since A_g is conformally covariant one has,

$$\ker\left(A_t\right) = \ker\left(A\right),$$

and $\dot{A}_t = \frac{d}{dt}\tilde{A}_t$, where the $\tilde{A}_g = A_g + \Pi_{A_g}$. Therefore in computing the variations of A_t one can replace A_g by $\tilde{A}_g$.

Definition 2.1.39. *Let* $\mathrm{Met}(M)$ *be the space of Riemannian metrics on* M *and* $\mathcal{F} : \mathrm{Met}(M) \to \mathbb{C}$ *be a Fréchet differentiable functional. The conformal anomaly of* $\mathcal{F}$ *at the reference metric* g *is defined as*

$$\delta_f \mathcal{F} := \frac{d}{dt}\, \mathcal{F}\left(e^{2tf} g\right)\Big|_{t=0}. \tag{2.62}$$

In the following, we restrict ourselves to self adjoint conformally covariant differential operators[1] and compute the conformal anomaly of the three functionals ,

$$\begin{aligned} g &\mapsto \zeta\left(A_g\right)(0), \\ g &\mapsto \zeta'\left(A_g\right)(0), \\ g &\mapsto \eta\left(A_g\right)(0). \end{aligned}$$

The following result computes the conformal anomaly of the above functionals [28].

Proposition 2.1.40. *Let* A_g *be a self adjoint conformally covariant differential operator of order* α*. One has the following conformal anomalies,*

$$\begin{aligned} \delta_f \zeta\left(A_g\right)(0) &= \delta_f \mathrm{tr}^{A_g}(I) = 0, \\ \delta_f \zeta'\left(A_g\right)(0) &= -\delta \mathrm{tr}^{A_g}\left(\log A_g\right) = (a-b)\mathrm{tr}^{A_g}(f), \\ \delta_f \mathrm{tr}^{A_g}\left(\frac{A_g}{|A_g|}\right) &= \frac{b-a}{\alpha}\mathrm{Wres}\left(f\frac{A_g}{|A_g|}\right). \end{aligned}$$

Proof. The proof is based on the result of Proposition 2.1.36 for the choices of $h = 1$, $h(\lambda) = \lambda$ and $h(\lambda) = \frac{\lambda}{\lambda}$ and the observation that for a conformally covariant operator A_g ,

$$\dot{A}_g = (a-b) f A_t - a[f, A_t].$$

□

[1] The results hold for the more general class of admissible operators [28]

Corollary 2.1.41. From the above result it follows that for a conformally covariant operator A_g , the spectral value $\zeta(A_g)(0)$ is conformal invariant. In dimension 2 and $A_g = \Delta_g$ this result is the spectral formulation of the Gauss-Bonnet theorem (see [17], [8]). Of course Gauss-Bonnet theorem proves that in dimension 2, $\zeta(\Delta_g)(0)$ is in fact a topological invariant.

Regarding the conformal anomaly of $\zeta'(A_g)(0)$ and $\eta(A_g)(0)$ we can say more by incorporating the asymptotic expansion of the heat kernel around zero of an admissible operator with non-negative leading symbol [17]. Let $\mathrm{ord}(A) = \alpha$, under these conditions the operator e^{-tA} is smoothing and has the following asymptotic expansion around zero,

$$\mathrm{Tr}\left(e^{-tA}\right) \sim \sum_{j=0}^{\infty} a_j(A) t^{\frac{j-n}{\alpha}} + \sum_{k=0}^{\infty} b_k(A) t^k \log t, \tag{2.63}$$

where $a_j(A)$ and $b_k(A)$ are given by integrals over M of local densities.

We need the two following results relating the coefficients in the asymptotic expansion of $\mathrm{Tr}(Ae^{-tQ})$ to the constant term in the Laurent expansion of $\mathrm{TR}\left(AQ^{-z}\right)$. We refer to [6] for a proof.

Lemma 2.1.42. *Let* $\mathrm{f.p.Tr}\left(Ae^{-tQ}\right)$ *be the constant term in the asymptotic expansion of* $\mathrm{Tr}\left(Ae^{-tQ}\right)$ *for the admissible operators* $A, Q \in \Psi^*_{cl}(m)$ *where* Q *has a non-negative leading symbol with* $\mathrm{ord}(Q) > 0$*. Then*

$$\mathrm{tr}^Q(A) = \mathrm{f.p.Tr}\left(Ae^{-tQ}\right) - \frac{\gamma}{\mathrm{ord}(Q)} \mathrm{Wres}(A), \tag{2.64}$$

where γ *is the Euler constant.* □

Again, consider an admissible operator A with non-negative leading symbol, one has the following relation between the coefficients of $\mathrm{Tr}(e^{-tA})$ in (2.63) and the Wodzicki residue of the powers A^k (see [28] for a proof).

Lemma 2.1.43. *Under the above conditions, one has*

$$\mathrm{Wres}\left(A^k\right) = \begin{cases} (-1)^{k+1} k! \alpha b_k(A) & k \in \mathbb{Z}^{\geqslant 0} \\ \frac{\alpha}{(-k-1)!} a_{n+\alpha k}(A) & k \in \mathbb{Z}^{-} \end{cases}$$

where $\alpha = \mathrm{ord}(A)$ *and* $a_{n+\alpha k} = 0$ *if* $\alpha k \notin \mathbb{Z}$*.* □

From the above lemmas the conformal invariance of $\zeta'(A_g)(0)$ for a conformally covariant differential operator A_g in odd dimensions follows.

Proposition 2.1.44. *Let A_g be a conformally covariant differential operator on an odd dimensional closed Riemannian manifold (M, g). Then,*

$$\delta_f \zeta'(A_g)(0) = (b-a)\int_M f(x)a_n(A_g)(x)\mathrm{dvol}_g. \tag{2.65}$$

Therefore $\zeta'(A_g)(0)$ is conformal invariant.

Proof. From Proposition 2.1.40 and Lemma 2.1.42 it follows that

$$\delta_f \zeta'(A_g)(0) = (a-b)\mathrm{tr}^{A_g}(f) = (a-b)\mathrm{f.p.Tr}\left(fe^{-tA_g}\right) = (a-b)\int_M f(x)\left(a_n(A_g)(x)\right)\mathrm{dvol}_g.$$

It is a known fact that for differential operators only the coefficients of the terms with even powers of t are nonzero (see [17]), therefore in odd dimensions,

$$\int_M f(x)a_n(A_g)(x)\mathrm{dvol}_g = 0.$$

□

And lastly, on has the following result on conformal invariance of $\eta(A_g)(0)$.

Proposition 2.1.45. *Let A_g be a conformally covariant differential operator of order α on an n-dimensional closed Riemannian manifold (M, g), the $\eta(A_g)(0)$ is conformal invariant if α and n have opposite parity.*

Proof. We consider the following asymptotic expansion,

$$\mathrm{Tr}\left(Ae^{-t|A|}\right) \sim \sum_{j=0}^{\infty} \tilde{a_j}(A)t^{\frac{j-n}{\alpha}} + \sum_{k=0}^{\infty} \tilde{b_k}(A)t^k \log t.$$

By using Lemma 2.1.43 we get,

$$\mathrm{Wres}\left(f\frac{A_g}{|A_g|}\right) = -(b-a)\int_M f(x)\tilde{a_n}(A_g, x)\mathrm{dvol}_g.$$

Again, careful analysis of the terms involved in above asymptotic expansion [33] shows that when α and n have opposite parity, the density $\tilde{a_n}(x)$ is identically zero and hence the result follows. □

Remark 2.1.46. The above result can be extended [33] and one can prove that for a conformally covariant differential operator A on a closed manifold M, the quantity $\eta(A)(0)$ is conformally covariant. We consider the case of an odd order conformally covariant operator A on an odd dimensional manifold M (the even case is similar). We choose an elliptic operator B of order zero with $\mathrm{index}(B) = -1$ on the circle and construct an operator $A\#B$ on $M \times S^1$. If A acts on the sections of a vector bundle E, $A\#B$ acts on the sections of $E \otimes \mathbb{R}$ and is given by

$$A\#B = \begin{bmatrix} A \otimes I & I \otimes B^* \\ I \otimes B & -A \otimes I \end{bmatrix}.$$

It can be shown that [17]

$$\eta(A\#B)(0) = \eta(A)(0).\left(\mathrm{index}(B)\right).$$

Now a slight modification of the proof of the Proposition 2.1.45 gives the conformal invariance of $\eta(A\#B)(0)$ and hence the conformal invariance of $\eta(A)(0)$ [33].

2.2 Framework of noncommutative geometry

In this section we review the basic tools and methods of noncommutative topology and noncommutative geometry. Our exposition closely follows [12].

2.2.1 Noncommutative topology

The duality between a space X and $\mathcal{F}(X)$, the algebra of functions on it is perhaps one of the oldest ideas in mathematics and has different incarnations in many areas. It is also considered as one of the cornerstones of Alain Connes' noncommutative geometry [9]. The first important theorem to point at is the following result of Gelfand and Naimark published in 1943 where the above duality is expressed in terms of equivalence of certain categories.

Theorem 2.5. *The category of unital commutative C^*-algebras is anti-equivalent to the category of compact Hausdorff topological spaces. The anti-equivalence is given by,*

$$X \longleftrightarrow C(X), \tag{2.66}$$

where $C(X)$ is the algebra of complex valued continuous functions on X. □

Remark 2.2.1. A C^*-algebra A is a *Banach algebra* equipped with an *involution* $* : A \longrightarrow A$ which is compatible with the norm in the following way,

$$\|aa^*\| = \|a\|^2.$$

The algebra of bounded operators on a Hilbert space $\mathcal{B}(\mathcal{H})$ is an example of a C^*-algebra. In fact another important theorem due to Gelfand and Naimark shows that any C^*-algebra can be embedded into some $\mathcal{B}(\mathcal{H})$. Also the correspondence in the above theorem can be extended to *locally compact Hausdorff* spaces as follows,

$$X \longleftrightarrow C_0(X),$$

where $C_0(X)$ is the algebra of complex valued continuous functions on X vanishing at infinity. Note that the algebra $C_0(X)$ does not have a unit unless X is compact.

The idea of a *noncommutative space* enters the picture when we remove the commutativity condition on C^*-algebras in Theorem 2.5. The category of C^*-algebras still makes sense and therefore a noncommutative topological space is merely a noncommutative C^*-algebra.

The goal of noncommutative topology is to look for further topological and geometrical structures that can be categorically formulated for algebraic structures . Among those, the results of R. G. Swan (1962) following the result of of J.-P. Serre (1957/58) gives a categorical equivalence between projective finitely generated $C(X)$-modules and vector bundles over a compact Hausdorff spaces X (see [22] for a detailed exposition). Therefore a *vector bundle* over a noncommutative space $\mathcal{A}$ is a finitely generated projective $\mathcal{A}$-module.

One can go further and ask for the analogues of differential forms, de Rham cohomology and characteristic classes for a noncommutative space. We should mention that the answer to this is part of the machinery of *Connes-Chern character* which is the extension of Chern character in the commutative case. Since these aspects are not related to the topic of this text, we skip these topics and refer the reader to [12] and [22].

2.2.2 Dirac operators and spectral triples

Beyond the noncommuattive topology, the program of noncommutative geometry in the spectral triple picture initiated by Alain Connes [9] is to extend the framework of Riemannian geometry to noncommutative spaces. The fundamental idea is that since the local properties of Riemann metric tensor are not accessible in noncommutative setting, one has to see the metric through the window of *spectral geometry* of elliptic operators, or more precisely the *Dirac operators*. Within this paradigm, those properties and invariants of Riemannian geometry which can be formulated in terms of spectral data of elliptic operators, stand a chance of being translated to noncommutaive language.

In this sense, a noncommutative geometry tries to encapsulate the geometric data of a Riemanninan spin geometry and generalize it to more general notions of spaces. In order to motivate the definition of a noncommutative Riemannian geometry (a spectral triple) we recall the construction of spinor bundles and Dirac operators over a $\mathrm{Spin}^{\mathbb{C}}$ manifold. We refer the reader to [25] and [12] for the details and proofs.

Let (M, g) be an n dimensional Riemannian manifold. Each fiber T_xM of the tangent bundle is equipped with the metric g_x and therefore one can assign to it the *real Clifford algebra* in the following way,

$$\mathrm{Cl}\,(T_xM) = \frac{\mathcal{T}(T_xM)}{(v \otimes v - g_x(v,v))},$$

where $\mathcal{T}(T_xM)$ is the tensor algebra over T_xM,

$$\mathcal{T}(T_xM) = \mathbb{C} \oplus T_xM \oplus (T_xM \otimes T_xM) \oplus \cdots (T_xM \otimes \cdots T_xM) \oplus \cdots .$$

These fibers patch together and form the *real Clifford bundle* $\mathrm{Cl}(TM)$ over the Riemannian manifold M. At each fiber, there exist a $\mathbb{Z}_2$-grading on $\mathrm{Cl}(T_xM, g_x)$ induced by the map $\chi : (x, v) \mapsto (x, -v)$. By patching the $+1$ and -1 eigenspace of this map we obtain the sub-bundles $\mathrm{Cl}^+(M)$ and $\mathrm{Cl}^-(M)$. Also note that there exist a natural isomorphism

$$\mathrm{Cl}(TM) \simeq \mathrm{Cl}(T^*M).$$

The fibers of the complexified Clifford bundle are defined as follows ,

$$\mathbb{C}l_x(M) = \mathrm{Cl}(T_xM) \otimes_{\mathbb{R}} \mathbb{C},$$

and one has the following characterization [25],

$$\mathbb{C}l_x(M) = \begin{cases} M_{2^m}(\mathbb{C}) & n = 2m \\ M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}) & n = 2m+1. \end{cases}$$

The smooth sections of the (complexified) Clifford bundle $C^\infty(\mathbb{C}l(M))$ form a $*$-algebra. In fact the set of continuous sections $C(\mathbb{C}l(M))$ is a C^*-algebra with the fiberwise product and the involution induced by $\otimes_{\mathbb{R}}\mathbb{C}$.

Now Consider the Clifford bundle $\mathbb{C}l(M)$ in even dimensions $n = 2m$ and the first matrix block part $\mathbb{C}l(M)^\dagger$ of the Clifford bundle $\mathbb{C}l(M)$ for $n = 2m+1$. For any $x \in M$ one can find a vector space S_x on which $\mathbb{C}l_x(M)$ acts linearly and irreducibly. Therefore, locally the Clifford bundle is isomorphic to $(U \times \mathrm{Hom}(S_x), U)$. One can equip S_x with a scalar product compatible with the C^*-algebra structure of $C(\mathbb{C}l_x(M)$. Also the spaces S_x is constant locally and $\dim(S_x) = 2^m$ where $m := [n/2]$.

We say that the tangent bundle TM admits a $\mathrm{Spin}^{\mathbb{C}}$ structure if one can glue together the local data $(U \times \mathrm{Hom}(S_x), U)$ and form a vector bundle over M.

Definition 2.2.2. *The Riemannian manifold (M, g) is $\mathrm{Spin}^{\mathbb{C}}$ if the tangent bundle TM admits a $\mathrm{Spin}^{\mathbb{C}}$ structure.*

Not every Riemannian manifold (M, g) admits a $\mathrm{Spin}^{\mathbb{C}}$ structure. In fact the exist a certain integral cohomology class of M associated to $\mathbb{C}l(M)$ in even dimensions and to $\mathbb{C}l^\dagger(M)$ in odd dimensions which is the obstruction for existence of $\mathrm{Spin}^{\mathbb{C}}$ structure. This cohomology class is called the *Dixmier-Douady class* (see [25] and references therein),

$$\delta(\mathbb{C}l(M)) \in H^3(M, \mathbb{Z}),\ n = 2m,$$

$$\delta(\mathbb{C}l^\dagger(M)) \in H^3(M, \mathbb{Z}),\ n = 2m+1.$$

One has the following proposition.

Proposition 2.2.3. *The Riemannian manifold (M, g) is $\mathrm{Spin}^{\mathbb{C}}$ if $\delta(\mathbb{C}l(M)) = 0$ in even dimensions and $\delta(\mathbb{C}l^\dagger(M)) = 0$ in odd dimensions.* □

Let (M, g) be a $\mathrm{Spin}^{\mathbb{C}}$ Riemannian manifold. The local data $(U \times \mathrm{Hom}(S_x), U)$ patch together and form the *Spinor bundle* over $S \longrightarrow M$ over which $\mathbb{C}l(M)$ or $\mathbb{C}^\dagger l(M)$ acts irreducibly. The space $C^\infty(S)$ of smooth sections of spinor bundle are called *spinors* or *chiral vector fields* in physics literature. Note that in even dimensions, the grading of the Clifford bundle induces a grading on spinor bundle,

$$S = S^{+} \oplus S^{-}.$$

One can always equip the space of spinors with an inner product and by completing it one gets the Hilbert space of L^2-spinors,

$$H = L^2(S).$$

In even dimensions, this Hilbert space is $\mathbb{Z}_2$ graded,

$$H = H^{+} \oplus H^{-}.$$

The smooth sections of the Clifford bundle act on H (since each fiber S_x is a representation space of $\mathbb{C}l(M)$ or $\mathbb{C}l^{\dagger}(M)$). Since

$$C^{\infty}(\mathbb{C}l(M)) = C^{\infty}(M) \oplus \Omega^1 M \oplus \cdots ,$$

it is seen that the smooth functions act on H by multiplication. The representation $\gamma : \mathbb{C}l(M) \to B(H)$ of the Clifford bundle on spinors is called the *spin representation*. Restricted to $1-$forms on M, one has the following identity,

$$\gamma(\alpha)\gamma(\beta) + \gamma(\beta)\gamma(\alpha) = 2g^{ij}\alpha_i\beta_j,$$

where α_i and β_j are the components of the $1-$forms α and β with respect to the orthonormal basis for T^*M.

Next we show that the spinor bundle over a $\mathrm{Spin}^{\mathbb{C}}$ Riemannian manifold (M, g) is equipped with a natural first order elliptic self adjoint differential operator called the *Dirac operator*. The Riemannian metric g on M gives rise to a unique connection called the *Levi-Civita connection*

$$\nabla^g : \Omega^1 M \to \Omega^1 M \otimes \Omega^1 M.$$

It can be extended to (contra-/covariant) C^{∞}-tensor fields over M and it is characterized by the properties of being symmetric and torsion free. The Levi-Civita connection lifts to the spinor bundle S and one obtains obtain the $\mathrm{Spin}^{\mathbb{C}}$ *connection*.

Definition 2.2.4. *A* $\mathrm{Spin}^{\mathbb{C}}$ *connection is a* $\mathbb{C}-$*linear*

$$\nabla^S : C^\infty(S) \to C^\infty(S) \otimes \Omega^1 M$$

and satisfies the two Leibniz rules

$$\nabla^S(\psi a) = \nabla^S(\psi)a + \psi \otimes da$$

and

$$\nabla^S(\gamma(\alpha)\psi) = \gamma(\nabla^g \alpha)\psi + \gamma(\alpha)\nabla^S(\psi),$$

where $a \in C^\infty(M)$, $\alpha \in \Omega^1 M$ *and* $\psi \in C^\infty(S)$.

It can be shown that there exists a $\mathrm{Spin}^{\mathbb{C}}$connection on the spinor bundle. Now, we are ready to define the *Dirac operator*.

Definition 2.2.5. *Let* $m : C^\infty(S) \otimes \Omega^1 M \to C^\infty(S)$ *be the Clifford multiplication defined by* $m(\psi \otimes \alpha) = \gamma(\alpha)\psi$ *for* $\alpha \in \Omega^1 M$ *and* $\psi \in C^\infty(S)$. *The Dirac operator* $\not{D} : C^\infty(S) \to C^\infty(S)$ *is defined by*

$$\not{D} := m \circ \nabla^S. \tag{2.67}$$

Below, we list the important properties if the Dirac operator.

- $\not{D} : C^\infty(S) \to C^\infty(S)$ is symmetric and extends to an unbounded self adjoint operator on $H = L^2(S)$.
- In even dimensions $n = 2m$ the Dirac operator is *odd* with respect to the grading of spinors $S = S^+ \oplus S^-$ and can be written as

$$\not{D} = \begin{bmatrix} 0 & \not{D}^+ \\ \not{D}^- & 0 \end{bmatrix}, \tag{2.68}$$

  where $\not{D}^+ : H^- \to H^+$ and $\not{D}^- : H^+ \to H^-$.
- $\not{D} : C^\infty(S) \to C^\infty(S)$ is a *first order, elliptic differential operator.*
- From the ellipticity of $\not{D}$ it follows that $\not{D}$ is a *Fredholm operator*, i.e. the $\ker(\not{D})$ is finite dimensional.

- The *parametrix*, $\not{D}^{-1}$ is compact. The eigenvalues λ_k of $\not{D}^{-1}$ counted with multiplicity satisfy the relation $\lambda_k \leqslant C.k^{-1/n}$ where C is a constant and $n = \dim(M)$. Therefore the spectral growth of the Dirac operator detects the dimension of the manifold.

- $[\not{D}, a]$ is bounded for any $a \in C^\infty(M)$. in fact

$$[\not{D}, a](\psi) = \not{D}(a\psi) - a\not{D}(\psi) = \gamma(da)\psi, \ a \in C^\infty(M), \ \psi \in C^\infty(S).$$

  Therefore $[\not{D}, a]$ is bounded by the sup-norm $\|\gamma(da)\|_\infty$.

- (Connes' distance formula) The Dirac operator encodes the *geodesic distance* between the points on the manifold. More precisely, for two points x, y on M we have

$$d(x, y) = \sup\{|f(x) - f(y)|, \ f \in C^\infty(M), \ \|[f, \not{D}]\| \leqslant 1\}. \tag{2.69}$$

The example of a compact Spin$^{\mathbb{C}}$ Riemannian manifold (M, g) gives the first example of a *spectral triple*. First we give the basic definition of a spectral triple [12].

Definition 2.2.6. *A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive unital (possibly noncommutative) algebra $\mathcal{A}$, a representation $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, and a self-adjoint densely defined operator $D : Dom(D) \subset \mathcal{H} \longrightarrow \mathcal{H}$ with compact resolvent and the property that $[D, \pi(a)]$ is bounded for any $a \in \mathcal{A}$.*

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called *even* if there exist a selfadjoint unitary operator $\Gamma : \mathcal{H} \to \mathcal{H}$, such that $a\Gamma = \Gamma a$, for $a \in \mathcal{A}$ and $D\Gamma = -\Gamma D$. The operator Γ induces a grading $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ with respect to which the Dirac operator is odd:

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}.$$

The following definitions abstract and generalize the fact that the spectral growth of the Dirac operator detects the dimension of the manifold.

Definition 2.2.7. *A spectral triple is finitely summable when the resolvent of D has characteristic values $\mu_n = O(n^{-\alpha})$, for some $\alpha > 0$ (see [18, Ch. 7] for details).*

Definition 2.2.8. *A finitely summable spectral triple is of metric dimension p if D^{-1} is of order $1/p$ (see [18, Ch. 7] for details).*

From the above list of properties of the Dirac operator it is seen that the triple $\big(C^\infty(M), L^2(S), \not{D}\big)$ encoding the data of a compact Riemmanian Spin$^{\mathbb{C}}$ manifold satisfies the requirements of a spectral triple. Note that in even dimensions the spectral triple $\big(C^\infty(M), L^2(S), \not{D}\big)$ is even.

Below, we explain the ingredients of a spectral triple in more detail.

- **The algebra.** We assume the the involutive algebra $\mathcal{A}$ is in fact a *pre-C^*-algebra*, i.e. $\mathcal{A}$ is a dense sub-algebra of a C^* algebra A and *stable under holomorphic functional calculus*. This means that for any $a \in \mathcal{A}$ and any holomorphic function defined on a neighborhood of spec(a),

$$f : \mathrm{spec}(a) \to \mathbb{C}$$

  the element $f(a)$ belongs to $\mathcal{A}$.

- **The Dirac operator.** The compact resolvent property of D means that $(D-\lambda)^{-1}$ is compact for $\lambda \notin \mathbb{R}$. Therefore D^{-1} defined over the orthogonal complement of $\ker(D)$ is compact and also $\ker(D)$ is finite dimensional. Also it follows that D has *discrete spectrum* and each eigenvalue has finite multiplicity. We see that this property generalizes the *ellipticity* of the Dirac operator on a compact Spin$^{\mathbb{C}}$ manifold.

- The boundedness of the commutators $[D, \pi(a)]$ is clearly the generalization of the fact that functions on M act on spinors as multiplication operators and

$$\|[\not{D}, a](\psi)\| \leqslant \|\gamma(da)\psi\|_\infty, \; a \in C^\infty(M), \; \psi \in C^\infty(S).$$

Below, we list a few examples of spectral triples.

Example 2.2.9. Finite spectral triples. A finite spectral triple is essentially a Riemannian geometry over a point. The triple $(\mathcal{A}, \mathcal{H}, D)$ is zero dimensional (Recall that the spectral growth of the Dirac operator detects the dimension) and both $\mathcal{A}$ and $\mathcal{H}$ are finite dimensional. The structure of the finite dimensional real involutive algebras which carry a faithful representation on a finite dimensional Hilbert space is known. They are of the form

$$\mathcal{A} = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{K})$$

where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The classification of finite spectral triples is done in [24].

Example 2.2.10. Almost commutative spectral triples. Almost commutative spectral triples are products of the commutative spectral triple $(C^\infty(M), L^2(S), \not{D})$ encoding the data of a compact Riemannian Spin$^{\mathbb{C}}$ manifold (M,g) and finite spectral triples. Recall that the product of the spectral triples $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ is defined by the triple $(\mathcal{A}, \mathcal{H}, D)$ where,

$$\begin{aligned}\mathcal{A} =& \mathcal{A}_1 \otimes_{\mathbb{C}} \mathcal{A}_2,\\ \mathcal{H} =& \mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2,\\ D =& D_1 \otimes I + I \otimes D_2.\end{aligned}$$

An example of this construction is the product of $(C^\infty(M), L^2(S), \not{D})$ with the finite spectral triple $(M_N(\mathbb{C}), \mathbb{C}^N, D = 0)$ wich plays a fundamental role in the approach of noncommutative geometry to standard model of elementary particles via *spectral action principle* (see [14]).

Example 2.2.11. Noncommutative tori. The noncommutative tori A_θ^n are perhaps the most popular platform for examining the methods of noncommutative geometry. The origin of these noncommutative algebras can be traced back to Heisenberg formulation of quantum mechanics. It was proposed by Hermann Weyl (see [34]) that the Heisenberg commutation relations should be replaced by their exponential form in order to obtain a bounded Hilbert space realization. The C^*-algebra generated by these exponential elements are in fact the noncommutative tori. From another point of view, the noncommutative tori can be thought as the *strict deformation quantization* of algebra of functions on tori ([32]) There are two main spectral triples on noncommutative tori, each reflecting a different aspect of their geometry. The *Spin spectral triples* on noncommutative tori (see [18]) are basically the deformation of the spin spectral triple on commutative tori. The other class of spectral triples on noncommutative tori are the so called *Dolbeault spectral triples* (see [15]) which are meant to reflect the complex geometry of tori.

Example 2.2.12. Isospectral deformations. The Connes-Landi [13] method of isospectral deformation, deforms the commutative spectral triple $(C^\infty(M), L^2(S), \not{D})$, where (M,g) is a compact Riemannian Spin manifold admitting an isometric action of a torus $\mathbb{T}^l$ for $l \geqslant 2$. The outcome is a $\theta-$deformed spectral triple $(C^\infty(M_\theta), \mathcal{H} = L^2(S), \not{D})$ with the same Hilbert space and the Dirac operator. Note that the deformed algebra $C^\infty(M_\theta)$ still has a compatible representation on the Hilbert space $\mathcal{H}$. The noncommutative tori fit into this picture as well. Another example of this construction is the spectral triple $(C^\infty(\mathbb{S}^4_\theta), L^2(S), \not{D})$ which is the isospectral deformation of the Riemannian geometry of the the round sphere $\mathbb{S}^4$.

At the end we say a few words about the characterization of commutative spaces by the spectral triples. It seems natural to ask if a general spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with the a commutative unital algebra $\mathcal{A}$ corresponds to a Spin$^{\mathbb{C}}$ (or spin) geometry $(C^\infty(M), L^2(S), \not{D})$ for a compact Riemannian Spin manifold (M, g). This is in fact the content of Connes' *reconstruction theorem* [10], [11]. First it turns out that in order to speak of a *noncommutative manifold*, the notion of spectral triple is not enough and one has to add extra properties and axioms (see [18]) to data of the spectral triple. Here we avoid the technicalities of the axioms and only give a list of them without going in depth.

- **Axiom 1, Dimension.** There is an integer p, the metric dimension of the spectral triple, such that $|D|^{-1}$ is of order $1/p$ (see Definition 2.2.8).

- **Axiom 2, Reality.** There exists an anti-unitary operator $J : \mathcal{H} \to \mathcal{H}$ such that $J(DomD) \subset DomD$, and $[a, Jb^*J^{-1}] = 0$ for all $a, b \in \mathcal{A}$. We say $(\mathcal{A}, \mathcal{H}, D)$ is of KO-dimension n if the operator J satisfies the following commutation relations

$$J^2 = \epsilon 1, \;\; JD = \epsilon' DJ, \;\; J\Gamma = \epsilon'' \Gamma J,$$

where the $\epsilon, \epsilon', \epsilon''$ depend on $n \in \mathbb{Z}_8$ according to the following table:

n	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

- **Axiom 3, First order condition.** For $a, b \in \mathcal{A}$, one has the following commutation relation:
$$[[D, a], Jb^*J^*] = 0. \tag{2.70}$$

- **Axiom 4, Orientability.** There is a Hochschild cycle $c \in Z_n(A, A \otimes A^\circ)$ whose representation on $\mathcal{H}$ is given by

$$\pi_D(c) = \begin{cases} \Gamma & p = 2k \\ 1 & p = 2k+1 \end{cases}$$

,where Γ is the grading operator for the spectral triple.

- **Axiom 5, Regularity.** For any $a \in \mathcal{A}$, $[D, a]$ is a bounded operator on $\mathcal{H}$, and both a and $[D, a]$ belong to the domain of smoothness $\bigcap_{k=1}^{\infty} Dom(\delta^k)$, where the derivation δ on $\mathcal{B}(\mathcal{H})$ given by $\delta(T) := [|D|, T]$.

- **Axiom 6, Finiteness** The space of smooth vectors $\mathcal{H}^{\infty} := \bigcap_{k=1}^{\infty} Dom(D^k)$ is a finitely generated projective left A-module with a Hermitian pairing $(.,.)$ implicitly given by
$$⨍ (\xi, \eta)|D|^{-p} = \langle \xi, \eta \rangle \,, \tag{2.71}$$
where for an element a in the algebra generated by $\mathcal{A}$, $[D; \mathcal{A}]$ and $|D|^{-z}$, $z \in \mathbb{C}$,
$$⨍ a := \mathrm{Res}_{z=0} \mathrm{TR}(a|D|^{-z}), \tag{2.72}$$
and $\langle .,. \rangle$ is the inner product of the Hilbert space $\mathcal{H}$.

- **Axiom 7, Poincaré duality.** The Fredholm index of the operator D yields a nondegenerate intersection form on the K-theory ring of the algebra $\mathcal{A} \otimes \mathcal{A}^{\circ}$.

Definition 2.2.13. *A noncommutative manifold is a real spectral triple* $(\mathcal{A}, \mathcal{H}, D, \Gamma, J)$ *or* $(\mathcal{A}, \mathcal{H}, D, J)$*, according as its dimension is even or odd, that satisfies the above seven axioms.*

We finish this chapter by giving a weaker version a the reconstruction theorem (see [7] for a proof and [11] for the stronger form of the theorem).

Theorem 2.6. *Let* $(\mathcal{A}, H, D)$ *be a noncommutative manifold with the unital commutative algebra* $\mathcal{A}$ *and spectral dimension* p*. There exists a compact oriented Riemannian manifold* M *of dimension* p*, a Hermitian vector bundle* E *on* M*, and an essentially self-adjoint Dirac-type operator* D_E [2] *such that*
$$(\mathcal{A}, H, D) \simeq (C^{\infty}(M), L^2(M, E), D_E), \tag{2.73}$$
where $\simeq$ *denotes a unitary equivalence of spectral triples.*

□

[2] i.e., a first-order differential operator such that the principal symbol of D_E^2 is given by $\sigma^L(D_E^2)(x, \xi) = g^{-1}(\xi, \xi).I$

Bibliography

[1] M.F. Atiyah, V.K. Patodi, I.M. Singer. Spectral asymmetry and Riemannian geometry.I. *Math. Proc. Cambridge Philos. Soc.*, Volume77:43–69, 1975.

[2] M.F. Atiyah, V.K. Patodi, I.M. Singer. Spectral asymmetry and Riemannian geometry.III. *Math. Proc. Cambridge Philos. Soc.*, Volume79:71–99, 1976.

[3] C. Bär. Conformal structures in noncommutative geometry. *J. Noncommut. Geom.* , 1 (2007), 385-395. .

[4] J. Brünning, M. Lesch. On the eta-invariant of certain nonlocal boundary value problems. *Duke Math. J.*, 96(2), 425?-468 (1999).

[5] A.A. Bytsenko, G. Cognola, E. Elizalde, Moretti. Analytic aspects of quantum fields. *World Scientific Publishing Co., Inc., River Edge, NJ*, 2003.

[6] A. Cardona, C. Ducourtioux, S. Paycha. From Tracial Anomalies to Anomalies in Quantum Field Theory. *Comm. Math. Phys.*, 242: 31–65, 2003.

[7] B. Ćaćić. A Reconstruction Theorem for Almost-Commutative Spectral Triples. *Letters in Mathematical Physics*, 100: 181–202, 2012.

[8] S-Y. A. Chang. Conformal invariants and partial differential equations. *Bull. AMS*, 42, 365–394 (2005).

[9] A. Connes. C^* algèbres et géométrie différentielle. *C. R. Acad. Sci. Paris Sér. A-B*, 290(13):A599–A604, 1980.

[10] A. Connes. Geometry from the spectral point of view. In *Lett.Math.Phys.*,34,203-238, 1995.

[11] A. Connes. On the spectral characterization of manifolds. In *J.Noncommut.Geom.*, 7, 1-82, 2013.

[12] A. Connes. *Noncommutative geometry.* Academic Press, Inc., San Diego, CA, 1994.

[13] A. Connes, G. Landi. Noncommutative manifolds, the instanton algebra and isospectral deformations. *Comm. Math. Phys.*,221(2001), no.1, 141-159.

[14] A. Connes, M. Marcolli. *Noncommutative Geometry, Quantum Fields and Motives.* American Mathematical Soc., 2007.

[15] A. Connes, P. Tretkoff. The Gauss-Bonnet theorem for the noncommutative two torus. In *Noncommutative geometry, arithmetic, and related topics*, pages 141–158. Johns Hopkins Univ. Press, Baltimore, MD, 2011.

[16] P. B. Gilkey. The residue of the local eta function at the origin. *Math. Ann.*, 240(2):183–189, 1979.

[17] P. B. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, volume 11 of *Mathematics Lecture Series.* Publish or Perish Inc., Wilmington, DE, 1984.

[18] J. M. Gracia-Bondia, J. C. Varilly, H. Figueroa. *Elements of Noncommutative Geometry.* Birkhäuser, 2001.

[19] V. Guillemin. Gauged Lagrangian distributions. *Adv. Math.*, 102;2(1993):184–201.

[20] V. Guillemin. Residue traces for certain algebras of Fourier integral operators. *J. Funct. Anal.*, 115;2(1993), 391–417.

[21] N. Hitchin. Harmonic Spinors. *Advances in Mathematics*, 14(1):1 - 55, 1974.

[22] M. Khalkhali. *Basic Noncommutative geometry.* European Mathematical Society Publishing House, 2013.

[23] M. Kontsevich and S. Vishik. Geometry of determinants of elliptic operators. In *Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993)*, volume 131 of *Progr. Math.*, pages 173–197. Birkhäuser Boston, Boston, MA, 1995.

[24] T. Krajewski. Classification of finite spectral triples. *Journal of Geometry and Physics*, 28:1-30, 1998.

[25] H. B. Lawson and M. L. Michelsohn. *Spin Geometry.* Princeton University Press, 1990.

[26] M. Lesch. On the noncommutative residue for pseudodifferential operators with log-polyhomogeneous symbols. *Ann. Global Anal. Geom.*, 17(2):151–187, 1999.

[27] S. Paycha. *Regularised integrals, sums and traces*, volume 59 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2012.

[28] S. Paycha and S. Rosenberg. Conformal anomalies via canonical traces. In *Analysis, geometry and topology of elliptic operators*, pages 263–294. World Sci. Publ., Hackensack, NJ, 2006.

[29] S. Paycha and S. Scott. A Laurent expansion for regularized integrals of holomorphic symbols. *Geom. Funct. Anal.*, 17(2):491–536, 2007.

[30] R. Ponge. Spectral asymmetry, zeta functions, and the noncommutative residue. *Internat. J. Math*, 17(9): 1065–1090, 2006.

[31] R. Ponge. Traces on pseudodifferential operators and sums of commutators. *Journal d'Analyse Mathmatique*, 110(1):1–30, 2010.

[32] M. Rieffel. Non-Commutative Tori- A Case Study of Non-Commutative Differentiable Monifolds. *Contemporary Mathematics*, 105: 191–211, 1990.

[33] S. Rosenberg. The Determinant of a Conformally Covariant Operator. *Journal of the London Mathematical Society*,Volumes 2-36(3):553-568, 1987.

[34] H. Weyl. *The Theory of Groups and Quantum Mechanics.* Dover Books on Mathematics, 1950.

[35] M. Wodzicki. Spectral asymmetry and noncommutative residue (in Russian) Thesis, Steklov Institute, (former) Soviet Academy of Sciences, Moscow, 1984.

[36] M. Wodzicki. Local invariants of spectral asymmetry. Invent. Math., 75 (1984) 143-177 .

[37] M. Wodzicki. Noncommutative residue. I. Fundamentals. K-theory, Arithmetic and Geometry (Moscow, 1984–1986), Lecture Notes in Math., 1289, Springer, Berlin, 1987, p. 320-399.

Chapter 3

On Certain Spectral Invariants of Dirac Operators on Noncommutative Tori

3.1 Introduction

In this paper we study the variations of spectral zeta and eta functions $\zeta_{|D|}(z) = \mathrm{TR}\,(|D|^{-z})$ and $\eta_D(z) = \mathrm{TR}(D|D|^{-z-1})$ associated to certain families of Dirac operators on noncommutative 3-torus. In the classical case the canonical trace TR [30] provides a unified method of studying various spectral functions of elliptic operators and their variations. Connes' pseudodifferential calculus for noncommutative tori makes it possible to define a suitable notion of noncommutative canonical trace [19], and translate some of the properties of the canonical trace on manifolds to noncommutative settings. Among these, the fundamental result is the explicit description of the Laurent expansion at zero of the function $\mathrm{TR}(AQ^{-z})$ where A and Q are classical elliptic operators [36]. This result enables us to prove the regularity of $\zeta_{|D|}(z)$ and $\eta_D(z)$ at $z = 0$, and also gives a local description for variations of $\eta_D(0)$ and $\zeta'_{|D|}(0)$. In particular, we show that $\eta_D(0)$ is constant over the family $\{e^{th}De^{th}\}$ and hence is a conformal invariant of noncommutative 3-torus. Also, using the local description for conformal variation of $\zeta'_{|D|}(0)$, we prove that this quantity is a conformal invariant of noncommutative 3-torus. This paper is organized as follows. In Section 2 we recall the definition of a spectral triple which is the basic ingredient in the definition of a noncommutative Riemannian space [12]. Our main example is the spin spectral triple for noncommutative tori and its conformal perturbation first proposed in [10, 16]. In Section 3 we give a brief review of Connes' pseudodifferential calculus for noncommutative tori from [11, 16], and recall the extension of the Kontsevich-Vishik canonical trace to the setting of noncommutative tori from [19]. It should be mentioned that this is also done in [33] where one works with toroidal symbols instead of Connes' symbols.

In Section 4 we study the eta function associated to the Dirac operators of the conformally perturbed spectral triples $(\mathrm{C}^\infty(\mathbb{T}^3_\theta), \mathcal{H}, e^{th}De^{th})$, and also to the coupled Dirac operator of the spectral triple $(\mathrm{C}^\infty(\mathbb{T}^3_\theta), \mathcal{H}, D + A)$. By exploiting the developed canonical trace, the regularity of the eta function at zero in above cases will be proved. Next, by using variational techniques we show that the value of the eta function at zero is constant over a conformally perturbed family. Also, by considering the spectral triple $(\mathrm{C}^\infty(\mathbb{T}^3_\theta), \mathcal{H}, D)$ and the family $D_t = D + tu^*[D, u]$ for a unitary element $u \in \mathrm{C}^\infty(\mathbb{T}^3_\theta)$, we relate the difference $\eta_{D_1}(0) - \eta_{D_0}(0)$ to the spectral flow of the family D_t and give a local formula for index of the operator PuP. This is the analogue of the result of Getzler [25], in the case of noncommutative 3-torus.

In Section 5 we consider the spectral zeta function $\zeta_{|D|}(z) = \mathrm{TR}(|D|^{-z})$ and study the conformal variation of the spectral value $\zeta'_{|D|}(0)$ within the framework of the canonical trace. We show that for the noncommutative 3-torus this quantity is a conformal invariant. In even dimensions though, the conformal variation is not zero and hence conformal anomaly exists. Following [15] we give a local formula for the conformal variation of $\zeta'_\Delta(0)$ in the case of noncommutative two torus.

Finally, in section 6 we consider the coupled Dirac operator $D + A$ and study the value $\zeta'_D(0)$ where $\zeta_D(z) = \mathrm{TR}(D^{-z})$. Since the spectrum of D is extended along the real line, there is an ambiguity in the definition of the complex power D^{-z} and hence in the value $\zeta'_D(0)$. In odd dimensions, this ambiguity can be expressed in terms of $\eta_{D+A}(0)$ and hence has a dependence on the coupled gauge field A. This dependence can be computed by a local formula and in physics literature it is usually referred to as the induced Chern-Simons term generated by the coupling of a massless fermion to a classical gauge field (cf. e.g. [6]). We give an analogue of this computation and the local formula in the case of noncommutative 3- torus.

Conformal and complex geometry of noncommutative two tori were first studied in the seminal work of Connes and Tretkoff [16] where a Gauss-Bonnet theorem was proved for a conformally perturbed metric (cf. [10] for a preliminary version). This result was extended in [20] where the Gauss-Bonnet theorem was proved for metrics in all translation invariant conformal structures. The problem of computing the scalar curvature of the curved noncommutative two torus was fully settled in [6], and in [21], and in [22] in the four dimensional case. Other related works on curved tori include [17, 18, 31, 40]. The computation of the curvature of the determinant line bundle in the sense of Quillen for certain families of Dirac operators on noncommutative tori was carried out in [19].

AF would like to thank Asghar Ghorbanpour for useful discussions on the subject of this paper. MK would like to thank the Hausdorff Institute in Bonn for its hospitality and support while this work was being completed.

3.2 Noncommutative geometry framework

In this section we recall the basic ingredients in the definition of a noncommutative geometry. Our main example is the spin spectral triple for noncommutative tori on which all the material in this paper is based. The data of a noncommutative Riemannian geometry is encoded in a spectral triple [11].

Definition 3.2.1. *A spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is given by an involutive unital (possibly noncommutative) algebra* $\mathcal{A}$*, a representation* $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ *on a Hilbert space* $\mathcal{H}$*, and a self-adjoint densely defined operator* $D : Dom(D) \subset \mathcal{H} \longrightarrow \mathcal{H}$ *with compact resolvent and the property that* $[D, \pi(a)]$ *is bounded for any* $a \in \mathcal{A}$*.*

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called *even* if there exist a self-adjoint unitary operator $\Gamma : \mathcal{H} \to \mathcal{H}$, such that $a\Gamma = \Gamma a$, for $a \in \mathcal{A}$ and $D\Gamma = -\Gamma D$. The operator Γ induces a grading $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ with respect to which the Dirac operator is odd:

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}.$$

It can be shown that the triple $(C^\infty(M), L^2(M, S), \not{D})$, consisting of $\mathcal{A} = C^\infty(M)$ the algebra of smooth functions on a closed Riemannian spin manifold (M, g), and the spin Dirac operator $\not{D}$ on the Hilbert space $\mathcal{H} = L^2(M, S)$ of L^2-spinors satisfies the requirements of a spectral triple (see [28]). In even dimensions, the spinor bundle admits a grading and we have $S = S^+ \oplus S^-$ with respect to which the Dirac operator is odd. Therefore in even dimensions the spectral triple $(C^\infty(M), L^2(M,S), D\!\!\!/)$ is even.

3.2.1 Geometry of noncommutative tori

Let $\Theta \in M_n(\mathbb{R})$ be a skew symmetric matrix. The noncommutative n-torus $C(\mathbb{T}^n_\Theta)$ is defined to be the universal C^*-algebra generated by the unitaries U_k for $k \in \mathbb{Z}^n$ and relations,

$$U_k U_l = e^{\pi i \Theta(k,l)} U_{k+l}, \quad k, l \in \mathbb{Z}^n.$$

Consider the standard basis $\{e_i\}$ for $\mathbb{R}^n$ and let $u_i = U_{e_i}$. Then it follows that

$$u_k u_l = e^{2\pi i \theta_{kl}} u_l u_k,$$

where $\theta_{kl} = \Theta(e_k, e_l)$. The smooth noncommutative n-torus $C^\infty(\mathbb{T}^n_\Theta)$ is defined to be the Fréchet $*-$subalgebra of elements with Schwartz coefficients in the Fourier expansion, that is all the $a \in C^\infty(\mathbb{T}^n_\theta)$ that can be written as

$$a = \sum_{k \in \mathbb{Z}^n} a_k U_k,$$

where $\{a_k\} \in \mathcal{S}(\mathbb{Z}^n)$. In fact, $C^\infty(\mathbb{T}^n_\theta)$ is a deformation of $C^\infty(\mathbb{T}^n)$ and consists of the smooth vectors under the periodic action of $\mathbb{R}^n$ on $C(\mathbb{T}^n_\Theta)$ given by

$$\alpha_s(U_k) = e^{is.k} U_k, \quad s \in \mathbb{R}^n, k \in \mathbb{Z}^n.$$

The algebra $C^\infty(\mathbb{T}^n_\theta)$ is equipped with a tracial state given by

$$\tau\Big(\sum_{p \in \mathbb{Z}^n} a_p U_p\Big) = a_0.$$

We also denote by δ_μ the analogues of the partial derivatives $\frac{1}{i}\frac{\partial}{\partial x^\mu}$ on $C^\infty(\mathbb{T}^n)$ which are derivations on the algebra $C^\infty(\mathbb{T}^n_\theta)$ defined by

$$\delta_\mu(U_k) = k_\mu U_k.$$

These derivations have the following property

$$\delta_\mu(a^*) = -(\delta_\mu a)^*,$$

and also satisfy the integration by parts formula

$$\tau(a\delta_\mu b) = -\tau((\delta_\mu a)b), \quad a, b \in C^\infty(\mathbb{T}^n_\theta).$$

By GNS construction one gets the Hilbert space $\mathcal{H}_\tau$ on which $C^\infty(\mathbb{T}^n_\theta)$ is represented by left multiplication denoted by $\pi(a)$ for $a \in C^\infty(\mathbb{T}^n_\theta)$. The spectral triple describing the noncommutative geometry of noncommutative n-torus consists of the algebra $C^\infty(\mathbb{T}^n_\theta)$, the Hilbert space $\mathcal{H} = \mathcal{H}_\tau \otimes \mathbb{C}^N$, where $N = 2^{[n/2]}$ with the inner product on $\mathcal{H}_\tau$ given by

$$\langle a, b \rangle_\tau = \tau(b^* a),$$

and the representation of $C^\infty(\mathbb{T}^n_\theta)$ given by $\pi \otimes 1$.

The Dirac operator is

$$D = \partial\!\!\!/ = \partial_\mu \otimes \gamma^\mu,$$

where $\partial_\mu = \delta_\mu$, is seen as an unbounded self-adjoint operator on $\mathcal{H}_\tau$ and γ^μs are Clifford (Gamma) matrices in $M_N(\mathbb{C})$ satisfying the relation

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta^{ij} I_N.$$

In 3−dimension the Clifford matrices are given by the Pauli spin matrices,

$$\gamma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \gamma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \gamma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Consider the chirality matrix

$$\gamma = (-i)^m \gamma^1 \cdots \gamma^n,$$

where $n = 2m$ or $2m+1$. It is seen that $\gamma \cdot \gamma = 1$, and γ anti-commutes with every γ^μ for n even. The operator $\Gamma = 1 \otimes \gamma$ on $\mathcal{H} = \mathcal{H}_\tau \otimes \mathbb{C}^{2^{[n/2]}}$ defines a grading on $\mathcal{H}$ and for even n one has

$$\Gamma D = -D\Gamma.$$

Therefore $(C^\infty(\mathbb{T}^n_\theta), \mathcal{H}, D)$ is an even spectral triple for n even.

3.2.2 Real structure

Definition 3.2.2. A real structure on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is an anti-unitary operator $J : \mathcal{H} \to \mathcal{H}$ such that $J(DomD) \subset DomD$, and $[a, Jb^*J^{-1}] = 0$ for all $a, b \in \mathcal{A}$. We say $(\mathcal{A}, \mathcal{H}, D)$ is of KO-dimension n if the operator J satisfies the following commutation relations

$$J^2 = \epsilon 1, \;\; JD = \epsilon' DJ, \;\; J\Gamma = \epsilon'' \Gamma J,$$

where the $\epsilon, \epsilon', \epsilon''$ depend on $n \in \mathbb{Z}_8$ according to the following table:

n	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

Note that the real structure operator switches left action to right action. More precisely, Since $\mathcal{A}$ commutes with $J\mathcal{A}J^*$ the real structure gives a representation of the opposite algebra $\mathcal{A}^{op}$ by $b^\circ = Jb^*J^*$ and turns the Hilbert space $\mathcal{H}$ into an $\mathcal{A}-$bimodule by

$$a\psi b = aJb^*J^*(\psi) \;\;\; a, b \in \mathcal{A}, \;\; \psi \in \mathcal{H}.$$

Example 3.2.3. The spectral triples $(C^\infty(\mathbb{T}^n_\theta), \mathcal{H} = H_\tau \otimes \mathbb{C}^N, D)$ for $N = 2^{[n/2]}$ are all equipped with real structure. The real structure operator $J : H_\tau \otimes \mathbb{C}^N \to H_\tau \otimes \mathbb{C}^N$ is given by

$$J = J_0 \otimes C_0,$$

where J_0 is the Tomita conjugation map associated to the GNS Hilbert space H_τ given by

$$J_0(a) = a^*, \;\; a \in C^\infty(\mathbb{T}^n_\theta)$$

and C_0 is the charge conjugation operator on $\mathbb{C}^N$ (cf. e.g. [28]). For $N = 2$ we have

$$J = \begin{bmatrix} 0 & -J_0 \\ J_0 & 0 \end{bmatrix}.$$

$J_0^2 = 1$ for all n-mod 8, so by extending the Hilbert space H_τ and coupling J_0 with C_0 one can check that the resulting operator J satisfies the requirements for a real structure (see [28]).

3.2.3 The coupled Dirac operator

In this section We consider the spectral triples obtained by coupling (twisting) the Dirac operator by a gauge potential. We begin by recalling the construction in commutative case.

Consider a compact Riemannian spin manifold (M, g) with the spin Dirac operator $D : L^2(S) \to L^2(S)$ on spinors. Let $V \to M$ be a Hermitian vector bundle equipped with a compatible connection ∇^V. Then one constructs the coupled (twisted) spinor bundle $S \otimes V \to M$ with the extended Clifford action

$$\pi(\omega).(\psi \otimes f) = (\omega.\psi) \otimes f,$$

where $\pi(\omega)$ is a section of the Clifford bundle over M represented on the space of spinors and ψ and f are sections of spinor bundle and the vector bundle V respectively. Also, the vector bundle $S \otimes V$ is equipped with the coupled (twisted) spin connection

$$\nabla^{S\otimes V} = \nabla^S \otimes 1 + 1 \otimes \nabla^V,$$

where ∇^S is the spin connection on S. This connection gives rise to the coupled Dirac operator D_{∇^V} acting on the sections of $S \otimes V$. When V is a trivial vector bundle, the connection ∇^V can be globally written as $\nabla^V = d + A$ where A is a matrix of one forms (vector potential) and an easy computation shows that

$$D_A = D + \pi(A).$$

The above construction can be generalized to the setting of spectral triples. Starting with a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, one can construct a new spectral triple $(\mathcal{A}, \mathcal{H}, D + A)$ by adding a gauge potential to the Dirac operator, this corresponds to picking a trivial projective module over the algebra $\mathcal{A}$.

More precisely, A is a self adjoint element of

$$\Omega^1_D = \left\{ \sum_j a^0_j [D, a^1_j],\ a^i_j \in \mathcal{A} \right\}.$$

In fact, Ω^1_D is the image inside $\mathcal{B}(\mathcal{H})$ of the noncommutative 1-forms on $\mathcal{A}$ under the induced map

$$\pi : a^0 da^1 \longrightarrow a^0[D, a^1].$$

Below, we explicitly write down the coupled Dirac operator for the spectral triple $(C^\infty(\mathbb{T}^n_\theta), \mathcal{H}, D + A)$. First note that for any element $a = \sum_{k\in\mathbb{Z}^n} a_k U_k$ in $C^\infty(\mathbb{T}^n_\theta)$ we have,

$$[D, a] = \partial_\mu(a) \otimes \gamma^\mu.$$

Any $A \in \Omega^1(C^\infty(\mathbb{T}^n_\theta))$ is of the form $A = \sum_i a_i db_i$ where a_i, b_i are in $C^\infty(\mathbb{T}^n_\theta)$ and we have

$$\pi(A) = \sum_i a_i \partial_\mu(b_i) \otimes \gamma^\mu.$$

We denote the elements $\sum_i a_i \partial_\mu b_i$ by A_μ and hence,

$$\pi(A) = A_\mu \otimes \gamma^\mu,$$

also self adjointness of A gives $A^*_\mu = A_\mu$. Therefore, the coupled Dirac operator is given by

$$D + A = \partial\!\!\!/ + A\!\!\!/,$$

where again by Feynman slash notation $A\!\!\!/ = A_\mu \otimes \gamma^\mu$.

3.3 Elliptic theory on noncommutative tori

Our aim in this section is to recall the extension of the Kontsevich-Vishik canonical trace to the setting of noncommutative tori from [19]. Alternatively, this is also done in [33] where they work with toroidal symbols instead of Connes' symbols. We begin by a brief review of the basics of Connes' pseudodifferential calculus for noncommutative tori from [11, 16].

3.3.1 Matrix pseudodifferential calculus on $C^\infty(\mathbb{T}^n_\theta)$

We shall use the multi-index notation $\alpha = (\alpha_1, .., \alpha_n)$, $\alpha_i \geqslant 0$, $|\alpha| = \alpha_1 + ... + \alpha_n$, $\alpha! = \alpha_1!...\alpha_n!$, $\delta^\alpha = \delta_1^{\alpha_1}...\delta_n^{\alpha_n}$ and $\partial^\beta_\xi = \partial^{\beta_1}_{\xi_1}...\partial^{\beta_n}_{\xi_n}$.

Definition 3.3.1. *A matrix valued symbol of order m on noncommutative $n-$torus is a smooth map*

$$\sigma : \mathbb{R}^n \longrightarrow C^\infty(\mathbb{T}^n_\theta) \otimes M_N(\mathbb{C}),$$

such that

$$||\delta^\alpha \partial_\xi^\beta \sigma(\xi)|| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|},$$

and there exists a smooth map $k : \mathbb{R}^n \setminus \{0\} \longrightarrow C^\infty(\mathbb{T}^n_\theta) \otimes M_N(\mathbb{C})$ such that

$$\lim_{\lambda\to\infty} \lambda^{-m}\sigma(\lambda\xi_1, \lambda\xi_2, ..., \lambda\xi_n) = k(\xi_1, \xi_2, ..., \xi_n).$$

We denote the symbols of order m by $\mathcal{S}^m(C^\infty(\mathbb{T}^n_\theta))$.

A matrix pseudodifferential operator associated with $\sigma \in \mathcal{S}^m(C^\infty(\mathbb{T}^n_\theta))$ is the operator $A_\sigma : C^\infty(\mathbb{T}^n_\theta) \otimes \mathbb{C}^N \longrightarrow C^\infty(\mathbb{T}^n_\theta) \otimes \mathbb{C}^N$ defined by

$$A_\sigma(a) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is.\xi}\sigma(\xi)\alpha_s(a)dsd\xi,$$

where α_s is the extended action of $\mathbb{R}^n$ on $C^\infty(\mathbb{T}^n_\theta) \otimes \mathbb{C}^N$.

Two symbols σ, $\sigma' \in \mathcal{S}^m(C^\infty(\mathbb{T}^n_\theta))$ are considered equivalent if $\sigma - \sigma' \in \mathcal{S}^m(C^\infty(\mathbb{T}^n_\theta))$ for all m. The equivalence of the symbols will be denoted by $\sigma \sim \sigma'$. We denote the collection of pseudodifferential operators by $\Psi^*(C^\infty(\mathbb{T}^n_\theta))$. The order gives a natural filtration on $\Psi^*(C^\infty(\mathbb{T}^n_\theta))$ and the following proposition [16] gives an explicit formula for the symbol of the product of pseudodifferential operators as operators on $\mathcal{H}$ modulo the above equivalence relation.

Proposition 3.3.2. *Let P and Q be pseudodifferential operators with the symbols σ and σ' respectively. The product PQ is a pseudodifferential operator with the following symbol,*

$$\sigma(PQ) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha(\sigma(\xi))\delta^\alpha(\sigma'(\xi)).$$

□

Definition 3.3.3. *A symbol $\sigma \in \mathcal{S}^m(C^\infty(\mathbb{T}^n_\theta))$ is called elliptic if $\sigma(\xi)$ is invertible for $\xi \neq 0$, and for some c*

$$||\sigma(\xi)^{-1}|| \leqslant c(1+|\xi|)^{-m},$$

for large enough $|\xi|$.

Example 3.3.4. Consider the Dirac operator of the spectral triple $\left(\mathrm{C}^\infty(\mathbb{T}^3_\theta), \mathcal{H}_\tau \otimes \mathbb{C}^2, D = \partial\!\!\!/\right)$,

$$\partial\!\!\!/ = \partial_\mu \otimes \gamma^\mu : \mathcal{H}_\tau \oplus \mathcal{H}_\tau \longrightarrow \mathcal{H}_\tau \oplus \mathcal{H}_\tau.$$

The symbol reads

$$\sigma(D)(\xi) = \xi_\mu \otimes \gamma^\mu = \begin{bmatrix} \xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & -\xi_3 \end{bmatrix},$$

which is clearly elliptic.

A smooth map $\sigma : \mathbb{R}^n \to C^\infty(\mathbb{T}^n_\theta) \otimes M_N(\mathbb{C})$ is called a classical symbol of order $\alpha \in \mathbb{C}$ if for any L and each $0 \leqslant j \leqslant L$ there exist $\sigma_{\alpha-j} : \mathbb{R}^n \backslash \{0\} \to C^\infty(\mathbb{T}^n_\theta) \otimes M_N(\mathbb{C})$ positive homogeneous of degree $\alpha - j$, and a symbol $\sigma^L \in \mathcal{S}^{\mathrm{Re}(\alpha)-L-1}(C^\infty(\mathbb{T}^n_\theta))$, such that

$$\sigma(\xi) = \sum_{j=0}^{L} \chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^L(\xi) \quad \xi \in \mathbb{R}^n. \tag{3.1}$$

Here χ is a smooth cut off function on $\mathbb{R}^n$ which is equal to zero on a small ball around the origin, and is equal to one outside the unit ball. The homogeneous terms in the expansion are uniquely determined by σ. The set of classical symbols of order α on noncommutative n-torus will be denoted by $\mathcal{S}^\alpha_{cl}(C^\infty(\mathbb{T}^n_\theta))$.

The analogue of the Wodzicki residue for classical pseudodifferential operators on the noncommutative n-torus is defined in [23].

Definition 3.3.5. *The noncommutative residue of a classical pseudodifferential operator A_σ is defined as*

$$\mathrm{Wres}(A_\sigma) = \tau\left(\mathrm{res}(A_\sigma)\right),$$

where $\mathrm{res}(A_\sigma) := \int_{|\xi|=1} \mathrm{tr}\, \sigma_{-n}(\xi) d\xi$.

It is evident from the definition that noncommutative residue vanishes on differential operators, operators of order $< -n$ as well as non-integer order operators

3.3.2 The canonical trace

In what follows, we recall the analogue of Kontsevich-Vishik canonical trace [30] on non-integer order pseudodifferential operators on the noncommutative tori from [19]. For an alternative approach based on toroidal noncommutative symbols see [33]. For a thorough review of the theory in the classical case we refer to [35, 37].

The existence of the so called cut-off integral for classical noncommutative symbols is established in [19].

Proposition 3.3.6. *Let $\sigma \in \mathcal{S}^{\alpha}_{cl}(C^{\infty}(\mathbb{T}^n_\theta))$ and $B(R)$ be the ball of radius R around the origin. One has the following asymptotic expansion*

$$\int_{B(R)} \sigma(\xi)d\xi \sim_{R\to\infty} \sum_{j=0,\alpha-j+n\neq 0}^{\infty} \alpha_j(\sigma)R^{\alpha-j+n} + \beta(\sigma)\log R + c(\sigma),$$

where $\beta(\sigma) = \int_{|\xi|=1} \sigma_{-n}(\xi)d\xi$ and the constant term in the expansion, $c(\sigma)$, is given by

$$\int_{\mathbb{R}^n} \sigma^L + \sum_{j=0}^{L} \int_{B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi - \sum_{j=0,\alpha-j+n\neq 0}^{L} \frac{1}{\alpha-j+n} \int_{|\xi|=1} \sigma_{\alpha-j}(\omega)d\omega. \tag{3.2}$$

Here we have used the notation of (3.1).

Definition 3.3.7. *The cut-off integral of a symbol $\sigma \in \mathcal{S}^{\alpha}_{cl}(C^{\infty}(\mathbb{T}^n_\theta))$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $⨍ \sigma(\xi)d\xi$.*

Now the canonical trace of a classical pseudodifferential operator of non-integer order on $C^{\infty}(\mathbb{T}^n_\theta)$ is defined as follows [19]:

Definition 3.3.8. *The canonical trace of a classical pseudodifferential operator A of non-integral order α is defined as*

$$\mathrm{TR}(A) = \tau\left(⨍ \mathrm{tr}\,\sigma_A(\xi)d\xi\right).$$

The relation between the TR-functional and the usual trace on trace-class pseudodifferential operators is established in [19]. Note that any pseudodifferential operator A of order less that $-n$ is a trace class operator and its trace is given by

$$\mathrm{Tr}(A) = \tau\left(\int_{\mathbb{R}^n} \mathrm{tr}\sigma_P(\xi)d\xi\right).$$

The symbol of such operators is integrable and we have

$$⨍ \sigma_A(\xi) = \int_{\mathbb{R}^n} \sigma_A(\xi)d\xi. \tag{3.3}$$

Therefore, the TR-functional and operator trace coincide on classical pseudodifferential operators of order less than $-n$.

The TR-functional is in fact the analytic continuation of the operator trace and using this fact we can prove that it is actually a trace.

Definition 3.3.9. *A family of symbols $\sigma(z) \in \mathcal{S}_{cl}^{\alpha(z)}(C^\infty(\mathbb{T}_\theta^n))$, parametrized by $z \in W \subset \mathbb{C}$, is called a holomorphic family if*

i) The map $z \mapsto \alpha(z)$ is holomorphic.

ii) The map $z \mapsto \sigma(z) \in \mathcal{S}_{cl}^{\alpha(z)}(\mathcal{A}_\theta)$ is a holomorphic map from W to the Fréchet space $\mathcal{S}_{cl}(C^\infty(\mathbb{T}_\theta^n))$.

iii) The map $z \mapsto \sigma(z)_{\alpha(z)-j}$ is holomorphic for any j, where

$$\sigma(z)(\xi) \sim \sum_j \chi(\xi)\sigma(z)_{\alpha(z)-j}(\xi) \in \mathcal{S}_{cl}^{\alpha(z)}(C^\infty(\mathbb{T}_\theta^n)). \tag{3.4}$$

iv) The bounds of the asymptotic expansion of $\sigma(z)$ are locally uniform with respect to z, i.e, for any $L \geqslant 1$ and compact subset $K \subset W$, there exists a constant $C_{L,K,\alpha,\beta}$ such that for all multi-indices α, β we have

$$\left\|\delta^\alpha \partial^\beta \left(\sigma(z) - \sum_{j<L} \chi\sigma(z)_{\alpha(z)-j}\right)(\xi)\right\| < C_{L,K,\alpha,\beta}|\xi|^{\mathrm{Re}(\alpha(z))-L-|\beta|}.$$

A family $\{A_z\} \in \Psi_{cl}(C^\infty(\mathbb{T}_\theta^n))$ is called holomorphic if $A_z = A_{\sigma(z)}$ for a holomorphic family of symbols $\{\sigma(z)\}$.

Complex powers of elliptic operators are an important class of holomorphic families. Let $Q \in \Psi_{cl}^q(C^\infty(\mathbb{T}_\theta^n))$ be a positive elliptic pseudodifferential operator of order $q > 0$. The complex power of such an operator, Q_ϕ^z, for $\mathrm{Re}(z) < 0$ can be defined by the following Cauchy integral formula.

$$Q_\phi^z = \frac{i}{2\pi}\int_{C_\phi} \lambda_\phi^z (Q-\lambda)^{-1} d\lambda. \tag{3.5}$$

Here λ_ϕ^z is the complex power with branch cut $L_\phi = \{re^{i\phi}, r \geqslant 0\}$ and C_ϕ is a contour around the spectrum of Q such that

$$C_\phi \cap \mathrm{spec}(Q)\backslash\{0\} = \emptyset, \qquad L_\phi \cap C_\phi = \emptyset,$$

$$C_\phi \cap \{\mathrm{spec}(\sigma(Q)^L(\xi)),\, \xi \neq 0\} = \emptyset.$$

Remark 3.3.10. More generally, an operator for which one can find a ray L_ϕ with the above property, is called an admissible operator with the spectral cut L_ϕ and its complex power can be defined as above. Self-adjoint elliptic operators are admissible (see [35] and [19]).

The following Proposition is the analogue of the result of Kontsevich and Vishik [30], for pseudodifferential calculus on noncommutative tori.

Proposition 3.3.11. *Given a holomorphic family $\sigma(z) \in \mathcal{S}_{cl}^{\alpha(z)}(C^\infty(\mathbb{T}_\theta^n))$, $z \in W \subset \mathbb{C}$, the map*

$$z \mapsto \fint \sigma(z)(\xi) d\xi,$$

is meromorphic with at most simple poles located in

$$P = \{z_0 \in W;\ \alpha(z_0) \in \mathbb{Z} \cap [-n, +\infty]\}.$$

The residues at poles are given by

$$\mathrm{Res}_{z=z_0} \fint \sigma(z)(\xi) d\xi = -\frac{1}{\alpha'(z_0)} \int_{|\xi|=1} \sigma(z_0)_{-n} d\xi.$$

Proof. By definition, one can write $\sigma(z) = \sum_{j=0}^{L} \chi(\xi)\sigma(z)_{\alpha(z)-j}(\xi) + \sigma(z)^L(\xi)$, and by Proposition (3.3.6) we have,

$$\begin{aligned}\fint \sigma(z)(\xi) d\xi = &\int_{\mathbb{R}^n} \sigma(z)^L(\xi) d\xi + \sum_{j=0}^{L} \int_{B(1)} \chi(\xi)\sigma(z)_{\alpha(z)-j}(\xi) \\ &- \sum_{j=0}^{L} \frac{1}{\alpha(z)+n-j} \int_{|\xi|=1} \sigma(z)_{\alpha(z)-j}(\xi) d\xi.\end{aligned}$$

Now suppose $\alpha(z_0) + n - j_0 = 0$. By holomorphicity of $\sigma(z)$, we have $\alpha(z) - \alpha(z_0) = \alpha'(z_0)(z - z_0) + o(z - z_0)$. Hence

$$\mathrm{Res}_{z=z_0} \fint \sigma(z) = \frac{-1}{\alpha'(z_0)} \int_{|\xi|=1} \sigma(z_0)_{-n}(\xi) d\xi.$$

□

Corollary 3.3.12. *The functional* TR *is the analytic continuation of the ordinary trace on trace-class pseudodifferential operators.*

Proof. First observe that, by the above result, for a non-integer order holomorphic family of symbols $\sigma(z)$, the map $z \mapsto ⨍ \sigma(z)(\xi)d\xi$ is holomorphic. Hence, the map $\sigma \mapsto ⨍ \sigma(\xi)d\xi$ is the unique analytic continuation of the map $\sigma \mapsto \int_{\mathbb{R}^n} \sigma(\xi)d\xi$ from $\mathcal{S}_{cl}^{<-n}(C^\infty(\mathbb{T}_\theta^n))$ to $\mathcal{S}_{cl}^{\notin\mathbb{Z}}(C^\infty(\mathbb{T}_\theta^n))$. By (3.3) we have the result. □

Corollary 3.3.13. *Let $A \in \Psi_{cl}^\alpha(C^\infty(\mathbb{T}_\theta^n))$ be of order $\alpha \in \mathbb{Z}$ and let Q be a positive elliptic classical pseudodifferential operator of positive order q. We have*

$$\mathrm{Res}_{z=0}\mathrm{TR}(AQ^{-z}) = \frac{1}{q}\mathrm{Wres}(A).$$

Proof. For the holomorphic family $\sigma(z) = \sigma(AQ^{-z})$, $z = 0$ is a pole for the map $z \mapsto ⨍ \sigma(z)(\xi)d\xi$ whose residue is given by

$$\mathrm{Res}_{z=0}\left(z \mapsto ⨍ \sigma(z)(\xi)d\xi\right) = -\frac{1}{\alpha'(0)}\int_{|\xi|=1} \sigma_{-n}(0)d\xi = -\frac{1}{\alpha'(0)}\mathrm{res}(A).$$

Taking τ-trace on both sides gives the result. □

Now we can prove the trace property of TR-functional.

Proposition 3.3.14. *We have* $\mathrm{TR}(AB) = \mathrm{TR}(BA)$ *for any* $A, B \in \Psi_{cl}^*(C^\infty(\mathbb{T}_\theta^n))$, *provided that* $ord(A) + ord(B) \notin \mathbb{Z}$.

Proof. Consider the families $\{A_z\}$ and $\{B_z\}$ such that $A_0 \sim A$, $B_0 \sim B$, $\mathrm{ord}(A_z) = \mathrm{ord}(A) + z$ and $\mathrm{ord}(B_z) = \mathrm{ord}(B) + z$. For $z \in W = -(\mathrm{ord}(A) + \mathrm{ord}(B)) + \mathbb{Z}$ the families $\{A_zB_z\}$ and $\{B_zA_z\}$ have non-integer order . For $\mathrm{Re}(z) \ll 0$, the two families are trace-class and $\mathrm{Tr}(A_zB_z) = \mathrm{Tr}(B_zA_z)$. now by analytic continuation we have $TR(A_zB_z) = TR(B_zA_z)$, for $z \in \mathbb{C} - W$. Putting $z = 0$ gives $TR(AB) = TR(BA)$. □

Remark 3.3.15. The above result provides another proof for the trace property of the non-commutative residue on $\Psi_{cl}^{\mathbb{Z}}(C^\infty(\mathbb{T}_\theta^2))$ given in [23], namely, for $A, B \in \Psi_{cl}^{\mathbb{Z}}(C^\infty(\mathbb{T}_\theta^2))$,

$$\mathrm{Wres}([A, B]) = 0.$$

On can write,

$$\mathrm{Wres}([A, B]) = \mathrm{Res}_{z=0}\mathrm{TR}([A, B]Q^{-z}) = \mathrm{Res}_{z=0}\mathrm{TR}(C_z) + \mathrm{Res}_{z=0}\mathrm{TR}([AQ^{-z}, B]),$$

where $C_z = ABQ^{-z} - AQ^{-z}B$. For $Re(z) \gg 0$, the operator AQ^{-z} is trace-class and $\mathrm{Tr}([AQ^{-z}, B]) = 0$, so by analytic continuation,$\mathrm{TR}([AQ^{-z}, B]) = 0$ and therefore, $\mathrm{Res}_{z=0}\mathrm{TR}([A, B]Q^{-z}) = \mathrm{Res}_{z=0}\mathrm{TR}(C_z)$. Finally, $C_0 = ABQ^0 - AQ^0B \in \Psi_{cl}^{-\infty}(C^\infty(\mathbb{T}^2_\theta))$, so

$$\mathrm{Wres}([A, B]) = \mathrm{Res}_{z=0}\mathrm{TR}(C_z) = \mathrm{Wres}(C_0) = 0,$$

where in the last equality we used the fact that the noncommutative residue of a smoothing operator is zero.

3.3.3 Log-polyhomogeneous symbols

In general, z-derivatives of a classical holomorphic family of symbols are not classical anymore and therefore we introduce log-polyhomogeneous symbols which include the z-derivatives of the symbols of the holomorphic family $\sigma(AQ^{-z})$.

Definition 3.3.16. *A symbol σ is called a log-polyhomogeneous symbol if it has the following form*

$$\sigma(\xi) \sim \sum_{j\geqslant 0} \sum_{l=0}^{\infty} \sigma_{\alpha-j,l}(\xi) \log^l |\xi| \quad |\xi| > 0, \tag{3.6}$$

with $\sigma_{\alpha-j,l}$ positively homogeneous in ξ of degree $\alpha - j$.

An important example of an operator with such a symbol is $\log Q$ where $Q \in \Psi^q_{cl}(C^\infty(\mathbb{T}^n_\theta))$ is a positive elliptic pseudodifferential operator of order $q > 0$. The logarithm of Q can be defined by

$$\log Q = Q \left.\frac{d}{dz}\right|_{z=0} Q^{z-1} = Q \left.\frac{d}{dz}\right|_{z=0} \frac{i}{2\pi} \int_C \lambda^{z-1}(Q-\lambda)^{-1} d\lambda.$$

It is a pseudodifferential operator with symbol

$$\sigma(\log Q) \sim \sigma(Q) \star \sigma\Big(\left.\frac{d}{dz}\right|_{z=0} Q^{z-1}\Big), \tag{3.7}$$

where $\star$ denotes the product of symbols. One can show that (3.7) is a log-polyhomogeneous symbol of the form

$$\sigma(\log Q)(\xi) = q \log |\xi| I + \sigma_{cl}(\log Q)(\xi),$$

where $\sigma_{cl}(\log Q)$ is a classical symbol of order zero (see [35]).

By adapting the proof of Theorem 1.13 in [37] to the noncommutative case, we have the following theorem for the family $\sigma(AQ^{-z})$.

Proposition 3.3.17. *Let $A \in \Psi_{cl}^{\alpha}(C^{\infty}(\mathbb{T}_{\theta}^{n}))$ and Q be a positive (or more generally, an admissible) elliptic pseudodifferential operator of positive order q. If $\alpha \in P$ then $z = 0$ is a possible simple pole for the function $z \mapsto \mathrm{TR}(AQ^{-z})$ with the following Laurent expansion around zero,*

$$
\begin{aligned}
\mathrm{TR}(AQ^{-z}) &= \frac{1}{q}\mathrm{Wres}(A)\frac{1}{z} \\
&+ \tau\left(⨍ \sigma(A) - \frac{1}{q}\mathrm{res}(A\log Q)\right) - \mathrm{Tr}(A\Pi_Q) \\
&+ \sum_{k=1}^{K}(-1)^k\frac{(z)^k}{k!} \\
&\times\left(\tau\left(⨍ \sigma(A(\log Q)^k)d\xi - \frac{1}{q(k+1)}\mathrm{res}(A(\log Q)^{k+1})\right) - \mathrm{Tr}(A\log^k Q\Pi_Q)\right) \\
&+ o(z^K).
\end{aligned}
$$

Where Π_Q is the projection on the kernel of Q.

□

Remark 3.3.18. The term $\mathrm{res}(A\log Q)$ appearing in above Laurent expansion is an extension of Wodzicki residue density to operators with Log-polyhomogeneous symbols [32]. For an operator P with log-polyhomogeneous symbol, by $\mathrm{res}(P)$ we mean,

$$\mathrm{res}(P) = \int_{|\xi|=1} \sigma_{-n,0}(\xi)d\xi,$$

(see (3.6)).

3.4 The spectral eta function

In this section we study the eta function associated with the family of spectral triples $(\mathrm{C}^{\infty}(\mathbb{T}_{\theta}^{3}), \mathcal{H}, e^{th}De^{th})$ where $h \in \mathrm{C}^{\infty}(\mathbb{T}_{\theta}^{3})$ is a self-adjoint element [14], and also the coupled spectral triple $(\mathrm{C}^{\infty}(\mathbb{T}_{\theta}^{3}), \mathcal{H}, D + A)$. By exploiting the developed pseudodifferential calculus, the regularity of the eta function at zero in above cases will be proved.

3.4.1 Regularity at zero

The spectral eta function was first introduced in [1] where its value at zero appeared as a correction term in the Atiyah-Patodi-Singer index theorem for manifolds with boundary. It is defined as

$$\eta_D(z) = \sum_{\lambda \in \mathrm{spec}(D), \lambda \neq 0} \mathrm{sgn}(\lambda)|\lambda|^{-z} = \mathrm{TR}\left(D|D|^{-z-1}\right),$$

where D is a self-adjoint elliptic pseudodifferential operator. Unlike the spectral zeta functions for positive elliptic operators, proving the regularity of eta function at zero is difficult. This was proved in [2] and [26] using K-theoretic arguments and in [5], Bismut and Freed gave an analytic proof of the regularity at zero of the eta function for a twisted Dirac operator on an odd dimensional spin manifold.

Remark 3.4.1. Note that for an even spectral triple $(\mathcal{A}, H, D)$ we have $D\Gamma = -\Gamma D$, therefore the spectrum of the Dirac operator is symmetric and $\eta_D(z)$ is identically zero. Also the same vanishing happens if the spectral triple admits a real structure with KO-dimensions 1 or 5 (see definition 3.2.2).

The meromorphic structure of eta function for Dirac operator can be studied by the pole structure of the canonical trace for holomorphic families.

Proposition 3.4.2. *Let D be an elliptic self-adjoint first-order differential operator on $C^\infty(\mathbb{T}^3_\theta)$. The poles of the eta function $\eta_D(z)$ are located among $\{3-i, \ i \in \mathbb{N}\}$, and*

$$\mathrm{Res}_{z=0}\eta_D(z) = \mathrm{Wres}(D|D|^{-1}).$$

Proof. By using the result of Proposition 3.3.11, the family $\left\{\sigma\left(D|D|^{-z-1}\right)\right\}$ has poles within the set $\{z; -z \in \mathbb{Z} \cap [-3, \infty]\}$ or $\{z = 3-i, i \in \mathbb{N}\}$. Also, by Proposition 3.3.17 we have

$$\eta_D(z) = \mathrm{TR}\left(D|D|^{-z-1}\right) = \mathrm{Wres}(D|D|^{-1})\frac{1}{z} + a_0 + a_1 z + \cdots,$$

Hence,

$$\mathrm{Res}_{z=0}\eta_D(z) = \mathrm{Wres}(D|D|^{-1}).$$

∎

We now prove the regularity at $z = 0$ of eta function for the 1-parameter family $\{e^{th}De^{th}\}$ for the spectral triple $(C^\infty(\mathbb{T}^3_\theta), H_\tau \otimes \mathbb{C}^2, D = \partial_\mu \otimes \gamma^\mu)$ on noncommutative 3-torus.

Proposition 3.4.3. *Consider the family of operators* $\{e^{th}De^{th}\}$ *on* $\mathcal{H}_\tau \otimes \mathbb{C}^2$ *where* $h = h^* \in C^\infty(\mathbb{T}^3_\theta)$*, then*

$$\mathrm{Res}_{z=0}\eta_{e^{th}De^{th}}(z) = 0.$$

Proof. By definition, $\eta_{D_t}(z) = \mathrm{TR}(D_t|D_t|^{-z-1})$. Using Proposition 3.4.2 we have

$$\mathrm{Res}_{z=0}\eta_{D_t}(z) = \mathrm{Wres}(D_t|D_t|^{-1}),$$

where the right hand side is the Wodzicki residue on noncommutative 3-torus.

Now for each element of the family $D_t = e^{\frac{th}{2}}De^{\frac{th}{2}}$, $D_t^2 = e^{\frac{th}{2}}De^{th}De^{\frac{th}{2}}$ and $|D_t| = \sqrt{D_t^2}$. By using the product formula for the symbols we have,

$$\sigma(D_t) \sim \xi_\mu e^{th} \otimes \gamma^\mu + e^{\frac{th}{2}}\delta_\mu(e^{\frac{th}{2}}) \otimes \gamma^\mu, \tag{3.8}$$

and

$$\begin{aligned}\sigma(D_t^2) = \sigma(e^{\frac{th}{2}}De^{th}De^{\frac{th}{2}}) &\sim \xi_\lambda\xi_\mu e^{2th} \otimes \gamma^\lambda\gamma^\mu \\ &+ \xi_\lambda e^{\frac{3th}{2}}\delta_\mu(e^{\frac{th}{2}})\gamma^\lambda\gamma^\mu + \xi_\mu e^{\frac{th}{2}}\delta_\lambda(e^{\frac{3th}{2}}) \otimes \gamma^\lambda\gamma^\mu \\ &+ e^{\frac{th}{2}}\delta_\lambda(e^{th})\delta_\mu(e^{\frac{th}{2}}) \otimes \gamma^\lambda\gamma^\mu + e^{\frac{3th}{2}}\delta_\lambda\delta_\mu(e^{\frac{th}{2}}) \otimes \gamma^\lambda\gamma^\mu.\end{aligned}$$

To compute the homogeneous terms in the symbol of $|D_t|$, we observe that $|D_t| = \sqrt{D_t^2}$ and hence,

$$\sigma(|D_t|) = \sqrt{\sigma(D_t^2)} \sim \sigma_1(\xi) + \sigma_0(\xi) + \sigma_{-1}(\xi) + \cdots . \tag{3.9}$$

We compute the first three terms, which we need for computing the symbol of $|D_t|^{-1}$.

$$
\begin{aligned}
\sigma_1(\xi) &= \lim_{k\to\infty} \frac{\sigma(|D_t|)(k\xi)}{k} = \sqrt{(\xi^2)} e^{th} \otimes I, \\
\sigma_0(\xi) &= \lim_{k\to\infty} \sigma(|D_t|)(k\xi) - k\sigma_1(\xi) \\
&= \left(\xi_\lambda e^{\frac{3th}{2}} \delta_\mu(e^{\frac{th}{2}})\right) (\frac{1}{2\sqrt{\xi^2}} e^{-th}) \otimes \gamma^\lambda\gamma^\mu + \left(\xi_\mu e^{\frac{th}{2}} \delta_\lambda(e^{\frac{3th}{2}})\right) (\frac{1}{2\sqrt{\xi^2}} e^{-th}) \otimes \gamma^\lambda\gamma^\mu, \\
\sigma_{-1}(\xi) &= \lim_{k\to\infty} \frac{(\sigma(|D_t|)(k\xi) - \sigma_1(\xi) - \sigma_0(\xi))}{k^{-1}} \\
&= \frac{1}{2\sqrt{\xi^2}} \left(e^{\frac{th}{2}} \delta_\lambda(e^{th}) \delta_\mu(e^{\frac{th}{2}})\right) e^{-th} \otimes \gamma^\lambda\gamma^\mu + \frac{1}{2\sqrt{\xi^2}} \left(e^{\frac{3th}{2}} \delta_\lambda\delta_\mu(e^{\frac{th}{2}})\right) e^{-th} \otimes \gamma^\lambda\gamma^\mu \\
&- \frac{1}{8\xi^2\sqrt{\xi^2}} \left(\xi_\lambda e^{\frac{3th}{2}} \delta_\mu(e^{\frac{th}{2}})\right) e^{-th} \left(\xi_\nu e^{\frac{3th}{2}} \delta_\rho(e^{\frac{th}{2}})\right) e^{-2th} \otimes \gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho \\
&- \frac{1}{8\xi^2\sqrt{\xi^2}} \left(\xi_\lambda e^{\frac{3th}{2}} \delta_\mu(e^{\frac{th}{2}})\right) e^{-th} \left(\xi_\rho e^{\frac{th}{2}} \delta_\nu(e^{\frac{3th}{2}})\right) e^{-2th} \otimes \gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho \\
&- \frac{1}{8\xi^2\sqrt{\xi^2}} \left(\xi_\mu e^{\frac{th}{2}} \delta_\lambda(e^{\frac{3th}{2}})\right) e^{-th} \left(\xi_\nu e^{\frac{3th}{2}} \delta_\rho(e^{\frac{th}{2}})\right) e^{-2th} \otimes \gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho \\
&- \frac{1}{8\xi^2\sqrt{\xi^2}} \left(\xi_\mu e^{\frac{th}{2}} \delta_\lambda(e^{\frac{3th}{2}})\right) e^{-th} \left(\xi_\rho e^{\frac{th}{2}} \delta_\nu(e^{\frac{3th}{2}})\right) e^{-2th} \otimes \gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho,
\end{aligned}
$$

where we have used the notation $\xi^2 = \sum_{k=0}^{3} \xi_k^2$.

Now by using the relation $|D_t|^{-1}|D_t| = 1$ and by recursive computation we find the homogeneous terms in the symbol of the inverse,

$$\sigma(|D_t|^{-1}) \sim \sigma_{-1}(|D_t|^{-1})(\xi) + \sigma_{-2}(|D_t|^{-1})(\xi) + \cdots , \tag{3.10}$$

where,

$$\sigma_{-1}(|D_t|^{-1})(\xi) = \frac{1}{\sqrt{\xi^2}} e^{-th} \otimes I,$$

$$\sigma_{-2}(|D_t|^{-1})(\xi) =$$
$$- \sigma_{-1}(|D_t|^{-1})\{\sigma_0(|D_t|)\sigma_{-1}(|D_t|^{-1})$$
$$+ \sum_{\alpha_1+\alpha_2+\alpha_3=1} \partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_1(|D_t|)\delta^{\alpha_1}\delta^{\alpha_2}\delta^{\alpha_3}\sigma_{-1}(|D_t|^{-1})\},$$

$$\sigma_{-3}\{|D_t|^{-1})(\xi) =$$
$$- \sigma_{-1}(|D_t|^{-1})\{(\sigma_{-1}(|D_t|)\sigma_{-1}(|D_t|^{-1}) + \sigma_0(|D_t|)\sigma_{-2}(|D_t|^{-1})$$
$$+ \sum_{\alpha_1+\alpha_2+\alpha_3=2} \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_1(|D_t|)\delta^{\alpha_1}\delta^{\alpha_2}\delta^{\alpha_3}\sigma_{-1}(|D_t|^{-1})$$
$$+ \sum_{\alpha_1+\alpha_2+\alpha_3=1} \partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_1(|D_t|)\delta^{\alpha_1}\delta^{\alpha_2}\delta^{\alpha_3}\sigma_{-2}(|D_t|^{-1})\},$$

$$\sigma_{-4}(|D_t|^{-1})(\xi) =$$
$$-\sigma_{-1}(|D_t|^{-1})\{\sigma_{-2}(|D_t|)\sigma_{-1}(|D_t|^{-1}) + \sigma_{-1}(|D_t|)\sigma_{-2}(|D_t|^{-1}) + \sigma_0(|D_t|)\sigma_{-3}(|D_t|^{-1})$$
$$+ \sum_{\alpha_1+\alpha_2+\alpha_3=1} \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_0(|D_t|)\delta^{\alpha_1}\delta^{\alpha_2}\delta^{\alpha_3}\sigma_{-2}(|D_t|^{-1})$$
$$+ \sum_{\alpha_1+\alpha_2+\alpha_3=1} \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_{-1}(|D_t|)\delta^{\alpha_1}\delta^{\alpha_2}\delta^{\alpha_3}\sigma_{-1}(|D_t|^{-1})$$
$$+ \sum_{\alpha_1+\alpha_2+\alpha_3=1} \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_1(|D_t|)\delta^{\alpha_1}\delta^{\alpha_2}\delta^{\alpha_3}\sigma_{-3}(|D_t|^{-1})$$
$$+ \sum_{\alpha_1+\alpha_2+\alpha_3=2} \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_0(|D_t|)\delta^{\alpha_1}\delta^{\alpha_2}\delta^{\alpha_3}\sigma_{-1}(|D_t|^{-1})$$
$$+ \sum_{\alpha_1+\alpha_2+\alpha_3=2} \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_1(|D_t|)\delta^{\alpha_1}\delta^{\alpha_2}\delta^{\alpha_3}\sigma_{-2}(|D_t|^{-1})$$
$$+ \sum_{\alpha_1+\alpha_2+\alpha_3=3} \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_1(|D_t|)\delta^{\alpha_1}\delta^{\alpha_2}\delta^{\alpha_3}\sigma_{-1}(|D_t|^{-1})\}.$$

Therefore, the symbol of $\sigma(D_t|D_t|^{-1})$ reads

$$\sigma(D_t|D_t|^{-1}) \sim (\sigma_1(D_t) + \sigma_0(D_t)) \star \big(\sigma_{-1}(|D_t|^{-1}) + \sigma_{-2}(|D_t|^{-1}) + \sigma_{-3}(|D_t|^{-1}) + \cdots\big)$$
$$\sim \big(\sigma_1(D_t) \star \sigma_{-1}(|D_t|^{-1})\big) + \big(\sigma_1(D_t) \star \sigma_{-2}(|D_t|^{-1})\big)$$
$$+ \big(\sigma_1(D_t) \star \sigma_{-3}(|D_t|^{-1})\big) + \big(\sigma_1(D_t) \star \sigma_{-4}(|D_t|^{-1})\big) + \cdots$$
$$+ \big(\sigma_0(D_t) \star \sigma_{-1}(|D_t|^{-1})\big) + \big(\sigma_0(D_t) \star \sigma_{-2}(|D_t|^{-1})\big) + \big(\sigma_0(D_t) \star \sigma_{-3}(|D_t|^{-1})\big) + \cdots .$$

For $a = 1, 0$ and $b = -1, -2, -3. \cdots$, one has

$$\sigma_a(D_t) \star \sigma_b(|D_t|^{-1}) = \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_a(D_t) \delta^\alpha \sigma_b(|D_t|^{-1}),$$

and each term is of order $a - |\alpha| + b$. By collecting the terms of order -3 we obtain,

$$\begin{aligned}
\sigma_{-3}(D_t|D_t|^{-1}) \sim & \big(\sigma_1(D_t)\sigma_{-4}(|D_t|^{-1})\big) + \big(\sigma_0(D_t)\sigma_{-3}(|D_t|^{-1})\big) \\
&+ \left(\sum_{\alpha_1+\alpha_2+\alpha_3=1} \partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_1(D_t)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-3}(|D_t|^{-1})\right) \\
&+ \left(\sum_{\alpha_1+\alpha_2+\alpha_3=2} \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_1(D_t)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-2}(|D_t|^{-1})\right) \\
&+ \left(\sum_{\alpha_1+\alpha_2+\alpha_3=1} \partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_0(D_t)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-2}(|D_t|^{-1})\right) \\
&+ \left(\sum_{\alpha_1+\alpha_2+\alpha_3=2} \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_0(D_t)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-1}(|D_t|^{-1})\right).
\end{aligned}$$

Now, one should notice that Wodzicki residue of a matrix pseudodifferential operator involves a trace taken over the matrix coefficients. By analyzing the terms involved in $\sigma_{-3}(D_t|D_t|^{-1})$ and using the trace identities for gamma matrices (cf. e.g. [24]), we see that the only contribution is from the the terms whose matrix coefficient consist of either one γ matrix or product of three. Next, we use the following identities,

$$\mathrm{tr}(\gamma^\lambda) = 0, \tag{3.11}$$

$$\mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu) = i\epsilon^{\lambda\mu\nu}\mathrm{tr}(I), \tag{3.12}$$

where $\epsilon^{\lambda\mu\nu}$ is the Levi-Civita symbol. By observing that

$$\sum_{\lambda,\mu,\nu} \epsilon^{\lambda,\mu,\nu} = 0,$$

we conclude that

$$\mathrm{Wres}\left(D_t|D_t|^{-1}\right) = \tau\left(\int_{|\xi|=1} \mathrm{tr}\left(\mathrm{res}(D_t|D_t|^{-1}\right) d\xi\right) = 0. \tag{3.13}$$

□

Remark 3.4.4. In [5], Bismut-Freed showed that in fact, for a twisted Dirac operator on a Riemannian Spin manifold, the *local eta-residue*, $\mathrm{tr}\left(\mathrm{res}_x\left(D|D|^{-1}\right)\right)$ vanishes. The above and the following results confirm the same vanishing in the noncommutative case for the family of operators $\{D_t\}$ on $\mathrm{C}^\infty(\mathbb{T}^3_\theta)$.

The following result proves the regularity at zero of eta function for the coupled Dirac operator $D+A$ on $\mathrm{C}^\infty(\mathbb{T}^3_\theta)$.

Proposition 3.4.5. *Consider the coupled Dirac operator* $\not{\partial}+\not{A}$ *on* $\mathcal{H}_\tau\otimes\mathbb{C}^2$ *over* $\mathrm{C}^\infty(\mathbb{T}^3_\theta)$, *then*

$$\mathrm{Res}_{z=0}\eta_{D+A}(z) = 0.$$

Proof. The coupled Dirac operator D and D^2 are given by

$$\begin{aligned} D &= \partial_\mu \otimes \gamma^\mu + A_\mu \otimes \gamma^\mu \\ D^2 &= \partial_\mu\partial_\lambda \otimes \gamma^\mu\gamma^\lambda + \partial_\mu(A_\lambda) \otimes \gamma^\mu\gamma^\lambda + A_\mu\partial_\lambda \otimes \gamma^\mu\gamma^\lambda + A_\mu A_\lambda \otimes \gamma^\mu\gamma^\lambda. \end{aligned}$$

Note that $\partial_\mu A_\lambda(a) = \partial_\mu(A_\lambda)a + A_\lambda\partial_\mu(a)$ and also $A_\mu A_\lambda \neq A_\lambda A_\mu$. Hence,

$$D^2 = \sum_\mu \partial_\mu^2 \otimes I + 2A_\mu\partial_\lambda \otimes \gamma^\mu\gamma^\lambda + \partial_\mu(A_\lambda) \otimes \gamma^\mu\gamma^\lambda + A_\mu A_\lambda \otimes \gamma^\mu\gamma^\lambda,$$

where $\partial_\mu(A_\lambda)$ in the third term above is a multiplication operator. The symbols are:

$$\sigma(D) = \not{\xi} + \not{A},$$

and

$$\sigma(D^2) = \xi^2 + 2A_\mu\xi_\lambda \otimes \gamma^\mu\gamma^\lambda + \partial_\mu(A_\lambda) \otimes \gamma^\mu\gamma^\lambda + A_\mu A_\lambda \otimes \gamma^\mu\gamma^\lambda,$$

where $\xi^2 = \sum_\mu \xi_\mu^2 \otimes I$. Now,

$$\begin{aligned} \sigma(|D|) &= \sqrt{\xi^2 + 2A_\mu\xi_\lambda \otimes \gamma^\mu\gamma^\lambda + \partial_\mu(A_\lambda) \otimes \gamma^\mu\gamma^\lambda + A_\mu A_\lambda \otimes \gamma^\mu\gamma^\lambda} \\ &\sim \sigma_1(\xi) + \sigma_0(\xi) + \sigma_{-1}(\xi) + \cdots, \end{aligned}$$

where

$$\begin{aligned}
\sigma_1(\xi) &= \sqrt{\xi^2} \otimes I, \\
\sigma_0(\xi) &= \frac{\not{A}\not{\xi}}{\sqrt{\xi^2}}, \\
\sigma_{-1}(\xi) &= \frac{1}{2\sqrt{\xi^2}} \left(\partial_\mu(A_\lambda) \otimes \gamma^\mu \gamma^\lambda + A_\mu A_\lambda \otimes \gamma^\mu \gamma^\lambda \right) \\
&\quad - \frac{1}{2\xi^2\sqrt{\xi^2}} \left(A_\mu \xi_\lambda A_\nu \xi_\rho \otimes \gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\rho \right).
\end{aligned}$$

By using $\sigma(|D|) \star \sigma(|D|^{-1}) \sim 1$ and by recursive computation, we get:

$$\sigma_{-1}(|D|^{-1}) = \frac{1}{\sqrt{\xi^2}} \otimes I.$$

$$\begin{aligned}
&\sigma_{-2}(|D|^{-1}) = \\
&-\frac{1}{\sqrt{\xi^2}} \otimes I \left(\sigma_0(|D|)\sigma_{-1}(|D|^{-1}) + \sum_{\alpha_1+\alpha_2+\alpha_3=1} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \partial_{\xi_3}^{\alpha_3} \sigma_1(|D|) \delta_1^{\alpha_1} \delta_2^{\alpha_2} \delta_3^{\alpha_3} \sigma_{-1}(|D|^{-1}) \right) \\
&\qquad = -\frac{1}{\sqrt{\xi^2}} \otimes I \left(\frac{\not{A}\not{\xi}}{\sqrt{\xi^2}} . \frac{1}{\sqrt{\xi^2}} \otimes I \right) = -\frac{\not{A}\not{\xi}}{\xi^2\sqrt{\xi^2}}.
\end{aligned}$$

$$\begin{aligned}
\sigma_{-3}(|D|^{-1}) &= -\frac{1}{\sqrt{\xi^2}} \otimes I \{ \sigma_{-1}(|D|)\sigma_{-1}(|D|^{-1}) + \sigma_0(|D|)\sigma_{-2}(|D|^{-1}) \\
&+ \sum_{\alpha_1+\alpha_2+\alpha_3=1} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \partial_{\xi_3}^{\alpha_3} \sigma_0(|D|) \delta_1^{\alpha_1} \delta_2^{\alpha_2} \delta_3^{\alpha_3} \sigma_{-1}(|D|^{-1}) \\
&+ \sum_{\alpha_1+\alpha_2+\alpha_3=1} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \partial_{\xi_3}^{\alpha_3} \sigma_1(|D|) \delta_1^{\alpha_1} \delta_2^{\alpha_2} \delta_3^{\alpha_3} \sigma_{-2}(|D|^{-1}) \\
&+ \sum_{\alpha_1+\alpha_2+\alpha_3=2} \frac{1}{\alpha_1!\alpha_2!\alpha_3!} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \partial_{\xi_3}^{\alpha_3} \sigma_1(|D|) \delta_1^{\alpha_1} \delta_2^{\alpha_2} \delta_3^{\alpha_3} \sigma_{-1}(|D|^{-1}) \} \\
&= -\frac{1}{\sqrt{\xi^2}} \otimes I \left(\frac{\not{A}\not{\xi}}{\sqrt{\xi^2}} . \frac{-\not{A}\not{\xi}}{\xi^2\sqrt{\xi^2}} + \partial_\xi^1(\sqrt{\xi^2} \otimes I).\delta^1(\frac{-\not{A}\not{\xi}}{\xi^2\sqrt{\xi^2}}) \right) \\
&= \frac{\xi_\mu \xi_\nu A_\rho A_\lambda}{\xi^4\sqrt{\xi^2}} \otimes \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\lambda + \frac{1}{\sqrt{\xi^2}} \partial_\xi^1(\sqrt{\xi^2}) \delta^1(\frac{A_\mu \xi_\nu}{\xi^2\sqrt{\xi^2}}) \otimes \gamma^\mu \gamma^\nu.
\end{aligned}$$

$$
\begin{aligned}
\sigma_{-4}(|D|^{-1}) = & -\frac{1}{\sqrt{\xi^2}}\otimes I\{\sigma_{-2}(|D|)\sigma_{-1}(|D|^{-1})+\sigma_{-1}(|D|)\sigma_{-2}(|D|^{-1})\\
& +\sigma_0(|D|)\sigma_{-3}(|D|^{-1})\\
& +\sum_{\alpha_1+\alpha_2+\alpha_3=1}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_{-1}(|D|)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-1}(|D|^{-1})\\
& +\sum_{\alpha_1+\alpha_2+\alpha_3=1}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_{0}(|D|)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-2}(|D|^{-1})\\
& +\sum_{\alpha_1+\alpha_2+\alpha_3=2}\frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_{0}(|D|)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-1}(|D|^{-1})\\
& +\sum_{\alpha_1+\alpha_2+\alpha_3=1}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_{1}(|D|)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-3}(|D|^{-1})\\
& +\sum_{\alpha_1+\alpha_2+\alpha_3=2}\frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_{1}(|D|)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-2}(|D|^{-1})\\
& +\sum_{\alpha_1+\alpha_2+\alpha_3=3}\frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_{1}(|D|)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-1}(|D|^{-1})\}\\
& = -\frac{1}{\sqrt{\xi^2}}\otimes I\{\sigma_0(|D|)\sigma_{-3}(|D|^{-1})+\sum_{\alpha_1+\alpha_2+\alpha_3=1}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_{0}(|D|)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-2}(|D|^{-1})\\
& +\sum_{\alpha_1+\alpha_2+\alpha_3=1}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_{1}(|D|)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-3}(|D|^{-1})\\
& +\sum_{\alpha_1+\alpha_2+\alpha_3=2}\frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_{1}(|D|)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-2}(|D|^{-1})\}.
\end{aligned}
$$

Therefore the symbol of $\sigma(D|D|^{-1})$ reads

$$
\begin{aligned}
& \sigma(D|D|^{-1})\sim(\sigma_1(D)+\sigma_0(D))\star\big(\sigma_{-1}(|D|^{-1})+\sigma_{-2}(|D|^{-1})+\sigma_{-3}(|D|^{-1})+\cdots\big)\\
& \sim\big(\sigma_1(D)\star\sigma_{-1}(|D|^{-1})\big)+\big(\sigma_1(D)\star\sigma_{-2}(|D|^{-1})\big)\\
& +\big(\sigma_1(D)\star\sigma_{-3}(|D|^{-1})\big)+\big(\sigma_1(D)\star\sigma_{-4}(|D|^{-1})\big)+\cdots\\
& +\big(\sigma_0(D)\star\sigma_{-1}(|D|^{-1})\big)+\big(\sigma_0(D)\star\sigma_{-2}(|D|^{-1})\big)+\big(\sigma_0(D)\star\sigma_{-3}(|D|^{-1})\big)+\cdots.
\end{aligned}
$$

For $a=1,0$ and $b=-1,-2,-3.\cdots$, one has

$$
\sigma_a(D)\star\sigma_b(|D|^{-1})=\sum_\alpha\frac{1}{\alpha!}\partial_\xi^\alpha\sigma_a(D)\delta^\alpha\sigma_b(|D|^{-1}),
$$

and each term is of order $a - |\alpha| + b$. By collecting the terms of order -3 we get

$$\begin{aligned}
\sigma_{-3}(D|D|^{-1}) \sim & \big(\sigma_1(D)\sigma_{-4}(|D|^{-1})\big) + \big(\sigma_0(D)\sigma_{-3}(|D|^{-1})\big) \\
& + \left(\sum_{\alpha_1+\alpha_2+\alpha_3=1} \partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_1(D)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-3}(|D|^{-1})\right) \\
& + \left(\sum_{\alpha_1+\alpha_2+\alpha_3=2} \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_1(D)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-2}(|D|^{-1})\right) \\
& + \left(\sum_{\alpha_1+\alpha_2+\alpha_3=1} \partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_0(D)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-2}(|D|^{-1})\right) \\
& + \left(\sum_{\alpha_1+\alpha_2+\alpha_3=2} \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_0(D)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-1}(|D|^{-1})\right).
\end{aligned}$$

Since $\sigma_0(D)$ has no ξ dependence, the last two terms vanish and therefore we have:

$$\begin{aligned}
\sigma_{-3}(D|D|^{-1}) \sim & \big(\sigma_1(D)\sigma_{-4}(|D|^{-1})\big) + \big(\sigma_0(D)\sigma_{-3}(|D|^{-1})\big) \\
& + \left(\sum_{\alpha_1+\alpha_2+\alpha_3=1} \partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_1(D)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-3}(|D|^{-1})\right) \\
& + \left(\sum_{\alpha_1+\alpha_2+\alpha_3=2} \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\sigma_1(D)\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\sigma_{-2}(|D|^{-1})\right).
\end{aligned}$$

Finally, similar to the proof of Proposition (3.4.3), one observes that the matrix coefficients of all terms in $\sigma_{-3}(D|D|^{-1})$ consist of either one γ matrix or product of three γ matrices. Again, by using the similar trace identities of γ matrices identities we obtain,

$$\mathrm{Wres}(\frac{D}{|D|}) = \tau\left(\mathrm{res}(D|D|^{-1})\right) = \tau\left(\int_{|\xi|=1}\mathrm{tr}(\sigma_{-3})d\xi\right) = 0.$$

□

3.4.2 Conformal invariance of $\eta_D(0)$

In this section we study the variations of $\eta_D(0)$ for the spectral triple $(\mathrm{C}^\infty(\mathbb{T}^3_\theta), \mathcal{H}, D = \partial\!\!\!/)$. We consider the conformal variation of the Dirac operator and show that $\eta_D(0)$ will remain unchanged.

Consider the commutative spectral triple $(C^\infty(M), L^2(M,S)_g, D_g)$ encoding the data of a closed n-dimensional spin Riemannian manifold with the spin Dirac operator on the space of spinors. By varying g within its conformal class, we consider $\tilde{g} = k^{-2}g$ for some $k = e^h > 0$ in $C^\infty(M)$. The volume form for the perturbed metric is given by $\mathrm{dvol}_{\tilde{g}} = k^{-n}\mathrm{dvol}_g$ and one has a unitary isomorphism

$$U : L^2(M,S)_g \longrightarrow L^2(M,S)_{\tilde{g}}$$

by

$$U(\psi) = k^{\frac{n}{2}}\psi,$$

It can be shown that (cf. [29]) $D_{\tilde{g}} = k^{\frac{n+1}{2}} D_g k^{\frac{-n+1}{2}}$ and hence

$$U^* D_{\tilde{g}} U = k^{\frac{-n}{2}}(k^{\frac{n+1}{2}} D_g k^{\frac{-n+1}{2}})k^{\frac{n}{2}} = \sqrt{k} D_g \sqrt{k}.$$

The above property of the Dirac operator is usually referred to as being conformally covariant. It is a known fact that $\eta_D(0)$ for a Dirac operator on an odd dimensional manifold is a conformal invariant [1], i.e. it is invariant under the conformal changes of the metric. In a more general context, in [34, 41] and [36], using variational techniques, it was shown that for a coformally covariant self adjoint differential operator A, $\eta_A(0)$ is a conformal invariant.

In the framework of noncommutative geometry, conformal perturbation of the metric is implemented by changing the volume form [10], namely, by fixing a positive element $k = e^h$ for $h^* = h$ in $C^\infty(\mathbb{T}^n_\theta)$, one constructs the following positive functional

$$\varphi_k(a) = \tau(ak^{-n}).$$

By normalizing the above functional one obtains a state which we denote by φ.
The state φ defines an inner product

$$\langle a, b\rangle_\varphi = \varphi(b^*a) \quad a, b \in C^\infty(\mathbb{T}^n_\theta)$$

and hence one obtains a Hilbert space $\mathcal{H}_\varphi$ by GNS construction. The algebra $C^\infty(\mathbb{T}^n_\theta)$ acts unitarilly on $\mathcal{H}_\varphi$ by left regular representation and the right multiplication operator

$R_{k^{n/2}}$ extends to a unitary map $U_0 : \mathcal{H}_\tau \longrightarrow \mathcal{H}_\varphi$. In fact one has,

$$\langle U_0 a, U_0 b\rangle_\varphi = \varphi(k^{n/2} b^* a k^{n/2}) = \tau(k^{n/2} b^* a k^{n/2} k^{-n}) = \tau(b^* a) = \langle a, b\rangle_\tau \,.$$

We put $\tilde{\mathcal{H}} = H_\varphi \otimes \mathbb{C}^N$, the action of $C^\infty(\mathbb{T}^n_\theta)$ on $\tilde{\mathcal{H}}$ is given by

$$a \longrightarrow a \otimes 1$$

and the map

$$U = U_0 \otimes I : \mathcal{H} \longrightarrow \tilde{\mathcal{H}}$$

is a unitary equivalence between the two Hilbert spaces.
Now we consider the operator $\tilde{D} = R_{k^{\frac{n+1}{2}}} D R_{k^{\frac{-n+1}{2}}}$.

Proposition 3.4.6. *$(C^\infty(\mathbb{T}^n_\theta), \tilde{\mathcal{H}}, \tilde{D})$ and $(C^\infty(\mathbb{T}^n_\theta), \mathcal{H}, R_{\sqrt{k}} D R_{\sqrt{k}})$ are spectral triples, and the map U is a unitary equivalence between them.*

Proof. Note that left multiplication by an element $a \in C^\infty(\mathbb{T}^n_\theta)$ commutes with right multiplication operators R_{k^2}, $R_{k^{-1}}$ and $R_{\sqrt{k}}$. Also the two norms coming from $<,>_\varphi$ and $<,>_\tau$ are equivalent. So both commutators $[a, \tilde{D}]$ and $[a, \sqrt{k} D \sqrt{k}]$ are bounded. The unitary equivalence easily follows from definition of U and $\tilde{D}$. $\square$

In next step, we convert the right multiplications in $R_{\sqrt{k}} D R_{\sqrt{k}}$ to left multiplication using the real structure on noncommutative tori (see example 3.2.3) . It is easily seen that

$$J_0 R_{\sqrt{k}} \partial_\mu R_{\sqrt{k}} J_0 = -\sqrt{k} \partial_\mu \sqrt{k},$$

and

$$J R_{\sqrt{k}} D R_{\sqrt{k}} J = \sqrt{k} D \sqrt{k}.$$

Since $\sqrt{k} D \sqrt{k}$ is iso-spectral to $R_{\sqrt{k}} D R_{\sqrt{k}}$ (being intertwined by J), the following definition is reasonable.

Definition 3.4.7. *The conformal perturbation of the Dirac operator D is the 1-parameter family $(C^\infty(\mathbb{T}^n_\theta), \mathcal{H}, D_t)$, where*

$$D_t = e^{\frac{th}{2}} D e^{\frac{th}{2}}, \quad h = h^* \in C^\infty(\mathbb{T}^n_\theta).$$

We need the following formula for variation of eta function.

Lemma 3.4.8. *Let $\{D_t\}$ be a smooth 1-parameter family of invertible self-adjoint elliptic operators of order d, then*

$$\frac{d}{dt}\eta_{D_t}(z) = -z\mathrm{TR}(\dot{D}_t(D_t^2)^{-(\frac{z+1}{2})}).$$

Proof. For $k > 0$ odd, $\eta_{D^k}(z) = \eta_D(kz)$, hence we can replace D by D^k for k large enough.
For d large enough, $(D-\lambda)^{-1}$ is trace class, so one can write

$$\eta_{D_t}(z) = \frac{1}{2\pi i}\int_{\Gamma_1} \lambda^{-z}\mathrm{Tr}(D_t-\lambda)^{-1}d\lambda - \int_{\Gamma_2}(-\lambda)^{-z}\mathrm{Tr}(D_t-\lambda)^{-1}d\lambda,$$

where Γ_1 and Γ_2 are appropriate contours around positive and negative eigenvalues of D.

Now,

$$\frac{d}{dt}(D_t-\lambda)^{-1} = -(D_t-\lambda)^{-1}\dot{D}_t(D_t-\lambda)^{-1},$$

so

$$\mathrm{Tr}(\frac{d}{dt}(D_t-\lambda)^{-1}) = -\mathrm{Tr}(\dot{D}_t(D_t-\lambda)^{-2}),$$

therefore

$$\frac{d}{dt}\eta_{D_t}(z) = \frac{1}{2\pi i}\mathrm{Tr}(\dot{D}_t(\int_{\Gamma_1} -\lambda^{-s}\mathrm{Tr}(D_t-\lambda)^{-2}d\lambda + \int_{\Gamma_2}(-\lambda)^{-z}\mathrm{Tr}(D_t-\lambda)^{-2}d\lambda)).$$

Now, integration by parts in both integrals and the formula $\frac{d}{d\lambda}(D_t-\lambda)^{-1} = (D_t-\lambda)^{-2}$ gives the result.

□

Remark 3.4.9. By applying the above lemma to the Dirac operator D on noncommutative 3-torus we get,

$$\begin{aligned}\partial\eta_D(0) = \left[\frac{d}{dt}\Big|_{t=0}\eta_{D_t}(z)\right]_{z=0} &= \left[-z\mathrm{TR}(\partial D|D|^{-1}(D^2)^{\frac{-z}{2}})\right]_{z=0}\\ &= -\mathrm{Wres}(\partial D|D|^{-1}).\end{aligned}$$

By the properties of Wodzicki residue, one sees that $\eta(0)$ is constant under smoothing perturbations, so if $\ker(D)\neq 0$, on can replace D by $D+\Pi$ where Π is the projection on the finite dimensional kernel of D and hence the result of above lemma still makes sense for an operator with nontrivial kernel.

Proposition 3.4.10. *Consider the spectral triple* $(\mathrm{C}^\infty(\mathbb{T}^3_\theta), \mathcal{H}, D = \partial\!\!\!/)$, *then* $\eta_D(0)$ *is invariant under the conformal perturbations.*

Proof. Consider the 1-parameter family of operators

$$D_t = e^{\frac{th}{2}} D e^{\frac{th}{2}}, \quad h = h^* \in \mathrm{C}^\infty(\mathbb{T}^3_\theta).$$

One computes

$$\dot{D}_t = \frac{1}{2}(hD_t + D_t h),$$

and

$$\partial D = \frac{1}{2}(hD + Dh).$$

Therefore by trace property of noncommutative residue we get

$$\partial\eta(D, 0) = -\mathrm{Wres}(hD|D|^{-1}),$$

and it is seen that

$$\mathrm{Wres}(hD|D|^{-1}) = \tau\left(\mathrm{tr}(\mathrm{res}(hD|D|^{-1})\right) = \tau\left(\mathrm{tr}(h\ \mathrm{res}(D|D|^{-1})\right).$$

Now by looking at the symbols

$$\sigma(D) \sim \xi_\mu \otimes \gamma^\mu,$$
$$\sigma(|D|) \sim |\xi|^2 \otimes I,$$

we get,

$$\sigma(D|D|^{-1}) \sim \frac{\xi_\mu}{|\xi|} \otimes \gamma^\mu.$$

Therefore, there is no homogeneous term of negative order in the symbol of $\sigma(D|D|^{-1})$ and hence,

$$\mathrm{Wres}(hD|D|^{-1}) = 0.$$

□

3.4.3 Spectral flow and odd local index formula

The value of eta function at zero is intimately related to another spectral quantity called the spectral flow. To motivate this relation, consider a family A_t of $N \times N$ Hermitian matrices. The spectral flow of the family A_t is defined as the net number of the eigenvalues of A_t passing zero. It is easily seen that the difference of the signatures of the end points is related to the spectral flow of the family by,

$$\eta_{A_1}(0) - \eta_{A_0}(0) = 2\mathrm{SF}(A_t).$$

The spectral flow of a family of self adjoint elliptic operators A_t on manifolds can also be defined ([1]) but the above equality holds along with a correction term (see [7] for a proof).

Proposition 3.4.11. *Let A_t be a self-adjoint family of elliptic operators of order a with A_0 and A_1 invertible, then*

$$\eta_{A_1}(0) - \eta_{A_0}(0) = 2\mathrm{SF}(A_t) - \frac{1}{a}\int_0^1 \frac{d}{dt}\eta_{A_t}(0)dt. \tag{3.14}$$

□

It's a remarkable fact due to Getzler ([25]) that the spectral flow of the family of operators $D_t = D + tg^{-1}[D, g]$ interpolating D and $g^{-1}Dg$ for the Dirac operator D on an odd dimensional spin manifold and $g \in C^\infty(M, GL(N))$ in fact gives the index of PgP where $P = \frac{1+F}{2}$ and $F = D|D|^{-1}$ is the sign operator. Therefore one obtains a pairing between the odd K-theory and K-homology:

$$\mathrm{SF}(D_t) = \mathrm{index}(PgP). \tag{3.15}$$

The above equality has been generalized to the framework of non commutative index theory [8, 9]. Consider a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with sufficiently nice regularity properties and let $u \in \mathcal{A}$ be a unitary element and $[u] \in K^1(A)$ the corresponding class in K-theory. The sign operator $F = \frac{D}{|D|}$ gives the projection $P = \frac{1+F}{2}$ and PuP is a Fredholm operator. For the family $D_t = D + tu^*[D, u]$ interpolating between D and u^*Du one can show that(see [8] and [9]):

$$\mathrm{SF}(D_t) = \mathrm{index}(PuP). \tag{3.16}$$

The odd local index formula of Connes-Moscovici in turn expresses the right side of the above equality as a pairing between the periodic cyclic cohomolgy and K-theory given by a local formula in the sense of non commutative geometry [13]. Note that for the spectral triple $(C^\infty(\mathbb{T}^3_\theta), \mathcal{H}, D = \not{\partial})$, by Proposition (3.4.5), the eta function $\eta_{D_t}(0)$ over the family of Dirac operators $D_t = D + tu^*[D, u]$ makes sense and with a minor modification of the proof of Proposition (3.4.11) in the commutative case we get,

$$\eta_{u^*Du}(0) - \eta_D(0) = 2\mathrm{SF}(D_t) - \int_0^1 \frac{d}{dt}\eta_{D_t}(0)dt. \tag{3.17}$$

Since u^*Du and D have the same spectrum, we have $\eta_{u^*Du}(0) = \eta_D(0)$ and therefore we obtain obtain the following integrated local formula for the index:

$$\mathrm{index}(PuP) = \frac{1}{2}\int_0^1 \frac{d}{dt}\eta_{D_t}(0)dt. \tag{3.18}$$

3.5 Bosonic functional determinant and conformal anomaly

Consider a positive elliptic differential operator A on a closed manifold M, the corresponding bosonic action is defined by

$$S_{\mathrm{bos}}(\phi) = (\phi, A\psi),$$

where $\phi : M \to \mathbb{C}^d$ is a bosonic field living on M. Upon quantization, one constructs the partition function

$$\mathcal{Z}_{\mathrm{bos}} = \int e^{\frac{-1}{2}S_{bos}(\phi)}[\mathcal{D}\phi],$$

where $[\mathcal{D}\phi]$ is a formal measure on the configuration of bosonic fields. The corresponding effective action is given by

$$\mathcal{W} = \log \mathcal{Z}.$$

One should think of $\mathcal{Z}_{\mathrm{bos}}$ formally as $(\det A)^{-\frac{1}{2}}$. Through zeta regularization scheme (see e.g. [6]) one defines,

$$\mathcal{Z} := e^{\frac{1}{2}\zeta'_A(0)} \tag{3.19}$$

and hence,

$$\mathcal{W} = \frac{1}{2}\zeta'_A(0), \tag{3.20}$$

where the spectral zeta function is defined by

$$\zeta_A(z) = \mathrm{TR}(A^{-z}),$$

and we have dropped the dependence on the spectral cut L_θ in definition of the complex power. Note that since A is a differential operator, by Proposition (3.3.17), $\zeta_A(z)$ is regular at $z = 0$ and $\zeta'_A(0)$ makes sense.

Let D be the Dirac operator on a closed odd dimensional spin manifold, it's a known fact that $\zeta'_{|D|}(0)$ is a conformal invariant (see [41] and [34]). We prove the analogue of this result for the Dirac operator on noncommutative n-torus $(C^\infty(\mathbb{T}^n_\theta), \mathcal{H}, D = \partial\!\!\!/)$ in odd dimensions.

In even dimensions the conformal variation is not zero and hence conformal quantum anomaly exists. In dimension two, the Polyakov anomaly formula [38] gives a local expression for conformal anomaly of the Laplacian on a Riemann surface. Following [6] we derive an analogue of this formula for noncommutative 2-torus using the formalism of canonical trace.

For proving the conformal invariance of $\zeta'_{|D|}(0)$ on $C^\infty(\mathbb{T}^{2p+1}_\theta)$ we will need to carefully analyze the coefficients in short time asymptotic of the trace of heat operator e^{-tD^2}.

Lemma 3.5.1. *Consider the spectral triple $(C^\infty(\mathbb{T}^n_\theta), \mathcal{H}, D)$. One has the following asymptotic expansion*

$$\mathrm{Tr}(e^{-tD^2}) \sim \sum_{i=0}^{\infty} a_i(D^2) t^{\frac{i-n}{2}} \quad t \to 0,$$

where coefficients a_i are computed by integrating local terms.

Proof. We have,

$$\mathrm{Tr}(e^{-tD^2}) = \tau\Big(\int_{\mathbb{R}^n} \mathrm{tr}\sigma(e^{-tD^2})(\xi)d\xi\Big).$$

The operator e^{-tD^2} is again a pseudodifferential operator of order $-\infty$ whose symbol can be computed using the symbol product rules. For any λ in the resolvent of D^2, $(\lambda - D^2)^{-1}$ is a pseudodifferential operator of order -2 with symbol $r(\xi, \lambda)$ which can be written as its homogeneous parts $r = r_0 + r_1 + \cdots$, with $r_k(\xi, \lambda)$ homogeneous of order $-k-2$ in (ξ, λ) i.e.

$$r_k(t\xi, t^2\lambda) = t^{-2-k} r_k(\xi, \lambda) \quad \forall t > 0 \tag{3.21}$$

The operator e^{-tD^2} is also a pseudodifferential operator with the symbol $e_0+e_1+e_2+\cdots$ where $e_n \in S^{-\infty}$ and are defined by

$$e_n(t,\xi) = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda} r_n(\xi,\lambda)d\lambda \quad t>0 \tag{3.22}$$

$$\int_{\mathbb{R}^n} \sigma(e^{-tD^2})(t,\xi)d\xi$$

$$= \sum_i \int e_i(t,\xi)d\xi$$

The integrands $\int_{\mathbb{R}^n} e_i(t,\xi)d\xi$ are homogeneous in ξ.

$$\begin{aligned}\int_{\mathbb{R}^n} e_n(t,\xi)d\xi &= \frac{1}{2\pi i}\int_{\mathbb{R}^n}\int_\gamma e^{-t\lambda} r_n(x,\xi,\lambda)d\lambda d\xi \\ &= \frac{1}{2\pi i}\int_{\mathbb{R}^n}\int_\gamma e^{-\lambda'} r_n(t^{-1/2}\xi', t^{-1}\lambda')\frac{d\lambda'}{t}\frac{d\xi'}{t^{n/2}} \\ &= t^{\left(\frac{i-n}{2}\right)}\frac{1}{2\pi i}\int_{\mathbb{R}^n}\int_\gamma e^{-\lambda} r_n(\xi,\lambda)d\lambda d\xi\end{aligned}$$

In identity second identity we have used the change of variable $t\lambda = \lambda'$ and $t^{1/2}\xi = \xi'$.

Hence we have

$$\int_{\mathbb{R}^n} \sigma(e^{-tD^2})(\xi)d\xi \sim \sum_{i=0} \beta_i t^{\frac{i-n}{2}}, \tag{3.23}$$

Where

$$\beta_i(x) = \frac{1}{2\pi i}\int_{\mathbb{R}^n}\int_\gamma e^{-\lambda} r_i(\xi,\lambda)d\lambda d\xi \tag{3.24}$$

and hence

$$\mathrm{Tr}(e^{-tD^2}) \sim \sum_{i=0}^{\infty} t^{\frac{i-n}{2}} a_i(D^2), \quad t\to 0,$$

Where $a_i(D^2) = \tau(\mathrm{tr}(\beta_i))$ □

Remark 3.5.2. The above asymptotic expansion holds for any self-adjoint elliptic differnetial operator $A \in \Psi^*_{cl}(C^\infty(\mathbb{T}^n_\theta))$. In fact, one has

$$\mathrm{Tr}(e^{-tA^2}) \sim \sum_{i=0}^{\infty} a_i(A^2) t^{\frac{i-n}{\alpha}} \quad t \to 0,$$

where $\alpha = ord(A)$.

Remark 3.5.3. One also has the following asymptotic expansion

$$\mathrm{Tr}(ae^{-tD^2}) \sim \sum_{i=0}^{\infty} a_i(D^2, a) t^{\frac{i-n}{2}} \quad t \to 0,$$

Where $a \in C^\infty(\mathbb{T}^n_\theta)$ is considered as a multiplication operator and

$$a_i(D^2, a) = \tau(\mathrm{tr}(a\beta_i))$$

in above computations.

Next, we have the following variational result.

Lemma 3.5.4. *Let A_t be a 1-parameter family of positive elliptic differential operators of fixed order α on noncommutaive n-torus, then*

$$\frac{d}{dt}\zeta_{A_t}(z) = -z\mathrm{TR}(\dot{A}_t(A_t)^{-z-1}).$$

Proof. By using the contour integral formula for the complex powers we have,

$$\begin{aligned}\frac{d}{dt}\mathrm{TR}(A_t^{-z}) &= \frac{1}{2\pi i}\int_\Gamma \lambda^{-z}\frac{d}{dt}\mathrm{TR}(\lambda - A_t)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma \lambda^{-z}\mathrm{TR}(\dot{A_t}(\lambda - A_t)^{-2})d\lambda \\ &= \mathrm{TR}\left(\dot{A}_t\frac{1}{2\pi i}\int_\Gamma \lambda^{-z}(\lambda - A_t)^{-2}d\lambda\right).\end{aligned}$$

Now using integration by parts formula gives the result. □

Remark 3.5.5. Note that by above lemma we have:

$$\frac{d}{dt}\zeta_{A_t}(0) = \Big[-z\mathrm{TR}(\dot{A}_t A^{-1}A^{-z})\Big]_{z=0} = -\mathrm{Wres}(\dot{A}_t A^{-1}), \tag{3.25}$$

and therefore, the value of zeta at zero is constant under smoothing perturbations and hence the above result makes sense for non invertible operators.

We also need the following result relating the constant term in Laurent expansion of $\mathrm{TR}(AQ^{-z})$ around $z=0$ to the constant term in the asymptotic expansion $\mathrm{Tr}(Ae^{-tQ})$ at $t=0$. Let f.p. $\mathrm{TR}(AQ^{-z})|_{z=0}$ denote the constant term in the Laurent expansion and also f.p. $\mathrm{Tr}(Ae^{-tQ})\big|_{t=0}$ denote the constant term in the heat trace asymptotic near zero. We refer the reader to [3] for a proof of the following result in a more general form in commutative case.

Lemma 3.5.6. *Consider the differential operators $A, Q \in \Psi^*_{cl}(C^\infty(\mathbb{T}^n_\theta))$, where Q is positive with positive order q. Then,*

$$\text{f.p. } \mathrm{TR}(AQ^{-z})\big|_{z=0} = \text{f.p. } \mathrm{Tr}(Ae^{-tQ})\big|_{t=0}. \tag{3.26}$$

□

Proposition 3.5.7. *Consider the spectral triple $(C^\infty(\mathbb{T}^{2p+1}_\theta), \mathcal{H}, D = \partial\!\!\!/)$. The value $\zeta'_{|D|}(0)$ is a conformal invariant.*

Proof. It is enough to show that $\zeta'_\Delta(0)$ for $\Delta = D^2$ is conformally invariant. By the Lemma (3.5.4) we have,

$$\partial\zeta_\Delta(z) = -z\mathrm{TR}(\partial\Delta \cdot \Delta^{-1}\Delta^{-z}),$$

so we have the following Laurent expansion around $z=0$:

$$\partial\zeta_\Delta(z) = -z\left(a_{-1}\frac{1}{z} + a_0 + a_1 z + \cdots\right).$$

Therefore

$$\partial\zeta'_\Delta(0) = \frac{d}{dz}\,\partial\zeta_\Delta(z)|_{z=0} = -a_0 = -\tau\left(⨍\sigma(\partial\Delta\Delta^{-1}) - \frac{1}{2}res(\partial\Delta\Delta^{-1}\log\Delta)\right). \tag{3.27}$$

Consider the conformal perturbation of the Dirac operator,

$$D_t = e^{\frac{th}{2}}De^{\frac{th}{2}} \qquad h = h^* \in \mathcal{A}_\theta.$$

An easy computation gives

$$\partial\Delta = \frac{d}{dt}D_t^2\Big|_{t=0} = \frac{h}{2}D^2 + DhD + \frac{1}{2}D^2h,$$

and therefore by inserting the above identity into (3.27) we get

$$\partial\zeta'_\Delta(0) = -2\tau\left(⨍\sigma(h) - \frac{1}{2}\mathrm{res}(h\log\Delta)\right) = -2\text{f.p. }\mathrm{TR}(h\Delta^{-z})\big|_{z=0}. \tag{3.28}$$

Now using the Lemma (3.5.6) we have

$$\partial\zeta'_\Delta(0) = -2\text{f.p. }\mathrm{Tr}(he^{-t\Delta})\big|_{t=0} = -2a_{2p+1}(h,\Delta) = -2\tau(\mathrm{tr}(h\beta_{2p+1})). \tag{3.29}$$

By examining the proof of Lemma (3.5.1), we see that $a_n = 0$ for odd n since the integrand involved is an odd function, hence the result is obtained. □

In the following we give an analogue of Polyakov anomaly formula [38] for the spactral triple $(C^\infty(C^\infty(\mathbb{T}_\theta^2), \mathcal{H}, \not{\partial})$. Note that although $\log\det(\Delta) = -\zeta'_\Delta(0)$ is not local[1] (see Proposition (3.3.17)), the difference between $\log\det_{\Delta_h}$ of the conformally perturbed Laplacian and $\log\det_\Delta$ can be given by a local formula. This is of course an example of the local nature of anomalies in quantum field theory. Here we only express this difference as an integrated anomaly and refer the reader to [6] for further computations and interpretation of the formula.

Proposition 3.5.8. (A conformal anomaly formula) *Consider the spectral triple* $(C^\infty(\mathbb{T}_\theta^2), \mathcal{H}, D = \not{\partial})$, $\Delta = D^2$ *and* $\Delta_h = D_h^2$ *where* $D = e^{\frac{h}{2}}De^{\frac{h}{2}}$. *The difference between* $\log\det_{\Delta_h}$ *of the conformally perturbed Laplacian and* $\log\det_\Delta$ *can be given by a local formula:*

$$\log\det(\Delta_h) - \log\det(\Delta) = -\left(\zeta'_{\Delta_h}(0) - \zeta'_\Delta(0)\right) = -\int_0^1 \frac{d}{dt}\zeta'_{\Delta_t}(0)dt, \tag{3.30}$$

where $\Delta_t = D_t^2$, $D = e^{\frac{th}{2}}De^{\frac{th}{2}}$.

Proof. The proof follows from the Lemma (3.5.4) and fundamental theorem of calculus along the family Δ_t. □

[1] Roughly, it means that it can not be written as an integral of finite number of homogenous terms of the symbol of Δ.

Remark 3.5.9. Note that, by using the similar argument used in derivation of the equations (3.27) and (3.29) we see that the integrand in equation (3.30) is given by

$$\frac{d}{dt}\zeta'_{\Delta_t}(0) = -\tau\left(⨍\sigma(\frac{d}{dt}\Delta_t\Delta_t^{-1}) - \frac{1}{2}\mathrm{res}(\frac{d}{dt}\Delta_t\Delta_t^{-1}\log\Delta_t)\right) =$$
$$-2a_2(h,\Delta_t) = -2\tau\left(\mathrm{tr}(h\beta_2(\Delta_t))\right).$$

of course, computing the density $\mathrm{tr}(\beta_2(\Delta_t))$ requires considerable amount of computations (see [6] and [11]).

3.6 Fermionic functional determinant and induced chern-Simons term

Consider a closed manifold M and the classical fermionic action defined by

$$S_{\mathrm{fer}}(\bar{\psi},\psi) = (\bar{\psi}, D\psi),$$

where $\bar{\psi}$ and ψ are fermion fields on M and $D = i\gamma^\mu\nabla_\mu$ is the Dirac operator . The fermionic partition function is given by

$$\mathcal{Z}_{\mathrm{fer}} = \int e^{-S_{\mathrm{fer}}(\bar{\psi},\psi)}[\mathcal{D}\bar{\psi}][\mathcal{D}\psi],$$

where $[\mathcal{D}\bar{\psi}]$ and $[\mathcal{D}\psi]$ are formal measures on the space of fermions. The partition function $\mathcal{Z}$ should be thought as the $\det(D)$. The one-loop fermionic effective action is defined as

$$\mathcal{W} := \log(\mathcal{Z}) = \log\det(D).$$

Similar to bosonic case, after choosing a spectral cut L_ϕ [2], by definition we have (see (4.10)):

$$\zeta_D(z) = TR(D_\phi^{-z}),$$

and hence one can define

$$\mathcal{W} = \log\det(D) := -\zeta'_D(0). \tag{3.31}$$

[2] There are two choices of spectral cut L_ϕ for the Dirac operator: in upper or lower half plane

Note that due to choice of a spectral cut, there is an ambiguity involved in above definition of one-loop effective action. Let $\zeta_D^{\uparrow}(z)$ and $\zeta_D^{\downarrow}(z)$ be the spectral zeta functions corresponding to a choice of ϕ in upper and lower half plane respectively. a quick computation shows that (cf. [39])

$$\zeta_D^{\uparrow}(z) - \zeta_D^{\downarrow}(z) = (1 - e^{-i\pi z})\zeta_D^{\uparrow}(z) - (1 - e^{-i\pi z})\eta_D(z),$$

and the measure of ambiguity in the effective action is given by

$$\zeta_D^{\uparrow}(0)' - \zeta_D^{\downarrow}(0)' = i\pi\zeta_D^{\uparrow}(0) + i\pi\eta_D(0). \tag{3.32}$$

In $2p+1$ dimensions $\zeta_D^{\uparrow}(0) = \zeta_D^{\downarrow}(0) = 0$ and hence

$$\zeta_D^{\uparrow}(0)' - \zeta_D^{\downarrow}(0)' = i\pi\eta_D(0).$$

Therefore the ambiguity is given by the non local quantity $\eta_D(0)$ which also depends on the gauge field coupled to the Dirac operator. The measure of this dependence can be given by a local formula which in physics literature is referred to as the induced Chern-Simons term generated by the coupling of a massless fermion to a classical gauge field(see [6]).

Here, we give an analogue of this local term for the coupled Dirac operator on noncommutative 3-torus. We consider the operator $D = \partial\!\!\!/ + A\!\!\!/$ on $C^\infty(\mathbb{T}_\theta^3)$ and compute the variation of the eta invariant $\eta_D(0)$ with respect to the vector potential.

First we state the following lemma. The proof in commutative case also works in noncommutative setting with minor changes and we will not reproduce it here (see [36]).

Lemma 3.6.1. *In the asymptotic expansion of the heat kernel for a positive elliptic differential operator $A \in \Psi_{cl}^*(C^\infty(\mathbb{T}_\theta^n))$ of $ord(A) = \alpha$,*

$$\int_{\mathbb{R}^n} \sigma(e^{-tA})(\xi) d\xi \sim \sum_{i=0}^{\infty} \beta_i t^{\frac{i-n}{\alpha}},$$

one has

$$\mathrm{res}(A^{-k}) = \frac{\alpha}{(k-1)!}\beta_{n-\alpha k},$$

where $k \in \mathbb{Z}^+$ and $\beta_{n-\alpha k} = 0$ if $\alpha k \notin \mathbb{Z}$.

□

Consider the family $D_t = \partial\!\!\!/ + A\!\!\!/_t$ on $C^\infty(\mathbb{T}^3_\theta)$. By performing the variation with respect to A_μ we have:

$$\partial D = \gamma^\mu \partial A_\mu. \tag{3.33}$$

For the family pf Dirac operators $\{D_t\}$, a proof similar to the proof of Proposition (3.4.5) shows that $\eta_{D_t}(z)$ is regular at $z = 0$ and therefore along this family, $\eta_{D_t}(0)$ makes sense. By using Lemma (3.4.8) we obtain:

$$\partial\eta(0) = \mathrm{Wres}\left(\partial D|D|^{-1}\right) = \mathrm{Wres}\left(\gamma^\mu \partial A_\mu |D|^{-1}\right). \tag{3.34}$$

Now by using Lemma (3.6.1) and the fact that $\gamma^\mu \partial A_\mu$ is a zero order operator it follows that the variation of the eta invariant with respect to the gauge field A_μ is given by the following local formula:

$$\partial\eta_D(0) = \mathrm{Wres}\left(\gamma^\mu \partial A_\mu |D|^{-1}\right) = \tau\left(\mathrm{tr}\left(\gamma^\mu \partial A_\mu \beta_2\right)\right). \tag{3.35}$$

Bibliography

[1] M.F. Atiyah, V.K. Patodi, I.M. Singer. Spectral asymmetry and Riemannian geometry.I. *Math. Proc. Cambridge Philos. Soc.*, Volume77:43–69, 1975.

[2] M.F. Atiyah, V.K. Patodi, I.M. Singer. Spectral asymmetry and Riemannian geometry.III. *Math. Proc. Cambridge Philos. Soc.*, Volume79:71–99, 1976.

[3] N. Berline, E. Getzler, M. Vergne. Heat kernels and Dirac operators. *Springer-Verlag, Grundlehren der Mathematischen Wissenschaften* , 1992.

[4] T. A. Bhuyain and M. Marcolli. The Ricci flow on noncommutative two-tori. *Lett. Math. Phys.*, 101(2):173–194, 2012.

[5] J.M. Bismut, D. Freed. The analysis of elliptic families. *Comm. Math. Phys.*, 107: 103-163, 1986.

[6] A.A. Bytsenko, G. Cognola, E. Elizalde, Moretti. Analytic aspects of quantum fields. *World Scientific Publishing Co., Inc., River Edge, NJ*, 2003.

[7] A. Cardona, C. Ducourtioux, S. Paycha . From Tracial Anomalies to Anomalies in Quantum Field Theory. *Comm. Math. Phys.*, 242: 31–65, 2003.

[8] A. Carey, J. Phillips. Unbounded Fredholm modules and spectral flow. *Canad. J. Math.*, 50(4):673–718, 1998.

[9] A. Carey, J. Phillips. Spectral flow in Fredholm modules, eta invariants and the JLO cocycle. *K-Theory*, 31(2):135–194, 2004.

[10] P. Cohen, A. Connes. Conformal geometry of the irrational rotation algebra. Preprint of the MPI, Bonn. no. MPI/92-23, 1992.

[11] A. Connes. C^* algèbres et géométrie différentielle. *C. R. Acad. Sci. Paris Sér. A-B*, 290(13):A599–A604, 1980.

[12] A. Connes. *Noncommutative geometry.* Academic Press, Inc., San Diego, CA, 1994.

[13] A. Connes, H. Moscovici. The local index formula in noncommutative geometry. *Geom. Funct. Anal.*, 5(2):174–243, 1995.

[14] A. Connes, H. Moscovici. Type III and spectral triples. In *Traces in number theory, geometry and quantum fields.* 57-71, Aspects Math., E38, Vieweg, 2008.

[15] A. Connes, H. Moscovici. Modular curvature for noncommutative two-tori. *J. Amer. Math. Soc.*, 27(3):639–684, 2014.

[16] A. Connes, P. Tretkoff. The Gauss-Bonnet theorem for the noncommutative two torus. In *Noncommutative geometry, arithmetic, and related topics*, pages 141–158. Johns Hopkins Univ. Press, Baltimore, MD, 2011.

[17] L. Dabrowski, A. Sitarz, *Curved noncommutative torus and Gauss–Bonnet,* J. Math. Phys. 54, 013518 (2013).

[18] L. Dabrowski, A. Sitarz. Asymmetric noncommutative torus. arXiv:1406.4645, 2014.

[19] A. Fathi, A. Ghorbanpour, M. Khalkhali. The Curvature of the Determinant Line Bundle on the Noncommutative Two Torus. *ArXiv e-prints*, Arxiv:1410.0475, 2014.

[20] F. Fathizadeh, M. Khalkhali. The Gauss-Bonnet theorem for noncommutative two tori with a general conformal structure. *J. Noncommut. Geom.*, 6(3):457–480, 2012.

[21] F. Fathizadeh, M. Khalkhali. Scalar curvature for the noncommutative two torus. *J. Noncommut. Geom.*, 7(4):1145–1183, 2013.

[22] F. Fathizadeh, M. Khalkhali. Scalar curvature for noncommutative four-tori. *J. Noncommut. Geom.*, To appear.

[23] F. Fathizadeh, M. Wong. Noncommutative residues for pseudo-differential operators on the noncommutative two-torus. *Journal of Pseudo-Differential Operators and Applications*, 2(3):289–302, 2011.

[24] G.B. Folland. *Quantum Field Theory: A Tourist Guide for Mathematicians.* AMS, Mathematical Surveys and Monographs, 2008.

[25] E. Getzler. The odd Chern character in cyclic homology and spectral flow. *Topology,* 32(3):489–507, 1993.

[26] P. B. Gilkey. The residue of the local eta function at the origin. *Math. Ann.*, 240(2):183–189, 1979.

[27] P. B. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.* Volume 11 of *Mathematics Lecture Series.* Publish or Perish Inc., Wilmington, DE, 1984.

[28] J.M. Gracia-Bondía, J. Várilly, H. Figueroa. *Elements of noncommutative geometry.* Birkhäuser Advanced Texts: Basler Lehrbücher, 2001.

[29] N. Hitchin. Harmonic Spinors. *Advances in Mathematics*, 14(1):1 - 55, 1974.

[30] M. Kontsevich, S. Vishik. Geometry of determinants of elliptic operators. In *Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993)*, volume 131 of *Progr. Math.*, pages 173–197. Birkhäuser Boston, Boston, MA, 1995.

[31] M. Khalkhali and A. Moatadelro. A Riemann-Roch theorem for the noncommutative two torus. *Journal of Geometry and Physics*, 86(0):19 – 30, 2014.

[32] M. Lesch. On the noncommutative residue for pseudodifferential operators with log-polyhomogeneous symbols. *Ann. Global Anal. Geom.*, 17(2):151–187, 1999.

[33] C. Levy, C. N. Jimnez, S. Paycha. The canonical trace and the noncommutative residue on the noncommutative torus. ArXiv:1303.0241 [math.AP], 2013.

[34] T. Parker, S. Rosenberg. Invariants of conformal Laplacians. *J. Differential Geom.*, 25:199–222, 1987.

[35] S. Paycha. *Regularised integrals, sums and traces.* Volume 59, *University Lecture Series.* American Mathematical Society, Providence, RI, 2012.

[36] S. Paycha, S. Rosenberg. Conformal anomalies via canonical traces. In *Analysis, geometry and topology of elliptic operators*, pages 263–294. World Sci. Publ., Hackensack, NJ, 2006.

[37] S. Paycha, S. Scott. A Laurent expansion for regularized integrals of holomorphic symbols. *Geom. Funct. Anal.*, 17(2):491–536, 2007.

[38] S. Polyakov. Quantum geometry of bosonic strings. *Phys. Lett. B*, 103:207–210, 1981.

[39] R. Ponge. Spectral asymmetry, zeta functions, and the noncommutative residue. *Internat. J. Math*, 17(9): 1065–1090, 2006.

[40] J. Rosenberg. Levi-Civita's theorem for noncommutative tori. SIGMA 9 (2013), 071, 9 pages.

[41] S. Rosenberg. The Determinant of a Conformally Covariant Operator. *Journal of the London Mathematical Society*,Volumes 2-36(3):553-568, 1987.

[42] S. Scott. *Traces and determinants of pseudodifferential operators.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2010.

[43] M. Wodzicki. Noncommutative residue. I. Fundamentals. K-theory, Arithmetic and Geometry (Moscow, 1984–1986), Lecture Notes in Math., 1289, Springer, Berlin, 1987, p. 320-399.

Chapter 4

Curvature of the Determinant Line Bundle for the Noncommutative Two Torus

4.1 Introduction

In this paper we compute the curvature of the determinant line bundle associated to a family of Dirac operators on the noncommutative two torus. Following Quillen's pioneering work [23], and using zeta regularized determinants, one can endow the determinant line bundle over the space of Dirac operators on the noncommutative two torus with a natural Hermitian metric. Our result computes the curvature of the associated Chern connection on this holomorphic line bundle. In the noncommutative case the method of proof applied in [23] does not work and we had to use a different strategy. To this end we found it very useful to extend the canonical trace of Kontsevich-Vishik [16] to the algebra of pseudodifferential operators on the noncommutative two torus.

This paper is organized as follows. In Section 2 we review some standard facts about Quillen's determinant line bundle on the space of Fredholm operators from [23], and about noncommutative two torus that we need in this paper. In Section 3 we develop the tools that are needed in our computation of the curvature of the determinant line bundle in the noncommutative case. We recall Connes' pseudodifferential calculus and define an analogue of the Kontsevich-Vishik trace for classical pseudodifferential symbols on the noncommutative torus. A similar construction of the canonical trace can be found in [20], where one works with the algebra of toroidal symbols. Section 4 is devoted to Cauchy-Riemann operators on $\mathcal{A}_\theta$ with a fixed complex structure. This is the family of elliptic operators that we want to study its determinant line bundle. In Section 5 using the calculus of symbols and the canonical trace we compute the curvature of determinant line bundle. Calculus of symbols and the canonical trace allow us to bypass local calculations involving Green functions in [23], which is not applicable in our noncommutative case.

The study of the conformal and complex geometry of the noncommutative two torus started with the seminal work of Connes and Tretkoff [7] (cf. also [5] for a preliminary version) where a Gauss-Bonnet theorem is proved for a noncommutative two torus equipped with a conformally perturbed metric. This result was refined and extended in [10] where the Gauss-Bonnet theorem was proved for metrics in all translation invariant conformal structures. The problem of computing the scalar curvature of the curved noncommutative two torus was fully settled in the work of Connes and Moscovici [6], and, independently, in [11], and in [12] for the four dimensional case. Other related works include [1, 8, 9, 15, 18].

4.2 Preliminaries

In this section we recall the definition of Quillen's determinant line bundle over the space of Fredholm operators. We also recall some basic notions about noncommutative torus that we need in this paper.

4.2.1 The determinant line bundle

Unless otherwise stated, in this paper by a Hilbert space we mean a separable infinite dimensional Hilbert space over the field of complex numbers. Let $\mathcal{F} = \mathrm{Fred}(\mathcal{H}_0, \mathcal{H}_1)$ denote the set of Fredholm operators between Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$. It is an open subset, with respect to norm topology, in the complex Banach space of all bounded linear operators between $\mathcal{H}_0$ and $\mathcal{H}_1$. The index map *index* : $\mathcal{F} \to \mathbb{Z}$ is a homotopy invariant and in fact defines a bijection between connected components of $\mathcal{F}$ and the set of integers $\mathbb{Z}$.

It is well known that $\mathcal{F}$ is a classifying space for K-theory: for any compact space X we have a natural ring isomorphism

$$K^0(X) = [X, \mathcal{F}]$$

between the K-theory of X and the set of homotopy classes of continuous maps from X to $\mathcal{F}$. In other words, homotopy classes of continuous families of Fredholm operators parametrized by X determine the K-theory of X. It thus follows that $\mathcal{F}$ is homotopy equivalent to $\mathbb{Z} \times BU$, the latter being also a classifying space for K-theory. Let $\mathcal{F}_0$ denote the set of Fredholm operators with index zero. By Bott periodicity, $\pi_{2j}(\mathcal{F}) \cong \mathbb{Z}$ and $\pi_{2j+1}(\mathcal{F}) = \{0\}$ for $j \geq 0$. So by Hurewicz's theorem, $H^2(\mathcal{F}_0, Z) \cong \mathbb{Z}$. Now the determinant line bundle DET defined below has the property that its first Chern class, $c_1(\mathrm{DET})$, is a generator of $H^2(\mathcal{F}_0, \mathbb{Z}) \cong \mathbb{Z}$. We refer to [2, 24] and references therein for details.

In [23] Quillen defines a line bundle $\mathrm{DET} \to \mathcal{F}$ such that for any $T \in \mathcal{F}$

$$\mathrm{DET}_T = \Lambda^{max}(\ker(T))^* \otimes \Lambda^{max}(\mathrm{coker}(T)).$$

This is remarkable if we notice that $\ker(T)$ and $\mathrm{coker}(T)$ are not vector bundles due to discontinuities in their dimensions as T varies within $\mathcal{F}$. Let us briefly recall the construction of this determinant line bundle DET. For each finite dimensional subspace F of $\mathcal{H}_1$ let $U_F = \{T \in \mathcal{F}_1 : \mathrm{Im}(T) + F = \mathcal{H}_1\}$ denote the set of Fredholm operators whose range is transversal to F. It is an open subset of $\mathcal{F}$ and we have an open cover $\mathcal{F} = \bigcup U_F$.

For $T \in U_F$, the exact sequence

$$0 \to \ker(T) \to T^{-1}F \xrightarrow{T} F \to \mathrm{coker}(T) \to 0 \tag{4.1}$$

shows that the rank of $T^{-1}F$ is constant when T varies within a continuous family in U_F. Thus we can define a vector bundle $\mathcal{E}^F \to U_F$ by setting $\mathcal{E}^F_T = T^{-1}F$. We can then define a line bundle $\mathrm{DET}^F \to U_F$ by setting

$$\mathrm{DET}^F_T = \Lambda^{max}(T^{-1}F)^* \otimes \Lambda^{max} F.$$

We can use the inner products on $\mathcal{H}_0$ and $\mathcal{H}_1$ to split the above exact sequence (4.1) canonically and get a canonical isomorphism $\ker(T) \oplus F \cong T^{-1}F \oplus \mathrm{coker}(T)$. Therefore

$$\Lambda^{max}(\ker(T))^* \otimes \Lambda^{max}(\mathrm{coker}(T)) \cong \Lambda^{max}(T^{-1}F)^* \otimes \Lambda^{max}F.$$

Now over each member of the cover U_F a line bundle $\mathrm{DET}^F \to U_F$ is defined. Next one shows that over intersections $U_{F_1} \cap U_{F_2}$ there is an isomorphism $\mathrm{DET}^{F_1} \to \mathrm{DET}^{F_2}$ and moreover the isomorphisms satisfy a cocycle condition over triple intersections $U_{F_1} \cap U_{F_2} \cap U_{F_3}$. This shows that the line bundles $\mathrm{DET}^F \to U_F$ glue together to define a line bundle over $\mathcal{F}$. It is further shown in [23] that this line bundle is holomorphic as a bundle over an open subset of a complex Banach space.

It is tempting to think that since $c_1(\mathrm{DET})$ is the generator of $H^2(\mathcal{F}_0, \mathbb{Z}) \cong \mathbb{Z}$, there might exits a natural Hermitian metric on DET whose curvature 2-form would be a representative of this generator. One problem is that the induced metric from $\ker(T)$ and $\ker(T^*)$ on DET is not even continuous. In [23] Quillen shows that for families of Cauchy-Riemann operators on a Riemann surface one can correct the Hilbert space metric by multiplying it by zeta regularized determinant and in this way one obtains a smooth Hermitian metric on the induced determinant line bundle. In Section 5 we describe a similar construction for noncommutative two torus.

4.2.2 Noncommutative two torus

For $\theta \in \mathbb{R}$, the noncommutative two torus A_θ is by definition the universal unital C^*-algebra generated by two unitaries U, V satisfying

$$VU = e^{2\pi i\theta}UV.$$

There is a continuous action of $\mathbb{T}^2$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, on A_θ by C^*-algebra automorphisms $\{\alpha_s\}$, $s \in \mathbb{R}^2$, defined by

$$\alpha_s(U^mV^n) = e^{is.(m,n)}U^mV^n.$$

The space of smooth elements for this action will be denoted by A_θ^∞. It is a dense subalgebra of A_θ which can be alternatively described as the algebra of elements in A_θ

whose (noncommutative) Fourier expansion has rapidly decreasing coefficients:

$$A_\theta^\infty = \left\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n : a_{m,n} \in \mathcal{S}(\mathbb{Z}^2) \right\}.$$

There is a normalized, faithful and positive, trace φ_0 on A_θ whose restriction on smooth elements is given by

$$\varphi_0(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n) = a_{0,0}.$$

The algebra A_θ^∞ is equipped with the derivations δ_1, $\delta_2 : A_\theta^\infty \to A_\theta^\infty$, uniquely defined by the relations

$$\delta_1(U) = U,\ \delta_1(V) = 0, \quad \delta_2(U) = 0,\ \delta_2(V) = V.$$

We have $\delta_j(a^*) = -\delta_j(a)^*$ for $j = 1, 2$ and all $a \in A_\theta^\infty$. Moreover, the analogue of the integration by parts formula in this setting is given by:

$$\varphi_0(a\delta_j(b)) = -\varphi_0(\delta_j(a)b), \ \ \forall a, b \in A_\theta^\infty.$$

We apply the GNS construction to A_θ. The state φ_0 defines an inner product

$$\langle a, b\rangle = \varphi_0(b^* a), \ \ a, b \in A_\theta,$$

and a pre-Hilbert structure on A_θ. After completion we obtain a Hilbert space denoted $\mathcal{H}_0$. The derivations δ_1, δ_2, as densely defined unbounded operators on $\mathcal{H}_0$, are formally selfadjoint and have unique extensions to selfadjoint operators.

We introduce a complex structure associated with a complex number $\tau = \tau_1 + i\tau_2$, $\tau_2 > 0$, by defining

$$\bar\partial = \delta_1 + \tau\delta_2, \ \ \bar\partial^* = \delta_1 + \overline{\tau}\delta_2.$$

Note that $\bar\partial$ is an unbounded operator on $\mathcal{H}_0$ and $\bar\partial^*$ is its formal adjoint. The analogue of the space of anti-holomorphic 1-forms on the ordinary two torus is defined to be

$$\Omega_\theta^{0,1} = \left\{ \sum a\bar\partial b\ , a, b \in A_\theta^\infty \right\}.$$

Using the induced inner product from φ_0, one can turn $\Omega_\theta^{0,1}$ into a Hilbert space which we denote by $\mathcal{H}^{0,1}$ (see [4] for details).

4.3 The canonical trace and noncommutative residue

In this section we define an analogue of the canonical trace of Kontsevich and Vishik [16] for the noncommutative torus. Let us first recall the algebra of pseudodifferential symbols on the noncommutative torus [3, 7].

4.3.1 Pseudodifferential calculus on $\mathcal{A}_\theta$

Using operator valued symbols, one can define an algebra of pseudodifferential operators on A_θ^∞. We shall use the notation $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial\xi_1^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial\xi_2^{\alpha_2}}$, and $\delta^\alpha = \delta_1^{\alpha_1}\delta_2^{\alpha_2}$, for a multi-index $\alpha = (\alpha_1, \alpha_2)$.

Definition 4.1. For a real number m, a smooth map $\sigma : \mathbb{R}^2 \to A_\theta^\infty$ is said to be a symbol of order m, if for all non-negative integers i_1, i_2, j_1, j_2,

$$||\delta^{(i_1,i_2)}\partial^{(j_1,j_2)}\sigma(\xi)|| \leqslant c(1+|\xi|)^{m-j_1-j_2},$$

where c is a constant, and if there exists a smooth map $k : \mathbb{R}^2 \to A_\theta^\infty$ such that

$$\lim_{\lambda\to\infty} \lambda^{-m}\sigma(\lambda\xi_1, \lambda\xi_2) = k(\xi_1, \xi_2).$$

The space of symbols of order m is denoted by $\mathcal{S}^m(\mathcal{A}_\theta)$.

Definition 4.2. To a symbol σ of order m, one can associate an operator on A_θ^∞, denoted by P_σ, given by

$$P_\sigma(a) = \int\int e^{-is\cdot\xi}\sigma(\xi)\alpha_s(a)\, ds\, d\xi.$$

Here, $d\xi = (2\pi)^{-2}d_L\xi$ where $d_L\xi$ is the Lebesgue measure on $\mathbb{R}^2$. The operator P_σ is said to be a pseudodifferential operator of order m.

For example, the differential operator $\sum_{j_1+j_2\leqslant m} a_{j_1,j_2}\delta^{(j_1,j_2)}$ is associated with the symbol $\sum_{j_1+j_2\leqslant m} a_{j_1,j_2}\xi_1^{j_1}\xi_2^{j_2}$ via the above formula.

Two symbols σ, $\sigma' \in \mathcal{S}^m(\mathcal{A}_\theta)$ are said to be equivalent if and only if $\sigma - \sigma' \in \mathcal{S}^n(\mathcal{A}_\theta)$ for all integers n. The equivalence of the symbols will be denoted by $\sigma \sim \sigma'$.

Let P and Q be pseudodifferential operators with the symbols σ and σ' respectively. Then the adjoint P^* and the product PQ are pseudodifferential operators with the following symbols

$$\sigma(P^*) \sim \sum_{\ell=(\ell_1,\ell_2)\geqslant 0} \frac{1}{\ell!}\partial^\ell \delta^\ell (\sigma(\xi))^*,$$

$$\sigma(PQ) = \sigma(P) \star \sigma(Q) \sim \sum_{\ell=(\ell_1,\ell_2)\geqslant 0} \frac{1}{\ell!}\partial^\ell(\sigma(\xi))\delta^\ell(\sigma'(\xi)).$$

Definition 4.3. A symbol $\sigma \in \mathcal{S}^m(\mathcal{A}_\theta)$ is called elliptic if $\sigma(\xi)$ is invertible for $\xi \neq 0$, and for some c

$$||\sigma(\xi)^{-1}|| \leqslant c(1+|\xi|)^{-m},$$

for large enough $|\xi|$.

A smooth map $\sigma : \mathbb{R}^2 \to \mathcal{A}_\theta$ is called a classical symbol of order $\alpha \in \mathbb{C}$ if for any N and each $0 \leqslant j \leqslant N$ there exist $\sigma_{\alpha-j} : \mathbb{R}^2\backslash\{0\} \to \mathcal{A}_\theta$ positive homogeneous of degree $\alpha - j$, and a symbol $\sigma^N \in \mathcal{S}^{\mathrm{Re}(\alpha)-N-1}(\mathcal{A}_\theta)$, such that

$$\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^N(\xi) \quad \xi \in \mathbb{R}^2. \tag{4.2}$$

Here χ is a smooth cut off function on $\mathbb{R}^2$ which is equal to zero on a small ball around the origin, and is equal to one outside the unit ball. It can be shown that the homogeneous terms in the expansion are uniquely determined by σ. We denote the set of classical symbols of order α by $\mathcal{S}^\alpha_{cl}(\mathcal{A}_\theta)$ and the associated classical pseudodifferential operators by $\Psi^\alpha_{cl}(\mathcal{A}_\theta)$.

The space of classical symbols $\mathcal{S}_{cl}(\mathcal{A}_\theta)$ is equipped with a Fréchet topology induced by the semi-norms

$$p_{\alpha,\beta}(\sigma) = \sup_{\xi\in\mathbb{R}^2}(1+|\xi|)^{-m+|\beta|}||\delta^\alpha\partial^\beta\sigma(\xi)||. \tag{4.3}$$

The analogue of the Wodzicki residue for classical pseudodifferential operators on the noncommutative torus is defined in [13].

Definition 4.4. The Wodzicki residue of a classical pseudodifferential operator P_σ is defined as

$$\operatorname{Res}(P_\sigma) = \varphi_0\left(\operatorname{res}(P_\sigma)\right),$$

where $\operatorname{res}(P_\sigma) := \int_{|\xi|=1} \sigma_{-2}(\xi) d\xi$.

It is evident from its definition that Wodzicki residue vanishes on differential operators and on non-integer order classical pseudodifferential operators.

4.3.2 The canonical trace

In what follows, we define the analogue of Kontsevich-Vishik trace [16] on non-integer order pseudodifferential operators on the noncommutative torus. For an alternative approach based on toroidal noncommutative symbols see [20]. For a thorough review of the theory in the classical case we refer to [19, 22]. First we show the existence of the so called cut-off integral for classical symbols.

Proposition 4.5. *Let $\sigma \in \mathcal{S}^\alpha_{cl}(\mathcal{A}_\theta)$ and $B(R)$ be the ball of radius R around the origin. One has the following asymptotic expansion*

$$\int_{B(R)} \sigma(\xi) d\xi \sim_{R\to\infty} \sum_{j=0,\alpha-j+2\neq 0}^{\infty} \alpha_j(\sigma) R^{\alpha-j+2} + \beta(\sigma)\log R + c(\sigma),$$

where $\beta(\sigma) = \int_{|\xi|=1} \sigma_{-2}(\xi) d\xi$ and the constant term in the expansion, $c(\sigma)$, is given by

$$\int_{\mathbb{R}^n} \sigma^N + \sum_{j=0}^{N} \int_{B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi) d\xi - \sum_{j=0,\alpha-j+2\neq 0}^{N} \frac{1}{\alpha - j + 2} \int_{|\xi|=1} \sigma_{\alpha-j}(\omega) d\omega. \tag{4.4}$$

Here we have used the notation of (4.2).

Proof. First, we write $\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^N(\xi)$ with large enough N, so that σ^N is integrable. Then we have,

$$\int_{B(R)} \sigma(\xi) d\xi = \sum_{j=0}^{N} \int_{B(R)} \chi(\xi)\sigma_{\alpha-j}(\xi) d\xi + \int_{B(R)} \sigma^N(\xi) d\xi. \tag{4.5}$$

For $N > \alpha + 1$, $\sigma^N \in \mathcal{L}^1(\mathbb{R}^2, \mathcal{A}_\theta)$, so

$$\int_{B(R)} \sigma^N(\xi) d\xi \to \int_{\mathbb{R}^2} \sigma^N(\xi) d\xi, \quad R \to \infty.$$

Now for each $j \leqslant N$ we have

$$\int_{B(R)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi = \int_{B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi + \int_{B(R)\setminus B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi.$$

Obviously $\int_{B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi < \infty$ and by using polar coordinates $\xi = r\omega$, and homogeneity of $\sigma_{\alpha-j}$, we have

$$\int_{B(R)\setminus B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi = \int_1^R r^{\alpha-j+2-1}dr \int_{|\xi|=1} \sigma_{\alpha-j}(\xi)d\xi. \tag{4.6}$$

Note that the cut-off function is equal to one on the set $\mathbb{R}^2\setminus B(1)$. For the term with $\alpha - j = -2$ one has

$$\int_{B(R)\setminus B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi = \log R \int_{|\xi|=1} \sigma_{\alpha-j}(\xi)d\xi.$$

The terms with $\alpha - j \neq -2$ will give us the following:

$$\begin{aligned} \int_{B(R)\setminus B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi \quad &= \\ \frac{R^{\alpha-j+2}}{m-j+2}\int_{|\xi|=1} \sigma_{\alpha-j}(\xi)d\xi \quad &- \quad \frac{1}{\alpha-j+2}\int_{|\xi|=1} \sigma_{\alpha-j}(\xi)d\xi. \end{aligned} \tag{4.7}$$

Adding all the constant terms in (4.5)-(4.7), we get the constant term given in (4.4). $\square$

Definition 4.6. The cut-off integral of a symbol $\sigma \in \mathcal{S}^{\alpha}_{cl}(\mathcal{A}_\theta)$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $⨍ \sigma(\xi)d\xi$.

Remark 4.7. Two remarks are in order here. First note that the cut-off integral of a symbol is independent of the choice of N. Second, it is also independent of the choice of the cut-off function χ.

We now give the definition of the canonical trace for classical pseudodifferential operators on $\mathcal{A}_\theta$.

Definition 4.8. The canonical trace of a classical pseudodifferential operator $P \in \Psi^{\alpha}_{cl}(\mathcal{A}_\theta)$ of non-integral order α is defined as

$$\mathrm{TR}(P) := \varphi_0\left(⨍ \sigma_P(\xi)d\xi\right).$$

In the following, we establish the relation between the TR-functional and the usual trace on trace-class pseudodifferential operators. Note that any pseudodifferential operator P of order less that -2, is a trace-class operator on $\mathcal{H}_0$ and its trace is given by

$$\mathrm{Tr}(P) = \varphi_0 \left(\int_{\mathbb{R}^2} \sigma_P(\xi) d\xi \right).$$

On the other hand, for such operator the symbol is integrable and we have

$$⨍ \sigma_P(\xi) d\xi = \int_{\mathbb{R}^2} \sigma_P(\xi) d\xi. \tag{4.8}$$

Therefore, the TR-functional and operator trace coincide on classical pseudodifferential operators of order less than -2.

Next, we show that the TR-functional is in fact an analytic continuation of the operator trace and using this fact we can prove that it is actually a trace.

Definition 4.9. A family of symbols $\sigma(z) \in \mathcal{S}_{cl}^{\alpha(z)}(\mathcal{A}_\theta)$, parametrized by $z \in W \subset \mathbb{C}$, is called a holomorphic family if

i) The map $z \mapsto \alpha(z)$ is holomorphic.

ii) The map $z \mapsto \sigma(z) \in \mathcal{S}_{cl}^{\alpha(z)}(\mathcal{A}_\theta)$ is a holomorphic map from W to the Fréchet space $\mathcal{S}_{cl}(\mathcal{A}_\theta^n)$.

iii) The map $z \mapsto \sigma(z)_{\alpha(z)-j}$ is holomorphic for any j, where

$$\sigma(z)(\xi) \sim \sum_j \chi(\xi) \sigma(z)_{\alpha(z)-j}(\xi) \in \mathcal{S}_{cl}^{\alpha(z)}(\mathcal{A}_\theta). \tag{4.9}$$

iv) The bounds of the asymptotic expansion of $\sigma(z)$ are locally uniform with respect to z, i.e, for any $N \geqslant 1$ and compact subset $K \subset W$, there exists a constant $C_{N,K,\alpha,\beta}$ such that for all multi-indices α, β we have

$$\left\| \delta^\alpha \partial^\beta \left(\sigma(z) - \sum_{j<N} \chi \sigma(z)_{\alpha(z)-j} \right)(\xi) \right\| < C_{N,K,\alpha,\beta} |\xi|^{\mathrm{Re}(\alpha(z))-N-|\beta|}.$$

A family $\{P_z\} \in \Psi_{cl}(\mathcal{A}_\theta)$ is called holomorphic if $P_z = P_{\sigma(z)}$ for a holomorphic family of symbols $\{\sigma(z)\}$.

The following Proposition is an analogue of a result of Kontsevich and Vishik[16], for pseudodifferential calculus on noncommutative tori.

Proposition 4.10. *Given a holomorphic family $\sigma(z) \in \mathcal{S}_{cl}^{\alpha(z)}(\mathcal{A}_\theta)$, $z \in W \subset \mathbb{C}$, the map*

$$z \mapsto \fint \sigma(z)(\xi) d\xi,$$

is meromorphic with at most simple poles located in

$$P = \{z_0 \in W;\ \alpha(z_0) \in \mathbb{Z} \cap [-2, +\infty]\}.$$

The residues at poles are given by

$$\mathrm{Res}_{z=z_0} \fint \sigma(z)(\xi) d\xi = -\frac{1}{\alpha'(z_0)} \int_{|\xi|=1} \sigma(z_0)_{-2} d\xi.$$

Proof. By definition, one can write $\sigma(z) = \sum_{j=0}^{N} \chi(\xi)\sigma(z)_{\alpha(z)-i}(\xi) + \sigma(z)^N(\xi)$, and by Proposition 4.5 we have,

$$\begin{aligned} \fint \sigma(z)(\xi) d\xi = \int_{\mathbb{R}^2} \sigma(z)^N(\xi) d\xi + \sum_{j=0}^{N} \int_{B(1)} \chi(\xi)\sigma(z)_{\alpha(z)-j}(\xi) \\ - \sum_{j=0}^{N} \frac{1}{\alpha(z)+2-j} \int_{|\xi|=1} \sigma(z)_{\alpha(z)-j}(\xi) d\xi. \end{aligned}$$

Now suppose $\alpha(z_0) + 2 - j_0 = 0$. By holomorphicity of $\sigma(z)$, we have $\alpha(z) - \alpha(z_0) = \alpha'(z_0)(z - z_0) + o(z - z_0)$. Hence

$$\mathrm{Res}_{z=z_0} \fint \sigma(z)(\xi) d\xi = \frac{-1}{\alpha'(z_0)} \int_{|\xi|=1} \sigma(z_0)_{-2}(\xi) d\xi.$$

□

Corollary 4.11. *The functional* TR *is the analytic continuation of the ordinary trace on trace-class pseudodifferential operators.*

Proof. First observe that, by the above result, for a non-integer order holomorphic family of symbols $\sigma(z)$, the map $z \mapsto \fint \sigma(z)(\xi) d\xi$ is holomorphic. Hence, the map $\sigma \mapsto \fint \sigma(\xi) d\xi$ is the unique analytic continuation of the map $\sigma \mapsto \int_{\mathbb{R}^2} \sigma(\xi) d\xi$ from $\mathcal{S}_{cl}^{<-2}(\mathcal{A}_\theta)$ to $\mathcal{S}_{cl}^{\notin\mathbb{Z}}(\mathcal{A}_\theta)$. By (4.8) we have the result. □

Let $Q \in \Psi^q_{cl}(\mathcal{A}_\theta)$ be a positive elliptic pseudodifferential operator of order $q > 0$. The complex power of such an operator, Q^z_ϕ, for $\mathrm{Re}(z) < 0$ can be defined by the following Cauchy integral formula.

$$Q^z_\phi = \frac{i}{2\pi} \int_{C_\phi} \lambda^z_\phi (Q - \lambda)^{-1} d\lambda. \tag{4.10}$$

Here λ^z_ϕ is the complex power with branch cut $L_\phi = \{re^{i\phi}, r \geqslant 0\}$ and C_ϕ is a contour around the spectrum of Q such that

$$C_\phi \cap \mathrm{spec}(Q) \backslash \{0\} = \emptyset, \qquad L_\phi \cap C_\phi = \emptyset,$$

$$C_\phi \cap \{\mathrm{spec}(\sigma(Q)^L(\xi)),\, \xi \neq 0\} = \emptyset.$$

In general an operator for which one can find a ray L_ϕ with the above property, is called an admissible operator with the spectral cut L_ϕ. Positive elliptic operators are admissible and we take the ray L_π as the spectral cut, and in this case we drop the index ϕ and write Q^z.

To extend (4.10) to $\mathrm{Re}(z) > 0$ we choose a positive integer such that $\mathrm{Re}(z) < k$ and define

$$Q^z_\phi := Q^k Q^{z-k}_\phi.$$

It can be proved that this definition is independent of the choice of k.

Corollary 4.12. *Let $A \in \Psi^\alpha_{cl}(\mathcal{A}_\theta)$ be of order $\alpha \in \mathbb{Z}$ and let Q be a positive elliptic classical pseudodifferential operator of positive order q. We have*

$$\mathrm{Res}_{z=0} \mathrm{TR}(AQ^{-z}) = \frac{1}{q} \mathrm{Res}(A).$$

Proof. For the holomorphic family $\sigma(z) = \sigma(AQ^{-z})$, $z = 0$ is a pole for the map $z \mapsto \fint \sigma(z)(\xi) d\xi$ whose residue is given by

$$\mathrm{Res}_{z=0} \left(z \mapsto \fint \sigma(z)(\xi) d\xi \right) = -\frac{1}{\alpha'(0)} \int_{|\xi|=1} \sigma_{-2}(0) d\xi = -\frac{1}{\alpha'(0)} \mathrm{res}(A).$$

Taking trace on both sides gives the result. □

Now we can prove the trace property of TR-functional.

Proposition 4.13. *We have* $\mathrm{TR}(AB) = \mathrm{TR}(BA)$ *for any* $A, B \in \Psi_{cl}(\mathcal{A}_\theta)$, *provided that* $\mathrm{ord}(A) + \mathrm{ord}(B) \notin \mathbb{Z}$.

Proof. Consider the families $A_z = AQ^z$ and $B_z = BQ^z$ where Q is an injective positive elliptic classical operator of order $q > 0$. For $\mathrm{Re}(z) \ll 0$, the two families are trace class and $\mathrm{Tr}(A_z B_z) = \mathrm{Tr}(B_z A_z)$. By the uniqueness of the analytic continuation, we have

$$\mathrm{TR}(A_z B_z) = \mathrm{TR}(B_z A_z),$$

for those z for which $2qz + \mathrm{ord}(A) + \mathrm{ord}(B) \notin \mathbb{Z}$. At $z = 0$, we obtain $\mathrm{TR}(AB) = \mathrm{TR}(BA)$. $\square$

4.3.3 Log-polyhomogeneous symbols

Proposition 4.10 can be extended and one can explicitly write down the Laurent expansion of the cut-off integral around each of the poles. The terms of the Laurent expansion involve residue densities of z-derivatives of the holomorphic family. In general, z-derivatives of a classical holomorphic family of symbols is not classical anymore and therefore we introduce log-polyhomogeneous symbols which include the z-derivatives of the symbols of the holomorphic family $\sigma(AQ^{-z})$.

Definition 4.14. A symbol σ is called a log-polyhomogeneous symbol if it has the following form

$$\sigma(\xi) \sim \sum_{j \geqslant 0} \sum_{l=0}^{\infty} \sigma_{\alpha-j,l}(\xi) \log^l |\xi| \quad |\xi| > 0, \tag{4.11}$$

with $\sigma_{\alpha-j,l}$ positively homogeneous in ξ of degree $\alpha - j$.

An important example of an operator with such a symbol is $\log Q$ where $Q \in \Psi^q_{cl}(\mathcal{A}_\theta)$ is a positive elliptic pseudodifferential operator of order $q > 0$. The logarithm of Q can be defined by

$$\log Q = Q \left.\frac{d}{dz}\right|_{z=0} Q^{z-1} = Q \left.\frac{d}{dz}\right|_{z=0} \frac{i}{2\pi} \int_C \lambda^{z-1}(Q-\lambda)^{-1} d\lambda.$$

It is a pseudodifferential operator with symbol

$$\sigma(\log Q) \sim \sigma(Q) \star \sigma\Big(\left.\frac{d}{dz}\right|_{z=0} Q^{z-1} \Big), \tag{4.12}$$

where $\star$ denotes the products of the pseudodifferential symbols. Using symbol calculus and homogeneity properties, we can show that (4.12) is a log-homogeneous symbol of the form

$$\sigma(\log Q)(\xi) = q \log |\xi| I + \sigma_{cl}(\log Q)(\xi),$$

where $\sigma_{cl}(\log Q)$ is a classical symbol of order zero. This symbol can be computed using the homogeneous parts of the classical symbol $\sigma(Q^z) = \sum_{j=0}^{\infty} b(z)_{qz-j}(\xi)$ and it is given by the following formula (see e.g. [19]).

$$\sigma_{cl}(\log Q)(\xi) = \tag{4.13}$$
$$\sum_{k=0}^{\infty} \sum_{i+j+|\alpha|=k} \frac{1}{\alpha!} \partial^\alpha \sigma_{q-i}(Q) \delta^\alpha \left[|\xi|^{-q-j} \left.\frac{d}{dz}\right|_{z=0} b(z-1)_{qz-q-j}\left(\xi/|\xi|\right) \right].$$

For an operator A with log-polyhomogeneous symbol as (4.11) we define

$$\mathrm{res}(A) = \int_{|\xi|=1} \sigma_{-2,0}(\xi) d\xi.$$

By adapting the proof of Theorem 1.13 in [22] to the noncommutative case, we have the following theorem which is written only for the families of the form $\sigma(AQ^{-z})$ which we will use in Section 4.5.

Proposition 4.15. *Let $A \in \Psi^{\alpha}_{cl}(\mathcal{A}_\theta)$ and Q be a positive , in general an admissible, elliptic pseudodifferential operator of positive order q. If $\alpha \in P$ then 0 is a possible simple pole for the function $z \mapsto \mathrm{TR}(AQ^{-z})$ with the following Laurent expansion around zero.*

$$\begin{aligned}
\mathrm{TR}(AQ^{-z}) = &\frac{1}{q}\mathrm{Res}(A)\frac{1}{z} \\
&+ \varphi_0\left(⨍ \sigma(A)(\xi) d\xi - \frac{1}{q}\mathrm{res}(A \log Q)\right) - \mathrm{Tr}(A\Pi_Q) \\
&+ \sum_{k=1}^{K} (-1)^k \frac{(z)^k}{k!} \\
&\times \left(\varphi_0 \left(⨍ \sigma(A(\log Q)^k)(\xi) d\xi - \frac{1}{q(k+1)}\mathrm{res}(A(\log Q)^{k+1})\right) - \mathrm{Tr}(A(\log Q)^k \Pi_Q)\right) \\
&+ o(z^K).
\end{aligned}$$

Where Π_Q is the projection on the kernel of Q.

For operators A and Q as in the previous Proposition, we define the *generalized zeta function* by

$$\zeta(A, Q, z) = \mathrm{TR}(AQ^{-z}). \tag{4.14}$$

From Proposition 4.10, it follows that $\zeta(A, Q, z)$ is meromorphic function with at most simple poles. Moreover, by Corollary 4.11, it is obvious that $\zeta(A, Q, z)$ is the analytic continuation of the zeta function $\mathrm{Tr}(AQ^{-z})$.

Remark 4.16. If A is a differential operator, the zeta function (4.14) is regular at $z = 0$ with the value equal to

$$\varphi_0 \left(⨍ \sigma(A)(\xi) d\xi - \frac{1}{q} \mathrm{res}(A \log Q) \right) - \mathrm{Tr}(A\Pi_Q).$$

4.4 Cauchy-Riemann operators on noncommutative tori

In [23], Quillen studies the geometry of the determinant line bundle on the space of all Cauchy-Riemann operators on a smooth vector bundle on a closed Riemann surface. To investigate the same notion on noncommutative tori, we first briefly recall some basic facts in the classical case on how Cauchy-Riemann operators are related to Dirac operators and spectral triples. Then by analogy we define our Cauchy-Riemann operator on $\mathcal{A}_\theta$, and consider the spectral triples defined by them.

Let M be a compact complex manifold and V be a smooth complex vector bundle on M. Let $\Omega^{p,q}(M, V)$ denote the space of (p, q) forms on M with coefficients in V. A $\bar{\partial}$-flat connection on V is a $\mathbb{C}$-linear map $D : \Omega^{0,0}(M, V) \to \Omega^{0,1}(M, V)$, such that for any $f \in C^\infty(M)$ and $u \in \Omega^{0,0}(M, V)$,

$$D(fu) = (\bar{\partial} f) \otimes u + f Du, \tag{4.15}$$

and $D^2 = 0$. Here to define D^2, note that any $\bar{\partial}$-connection as above has a unique extension to an operator $D : \Omega^{p,q}(M, V) \to \Omega^{p,q+1}(M, V)$, defined by

$$D(\alpha \otimes \beta) = \bar{\partial}\alpha \otimes u + (-1)^{p+q} \alpha \wedge Du, \quad \alpha \in \Omega^{p,q}(M), \ u \in C^\infty(V).$$

We refer to $\bar{\partial}$-flat connections as Cauchy-Riemann operators. A holomorphic vector bundle V has a canonical Cauchy-Riemann operator $\bar{\partial}_V : \Omega^0(M, V) \to \Omega^{0,1}(M, V)$, whose extension to $\Omega^{0,*}(M, V)$ forms the Dolbeault complex of M with coefficients in

V. In fact there is a one-one correspondence between Cauchy-Riemann operators on V up to (gauge) equivalence, and holomorphic structures on V. We denote by $\mathcal{A}$ the set of all Cauchy-Riemann operators on V.

Any holomorphic structure on a Hermitian vector bundle V determines a unique Hermitian connection, called the Chern connection, whose projection on $(0,1)$-forms, $\nabla^{0,1}(M,V)$, is the Cauchy-Riemann operator coming from the holomorphic structure.

Now, if M is a Kähler manifold, the tensor product of the Levi-Civita connection for M with the Chern connection on V defines a Clifford connection on the Clifford module $(\Lambda^{0,+} \oplus \Lambda^{0,-}) \otimes V$ and the operator $D_0 = \sqrt{2}(\bar{\partial}_V + \bar{\partial}_V^*)$ is the associated Dirac operator (see e.g. [14]). Any other Dirac operator on the Clifford module $(\Lambda^{0,+} \oplus \Lambda^{0,-}) \otimes V$ is of the form $D_0 + A$ where A is the connection one form of a Hermitian connection. This connection need not be a Chern connection. However, on a Riemann surface (with a Riemannian metric compatible with its complex structure) any Hermitian connection on a smooth Hermitian vector bundle is the Chern connection of a holomorphic structure on V. Therefore, the positive part of any Dirac operator on $(\Lambda^{0,0} \oplus \Lambda^{0,1}) \otimes V$ is a Cauchy-Riemann operator, and this gives a one to one correspondence between all Dirac operators and the set of all Cauchy-Riemann operators.

Next we define the analogue of Cauchy-Riemann operators for the noncommutative torus. First, following [7, 10], we fix a complex structure on $\mathcal{A}_\theta$ by a complex number τ in the upper half plane and construct the spectral triple

$$(\mathcal{A}_\theta, \mathcal{H}_0 \oplus \mathcal{H}^{0,1}, D_0 = \begin{pmatrix} 0 & \bar{\partial}^* \\ \bar{\partial} & 0 \end{pmatrix}), \tag{4.16}$$

where $\bar{\partial} : \mathcal{A}_\theta \to \mathcal{A}_\theta$ is given by $\bar{\partial} = \delta_1 + \tau\delta_2$. The Hilbert space $\mathcal{H}_0$ is obtained by GNS construction from $\mathcal{A}_\theta$ using the trace φ_0 and $\bar{\partial}^*$ is the adjoint of the operator $\bar{\partial}$.

As in the classical case, we define our Cauchy-Riemann operators on $\mathcal{A}_\theta$ as the positive part of twisted Dirac operators. All such operators define spectral triples of the form

$$(\mathcal{A}_\theta, \mathcal{H}_0 \oplus \mathcal{H}^{0,1}, D_A = \begin{pmatrix} 0 & \bar{\partial}^* + \alpha^* \\ \bar{\partial} + \alpha & 0 \end{pmatrix}),$$

where $\alpha \in \mathcal{A}_\theta$ is the positive part of a selfadjoint element

$$A = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix} \in \Omega^1_{D_0}(\mathcal{A}_\theta).$$

We recall that $\Omega^1_{D_0}(\mathcal{A}_\theta)$ is the space of quantized one forms consisting of the elements $\sum a_i[D_0, b_i]$ where $a_i, b_i \in \mathcal{A}_\theta$ [4]. Note that the in this case, the space $\mathcal{A}$ of Cauchy-Riemann operators is the space of $(0,1)$-forms on $\mathcal{A}_\theta$.

We should mention that in the noncommutative case, in the work of Chakraborty and Mathai [2] a general family of spectral triples is considered and, under suitable regularity conditions, a determinant line bundle is defined for such families. The curvature of the determinant line bundle however is not computed and that is the main object of study in the present paper, as well as in [23].

4.5 The curvature of the determinant line bundle for $\mathcal{A}_\theta$

For any $\alpha \in \mathcal{A}$, the Cauchy-Riemann operator

$$\bar{\partial}_\alpha = \bar{\partial} + \alpha : \mathcal{H}_0 \to \mathcal{H}^{0,1}$$

is a Fredholm operator. We pull back the determinant line bundle DET on the space of Fredholm operators $\mathrm{Fred}(\mathcal{H}_0, \mathcal{H}^{0,1})$, to get a line bundle $\mathcal{L}$ on $\mathcal{A}$. Following Quillen [23], we define a Hermitian metric on $\mathcal{L}$ and compute its curvature in this section. Let us define a metric on the fiber

$$\mathcal{L}_\alpha = \Lambda^{max}(\ker \bar{\partial}_\alpha)^* \otimes \Lambda^{max}(\ker \bar{\partial}^*_\alpha).$$

as the product of the induced metrics on $\Lambda^{max}(\ker \bar{\partial}_\alpha)^*$, $\Lambda^{max}(\ker \bar{\partial}^*_\alpha)$, with the zeta regularized determinant $e^{-\zeta'_{\Delta_\alpha}(0)}$. Here we define the Laplacian as $\Delta_\alpha = \bar{\partial}^*_\alpha \bar{\partial}_\alpha : \mathcal{H}_0 \to \mathcal{H}_0$, and its zeta function by

$$\zeta(z) = \mathrm{TR}(\Delta_\alpha^{-z}).$$

It is a meromorphic function and by Remark 4.16 it is regular at $z = 0$. Similar proof as in [23] shows that this defines a smooth Hermitian metric on $\mathcal{L}$.

On the open set of invertible operators each fiber of $\mathcal{L}$ is canonically isomorphic to $\mathbb{C}$ and the nonzero holomorphic section $\sigma = 1$ gives a trivialization. Also, according to the definition of the Hermitian metric, the norm of this section is given by

$$\|\sigma\|^2 = e^{-\zeta'_{\Delta_\alpha}(0)}. \tag{4.17}$$

4.5.1 Variations of LogDet and curvature form

We begin by explaining the motivation behind the computations of Quillen in [23]. Recall that a holomorphic line bundle equipped with a Hermitian inner product has a canonical connection compatible with the two structures. This is also known as the Chern connection. The curvature form of this connection is computed by $\bar{\partial}\partial \log \|\sigma\|^2$, where σ is any non-zero local holomorphic section.

In our case we will proceed by analogy and compute the second variation $\bar{\partial}\partial \log \|\sigma\|^2$ on the open set of invertible index zero Cauchy-Riemann operators. Let us consider a holomorphic family of invertible index zero Cauchy-Riemann operators $D_w = \bar{\partial} + \alpha_w$, where α_w depends holomorphically on the complex variable w and compute

$$\delta_{\bar{w}}\delta_w \zeta'_\Delta(0).$$

One has the following first variational formula,

$$\delta_w \zeta(z) = \delta_w \mathrm{TR}(\Delta^{-z}) = \mathrm{TR}(\delta_w \Delta^{-z}) = -z\mathrm{TR}(\delta_w \Delta \Delta^{-z-1}),$$

where in the second equality we were able to change the order of δ_w and TR because of the uniformity condition in the definition of holomorphic families (cf. [21]).

Note that, although $\mathrm{TR}(\Delta^{-z})$ is regular at $z = 0$, $\mathrm{TR}(\delta_w \Delta\Delta^{-z-1})$ might have a pole at $z = 0$ since $\delta_w \Delta\Delta^{-z-1}|_{z=0} = \delta_w \Delta\Delta^{-1}$ is not a differential operator any more and may have non-zero residue. Around $z = 0$ one has the following Laurent expansion:

$$-z\mathrm{TR}(\delta_w \Delta\Delta^{-z-1}) = -z(\frac{a_{-1}}{z} + a_0 + a_1 z + \cdots).$$

Hence,

$$\delta_w \zeta(z)|_{z=0} = -a_{-1}, \qquad \frac{d}{dz}\delta_w \zeta(z)\Big|_{z=0} = -a_0.$$

Using Proposition 4.15 we have

$$\delta_w \zeta'(0) = \frac{d}{dz}\delta_w \zeta(z)\Big|_{z=0} = -\varphi_0 \left(⨍ \sigma(\delta_w \Delta\Delta^{-1})(\xi) d\xi - \frac{1}{2}\mathrm{res}_x(\delta_w \Delta\Delta^{-1} \log \Delta) \right).$$

To compute the right hand side of the above equality, we need to note that since D_w depends holomorphically on w, $\delta_w D^* = 0$ and hence

$$\delta_w \Delta = \delta_w D^* D + D^* \delta_w D = D^* \delta_w D.$$

Since $\delta_w D$ is a zero order differential operator, we have

$$\begin{aligned}\delta_w \zeta'(0) &= -\varphi_0 \left(⨍ \sigma(D^* \delta_w D \Delta^{-1})(\xi) d\xi - \frac{1}{2}\mathrm{res}(D^* \delta_w D \Delta^{-1} \log \Delta) \right) \\ &= -\varphi_0 \left(⨍ \sigma(\delta_w D \Delta^{-1} D^*)(\xi) d\xi - \frac{1}{2}\mathrm{res}(\delta_w D \log \Delta \, \Delta^{-1} D^*) \right) \\ &= -\varphi_0 \left(\delta_w D \left(⨍ \sigma(D^{-1})(\xi) d\xi - \frac{1}{2}\mathrm{res}(\log \Delta \, D^{-1}) \right) \right) \\ &= -\varphi_0 \left(\delta_w D \, J \right),\end{aligned}$$

where

$$J = ⨍ \sigma(D^{-1})(\xi) d\xi - \frac{1}{2}\mathrm{res}(\log(\Delta) D^{-1}).$$

The reader can compare this to the term J in Quillen's computations [23].

Now we compute the second variation $\delta_{\bar{w}} \delta_w \zeta'(0)$. Since D_w is holomorphic we have

$$\delta_{\bar{w}} \delta_w \zeta'(0) = -\varphi_0 \left(\delta_w D \delta_{\bar{w}} J \right).$$

Next we compute the variation $\delta_{\bar{w}} J$. Note that since D_w is invertible, D_w^{-1} is also holomorphic and hence $\delta_{\bar{w}} ⨍ \sigma(D^{-1})(\xi) d\xi = 0$. Therefore

$$\delta_{\bar{w}} J = \delta_{\bar{w}} \left(⨍ \sigma(D^{-1})(\xi) d\xi - \frac{1}{2}\mathrm{res}(\log \Delta \, D^{-1}) \right) = -\frac{1}{2}\delta_{\bar{w}} \mathrm{res}(\log \Delta \, D^{-1}).$$

Thus, we have shown that

Lemma 4.17. *For the holomorphic family of Cauchy-Riemann operators D_w, the second variation of $\zeta'(0)$ reads:*

$$\delta_{\bar{w}} \delta_w \zeta'(0) = \frac{1}{2}\varphi_0 \left(\delta_w D \delta_{\bar{w}} \mathrm{res}(\log \Delta \, D^{-1}) \right).$$

$\square$

Our next goal is to compute $\delta_{\bar{w}} \mathrm{res}(\log \Delta \, D^{-1})$. This combined with the above lemma shows that the curvature form of the determinant line bundle equals the Kähler form on the space of connections.

Lemma 4.18. *With above definitions and notations, we have*

$$\sigma_{-2,0}(\log\Delta\, D^{-1}) = \frac{(\alpha+\alpha^*)\xi_1 + (\bar{\tau}\alpha+\tau\alpha^*)\xi_2}{(\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)(\xi_1+\tau\xi_2)}$$

$$-\log\left(\frac{\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2}{|\xi|^2}\right)\frac{\alpha}{\xi_1+\tau\xi_2},$$

and

$$\delta_{\bar{w}}\mathrm{res}(\log(\Delta)D^{-1}) = \frac{1}{2\pi\Im(\tau)}(\delta_w D)^*.$$

Proof. By writing down the homogeneous terms in the expansion of $\sigma_{\bullet,0}(\log\Delta)$ and $\sigma(D^{-1})$ and using the product formula of the symbols we see that

$$\sigma_{-2,0}(\log\Delta D^{-1}) \sim \sigma_{-1,0}(\log\Delta)\sigma_{-1}(D^{-1}) + \sigma_{0,0}(\log\Delta)\sigma_{-2}(D^{-1}).$$

Starting with the symbol of Δ, we have

$$\sigma(\Delta) = \xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2 + (\alpha+\alpha^*)\xi_1 + (\bar{\tau}\alpha+\tau\alpha^*)\xi_2 + \bar{\partial}^*(\alpha).$$

Then, the homogeneous parts of $\sigma((\lambda-\Delta)^{-1}) = \sum_j b_{-2-j}$ is given by the following recursive formula

$$b_{-2} = (\lambda - \sigma_2(\Delta))^{-1},$$
$$b_{-2-j} = -b_{-2}\sum_{k+l+|\gamma|=j,\, l<j} \partial^\gamma\sigma_{2-k}(\Delta)\delta^\gamma b_{-2-l}/\gamma!,$$

which gives us

$$b_{-2} = \frac{1}{\lambda - (\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)},$$

and

$$b_{-3} = \frac{1}{(\lambda - (\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2))^2}\left((\alpha+\alpha^*)\xi_1 + (\bar{\tau}\alpha+\tau\alpha^*)\xi_2\right).$$

Also, Δ^z is a classical operator defined by

$$\Delta^z = \frac{1}{2\pi i}\int_C \lambda^z(\lambda-\Delta)^{-1}d\lambda,$$

with the homogeneous parts of the symbol given by

$$b(z)_{2z-j} := \sigma_{2z-j}(\Delta^z) = \frac{1}{2\pi i}\int_C \lambda^z b_{-2-j} d\lambda.$$

Hence we have

$$b(z)_{2z} = \frac{1}{2\pi i}\int_C \lambda^z \frac{1}{\lambda - (\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)} d\lambda$$

$$= (\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)^z$$

$$b(z)_{2z-1} = \frac{1}{2\pi i}\int_C \lambda^z \frac{((\alpha+\alpha^*)\xi_1 + (\bar\tau\alpha + \tau\alpha^*)\xi_2)}{(\lambda - (\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2))^2} d\lambda$$

$$= z(\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2))^{z-1}\left((\alpha+\alpha^*)\xi_1 + (\bar\tau\alpha + \tau\alpha^*)\xi_2\right).$$

Using (4.13) and what we have computed up to here, it is clear that

$$\sigma_{0,0}(\log\Delta)(\xi) = \sigma_2(\Delta)|\xi|^{-2}\left.\frac{d}{dz}\right|_{z=0} b(z-1)_{2z-2}\left(\xi/|\xi|\right)$$

$$= \sigma_2(\Delta)|\xi|^{-2}\left.\frac{d}{dz}\right|_{z=0}((\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)/|\xi|^2)^{z-1}$$

$$= \log((\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)/|\xi|^2).$$

Note that the above term is homogeneous of order zero in ξ.

$$
\begin{aligned}
&\sigma_{-1,0}(\log\Delta)(\xi) \\
&= \sum_{i+j+|\alpha|=1} \frac{1}{\alpha!}\partial^\alpha \sigma_{2-i}(\Delta)\delta^\alpha |\xi|^{-2-j} \left.\frac{d}{dz}\right|_{z=0} b(z-1)_{2z-2-j}\left(\xi/|\xi|\right) \\
&= \sigma_2(\Delta)|\xi|^{-3} \left.\frac{d}{dz}\right|_{z=0} b(z-1)_{2z-3}\left(\xi/|\xi|\right) \\
&+ \sigma_1(\Delta)|\xi|^{-2} \left.\frac{d}{dz}\right|_{z=0} b(z-1)_{2z-2}\left(\xi/|\xi|\right) \\
&= \frac{1-\log(\xi_1^2+2\mathrm{Re}(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2)/|\xi|^2)}{(\xi_1^2+2\mathrm{Re}(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2)}\left[(\alpha+\alpha^*)\xi_1+(\bar\tau\alpha+\tau\alpha^*)\xi_2\right] \\
&+ \frac{\log(\xi_1^2+2\mathrm{Re}(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2)/|\xi|^2)}{\xi_1^2+2\mathrm{Re}(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2}\left[(\alpha+\alpha^*)\xi_1+(\bar\tau\alpha+\tau\alpha^*)\xi_2\right] \\
&= (\xi_1^2+2\mathrm{Re}(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2)^{-1}\left[(\alpha+\alpha^*)\xi_1+(\bar\tau\alpha+\tau\alpha^*)\xi_2\right].
\end{aligned}
$$

Next we compute the symbol of D^{-1}. The symbol of D reads

$$\sigma(D) = \xi_1 + \tau\xi_2 + \alpha.$$

We need to compute the homogeneous parts of order -1 and -2 of D^{-1}. By using recursive formula for the symbol of the inverse we get:

$$
\begin{aligned}
\sigma_{-1}(D^{-1}) &= \sigma_1(D)^{-1} = (\xi_1+\tau\xi_2)^{-1} \\
\sigma_{-2}(D^{-1}) &= -\sigma_{-1}(D^{-1}) \sum_{k+|\gamma|=1} \partial^\gamma \sigma_{1-k}(D)\delta^\gamma \sigma_{-1}(D^{-1})/\gamma! \\
&= -\sigma_{-1}(D^{-1})^2\sigma_0(D) \\
&= -(\xi_1+\tau\xi_2)^{-2}\alpha.
\end{aligned}
$$

Finally, we have

$$\begin{aligned}\sigma_{-2,0}(\log\Delta\, D^{-1}) &= \sigma_{-1,0}(\log\Delta)\sigma_{-1}(D^{-1}) + \sigma_{0,0}(\log\Delta)\sigma_{-2}(D^{-1})\\ &= (\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)^{-1}(\xi_1+\tau\xi_2)^{-1}\left[(\alpha+\alpha^*)\xi_1 + (\bar\tau\alpha+\tau\alpha^*)\xi_2\right]\\ &- \log((\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)/|\xi|^2)(\xi_1+\tau\xi_2)^{-2}\alpha.\end{aligned}$$

Therefore, we compute the variation:

$$\begin{aligned}\delta_{\bar w}\sigma_{-2,0}(\log\Delta\, D^{-1}) &= (\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)^{-1}\left[(\delta_{\bar w}\alpha^*)\xi_1 + (\tau\delta_{\bar w}\alpha^*)\xi_2\right](\xi_1+\tau\xi_2)^{-1}\\ &= (\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)^{-1}(\delta_{\bar w}\alpha^*)\\ &= (\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)^{-1}(\delta_w D)^*. \end{aligned}\tag{4.18}$$

In order to compute the variation of the residue density, we need to integrate (4.18) with respect to ξ variable:

$$\delta_{\bar w}\mathrm{res}(\log(\Delta)D^{-1}) = \int_{|\xi|=1}(\xi_1^2 + 2\mathrm{Re}(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)^{-1}(\delta_w D)^* d\xi = \frac{1}{2\pi\Im(\tau)}(\delta_w D)^*.$$

Note that we have used the normalized Lebesgue measure in the last integral (see (4.2)).

□

We record the main result of this paper in the following theorem. It computes the curvature of the determinant line bundle in terms of the natural Kähler form on the space of connections.

Theorem 4.19. *The curvature of the determinant line bundle for the noncommutative two torus is given by*

$$\delta_{\bar w}\delta_w\zeta'(0) = \frac{1}{4\pi\Im(\tau)}\varphi_0\left(\delta_w D(\delta_w D)^*\right).\tag{4.19}$$

□

Remark 4.20. In order to recover the classical result of Quillen for $\theta = 0$, we have to take into account the change of the volume form due to a change of the metric. This means we have to multiply the above result by $\Im(\tau)$.

Bibliography

[1] T. A. Bhuyain and M. Marcolli. The Ricci flow on noncommutative two-tori. *Lett. Math. Phys.*, 101(2):173–194, 2012.

[2] P. S. Chakraborty and V. Mathai. The geometry of determinant line bundles in noncommutative geometry. *J. Noncommut. Geom.*, 3(4):559–578, 2009.

[3] A. Connes. C^* algèbres et géométrie différentielle. *C. R. Acad. Sci. Paris Sér. A-B*, 290(13):A599–A604, 1980.

[4] A. Connes. *Noncommutative geometry.* Academic Press, Inc., San Diego, CA, 1994.

[5] A. Connes and P. B. Cohen. Conformal geometry of the irrational rotation algebra. Preprint of the MPI, Bonn. no. MPI/92-23, 1992.

[6] A. Connes and H. Moscovici. Modular curvature for noncommutative two-tori. *J. Amer. Math. Soc.*, 27(3):639–684, 2014.

[7] A. Connes and P. Tretkoff. The Gauss-Bonnet theorem for the noncommutative two torus. In *Noncommutative geometry, arithmetic, and related topics*, pages 141–158. Johns Hopkins Univ. Press, Baltimore, MD, 2011.

[8] L. Dabrowski, A. Sitarz, *Curved noncommutative torus and Gauss–Bonnet,* J. Math. Phys. 54, 013518 (2013).

[9] L. Dabrowski, A. Sitarz. Asymmetric noncommutative torus. arXiv:1406.4645, 2014.

[10] F. Fathizadeh and M. Khalkhali. The Gauss-Bonnet theorem for noncommutative two tori with a general conformal structure. *J. Noncommut. Geom.*, 6(3):457–480, 2012.

[11] F. Fathizadeh and M. Khalkhali. Scalar curvature for the noncommutative two torus. *J. Noncommut. Geom.*, 7(4):1145–1183, 2013.

[12] F. Fathizadeh and M. Khalkhali. Scalar curvature for noncommutative four-tori. *J. Noncommut. Geom.*, To appear.

[13] F. Fathizadeh and M. Wong. Noncommutative residues for pseudo-differential operators on the noncommutative two-torus. *Journal of Pseudo-Differential Operators and Applications*, 2(3):289–302, 2011.

[14] P. B. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, volume 11 of *Mathematics Lecture Series*. Publish or Perish Inc., Wilmington, DE, 1984.

[15] M. Khalkhali and A. Moatadelro. A Riemann-Roch theorem for the noncommutative two torus. *Journal of Geometry and Physics*, 86(0):19 – 30, 2014.

[16] M. Kontsevich and S. Vishik. Geometry of determinants of elliptic operators. In *Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993)*, volume 131 of *Progr. Math.*, pages 173–197. Birkhäuser Boston, Boston, MA, 1995.

[17] M. Lesch. On the noncommutative residue for pseudodifferential operators with log-polyhomogeneous symbols. *Ann. Global Anal. Geom.*, 17(2):151–187, 1999.

[18] M. Lesch. Divided differences in noncommutative geometry: rearrangement lemma, functional calculus and Magnus expansion. arXiv:1405.0863, 2014.

[19] S. Paycha. *Regularised integrals, sums and traces*, volume 59 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2012.

[20] C. Levy, C. N. Jimnez, and S. Paycha. The canonical trace and the noncommutative residue on the noncommutative torus. arXiv:1303.0241 [math.AP], 2013.

[21] S. Paycha and S. Rosenberg. Conformal anomalies via canonical traces. In *Analysis, geometry and topology of elliptic operators*, pages 263–294. World Sci. Publ., Hackensack, NJ, 2006.

[22] S. Paycha and S. Scott. A Laurent expansion for regularized integrals of holomorphic symbols. *Geom. Funct. Anal.*, 17(2):491–536, 2007.

[23] D. Quillen. Determinants of Cauchy-Riemann operators on Riemann surfaces. *Funktsional. Anal. i Prilozhen.*, 19(1):37–41, 96, 1985.

[24] S. Scott. *Traces and determinants of pseudodifferential operators*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2010.

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