Gravitational Darkening of Classical Be Stars

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Gravitational Darkening of Classical Be Stars

by

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Faculty of Science
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Submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

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Abstract

Be stars are rapidly rotating B-type stars that possess a gaseous disk formed by matter released from the rapidly rotating, central star. The disk produces the characteristic observational features of these objects: hydrogen emission lines, particularly Hα, infrared excess, and continuum polarization. Gravitational darkening is a phenomenon associated with stellar rotation. It causes a reduction of the stellar effective temperature towards the equator and a redirection of energy towards the poles. Rotation also distorts the star, increasing its equatorial radius. It is an important physical effect in these star-disk systems because the photoionizing radiation from the star is essentially the sole energy source for the disk. The effect of gravitational darkening on models of the thermal structure of Be star disks is systematically studied for a wide range of Be star spectral types, rotation rates, and disk densities. Additionally, the effects of rapid rotation on model Hα lines, spectral energy distributions (SED) and photometric colours are investigated. To achieve this, gravitational darkening has been added to the bedisk and beray codes to produce circumstellar disk models and synthetic observables that include the change in the effective temperature with latitude and the non-spherical shape of the star. The effect of gravitational darkening on disk temperature is generally significant for rotation rates above 80 % of critical rotation. The bulk of the disk material becomes cooler with rotation. For example in a dense disk surrounding a B0V central star rotating at 95 % of its critical velocity, the density-averaged disk temperature is $\approx 2500$ K cooler than its non-rotating counterpart. The effect of gravity darkening on observables such as the Hα line, spectral energy distributions, and colours is found to be primarily through the change in the stellar surface brightness with rotation.
Dedication

I would like to dedicate this thesis to the people whose support I could not have done this without, my parents Judith and Gary McGill and my partner Brian Srivastava.

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I would like to express my sincere gratitude to my supervisors Aaron Sigut and Carol Jones for their expertise and patience.

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A.1 $\omega_{frac}$ versus $v_{frac}$
Opening Remarks

A hundred years ago astronomers could not explain how the sun shines. Now, the behaviour of normal stars and their lives are fairly well understood. But not all stars are normal; there still remain many collections of outliers and “oddball” stellar types that exhibit unusual behaviour not yet fully explained. The classical Be stars are one such group. While normal B stars in many respects, they possess circumstellar material in the form of a dense disk of gaseous material released from the central star.

The process driving this mass loss is not yet understood. The primary goal of research into classical Be stars is to understand the disk formation process and to determine what properties of the central B star allow this process to occur. But before we can completely answer these questions, we must understand the properties of these disks—size, composition, density and temperature distributions, velocities across the disk—in order to identify what this unknown process must be able to build.

Because the disks of classical Be stars cannot be resolved by single telescopes, and the nearest classical Be star systems are just barely resolvable by interferometry, most of the information about their structure must be inferred from the combined spectrum of the disk and the star. We need a way to explore the implications of these observables to understand the disk. Computer codes exist that can generate combined spectral energy distributions and photometric colours for various models of classical Be stars and their circumstellar disks (as well as other observables). These codes are important to determine if a particular configuration of circumstellar gas is a good model for the true system.

In this thesis, the Bedisk (Sigut & Jones, 2007) and BERAY Sigut (2011)
codes have been extended to include the important physical effects of gravitational darkening caused by rapid rotation and then are used to illuminate the nature of these disks and the stars they surround.

This thesis is organized as follows: Chapter 1 is an introduction to Be stars, emphasizing the observations used to understand them. Chapter 2 discusses stellar rotation, von Zeipel’s Theorem, and gravitational darkening. Chapter 3 describes the modifications to BEDISK and BERAY codes required to include gravitational darkening. Chapters 4 and 5 describe in detail the effect gravitational darkening has on the thermal structure of Be star disks. Chapter 6 discusses the effect gravitational darkening has on the observable colours and Hα line profiles of Be stars. Chapter 7 provides the conclusions. Lastly, mathematical derivations can be found in the Appendix.
Chapter 1

Introducing the Be Stars

1.1 What is a Classical Be star?

Table 1.1: Standard properties of B-type, luminosity class V and some III stars. (Class V Data Interpolated from Cox 2000; de Jager & Nieuwenhuijzen 1987 and class III is from Carroll&Ostlie 1996).

<table>
<thead>
<tr>
<th>Type</th>
<th>Radius ( (R_\odot) )</th>
<th>Mass ( (M_\odot) )</th>
<th>( T_{\text{eff}} ) ( (K) )</th>
<th>Luminosity ( (L_\odot) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>B0V</td>
<td>7.40</td>
<td>17.50</td>
<td>30000.</td>
<td>( 3.98 \times 10^4 )</td>
</tr>
<tr>
<td>B1V</td>
<td>6.42</td>
<td>13.21</td>
<td>25400.</td>
<td>( 1.54 \times 10^4 )</td>
</tr>
<tr>
<td>B2V</td>
<td>5.33</td>
<td>9.11</td>
<td>20800.</td>
<td>( 4.76 \times 10^3 )</td>
</tr>
<tr>
<td>B3V</td>
<td>4.80</td>
<td>7.60</td>
<td>18800.</td>
<td>( 2.58 \times 10^3 )</td>
</tr>
<tr>
<td>B4V</td>
<td>4.32</td>
<td>6.62</td>
<td>16800.</td>
<td>( 1.33 \times 10^3 )</td>
</tr>
<tr>
<td>B5V</td>
<td>3.90</td>
<td>5.90</td>
<td>15200.</td>
<td>( 7.28 \times 10^2 )</td>
</tr>
<tr>
<td>B6V</td>
<td>3.56</td>
<td>5.17</td>
<td>13800.</td>
<td>( 4.13 \times 10^2 )</td>
</tr>
<tr>
<td>B7V</td>
<td>3.28</td>
<td>4.45</td>
<td>12400.</td>
<td>( 2.28 \times 10^2 )</td>
</tr>
<tr>
<td>B8V</td>
<td>3.00</td>
<td>3.80</td>
<td>11400.</td>
<td>( 1.36 \times 10^2 )</td>
</tr>
<tr>
<td>B9V</td>
<td>2.70</td>
<td>3.29</td>
<td>10600.</td>
<td>( 8.27 \times 10^1 )</td>
</tr>
<tr>
<td>B0III</td>
<td>13.</td>
<td>20.</td>
<td>29000.</td>
<td>( 1.1 \times 10^5 )</td>
</tr>
<tr>
<td>B5III</td>
<td>6.3</td>
<td>7.</td>
<td>15000.</td>
<td>( 1.8 \times 10^3 )</td>
</tr>
</tbody>
</table>
Be stars are a category of stars primarily composed of B-type stars, but also including some members of the adjacent spectral types, such as hot A-type stars and cool O-type stars. Such stars have surface temperatures ranging from 10,000 to 30,000 K and masses from 2 to 18 $M_\odot$ (solar masses) (de Jager & Nieuwenhuijzen, 1987; Cox, 2000). The typical stellar parameters for the B-type stars are given in Table 1.1. Be stars are different from typical B stars in a number of ways. They are a group of emission line stars and carry the designation ‘e’ due to the presence of an emission line in their spectra at some point within their observational record. Spectral lines of stars are generally dark absorption lines (due to the decrease of the photospheric temperature with height), rather than bright emission lines. The first observation of such emission lines from a star was the detection of the hydrogen Balmer line in emission in the spectrum of the Be star γ Cassiopeia by Anglo Secchi in 1866. Emission lines usually indicate the presence of material surrounding the star, outside the stellar photosphere. The emission is caused by this material re-processing the stellar radiation (absorbing and re-emitting, as well as scattering photons).

A number of other categories of stars are also emission line stars with circumstellar material and several of these occur within the B spectral type. These are generally separated based on the kind of star embedded within the circumstellar material, the geometry of this material (spherical shells, disks, etc), and/or type of lines observed (allowed or forbidden (see Chapter 2 Page 27, of Tielens, 2005), atomic, molecular, or dust features). Each category (Be stars, Herbig Ae/Be stars, P Cygni stars, B[e] stars, and others) has its own set of characteristic features. The term “classical” Be star has often been adopted in order to better distinguish the the Be stars as defined above from B-type emission line stars of other groups, but it has yet to be universally adopted.
A classical Be star is currently thought of as a three component system:

1. A rapidly-rotating, central B star.

2. A dense, cool, and slowly out-flowing equatorial disk of gas (number density $10^{10}$ to $10^{12}$ cm$^{-3}$, $T = 6000$ to 10,000 K, and $v_{\text{radial}} \leq 10$ km/s). This disk has a roughly Keplerian velocity structure. Mass loss rates through the disk are estimated to be $\approx 10^{-8} M_\odot$ of material per year\(^1\) (Carciofi et al., 2012).

3. A polar wind, which is hot, thin and rapidly out-flowing, that is believed to be located above the stellar poles ($N \approx 10^9$ cm$^{-3}$, $T \approx 15000$ K, $v_{\text{radial}} \approx 1000$ km/s). The star looses only $\approx 10^{-10} M_\odot$ of material per year though the polar wind (Kervella & de Souza, 2006).

Be stars have a very long observation history. The Be phenomenon was first noticed in 1866 when Padre Angelo Secchi, the inventor of the stellar spectrograph, started a catalogue of stellar spectra. Normal stars have Balmer absorption lines; however, he reported “une particularité de l’étoile γ Cassiopée,” (a particularity in the star γ Cassiopeiae); in place of the Balmer absorption line, there was “une ligne lumineuse très belle et bien plus brillante que tout le reste du spectre” (a very beautiful, bright line which was much more brilliant than all the rest of the spectrum).

The light from Be stars is observed primarily in four ways: continuum light, spectral lines, polarization, and interferometry. Broad-band, low resolution spectroscopy and photometry give information about continuum flux levels. Observations of the continuum show an infrared excess with a peak at

\(^{1}\)It has been suggested that disk material fall backs onto the star, thus this may not represent an mass loss accumulative mass loss on evolutionary timescale
much longer wavelengths than the peak due to the stellar photosphere. This indicates a significant temperature difference between the star and the disk. Higher resolution spectra show lines within the spectrum, some from the star (absorption lines) and some from the disk (usually emission). The shape of these lines provides important information about the geometry of the system. The observation of continuum polarization from classical Be stars confirmed that their circumstellar material is non-spherical, and interferometry has allowed the disk structure to be resolved.

1.2 Spectral Lines from Classical Be Stars

![Sample of emission lines of Hα, Hβ and FeII 5169 for four classical Be stars including o Centauri Aquilae, HR4823, μ Cen and HR5223 (from Figure 3 of Porter & Rivinius, 2003).](image)

Classical Be star systems produce spectral lines from a variety of atoms
and ions across a wide range of wavelengths (both in emission and absorption), which have been observed from the ultraviolet (UV) down to \( \approx 1 \) millimetre (Houck et al., 2004; Hony et al., 2000; Porter & Rivinius, 2003). Examples of these spectra can be seen in Figure 1.1. The source of these lines can be the stellar photosphere, the dense circumstellar disk, the thin polar wind, or combinations of these components. The defining observational features of a classical Be star are hydrogen emission lines, in particular H\(_\alpha\), the \( 3 \rightarrow 2 \) jump in H\(_1\) at 6564 Å. Classical Be stars of all sub-types usually show emission in the first members of the Balmer series (H\(_\alpha\), H\(_\beta\), H\(_\gamma\)) (Arias et al., 2006). These emission lines are produced primarily through recombination: the stellar radiation ionizes the disk, and electrons that recombine above the ground state create emission lines.

A number of different helium and metal lines are also observed in certain classical Be stars. \( \gamma \) Cassiopeia, in particular, emits a particularly large number of helium and various metal lines (Arias et al., 2006; Jones et al., 2004). Not all lines from a classical Be stars share the same basic line shape or indicate an emission disk. The UV resonance lines, affected by the polar wind, have P Cygni shapes and indicate a different geometry and a different source region for these lines. Most spectral lines are formed in the stellar photosphere and are in absorption.

\subsection{Absorption Lines from the Star}

The lines seen in the photospheric spectrum of classical Be stars are similar to those of typical B-type stars and indicate normal ranges of gravity, temperature and abundances. However these photospheric lines have very large line
widths (Porter & Rivinius, 2003; Cranmer, 2005). The lines in B-type stellar spectra include neutral hydrogen (H\textsc{i}), and neutral helium (He\textsc{i}), and single ionized metals (Harwit, 2000).

Details in the line shape and strength can yield the effective temperature, surface gravity, and in the case of rotating stars, the projected equatorial rotation speed, \( v \sin i \), where \( v \) is the equatorial rotation speed, often written, \( v_{eq} \), and \( i \) is the inclination angle, the angle between stellar pole and direction of the observer (Gray, 1992). The He\textsc{i} \( \lambda 4471\text{Å} \) and Mg\textsc{ii} \( \lambda 4481\text{Å} \) lines are common choices when analyzing the line widths for rotation (Townsend et al., 2004; Chauville et al., 2001). The line \( \lambda 4471 \) is often used for estimates of the effective temperatures and surface gravities (Chauville et al., 2001). Unfortunately, rotational broadening of spectral lines can sometimes make spectral typing and luminosity classification problematic because it can be difficult to correctly estimate the strength of the wide and shallow absorption lines from fast rotators. Due to this, there are often more uncertainties about the spectral type and luminosity classification of classical Be stars than normal stars (Porter & Rivinius, 2003). The photospheric spectral lines are the key to understanding the central star, which is the source of energy for the disk emission.

1.2.2 Emission Lines from the Disk

Struve’s initial work on Be stars in 1931 was based on the various types of shapes observed in the emission lines. This work provided the first real insight into the cause of Secchi’s “very bright and beautiful” spectral line. Struve noticed that there were essentially two extremes in the emission line shapes from Be stars: (1) Be-shell stars which exhibit strong and enhanced absorption
in the centre of the line, emission in widely separated peaks, and wide photospheric lines; (2) Be stars which show emission in fairly narrow peaks, which also have narrow photospheric lines. Other stars are in-between cases connecting these two extremes. Struve postulated that these different line shapes did not represent different geometries of circumstellar material. Rather, he suggested that the same essential geometry, a rotating disk or equatorial envelope, was being viewed from different angles, as shown in Figure 1.2. A system which is viewed from the stellar pole will have no absorption components as none of the circumstellar material obscures the star. There is no relative velocity between the observer and the medium so the line is unshifted and contributes enough photons to both fill-in the natural photospheric absorption line and produce emission above it. A line with enhanced central absorption and emission in the wings corresponds to an edge-on view, towards the stellar equator where the gas directly in front of the star has no relative velocity to the observer; the enhanced central absorption is unshifted. The light coming from one edge of the disk is red-shifted and the other, blue-shifted. As this gas is
not obscuring the star, it contributes emission in the wings, resulting in two peaks.

Struve noted an additional correlation: the size of the projected rotational velocity of the photospheric lines ($v \sin i$) was connected to the width and shape of the emission lines: a star which showed narrow and (nearly) pure emission in the hydrogen lines also had lower projected rotational velocities than the average Be star. Further, the highest projected velocities of Be stars tended to be the Be shell stars (Slettebak, 1979). Struve concluded that the star and the disk shared a common rotational axis. These facts, and the unusually high projected velocities, prompted Struve to suggest that these stars are rotating fast enough that material could leave the stellar surface and form a disk. The idea that material is simply “thrown off” the star by rotation is likely incomplete at best (see Section 2.2), but the general picture of a flattened, rotating envelope surrounding the equator of a rapidly rotating central B star remains the basic picture of a classical Be star system today (Sigut & Jones, 2007; Carciofi & Bjorkman, 2006).

The presence of emission lines in the spectra of classical Be stars continues into the infrared. Spectra of several Be stars were taken by the Infrared Space Observatory (ISO) between 2.4 and 4.1 μm (Lenorzer et. al., 2002). See Figure 1.3 for a detailed spectrum of γ Cassiopeiae from ISO. In the infrared (IR), Be stars show emission in several hydrogen line series, including the Humphreys, Pfund and Bracket, as well as a few Helium I emission lines and some currently unidentified lines (Hony et al., 2000).

Some of the most important information that can be derived from emission lines is about the kinematics in the wind and disk. The shape of the emission lines provide information about the velocity structure of the disk. Different disk formation theories can be tested using velocity profiles: viscous disks
are predicted to be in Kelperian rotation; regions of magnetic rotator winds rotate at constant angular velocity; and the wind compressed disks are angular momentum conserving (Porter & Rivinius, 2003). Hummel & Vrancken (2000) simulated the line shapes for various velocity profiles of the form $v(r) \propto r^{-j}$ and concluded that classical Be stars possessed a disk with near Keplerian velocity structure (i.e. $j \approx 1/2$) suggesting that viscous disks surround these stars.

1.2.3 The Polar Wind

Not all emission lines originate in the disk. UV spectral lines indicating a high velocity wind were first detected in $\gamma$ Cassiopeia using a rocket-born spectrograph by Bohlin (1970) and Snow (1981). These lines had strong P Cygni-like
profiles with minimum outflow velocities of 450 km/s. Winds with velocities as high as 1000 km/s have been detected (Snow, 1981). The Copernicus and International Ultra-violet Explorer (IUE) satellites have made important contributions to our understanding of the winds of Be stars and their mass loss rates (Grady, 1989; Snow, 1981). (Snow, 1981) presents Copernicus observations of Si III and SiIV of 22 B type stars, 19 of which have Be star characteristics and reports mass loss rates between $10^{-11}$ and $3 \times 10^{-9} M_\odot$ yr$^{-1}$.

There is strong variability in the UV spectra from these objects (Doazan et al., 1986). There are species present that indicate higher temperatures and ionization. These lines have quite different shapes from the optical emission lines, with high blue shifts corresponding to high outflow velocities with only weak indications of possible rotation. This suggests that the UV resonance lines and the optical lines come from regions of different kinematic behaviour. These wide, optically thin, blue-shifted, UV resonance lines (in absorption) are from highly-ionized species such as SiIV, and C V and are characteristic of a low-density, high-velocity plasma (Dougherty et al., 1994). Estimates of the terminal velocities of these winds are 1000 km/s with densities of $N \approx 10^9$ cm$^{-3}$ at the base of the wind (Kervella & de Souza, 2006). The Be star $\gamma$ Cas has usually strong and high variable wind. It has variable, but non-transient, discrete absorption components with maximum absorption occurring at $-1280$ km$^{-1}$ and X-ray flaring Cranmer et al. (2000). Other Be star also show discrete absorption components Cranmer et al. (2000).
1.3 Continuum Photometry of Be Stars

Classical Be stars typically show an infrared excess, but this is not universally true. α Eridani, the brightest and closest classical Be star, has no measurable infrared excess. However, many classical Be stars have a flattened and redder spectrum than a B star of equivalent spectral type. This “infrared excess” is more accurately described as a “long wavelength excess,” as the excess remains noticeable for some classical Be stars into the radio band. It begins as a weak excess in the optical, resulting in contributions to the (B-V) colour index of around a tenth of a magnitude (Howells et al., 2001) and becomes more significant towards longer wavelengths. It peaks at about 10 microns and, while still present, is dimmer at radio wavelengths (Waters et al., 1987; Dougherty et al., 1991). In the radio band, all the radiation is essentially due to the circumstellar gas. The longest wavelength detection of a Be star is at 6 cm (Dougherty et al., 1991).

There are two primary sources of continuum radiation within the disk of classical Be stars. The disk is a partially ionized gas and free-free (or Bremsstrahlung) radiation is emitted by charged particles scattering off each other. The most significant contribution is free electrons scattering off protons. Free-free emission is proportional to the square of the electron density, making it a sensitive density diagnostic. The opacity of an isothermal gas producing free-free emission goes as \( \rho^2 \lambda^2 \) (Waters, 2000). The second source of continuum emission is the free-bound emission due to electrons in the continuum re-combining into bound states.

Coté & Waters (1987) reports infrared photometry from the Infrared Astronomical Satellite (IRAS) at three different band passes centred at 12, 25 and 60 microns for 101 Be stars, providing a statistically significant sample.
Photometry measurements and some spectra (see Section 1.2) of a limited number of Be stars were taken with the infrared space observatory (ISO). The ISOPHOT instrument performed measurements at 12, 25, 60, 90, 135 and 160 microns. Moderate resolution spectra were taken for 3 Be stars and a single high resolution spectra was taken for $\gamma$ Cassiopeiae between 2 and 12 microns. While analyzing the line strengths (see Section 1.2), Hony et al. (2000) used the size of the Humphreys jump at 3.28 microns to derive the average electron temperature of the emitting region. This was done by calculating the normalized excess due to the presence of the emission of the disk alone and then identifying the wavelength red-ward of the Humphreys jump at which the normalized excess is equal to that at the Humphreys jump. This allows the temperature to be found from the bound-free gaunt factor, which is sensitive to temperature. The electron temperature was found to be $9500 \pm 1000$ K, in agreement with the theoretical computed temperatures of Millar & Marlborough (1998).

### 1.4 Polarization in Classical Be stars

Classical Be stars have been known to possess variable continuum polarizations since the 1960’s (Coyne & Kruszewski, 1969). Polarization within continuum light is observed to be between 0 to 2% (Wood et al., 1997). Because the presence of linear polarization indicates scattering in a non-spherical region, polarization studies confirm that classical Be stars possess non-spherical, circumstellar material, strengthening the long held idea that classical Be stars are disks (Waters & Marlborough, 1992). Polarization measurements also provide information about electron density within the circumstellar material as scattering off free electrons (Thomson scattering) is the cause of the continuum
polarization (Waters & Marlborough, 1992). Waters & Marlborough (1992) used linear polarization, in combination with IR excess measurements, and a simple disk emission model (which assumes polarization is by single scattering events), to constrain the geometries of Be star disks, in particular their density parameters. Be star disks are treated as having a radial density structure decreasing according to a power-law. Waters & Marlborough (1992) determined that the index of this power-law must be greater than 2.5. This type of radial structure is consistent with hydrodynamical models, such as Okazaki (2012). Waters & Marlborough (1992) also note that the bulk of the polarization is produced close to the star, within 2 or 3 stellar radii due to the higher local density.

The spectral lines of 58 classical Be stars were examined as part of the MiMeS survey for circular polarization signatures of Zeeman splitting, indicating the presence of magnetic fields. However no stars were found to have a detectable field (Wade et al., 2012).

1.5 Interferometry

When interferometry is performed, radiation from a source is gathered by two or more telescopes and is recombined in pairs to produce a fringe pattern of alternating constructive and deconstructive interference. The location of the brightest band due to constructive interference defines the phase of the fringe pattern, and the strength of the intensity change between constructive and deconstructive interference is called the visibility. Phases and visibilities are measured for an individual baseline as projected across the sky. A single baseline responds to a single spatial frequency within the brightness distribution of the source, and this is a single piece of the two dimensional Fourier transfor-
mation of the sky brightness distribution. By making many measurements of various baselines with different projections on the sky, the Fourier transform of the image can be built and potentially, if sufficient baselines are measured, an image itself can be found by taking the inverse Fourier transform.

Figure 1.4: The left panel is a constructed K'-band image of ψ Persei, with its disk and its companion (the small white dot in the lower centre). Both the x-axis and the y-axis are in mas. The right panel is the measurements used to create this image, it is the Fourier transform of the image in the left panel. Both the x-axis and the y-axis are in cycles arcsec\(^{-1}\). The fringe pattern across the right panel is due to the presence companion. The white crosses indicate the location of a measured base-line projected on the inverse sky and at each of these points a visibility (plotted as visibility squared) was measured. (from Figure 8 of Gies et al., 2007).

Currently interferometry can be used to resolve the closest of the Be stars. Achernar (α Eridani) has been imaged in the K' band, an infrared photometric band with an effective wavelength of 2.1501 μm, with CHARA (Domiciano de Souza et al., 2003; Kervella & de Souza, 2006). This has allowed the ratio between equatorial radius and the polar radius of the star to be estimated at ≈1.6 by treating the stellar sky brightness function as a top hat function at different orientations (Domiciano de Souza et al., 2003). A ratio of 1.6 is
higher than the maximum of 1.5 predicted by Von Zeipel’s Theorem (see 2.3). This estimate has been refined to 1.51 ± 0.02 using a uniform ellipse as the stellar sky brightness function (Kervella & de Souza, 2006). The polar wind of Achernar has been resolved in the K band as well, showing two diffuse streams above each polar region (Kervella & de Souza, 2006).

From a technological perspective, radio interferometry is the most straightforward as the signals for the various telescopes can be electronically (as opposed to optically) combined. Unfortunately, Be stars are not particularly bright in the radio, and so far, only one disk has been resolved in the radio, ψ Persei (Dougherty & Taylor, 1992) (see Figure 1.5) which has the strongest radio brightness of any Be star. From this measurement the size of the emitting region was estimated to be 17 AU assuming a distance to ψ Persei of 155

The circumstellar disk has also been resolved in the infrared and the optical. Because the disk is so much brighter than the star in Hα, narrow band optical interferometry centered at these wavelengths is quite effective. The extent of the H α emitting region of Be stars γ Cas and ψ Per was measured with the Navy Prototype Optical Interferometer, and it was determined that the disk is consistent with a Gaussian (Tycner et al., 2006). Chesneau et al. (2005) presents VLTI/MIDI (Very Large Telescope Interferometer MID-InfraRed Interferometer) observations of the Be star α Arae in the N band, a mid infrared band centered at 10 μm. The disk is estimated to be ≈ 14 Rₚ at this wavelength.

1.6 The Variability of Be stars

Classical Be stars are variable objects, but they show considerably different levels of variability from one object to another. Some classical Be stars, such
as α Eridani, 1 Delphini and β Canis Minoris, have shown essentially stable Hα emission, with only small and slow changes, for as long as they have been observed (Saio et al., 2007; Marlborough & Cowley, 1997). However others, such as θ Coronae Borealis or HD 6226, show emission that appears and disappears over six months (Šlechta, 2004). Generally Hα emission variations are associated with the circumstellar disk and are variable over time scales of a few years to decades (Porter & Rivinius, 2003; Carciofi, 2009). Variability has been shown in virtually all aspects of the emission from these systems (i.e. line, continuum, and polarization; see Porter & Rivinius 2003). This makes it difficult to get a complete picture of the emission from classical Be stars unless many different wavelengths and types of measurements are taken contemporaneously. Not only is Hα variability observed, but variability is also seen in the metal lines and in almost the entire background continuum (optical, UV, IR and possibly the radio) (Porter & Rivinius, 2003; Arias et al., 2006;
Dougherty et al., 1991). The various forms of variability seen in Be stars have been associated with changes in the star, the disk, and the polar wind.

### 1.6.1 The Star: Short-Term Variability

Short term variability, on timescales between 0.5 to 2 days, is observed in the photospheric line shapes and photometry of most classical Be stars (Percy et al., 1994) and seems associated with changes in photosphere/near photosphere. These short-term photospheric variations are more easily monitored in the stellar lines than in photometric observations because other effects (produced within the disk) are superimposed on the photometric variability. However there exist Be stars with short term photometric variability with no photospheric line profile variability (Rivinius et al., 2003). A sample of the line profile variability (LPV) seen in classical Be stars is shown in the right panel of Figure 1.6. This short-term LPV tends to occur in early type classical Be stars (Aerts, 2000). Two theories have been put forward to explain the short term LPV, rotational modulation and non-radial pulsation, with the pulsation model currently favoured (Porter & Rivinius, 2003).

For a number of Be stars, including $\mu$ Centauri, $\omega$ Canis Majoris, $\theta$ Coro- nae Borealis, and $\lambda$ Eridani, variability has been linked to pulsation. Sometimes $\lambda$ Eridani is used to define a sub-class of classical Be stars with short term photometric variability and pulsation (Aerts, 2000). Some of these pulsating stars have “multiple modes” or “multiple periods,” including $\mu$ Centauri, EW Lacertae, $\zeta$ Ophiuchi (Aerts, 2000). This means that they have two or more pulsation frequencies and associated motion modes with the total motion of the stellar surface a combination of these modes. It has been suggested that outburst (disk building) events could be related to overlapping maxima.
between multiple modes (Porter & Rivinius, 2003; Aerts, 2000). Many Be stars with LPV undergo line emission outbursts (Rivinius et al., 2003); however, as yet only a small number of Be stars have been shown to have multiple pulsation modes.

Figure 1.6: The left panel shows the observed Hα line profiles of the classical Be star HD 6226, from 1997 to 2003. Hα goes from an absorption line on 2450714.0546 to an emission line, on 2450658.2815, reaching peak emission somewhere between dates 2452679.3053 and 2452706.6355, ≈ 5 years later. The line has returned an absorption line by 245868.9338, 5 months after this peak. The right panel shows the observations of the He I 6678 line profiles of HD 6226 over the same period. Note the rapid variation of the asymmetry of the line in the three observations from 2452834.4792 to 2452836.5613, a little more than two days (from Figure 7 & 8 of Bozic et al., 2004).
1.6.2 The Disk: Long-Term Variability

The most noticeable variability within classical Be stars is associated with the complete loss of the disk and its rebuilding sometime later. When a classical Be star has a disk, it also has Hα emission and the other Be star characteristics, and is considered to be in an “active phase.” When the disk is absent, its spectra is essentially that of the underlying B star, and it is considered to be “inactive.” The most notable features of shifting “out of” and “back into” active phases is the fading and re-emergence of the emission lines as seen in the left panel of Figure 1.6. An emission line can emerge rather quickly, on the scale of weeks and can persist for years and then fade again over the course of months (Šlechta, 2004).

There are more subtle variabilities observed in classical Be stars. In particular classical Be stars have been known to shift from shell type emission, with a central absorption beneath the stellar continuum, to an ordinary Be star with double peaked emission (Porter & Rivinius, 2003; Doazan et al., 1986). This is associated with disk-building phenomena (Rivinius et al., 1998), as the disk density changes. This can create a shift in line profiles without a change in disk orientation (Silaj et al., 2009).

Another subtle but significant variability in classical Be stars is “V/R variations.” The emission lines of classical Be stars often exhibit double-peaked emission, but these two peaks are not always symmetric: one peak is often larger than the other. The short wavelength half of the line is termed the violet side (V) and the long wavelength half, the red side (R).

A V/R variation refers to the shape of the line changing over time, from the red peak being larger, then both peaks being equal, followed by the violet peak being larger, and so on. This variation is semi-periodic and long term,
taking from a few years to a decade to finish a full cycle (Porter & Rivinius, 2003; Carciofi, 2009). These changes are on the viscous time scale of the disk (Kroll & Hanuschik, 1997). This phenomenon has been associated with a density perturbation orbiting or precessing around the star, often described as a “one-armed density wave” (Carciofi, 2009). These V/R variations are also connected to variations in the polarization (McDavid et al., 2000). For example, in order to model both the polarization and line changes for 48 Librae, a “spiral arm,” rather like a galactic arm, is required, rather than a radial enhancement, because of the phase difference between these polarization and line shape cycles (McDavid et al., 2000). Such phase lags are observed in other stars as well (Porter & Rivinius, 2003).

1.6.3 Other Variability

Many extremely short-term spectral features occur in early type classical Be stars. Examples are brief transient features and X-ray flares (Porter & Rivinius, 2003; Smith & Robinson, 2003; Rivinius et al., 1998). Blue shifted absorption features, which form in less than 10 minutes and have lifetimes on order of one hour, have been reported in the spectra of a number of classical Be stars (Peters, 1986; Penrod, 1986). Such features have been observed within photospheric lines of λ Eridani and other Be stars such as γ Cassiopeiae (Porter & Rivinius, 2003; Smith, 1989; Smith & Robinson, 1996). These events are believed to relate to active star-to-disk mass transfer (Rivinius et al., 1998). Between 1992 and 1997 the classical Be star μ Centauri was building stronger Hα emission (and presumably a disk) (Rivinius et al., 1998), and during this period, sudden changes in line strength and shape would occur over the course
of approximately 20 days, termed outburst cycles. Emission strength would remain relatively stable, then decrease in a precursor phase, build up to its highest emission in an outburst and decrease back to fairly stable emission (Rivinius et al., 1998).

### 1.7 Be stars

Although classical Be stars are viewed as a one group of stars, oddities still exist amongst this group: for example α Eridani does not have a noticeable IR excess; γ Cassiopeiae has a strong and variable X-ray spectra; ψ Persei is unusually bright in the radio; and 1 Delphini is considerably less variable then other classical Be stars.

Any attempts understand Be stars systems necessitate the understand of the central Be star which is the source of the gas and powers the circumstellar emission. In Chapter 2, we turn our attention to the star, what is it about the star that causes it to possess circumstellar material during its main sequence lifetime? How is it different from normal B stars?
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Chapter 2

Stellar Rotation

2.1 What makes a star a Be star?

We know that a little under 20\% of all B-type stars are classical Be stars, but this percentage is not constant across stellar type and evolutionary age (Porter & Rivinius, 2003). It is important to understand how the fraction of Be stars varies with stellar properties, including, but not limited to, spectral type, luminosity classification, age, metallicity, binarity, rotation rates and cluster density. These details can provide important clues that can constrain any proposed formation model. The factors that are associated with increased Be star fractions may be connected to what causes these stars to produce their decretion disks.

Stars in clusters are essentially the same age and metallicity. Spectral type, binarity and \( v \sin i \) are the only significant causes for variation in a star cluster. One particular question that can be addressed in cluster studies is how evolutionary factors (for example, stellar age) can be associated with the classical Be star phenomenon. It has been proposed that the classical Be star
phenomenon is associated with the later half of a B-type star’s main sequence lifetime (Fabregat & Torrejon, 2000). The estimated main sequence lifetimes of B-stars vary from 7 million to 1 billion years, depending on mass. This relationship can be explored by finding the fraction of classical Be stars within clusters of different ages. If classical Be stars exist within very young clusters, where most B stars are still near the zero age main sequence (ZAMS) line, it would imply that some stars are simply formed as Be stars. However, if there is a minimum age below which classical Be stars are not found, the Be phenomenon must be acquired during a star’s lifetime. Real data suggests both scenarios occur. Very young star clusters have measurable fractions of classical Be stars. The percentage of classical Be stars then increases with cluster age, reaching a maximum of 46% for early Be stars (B0-B3) at 12-20 Myr (Tarasov & Malchenko, 2012). Since classical Be stars are also strongly associated with rotation, this might be explained by some classical Be stars being born as rapid rotators, with other classical Be stars acquiring their rotation rates during their main sequence lifetime, possibly due to an evolutionary process or binary spin-up (Ekström et al., 2008).

Higher fractions of classical Be stars are also associated with lower metallicity clusters and higher cluster density (McSwain & Gies, 2005). The higher cluster density may be related to a high incidence of binarity found within classical Be stars. A large percentage of classical Be stars are known, or suspected, members of binary systems: approximately 30-50% of classical Be stars are members of binary systems (Gies, 2000), however, binarity is more frequent within early type stars in general, some studies suggest it is somewhat more prevalent among classical Be stars but there remains debate (Berger & Gies, 2001; Oudmaijer & Parr, 2010). It has been suggested that all classical Be stars have a binary companion even if this companion is as yet undiscovered (Gies,
Binarity has three significant implications: (1) A binary companion can affect the Be system directly by truncating or warping the outer regions of the disk (Okazaki, 2012); (2) Binary spin up early in the system’s lifetime may explain the high rotation rates of classical Be stars; (3) Mass transfer in the system may explain the origin of some Be star disks. These phenomena only occur in fairly close binary systems.

2.2 Rotation and the Classical Be star phenomenon

One of the longest recognized characteristics of classical Be stars is the presence of extremely rotationally broadened photospheric lines. Classical Be stars are known to rotate faster than any other group of non-degenerate stars (i.e. excluding white dwarfs or neutron stars) (Porter & Rivinius, 2003). The correlation between rotation and the classical Be star phenomenon prompted the idea, as described earlier, that these stars are rotating so fast that their atmospheres near the equator could drift off the surface of the star and form a disk (Struve, 1931). However, the currently accepted view (Yudin, 2001) is that classical Be stars rotate 20-30% below the speeds required for material to freely leave the stellar surface. While rotation certainly lowers the amount of energy and angular momentum required to build a rotating disk, it is currently believed that some other mechanism (or mechanisms) must provide additional energy and, especially, angular momentum in order to produce the Keplerian disks seen in Be stars. Pulsation and magnetic fields have been put forth as possible mechanisms to build the disk (Cranmer, 2009; Grundstrom et al., 2000).
2011; Smith & Balona, 2006), however magnetic fields have not been found in classical Be stars (Wade et al., 2012).

2.2.1 Measuring Rotation

The term critical velocity is used to describe the rotational speed (or angular speed) that produces unbound material at the equator of a rotating star. Within this document, $v_{\text{crit}}$ shall be used to indicate the velocity of the stellar equator for a critically rotating star, and $\omega_{\text{crit}}$, to describe the corresponding angular speed. They are defined in terms of the stellar parameters, $R_{\text{eq}}$, the equatorial radius, $R_p$, the polar radius, and $M_*$, the stellar mass, such that:

$$\omega_{\text{crit}} = \sqrt{\frac{GM_*}{R_{\text{eq}}^3}} = \sqrt{\frac{8GM_*}{27R_p^3}}, \quad (2.1)$$

and

$$v_{\text{crit}} = R_{\text{eq}}\omega_{\text{crit}} = \sqrt{\frac{2GM_*}{3R_p}}, \quad (2.2)$$

where we have used the result that $R_{\text{eq}} = 3/2R_p$ for a critically rotating star using the Roche model. Understanding how fast a star is rotating compared to this critical velocity is important to determine how easily material can be launched from the stellar surface. For this reason, the rotation speed of a star is often described in fractions of the critical speeds as either $w_{\text{frac}} = \omega/\omega_{\text{crit}}$ or $v_{\text{frac}} = v_{\text{eq}}/v_{\text{crit}}$. Determining these values can be difficult as there are uncertainties from a wide variety of sources (see below Sections 2.2.1 & 2.3.2).

Determining both the typical fractional rotational velocities and the range of velocities found amongst classical Be stars is important to understand the
Figure 2.1: Relationship between spectral type and rotation speed at which the critical velocity becomes obtainable by a disturbance equal to the sound speed of the stellar atmosphere, \((v_{\text{crit}} - v_{\text{sound}})/v_{\text{crit}}\).

feasibility of disk formation scenarios. Models have predicted that in order for a weak process (such as pulsation) to lift material off the star, the central star must be rotating with a speed of approximately 0.95 \(v_{\text{crit}}\) (Townsend et al., 2004) or higher. Knowing the rotation rates of these stars allows us to calculate how much additional energy and angular momentum is required to build the disk.

The inner edge of the Keplerian disk is believed to be orbiting at the critical speed. Pulsations or linear disturbances in the stellar atmosphere can generate flows of the same order as the sound speed, \(V_{\text{sound}} = 12.85\text{km/s}\sqrt{(T_{\text{eff}}/10000\text{K})}\). Figure 2.1 shows how quickly a star must rotate for a push equal to the sound speed to achieve the orbital/critical speed for a range of spectral types. The value of this speed changes only slightly over the B type stars. For a B0V star it is just under 96% of the critical velocity and increases to below 97% of the critical velocity for an A0V star. These rotation rates are much higher than those normally ascribed to Be stars, particularly for the early-type stars.

In order to estimate the average \(v_{\text{frac}}\) for classical Be stars, we must first
know $v_{\text{crit}}$. This means it is necessary to know both the mass and the radius of a star. Measuring the mass of a star is only possible in binaries, so only a very small percentage of stars have well known masses. Directly measuring the radius of a star is also a difficult prospect, possible only for eclipsing binaries or with interferometry for single stars. For most stars, radii and masses are estimated from standards of their spectral type and luminosity classification (see Tables 1.1 & 2.1). Rotation is not taken into account in these standard models, despite the fact that rotation is believed to affect the apparent spectral type of a star due to gravitational darkening, depending on inclination angle, and the relationship between mass, spectral types and radius (Maeder & Meynet, 2000). This introduces errors in fitting real stars to these simple standards (Howarth, 2007). This is more difficult in the case of stars with persistent circumstellar material because a purely photospheric spectrum can never be measured. Broadening of the photospheric lines can also make assigning spectral types difficult as lines become blended into adjacent lines or the continuum (Slettebak et al., 1980). Certain individual Be stars have been assigned to various spectral types and luminosity classification by different authors. As an illustration of these difficulties, α Eridani, is classed from B3-B4IIIe to B4Ve (e.g. Slettebak, 1982; Balona et al., 1987), and γ Cassiopeiae is also seen with luminosity class III-V.

Measurements of line widths for classical Be stars typically imply equatorial rotation speeds of a few hundred kilometres per second. Table 2.1 contains measurements of line widths for a small sample of individual Be stars (Yudin, 2001). Measurements of line widths only give us the projected rotational velocity along the line of sight, $v \sin i$. It is not possible to directly measure the equatorial velocity of a star without knowing its inclination angle, $i$. How-
Table 2.1: $v_{\text{frac}} = v_{\text{eq}}/v_{\text{crit}}$ for several prominent Be stars.

<table>
<thead>
<tr>
<th>Star</th>
<th>Stellar Line Type</th>
<th>Mass ($M_\odot$)</th>
<th>$R_p$ ($R_\odot$)</th>
<th>$v_{\text{crit}}$ ($\text{km}/\text{s}$)</th>
<th>$\frac{v_{\text{sin},i}}{v_{\text{crit}}}$</th>
<th>$\sin i$</th>
<th>$\frac{v_{\text{eq}}}{v_{\text{crit}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$ Per</td>
<td>B2Vpe</td>
<td>9.6</td>
<td>5.4</td>
<td>477</td>
<td>0.83</td>
<td>0.85</td>
<td>0.94</td>
</tr>
<tr>
<td>$\zeta$ Tau</td>
<td>B4IVe</td>
<td>6.4</td>
<td>4.2</td>
<td>440</td>
<td>0.55</td>
<td>0.95</td>
<td>0.57</td>
</tr>
<tr>
<td>$\gamma$ Cas</td>
<td>B0IVpe</td>
<td>17.5</td>
<td>7.7</td>
<td>538</td>
<td>0.47</td>
<td>0.60</td>
<td>0.78</td>
</tr>
<tr>
<td>$\beta$ CMi</td>
<td>B8Ve</td>
<td>3.8</td>
<td>3.0</td>
<td>401</td>
<td>0.61</td>
<td>0.70</td>
<td>0.87</td>
</tr>
</tbody>
</table>

$a$ Yudin (2001)  
$b$ Tycner et al. (2005)  
$c$ Townsend et al. (2004)

However, if a large number of stars are measured, it can be assumed that their rotational axes are randomly oriented in space, and then the true average of the equatorial velocities is related to the average of $v \sin i$ measurements by $\langle v_{\text{eq}} \rangle = (4/\pi) \langle v \sin i \rangle$. The $(4/\pi)$ coefficient comes from random distribution of the angle $i$. Thus the difficulties associated with the projected angle affects only the measurement of individual stars, not our ability to understand the distribution of rotation rates among a large number of classical Be stars. Attempts have also been made to determine the inclination from atmospheric models which include rotation and gravitational darkening, but the accuracy of these methods is unclear (Frémat et al., 2005). In many ways it is easier to estimate $\sin i$ for classical Be stars rather than for normal B-type stars because emission line shapes can provide information about the general orientation of the polar axis in the sky (see Section 1.2). Single peaked emission lines indicate systems nearly face on; double peaked emission lines indicate moderate inclination angles; and shell lines indicated systems that are nearly edge on. While this is only an estimate and has a large error (on the order of $15^\circ$) (Silaj et al., 2009), no such estimate is possible for a normal star.

Interferometric imaging, which allows the shape of the star to be deter-
mined, combined with photospheric line measurements of $\nu \sin i$, is the most accurate method of determining the inclination, particularly if it is rotating quickly, but this can only be done for fairly close systems. The current list of seven interferometrically resolved rotating star range in distance from 5 to 45 parsec (van Belle, 2010; van Leeuwen, 2007).

Only when estimates of $v_{\text{crit}}$ are combined with measurements of the widths of atmospheric lines can $v_{\text{frac}}$ be estimated. Yudin (2001) gives the current canonical value of 70-80% of the critical speed, while Townsend et al. (2004) suggests that values may be as high as 80-90% of the critical speed. Cranmer (2005) contends that early type Be stars may rotate at speeds as low as 40% of the critical speed and that most late type Be stars rotate quite close to their critical speeds. The difference between the rotation rates of early stars and late stars means that there may be a difference in the disk formation mechanism. This may be supported by evidence that early-type Be stars are more variable and later type Be stars have more stable disk emission (Cranmer, 2005).

2.3 Modelling Rotation

At the centre of a Be star disk is a rapidly rotating B-type star. Treating this embedded star as a normal B-type star, and using the standard atmospheric results for its spectrum, does not take into account the most extraordinary property of the star, its rotation. No complete, toneyanalytic, self-consistent treatment of stellar rotation exists (Tassoul, 1978) and compounding the problem is how evolution and rotation interplay (Maeder & Meynet, 2000). The addition of a centrifugal term to the gravitational potential of a rotating star changes the local pressures and pressure gradients by reducing the effective local gravity (and changing its direction as well). This alters both energy gen-
eration within a rotating core and the structural support throughout the star. Because rotation reduces the ability of gravity to compress the material, a rotating core will burn slower and longer than a non-rotating core of equivalent mass. The changes in energy generation and energy transport have a dramatic impact on the atmosphere above. A rotating star is no longer spherical, but oblate.

True solid body rotation cannot occur within a star (Clayton, 1968, Chapter 6) and a rotating star should naturally have some form of differential rotation in either latitude or depth. Gradients in the rotational speeds will lead to complex hydrodynamic circulations occurring within the star. Because angular momentum can be transferred between different internal regions over time, a rotating star may not be a time-independent system (Maeder & Meynet, 2000). A full model of a rotating star is an extremely difficult problem requiring nuclear synthesis, structural and hydrodynamic calculations. Progress is being made despite these challenges, but it will be some time before the full effects are known (Rieutord & Espinosa Lara, 2013). Since no complete model exists, we must content ourselves with using an incomplete and inconsistent model of rotation at least until better models are developed and the results standardized. A straightforward method is to use a combination of the Roche model for the rotating potential and Von Zeipel’s theorem for the effects of this potential on the temperature across the stellar surface. While incomplete and inconsistent, the assumptions are mathematically simple and well understood.
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2.3.1 Gravity Darkening: von Zeipel’s Theorem and the Roche Model

Light from a rotating photosphere is affected by rotation in two ways: (1) the star’s shape is distorted, the radius is larger at the equator than at the poles; (2) the effective temperature of the star is a function of latitude, with the cooler material at the equator. These phenomena together are commonly called gravitational darkening. The von Zeipel theorem is a model of the effects of rotation on a stellar atmosphere and the Roche model a treatment of the effect of rotation on stellar structure. The effect is illustrated in Figure 2.2.
Figure 2.3: Image of $v \sin i$ across the surface of the star for a star of 17 $M_\odot$ with a radius of 10 $R_\odot$ and $\omega_{\text{frac}} = 0.95$. All distances are shown in $R_\ast$. The most shifted parts of the spectra are from the edge of the star, which do not include the bright pole of the star.

The Roche Potential

The simplest treatment of rotation is to assume that the star rotates as a solid body. If we begin with a self-gravitating sphere and add a rotational term to the spherical gravitational potential, the location of the equi-potential surfaces are changed. This adjusts the distribution of mass within the star. The change in the mass distribution changes gravitational potential resulting in further changes to the equipotentials. The Roche model sidesteps this feedback and assumes that the potential associated with the mass can still be treated as a sphere. This is a safe assumption near the surface of a centrally concentrated star. For simplicity we require that the polar radius remains fixed (strictly speaking this is not true, but the effect has been shown to be small (Collins & Harrington, 1966)). The Roche potential of a star rotating at fixed angular
Figure 2.4: Calculation of the line width of a star rotating at \( v_{frac} = 0.95 \) with (solid line) and without (dotted line) the effect of gravitational darkening on the continuum (from Figure 3b of Cranmer, 2005).

speed \( \omega \) in spherical coordinates, \((R, \theta, \phi)\) is given by:

\[
\Phi (R, \theta, \omega) = -\frac{GM_*}{R} - \frac{1}{2} \omega^2 R^2 \sin^2 \theta, \tag{2.3}
\]

where \( G \) is the gravitational constant, \( M_* \) is the stellar mass and \( \theta \) is the co-latitude angle measured from the polar axis. By taking the negative gradient of the potential, one produces the local gravity,

\[
\vec{g} (\theta, \omega) = \left( \omega^2 R_\theta \sin^2 \theta - \frac{GM_*}{R_\theta^2} \right) \hat{r} + \left( \omega^2 R_\theta \sin \theta \cos \theta \right) \hat{\theta}, \tag{2.4}
\]

where \( R_\theta \) is the radius at co-latitude \( \theta \). When \( \omega \) is equal to \( \omega_{crit} \) or \( \sqrt{\frac{GM_*}{R_{eq}}} \), the radial term of the local gravity is zero at the equator and critical rotation occurs. The critical angular velocity \( \omega_{crit} \) and the corresponding critical velocity are given by Equations (2.1) and (2.2). A critically rotating star has a single unbound latitude at the equator \( (\theta = \pi/2) \) and super-critical rotation within the Roche model would imply an increasingly large unbound band around the middle of the star between latitudes \( \theta_{crit} \) and \( \pi - \theta_{crit} \) where \( \theta_{crit} = \arcsin(1/v_{frac}) \). The change in the potential causes the surface to be
distorted from a perfect sphere. As mentioned previously, the radius is largest at the equator. By requiring the potential across the surface to be equal to that of the pole, the radius is found to follow:

\[ R_{\theta, \omega_{frac}} = \left( \frac{-3R_p}{\omega_{frac} \sin \theta} \right) \cos \left( \frac{\arccos (\omega_{frac} \sin \theta) + 4\pi}{3} \right). \] (2.5)

If \( \omega \leq \omega_{\text{crit}} \), the largest value \( R_{\theta, \omega_{frac}} \) can take is \( 3/2R_p \) corresponding to the equator of a critically rotating star. It is useful to express properties in scaled variables \( \omega_{frac}, v_{frac} \) and \( x(\theta, \omega) = R/R_p \). The magnitude of the local gravity expressed in terms of these scaled variables is given by:

\[ g(\theta, \omega_{frac}) = \frac{4}{9} \frac{G M}{R_p^2} \sqrt{ \left( \frac{1}{x^2} - \omega_{frac}^2 \sin^2 \theta \right)^2 + w_{frac}^4 x^2 \sin^2 \theta \cos^2 \theta} \] (2.6)

\[ = \frac{G M}{R_p^2} f(x, \omega_{frac}) \] (2.7)

The change in shape is important for disk models because it effects the distribution of light around the star by tipping the mid-latitudes towards the pole and away from the equatorial regions. It also causes the surface area of the star to increase.

Von Zeipel’s theorem results from the combination of hydrostatic equilibrium and radiative energy transport. This results in a radiative flux that is proportional to the surface gravity and an effective temperature to the fourth power. It is easier for radiation to escape a more loosely held atmosphere, because the density gradient is smaller. On a rotating star surface the decrease in gravity with increasing stellar co-latitude, results in a corresponding temperature decrease. For solid body rotation, only the local gravity can depend on the stellar co-latitude and all other term are constant across the stellar
surface. The resulting expressions are

\[ F_{\text{rad}} = C_1 g(\theta, \omega_{\text{frac}}), \]  
(2.8)

and

\[ T_{\text{eff}} = (C g(\theta, \omega_{\text{frac}}))^\frac{1}{2}. \]  
(2.9)

This defines the local radiative flux, \( F_{\text{rad}} \), and the local effective temperature, \( T_{\text{eff}} \), in terms of the local effective gravity, and the constants \( C_1 \) and \( C \), which differ by the Stephan-Boltzmann constant. These constants are constructed by specifying the stellar luminosity such that,

\[ C_\omega = \frac{L}{M \Lambda(\omega_{\text{frac}})}, \]  
(2.10)

where

\[ \Lambda(\omega_{\text{frac}}) = G \sigma \left( \int_A f(x, \omega_{\text{frac}}) x^2 d\Omega \right). \]  
(2.11)

The stellar luminosity can be treated as a constant or some function of rotation may be defined. \( \Lambda(\omega_{\text{frac}}) \) is, except for some constants, the integration of the normalized local gravity, \( f(x, \omega_{\text{frac}}) = g(\theta, \omega_{\text{frac}}) R_\text{p}^2 / GM \) over the normalized stellar surface. Since the term \( \Lambda \) is only a function of \( \omega_{\text{frac}} \) if \( C_1 \) and \( C \) have been calculated for one star with some \( \omega_{\text{frac}} \) value, \( \omega_\text{\|} \), they can be re-scaled for any other also with \( \omega_{\text{frac}} = \omega_\text{\|} \) using the following,

\[ \frac{C_{\text{star}_1}(\omega_\text{\|})}{C_{\text{star}_2}(\omega_\text{\|})} = \frac{L_{\text{star}_1}(\omega_\text{\|})}{L_{\text{star}_2}(\omega_\text{\|})} \frac{M_{\text{star}_2}(\omega_\text{\|})}{M_{\text{star}_1}(\omega_\text{\|})}. \]  
(2.12)
Figure 2.5: An illustration of change in the shape of a star with increasing $v_{frac}$ for a B2V star. At critical rotation the equator is 1.5 times the polar radius and the unbound equator is a cusp.

An example of the results of the effects of gravity darkening are shown in Figures 2.2, 2.5, & 2.6. Figure 2.5 shows the changing shape of the star as rotation increases. Figure 2.6 shows the divergence of the surface temperatures with rotation for various stellar co-latitudes on a B2V star. The reader should notice that there is an increase in the polar temperatures as well as the more often described drop in temperature near the equator. This occurs because the luminosity remains constant. At critical rotation, the effective gravity of the equator is zero and so is the corresponding effective temperature. Figure 2.2 shows both the temperature variation and the distorted shape of a B2V star.

There are definite flaws in von Zeipel’s theorem which come from side-stepping the calculation of the structure of a rotating star - see the discussion in Appendix 4. However these structure models are not sufficiently developed to allow us to use them in this work (Rieutord & Espinosa Lara, 2013). In particular, they have not been performed for a wide enough group of stars to be of use in fitting the observations of Be stars. Despite the approximations, von
Figure 2.6: Change in the local effective temperature with increasing $v_{\text{frac}}$ (lower axis) for a B2V stellar model. The upper axis shows the corresponding $\omega_{\text{frac}}$. The darkest solid line gives the polar temperature ($\theta = 0^\circ$), while the lightest grey dash-dot line shows the equatorial temperature ($\theta = 90^\circ$). Various intermediate stellar latitudes are also shown. The right vertical axis shows the standard main sequence spectral type (with no rotation) for the same surface temperature as the left axis (Cox, 2000).
Zeipel’s theorem remains the standard treatment of rotation and will remain so until more detailed modern treatments reach a consensus and produce gravity darkening laws usable for a wide range of stellar spectral and luminosity classes.

Note that a modification to Von Zeipel’s theorem for low $T_{\text{eff}}$ where convection commences (Lucy’s law, see Lucy 1967) has been suggested. This is discussed in Appendix 5. However, it is not applicable to the case of Be stars.

### 2.3.2 Effects of Gravitational Darkening on the Photospheric Spectrum

We now return to the problem of measuring rotation using the widths of stellar absorption lines. Previously, it was assumed that a measurement of photospheric line width accurately measures the projected velocity of the stellar equator. This is true for slow to moderately rotating stars, but for fast rotators, the regions of the photosphere which are rotating the fastest are also the dimmest, making it difficult to directly measure rotation from line widths. The most strongly shifted wings of a line come from the limb of the star (see Figure 2.3), which sample more of the lower temperature regions. The bright, unshifted continuum from the poles can overwhelm the absorption component from the limb of the star. As a result, the line is narrower than one would expect from a star of constant surface temperature (see Figure 2.4). This effect makes measurements of the equatorial velocity for a fast rotating star difficult. Consequently, how fast Be stars rotate is still under debate (Townsend et al., 2004; Cranmer, 2005). Townsend et al. (2004) modelled the shape of the atmospheric absorption lines He I $\lambda 4471$ and Mg II $\lambda 4481$ and included the effect of gravitational darkening. Their results indicate that line widths become saturated as equatorial velocities increase. This is shown in Figure 2.7.
Figure 2.7: Line width versus the projected fractional velocity ($v_{frac} \sin i$), for four different inclinations corresponding to $\sin i$ values of 0.25, 0.50, 0.75 and 1.00 with gravitational darkening. The dotted line represents the linear relationship between line width and $v_{frac}$ which occurs when gravitational darkening is not included in the models (from Figure 1 of Townsend et al., 2004).

This line saturation depends on the inclination angle, $i$. Saturation occurs at $\approx 0.8 \, v_{crit}$ for $\sin i = 1.00$ and at even lower equatorial velocities for low values of $\sin i$. This suggests that near critical rotation could be occurring in some fast rotating stars without being reflected in the measured line widths. The range often given for the equatorial velocities of classical Be stars, 0.7 to 0.8 of $v_{crit}$, is tantalizingly close to this saturation point, prompting Townsend et al. (2004) to suggest that near critical rotation is still an open possibility in classical Be stars, despite the canonical view.

The approach used by Cranmer (2005) was to predict the effects of gravitational darkening on the distribution of rotational speeds amongst a group of stars and then fit the line width distributions resulting from these model distributions to the $v \sin i$ measurements of Yudin (2001). The advantage of using measurements of a large number of stars is that for independent and well separated stars, the stellar rotation axis have random orientation in space,
so one can to find the distribution of the equatorial velocities for a set of stars from measured \( v \sin i \)'s. Although often described as having opposite view points, both Townsend et al. (2004) and Cranmer (2005) explain that a direct measurement of \( v \sin i \) past 0.8 \( v_{\text{crit}} \) using spectroscopy may not be possible (see Figures 2.7 and 2.8), and there may be a systematic under measurement of the equatorial velocities of classical Be stars. However, when this underestimate is taken into account by Cranmer (2005), early-type classical Be stars are still found to rotate too slowly to allow a weak process such as pulsation to significantly aid disk formation. Most early-type, classical Be stars rotate at significantly sub-critical speeds, with an average of approximately 0.5 \( v_{\text{crit}} \) (see Figure 2.8). The same is not true of the results for late-type, classical Be stars. Cranmer (2005) finds that near critical rotation is likely occurring in most classical Be stars later then B6 and in a significant fraction between B3.5 to B6. This may explain the differences in variability between the generally more variable early type classical Be stars and the more stable late-type, classical Be stars. Howarth (2007) suggests that the sample set used by Cranmer (2005), Yudin (2001), underestimates the rotation rates of Be stars and claims that an average rotation rate of 0.95 \( v_{\text{crit}} \) is consistent with the data set of Chauville et al. (2001), which while smaller than Yudin (2001), may be of better quality. Howarth (2007) also emphasizes the difficulty in knowing the mass and polar radius of a rotating star. Thus \( v_{\text{crit}} \) many not be well known. Rivinius (2013) discusses the debate of the rotation rates of Be stars and suggests a minimum of 0.70 \( v_{\text{crit}} \) and 0.80 \( v_{\text{crit}} \) as a tentative average, but acknowledges the uncertainty.
Figure 2.8: The grey histogram represents the measured distribution of projected fractional velocities for the very early-types of classical Be stars. Lines represent the results from model calculations for the distribution of projected fractional velocities assuming all classical Be stars rotate at critical speeds, including random axis orientation and gravitational darkening effects with different assumed measurement errors. Note the solid black line represents the case of 100% critical rotation and perfect data, with no measurement errors, yet no stars are measured with $v \sin i > v_{\text{crit}}$. This cut off occurs at between 0.7 and 0.8 $v_{\text{crit}}$ (the two extrema) depending on the exact rotation rate of the model (from Figure 5 of Cranmer, 2005).

2.3.3 Observing Gravitational Darkening

The challenges in observing the details of rotation in stars other than the sun meant that von Zeipel’s theorem remained essentially un-tested for over seventy years, as we could only measure the integrated flux from across the stellar surface, and surface features could only be inferred. The shape of a spectral line can be subtly affected by the temperature variation across the surface of a rapidly rotating star. By combining detailed measurements of
multiple spectral lines including, Si II λ 6371 Å and O III λ 6383 Å, with atmospheric models that include gravitational darkening, Vinicius et al. (2007) determine the rotation rates, average temperatures and inclination angles of 5 early Be stars: α Eri, η Car, ε Cap, α Ara and λ Pav. Rotation speeds found were between 0.5-0.8 $v_{\text{crit}}$. Using this technique, α Eri was found to have a rotation speed of 0.75 $v_{\text{crit}}$. Extracting rotational information from a multi-line analysis of spectral lines is challenging; it necessitates high resolution spectra, detailed models, and is unfortunately model dependent. The need for models is problematic in a star that has an unclear spectral type/luminosity class (which as stated above, is typical in rotating stars, including α Eri). It also requires that the lines present within the spectrum have intrinsic strengths that peak at temperatures far enough apart to characterize the temperature variations across the surface. These complications may explain why these measurements, which have been possible for quite some time, are rarely attempted (Slettebak et al., 1980).

Currently well established interferometrically derived numbers (see below) put α Eri as rotating very close to its critical limit (Domiciano de Souza et al., 2003; Kervella & Domiciano de Souza, 2006) which calls into question the low velocities reported for Be stars in Vinicius et al. (2007).

Direct imaging is the best way to understand these stars. The shape of the star and any intensity variations across its surface can be seen directly without modelling. Interferometric measurements are beginning to approach this goal (van Belle, 2010). Each measurement for a pair of telescopes fills in part of the Fourier image plane on the sky. When the number of telescopes in an array is small, modelling is still needed to interpret the data. As the number of baselines increase and techniques improve, however, optical interferometers approach model-free imaging of single telescopes and radio arrays.
The first measurement of oblateness in a star was Altair by van Belle et al. (2001) using the Palomar Testbed Interferometer (PTI) with two baselines in different orientations. Each baseline gave different size measurements, while Vega, the comparison star had size measurements which did agree. The $v \sin i$ of Altair was found to be 210 km/s. Currently seven stars have been measured interferometrically: Altair ($\alpha$ Aql), Vega ($\alpha$ Leo), Achernar ($\alpha$ Eri), Regulus ($\alpha$ Leo), Rasalhague ($\alpha$ Oph), Alderamin ($\alpha$ Cep) and Caph ($\beta$ Cas). Achernar, B6 Vep, is the only Be star which has been resolved interferometrically. However Regulus, B7V, is a normal B type star that is a rapid rotator, and is likely similar to the central stars of Be star systems. Initial investigations used uniform disk models which finds the size of the star in various directions across the sky (van Belle et al., 2001). The dimming across the star was not resolved. Measurements of $\alpha$ Eri presented in Domiciano de Souza et al. (2003) indicate that this star is extremely distorted, with an equatorial to polar radius ratio of 1.5 or beyond, implying rotation near the critical limit. Measurements of Altair, $\alpha$Cep and $\alpha$Oph performed with multi-baseline beam combiners (MIRC, AMBER, PIONIER) allow for more baselines to be measured, filling in more of the inverse image, so that the variation in flux across the star can be detected (Monnier et al., 2007; Zhao et al., 2010).

Von Zeipels theorem predicts that the temperatures should be proportional to the gravity to the 0.25 power, called the standard $\beta$ value. These new detailed data sets allow this to be investigated. An image of Altair using CHARA/MIRC (Monnier et al., 2007) suggests a $\beta$ value for Altair of 0.19. Zhao et al. (2010) presents similar reconstructed images of $\alpha$ Cep and $\alpha$ Oph by aperture synthesis imaging. For $\alpha$ Cep, there is a bright region near the pole which has a brightness temperature above 7000 K and a darker region surrounding the equator with a brightness temperature of 6500 K. Gravitational
darkening models constrained by these observations show that the star has a rotation rate of 0.94 $v_{\text{crit}}$, an equatorial to polar radius ratio of 1.27, inclination of 55°, and a temperature profile of $\beta = 0.22$, which differs by 12% from the expected value. α Oph has a polar temperature of 9300 K and an equatorial region that is 1840 K cooler. It has a rotation rate of 0.885 $v_{\text{crit}}$ and an inclination very close to 90°. The high inclination makes simultaneously fitting both the inclination and $\beta$ difficult and these values remain undetermined. 

van Belle (2010) gives 0.21 as the typical value of $\beta$ determined by interferometry. This has prompted the description of empirically derived rotation laws of the form

$$T = (C_{\omega,\beta} g)^\beta.$$  \hfill (2.13)

The use of such laws require modification of the normalization:

$$C_{\omega,\beta} = L_\omega/\int g^{4\beta} dA,$$  \hfill (2.14)

which is simply Equation 2.9 changed to account for a variable $\beta$ in order to ensure that the star has the correct total luminosity. These formulations, while empirically derived, unfortunately introduce new parameters with no physical explanation. Von Zeipel’s theorem and empirically derived laws use constant, normalization factors, $C(\omega, \beta)$ and constant $\beta$’s. More complex forms such as those presented in Espinosa Lara & Rieutord (2011) and Claret (2012), present function forms of either $C(\theta)$ or $\beta(\theta)$, which are equivalent.

### 2.3.4 Beyond von Zeipel

Interferometric data presents the possibility of not only confirming or disproving von Zeipels theorem but also refining it. When considering the above
measurements of “non-standard” $\beta$ values, it is important to note that interferometric measurements do not directly find $\beta$ because the local gravity cannot be measured. Rather, the variation of the brightness temperatures and the shape of the star can be seen. This data indicates departures from von Zeipel’s theorem. Von Zeipel’s theorem is based on several idealizations, some of which are actually inconsistent (see Appendix 4) and does not include the details of how rotation affects the structure of the star. It would be revealing to see if the measurements of detailed spectra, combined with atmospheric models as done in Vinicius et al. (2007) using non-standard $\beta$’s where appropriate, would produce results consistent with the interferometric measurement.

**The Espinosa Lara-Rieutord formulation**

Espinosa Lara & Rieutord (2011) presents a refinement to Von Zeipel’s theorem motivated by the current observations. Both Espinosa Lara & Rieutord (2011) and the method of Collins (1963) begin with the expression for radiative energy transport. The equation of hydrostatic equilibrium is used to relate the radiative flux to the local gravity, resulting in expressions of the form, $F_{\text{rad}} \propto g$, which becomes $T_{\text{eff}} \propto |g|^{1/4}$ using the Stephan-Boltzmann law. The difference in these approaches is how the other terms in the equation are handled. Espinosa Lara & Rieutord (2011) make no assumptions about the other terms necessarily being constant, letting $F_{\text{rad}} = f(r, \theta)g$. The unknown function, $f(r, \theta)$, is found by requiring that $\nabla \cdot F_{\text{rad}} = 0$ as implied by radiative diffusion. Using the Roche model, solid body rotation, and solving the resulting partial differential equation gives

$$T_{\text{eff}} = \left(\frac{L_\omega}{\sigma G M}\right)^{1/4} \sqrt{\frac{\tan \theta_w}{\tan \theta}} g^{1/4},$$

(2.15)
where $\theta_w$ is defined by the requirement that

$$
\cos \theta_w + \ln \tan \frac{\theta_w}{2} = \frac{1}{3} x^3 w^2 \cos^3 \theta + \cos \theta + \ln \tan \frac{\theta}{2}.
$$

(2.16)

In Equation 2.16, $\theta$ is the spherical coordinate, $x$ is a scaled radius $r/r_{eq}$, and $w$ is a different fractional angular velocity given by $\omega_{\text{star}} \sqrt{r^3_{eq}} / (\omega_{\text{star}}) / GM$ (which is not $\omega_{\text{frac}}$).

While both approaches (Espinosa Lara & Rieutord (2011) and von Zeipel (1924)) used the same assumptions: radiative energy transport, hydrostatic equilibrium, solid body rotation and the Roche potential, the main difference is the order in which the restrictions are placed. Von Zeipel’s theorem uses the Wavre-Poincaré theorem, (Tassoul, 1978) implied by solid body rotation, as its first restriction, and uses it to argue that $C(\theta)$ is constant. $\nabla \cdot F_{\text{rad}} = 0$, is considered last in von Zeipel’s treatment and found to be impossible. If this is reversed and $\nabla \cdot F_{\text{rad}} = 0$ is considered first and the result cannot be consistent with Wavre-Poincaré theorem because the $f(\theta)$ is not constant across an equi-potential. The differences result from the different orders in which these approximations are applied. Because the assumptions are inconsistent, the last constraint considered will be found not to be possible. Indeed, if the assumptions were consistent, the order would not matter and both methods would re-construct the same function. Neither approach is better than the other. However, the Espinosa Lara & Rieutord (2011) treatment seems to be closer to that which is observed in stars (see Figure 4 in Espinosa Lara & Rieutord (2011) ). This suggests that the idealization of radiative energy transport and $\nabla \cdot F_{\text{rad}} = 0$ is a good approximation for real stars. Claret (2012) confirms this.

Equation (2.15) predicts lower polar temperatures and higher equatorial
temperatures than Collins (1963), similar to observationally derived rotation laws (van Belle, 2010). However, “β” changes both with rotation and latitude in Equation (2.15). The formulation of Espinosa Lara & Rieutord (2011) is an excellent alternative to simply lowering β, as it offers a physical explanation, avoids adding an artificial parameter, and is an analytic solution which can be utilized for any star and any rotation rate lower than critical.

**Hydrodynamical Simulations of Stellar Rotation**

Stellar structure models which include rotation indicate similar behaviour to von Zeipel’s theorem, namely increasing flux from the poles and decreasing flux from the equator as rotation increases (Lovekin et al. (2006), Gillich et al. (2008)). Lovekin et al. (2006) and Gillich et al. (2008), both two dimensional \((r,\theta)\) calculations, find relatively good agreement between their models and von Zeipel’s theorem. However, there are departures. von Zeipel theorem predicts a stronger drop in temperature in the equatorial region than Lovekin et al. (2006). Lovekin et al. (2006) attributes this difference to the independence of the flux from the surface temperatures in von Zeipel’s model.

Calculating the effects of rotation on the structure and evolution of a star requires complex codes. Examples of such currently being used are the two dimensional codes ROTOR and ESTER and the one dimensional codes used by Maeder & Meynet (Lovekin, 2011; Espinosa Lara & Rieutord, 2011; Maeder & Meynet, 1997). One of the important effects studied in these works is rotational evolution: how the rotation of a star changes over time and how angular momentum and material is transferred between different sections of a star (Maeder, 1997). This is typically done by examining differential rotation and the evolution of differential rotation, often using models that treat the
star as a series of rotating shells. Some of the most important work about the effects of rotation on stellar structure and evolution is a series of 13 papers beginning with Maeder & Meynet (1997) and ending with Hirschi et al. (2005). These examine the effects of rotation and mass loss on the evolution of stars in great detail, including rotational effects on the supernova stage, inclusion of magnetic fields, and the effects of rotation on Wolf-Rayet stars (Georgy et al., 2009; Maeder & Meynet, 2003; Meynet & Maeder, 2005). Generally, rotation causes cooler and dimmer stars in the main sequence and brighter giants (Maeder & Meynet (2000)). Meynet & Maeder (2000) shows that surface velocities drop during main sequence evolution, but the critical velocities drop as well. Maeder & Meynet (2008) (two dimensional calculations), indicates that the internal rotation profile varies on evolutionary timescales due to angular momentum transfer within these stars. Models presented in Rieutord & Espinosa Lara (2009) describe barotropic flows which are caused by the surface distortions and differences between the pressure and temperature gradients and a stellar core rotating 1.5 times faster than the envelope. Rieutord & Espinosa Lara (2009) describes equators rotating faster than the poles, which is the opposite of what is observed in the Sun with the angular velocity of the core is 50% larger than that of the envelope. Near the critical limit, convection is also favoured in the outer layers (Meynet et al., 2010).

The one dimensional calculations of Ekström et al. (2008) studied the effect of rotation on the structure and evolution of B type stars. Their results show small deviations from the simple assumptions of constant luminosity and polar radius; however, they find a smaller temperature difference between the pole and the equator than expected by von Zeipel. Their evolutionary calculations indicate that a spin up of the outer layers occurs for stars born as moderate rotators during their main sequence lifetimes. This occurs because meridional
circulation transports angular momentum to the surface of the star. This may explain the association between cluster age and Be star fractions (McSwain & Gies, 2005). For low mass loss rates, where stars lose less of their angular momentum by their winds, critical velocities can be reached on the surface. To match the observational statistics, they found it was necessary to consider a star to be a Be star when it reached 70% of the critical velocity, consistent with estimate of Yudin (2001). They also found longer main sequence lifetimes with factors of \( \approx 9/8 \) times longer near the breakup velocity.

While there is much work being produced which explains aspects of rotating stars, there has yet to be a full consensus on the effects of rotation on stellar structure and the evolution grids are often quite sparse (Maeder & Meynet, 1997; Ekström et al., 2008). This means that while these results inform our understanding of stellar rotation, they are not yet robust enough to define a consistent gravitational darkening law(s) valid for a range of stellar classes.
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Chapter 3

Implementing Gravitational Darkening in Bedisk and Beray
The purpose of this chapter is to describe the inclusion of gravitational darkening of the central B star in the BEDISK and BERAY codes. For both codes, the main effects are the distortion of the stellar surface and the latitude dependence of the local effective temperature and surface gravity caused by rapid rotation, as described in Chapter 1. In the BEDISK code, which solves for the thermal structure of the disk, these effects alter the photoionizing radiation reaching each point in the circumstellar disk, which in turn changes the thermal balance and all of the atomic level populations. In the BERAY code, which performs a formal solution of the equation of radiative transfer to predict observables (such as line profiles and colours), the distorted star and latitude dependent temperature and gravity must be incorporated into the boundary conditions for rays terminating on the stellar surface.

3.1 Modifications to Bedisk

The BEDISK code (Sigut & Jones, 2007) enforces radiative equilibrium in a circumstellar disk of a prescribed density distribution. Solar abundances are assumed for the circumstellar gas, although other abundances can be used. For example, Ahmed & Sigut (2012) presents an analysis of the differences in disk temperatures when the much lower abundances appropriate to stars in the Small Magellanic Cloud are used.

In this thesis, the disk density law is taken to be a simple power-law drop-off with radial distance. This form is suggested by the results of hydrodynamical calculations (Porter, 1999; Okazaki, 2001) that finds that an isothermal disk has a radial density structure which follows $R^{-3/2}$, suggesting a parametrization of $R^{-n}$ for non-isothermal disks; this point is further discussed by Jones et al. (2008) within the context of hydrodynamical simulations based on viscous
disks. Be star disks are believed to be rotationally supported in the radial direction and pressure supported in the vertical (i.e. in hydrostatic equilibrium in the direction perpendicular to the plane of the disk). These assumptions result in a simple specification of the disk density of the form

$$\rho(R, Z) = \rho_0 \left( \frac{r_*}{R} \right)^n e^{-\left( \frac{Z}{H} \right)^2}. \quad (3.1)$$

Here $R$ and $Z$ are the cylindrical coordinates specifying the location in the (assumed axi-symmetric) disk; $R$ is the distance from the rotation axis of the star and $Z$ is the height above or below the equatorial plane. $r_*$ is the stellar radius. The model parameters are $\rho_0$, which is the density (in g cm$^{-3}$) at the inner edge of the disk in the equatorial plane, and $n$, which is the power-law index of the density drop-off in the equatorial plane. In this expression, the function $H$ determines the vertical extent of the disk and is given by

$$H = \left[ \frac{2R^3 kT_d}{GM_* \mu m_H} \right]^{1/2}, \quad (3.2)$$

where $T_d$ is an assumed average isothermal disk temperature chosen to fix $H$ and is usually taken to be $\approx 0.6 T_{\text{eff}}$, where $T_{\text{eff}}$ is the effective temperature of the star. This form of the disk scale-height follows from the assumption that the gas is in vertical, isothermal, hydrostatic equilibrium satisfying

$$\frac{dP}{dZ} = -\rho g_Z \quad (3.3)$$

at each location $R$ in the disk. Here $g_Z$ is the vertical component of the central star’s gravitational acceleration at $R$. Using Equation (3.3) produces very thin disks as, physically, $(H/R)$ is the ratio of the local sound speed (of order ten
km s\(^{-1}\)) to the local orbital speed (several hundreds of km s\(^{-1}\)). Note that \(T_d\) is specified independently of the radiative equilibrium solution for \(T(R, Z)\) performed by BEDISK. This is, of course, an inconsistency. Consistent modes in which Equation (3.3) is numerically integrated and made consistent with \(T(R, Z)\) are available from Sigut et al. (2009). This point is also discussed in Chapter 5.

In a rapidly rotating star, the equatorial and polar radius of the star are no longer equal. Hence the value for stellar radius \(r\) to be used in Equation (3.1) becomes ambiguous. In all calculations in this thesis, the disk is started at 1.5 \(r_p\), which is the maximum extent of the star’s equatorial radius in the case of critical rotation. This forces the density structure of the disk to remain fixed for all central stellar rotation rates, which aids in the comparison of the models. This subtle point is discussed in Section 4.6 of Chapter 4.

The structure and operation of BEDISK is discussed by Sigut & Jones (2007). Basically, BEDISK is a stellar atmospheres code with different geometry. Instead of spherical shells or parallel planes being illuminated from below, the computational grid is located in a disk and is illuminated by the central star. Despite the change in geometry, the physics and computational procedure is essentially the same, implementing the physics of radiative transfer and statistical equilibrium as outlined in Mihalas (1978). The general flow of a BEDISK calculation is shown in Figure 3.1.

With the fixed disk density structure above, the photoionizing radiation reaching each grid point \((R, Z)\) can be found, and the atomic level populations calculated by solving the equations of statistical equilibrium. Radiative equilibrium is then enforced at each point in the disk by balancing the microscopic rates of heating and cooling to determine the local temperature. This procedure is repeated for all grid point in the circumstellar disk. BEDISK
Figure 3.1: Flow chart showing the order of the operations in Bedisk. The code is a series of nested loops: an inner population loop, an outer temperature correction loop and, if self-consistent hydrostatic equilibrium is included, an outer density correction loop includes atomic data for nine of the most abundance elements over several ionization stages. Details can be found in Sigut & Jones (2007).

With the addition of rapid rotation of the central B star, the stellar surface is no longer spherical in shape and is no longer described by a single $T_{\text{eff}}$ and $\log(g)$. As this changes the photoionizing radiation field, the expectation is the gravitational darkening could have an important influence on the radiative solution in the disk. The changes to implement gravitational darkening in BEDISK will now be described.
### 3.1.1 Photoionizing Radiation from the Central B star

In order to compute the photoionizing radiation field at each grid point in the disk, rays must be traced back to the stellar surface and the transfer equation solved along each ray. As each ray terminates on the stellar surface, the correct, latitude-dependent $T_{\text{eff}}$ and $\log(g)$ must be used for the assumed stellar rotation rate. The photoionizing radiation from the central star is represented by a set of Kurucz stellar atmosphere files from Kurucz (1996) which specify the intensity of radiation for each $(T_{\text{eff}}, \log g)$ combination. The original bedisk Kurucz model grid consisted of thirty-six models for three values of $\log_{10}(g)$ (3.5, 4.0, and 4.5) and twelve values of the stellar effective temperature (10000, 12000, 14000, 15000, 17000, 19000, 21000, 23000, 25000, 27000, 29000, and 31000 K). For a rotating star these models specify the local atmosphere. To implement gravitational darkening, the number of stellar models was greatly enlarged to increase the range and decrease the sampling step of temperature and gravity. To account for the cool equator of a rapidly rotating B star, the Kurucz models were extended down to 7000 K. MATLAB was then used to interpolate a fine grid of 1331 models at 11 different gravities (at intervals of 0.1 between 3.5 and 4.5) and 121 different temperatures (at intervals of 200 K between 7000 K and 31000 K). The interpolation was done in log space using the MATLAB routine “cubic” which is a piece-wise, cubic Hermite polynomial. This method produces a smooth interpolation without oscillations at the end of the function.

### 3.1.2 Finding the Visible Region of the Star

Once the distorted stellar surface has been correctly “tiled” with latitude dependent $T_{\text{eff}}$ and $\log(g)$, the region of the stellar surface visible from each
computational grid point \((R_{\text{disk}}, Z_{\text{disk}})\) must be found. This sets the limits on the rays traced back to the stellar surface from the grid point.

In the case of a spherical star, the visible region is a simple spherical cap subtending angle

\[
\sin \theta_{\text{cap}} = \frac{r_{\ast}}{\sqrt{R_{\text{disk}}^2 + Z_{\text{disk}}^2}}
\]

at the location of the disk grid point. However, for computational purposes, this spherical cap is generally specified in spherical coordinates centred on the centre of the star, as described below.

For a star distorted by rotation, the visible region is extended (along with the star), becoming shaped like an American football instead of a simple, spherical sector. This section can be found using the star’s coordinate system, where the \(z\) axis passes through the poles of the star. The disk grid point can be assumed to lie in the \(x - z\) plane as the disk is axis symmetric disk. The \(\theta_{\ast}\) limits of the visible region, top and bottom, are found by solving the non-linear equation

\[
\frac{8}{27} \frac{r_{\text{frac}}^2}{r_p^3} R_{\text{disk}} \sin \theta_{\ast} - r_{\ast}(\omega_{\text{frac}}, \theta_{\ast}) - R_{\text{disk}} \sin \theta_{\ast} + Z_{\text{disk}} \cos \theta_{\ast} = 0 \tag{3.5}
\]

for the critical angles \(\theta_{\ast}\). Here \(R_{\text{disk}}\) and \(Z_{\text{disk}}\) are the cylindrical coordinates of the grid location. \(r_p\) is the polar radius of the disk and \(r_{\ast}(\omega_{\text{frac}}, \theta_{\ast})\) is the stellar radius, given in Chapter 2, Equation 2.5. Note that the stellar radius is a function of both the co-latitude on the star and the rotation rate. This equation was solved numerically using the bi-section method. Care needs to be taken to ensure that solutions behind pole, where \(\theta_{\ast}\) is near 2 \(\pi\), (as seen from the upper disk) are found properly.

Equation (3.5) produces two real roots, \(\theta_{\ast 1}\) and \(\theta_{\ast 2}\). If the disk grid point
is taken to be in the $\phi_s = 0$ plane, the $\theta_s$ limits of the visible regions are located at $(r_s(\omega_{\text{frac}}, \theta_1^s), \theta_1^s, 0)$ and $(r_s(\omega_{\text{frac}}, \theta_2^s), \theta_2^s, 0)$ in the coordinate system of the star. To find the $\phi_s$ values of the edge between $\theta_1^s$ and $\theta_2^s$, the following equation can be used

$$\phi_{\text{edge}} = \pm \cos^{-1} \left[ \frac{r_s(\omega_{\text{frac}}, \theta_s) \sin^2 \theta_s (1 - \kappa_{\theta_s}) + r_s(\omega_{\text{frac}}, \theta_s) \cos^2 \theta_s - Z_{\text{disk}} \cos \theta_s}{(r_s(\omega_{\text{frac}}, \theta_s) \sin \theta_s - \kappa_{\theta_s} R_{\text{disk}} \sin \theta_s)} \right]$$

(3.6)

where

$$\kappa_{\theta_s} = \frac{8}{27} \omega_{\text{frac}} \left( \frac{r_s(\omega_{\text{frac}}, \theta_s)}{r_p} \right)^3.$$  

(3.7)

This equation produces real values between $\theta_1^s$ and $\theta_2^s$. The derivation of these equations are given in Appendix I. A diagram of the visible region of the disk is shown in Chapter 4, Figure 2.

The subroutine in BEDISK that calculates the amount of photoionizing radiation from the star reaching a grid point defines a series of rays from the $(R_{\text{disk}}, Z_{\text{disk}})$ grid point back to the visible surface of the star. It works in a disk coordinate system such that the line from the centre of the star and the grid point of interest define $\theta_{\text{ray}} = 0$. On a spherical star, the rays are built by varying $\theta_{\text{ray}}$ from zero to $\theta_{\text{ray}}^{\text{max}} \equiv \theta_{\text{cap}}$ (Equation 3.4), the edge of the visible region, and then circling around $\phi_{\text{ray}}$ to build the visible region. When this is extended to a non-spherical cap, it is the largest value of $\theta_{\text{ray}}$, $\theta_{\text{ray}}^{\text{max}}$ that is needed to effectively grid the star. Note that $\theta_{\text{ray}}^{\text{max}}$ is not the same as $\theta_1^s$ or $\theta_2^s$ because the former is in the disk co-ordinate system while the later are in the star’s co-ordinate system.

Given these bounds on the visible portion of the stellar surface, the tracing of rays back to the stellar surface and the numerical integration of the transfer equation along these rays proceeds in a manner similar to the spher-
Figure 3.2: Diagram of the relationship between the angles ($\theta_{\text{ray}}, \phi_{\text{ray}}$) which mathematically define a ray and its location in space. The angle $\phi_{\text{ray}}$ specifies the location of a ray around the axis defined by the line from the center of the star to the observation point, i.e. into or out of the page.

The angle $\theta_{\text{ray}}$ specifies the location of a ray around the axis defined by the line from the center of the star to the observation point, i.e. into or out of the page. However, one difference is that when a ray reaches the stellar surface, its inclination to the local surface normal is different in the rotating case because of the distortion of the stellar surface. The inclination angle is required because of the inclusion of limb-darkening in the stellar surface intensities, i.e. rays at larger angles to the local surface normal have lower intensities. In the BEDISK code, wavelength-dependent (quadratic) limb darkening coefficients are used.

As the inclination angle is defined relative to the local surface normal, an expression for the direction of the local normal is required. For a rotating star, the Cartesian coordinates of the local gravitational acceleration (in the star’s co-ordinate system) are given by

$$\tilde{g} = -\frac{GM}{R_\theta^2} \left( \sin \theta \cos \phi \left[ 1 - \frac{8}{27} \omega^2_{\text{rad}} \frac{R_\theta^3}{r_p^3} \right], \sin \theta \sin \phi \left[ 1 - \frac{8}{27} \omega^2_{\text{rad}} \frac{R_\theta^3}{r_p^3} \right], \cos \theta \right). \quad (3.8)$$

With this, the surface normal is $\hat{n}_{\text{surf}} \equiv -\frac{\tilde{g}}{|\tilde{g}|}$. 

$$\hat{n}_{\text{surf}} \equiv -\frac{\tilde{g}}{|\tilde{g}|}. $$
Figure 3.3: Diagram of the differential area on a rotationally distorted star showing the reason for the $\cos \eta$ term.

In order to calculate the projection of the surface area on the ray, $\mu$, we need to find $\hat{n}_{\text{pointer}}$, the unit vector which points from the patch of the star at the base of ray ($\theta_{\text{disk}}^{\text{ray}}, \phi_{\text{disk}}^{\text{ray}}$) to the grid location. This is given by

$$\hat{n}_{\text{pointer}} \equiv \frac{\vec{r}_{\text{grid}} - \vec{R}_{\text{surf}}^{\text{ray}}}{|\vec{r}_{\text{grid}} - \vec{R}_{\text{surf}}^{\text{ray}}|} \quad (3.9)$$

where

$$\vec{r}_{\text{grid}} = (0, R_{\text{disk}}, Z_{\text{disk}}) \quad (3.10)$$

$$\vec{R}_{\text{surf}}^{\text{ray}} = R_{\theta_x}^{\text{ray}} (\sin \theta_x^{\text{ray}} \cos \phi_x^{\text{ray}}, \sin \theta_x^{\text{ray}} \sin \phi_x^{\text{ray}}, \cos \theta_x^{\text{ray}}). \quad (3.11)$$

The projection of the surface orientation on the ray is then just $\mu = \hat{n}_{\text{pointer}} \cdot \hat{n}_{\text{surf}}$.

In addition, as the patches on which rays terminate do not have the same area on the star, we need the $\Delta A$ represented by each patch. For a sphere, in
the discrete form, we have

\[ \Delta A = r^2 \sin \theta \Delta \theta \Delta \phi . \quad (3.12) \]

Including gravity darkening, this becomes

\[ \Delta A = \frac{r^2 \sin \theta \Delta \theta \Delta \phi}{\cos \eta} , \quad (3.13) \]

where \( \cos \eta \) is the cosine of the angle between the surface normal and \( \hat{r} \) as shown in Figure 3.3. The size of the solid angle of the ray is given by

\[ \Delta \Omega = \frac{|\mu| \Delta A}{4\pi (R^2_{\text{grid}} + z^2_{\text{grid}})} . \quad (3.14) \]

Because we consider all points with \( \theta_{\text{disk}} \) less than \( \theta_{\text{ray}}^{\text{max}} \), and the region is not a spherical cap, some of the points considered will actually be outside the visible region. A check is done and any locations with \( \mu \) less than zero has its area set to zero.

When the geometry calculations complete, the local temperature of the star is found using

\[ T_{\theta} = [C(\omega, \beta) g_{\text{eff}}]^\beta , \quad (3.15) \]

where \( g_{\text{eff}} \) is given as the magnitude of

\[ g_{\text{eff}}(\theta, \omega) = \left( \omega^2 R_\theta \sin^2 \theta - \frac{G M_\star}{R_\theta^2} \right) \hat{r} + (\omega^2 R_\theta \sin \theta \cos \theta) \hat{\theta} . \quad (3.16) \]

As discussed in Chapter 1, the default value for the exponent is \( \beta = 0.25 \), but it can be varied as discussed in Chapter 5. The temperature can also be specified using Equation 2.15 if we wish to use the Espinosa Lara & Rieutord
(2011) formulation.

Finally, we compare the temperatures and gravities of the surface to those of the Kurucz model grid and select the closest model \((T_{\text{closest}}, g_{\text{closest}})\). This closest model is then corrected by the ratio of the black body of the actual surface temperature to that of the model,

\[
I_\nu(\theta_*) = \frac{B_\nu(T_\theta)}{B_\nu(T_{\text{closest}})} I_\nu_{\text{grid}}(T_{\text{closest}}, g_{\text{closest}}).
\]

As the stellar atmosphere grid is fairly dense, with no more than 200 K between adjacent models, this black body scaling is perfectly adequate because this corrects for any differences in the bulk flux.

### 3.1.3 Refinements to the Transfer Solution

Given the emergent intensity from the stellar surface for each patch, the equation of radiative transfer needs to be solved along a ray connecting the current grid location to that patch on the stellar surface. Integrating over the surface of the star then gives the stellar contribution to the photoionizing radiation field at that disk location. To solve the transfer equation, the gas opacity and emissivity are required and, therefore, interpolation in the physical conditions (such as local temperature, electron density, hydrogen level populations, ...) are required for each point along the ray. In BDISK, the order of the temperature solution proceeds outward in \(R_i\), for i.e. \(i = 1, \ldots, N_R\), from the stellar surface to the outer edge of the disk. At each \(R_i\), the solution starts at the topmost \(Z_{ij}\) location, i.e. \(j = N_Z\), the point furthest from the equatorial plane. This ordering ensures that for points along each ray, interpolation is done using grid points that already have converged solutions for the temperature.

However, it is not always possible to use only grid points with converged
temperature and populations. Rays going “downward” towards the equatorial plane can require the grid point just below that being calculated, which does not have a converged solution. Hence some sort of guess must be made. Previously in Sigut & Jones (2007), the guess was the initial isothermal disk temperature used to start the solution and the level populations were assumed to be the LTE values at this temperature. Since this initial temperature can be quite inaccurate, errors in the transfer solution can result. This issue was exacerbated by the inclusion of gravitational darkening because of the distorted star. The solution to this problem was to improve the guess. The best estimate of the temperatures and level populations at grid point \((R_i, Z_{ij})\) is actually the values at the grid point directly in front of it at \((R_{i-1}, Z_{i-1,j})\). Although the disk does change temperature with radius, these changes are smooth and the temperature difference between adjacent columns is quite small. This was further improved for most grid points by a scaling ratio. If the temperature, for example, is required at the unsolved location \(T_{ij} = T(R_i, Z_{ij})\), then the expression
\[
T_{ij} = T_{i-1,j} \left[ \frac{T_{i,j+1}}{T_{i-1,j+1}} \right].
\] (3.18)
was used. Here all the terms on the RHS are from the converged solution. The populations are defined in a similar way, but the interpolation occurs in log space. If the unsolved grid point is the upper edge of the disk, \(j = n_z\) the above ratio cannot be found and the value at \(R_{i-1}, Z_{i-1,n_z}\) must be used as the guess. For the first column no improved guess can be made and the original guess, the initial isothermal disk temperature and the LTE level populations is assumed.
3.1.4 Other Gravitational Darkening Modes

The shape and temperature changes of gravity darkening have been implemented independently by defining $\omega_{\text{frac}}^{\text{shape}}$ as the fractional angular velocity used in all functions related to the shape of the star and $\omega_{\text{frac}}^{\text{temp}}$ used in calculations related to the temperature and surface gravity of the stellar star atmosphere. $\Omega^{\text{shape}}$ and $\Omega^{\text{temp}}$ are the equivalent angular velocities. This allows two test modes in BEDISK:

- Spherical Gravitational Darkening (SGD): Here the shape distortion of the star due to rotation is not implemented (i.e. the star remains spherical), but the temperature variation over the stellar surface is retained.

- Pure Shape Distortion (PSD): Here the star is distorted according to the Roche model, but the temperature is constant across the star and equal to the non-rotating, stellar effective temperature.

The two modes can be used to investigate the temperature and geometry effects of gravitational darkening separately. In addition, the validity of SGD is assessed in Chapter 4 as the temperature variation is much more easily implemented in existing codes than the distortion of the stellar surface.
3.2 Modifications to Beray

The program BERAY (Sigut, 2011) generates the observables from the disk solution found by BEDISK. The observables include spectral energy distributions, spectral line profiles, and monochromatic images on the sky which are required to interpret interferometric visibilities. BERAY computes all of these quantities by solving the radiative transfer equation along a series of rays. The exact geometry is set by the inclination of the system to the line of sight, $i$. The convention is $i = 0$ degrees represents a pole-on star with the disk in the plane of the sky. Conversely, $i = 90$ degrees represents an equator-on star seen (partially) through the circumstellar disk. In this section, the changes to BERAY necessary to include the gravitationally darkened BEDISK solution are described. The required changes are very similar to those required to BEDISK except that the rays are no longer internal to the disk but are parallel and point in a common direction at a distant observer.

3.2.1 Stellar Flux and Line Shape Files

As for BEDISK in the previous sections, the number of models used to define the stellar intensity must be increased to properly account for the range of stellar temperature and gravities found over the stellar surface. In the case of BERAY, this includes photospheric line profiles as well in order to properly represent stellar boundary condition. For example, in a calculation of a Hα line profile or image, a proper set of photospheric Hα line profiles are required for the boundary condition for rays that terminate on the stellar surface. Table 3.1 lists the features and the number of increased models. The continuum models listed in Table 3.1 are the same as used in the BEDISK code.
Table 3.1: Changes to input Atmospheric files for BERAY.

<table>
<thead>
<tr>
<th>Feature</th>
<th>Old Ranges of $T_{\text{eff}}$ &amp; log($g$)</th>
<th>Num. Models</th>
<th>New Range of $T_{\text{eff}}$ &amp; log($g$)</th>
<th>Num. Models</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuum</td>
<td>10000 - 31000, 3.5 - 4.5</td>
<td>36</td>
<td>7000 -31000, 3.5 - 4.5</td>
<td>121</td>
</tr>
<tr>
<td>Hα</td>
<td>10000 - 31000, 3.5 - 4.5</td>
<td>36</td>
<td>10000 -31000, 3.5 - 4.5</td>
<td>106</td>
</tr>
<tr>
<td>MgII 4481</td>
<td>10000 - 31000, 3.5 - 4.5</td>
<td>36</td>
<td>10000 -31000, 3.5 - 4.5</td>
<td>106</td>
</tr>
</tbody>
</table>

3.2.2 Radiative Transfer Solution

BERAY works in the coordinates defined by the plane of the sky, which will be taken to define the $x_{\text{sky}} - y_{\text{sky}}$ plane. The $z_{\text{sky}}$ co-ordinate, directed at the observer located at an infinite distance, then defines the rays along which the equation of radiative transfer needs to be solved. Before this can be done, however, the intersections and boundary conditions for each ray are needed. Figure 3.4 shows the relationship between the coordinate system defined by the sky plane, $(x_{\text{sky}}, y_{\text{sky}}, z_{\text{sky}})$, that of the disk and star $(x_{\text{sky}}, y_{\text{sky}}, z_{\text{sky}})$ and the inclination angle $i$.

The bulk of the changes to BERAY involve changing the routines which determine if a ray terminates on the stellar surface (because of its distorted shape) and the local conditions at the end of such rays (as $T_{\text{eff}}$ and log $g$ now vary over the surface). Consider the ray defined by $(x_{\text{sky}}, y_{\text{sky}})$.

In the case of a spherical star, it is easy to determine if this ray terminates on the stellar surface. If $x_{\text{sky}}^2 + y_{\text{sky}}^2$ is less the stellar radius, then the ray intersects the star. If it is larger, the ray misses the star, and if it is equal, the ray is tangent to the star. The intersection point can be found by $z_{\text{sky}} = \sqrt{r_{\text{star}}^2 - (x_{\text{star}}^2 + y_{\text{star}}^2)}$. Several other variables are then needed to
Figure 3.4: Diagram showing the relationship between the coordinate systems \((x_{\text{sky}}, y_{\text{sky}}, z_{\text{sky}}), (x_{\text{star}}, y_{\text{star}}, z_{\text{star}})\) and the inclination angle, \(i\). \(x_{\text{sky}}/x_{\text{star}}\) is coming out of the page.

set the stellar intensity correctly: the surface inclination, \(\mu\), and the local projected surface velocity, \(v_z\). These are easily found in the spherical case as \(v_z = -(x_{\text{star}}/r_{\text{star}})v \sin i\) and \(\mu = z_{\text{star}}/r_{\text{star}}\), where \(v \sin i\) is the projected equatorial velocity of the star. Gravity and temperatures are also need but these are constant over the stellar surface.

In the case of a rapidly rotating, distorted star, all of the above quantities are still needed, but are more complex to compute. For example, even finding if a ray terminates on the star requires many more calculations. There are a few are simple cases:

- When the radius on the sky, ie. in the \(x_{\text{sky}} - y_{\text{sky}}\) plane, \(r_{2D\text{sky}}\) given by \(\sqrt{x_{\text{sky}}^2 + y_{\text{sky}}^2}\) is more than \(r_{\text{equat}}\), the equatorial radius, the ray cannot intersect the star as it is beyond the largest possible radius of the star.

- When \(i = 0\) degrees, the star is being view from directly above the pole.
Figure 3.5: Figure A shows a top down view of the star with $z_{\text{sky}}$ pointing out of the page. Figure B is a cut though the star at some $y_{\text{sky}}$. The location of $\theta^1$ has been highlighted. Figure C is a cut though the star at some $x_{\text{sky}}$, the overhanging and underhanging regions at the edge of the star.

In this case the (stellar co-ordinates) of the intersection are

$$\sin \theta_{\text{star}} = \frac{\cos(3 \arccos(\frac{\omega_{\text{frac}}}{\omega_{\text{frac}} - \frac{r_{2D_{\text{sky}}}}{r_p}}) - 4\pi)}{\omega_{\text{frac}}}$$  \hfill (3.19)

$$\cos \theta_{\text{star}} = \sqrt{1 - \sin^2 \theta_{\text{star}}}$$  \hfill (3.20)

$$z_{\text{star}} = r_p x(\omega_{\text{frac}}, \sin \theta_{\text{star}} \cos \theta_{\text{star}})$$  \hfill (3.21)

However, unlike these simple cases, the general cases have no algebraic solution and the intersection must be determined numerically. A straightforward and stable method for root finding is the bisection method. It is of sufficient speed for our purposes because with each step, the error in the approximation is halved. However to implement the bisection method, a bound on the root is required, namely two points along the ray, one “above” and outside the star and one inside. The search for the stellar surface occurs along a ray looking
between these known points. For a point in space, \((x_{\text{sky}}, y_{\text{sky}}, z_{\text{guess}})\), we can determine if it is inside or outside the star by the following algorithm:

1. Calculate \(R_{\text{guess}}^2 = x_{\text{sky}}^2 + y_{\text{sky}}^2 + z_{\text{guess}}^2\);

2. Convert to guess position to the stellar coordinate system;

3. Find \(\theta_{\text{star}}^{\text{guess}} = \arctan(\frac{\sqrt{x_{\text{star}}^2 + y_{\text{star}}^2}}{z_{\text{star}}})\);

4. Find the stellar radius, \(R(\theta_{\text{star}}^{\text{guess}})\).

5. If \(R(\theta_{\text{star}}^{\text{guess}})^2 > R_{\text{guess}}^2\) then the guess position is within the star. If \(R(\theta_{\text{star}}^{\text{guess}})^2 < R_{\text{guess}}^2\), the guess position is outside the star.

Choosing a location outside the star is simple by taking \(z_{\text{guess}}^{\text{sky}} = 2r_p\). The difficulty is finding a point on the ray within the star because this only occurs if there is an intersection. For the bulk of the rays which intersect the star (all them when \(i = 0\) or \(i = 90\)), the \(z_{\text{sky}} = 0\) plane is within the star while only the edges of the star hang over or under the \(z_{\text{sky}} = 0\) plane, the overhanging and underhanging regions, see illustration C of Figure 3.5. For a given ray \((x_{\text{sky}}, y_{\text{sky}})\), if the point \((x_{\text{sky}}, y_{\text{sky}}, 0)\) is within the star, the search occurs between \(z_{\text{sky}} = 0\) and \(z_{\text{sky}} = 2r_p\).

If the point \((x_{\text{sky}}, y_{\text{sky}}, 0)\) is not within the star the ray may intersect the star in the overhang or not intersect it at all. For these rays it must be determined if an intersection occurs. The edge of the star \(y_{\text{edge}}\) and its height above (or below) the \(z_{\text{sky}} = 0\) plane is found for the ray. This search occurs in the \(x_{\text{sky}}-z_{\text{sky}}\) plane though the coordinate \(\theta_{\text{star}}\) between \(\theta_1\), the lowest possible value of \(\theta_{\text{star}}\) for a particular \(x_{\text{sky}}\), given by

\[
\sin(\theta_1) = \frac{\cos(3\arccos(\frac{\frac{\text{shape}}{\text{frac}}}{x_{\text{star}}} - \frac{3\pi}{4})) - 4\pi}{\frac{\text{shape}}{\text{frac}}} \tag{3.22}
\]
as shown in Figure 3.5 B), and $\theta_2$ which is slightly greater than $\pi/2$. The bisection method is used to find the location where $\mu$ is zero, which indicates the surface is tangent to the $z_{\text{sky}}$-axis and thus the edge of the star for that $x_{\text{star}}$. For each $\theta_{\text{star}}^{\text{guess}}$, the remaining coordinates are determined by:

$$\phi(\theta_{\text{star}}^{\text{guess}}) = \arccos\left(\frac{x_{\text{star}}}{R(\theta_{\text{star}}^{\text{guess}}) \sin(\theta_{\text{star}}^{\text{guess}})}\right),$$

(3.23)

$$y(\theta_{\text{star}}^{\text{guess}}) = R(\theta_{\text{star}}^{\text{guess}}) \sin(\theta_{\text{star}}^{\text{guess}}) \sin(\phi(\theta_{\text{star}}^{\text{guess}})),$$

(3.24)

and

$$z(\theta_{\text{star}}^{\text{guess}}) = R(\theta_{\text{star}}^{\text{guess}}) \cos(\theta_{\text{star}}^{\text{guess}}).$$

(3.25)

Once $y_{\text{edge}}$ has been found if $|y_{\text{sky}}| < y_{\text{edge}}$ the ray intersects the star. For the portion of the star which over-hangs or under-hangs the $z_{\text{sky}} = 0$ plane, a new lower bound is needed. $z_{\text{sky}}$ is found using the bisection method between $z_{\text{edge}}$ and $2r_p$ if $y_{\text{sky}} < 0$ or between $-z_{\text{edge}}$ and 0 if $y_{\text{sky}} > 0$.

Once an intersection is located, it is straightforward to calculate the quantities on the stellar surface required to implement the boundary condition for the integration of the transfer equation:

$$R = \sqrt{x_{\text{star}}^2 + y_{\text{star}}^2 + z_{\text{star}}^2}$$

(3.26)

$$\vec{g} = -\frac{GM}{R^2}\left(\frac{x_{\text{star}}}{R}, \frac{y_{\text{star}}}{R}, \frac{z_{\text{star}}}{R}\right) + (\Omega_{\text{shape}}^2 x_{\text{star}}, \Omega_{\text{shape}}^2 y_{\text{star}}, 0)$$

(3.27)

$$\mu = \frac{\vec{g}}{|\vec{g}|} \cdot \hat{z}_{\text{sky}} = \frac{(g_x, g_y, g_z)}{\sqrt{g_x^2 + g_y^2 + g_z^2}} \cdot (0, -\sin(i), \cos(i))$$

(3.28)

$$\log |\vec{g}| = \log \sqrt{g_x^2 + g_y^2 + g_z^2}$$

(3.29)

$$T_{\text{eff}} = (C_w |\vec{g}|)^{\frac{1}{4}}$$

(3.30)

$$v_z_{\text{star}} = \hat{z}_{\text{sky}} \cdot \vec{v}_{\text{star}} = -\sin(i) x_{\text{star}} \Omega_{\text{shape}}.$$ 

(3.31)
If only spherical gravitational darkening is implemented (as discussed above), these results become

\[ R = \sqrt{x_{\text{star}}^2 + y_{\text{star}}^2 + z_{\text{star}}^2} \]  
\[ \bar{g} = -\frac{GM}{R^2} \left( \frac{x_{\text{star}}}{R}, \frac{y_{\text{star}}}{R}, \frac{z_{\text{star}}}{R} \right) \]  
\[ \mu = \frac{\bar{g} \cdot \vec{z}_{\text{sky}}}{|\bar{g}|} = \frac{(g_x, g_y, g_z)}{\sqrt{g_x^2 + g_y^2 + g_z^2}} \cdot (0, -\sin(i), \cos(i)) \]  
\[ g^2 = -\frac{GM}{R^2} \left( \frac{x_{\text{star}}}{R}, \frac{y_{\text{star}}}{R}, \frac{z_{\text{star}}}{R} \right) \]  
\[ + (\Omega_{\text{temp}}^2 x_{\text{star}}, \Omega_{\text{temp}}^2 y_{\text{star}}, 0) \]  
\[ \log |\bar{g}| = \log \sqrt{g^2_x + g^2_y + g^2_z} \]  
\[ T_{\text{eff}} = \left( C_{\omega} |\bar{g}| \right)^{\frac{1}{4}} \]  
\[ v_{z_{\text{star}}} = \vec{z}_{\text{sky}} \cdot \vec{v}_{\text{star}} = -\sin(i) x_{\text{star}} \Omega_{\text{temp}}. \]
Bibliography


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Chapter 4

The Thermal Structure of Gravitationally-Darkened Disks


A classical Be-star is a non-supergiant B star that possesses a gaseous equatorial disk (Porter & Rivinus, 2003). This circumstellar material produces an emission line spectrum including a prominent H$_\alpha$ emission line. Some Be stars, for example $\gamma$ Cassiopeia, produce a rich emission line spectrum with many hydrogen lines and some metal lines. Be star disks also produce a continuum excess due to bound-free and free-free emission. This excess has been observed at a variety of wavelengths from the visible to the radio (Coté & Waters, 1987; Waters & Coté, 1987). Linear polarization of up to 2% has been observed in the continuum emission and is caused by electron scattering in the non-spherical circumstellar envelope (McLean & Brown, 1978; Poeckert et al., 1979). Be star disks have been partially resolved by long-
baseline interferometric measurements, initially in the radio (Doughterty & Taylor, 1992), and later at infrared and optical wavelengths (Stee & Bittar, 2001; Tycner, 2004).

The stellar absorption lines of Be stars indicate rapid rotation of the central star, and this property is thought to play an important role in the release of material into the disk (Porter, 1996). There is strong evidence that Be star disks are in Keplerian rotation (Hummel & Vrancken, 2000; Oudmaijer et al., 2008); however, it is still uncertain if the equatorial rotation speeds of Be stars are close to “critical”, i.e. close to the Keplerian orbital speed at the inner edge of the disk (Cranmer, 2005). In addition, another mechanism must somehow allow for the wide range of behaviours observed in Be stars: some systems have exhibited essentially stable Hα emission for as long as they have been observed, while others are highly variable with timescales ranging from less than a day to decades (Porter & Rivinus, 2003).

A rapidly rotating star is changed in two ways: the stellar surface is distorted with the equatorial radius becoming larger than the polar radius, and the stellar surface temperature acquires a dependence on stellar latitude with cooler gas at the equator compared to the hotter pole. These phenomena together are commonly called gravitational darkening. The rotational distortion and surface intensity variation with latitude have direct observational support from interferometric observations, including the stars Achernar (Domiciano de Souza et al., 2003) and Altair (Monnier et al., 2007).

Because the equatorial region of the photosphere is both the fastest rotating and the dimmest, it can be difficult to detect this maximally Doppler-shifted region in integrated light. This problem can be compounded by the obscuration of the equatorial regions by disk material. As a result, the bounds on Be star rotation rates remain contentious (Chauville et al., 2001; Townsend et al.,
Nevertheless, the canonical result is that most Be stars rotate at \( \approx 80\% \) of their critical speed (as defined in Section 4.1). Unfortunately, it seems that directly measuring projected rotational velocities above 80\% of the critical speed may not be possible due to the dimming of the equatorial regions (Townsend et al., 2004). Cranmer (2005) presents a detailed statistical analysis of Be star \( v \sin i \) values in which he attempts to match the observed distributions using a parametrized distribution of equatorial velocities. He concludes that early-type Be stars, O7–B3.5, must rotate at rates significantly less than their critical speeds (peaking at 40-60\%) while later type Be stars, B3.5–A0, rotate much closer to their critical speeds (peaking at 70-90\%).

Previously, the effects of gravitational darkening have been included in disk models for the stars Achernar, or \( \alpha \) Eridani, (Carciofi et al., 2008) and \( \zeta \) Tauri (Carciofi et al., 2009). However, the rotating star was assumed to be a spheroid rather than following a Roche profile (see Section 4.1). In the literature, gravitational darkening has been included only as a fit parameter in models of individual stars. A comprehensive study of the how the inclusion of gravitational darkening affects models of Be star disks has not yet been performed.

In this paper, we systematically examine the effect of gravitational darkening on the thermal structure of a circumstellar disk. We restrict our investigation to changes in the energy reaching the disk and the resulting changes in the temperature distribution. Section 4.1 outlines the basic theory behind gravitational darkening and also gives computational details. Section 4.2 describes the stellar and disk models chosen for this investigation. Results are presented in Section 4.3 where the consequences of rotation on the energy reaching the circumstellar disk (Section 4.3.1) and the corresponding changes in the disk
temperature structure (Section 4.3.2) are investigated. In Section 4.4, we examine the two principle ingredients of gravitational darkening (temperature variation with latitude and distortion of the stellar surface) separately in order to gauge their relative importance. We also evaluate the effectiveness of the simpler spherical gravitational darkening approximation in which the shape distortion produced by rapid rotation is ignored. Section 4.5 gives the conclusions, and in Appendix A, we discuss the subtle aspects of how varying the equatorial radius can change the inherent properties of our disk models.

4.1 Theory of Gravitational Darkening

4.1.1 The Central Star

Gravitational darkening, first described by von Zeipel (1924), is due to a rotational, centrifugal term included in the classical gravitational potential. For a star rotating at a fixed angular speed $\omega$, the potential, $\Phi$, in spherical coordinates $(r, \theta, \phi)$, is given by

$$\Phi (r, \theta) = -\frac{GM}{r} - \frac{1}{2} \omega^2 r^2 \sin^2 \theta, \quad (4.1)$$

where $G$ is the gravitational constant, $M$ is the stellar mass, and $\theta$ is the stellar co-latitude ($\theta = 0^\circ$ on the polar axis). The local gravitational acceleration is given by

$$\vec{g} (\theta) = \left( \omega^2 r \sin^2 \theta - \frac{GM}{r^2} \right) \hat{r} + \left( \omega^2 r \sin \theta \cos \theta \right) \hat{\theta}. \quad (4.2)$$

The angular speed at which the local value of gravity at the equator becomes zero defines the critical angular speed, $\omega_{\text{crit}}$, and the corresponding critical
rotation velocity, $v_{\text{crit}}$. The angular speeds and the equatorial velocities of stars are often expressed as fractions of these critical values, $\omega_{\text{frac}}$ and $v_{\text{frac}}$. The relevant equations are

$$\omega_{\text{crit}} = \sqrt{\frac{8GM}{27r_p^3}}, \quad \omega_{\text{frac}} = \frac{\omega}{\omega_{\text{crit}}}, \quad v_{\text{frac}} = \frac{v_{\text{eq}}}{v_{\text{crit}}},$$

(4.3)

where $r_p$ is the polar radius of the star. In our calculations we have assumed that the polar radius remains constant following Collins (1966).

By requiring the potential across the surface to be constant, the radius as a function of $\theta$ is found to be (see again Collins, 1966)

$$r(\theta) = \left( \frac{-3r_p}{\omega_{\text{frac}} \sin \theta} \right) \cos \left( \frac{\arccos (\omega_{\text{frac}} \sin \theta) + 4\pi}{3} \right).$$

(4.4)

When the equator of the star becomes unbound at $\omega = \omega_{\text{crit}}$, $r(\theta = \frac{\pi}{2})$ takes on its largest value, $\frac{3}{2}r_p$. The Roche distortion of a stellar surface has direct interferometric confirmation. For example, Zhao et al. (2010) present reconstructed images of $\alpha$ Cep and $\alpha$ Oph which clearly reveal rotational distortion consistent with the von Zeipel theory. The shape of the distorted surface affects the stellar radiation intercepted by the disk as the mid-latitudes are tipped towards the pole and away from the equatorial regions. In addition, rapid rotation increases the surface area and projected surface area of the star. The physical effects associated with near critical rotation have been recently reviewed by Meynet et al. (2010).

The local effective temperature of the stellar atmosphere, $T_{\text{eff}}$, and the local surface gravity, $|\vec{g}|$, are related by von Zeipel’s theorem,

$$T_{\text{eff}}(\theta) = (C_\omega |\vec{g}(\theta)|)^{1/4},$$

(4.5)
where $C_\omega$ is von Zeipel’s constant. Vinicius et al. (2007) find that the von Zeipel variation of $T_{\text{eff}}$ with stellar latitude is required to simultaneously fit several He I photospheric absorption profiles from five rapidly rotating, early-type Be stars. Nevertheless, the power of $1/4$ appearing in the standard gravity-darkening law has been observationally questioned by Monnier et al. (2007) who find a better fit to their interferometric observations of Altair with an exponent of $0.19 \pm 0.012$. van Belle (2010) reviews all current interferometric observations of rapidly rotating stars and finds that of six observed stars, four are well fit by the standard $1/4$ exponent and two (including Altair) are not. Despite these results, we will retain the standard exponent in all calculations to follow.

The constant, $C_\omega$, is defined using the assumption that the total luminosity of the star remains fixed (for discussion of this point, see Lovekin et al., 2006). Therefore, while the equatorial temperature of a star decreases with rotation rate, the polar temperature must increase to maintain a constant luminosity. This effect is illustrated in Figure 4.1 which shows the change in temperature with increasing rotation for five stellar co-latitudes: $0^0$, $30^0$, $60^0$, $70^0$, and $90^0$. In the absence of rotation, the stellar surface has a uniform temperature, but as the rotation speed increases the surface temperatures begin to diverge. Note that the temperatures of the middle co-latitudes, such as $60^0$, do not monotonically increase or decrease in temperature. For example as rotation increases from zero, the temperature at $60^0$ decreases until \( \sim 0.80 v_{\text{crit}} \) where it reaches a minimum before increasing slightly with increased rotation.

The constant $C_\omega$ can be found for a specific set of stellar parameters ($r_p$, $M$ and $L$), given a value of $\omega$, by integrating the local gravity over the stellar surface and then multiplying by the luminosity divided by the Stephan-Boltzmann constant. Once $C_\omega$ has been determined for one set of stellar parameters, the
Figure 4.1: Change in the local effective temperature with increasing \( v_{\text{frac}} \) (lower axis) for a B2V stellar model (see Table 4.1). The upper axis shows the corresponding \( \omega_{\text{frac}} \). The darkest solid line gives the polar temperature \( (\theta = 0^0) \), while the lightest grey dash-dot line shows the equatorial temperature \( (\theta = 90^0) \). Various intermediate stellar latitudes are also shown. The right vertical axis shows the standard main sequence spectral type (with no rotation) for the same surface temperature as the left axis (Cox, 2000).

results can be rescaled for any other star following Collins (1966) using

\[
C_\omega' C_\omega = \frac{M L'}{M' L}. \tag{4.6}
\]

Here \( M', L', \) & \( C_\omega' \) are the mass, luminosity and von Zeipel’s constant of a particular star, and \( M, L, \) & \( C_\omega \) are for another star with an identical rotation rate.

The local gravity and surface orientation can be established from Equation 4.2, and the temperature from Equation 4.5. This allows the properties of the local stellar atmosphere, \( g \) and \( T_{\text{eff}} \), to be defined for each point on the surface of the star.
4.1.2 The Circumstellar Disk

We now turn to the Be star circumstellar disk. From a particular vantage point within the disk, labelled $\mathbf{s}$, defined using the cylindrical coordinates, $(s_R, s_z, s_\phi)$, only a portion of the star is visible. For a section of the stellar surface specified by $\mathbf{r}'(\theta, \phi)$ to be visible from the vantage point, the dot product between the unit vector pointing from the stellar surface to the vantage point, $\hat{n}_{vp} = \mathbf{s} - \mathbf{r}'$, and the local surface normal, $\hat{n}_{surf} = \frac{-\hat{\mathbf{g}}}{|\hat{\mathbf{g}}|}$, must be greater than zero. The boundary of this region where $\hat{n}_{surf} \cdot \hat{n}_{vp} = 0$ is important because it allows the lines of sight between the star and the vantage point to be efficiently chosen. The upper and lower edges of the visible region correspond to the maximum and minimum values of $\theta$ such that $\hat{n}_{surf} \cdot \hat{n}_{vp} \geq 0$. These are found by solving

\[
\frac{8}{27} \frac{\omega_{frac}^2}{r_p^3} \frac{r^3}{r_p^3} s_R \sin \theta - r_\theta - s_R \sin \theta + s_z \cos \theta = 0 \tag{4.7}
\]

for its two solutions, $\theta_1$ and $\theta_2$. For values of $\theta$ between $\theta_1$ and $\theta_2$, the visible portion of the surface lies between the bounds on $\phi$ given by

\[
\phi_{edge} = s_\phi \pm \arccos \left( \frac{r(\theta) \sin^2 \theta (1 - \kappa_\theta) + r(\theta) \cos^2 \theta - s_z \cos \theta}{s_R \sin \theta - \kappa_\theta s_R \sin \theta} \right), \tag{4.8}
\]

with

\[
\kappa_\theta = \frac{8}{27} \omega_{frac}^2 \left( \frac{r_\theta}{r_p} \right)^3. \tag{4.9}
\]

Given this geometry, we can now take into account the temperature changes and the radial distortion of the star to determine the photoionizing radiation field reaching each point in the disk. The effects of gravitational darkening are illustrated in Figure 4.2. This figure shows a star rotating at 80% of $v_{crit}$ with
Figure 4.2: The temperature structure of a B2V star rotating at $v_{\text{rot}} = 0.8$ viewed with an inclination of $45^\circ$ to the polar axis (marked by a small dot in uppermost oval). The bold, solid line is the equator. The local temperatures shown are determined by Equation (4.5). The dashed line is the limit of the region visible from a vantage point located within the disk at a radius of $s_R = 2r_p$ and a height of $s_z = 1.2r_p$, as defined by Equation (4.8). The square and diamond symbols mark the location of $\theta_1$ and $\theta_2$ respectively which are found by solving Equation (4.7). The direct line of sight from $s_R = 2r_p$ and $s_z = 1.2r_p$ is indicated by the symbol $\odot$.

its shape distortion and associated surface temperature variations. Figure 4.2 also shows the visible region, $\theta_1$, $\theta_2$ and $\phi_{\text{edge}}$, for a vantage point at a radius of $s_R = 2r_p$ and a height above the disk of $s_z = 1.2r_p$. The disk is assumed to be axisymmetric so any $\Phi$ dependence is not included. This figure demonstrates how the visible region changes from a simple spherical cap for a non-rotating star to an elongated region.
The circumstellar gas is exposed to photoionizing radiation from the various latitudes of the star and this radiation changes with increasing rotation. Different locations within the disk have different regions of the photosphere within their field of view. The polar temperature increases with rotation in contrast to the decrease in the equatorial temperature (see Figure 4.1). The drop in the equatorial temperature occurs at a faster rate than the rise in the polar temperature. This means the effects of the hot stellar pole become significant at speeds above 80% of $v_{\text{crit}}$. In the equatorial plane up to a distance of $4 r_p$, stellar latitudes above $25^\circ$ are not visible for a circular star. For a critically rotating star this increases to $\approx 30^\circ$. As the gas density in the disk is assumed to fall off with height above the equatorial plane, the light reaching upper parts of the disk from the cooler stellar equatorial region passes through a greater optical depth than that from the hotter pole. Thus the effects of gravitational darkening can vary in strength depending on location within the disk and can have very different effects on the temperature depending on what part of the disk is considered. The effect of gravitational darkening cannot be considered to be simply a reduction of the effective temperature of the star.

### 4.2 Calculations

A gravitationally darkened version of BEDISK (Sigut & Jones, 2007), incorporating the theory of Section 4.1, was created and run for the stellar parameters given in Table 4.1 and the rotation rates given in Table 4.2. These stellar parameters (adopted from Cox, 2000) were chosen to include a model from four of the five bins used by Cranmer (2005) to analyze the effects of spectral type on Be star rotational statistics. For reasons discussed below, we have not considered models from the coolest bin (the bin with characteristic spectral type...
B8) considered by Cranmer (2005). Bedisk solves the statistical equilibrium equations for the atomic level population equations and then enforces radiative equilibrium at each point of the computational grid (Sigut & Jones, 2007). The density structure of the disk is assumed to be of the form

$$\rho(R, z) = \rho_0 \left( \frac{R_*}{R} \right)^n e^{-\frac{(z/H)^2}{2}}, H(R) = \sqrt{\frac{2R^3kT_{iso}}{GM\mu}}. \quad (4.10)$$

Here $R_*$ is the stellar radius (see next paragraph). In the second equation for the disk scale height, $H(R)$, $\mu$ is the mean-molecular weight of the gas, and $T_{iso}$ is an assumed isothermal temperature used for the sole purpose of setting the density scale height. The models are constructed with the density parameters set to $n = 3$ and $\rho_0 = 5 \times 10^{-11} \text{ g/cm}^3$. These density parameters are kept constant. The form of the disk density given by Equation 4.10 follows from the assumption of a radial power-law drop in the equatorial plane ($z = 0$) coupled with the assumption that the disk is in vertical hydrostatic equilibrium set by the $z$-component of the star’s gravitational acceleration. Note that Equation 4.10 results in a flaring disk with $H \propto R^{3/2}$.

If the equatorial radius expands with rotation, this can change the disk’s density structure if the $R_*$ appearing in Equation 4.10 is associated with the star’s equatorial radius. This is undesirable because we wish to investigate changes in the disk temperatures resulting solely from the changing photoionizing radiation field seen by the disk and not from an unintended change to the underlying density structure of the disk. Unfortunately, there is no ideal solution to this complication and several approaches are outlined in Appendix A. In the notation of Appendix A, all of the calculations were computed with an unchanging grid in which the inner edge of the disk is set to $3/2 r_p$. This approach preserves the disk density structure and is best for temperature com-
parisons. A physical grid, in which the density structure of the non-rotating star is shifted outward as $R_\star$ increases, is more useful when calculating observables, and this grid will be used in a subsequent paper on predicted H\alpha line profiles and infrared excesses.

Table 4.1: Adopted stellar parameters.

<table>
<thead>
<tr>
<th>Spectral Type</th>
<th>Mass ($M_\odot$)</th>
<th>Polar Radius ($R_\odot$)</th>
<th>Luminosity ($L_\odot$)</th>
<th>$\omega_{\text{crit}}$ (s$^{-1}$)</th>
<th>$v_{\text{crit}} = \frac{2}{5} r_p \omega_{\text{crit}}$ (km s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B0V</td>
<td>17.5</td>
<td>7.40</td>
<td>$3.98 \times 10^4$</td>
<td>$7.10 \times 10^{-5}$</td>
<td>548</td>
</tr>
<tr>
<td>B2V</td>
<td>9.11</td>
<td>5.33</td>
<td>$4.76 \times 10^3$</td>
<td>$8.38 \times 10^{-5}$</td>
<td>466</td>
</tr>
<tr>
<td>B3V</td>
<td>7.60</td>
<td>4.80</td>
<td>$2.58 \times 10^3$</td>
<td>$8.95 \times 10^{-5}$</td>
<td>449</td>
</tr>
<tr>
<td>B5V</td>
<td>5.90</td>
<td>3.90</td>
<td>$7.28 \times 10^2$</td>
<td>$1.08 \times 10^{-4}$</td>
<td>439</td>
</tr>
</tbody>
</table>

Notes: Stellar parameters adopted from Cox (2000).

Table 4.2: Adopted rotation rates and example equatorial speeds.

<table>
<thead>
<tr>
<th>$v_{\text{frac}}$</th>
<th>$\omega_{\text{frac}}$</th>
<th>$r_{\text{eq}}/r_p$</th>
<th>$v_{\text{eq}}$ B0 (km s$^{-1}$)</th>
<th>$v_{\text{eq}}$ B3 (km s$^{-1}$)</th>
<th>$v_{\text{eq}}$ B5 (km s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>0.000</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.001</td>
<td>0.002</td>
<td>1.00</td>
<td>0.55</td>
<td>0.45</td>
<td>0.40</td>
</tr>
<tr>
<td>0.200</td>
<td>0.296</td>
<td>1.01</td>
<td>110</td>
<td>90</td>
<td>80</td>
</tr>
<tr>
<td>0.400</td>
<td>0.568</td>
<td>1.06</td>
<td>219</td>
<td>180</td>
<td>160</td>
</tr>
<tr>
<td>0.600</td>
<td>0.792</td>
<td>1.14</td>
<td>329</td>
<td>269</td>
<td>241</td>
</tr>
<tr>
<td>0.800</td>
<td>0.944</td>
<td>1.27</td>
<td>438</td>
<td>359</td>
<td>321</td>
</tr>
<tr>
<td>0.950</td>
<td>0.996</td>
<td>1.43</td>
<td>521</td>
<td>427</td>
<td>381</td>
</tr>
<tr>
<td>0.990</td>
<td>0.999</td>
<td>1.49</td>
<td>544</td>
<td>445</td>
<td>397</td>
</tr>
</tbody>
</table>

The photoionizing radiation field incident on the disk was calculated as follows: first, von Zeipel’s constant is found by using Equation 4.6 to rescale an interpolated value of $C_{\omega}$ from Collins (1966) to match the stellar parameters. Next, for each point in the disk, the visible region of the stellar surface is
determined, and a grid is constructed across this surface. At each point in this surface grid, the local temperature, local gravity, and the viewing angle are found. Finally, the specific intensity, \( I_{\nu}(T_{\text{eff}}, g) \), for each point is scaled from the Kurucz (1993) model atmosphere closest to the effective temperature and gravity at each location, \( I_{\nu}^K(T_{\text{best}}, g_{\text{best}}) \), to the required temperature using a blackbody function,

\[
I_{\nu}(T_{\text{eff}}, g) = I_{\nu}^K(T_{\text{best}}, g_{\text{best}}) \frac{B_{\nu}(T_{\text{eff}})}{B_{\nu}(T_{\text{best}})}.
\]  

(4.11)

Here \( T_{\text{eff}} \) is the local effective temperature of the stellar surface, and \( T_{\text{best}} \) and \( g_{\text{best}} \) are the parameters of the closest matching Kurucz model. The available Kurucz grid of model atmosphere intensities was interpolated to a finer grid of spacing of \( \Delta T_{\text{eff}} = 200 \text{K} \) and \( \Delta \log(g) = 0.10 \). Equation 4.11 was used to correct the intensity only in these very narrow temperature bins.

Most of the calculations performed in this work implement the above description of gravitational darkening and are referred to as full gravitational darkening (FGD) calculations. All of the models presented in Section 4.3 are FGD calculations. However in Section 4.4, two simplified approaches are described which isolate one of the two main effects of gravitational darkening: spherical gravitational darkening (SGD), accounting only for the temperature change across the stellar surface, and pure shape distortion (PSD), accounting only for the changing the shape of the star.
4.3 Results for Gravitationally-Darkened Be Star Disks

4.3.1 Energy reaching the Circumstellar Disk

The stellar radiation field is modified by gravitational darkening, and this directly affects the disk temperatures through the photoionization rates. Figure 4.3 shows the total photoionizing mean intensity at each location in the computation grid for a B2 star both with and without a disk, and with and without rotation. The mean intensity ratios between the various cases are also shown. Consider the left-hand panels A and B in which there is no disk material. In the rotating case ($v_{\text{frac}} = 0.99$), less radiation from the poles reaches the equatorial regions of the disk, and this results in the hour glass shape in the mean intensity of panel B. In the bottom-left panel which gives the ratio of the rotating to non-rotating case, the stellar radiation field reaching the disk has been reduced by approximately 50% compared to the non-rotating star. However, for regions close to the star and near the equatorial plane, there can be reductions as great as 90%. Now consider the right-hand panels C and D in which the opacity and emissivity of the disk gas have been included. Again the cases of rotation at $v_{\text{frac}} = 0.99$ and no rotation are compared. Most of the radiation reaching various positions within the disk is re-processed by the circumstellar gas between a particular location and the star. In general, the total irradiance reaching the shielded parts of the disk is reduced down with gravitational darkening by about 50%. However, there are locations within the mid-regions of the inner disk where the energy actually increases with rotation (see Figure 4.3) due to changes in the disk’s opacity.
Figure 4.3: Left panels A and B show the log of the mean intensity at each location in the computational grid without any disk material present for a star with $v_{\text{frac}} = 0$ (panel A) and $v_{\text{frac}} = 0.99$ (panel B). The ratio of the mean intensity for the rotating case with the non-rotating case is given in the bottom left panel (labelled B/A). The right panels C and D show the mean intensity at each location in the disk including the opacity effects of the disk material for a star with $v_{\text{frac}} = 0$ (panel D) and $v_{\text{frac}} = 0.99$ (panel E). The ratio of the shielded rotating case with the non-rotating case is shown in bottom right panel (labelled D/C).
4.3.2 Temperatures in the Circumstellar Disk

Any change in the energy received from the star produces a change in the radiative equilibrium temperature. The changes produced by increasing rotation on the general temperature structure of the disk are illustrated in Figures 4.4 through 4.7 for the spectral types given in Table 4.1. Shown in all figures are four disk temperature diagnostics: the maximum temperature, the minimum temperature, the density-weighted temperature, defined as

\[ T_{\rho} = \frac{1}{M_{\text{disk}}} \int T(R, z) \rho(R, z) \, dV, \]  

(4.12)

and the volume-averaged temperature, defined as

\[ T_V = \frac{1}{V_{\text{disk}}} \int T(R, z) \, dV. \]  

(4.13)

In order to avoid numerical effects, the maximum and minimum temperatures are averages over the twenty hottest and twenty coolest disk locations, respectively. In Figures 4.4 through 4.7, all of these temperature diagnostics are plotted as a function of \( v_{\text{frac}} \) and \( \omega_{\text{frac}} \). Because their behaviour depends somewhat on spectral type, the results for each model are discussed separately.

For the B0 model, seen in Figure 4.4, both the density and volume-averaged disk temperatures decrease steadily with increasing rotation. Small declines in the density-weighted, maximum, and minimum disk temperatures are apparent even at \( v_{\text{frac}} \approx 0.4 \). By critical rotation, the density-weighted temperature has fallen to just under 11,000 K, a decline of 2500 K compared to the non-rotating case. The decline is not as large in the volume-average temperature, and this is likely due to the influence of the hotter stellar pole that develops for increased rotation (see Figure 4.1); this is also reflected in the maximum and
Figure 4.4: Change in disk temperature diagnostics with increasing rotation for the B0 model.

Figure 4.5: Same as Figure 4.4 for a B2 star.
Figure 4.6: Same as Figure 4.4 for a B3 star.

Figure 4.7: Same as Figure 4.4 for a B5 star.
minimum disk temperatures: while the minimum temperature has a steep decline, the maximum temperature is much flatter and actually reaches a shallow minimum at an intermediate rotation rate.

For the B2 model, shown in Figure 4.5, all temperatures initially decline slowly. The density-weighted disk temperature falls from over 9000 K in the non-rotating case to under 8000 K by critical rotation. However, unlike the previous case, the volume-averaged temperature reaches a minimum near \( v_{\text{frac}} \approx 0.8 \) and then begins to increase for extremely rapid rotation. The lack of an increase in the density-weighted temperature indicates that the additional heating occurs mainly in the upper disk, where there is less gas, and the heating is likely due to the effect of the hot stellar pole. We also note that minimum temperature in the disk, \( \approx 6000 \text{ K} \), is approximately constant with rotation, with only a small decline for \( v_{\text{frac}} \geq 0.7 \).

Spectral type B3 is shown in Figure 4.6. Its behaviour is very similar to the B2 case above, although the temperature rise in the volume-averaged average temperature above \( v_{\text{frac}} \geq 0.8 \) is not as large as in the B2 case. Again the minimum disk temperature is around 6000 K and is not strongly affected by rotation. For B5, shown in Figure 4.7, both the density and volume-averaged temperatures show a similar decline with rotation and there is no rise in the volume-averaged temperature above \( v_{\text{frac}} \geq 0.9 \). By this spectral type, the minimum disk temperature has fallen to \( \approx 5000 \text{ K} \) and shows no clear trend with rotation. The low temperatures reached in this model have implications for later spectral types that we shall now discuss.

As noted in the introduction to Section 4.2, we have not considered a model of spectral type B8 from the VL (very late) spectral bin of Cranmer (2005). Our test calculations have indicated that very cool disk temperatures are reached in this model, particularly for near critical rotation, and these may
Table 4.3: Change in density-weighted average disk temperature for near-critical rotation.

<table>
<thead>
<tr>
<th>Spectral Type</th>
<th>$T(\rho)(K)$</th>
<th>$v_{frac} = 0.00$</th>
<th>$v_{frac} = 0.99$</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>B0</td>
<td>13280</td>
<td>10780</td>
<td>-19</td>
<td></td>
</tr>
<tr>
<td>B2</td>
<td>9190</td>
<td>7870</td>
<td>-14</td>
<td></td>
</tr>
<tr>
<td>B3</td>
<td>8690</td>
<td>7460</td>
<td>-14</td>
<td></td>
</tr>
<tr>
<td>B5</td>
<td>7660</td>
<td>6760</td>
<td>-12</td>
<td></td>
</tr>
</tbody>
</table>

fall outside of the domain of applicability of the current version of bedisk. In particularly while the global density-weighted disk temperature still exceeds 6000 K for a B8 model, there is a significant volume of the disk which falls well below 5000 K by $v_{frac} = 0.99$. While our calculations include abundant metals with low ionization potentials (such as Mg, Ca, and Fe) that provide sources of free electrons at low temperatures, molecule formation is not included. In addition for such cool disks, the treatment of the diffuse radiation field generated within the disk may require a more careful treatment. For these reasons, we have chosen not to include B8 models in this work. We note that in the analysis of Cranmer (2005), the four bins we do account for include the full range of inferred threshold rotation rates, from $v_{frac} \approx 0.6$ for early spectral types B0–B2 to $v_{frac} \approx 1.0$ for the late spectral type B5.

The effect that near critical rotation ($v_{frac} = 0.99$) has on the global, density-weighted disk temperature is summarized in Table 4.3. Compared to non-rotating models, near critical rotating models are between 15 and 20% cooler with the largest difference occurring for the earliest spectral type considered (B0).

Next we discuss how the detailed temperature structure of a Be star disk, $T(R, z)$, changes with rotation. In general, Be star circumstellar disks are
Figure 4.8: Temperatures for both a non-rotating B0 star and its circumstellar disk (upper left) and a B0 star rotating at $v_{\text{frac}} = 0.95$ with an identical disk (lower left). In each of these panels, the colour bar on the left is for the star and the colour bar on the right, for the disk. The lower right panel shows the temperature differences in the disk. Positive differences mean that the non-rotating star is hotter. The upper-right panel shows a histogram of the temperature differences.
Figure 4.9: Same as Figure 4.8 for a B5 star.
highly non-uniform in temperature as disks of sufficient density develop a cool, inner zone surrounded by a larger warm region (Millar & Marlborough, 1998; Sigut et al., 2009). Between these two regions is the hottest part of the disk, forming a narrow sheath. Figures 4.8 and 4.9 show the effect of rotation on the temperature of disks surrounding stars of spectral types B0 and B5 respectively. In each plot, temperature distributions for two models are shown, one with $v_{\text{frac}} = 0.0$ and the other with $v_{\text{frac}} = 0.95$. Also shown is the temperature difference between the rotating and non-rotating models and a histogram of these differences. The value of $v_{\text{frac}} = 0.95$ was chosen for the rotating model because this is approximately the rotation rate required for a velocity perturbation on the order of the local sound speed in the star’s photosphere to be sufficient to feed material into a rotationally supported disk.

For the B0 model, shown in Figure 4.8, the most dramatic changes in temperature occur because of the expansion of the cool, inner zone to larger radii with increased rotation. In the non-rotating case, this cool region does not extend past $\approx 7 R_*$ while in the $v_{\text{frac}} = 0.95$ model, it extends well past $10 R_*$. This results in some computational grid points that are more than 4,000 K cooler in the rotating model (compared to the 2500 K difference in the density-weighted average) as illustrated by the histogram of temperature differences in Figure 4.8. However, there are also temperature changes in the hot sheaths above and below the inner cool zone close to the star. For the $v_{\text{frac}} = 0.95$ model, these hot sheaths occur only beyond $\approx 3 R_*$ whereas in the non-rotating case, these sheaths extend to the inner edge of the disk.

Despite these changes, the temperature in the equatorial plane of the cool, inner zone does not drop dramatically with rotation, likely due to the high optical depths along the rays back to the star. This explains why the density-weighted average temperature of the disk shows only a modest decrease com-
pared to the non-rotating case (see Figure 4.4). Interestingly, the rotating model is actually hotter than the non-rotating model in a narrow region a few scale heights above the equatorial plane for $R > 10R_*$, although the number of computational grid points involved is quite small. The optically thin gas above and below the plane of the disk shows only a very small decrease in temperature and thus the volume-averaged temperature of Figure 4.4 is essentially unaffected by rapid rotation.

Figure 4.8 clearly illustrates that the temperature distribution surrounding a rotating star may differ dramatically from the non-rotating case, even when global measures of disk temperature such as $\overline{T}_p$ and $\overline{T}_V$ are similar. From the point of view of computing the strength of emission lines or the infrared excess, the increased extent of the inner cool zone and the changes to the temperature of the hot sheaths above and below this zone have the potential to produce large changes in these observational diagnostics. Such changes will be discussed in a subsequent paper.

Figure 4.9 shows a similar comparison between a non-rotating model and one rotating at $v_{frac} = 0.95$ for spectral type B5. As in the B0 case, the main effect is the extension of the inner, cool zone to larger radii. However the B5 model also has a significant extension of this cool zone to larger distances above (and below) the plane of the disk. As these regions occupy a significant volume, the volume-weighted disk temperature is significantly decreased (unlike the previous case of a B0 star). As shown in Figure 4.7, both the density and volume-weighted disk temperatures show a similar drop with rotation.

While most disk locations are, as before, cooler, there is a much larger fraction of disk locations that are hotter in the B5 rotating model than in the previous B0 case. Most of these hotter regions are associated with a cool zone at large radii, $R > 10R_*$, far above (and below) the equatorial plane of the
Figure 4.10: Variation of the vertically-averaged, density-weighted temperature with disk radius for various stellar rotation rates for the B0 model.

disk. In these regions, the lines of sight back to the central star have significant optical depths whereas the optical depths (in the $z$-direction) to the nearest disk edge have become less than unity. Hence heating via photoionization is suppressed whereas cooling via escaping line radiation becomes effective and there is a reduction in temperature due to this cooling. This low temperature zone disappears for higher $z$ because the optical depths back to the central star eventually drop and the heating rate increases. The exact location and detailed form of the zones is influenced by rotation and some locations with significantly hotter temperatures are predicted. It should be kept in mind that at large radii and far above or below the plane of the disk, the gas densities are very low, and the detailed structure of these cool zones will have little effect on observable diagnostics such as emission line strengths, polarization, or infrared excess.

While the detailed $T(R, z)$ comparisons of the previous paragraphs are
Figure 4.11: Same as Figure 4.10 for the B3 model.

Figure 4.12: Same as Figure 4.10 for the B5 model.
instructive, it is difficult to see general trends over a wide range of rotation
rotates. For this reason, it is useful to have temperature diagnostics that are
intermediate between the disk-averaged, global measures of Figures 4.4 to 4.7
and the detailed temperature distributions of Figures 4.8 and 4.9. To this end,
we have performed both vertical ($z$) and radial ($R$) temperature averages over
the disks of three stellar models (B0, B3, and B5) and considered how these
averages are affected by a wide range of rotation rates.

The vertical temperature averages (yielding an average disk temperature
as a function of distance from the star) were computed via

$$
\overline{T}(R) = \frac{\int_0^{z_{max}} T(R, z) \rho(R, z) \, dz}{\int_0^{z_{max}} \rho(R, z) \, dz}, \tag{4.14}
$$

and the change in vertically-averaged, density-weighted temperature as a func-
tion of radius is shown in Figures 4.10 through 4.12 for three stellar models,
B0, B3, and B5. Five rotation rates, $v_{frac} = 0.00, 0.40, 0.80, 0.90, 0.95$ and
0.99, were considered. The behaviour of the B2 model of Table 4.1, not shown
for brevity, is similar to the B3 case. In all cases considered, the $v_{frac} = 0.40$
models show vertically averaged temperatures that are very close to the non-
rotating case while significant differences develop by $v_{frac} = 0.80$.

These vertical averages show that the region next to the central star is
hottest, followed by a rapid decrease towards a minimum temperature near $\approx
4 \, R_\ast$ (for B0) to $\approx 6 \, R_\ast$ (for B5). Further out, the temperature increases again
towards the outer region of the disk. In the B0 model, the temperature then
plateaus, whereas in the B3 and B5 models the temperature is still increasing
by $R \approx 20 \, R_\ast$. From these figures, it can be seen that increasing rotation tends
to: (1) decrease the average temperature at all radii, (2) move the location
of the temperature minimum in the cool zone inward (although not in the
Figure 4.13: Variation of the radially averaged temperature profile in the vertical direction for the B0 model. Here the gas scale height is the variable $u$ defined by $u \equiv z/H(R)$.

The B0 model, and (3) broaden the extent of the cool, inner zone. Rotation also tends to flatten the temperature increase occurring in the outer regions of the disk.

Turning to the vertical ($z$) structure of the disks, we define a density-weighted, radially-averaged temperature (yielding an average temperature as a function of height above the equatorial plane) as

$$
\overline{T}(u) = \frac{\int_0^{R_{\text{max}}} T(R, u) \rho(R, u) A(R, u) dR}{\int_0^{R_{\text{max}}} \rho(R, u) A(R, u) dR},
$$

(4.15)

where $u \equiv z/H(R)$ and $H(R)$ is the vertical scale height given by Equation 4.10. The vertical variable, $z$, was rescaled by $H(R)$ because lines of constant $u$ better define a radial average due to the increase in the disk scale height with $R$ (see Equation 4.10). The function $A(R, u)$ arises because Equa-
Figure 4.14: Same as Figure 4.13 for the B3 model.

Figure 4.15: Same as Figure 4.13 for the B5 model.
tion 4.15 is effectively a line integral through the computational grid. At each \( R_i \) in the discrete sum used to compute this integral, \( z \) must be found from \( R_i \) and the required \( u \). As different \( z \) occur in the sum and the spacing of the \( z \)-grid increases with \( R \), we have weighted each \( z \) point by the width of the strip it effectively represents as based on the local grid spacing; this is the origin of the \( A(R, u) \) function. Such considerations do not occur for Equation 4.14 because this integral is performed over all \( z_j \) for a fixed \( R_i \).

The changes in the vertical temperature structure with rotation are shown in Figures 4.13 through 4.15 for spectral types B0, B3, and B5, respectively. Again, the B2 model of Table 4.1 is similar to B3 but is not shown for brevity. Note that for each spectral type, the set of weights \( A(R, u) \) used in Equation 4.15 is always the same as the computational grid is fixed and independent of \( v_{\text{frac}} \) (i.e. we have chosen to use the unchanging grid of Appendix A).

In general with increasing vertical height (i.e. perpendicular to the equatorial plan), the temperature is coolest in the plane of the disk \( (z = 0) \) and then rises to a maximum near \( u \approx 2 \). This maximum corresponds to the hot sheath seen in the plots of the temperature distributions in these disks (for example, see Figure 4.8). At larger heights, the temperature drops below the mid-plane maximum and is nearly constant. With increasing rotation, several trends are seen: (1) rotation reduces the temperature at all scale heights, (2) rotation increases the vertical extent of the cool, inner zone, (3) rotation decreases the temperature maximum in the mid-plane, and (4) rotation shifts the maximum temperature to larger scale heights, although this shift is not large.

Nevertheless, there are exceptions to these general trends, particularly in the temperatures at large \( u \). In the B0 model (Figure 4.13), the temperature at the upper edge of the disk decreases monotonically with rotation. However, for the B3 model, the temperature at the upper edge of the disk is actually largest
Figure 4.16: Density-weighted temperature average plotted against spectral type for six different rotation rates.

for the greatest rotation rate. The location of the hot sheath (the temperature maximum in $u$) moves outward in $u$ for later spectral types; it occurs well below $u = 2$ for the B0 model at all rotation rates, whereas it is above $u = 2$ for the two later spectral types, and it moves higher with increased rotation. Hence in the later spectral types, the temperature at the upper edge of the disk is affected by the presence of the hot sheath.

Finally, turning back to global measures of disk temperature, Figure 4.16 summarizes the change in the density-weighted average disk temperature with rotation for all four spectral types. In general, the non-rotating models define the upper envelope of the temperatures while the extreme rotators ($v_{\text{frac}} \geq 0.95$) define the lower temperature envelope. In constructing this figure, any change in the overall spectral type of the star due to increasing rotation has been ignored.
4.4 Approximate Treatments of Gravitational Darkening

In this section, we assess the effectiveness of partial or approximate treatments of gravitational darkening. In one approach, spherical gravitational darkening (SGD), we have treated the star as a sphere and implemented only the variation of the photospheric temperature across the stellar surface. As this procedure is very simple to implement, it is of interest to see how the predicted disk temperature structure compares to the full treatment. In another approach, pure shape distortion (PSD), we include only the rotational distortion of the central star, and not the temperature variation, to understand the effects of geometry alone. Global density and volume-weighted disk temperatures for both approximations (SGD and PSD), and the full treatment (FGD), are shown in Figure 4.17 for the B2 stellar model.

The differences between FGD and SGD are subtle for $v_{\text{frac}} \leq 0.7$, with only small differences in the density-weighted and volume-weighted average temperatures. For the largest rotational rates, $v_{\text{frac}} \geq 0.95$, SGD predicts density-weighted temperatures that level off for higher rotation rates while the full treatment gives temperatures that continue to drop. The difference amounts to $\approx +500\,\text{K}$ for the SGD model at $v_{\text{frac}} = 0.99$ in the density-weighted average temperature. In the volume-averaged temperature, the FGD temperature actually falls below the SGD prediction in the vicinity of $v_{\text{frac}} \approx 0.80$, but then increases above the SGD prediction for larger rotation rates. By $v_{\text{frac}} = 0.99$, the difference is again about 500 K.

To further illuminate this result, these calculations were repeated with PSD. As evident from Figure 4.17, PSD results in essentially no change to
the average disk temperatures for $v_{\text{frac}} \leq 0.7$ and only very small changes for faster rotation rates. This, of course, does not mean that the effect of the rotational distortion of the stellar surface is unimportant; indeed the difference between the FGD treatment and SGD noted above is the neglect of the stellar distortion. The PSD treatment simply illustrates that the distortion of the star alone produces almost no change in global disk temperatures.

However global temperature diagnostics tell only part of the story. To illustrate how the temperature structure of the disk is reproduced by the SGD approximation, Figure 4.18 compares the vertically-averaged (Eq. 4.14) and radially-averaged (Eq. 4.15) disk temperatures for a B2 stellar model computed with three rotation rates, $v_{\text{frac}} = 0, 0.80$, and 0.95. As can be seen from the Figure, there are significant differences between all the rotating models (either FGD or SGD) and the non-rotating model, and all of the previously discussed effects can be seen. Interestingly both SGD profiles for $v_{\text{frac}} = 0.8$ are close to the FGD prediction. However by $v_{\text{frac}} = 0.95$, the SGD temperature profiles remain close to the $v_{\text{frac}} = 0.8$ SGD predictions and do not follow the trend of the full (FGD) treatment; this is particularly noticeable in the radially-averaged profile (bottom panel) where the (average) location of the hot sheath is very poorly predicted by the SGD model. We conclude that SGD is an acceptable approximation to the temperature structure only for $v_{\text{frac}} \leq 0.8$, and that for higher rotation rates, the full treatment is required.
Figure 4.17: The global volume-averaged and density-averaged disk temperature as a function of $\nu_{\text{frac}}$ for the B2 model. Three different treatments of gravitational darkening considered: FGD, SGD, and PSD.
Figure 4.18: Vertically-averaged (top panel) and radially-averaged (bottom panel) temperature profiles (see Equations 4.14 and 4.15) for a B2 stellar model that compare the SGD approximation with the full treatment (FGD). Models rotating at $v_{\text{frac}} = 0$ (solid lines), 0.80 (dot-dashed lines) and 0.95 (dashed lines) are shown.
4.5 Conclusions

Gravitational darkening produces noticeable changes in the temperature structure of Be star circumstellar disks. Rotation causes the stellar surface temperature to diverge from a single value into a range of temperatures, and this changes the photoionizing radiation field incident on the disk. Three measures of disk temperature were used to quantify the effect of rapid rotation: detailed temperature distributions, $T(R, z)$, vertically and radially-averaged disk temperatures, and global density or volume-weighted average disk temperatures that reduced the complex temperature variations of the disk to a single, average temperature. Among these choices, the vertically and radially-averaged temperatures are perhaps the most useful for illustrating the effect of rapid rotation over a wide range of rotation rates and spectral types.

Rapid rotation of the central star and the accompanying gravitational darkening cause most of the disk gas to systematically decrease in temperature, and this is reflected in lower density-weighted, average disk temperatures for increasing rotation rates. However volume-averaged disk temperatures indicate that upper regions of the disks can experience additional heating at rotational speeds above 80% of the critical rate. Such changes cannot be produced by simply lowering the effective temperature of the star with rotation.

All spectral types considered, B0–B5, show a decline in the global, density-weighted averaged disk temperature of $\approx 15$–$20\%$ for near critical rotation. The decline is largest for the earliest spectral type considered, B0. Changes in the global, volume-weighted average disk temperature and the minimum and maximum temperatures in the disk also occur, although results are dependent on spectral type. The volume-weighted average temperature shows the effect of the hot stellar pole (that develops for rapid rotation) at intermediate spectral
types B2 – B3 (see Figures 4.5 and 4.6).

The most important change to the temperature structure of the disks is the expansion of the inner cool zone close to the central star. This zone develops in all disks of sufficient density, and this cool region generally expands in both radius and height as the rotation rate is increased. Also important are the hot sheaths above and below the cool, inner zone. These generally decrease in temperature and move to larger scale heights with increased rotation. It is these complex changes that will most affect physical observables such as the Hα emission line strength or the infrared excess. These issues will be explored in a subsequent paper.

The temperature effects of gravitational darkening on global measures of Be star circumstellar disks can be adequately approximated by spherical gravitational darkening (SGD, which ignores the distortion of the stellar surface) for rotation rates less than 80% of critical. For higher rotation rates, the distortion of the stellar surface must be included to accurately compute the disk’s global average temperature. Comparisons of the vertically-averaged and radially-averaged disk temperature profiles for models with different rotation rates suggests that the general temperature structure of the circumstellar disk can be reasonably predicted using the SGD approximation for rotation rates of also less than 80% of critical.

While the effect of rapid rotation of the central star can produce complex effects in the temperature structure of Be star circumstellar disks, temperature is, of course, not directly accessible to observers. In a subsequent paper, we will quantify the effect of rapid stellar rotation on the classic diagnostics of circumstellar material, namely emission line strengths and profiles and the predicted near-IR excess in the system’s spectral energy distribution.
4.6 Construction of the Computational Grid

The primary goal of this paper is to explore changes in the thermal structure of a circumstellar disk caused by changes in the stellar flux due to rotation. Ideally the only difference between different models should be the rotation speed of the star. Rapid rotation has two effects on the central star; the dependence of the photospheric temperature on stellar latitude, and the Roche distortion of the stellar surface. If the latter effect (distortion) did not occur, creating a series of comparison models would be a simple matter of swapping out the central star and examining the corresponding changes in the thermal structure of the disk. Indeed, this is what occurs in the SGD approximation. Nevertheless, the stellar surface is distorted by rotation, and this geometrical change can be propagated into the physical properties of the disk. The simple fact that the distance between the stellar pole and equator changes with rotation is the central issue. For a given point within the disk it is possible to preserve the distance to one, but not both of these locations. Given this, there are several different ways one could attempt to mitigate the effects, and these approaches are discussed in this Appendix. All of the various choices are illustrated in Figure 4.19 which shows in the left hand panels, a non-rotating B2 star and its associated \((R, z)\) grid and in the right hand panels, a B2 star rotating at \(v_{\text{frac}} = 0.99\) and its associated \((R, z)\) grid. In each panel, the \((R, z)\) grid points in the circumstellar disk are shown as small dots.

In the most straight forward description, the grid locations \((R, z)\) within the disk are specified in terms of the equatorial radius. In the absence of rotation, this is same as the polar radius. However when \(r_{\text{eq}}\) increases due to rotational distortion, the computation grid is stretched in both \(R\) and \(z\) (see Eq. 4.10). This increases the volume and mass of the disk. The computational grid
points are also moved further from both the stellar pole, the stellar equator, and each other, as rotation increases. Hence with grids constructed in this way, any comparison between global temperature averages and total emission for increasing rotation must be considered carefully. Because of the simplicity of this approach, it will be referred to it as the naive grid. It is illustrated in panels A and B of Figure 4.19.

A simple alternative would be to define the grid spacing and density structure to be that of an non-rotating star, keep the grid fixed, and then shift the whole grid and density structure outward as $r_{eq}$ increases. This preserves the distances of all grid points from the stellar equator and between each other. While the cross-sectional area and density structure remains unchanged, both the inner and outer radii of the disk are systematically increased causing the volume and mass of the disk to also increase with rotation. This effect can be noticed when the disk emission measure is calculated. The distance to the pole of the star also increases systematically with rotation, and this could bias any global temperature averages to lower temperatures. This approach will be referred to as the physical grid and is illustrated in panels C and D of Figure 4.19.

An alternative to maintaining the distance between the grid points and the equator is to maintain the distance between grid points and the polar axis. Once again the grid spacing and density structure is defined to be that of an non-rotating star, but instead of shifting it outward as $r_{eq}$ increases we simply remove sections of the grid that would be swallowed by the star and leave the rest of the disk unchanged. Unfortunately, the inner region which gets swallowed is also densest region of the disk. More of this region is removed as rotation increases, and this systematically reduces all optical depths within the disk which in turn increases the stellar flux reaching the outer regions.
This approach will be referred to as the swallowed grid and is illustrated in panels E and F of Figure 4.19.

One potential solution to the swallowed grid is to start the disk at $R = (3/2) r_p$ which allows the central star to swell to the inner disk boundary by critical rotation; the disk remains truly unchanged. This preserves disk volume, mass and density structure, and keeps the distances between each grid point and the stellar pole, the hottest part of the star unchanged. However, the distances to the stellar equator still systematically change. We call this the unchanging grid and it is illustrated in panels G and H of Figure 4.19. This method, unfortunately, makes important regions of the disk empty that were previously full and while comparisons between runs are unbiased any comparison to previous work becomes difficult. The dense inner region of the disk is the source of the IR lines and a significant amount of the mass in this region is no longer included.

In conclusion, there are at least four ways to define a grid surrounding an expanding star: a naive approach where everything increases, a physical grid which shifts the disk, a swallowed grid in which mass is lost from the disk; and an unchanging grid in which the star closes an inner gap. The results included in this paper were computed with the unchanging grid to allow the most straightforward temperature comparisons between the different models.
Figure 4.19: Each panel shows a cross section of the disk with the \((R, z)\) grid points shown as small dots. The spherical star is not rotating, and the distorted star is rotating at \(\omega_{\text{frac}} = 0.99\). Panels A and B depict the naive grid, panels C and D, the physical grid, panels E and F, the swallowed grid, and G and H, the unchanging grid. The colours represent the log of the disk density.
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Chapter 5

The Effect of Density on the Thermal Structure of Gravitationally-Darkened Be Star Disks


5.1 Introduction

Jaschek et al. (1981) gives the current working definition of a Be star as “a nonsupergiant B star whose spectrum has, or had at some time, one or more Balmer lines in emission.” The source of these emission lines is a geometrically thin, gaseous disk surrounding the star. These disks are often tempo-
rary, building up and dissipating over approximately a decade, although some Be stars have shown persistent Balmer emission over their whole observational history (Wisniewski et al., 2010; Carciofi et al., 2012). When present, these disks produce the other characteristic features of a Be star system. (1) A continuum excess occurs due to the cooler disk material (Dougherty et al., 1991). It begins as a weak, but noticeable, excess in the optical, resulting in contributions to the (B-V) colour index of about a tenth of a magnitude (Howells et al., 2001), peaking at \( \approx 10\mu m \) and continuing well past the infra-red (Waters et al., 1987). The longest wavelength detection of a Be star is 6 cm (Dougherty et al., 1991). (2) Continuum polarization is present due to scattering in the non-spherical circumstellar material (Waters & Marlborough, 1992). (3) Emission occurs in other hydrogen line series, Lyman, Paschen, Brackett, Pfund, and Humphreys (Hony et al., 2000; Houck et al., 2004). (4) Some Be stars show emission lines in elements other than hydrogen, such as HeI, FeII, and sometimes SiII and MgII (Slettebak et al., 1992; Porter & Rivinius, 2003).

The formation mechanism(s) of these disks is still unclear, but it is likely facilitated by the rapid rotation of these stars which could allow a relatively weak process to drive mass-loss in a main sequence or just post-main sequence star (Porter & Rivinius, 2003). Short term variability, on timescales between 0.5 to 2 days, is observed in the photospheric line shapes and photometry of most classical Be stars (Percy et al., 1994), and Be stars have also been known to exhibit non-radial pulsations, indicating disturbances on and above the surfaces of these stars (Porter & Rivinius, 2003). Disk building appears to be a stochastic process occurring in a series of outbursts associated with increased emission (Štefl et al., 2011).

The B type stars that produce these disks are rapid rotators (Yudin, 2001). Their rotation rates are still a matter of debate (Cranmer, 2005; van Belle,
but they are fast enough that the effects of rotational distortion and gravitational darkening should be considered. As described in § 2 of our previous paper, McGill et al. (2011), gravitational darkening is the rotationally induced reduction of the effective gravity towards the stellar equator causing a corresponding decrease in the local temperature. The traditional formulation is found in von Zeipel (1924) and Collins (1963), while newer results are discussed in van Belle (2012). Due to the small effect of rotation on the total luminosity, the pole of a rotating star is actually hotter than an equivalent, non-rotating star. Rapid rotation causes the star to become distorted with the radius becoming larger at the equator than the pole, to a maximum of 1.5 times the polar radius.

Interferometric observations of these stars have confirmed a variation in brightness across the surface of rapidly rotating stars (van Belle, 2012). The polar regions of such stars are noticeably brighter than the equator (van Belle, 2012). However the temperature contrast between the pole and equator is not as strong as expected for the classical formulation of gravitational darkening (van Belle, 2012). This is parameterized by $\beta$, as $T_{\text{eff}} \propto g_{\text{eff}}^\beta$. The canonical value of $\beta$ is 0.25 (von Zeipel, 1924).\textsuperscript{1} Smaller values of $\beta$ indicate a smaller temperature difference between the pole and the equator. Che et al. (2011) find their interferometric observations of Regulus can be best explained by $\beta = 0.188$. Observationally determined values for $\beta$ are typically between 0.25 and 0.18, and van Belle (2012) gives $\beta = 0.21$ as the typical value seen from interferometry. Interferometric data seems to suggest that this only affects the temperature profile, while the shape of the star is consistent with a Roche

\textsuperscript{1}von Zeipel (1924) defines $\beta = 0.25$ for radiative energy transport. Convective transport occurs at $T_{\text{eff}} < \sim 7000$ K which is less than 5 % of the total area our B5V model rotating at 0.99 $v_{\text{crit}}$ so neglecting this should have no effect on our results.
Model (van Belle, 2012).

As the star is the energy source for the disk, changes introduced by rotation have an effect on the disk. As described in McGill et al. (2011), the effects of gravitational darkening on models of classical Be stars is to reduce the temperature in the mid-plane of the disk while causing some temperature increases in the upper disk. McGill et al. (2011) presented models for four different spectral types: B0V, B2V, B3V, and B5V at ten different rotation rates. However only a single disk density scale of $\rho_o = 5.0 \times 10^{-11}$ g cm$^{-3}$ (see Equation (5.1)) was considered. In this paper, we examine the combined effects of variations in disk density and stellar rotation. We also include discussions on the effects of rotation on hydrostatically converged models and on the inclusion of different formulations of gravitational darkening. The organization of the paper is as follows: § 5.2 briefly outlines the models presented in this paper; § 5.3 provides our results; § 5.3.1 explores the effects of rotation when combined with changes in the density of the disk; § 5.3.2 looks at the effects of rotation on disk models that have been hydrostatically converged; § 5.3.3 examines the effects of using different formulations of gravity darkening on the disk temperatures; and conclusions are presented in § 5.4.

### 5.2 Calculations

The modelling program, Bedisk, was used (Sigut & Jones, 2007). Bedisk solves the statistical equilibrium equations for the atomic level populations and then enforces radiative equilibrium at each point of the computational grid representing the disk. The version of Bedisk described in McGill et al. (2011) includes gravitational darkening and was run for the stellar parameters given in Table 5.1 and the rotation rates given in Table 5.2. In our calculations, we have
Table 5.1: Adopted Stellar parameters

<table>
<thead>
<tr>
<th>Type</th>
<th>Mass ($M_\odot$)</th>
<th>Polar Radius ($R_\odot$)</th>
<th>Luminosity ($L_\odot$)</th>
<th>$T_{\text{eff}}$ (K)</th>
<th>$\omega_{\text{crit}}$ (rad s$^{-1}$)</th>
<th>$v_{\text{crit}}$ (km s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B0V</td>
<td>17.5</td>
<td>7.40</td>
<td>3.98 x 10$^4$</td>
<td>30000</td>
<td>7.10 x 10$^{-5}$</td>
<td>548</td>
</tr>
<tr>
<td>B2V</td>
<td>9.11</td>
<td>5.33</td>
<td>4.76 x 10$^3$</td>
<td>20800</td>
<td>8.38 x 10$^{-5}$</td>
<td>466</td>
</tr>
<tr>
<td>B3V</td>
<td>7.60</td>
<td>4.80</td>
<td>2.58 x 10$^3$</td>
<td>18800</td>
<td>8.95 x 10$^{-5}$</td>
<td>449</td>
</tr>
<tr>
<td>B5V</td>
<td>5.90</td>
<td>3.90</td>
<td>7.28 x 10$^2$</td>
<td>15200</td>
<td>1.08 x 10$^{-4}$</td>
<td>439</td>
</tr>
</tbody>
</table>

Notes: $\omega_{\text{crit}} = \sqrt{\frac{8GM}{27r_p^3}}$, where $r_p$ is the polar radius of the star and $M$ is its mass.  
$v_{\text{crit}} = r_{\text{max}} \omega_{\text{crit}}$, where $r_{\text{max}} = 1.5 r_p$. 
Stellar parameters adopted from Cox (2000).

assumed that the polar radius remains constant following Collins (1966). The stellar temperatures are defined using the assumption that the total luminosity of the star remains fixed which is a reasonable approximation (for discussion of this point, see Lovekin et al. (2006)). These stellar parameters were chosen to include a model from each of the five bins adopted by Cranmer (2005) to analyse the effects of spectral type on Be star rotational statistics. We have calculated sets of models for the spectral types B0V, B2V, B3V, and B5V. Unfortunately, B8V models are too cool for BeDISK when run for gravitational darkening near critical rotation without explicitly improving the treatment of the diffuse radiation field (i.e. disk self-heating) beyond the modified on-the-spot approximation used by Sigut & Jones (2007).

Models presented in § 5.3.1 and § 5.3.3 assumed a fixed density structure for the disk in the form,

$$\rho(R, z) = \rho_o \left(\frac{R_o}{R}\right)^n e^{-\left(z/H(R)\right)^2},$$  \hspace{1cm} (5.1)
Table 5.2: Rotation rates.

<table>
<thead>
<tr>
<th>$v_{\text{frac}}$</th>
<th>$\omega_{\text{frac}}$</th>
<th>$r_{\text{eq}}/r_p$</th>
<th>$v_{\text{eq}}$ (km s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>B0V</td>
</tr>
<tr>
<td>0.0000</td>
<td>0.0000</td>
<td>1.00</td>
<td>0</td>
</tr>
<tr>
<td>0.2000</td>
<td>0.2960</td>
<td>1.01</td>
<td>110</td>
</tr>
<tr>
<td>0.4000</td>
<td>0.5680</td>
<td>1.06</td>
<td>219</td>
</tr>
<tr>
<td>0.6000</td>
<td>0.7920</td>
<td>1.14</td>
<td>329</td>
</tr>
<tr>
<td>0.8000</td>
<td>0.9440</td>
<td>1.27</td>
<td>439</td>
</tr>
<tr>
<td>0.9000</td>
<td>0.9855</td>
<td>1.37</td>
<td>493</td>
</tr>
<tr>
<td>0.9500</td>
<td>0.9963</td>
<td>1.43</td>
<td>521</td>
</tr>
<tr>
<td>0.9900</td>
<td>0.9999</td>
<td>1.49</td>
<td>544</td>
</tr>
</tbody>
</table>

where

$$H(R) = \sqrt{\frac{2R^3kT_{\text{iso}}}{GM\mu}}. \quad (5.2)$$

In Equation (5.2) for the disk scale height, $\mu$ is the mean-molecular weight of the gas (treated as a constant and set to half the mass of hydrogen) and $T_{\text{iso}}$ is an assumed isothermal temperature used for the sole purpose of setting the density scale height; $T_{\text{iso}}$ is set equal to 60% of the effective temperature of the central star, listed in Table 5.1 (Sigut et al., 2009). The form of the disk density given by Equation (5.1) follows from the assumption of a radial power-law drop in the equatorial plane ($z = 0$) coupled with the assumption that the disk is in vertical hydrostatic equilibrium set by the $z$-component of the star’s gravitational acceleration and the assumed temperature $T_{\text{iso}}$. This leads to a “flaring disk” in which $H \propto R^{3/2}$.

Models were made with the density fall off parameter, $n$, set to 3.0. The disks were constructed with the unchanging grid described in McGill et al. (2011) which keeps the disk empty for $r \leq 1.5r_p$. This allows a disk of
fixed density structure to be used for all calculations as the star expands with rotation up to a maximum of 1.5 $r_p$ at critical rotation. The filled region begins at $r = 1.6 r_p$ and the density in the mid-plane of the disk at the beginning of this filled region is given by $\rho_{bd} = \rho_o (1/1.6)^3 = 0.24 \rho_o$ using Equation (5.1). The parameters $\rho_o$ and $\rho_{bd}$ will be referred to as the density scale and the base density respectively. The models were constructed for four different values of the density scale, $\rho_o$: 5.0 $\times$ 10$^{-11}$; 2.5 $\times$ 10$^{-11}$; 1.0 $\times$ 10$^{-11}$; and 5.0 $\times$ 10$^{-12}$ g cm$^{-3}$.

Models presented in § 5.3.2 have been hydrostatically converged which ensures that the vertical density structure is consistent with the temperature structure by enforcing hydrostatic equilibrium over an additional loop as described in Sigut et al. (2009). Only eight hydrostatically converged models were made due to the long computation time required, $\approx$ 10 times longer than an unconverged model. Using the B0V stellar parameters, two hydrostatically converged models were calculated for each density scale, $(\rho_o$: 5.0 $\times$ 10$^{-11}$; 2.5 $\times$ 10$^{-11}$; 1.0 $\times$ 10$^{-11}$; and 5.0 $\times$ 10$^{-12}$ g cm$^{-3}$). The first model without rotation and the second, with a rotation rate of $v = 0.99 v_{\text{crit}}$.

Models presented in § 5.3.3 use a version of BEDISK that allows for variations in the formulation of gravitational darkening. Results for the B2V model were recalculated for the four fastest velocities in Table 5.2 at the four different density scales, $\rho_o$: 5.0 $\times$ 10$^{-11}$; 2.5 $\times$ 10$^{-11}$; 1.0 $\times$ 10$^{-11}$; and 5.0 $\times$ 10$^{-12}$ g cm$^{-3}$). Two variants of gravitational darkening were used: (1) reducing the value of the exponent on the magnitude of gravity in von Zeipel’s theorem, $\beta$, from the standard value of 0.25 to 0.18; (2) changing the expression for the temperature structure based on von Zeipel (1924) to that given by Espinosa Lara & Rieutord (2011).
Figure 5.1: Change in four disk temperature diagnostics with increasing stellar rotation and varying densities for the B0V model. Shown are the volume-weighted and the density-weighted average temperatures as defined in Equations (5.3) and (5.4) (upper left and upper right panels) and the maximum and minimum temperatures (lower left and lower right panels). Four different disk densities are shown with $\rho_0 = 5.0 \times 10^{-12}; 1.0 \times 10^{-11}; 2.5 \times 10^{-11};$ and $5.0 \times 10^{-11}$ g cm$^{-3}$. Darker lines indicate higher $\rho_0$. The lower horizontal axis indicates the fractional rotational velocity at the equator and the upper horizontal axis indicates the corresponding fractional angular velocity.

5.3 Results

5.3.1 Effects of Density Changes

Figures 5.1, 5.2, 5.3, and 5.4 show the changes produced by increasing rotation on the temperature of disks for a variety of densities using the same four temperature diagnostics as McGill et al. (2011): the density-weighted average temperature, the volume-weighted average temperature, the maximum temperature, and the minimum temperature. The density-weighted average
temperature is defined as

$$T_{\rho} = \frac{1}{M_{\text{disk}}} \int T(R, z) \rho(R, z) \, dV, \quad (5.3)$$

and the volume-weighted average temperature is defined as

$$T_V = \frac{1}{V_{\text{disk}}} \int T(R, z) \, dV. \quad (5.4)$$

In order to avoid numerical effects, the median of the twenty hottest and twenty coolest disk locations are taken to represent the maximum and minimum disk temperatures.

The most important trend seen in Figures 5.1 through 5.4 is in the density-weighted average temperatures. There is a decrease in this temperature with both increasing rotation rate and increasing density. This trend is seen in all spectral types. In addition, for all spectral types, the curves for different $\rho_o$ are well separated and decrease smoothly. For spectral types B2V, B3V,
and B5V, the slope increases with increasing rotation but flattens out near critical rotation. For spectral types B2V and B3V, the four curves for each $\rho_o$ are essentially parallel and for B5V are nearly parallel. Parallel curves are not seen in the B0V model, and the decrease in temperature with rotation is greatest for the largest density model.

The temperature minimums also decrease with both increasing rotation rate and increasing density, analogous to the density-weighted average temperatures. The size of the decrease in the minimum temperature is also larger for the least dense models. For spectral types B2V, B3V, and B5V, the curve for the the most dense model, $\rho_o = 5.0 \times 10^{-11}$ g cm$^{-3}$, is essentially flat. This causes the temperature minimums to converge at $\nu_{\text{crit}}$ to values between 5000 and 6000 K for all spectral types. The curves for different $\rho_o$ are distinct, but there is significant overlap. The temperature minimums in the non-rotating models are larger for the earlier spectral types, and the earlier models experience a larger decrease in temperature with both rotation and density. The drop in the value of the minimum temperatures with rotation and density is
barely noticeable in the B5V models as all curves are nearly flat and cluster at $\approx 5000$ K.

One of the most interesting effects produced by rotation on Be star disk temperatures is seen in the volume-weighted average temperatures of the B2V, B3V, and B5V models. These values initially decrease with moderate rotation but begin to increase approaching critical rotation. The strength of both the initial decrease and the increase near critical rotation is larger for early spectral types, excluding B0V. For the two highest densities of the critically rotating B2V models, the volume-weighted average temperatures are essentially the same as the non-rotating models. Generally, the volume-weighted average temperatures are hotter for the least dense models, but the separations between the averages are not as large as for the density-weighted average temperatures or the temperature minimums, and there is a great deal of overlap. The effect of density is strongest in the B2V model and very small by B5V. For moderate rotation, lower density models generally have smaller volume-weighted temperature averages than denser models, but the lowest density B2V models

Figure 5.4: Same as Figure 5.1 but for the B5V model.
(ρ = 5.0 \times 10^{-12})$, and the non-rotating and slowest rotating models of the second lowest density B2V models (ρ = 1.0 \times 10^{-11}) do not follow this trend. For rapid rotation, lower density models generally have larger volume-weighted temperature averages than denser models, but this does not occur for the B5V models.

The volume-weighted temperature averages of the B0V model does not behave like the other spectral types. There is only a very small change in temperature with rotation, which is somewhat larger for larger densities. The least dense B0V model is of nearly constant temperature. The densest, non-rotating models are hotter than the least dense models, but by critical rotation, this trend has reversed. All of the values are tightly clustered between 13900 to 14500 K.

The maximum disk temperatures change very little with rotation or density. This is because these temperatures are found in parts of the disk that are not shielded by the dense, mid-plane and are directly illuminated by the pole of the star. The maximum temperatures are determined essentially by spectral type. There is a small but noticeable decrease in temperature with rotation in all models, which flattens out near critical rotation. Like the volume-weighted average temperatures, the temperature maximums begin to increase for extreme rotation, significantly for the B3V models but very weakly for the other spectral types.

One of the most complex behaviours seen in Figures 5.1 through 5.4 is the how the relationship between the volume-weighted and the density-weighted average temperatures is affected by rotation and disk density. These effects can be separated into three categories based on density, the high and low density extremes and the case of intermediate density. (1) For the highest density B0V model, the two highest density B2V and B3V models, and the highest density
B5V model, the volume-weighted average temperatures are always higher than the density-weighted average temperatures. This is due to the presence of a large, cool region in the equatorial plane at high density. (2) For the lowest density B0V models and the two lowest density B5V models, the volume-weighted average temperatures are always lower than the density-weighted average temperatures. This is due to the absence of a significant cool region in the equatorial plane at low density. (3) All models of intermediate density experience a transition: for low rotation rates there is either not a cool region in the equatorial plane or it is too small to cause the density-weighted average temperatures to be lower than the volume-weighted average temperatures. At higher rotation rates, gravitational darkening causes the development of a larger cool region and the density-weighted average temperatures become less than the volume-weighted average temperatures.

Figures 5.5, 5.6, 5.7, and 5.12 show radial temperature profiles of these disks obtained by averaging the temperatures within each vertical column perpendicular to the mid-plane of the disk. The average temperature at each radial distance is density-weighted and is found using

$$T(R) = \frac{\int_0^{z_{\text{max}}} T(R, z) \rho(R, z) dz}{\int_0^{z_{\text{max}}} \rho(R, z) dz}.$$  \hspace{1cm} (5.5)$$

As seen in the previous figures, lower density models have higher temperatures. The profiles follow two patterns: (1) for low density models, the temperature at the inner edge of the disk begins fairly hot, increases to a maximum and then the temperature decreases as the radius increases, sometimes reaching a constant value; (2) for sufficiently large densities, the temperature at the front of the disk still begins fairly hot, but the temperature drops to a minimum and then increases again at larger radii, sometimes reaching a maximum value.
and then dropping again. For all spectral types, the temperature minimum becomes cooler and the extent of the cool zone increases in size with rotation. The slope of the temperature increase at large radii becomes smaller with increasing rotation. If a temperature maximum occurs, its size is reduced with rotation and the slope of the temperature drop off at large radii becomes shallower.

Figures 5.8, 5.9, 5.10, and 5.11 show vertical temperature profiles of these disks obtained by averaging the temperatures at different radii but at the same scale height, $u = z/H(R)$. The average temperature is density-weighted and
is found using

$$T(u) = \frac{\int_0^{R_{\text{max}}} T(R, u) \rho(R, u) A(R, u) dR}{\int_0^{R_{\text{max}}} \rho(R, u) A(R, u) dR}, \quad (5.6)$$

where $A(R, u)$ is the area function of the disk. $A(R, u)$ is included in Equation 5.6 to account for the non-uniform spacing in $r$ and $z$ of the computational grid (i.e. $\Delta r$ and $\Delta z$ both increase with $R$). The exact nature of this weighting matters less than the fact that all the models have been averaged in the same way. Comparing these plots illustrates the effects of rotation and density on the vertical structure of these disks.

In these plots, the mid-plane of the disk is at $u=0$. All vertical temperature profiles show a temperature maximum occurring between one and two scale heights above the mid-plane. The location of this maximum moves higher above the mid-plane with both increasing density and increasing rotation. Rotation causes the width of the temperature peak to increase and become broader, while increasing density causes width of the peak to become narrower.
and sharper. Low density models without rotation have mid-plane regions either at the same temperature or hotter than the upper disk. Increasing the density causes the development of a cool region in the mid-plane. Rotation causes this cool region to form in the mid-plane at lower densities. At high rotation rates, pronounced cool regions are seen even in the least dense models.

There is little variation in the temperature of the upper edge of the disk with either rotation or density in the B5V models and no significant change with density for the non-rotating B0V models. The upper edges of the disk increase in temperature with rapid rotation in the B2V and B3V models and have a minimum value at $v_{\text{frac}} = 0.80$. In the B2V, B3V and the rotating B0V models, the upper edges of the disk are hotter for lower densities. While there is a change in the temperature of the upper edges of the disk with density in the rotating B0V models, which increases in size with rotation, the median of this range is not significantly affected by rotation and remains at $\approx 13000$ K.

Figure 5.13 shows the two dimensional temperature structure for a selection of eight B0V models. Two models are shown for each density, one without
Figure 5.8: Variation of the radially-averaged temperature profile for various stellar rotation rates and disk densities for the B0V model. Here the temperature is plotted versus the scale height, \( u \), defined by \( u \equiv z/H(R) \). Each panel shows a different rotation rate: no rotation (upper left); \( v = 0.80 v_{\text{crit}} \) (upper right); \( v = 0.90 v_{\text{crit}} \) (lower left); and \( v = 0.99 v_{\text{crit}} \) (lower right). Four different disk densities are shown for each rotation rate with \( \rho_0 = 5.0 \times 10^{-11} \); \( 1.0 \times 10^{-11} \); \( 2.5 \times 10^{-11} \); and \( 5.0 \times 10^{-11} \) g cm\(^{-3} \). Darker lines indicate higher \( \rho_0 \).

rotation and one for \( v_{\text{frac}} = 0.95 v_{\text{crit}} \). Clearly the temperature in the equatorial plane becomes cooler as the density increases; this has been noted many times. What is interesting is that this cool region appears at smaller densities when rotation is included.

Finally a useful way to illustrate the effect of density and rotation on the range of disk temperature is to construct histograms of \( T(R,z) \) for each model. Figures 5.14 and 5.15 each show a set of histograms for increasing rotation. Only two sets of histograms are shown for brevity, the B2V model with \( \rho_0 = 5.0 \times 10^{-12} \) g cm\(^{-3} \) (the lowest density considered) and another with \( \rho_0 = 5.0 \times 10^{-11} \) g cm\(^{-3} \) (the highest density considered). In Figure 5.14, the model without rotation is fairly warm and as rotation increases, a low
temperature tail forms in the distribution. In addition, the fraction of disk temperatures in the highest temperature bins also increases at high rotation rates. In Figure 5.15 the non-rotating model has a two strong peaks, one quite cool and a second middle peak. There is also a large and long high temperature tail and a weak high temperature peak because this model is dense enough to possess a cool region in the equatorial plane, with warmer regions above. As rotation increases, the middle temperature peak weakens and eventually disappears, while the low temperature peak becomes stronger and cooler. The weak, higher temperature peak becomes stronger. The distribution of temperatures in the fastest rotating model has two strong peaks at the temperature extremes. Overall, these histograms demonstrate that gravitational darkening increases the amount of very cool and very hot material in the disk and decreases the amount of disk material of intermediate temperatures.
Figure 5.10: Same as Figure 5.8 for the B3V model.

Figure 5.11: Same as Figure 5.8 for the B5V model.
Figure 5.12: Same as Figure 5.5 for the B5V model.
Figure 5.13: Temperature profiles of the disk for a selection of B0V models. The left column shows models with no rotation. The right column shows models with a rotational speed of $v_{\text{frac}} = 0.95$. The top row shows models with $\rho_0 = 5.0 \times 10^{-12}$ g cm$^{-3}$; the second row, $\rho_0 = 1.0 \times 10^{-11}$ g cm$^{-3}$; the third row, $\rho_0 = 2.5 \times 10^{-11}$ g cm$^{-3}$; and the bottom row, $\rho_0 = 5.0 \times 10^{-11}$ g cm$^{-3}$. The colour-map indicates the disk temperatures. The black line outlines the star.
Figure 5.14: Series of histograms of disk temperatures. All histograms plotted are for the B2V model with $\rho_o = 5.0 \times 10^{-12}$ g cm$^{-3}$. Each panel shows a model with a different velocity as indicated in the left of each panel. The upper most panel is the non-rotating model and the rotation rates increase downward. The lower most panel is the fastest rotating model with $v_{\text{frac}} = 0.99$ $v_{\text{crit}}$. The filled circles indicate the density-weighted average temperatures and the filled squares indicate the volume-weighted average temperatures for each model.

Figure 5.15: Same as Figure 5.14 but with $\rho_o = 5.0 \times 10^{-11}$ g cm$^{-3}$. 
It is often useful in the study of circumstellar material to compare the temperatures found in the disk to the effective temperature of the star. However, when the star is a rapid rotator, there is a range of surface temperatures, and some representative value must be found. The most basic definition of effective temperature is from the Stephan-Boltzmann law, namely $T_{\text{eff}} \equiv (L/A)^{1/4}$ where $A$ is the surface area of the star. Extending this to a rotating star gives, $T_{\text{eff}}(v_{\text{frac}}) = (L/\sigma A(v_{\text{frac}}))^{1/4}$. If the luminosity is considered unaffected by rotation we have,

$$T_{\text{eff}}(v_{\text{frac}}) = T_{\text{eff}}(v_{\text{frac}} = 0) \left( \frac{A(v_{\text{frac}} = 0)}{A(v_{\text{frac}})} \right)^{1/4}. \quad (5.7)$$
Figure 5.16 shows the density-weighted average temperatures divided by the stellar effective temperatures defined by Equation (5.7) versus rotation rate for all models. This plot summarizes the predicted global temperatures of the disks around rapidly rotating stars of a wide range of disk densities. The temperature ratio of the disk to the star is between 0.40 - 0.65 for all models. Denser disks are proportionally cooler. The disks around cooler stars all have temperatures which are larger fractions the stellar effective temperatures than those around hot stars. The ratio between the density-weighted average temperatures in the disk and the effective stellar temperatures drops with moderate rotation. Approach to critical rotation causes an increase in the ratios with rotation in both cooler stars and less dense disks. The slope of the ratio flattens out for rapid rotation in hot and dense disks. In all cases, the trends are dominated by the variations in disk density, not rotation.

Table 5.3 gives the density-weighted temperatures for all density scales and spectral types without rotation and at maximum rotation \( (v = 0.99 v_{\text{crit}}) \). This table shows the full range of temperature changes that occur with rotation for different densities. Increases in either density or rotation cause decreases in the density-weighted temperatures. For the B0V models, these two effects strengthen each other; denser disks are more strongly affected by increasing rotation and disks surrounding critically rotating stars are more strongly affected by density changes than disks surrounding non-rotating stars. The density-weighted average temperatures change by \(-6\%\) as rotation varies from zero to \(0.99 v_{\text{crit}}\) for the least dense models, and change by \(-21\%\) for the densest model over the same range in velocity. When we look at the results with rotation rates fixed, we find that the density-weighted average temperatures change by \(-14\%\) from \(\rho_o = 5.0 \times 10^{-12} \text{ g cm}^{-3}\) to \(\rho_o = 5.0 \times 10^{-11} \text{ g cm}^{-3}\) without rotation, and change by \(-28\%\) for the fastest rotating models. In the
Table 5.3: Density-weighted average temperatures without rotation and with $v_{\text{frac}} = 0.99$

<table>
<thead>
<tr>
<th>Type</th>
<th>$\rho_0$ (g cm$^{-3}$)</th>
<th>$T_{\rho}(0.00)$ (K)</th>
<th>$T_{\rho}(0.99)$ (K)</th>
<th>$% \Delta(v_{\text{frac}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>B0V</td>
<td>$5.0 \times 10^{-12}$</td>
<td>15200.</td>
<td>14250.</td>
<td>-6.</td>
</tr>
<tr>
<td></td>
<td>$1.0 \times 10^{-11}$</td>
<td>15180.</td>
<td>13920.</td>
<td>-9.</td>
</tr>
<tr>
<td></td>
<td>$2.5 \times 10^{-11}$</td>
<td>14520.</td>
<td>13010.</td>
<td>-11.</td>
</tr>
<tr>
<td></td>
<td>$5.0 \times 10^{-11}$</td>
<td>13280.</td>
<td>10780.</td>
<td>-21.</td>
</tr>
<tr>
<td></td>
<td>$% \Delta(\rho)$</td>
<td>-14.</td>
<td>-28.</td>
<td></td>
</tr>
<tr>
<td>B2V</td>
<td>$5.0 \times 10^{-12}$</td>
<td>11620.</td>
<td>10220.</td>
<td>-13.</td>
</tr>
<tr>
<td></td>
<td>$1.0 \times 10^{-11}$</td>
<td>11090.</td>
<td>9690.</td>
<td>-13.</td>
</tr>
<tr>
<td></td>
<td>$2.5 \times 10^{-11}$</td>
<td>10200.</td>
<td>8800.</td>
<td>-15.</td>
</tr>
<tr>
<td></td>
<td>$5.0 \times 10^{-11}$</td>
<td>9190.</td>
<td>7870.</td>
<td>-15.</td>
</tr>
<tr>
<td></td>
<td>$% \Delta(\rho)$</td>
<td>-23.</td>
<td>-26.</td>
<td></td>
</tr>
<tr>
<td>B3V</td>
<td>$5.0 \times 10^{-12}$</td>
<td>11000.</td>
<td>9520.</td>
<td>-15.</td>
</tr>
<tr>
<td></td>
<td>$1.0 \times 10^{-11}$</td>
<td>10590.</td>
<td>9110.</td>
<td>-15.</td>
</tr>
<tr>
<td></td>
<td>$2.5 \times 10^{-11}$</td>
<td>9670.</td>
<td>8230.</td>
<td>-16.</td>
</tr>
<tr>
<td></td>
<td>$5.0 \times 10^{-11}$</td>
<td>8690.</td>
<td>7460.</td>
<td>-15.</td>
</tr>
<tr>
<td></td>
<td>$% \Delta(\rho)$</td>
<td>-24.</td>
<td>-24.</td>
<td></td>
</tr>
<tr>
<td>B5V</td>
<td>$5.0 \times 10^{-12}$</td>
<td>9710.</td>
<td>7970.</td>
<td>-20.</td>
</tr>
<tr>
<td></td>
<td>$1.0 \times 10^{-11}$</td>
<td>9290.</td>
<td>7760.</td>
<td>-18.</td>
</tr>
<tr>
<td></td>
<td>$2.5 \times 10^{-11}$</td>
<td>8450.</td>
<td>7230.</td>
<td>-16.</td>
</tr>
<tr>
<td></td>
<td>$5.0 \times 10^{-11}$</td>
<td>7660.</td>
<td>6760.</td>
<td>-12.</td>
</tr>
<tr>
<td></td>
<td>$% \Delta(\rho)$</td>
<td>-24.</td>
<td>-16.</td>
<td></td>
</tr>
</tbody>
</table>

B2V and B3V models, the effect of rotation is nearly the same for all $\rho_0$, and the percent differences range between $-13$ and $-15\%$ for the B2V models and between $-15$ and $-16\%$ for the B3V models. The effect of density changes is also nearly same on models without rotation and those with a rotation rate of $0.99 v_{\text{crit}}$, with the percent differences ranging between $-23$ and $-26\%$ for the B2V models and remaining unchanged at $-24\%$ for the B3V models. By B5V the these two effects weaken each other. The least dense models are more strongly affected by rotation and the non-rotating models are most affected by
density changes. The density-weighted average temperatures change by \(-20\%\) from zero to 0.99 $v_{\text{crit}}$ for the least dense models, and change by \(-12\%\) for the densest model. Holding rotation rates fixed, we find that the density-weighted average temperatures change by \(-24\%\) from $\rho_o = 5.0 \times 10^{-12}$ g cm$^{-3}$ to $\rho_o = 5.0 \times 10^{-11}$ g cm$^{-3}$ without rotation, and change by \(-16\%\) for the fastest rotating models.

Taking all the results into consideration, we see that while the bulk of the disk becomes cooler, there is evidence of heating in some parts of the disk (either from the volume-weighted average temperatures or the maximum temperatures or both). Heating of the upper edge of the disk is seen in the vertical profiles of the B2V and the B3V models, as shown is Figures 5.9 and 5.10. This was also seen in Figures 8, 9 and 13-15 in McGill et al. (2011). When density changes are combined with rotation, we see that the cooling associated with increasing density occurs at lower $\rho_o$ at higher rotation rates. This suggests that if rotation is not taken into account, the densities of these disks could be over estimated.

5.3.2 Self-Consistent Vertical Hydrostatic Equilibrium

In this section we describe the effects of rotation on disk models that have been hydrostatically converged, as described in Sigut et al. (2009). The vertical structure of classical Be star disks are believed to be in hydrostatic equilibrium. Equation (5.1) governs the density structure of a hydrostatically supported isothermal disk, but real Be star disks are not isothermal. Temperatures throughout the disks typically vary by factors of 2 to 3, as seen in Figures 5.1 through 5.4 and Figures 5.14 and 5.15. This means that Equations (5.1) and (5.2) are inconsistent with the detailed $T(R,Z)$ distributions
Figure 5.17: Ratio of disk scale height to disk radius versus disk radius for the fixed density structure shown in Equation (5.1), and two hydrostatically converged models, one without rotation (\(v_{\text{frac}} = 0\)) and one with rotation including gravitational darkening (\(v_{\text{frac}} = 0.99\)) for the B0V model. Four different disk densities are shown: \(\rho_0 = 5.0 \times 10^{-12}\) (upper left); \(1.0 \times 10^{-11}\) (upper right); \(2.5 \times 10^{-11}\) (lower left); and \(5.0 \times 10^{-11}\) g cm\(^{-3}\) (lower right).

for the models computed from radiative equilibrium. Vertical pressure support in the disk requires that the vertical density profile follow \(-dP/dz = \rho(z, r)g_z\), in which \(g_z\) is the vertical component of the star’s gravitational acceleration. The local pressure is strongly affected by the local temperature of the disk via the equation of state, \(P = \rho kT/\mu\). This means that in order to create a density profile consistent with the temperature solution, an additional loop is required within the BEDISK code to enforce hydrostatic equilibrium in each column, as described by Sigut et al. (2009). Because of the extra loop, a converged model takes approximately ten times longer to calculate than an unconverged model.

Figure 5.17 shows the ratio of the disk scale height to the disk radius, \(H(R)/R\), versus disk radius, \(R\), in the inner disk \((r \leq 5r_p)\) for B0V models of both the fixed density structure described by Equation (5.1) and those that have been hydrostatically converged. \(H(R)\) is the height where the den-
sity drops by a factor of $1/e$. For an isothermal disk, $H(R)/R$ is equal to the ratio of the sound speed, $c_{\text{sound}} = \sqrt{P/\rho}$, to the Keplerian orbital speed, $v_{\text{orbit}} = \sqrt{GM/R}$. Because of the nature of the fixed isothermal density structure, it always produces the same $H(R)/R$ profile that increases as $R^{1/2}$ regardless of $\rho_o$ or $v$; as it is a function of only $T_{\text{iso}}$. In the current set of models, $H(R)/R$ is always larger for the fixed structure compared to the hydrostatically converged structure because we have adopted the usual result that $T_{\text{iso}} = 0.6T_{\text{eff}}$. However this temperature is often higher than the temperatures actually found in the mid-plane of dense disks. Therefore, Equation (5.1) over-estimates the pressure support of the mid-plane, (see Sigut et al. (2009) for more details). The difference between $T_{\text{iso}}$ and the actual disk temperature becomes smaller as the disk radius increases and this is why the difference in $H(R)/R$ between the hydrostatically converged models and those with the fixed density structure decreases with increasing radius. The difference in $H(R)/R$ between the fixed structure and the hydrostatically converged models also increases as $\rho_o$ increases because the temperatures in the mid-plane decrease strongly as $\rho_o$ increases. Note that there is no single choice for $T_{\text{iso}}$ that would be able to match the correct vertical density structures of the hydrostatically converged models shown in Figure 5.17.

We now demonstrate that rotation, in addition to density, affects the scale heights computed for these disks. Figure 5.17 demonstrates that $H(R)/R$ for $v = 0.99 \, v_{\text{frac}}$ is always lower than that for $v = 0$ for the same radius and density. This is because rapidly rotating models have cooler temperatures in the mid-plane, resulting in smaller scale heights: rapid rotation causes models with thinner disks. The model with $\rho_o = 5.0 \times 10^{-11} \, \text{g cm}^{-3}$ and $v = 0.99 \, v_{\text{crit}}$ has a very thin disk with a maximum value in the inner disk of $H(R)/R = 0.033$ at 2.6 stellar radii and a minimum of 0.027 at 3.75 stellar
Table 5.4: Density-weighted average temperatures (K) with and without self-consistent hydrostatic equilibrium

<table>
<thead>
<tr>
<th>ρ₀ (g cm⁻³)</th>
<th>vfrac = 0.</th>
<th>Fixed</th>
<th>% Δ(vfrac)</th>
<th>vfrac = 0.</th>
<th>Hydro</th>
<th>% Δ(vfrac)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0 × 10⁻¹²</td>
<td>15200</td>
<td>14250</td>
<td>-6</td>
<td>15170</td>
<td>14300</td>
<td>-6</td>
</tr>
<tr>
<td>1.0 × 10⁻¹¹</td>
<td>15190</td>
<td>13920</td>
<td>-9</td>
<td>15120</td>
<td>14070</td>
<td>-7</td>
</tr>
<tr>
<td>2.5 × 10⁻¹¹</td>
<td>14520</td>
<td>13010</td>
<td>-11</td>
<td>14590</td>
<td>13500</td>
<td>-8</td>
</tr>
<tr>
<td>5.0 × 10⁻¹¹</td>
<td>13280</td>
<td>10780</td>
<td>-21</td>
<td>14020</td>
<td>12570</td>
<td>-11</td>
</tr>
<tr>
<td>% Δ(ρ)</td>
<td>-14</td>
<td>-28</td>
<td>-8</td>
<td>-8</td>
<td>-13</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.4 shows the density-weighted average temperatures without rotation and with rotation at 0.99 vfrac for the fixed density structure and the hydrostatically converged structure for the four different densities shown in Figure 5.17. Increasing density or rotation causes a decrease in the density-weighted average temperature with or without hydrostatic equilibrium. Requiring self-consistent hydrostatic equilibrium reduces the effects of changing rotation and density and also increases the temperatures found in the dense B0V models.
Figure 5.18: Surface temperature versus rotation rate for four different colatitudes, $0^\circ$, $30^\circ$, $60^\circ$, and $90^\circ$, for the B2V model. The solid lines indicate $\beta = 0.25$ while dots indicate $\beta = 0.18$. The lower horizontal axis indicates the fractional rotational velocity at the equator and the upper horizontal axis indicates the corresponding fractional angular velocity.

5.3.3 Different forms of gravitational darkening

As described in van Belle (2012), gravitational darkening has been interferometrically confirmed, but the difference in brightness over the stellar disk is not as strong as that predicted by the standard formulation of gravitational darkening presented by Collins (1963). This formulation states that

$$F = C_\omega g^{4\beta},$$

(5.8)

where $F$ is the local radiative flux and $C_\omega$ is constant across a star and determined by the luminosity such that

$$C_\omega = L_\omega / \int g^{4\beta} dA, \quad (5.9)$$
where the integration is over the surface of the star. The luminosity, $L_\omega$, can be treated as a constant or as a function of rotation. The canonical value of $\beta$, defined in von Zeipel (1924), is 1/4, but observations suggest a range of values between 0.25 and 0.19 for B type stars (van Belle, 2012). Claret (2012) presents a calculation for the variation of $\beta$ with local temperature and optical depth in a rotating star of $4 \, M_\odot$ and describes departures from von Zeipel in the upper layers where $\beta$ can be as low as $\sim 0.18$ when energy transport is radiative. Figure 5.18 shows the variation in temperature with rotation for $\beta = 0.18 \, \text{and} \, 0.25$ at four different stellar co-latitudes. The constant $C_\omega$ was recomputed in each case. Using $\beta = 0.18$ weakens both the temperature increase in the polar region and the temperature decrease in the equatorial regions, reducing temperature difference between the equator and the pole.

We choose $\beta = 0.18$ as a lower limit and re-ran Bdisk for the four highest rotation rates assuming the B2V star surrounded by a disk with $\rho_o = 5.0 \times 10^{-11} \, \text{g cm}^{-3}$. The results are shown in Figure 5.19. Lowering $\beta$ reduces the strength of gravitational darkening and the corresponding effect of the stellar flux on the temperatures in the disk. The density-weighted average temperature decreases with rotation and this decrease is less with $\beta = 0.18$. The volume-weighted average temperatures in the disk have a more complex behaviour, initially decreasing then increasing with rapid rotation, as $\beta$ is reduced to 0.18. The initial drop in temperature becomes smaller and the increase in temperature at higher rotation rates is also smaller. The value of $\beta$ does not effect the maximum or minimum temperatures found in the disk.

The discrepancy in $\beta$ between theory and observation has motivated a closer look at gravitational darkening and possible refinements to the theory. Espinosa Lara & Rieutord (2011) presents such a refinement. Both Espinosa Lara & Rieutord (2011) and the canonical method begin with the expression
Figure 5.19: Change in disk temperature diagnostics with increasing rotation for the B2V model with $\rho_o = 5.0 \times 10^{-11}$ g cm$^{-3}$. The lines marked with upward pointing triangles indicate the maximum temperature and the lines marked with downward pointing triangles indicate the minimum temperature. The lines marked with circles indicate the volume-weighted average temperature and the lines marked with squares indicate the density-weighted average temperatures. The black and grey lines show models with $\beta = 0.25$ and $\beta = 0.18$ respectively. The lower horizontal axis indicates the fractional rotational velocity at the equator and the upper horizontal axis indicates the corresponding fractional angular velocity.

for radiative energy transport. The equation of hydrostatic equilibrium is used to relate the radiative flux to the local gravity, resulting in expressions of the form, $F_{\text{rad}} \propto g$, which becomes $T_{\text{eff}} \propto |g|^{1/4}$ using the Stephan-Boltzmann law (von Zeipel, 1924; Clayton, 1983). The difference in these approaches is how the other terms in the equation are handled. von Zeipel (1924) treats the equipotentials as isobaric surfaces which implies that all other terms in the equation are constant over the surface of a rotating star (for this type of argument see §6.8 of Clayton (1983)). This leads to a contradiction: the gas temperature is taken as a constant across the surface of a rotating star, while the effective temperature decreases with increasing stellar co-latitude.
Although this is not truly a paradox (as the effective temperature and the gas temperature are not the same quantities), it would be reasonable for a higher radiative flux at the poles to produce higher local temperatures.

Alternatively, Espinosa Lara & Rieutord (2011) make no assumptions about the other terms necessarily being constant, letting \( F_{\text{rad}} = f(r, \theta)g \). The unknown function, \( f(r, \theta) \), is found by requiring that \( \nabla \cdot F_{\text{rad}} = 0 \). Using the Roche model and solving the resulting partial differential equation gives

\[
T_{\text{eff}} = \left( \frac{L}{\sigma G M} \right)^{1/4} \sqrt{\tan \frac{\theta_w \tan \theta}{\tan \theta}} g^{1/4}, \quad (5.10)
\]

where \( \theta_w \) is defined by the requirement that

\[
\cos \theta_w + \ln \tan \frac{\theta_w}{2} = \frac{1}{3} x^3 w^2 \cos^3 \theta + \cos \theta + \ln \tan \frac{\theta}{2}. \quad (5.11)
\]

In Equation 5.11, \( \theta \) is the spherical coordinate, \( x \) is a scaled radius \( r/r_{\text{eq}}(\omega_{\text{star}}) \), and \( w \) is a different fractional angular velocity given by \( \omega_{\text{star}} \sqrt{r_{\text{eq}}^3(\omega_{\text{star}})/GM} \) which is not \( \omega_{\text{frac}} \). Equations of this form which describe a variable \( f(\theta) \) are sometimes written in the equivalent form of \( T = f_o g^{3\omega+\delta(\theta)} \), where \( f_o \) is now constant with all variation accounted for in \( \delta(\theta) \). Gravity darkening laws expressed in this form are found in Claret (2012) and Zorec et al. (2011).

Equation (5.10) predicts lower polar temperatures and higher equatorial temperatures than von Zeipel (1924). This is similar to, but not the same as, reducing \( \beta \). At slow rotation rates there is very little difference between the temperatures predicted by Espinosa Lara & Rieutord (2011) and von Zeipel (1924), but the differences increase with rotation and the ratio between them actually diverges at the equator for critical rotation because while both functions find a temperature of zero at the equator for critical rotation, the predic-
Figure 5.20: Ratio of the temperatures predicted by Espinosa Lara & Rieutord (2011), $T_{ELR}$, to those of standard gravitational darkening, $T_{VZ}$, (von Zeipel, 1924; Collins, 1963) versus rotation rate for four different co-latitudes, 0°, 30°, 60°, and 90°. The lower horizontal axis indicates the fractional rotational velocity at the equator and the upper horizontal axis indicates the corresponding fractional angular velocity. For readability we only show rotation rates faster than 0.75$v_{f\text{rac}}$, since this ratio is $\approx 1$ for low rotation rates.

The ratio of Espinosa Lara & Rieutord (2011) goes to zero with rotation more slowly (see Figure 5.20). Figure 5.21 shows the effect of using the Equation (5.10) compared to traditional gravity darkening of von Zeipel (1924) and Collins (1963). It is similar to Figure 5.19, predicting a weakening of the effects of gravitational darkening. However, Equation (5.10) reduces the heating of the pole more than the cooling of the equator, in comparison to simply reducing $\beta$. This is why using Equation (5.10) does not affect the density-weighted average temperatures as much as setting $\beta = 0.18$. Using Equation (5.10) has a similar effect on the volume-weighted average temperatures as setting $\beta = 0.18$, reducing the size of the initial drop in temperatures as well as reducing the heating at high rotation rates but the effect of setting $\beta$ to 0.18 is stronger.
Figure 5.21: Same as Figure 5.19 except the black lines show standard gravitational darkening with $\beta = 0.25$ while the grey lines show results using Espinosa Lara & Rieutord (2011).

5.4 Conclusions

Both density and rotation significantly affect disk temperatures. This is clearly seen in the density-weighted average temperatures in Figures 5.1 through 5.4 and in Table 5.3. Both cause disks to become cooler. Density is a stronger controller of the temperatures of these disks than rotation. This is as expected. Increasing the optical depth through the disk decreases the amount of photoionization radiation able to penetrate the disk by a large factor, especially in the mid-plane where the bulk of the material is located. Alternatively, rotation does not change the luminosity of the star, but only redirects it away from the mid-plane. The effects of rotation on the thermal structure of disks are not noticeable below $0.20 v_{\text{frac}}$. For moderate rotation, $0.20 v_{\text{crit}}$ to $0.60 v_{\text{crit}}$, there are small but noticeable changes in the disk temperatures. From $0.60 v_{\text{crit}}$ to $0.80 v_{\text{crit}}$ the effects of rotation become stronger as the equator cools and from $0.80 v_{\text{crit}}$ to $0.99 v_{\text{crit}}$ the stellar pole becomes hot enough to influence the disk.
while the equatorial cooling continues.

Increasing rotation from zero to 0.80 \( v_{\text{frac}} \) can have the same effect on the density-weighted average temperatures as increasing the density by a factors of 1.5 to 5, depending on the model. The effects become even larger closer to critical rotation. Increasing rotation from zero to 0.99 \( v_{\text{frac}} \) can have the same effect on the density-weighted average temperatures as increasing the density by 2.5 to 7.5 times, depending on the model. Therefore, while not as strong a temperature controller as density, rotation can be very significant for moderate to strong rotation and should not be neglected. Because classical Be stars rotate faster than 0.40 \( v_{\text{frac}} \), and many rotate faster than 0.80 \( v_{\text{frac}} \) (Cranmer, 2005), gravitational darkening should be included in models of classical Be stars disks.

With increasing rotation and density, Be star disks become less isothermal. This is can be clearly seen in Figures 5.14 and 5.15. These histograms show an increase in the amount of very cool gas as rotation increases, and at high rotation rates, there is an increase in the amount of hot gas also. Increasing the density of these disks increases the amount of cool gas, but does not significantly increase the temperatures in the upper disk. Therefore, only strong rotation causes an noticeable increase in the temperatures of the upper disk due to the hotter stellar pole.

Classical Be stars are known to be rapid rotators with geometrically thin disks (Porter & Rivinius, 2003). The inclusion of gravitational darkening in hydrostatically converged models shows that these disks are predicted to be very thin around rapidly rotating stars because the large equatorial cool region reduces pressure support in the vertical direction. The scale heights predicted for the densest model considered are very small, with \( H/R \) reaching as low as 0.027 at 3.75 stellar radii.
Unsurprisingly, using weaker forms of gravitational darkening weakens the effects of gravitational darkening on the disk. Simply reducing $\beta$ decreases the effects of rotation on both the volume-weighted average temperatures and the density-weighted average temperatures in the disk, but also introduces a new free parameter. The formulation of Espinosa Lara & Rieutord (2011) is an excellent alternative as it offers a physical explanation, avoids adding an artificial parameter, and is an algebraic solution so that it can be utilized for any star and any rotation rate lower than critical. Replacing standard gravitational darkening with the formulation of Espinosa Lara & Rieutord (2011) does not change the density-weighted average temperatures in the disk very much, but it does reduce the heating of the upper disk due to the stellar pole. There are other physical phenomena which could further affect the temperature structure of a rotating star, such as horizontal transport of energy as described in Hadrava (1992) which would re-distribute energy from the bright poles to the dimmer equator, but this effect is expected to be small.

In the future, we plan to produce observables for our rotating Be star disk models. It will be interesting to see how rotation and density changes affect the colours and H$\alpha$ emission from these systems, which are often used as the primary diagnostics of our circumstellar disk. For disks with large optical depths, temperature changes can strongly influence disk emission so effects due to rotation are likely to be important.
Bibliography


Jaschek, M., Slettebak, A., & Jaschek, C. 1981, Be Star Newsletter, 4, 9
von Zeipel, H. 1924, MNRS 84, 48
Chapter 6

Observational Diagnostics of Gravity Darkening
McGill et al. (2011) and McGill et al. (2013) discuss the temperature changes in the disk due to the rapid rotation of the central star. Temperature, however, cannot be directly observed, though it can be inferred from observations such as line ratios. In this Chapter, we present specific observables for the disk temperature models described in McGill et al. (2013), namely spectral energy distributions (SEDs), optical and IR colours and Hα line profiles. We divide our results into two sections: § 6.1 includes the results of models produced without a disk; and § 6.2 includes the results of models with circumstellar disks of varying densities.

6.1 Results for Gravitationally-Darkened Central Stars

We first wish to verify the gravitational darkening implementation in Beray. To do so, two sets of models were made without a disk in order to verify that the gravitationally darkened stellar model reproduced the following results:

1. A set of calculations where made for the stellar Mg II $\lambda 4481$ line for the B2V model of Table 6.2 to confirm the behaviour described in Townsend et al. (2004) and Cranmer (2005). Townsend et al. (2004) describes a saturation of line widths found at high rotation rates and points out that the highest velocities measurable are $\approx 0.8 \, v_{\text{crit}}$. Cranmer (2005) reiterates this point and describes effects of this phenomenon on the statistics of stellar rotation rates. Models of the stellar Mg II $\lambda 4481$ line were made for the following rotational velocities, ($0.00 \, 0.001 \, 0.01 \, 0.25 \, 0.5 \, 0.75 \, 0.85 \, 0.9 \, 0.95 \, 0.97 \, 0.99 \, v_{\text{crit}}$) both with and without gravitational darkening.
Table 6.1: Adopted stellar parameters for comparisons to Collins et al. (1991).

<table>
<thead>
<tr>
<th>Type</th>
<th>Mass ($M_\odot$)</th>
<th>Polar Radius ($R_\odot$)</th>
<th>Luminosity ($L_\odot$)</th>
<th>$T_{\text{eff}}$ (K)</th>
<th>$\omega_{\text{crit}}$ (rad s$^{-1}$)</th>
<th>$v_{\text{crit}}$ (km s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1V</td>
<td>11.00</td>
<td>5.80</td>
<td>$1.20 \times 10^4$</td>
<td>25200.</td>
<td>1.115</td>
<td>490.</td>
</tr>
<tr>
<td>B3V</td>
<td>6.10</td>
<td>4.60</td>
<td>$2.47 \times 10^3$</td>
<td>19060.</td>
<td>1.176</td>
<td>410.</td>
</tr>
<tr>
<td>B5V</td>
<td>4.50</td>
<td>3.80</td>
<td>$7.50 \times 10^2$</td>
<td>15570.</td>
<td>1.345</td>
<td>390.</td>
</tr>
<tr>
<td>B1III</td>
<td>11.00</td>
<td>9.19</td>
<td>$3.01 \times 10^4$</td>
<td>25200.</td>
<td>0.559</td>
<td>390.</td>
</tr>
<tr>
<td>B3III</td>
<td>6.10</td>
<td>7.29</td>
<td>$6.20 \times 10^3$</td>
<td>19060.</td>
<td>0.589</td>
<td>330.</td>
</tr>
<tr>
<td>B5III</td>
<td>4.50</td>
<td>6.02</td>
<td>$1.88 \times 10^3$</td>
<td>15570.</td>
<td>0.675</td>
<td>310.</td>
</tr>
</tbody>
</table>

2. A set of optical colours was produced by Beray in order to compare to the UBV colours of Collins et al. (1991). This set of calculations has identical stellar parameters to Collins et al. (1991) (see Table 6.1) and used the same set of inclinations (0° 30° 45° 60° 90°) and rotational velocities (0.0 0.5 0.8 0.9 1.0 $w_{\text{crit}}$).

### 6.1.1 Stellar Spectral Energy Distributions

Figure 6.1 shows the full spectra of our B2V models without a disk for four rotation rates, grouped by inclination. At $i = 0°$, the spectra are approximately parallel in log-log space and show an increase in luminosity with increasing rotation at all wavelengths. In the model spectra with $i = 30°$, the same increase and parallel behaviour is seen as in the $i = 0°$ models; however, the brightening with rotation is not as strong.

At $i = 60°$, the spectra become dimmer with increasing rotation for wavelengths shorter than the Balmer limit (3646 Å). The slope of the Balmer continuum is similar in all rotating models but is largest in the $v = 0$ model (crossing the other spectra). For wavelengths longer than the Balmer limit,
Figure 6.1: Full spectral energy distributions of our B2V models without a disk for four different rotation rates. The panels are for different inclinations: upper left $i = 0^\circ$; upper right $i = 30^\circ$; lower left $i = 60^\circ$; and lower right $i = 90^\circ$.

The brightness increases with moderate rotation, peaking at $v = 0.95 v_{\text{crit}}$. The $v = 0.99 v_{\text{crit}}$ model is actually dimmer than the $v = 0.80 v_{\text{crit}}$ model.

For $i = 90^\circ$, the spectra decrease in flux with increasing rotation for the shorter wavelengths (ultraviolet - optical). By 2 $\mu$m, there is very little change with rotation, and the lines begin to cross each other at about this point. Longward of 2 $\mu$m, there is a small increase in brightness with mild to moderate rotation, and the luminosity at $v = 0.99 v_{\text{crit}}$ is dimmer than that at $v = 0.95 v_{\text{crit}}$. The behaviour of the other spectral types is very similar to these
results.

### 6.1.2 Optical Colours

![Colour-magnitude diagram showing the rotational displacement fans made by our models and those of Collins et al. (1991). Each of these three displacement fans contains 25 models from our data set and 25 models from Collins et al. (1991) made for the same 5 choices of inclination angles (0° 30° 45° 60° 90°) and rotational velocities (0.0 0.5 0.8 0.9 1.0 \( w_{\text{crit}} \)) for one spectral type (B1V, B3V, or B5V). The location of each model is highlighted with a point. The largest point is the non-rotating model at the base of each fan, while the far end of each track is the fastest rotating model for that inclination (1.0 \( w_{\text{crit}} \)). The NRMS (dashed line) and the NRGS (dash-dotted line) are also shown for reference.](image)

Figure 6.2: Colour-magnitude diagram showing the rotational displacement fans made by our models and those of Collins et al. (1991). Each of these three displacement fans contains 25 models from our data set and 25 models from Collins et al. (1991) made for the same 5 choices of inclination angles (0° 30° 45° 60° 90°) and rotational velocities (0.0 0.5 0.8 0.9 1.0 \( w_{\text{crit}} \)) for one spectral type (B1V, B3V, or B5V). The location of each model is highlighted with a point. The largest point is the non-rotating model at the base of each fan, while the far end of each track is the fastest rotating model for that inclination (1.0 \( w_{\text{crit}} \)). The NRMS (dashed line) and the NRGS (dash-dotted line) are also shown for reference.

Figure 6.2 shows three pairs of “rotational displacement fans” in the (B-V) versus V colour-magnitude diagram. Three of the fans are models from Collins et al. (1991), and three are our own calculations using BERAY for the same B1V, B3V and B5V spectral types (parameters shown in Table 6.1). Also shown for reference is the non-rotating main sequence (NRMS) and the
The effect of gravitational darkening on the position of a rotating star in a (B-V), V colour magnitude diagram is to move the star away from the NRMS and towards the NRGS. This means that a rotating star will generally appear more evolved than it truly is on such a colour-magnitude diagram. A track across the diagram is produced for each inclination as rotation speed is increased. The different tracks form a rotational displacement fan, starting at the non-rotating (NR) model whose spectrum is not affected by inclination. The distance from the NR model increases with rotational velocity, and the details and direction of these tracks depend on the inclination and velocity. In addition, the size of the rotational fan can be quite large. The nearly vertical \( i = 0^\circ \) tracks almost reach the brightness of type III stars, and the nearly horizontal \( i = 90^\circ \) tracks are approximately as wide as two spectral classes (ie. the B1V fan almost reaches the B3V fan, the B3V fan overlaps the B5V fan, etc ... )

These tracks spiral out from the NR model value in a clockwise manner, with an initial direction that depends on inclination. The \( 0^\circ \) tracks move vertically (decreasing in V magnitude), away from the NR model value, while the \( 90^\circ \) tracks depart to the right (increasing (B-V) colour index). This direction varies smoothly between these extremes for intermediate angles. The size of the track is largest for the extreme inclinations (\( 0^\circ \) and \( 90^\circ \)) and shortest for \( 45^\circ \). In the B0V and B3V fans, the \( 45^\circ \) track is the straightest and \( 60^\circ \) track has the most pronounced spiral, while in the B5V fan, the \( 45^\circ \) track has the most pronounced spiral and the \( 0^\circ \) track is the straightest.

For a pole on star \( (i = 0^\circ) \), slow rotation causes virtually no change in the (B-V) colour index and a decrease in V magnitude as the track moves nearly vertically towards the NRGS. By \( v_{crit} \), the total decrease in V magnitude is
\( \approx 0.7 \text{ mag}, \) while the (B-V) colour index has increased slightly (\( \approx 0.002 \text{ mag}. \)). This behaviour is seen in all three spectral types.

For a star viewed along its equator (90\(^\circ\)), slow rotation causes a decrease in V magnitude and an increase in the (B-V) colour index. As rotation becomes larger, the (B-V) colour index continues to increase while V magnitude reaches a minimum at around 80\% of \( v_{\text{crit}} \) and then begins to increase. By \( v_{\text{crit}} \) there has been a large shift in the (B-V) colour index, and the V magnitude has increased beyond the NR model. The mid-inclinations behave between the extremes of the 0\(^\circ\) and 90\(^\circ\) tracks.

The star becomes brighter when seen at low inclinations because the bright, hot pole is in full view, the area increase with rotation is largest, and the change in the local gravity points the stellar surface towards the pole. However, because the whole star is seen, there is little change in colour. When viewed at higher inclinations, the increase in stellar area with rotation is smaller, the surface is pointed away from the viewer, and the cool equator is in full view; hence significant brightening is not seen. Because of the influence of the cool equator, rotation causes the star to appear redder when seen from high inclinations.

As shown in Figure 6.2, the BERAY results and those of Collins et al. (1991) agree quite well. The agreement is best for angles 45\(^\circ\) and 60\(^\circ\) and is the worst for 90\(^\circ\). The agreement is generally better for the B3V spectral type and poorer for the B5V model. As the sets of model atmospheres used are not the same, and the computational approaches are somewhat different, exact agreement is perhaps not expected.
Figure 6.3: FWHM of MgII 4481 Å versus rotation with gravity darkening (solid line) and without the inclusion of gravitational darkening (dashed line) from our stellar models. The FWHM for a pure rotation profile from Gray (2005) (dotted line) is also shown. All models are viewed at 90°

6.1.3 The Widths of Stellar Absorption Lines

Figure 6.3 shows the change in the MgII λ4481 Å line width (as measured by its full width at half-maximum, FWHM) with, and without, gravitational darkening for increasing stellar rotation. Linear limb darkening of the continuum is included in these models. The FWHM of a pure rotational profile, given by \( \approx 1.7(v_{eq} \sin i/c)\lambda_0 \), is also shown (Gray, 2005, Equation 18.14 with \( \epsilon = 0 \)). Without gravitational darkening, the line width increases linearly with rotation for both BERAY (with limb darkening) and the pure rotational profile from Gray (2005). When gravitational darkening is included, the bright unshifted continuum overwhelms the rapidly rotating and cooler limbs, effectively reducing the width of the spectral line. This causes the line widths to saturate, with rotational speeds above \( v_{frac} \approx > 0.8 \) giving little increase in the line FWHM.
This phenomenon is well described in Townsend et al. (2004) and Cranmer (2005). Note that the deviation from linearity at very low velocities in the Beray calculation is caused by the intrinsic width of the MgII line, due to thermal and pressure broadening and the multiplet structure of the line, all of which remain without rotational Doppler broadening.

Figure 6.3 shows the same behavior as Figure 1 of Townsend et al. (2004), but we do not reproduce the identical widths. Note that (1) our stellar models have different $v_{\text{crit}}$; (2) there are differences in the intrinsic widths of the different spectral lines (He II 4471 versus MgII 4481), (3) possible differences in the details of limb darkening calculations, and (4) the different method of assigning the line widths (Fourier transform versus FWHM). We are confident that our code correctly characterizes a gravity darkened stellar line.

### 6.2 Results Including a Circumstellar Disk

We now consider the changes caused by varying rotation, density, and inclination angle on the observables of a Be star system. Then synthetic Hα line profiles, continuum spectra, and magnitudes (U, B, V, R, I, J, H, K) were made with Beray for a selection of the Bedisk solutions presented in McGill et al. (2011) and McGill et al. (2013). The models are constructed with the density fall off $n = 3$ and density scales (see Chapter 4) of $\rho_o = 5.0 \times 10^{-11}$, $2.5 \times 10^{-11}$, $1.0 \times 10^{-11}$, and $5.0 \times 10^{-12} \frac{g}{cm^2}$, using the unchanging grid described in McGill et al. (2011). The case of an empty disk was also computed for all four spectral types (stellar parameters are listed in Table 6.2). Model observables were produced by Beray for four of the eleven velocities used in McGill et al. (2011) and McGill et al. (2013), (0.0, 0.80, 0.95, and 0.99 $v_{\text{crit}}$) at four different inclination angles (0°, 30°, 60°, and 90°) making a total of
Table 6.2: Adopted Stellar parameters for model observables.

<table>
<thead>
<tr>
<th>Type</th>
<th>Mass</th>
<th>Polar Radius</th>
<th>Luminosity</th>
<th>( T_{\text{eff}} )</th>
<th>( \omega_{\text{crit}} )</th>
<th>( v_{\text{crit}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>B0V</td>
<td>17.5</td>
<td>7.40</td>
<td>3.98 x 10^4</td>
<td>30000</td>
<td>7.10 x 10^{-5}</td>
<td>548</td>
</tr>
<tr>
<td>B2V</td>
<td>9.11</td>
<td>5.33</td>
<td>4.76 x 10^3</td>
<td>20800</td>
<td>8.38 x 10^{-5}</td>
<td>466</td>
</tr>
<tr>
<td>B3V</td>
<td>7.60</td>
<td>4.80</td>
<td>2.58 x 10^3</td>
<td>18800</td>
<td>8.95 x 10^{-5}</td>
<td>449</td>
</tr>
<tr>
<td>B5V</td>
<td>5.90</td>
<td>3.90</td>
<td>7.28 x 10^2</td>
<td>15200</td>
<td>1.08 x 10^{-4}</td>
<td>439</td>
</tr>
</tbody>
</table>

Notes: Stellar parameters adopted from Cox (2000).

320 different combinations of spectral type, density, velocity, and inclination angle.

6.2.1 Spectral Energy Distributions

In order to provide a context for the changes in photometric colours of the next section it is necessary to understand how the SED of a Be star changes with rotation, density, and inclination. The SED of the star and disk system reflects the Kurucz (1993) stellar atmospheres used to represent the central star and the disk emission from free-free and free-bound processes. Note that because these models focus on the continuum, no line processes are included. Our models are shown in Figures 6.4 and 6.5.

Figure 6.4 shows the change in the SED with density scale, \( \rho_o \), for models without rotation viewed at \( i = 0^\circ \) and \( i = 90^\circ \). At long wavelengths (\( \geq 3.2 \mu m \)), the system becomes, brighter regardless of inclination. However, the increase is larger for lower inclinations. This due to the thermal emission of the disk, which is optically thick at these wavelengths and, hence, has a larger projection on the sky at lower inclinations.

In the UV/optical increasing the density at low inclinations causes an in-
crease in brightness as the disk emits, and scatters stellar light towards the observer. In the optical, this brightening is strongly effected by free-bound emission (recombination), which can cause large increases and decreases in brightness across the hydrogen series limits. At low inclinations, the series jumps are in emission at higher densities. Within a free-bound continuum, the largest increase in brightness resulting from density to density is found immediately to the blue edge of a limit series limit, such as the Balmer jump at 3646 Å \((\log_{10}(\lambda_{BJ}) = 3.562)\) and the Paschen jump at 8200 Å \((\log_{10}(\lambda_{PJ}) = 3.914)\).

The slopes of the free-bound continua decrease as the disk density increases, and this is strongest for the Balmer continuum. This causes the disk to appear redder at low inclinations.

At high inclinations in the UV/optical, the system becomes dimmer as the density increases due to absorption and scattering processes by the disk. Nevertheless, the continuum is still affected by free-bound processes. At \(i = 90^\circ\), there is an increase in brightness, which becomes larger with increasing density, going across a series limit from blue to red (jumps are in absorption).
Within a free-bound continuum (i.e. the Balmer continuum), the largest decrease in brightness resulting from density changes is found immediately to the blue edge of a limit series limit. The slopes of the free-bound continua increase as the disk density increases, and this change is strongest for the Balmer continuum; in this case, the increase in disk density causes the system to appear bluer at high inclinations.

Figure 6.5: Effects of rotation on the spectra of a Be system for $\rho_o = 5.0 \times 10^{-11}$ g cm$^3$.

Figure 6.5 show the effect of rotation on the SEDs of Be star models with $\rho_o = 5.0 \times 10^{-11}$ g cm$^3$ at four different inclination angles. In general, the effects of rotation on the SEDs are complex and subtle. At long wavelengths
(past 2 $\mu$m), rotation causes the systems to become dimmer. This occurs at all inclinations, but is stronger for lower inclination where the disk faces the viewer. The dimming is strongest at $\approx 10 \, \mu$m, and this produces a small “kink” in the long wavelength spectrum of the system.

The effects of rotation are different at shorter wavelengths and vary for different inclinations. At low inclinations ($i = 0^\circ$, $i = 30^\circ$), rotation causes a brightening at short wavelengths, due to the hot stellar pole. The size of the Paschen jump (which is in emission at $\rho_o = 5.0 \times 10^{-11}$ g cm$^3$) decreases with rotation due to lower disk emission; and the size of the Balmer jump (which is in absorption) increases with rotation. The slopes of the Balmer and Paschen continuum also increase with rotation. Typically, these effects are stronger at $i = 0^\circ$ than at $i = 30^\circ$.

The effects of rotation at $i = 60^\circ$ and $90^\circ$ are different than those at lower inclinations. At $i = 60^\circ$, the Balmer continuum becomes dimmer with rotation and increases slightly in slope as rotation increases. This also occurs at $i = 90^\circ$, but the effect is larger. At $i = 60^\circ$, the Paschen continuum becomes somewhat brighter with rotation, and this is largest for wavelengths immediately red-ward of the Balmer jump. In contrast, $i = 90^\circ$ has very little change in the brightness of the Paschen continuum. It is slightly dimmer immediately red-ward of the Balmer jump, slightly brighter immediately blue-ward of the Paschen jump, and, like the Balmer continuum, increases in slope by a small amount.
6.2.2 Optical and Infrared Colours including the Circumstellar Disk

We have 80 disk models for each of the four spectral types we consider: four velocities, four inclination angles, and five densities, including one with a very low $\rho_0$ to simulate the absence of a disk. For all these models we have produced 8 broadband photometric colours: $U$, $B$, $V$, $R$, $I$, $J$, $H$, and $K$ centred at 3650, 4450, 5510, 6580, and 8060 Å, and 1.22, 1.63, and 2.19 $\mu$m respectively.

We have chosen to provide two types of colour-magnitude diagrams: $V$ magnitude versus (B-V) colour and $H$ magnitude versus (H-J) colour for three spectral types, B0V, B2V, and B5V. We plot the change in position of a model on the colour-magnitude diagram with a track for each of the five possible densities for each inclination. When looking at these figures, one should be aware that in the optical continuum, the dominate processes in the disk are free-bound absorption and emission with some scattering. Free-bound emission and absorption are particularly important in determining the shape of the continuum. Finally, there is thermal emission from the disk but in the optical it is in the Wien limit of the emission which peaks in the infrared (Waters & Coté, 1987).

Figures 6.6 through 6.8 show the results for the optical colours. Each panel contains tracks for one inclination and is for one spectral type. The NR (non-rotating) model is marked with a circle and rotation increases along the track to the fastest rotating model, $v_{\text{frac}} = 0.99$, marked with a cross at the far end of the track. Each track is for a different density, no disk ($\rho_0 = 0$), $\rho_0 = 5.0 \times 10^{-12}$, $1.0 \times 10^{-11}$, $2.5 \times 10^{-11}$, and $5.0 \times 10^{-11}$ g cm$^3$. The NR model without a disk is marked with a star. The non-rotating MS and non-rotating GS are shown for reference in each figure.
The tracks without disks are essentially the same as those shown in the rotational displacement fans of Figures 6.2, except that the standard models of Cox (2000) were used. Rotational displacement fans are quite large (Figure 6.2) and by adding disks of various densities, they become even larger. The densest models cross the giant sequence in the B0V and B2V models, but the B5V models do not touch the giant for the range of densities considered. The changes with density are smaller for the rapidly rotating models compared to the models without stellar rotation, and the tracks for various densities somewhat converge with increasing rotation. Many of the changes in colour are controlled by changes in the Paschen continuum and relate to free-bound absorption and emission which are stronger at wavelengths closer to the Paschen limit. This can be seen directly in the SEDs in section 6.2.1.

For inclinations $i = 0^\circ$, $30^\circ$, and $60^\circ$, the tracks for higher density disks are redder than those of lower density, and the change in V magnitude with increasing density is small (very small for inclinations $0^\circ$ and $30^\circ$). Tracks at $i = 0^\circ$, $30^\circ$ show a decrease in V magnitude as rotation increases. For the B0V and B2V models at low densities, there is little change in colour. However, for high-density disks, rotation causes the system to become bluer. The B5V models are different. Rotation initially causes the system to become bluer, the colour index then reaches a minimum at $v_{\text{frac}} = 0.80$, and the system becomes redder for higher rotation rates. Like the B0V and B2V models, increasing the disk density causes the colour change with rotation to become larger. At $i = 60^\circ$, rotation initially causes a decrease in V magnitude before reaching a maximum at $v_{\text{frac}} = 0.80$ (0.95 for B2V) and then V increases for higher rotation rates. With the exception of the densest B0V and B2V models, at $i = 60^\circ$, rotation causes an increase in colour index. At $i = 90^\circ$, the tracks for higher density disks are bluer and dimmer than those of lower density.
Increasing rotation at $i = 90^\circ$ causes the system increase in V magnitude and become bluer.

In summary, in an optical colour magnitude diagram, the changes in magnitudes are dominated by the changes in the stellar flux at low inclination. Rapid rotation decreases the effect of density changes on colours and magnitudes. The critically rotating models of different densities are closer together than those without rotation. However, rapid rotation also increases the impact of inclination.

Figures 6.9 through 6.11 show the (J-H,H) colour magnitude diagram for the B0V, B2V and B5V models. There are similarities with the (B-V,V) diagrams. Rotation typically produces a clockwise arc across the diagram, but the infrared colours are generally more strongly affected by density changes than are the optical colours. At $i = 60^\circ$ and $90^\circ$, the tracks for different densities at B0V are particularly well separated. For inclinations $i = 0^\circ$, $30^\circ$, and $60^\circ$, like the (B-V,V) diagram, the tracks for higher density disks are redder than those of lower density, but unlike the (B-V,V) diagram, the track for $i = 90^\circ$ also becomes redder with increasing density because the disk emission is more important in the IR.

The rotation tracks in the infrared have the same general pattern as in the optical. At low inclinations, rotation causes decreases in magnitude, and for higher densities, a decrease in colour index. At $i = 60^\circ$, rotation initially causes a decrease in H magnitude, which passes through a minimum and increases at higher rotation rates. This minimum generally occurs at $v_{\text{frac}} = 0.95$ in (J-H,H) diagrams. The densest B0V and B2V models show H magnitude increases at $i = 60^\circ$ and a decrease in colour index. The (J-H) colour index change is small for B0V and B2V for low densities. In the B5V models, there is an increase in colour index with rotation at low densities, but in the highest
density models, there is an initial decrease in colour index at small rotation rates and an increase at higher rates. For $i = 90^\circ$, rotation causes an increase in (J-H) colour index and a decrease in H magnitude for B2V and B5V models. However, this is not seen in the B0V models. Generally, the IR colours are more strongly affected by the changes in disk temperature with rotation than the optical colours.
Figure 6.6: (B-V) versus V colour-magnitude diagram for the B0V model, showing the rotational displacements made by our models without a circumstellar disk and with a disk at four different densities: \( \rho_0 = 5.0 \times 10^{-12} \); \( 1.0 \times 10^{-11} \); \( 2.5 \times 10^{-11} \); and \( 5.0 \times 10^{-11} \) g cm\(^{-3}\). Circles mark the models without rotation. Plus signs indicate the models with rotation rates at \( 0.99 v_{\text{crit}} \). A star marks the NR model, which has the same (B-V,V) values for all inclinations of (-0.3,4.0). The NRMS (solid grey line) and the NRGS (dash grey line) are also shown for reference. Each panel shows a different inclination angle: the upper left panel, \( i = 0^\circ \); the upper right panel, \( i = 30^\circ \); the lower left panel, \( i = 60^\circ \); the lower right panel, \( i = 90^\circ \).
Figure 6.7: Same as Figure 6.6 but for the B2V model.

Figure 6.8: Same as Figure 6.6 but for the B5V model.
Figure 6.9: Same as Figure 6.6 but for ((J-H),H).

Figure 6.10: Same as Figure 6.6 but for ((J-H),H) for the B2V model.
Figure 6.11: Same as Figure 6.6 but for ((J-H),H) for the B5V model.
Figure 6.12: Scatter plot of (B-V) colour versus V magnitude for models of all densities and inclinations separated by rotation rate. The upper left panel shows non-rotating models; the upper right panel shows models rotating at $v_{\text{frac}} = 0.80 \, v_{\text{crit}}$; the lower left panel shows models rotating at $v_{\text{frac}} = 0.95 \, v_{\text{crit}}$; the lower right panel shows models rotating at $v_{\text{frac}} = 0.99 \, v_{\text{crit}}$. A parabolic fit to the data is shown in each panel (dotted line). The MS and GS for reference (solid and dashed lines, respectively).

To attempt to put all of these models in context, Figure 6.12 shows the locations of all models on a (B-V),V diagram. The four panels of Figure 6.12 represent the four rotation rates considered ($v_{\text{frac}}=0.0$, 0.80, 0.95, and 0.99). A parabolic fit has been made to each of the four sets of models in order to define the average position of Be stars with disks of various densities as observed from various inclinations. In this fit the data points have been weighted by
sin $i$ in order to account for the random orientation of stellar axis in the sky.

The standard MS and the GS are also shown for reference. The fit for the models with non-rotating stars is just red-ward of the standard MS. As stellar rotation increases, the average position of Be stars moves toward the GS. By $v_{\text{frac}} = 0.99$, the average position of early Be stars has moved past the GS. This is only an apparent change due to rotation, disk density and inclination effects combined as all models treat the central B star as class V main sequence stars. Density has a much stronger effect on the (B-V),V position of the non-rotating models, causing these models to lie further from their fits than the rotating models which are more tightly clustered.

Figure 6.13 shows the residuals from the fits shown in Figure 6.12 for our four different rotational velocities, $v_{\text{frac}} = 0.00$, 0.80, 0.95, and 0.99. For $v_{\text{frac}} = 0.00$ and $v_{\text{frac}} = 0.80$, the scatter is larger for magnitudes below the
fit, but for $v_{frac} = .95$ and $v_{frac} = 0.99$ the scatter is roughly symmetric about zero. The scatter decreases with both velocity and spectral type. The range of the residuals goes from +1 V mag to -1.5 mag without rotation to +0.7 V mag to -0.6 mag at $v=0.99$. Typical residuals for $v_{frac} = 0.0$ are from +0.4 to -0.5 mag and for $v_{frac} = 0.99$ from +0.3 to -0.25 mag.

Figure 6.14 is a scatter plot of a selection of models across the (B-V),V diagram to shown how the changes with rotation, density and inclination scale with spectral type and luminosity classification. Generally rotation causes the effects of disk density changes to become smaller, while inclination causes the reverse. Increasing stellar rotation causes the lower inclinations to be much brighter than higher inclinations. The size of the spread of models is quite large, covering around two spectral types in colour and reaching over the giant sequence in many cases. Note that the B3V models are not included on the plot because they overlap with the B2V and B5V models. Disk density increases cause the Be stars to become redder at 30° and 60° but bluer by 90°. The change in colour with density is smaller for models at $v_{crit}$. Rotation to $v_{crit}$ causes decreases in V magnitude at 30° with increases seen by 90°. For $i = 30°$ at low densities, rotation causes the systems to appear redder, but for larger densities they appear bluer. Systems at $v_{crit}$ are redder than the non-rotating models for 90°. The behaviour is more complex for 60° and depends on density; without rotation, increasing density causes the system to become redder and remains of similar magnitude, but for rapid rotation, increasing the density causes the disk to become bluer and dimmer. This is because the rotation track spiral at 60°. Rapid rotation generally moves the systems away from the main sequence toward the giant sequence by increasing the brightness of the system or making it redder or both. Only for the densest models for B0V and B2V at 60° does the system move towards the main sequence with rotation.
Figure 6.14: Scatter plot of \((B-V),V\) colours for B0V, B2V and B5V models without rotation and rotating at \(v_{\text{frac}} = 0.99 v_{\text{crit}}\) for all densities and at inclinations of 30°, 60°, 90°. Shape indicates rotational speed and spectral type. Upwards pointing triangles are non-rotating B0V models; pluses are B0V models rotating at \(v_{\text{frac}} = 0.99 v_{\text{crit}}\); diamonds are non-rotating B2V models; stars are B2V models rotating at \(v_{\text{frac}} = 0.99 v_{\text{crit}}\); downward pointing triangles are non-rotating B5V models; crosses are B5V models rotating at \(v_{\text{frac}} = 0.99 v_{\text{crit}}\). Marker sizes correspond to the disk density, \(\rho_o\). The smallest marker the models without disks. The four largest markers indicate systems with disks of increasing density \(\rho_o = 5.0 \times 10^{-12}; 1.0 \times 10^{-11}; 2.5 \times 10^{-11};\) and \(5.0 \times 10^{-11}\) \(\text{g cm}^3\), respectively. Colour indicates inclination angle, dark blue is \(i = 30°\), light blue is \(i = 60°\) and red is \(i = 90°\). The MS and GS are also shown.
6.2.3 \( \text{H} \alpha \) Lines Including the Circumstellar

Figures 6.15 through 6.22 show the effects of rotation on synthetic \( \text{H} \alpha \) profiles for the B0V and B2V models for two different densities and four inclinations. \texttt{BERAY} produces the changes expected in the \( \text{H} \alpha \) lines with inclination and density: lines become lower in continuum contrast and wider as inclination increases. The profiles go from strong, single-peaked, narrow emission lines to doubled-peaked, emission lines, and finally to wide, low, double-peaked lines not rising much above the continuum with the line core in absorption (eg. shell lines see Figure 1.1). Denser disks show a small dimming of the continuum with increasing inclination as more of the disk blocks the star. At low inclinations, increasing the disk density increases the continuum level, while at \( i = 90^\circ \), increasing disk density reduces the continuum level. Increasing density increases the heights of the line emission peaks, increases the depth of the gap between the peaks. At \( 90^\circ \), increasing density increases the depth of the absorption core. This does not occur for the densest B0V models at \( 90^\circ \), which have and line cores in emission.

Rotation can have subtle effects on the \( \text{H} \alpha \) line above the continuum. The shape of the lines is mostly determined by inclination and density, but rotation can cause changes to the line shape, especially in line center. The height of the line is dominated by density effects, but at low inclinations, rotation also impacts the line heights noticeably. The height of the line is reduced with rotation at low inclinations. At high inclinations, the effect of rotation on the line is very small for the B2V models. The change in maximum flux is not large, but the continuum level rises significantly with rotation causing a decrease in the line height. This is increase is larger for low inclinations. Rotation causes an increase of the small inner emission peak at line centre for
the B0V lines at high inclinations and density.

Useful parameters for emission lines are the equivalent width (EQW), full width at half maximum (FWHM), peak separation (for doubly peaked profiles), continuum flux level ($F_{\text{cont}}$), and the maximum flux level ($F_{\text{max}}$). Because of the changes in width and height with inclination, EQW is the best way to measure the strength of an emission line, but it is measured relative to the continuum. For a simple emission line with no central absorption, the EQW scales roughly in terms of the other parameters as $\propto FWHM \times (F_{\text{max}}/F_{\text{cont}} - 1)$, when differences in the line shapes are small. The affects of rotation and density are found mostly in the term $F_{\text{max}}/F_{\text{cont}}$. Rotation causes a brightening of the continuum at low inclination. Increasing rotation from zero to critical causes an increase in the continuum flux by a factor of 2.5 and 2.8 at $i = 0^\circ$ for the most dense and least dense B2V models respectively, and 1.9 and 2.3 at $i = 30^\circ$ for the same. At $i = 60^\circ$, rotation causes a moderate brightening up to $v_{\text{frac}} = 0.80$ (which is a maximum) and than a decreases. For $i = 90^\circ$, rotation causes a small drop in the continuum level, although a small increase in the continuum is seen in the densest B0V models. The changes in $F_{\text{max}}$ are smaller than the changes in $F_{\text{cont}}$. Rotation generally causes a small drop in the $F_{\text{max}}$ with rotation. The largest drop occurs in the densest $i = 30^\circ$ models, where rotation causes a decrease with rotation of a factor of $\approx 1.4$ by critical rotation in the B2V model. At $i = 90^\circ$, the densest B0V model shows an increase in $F_{\text{max}}$ with rotation, of $\approx 1.4$ by critical rotation. For other inclinations ($i = 0^\circ$, $30^\circ$, and $60^\circ$), there is very little change in $F_{\text{max}}$ and an increase in $F_{\text{cont}}$ is what effects the EQW.

Figure 6.23 and 6.24 show the change in the EQW of the H$\alpha$ line with rotation for the B0V and B2V models for four inclinations and all disk densities. The EQW drop considerably with rotation for models at $i = 0^\circ$ and $i = 30^\circ$, 

and the size of this drop is larger for higher densities. At $i = 60^\circ$, the change in EQW is flat. No change is seen in EQW with rotation at $i = 90^\circ$, except in the densest B0V model, which shows a small increase in EQW with rotation. Rotation also cause an increase in the FWHM, but this is caused by of the height of the line dropping and no real change in the width of the base of the line, thus, the FWHM increases. Some narrowing of the distance between the peaks is seen as rotation increases, but this is quite small.

Figure 6.25 shows the effects of rotation on the H $\alpha$ line EQW for the B2V models for four inclinations and for all densities represented as a histogram of line strength. Without rotation, most of the equivalent widths are fairly small, less than 5 Å, but there is a long high value tail on the distribution, going as high as 70 Å. With rotation, the number of models with small equivalent widths increases, and the high value tail becomes smaller. The largest values are now found at 65 Å, but the count is smaller and there is a gap in the distribution which forms due to rotation. The size of the gap increases with rotation. By $v_{\text{frac}} = 0.99$, the gap in the distribution begins at 35 Å.
Figure 6.15: Change in the Hα line of the B0V models with rotation for $\rho_o = 5.0 \times 10^{-12}$ g cm$^{-3}$ at four different inclination angles: upper left panel, $i = 0^\circ$; lower left panel, $i = 30^\circ$; upper right panel, $i = 60^\circ$; and lower right panel, $i = 90^\circ$.

Figure 6.16: Same as Figure 6.15 but for $\rho_o = 1.0 \times 10^{-11}$ g cm$^{-3}$. 
Figure 6.17: Same as Figure 6.15 but for $\rho = 2.5 \times 10^{-11} \text{ g cm}^{-3}$.

Figure 6.18: Same as Figure 6.15 but for $\rho = 5.0 \times 10^{-11} \text{ g cm}^{-3}$.
Figure 6.19: Same as Figure 6.15 but for the B2V models.

Figure 6.20: Same as Figure 6.15 but for the B2V model with $\rho_0 = 1.0 \times 10^{-11}$ g cm$^{-3}$.

Figure 6.21: Same as Figure 6.15 but for the B2V model with $\rho_o = 2.5 \times 10^{-11}$ g cm$^{-3}$.

Figure 6.22: Same as Figure 6.15 but for B2V models with $\rho_o = 5.0 \times 10^{-11}$ g cm$^{-3}$. 
Figure 6.23: Change in the Equivalent width of the Hα line of the B0V models with rotation for four different disk densities at four different inclination angles: upper left panel shows $i = 0^\circ$; upper right panel shows $i = 30^\circ$; lower left panel shows $i = 60^\circ$; and lower right panel shows $i = 90^\circ$.

Figure 6.24: Same as Figure 6.24 but for B2V.
Figure 6.25: Histograms over equivalent width of the Hα line of the ensemble of B2V models divided by rotation: without rotation (upper panel); rotating at 0.80 $v_{\text{crit}}$ (2nd panel); rotating at 0.95 $v_{\text{crit}}$ (3rd panel); and rotating at 0.99 $v_{\text{crit}}$ (lower panel).
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Chapter 7

Conclusions

Most published models of classical Be star disks do not include the effects of gravity darkening with the exceptions of Carciofi et al. (2009) and Levenhagen et al. (2011), who only present a single model, and Sigut & Patel (2013) who presents only one rotation rate in the spherical gravity darkening approximation (SGD). The primary purpose of this thesis has been to add gravity darkening to BEDISK/BERAY and then to systematically examine the effects on temperature models and synthetic observables of classical Be stars to understand the differences in disks surrounding stars with different rotation rates. This will help us to discern the effects of gravity darkening on the circumstellar disks.

7.1 Tools to Analyze the Temperature Structure of Be Star Disks

Gravity darkening can have a significant affect on disk temperatures, and it is important to understand the changes in disk temperatures as a function of
rotation rate and density. Changes to the detailed disk temperature structure are shown on two dimensional temperature plots (Figures 4.8, 4.9, and 5.13). On these plots, the development of the inner cool region in the equatorial plane as density increases can be clearly seen and the extent of the cool region expands with increasing rotation.

Due to the number of models, a more efficient means of understanding the temperature structure of these disks is needed. Therefore, we also investigate effects on disk temperature averages using the density-weighted and the volume-weighted disk temperature averages. Density-weighted averages reflect the temperatures found in the equatorial plane of the disk, and the regions closer to the star, while the volume-weighted averages sample mostly the upper regions of the disk. Because of the nature of the grid, a numerical average is somewhere between. Other useful temperature parameters are measures of the maximum and minimum temperatures, taking the mode of the hottest and coolest 1% of disk temperatures. By looking at multiple temperature parameters, one gets a sense of how different regions of the disk respond to changes in stellar rotation and disk density (see Figures 4.4 through 4.7, and also Figure 4.17 and Figures 5.1 through 5.4, and also Figures 5.19 and 5.21).

We also investigated various one-dimensional representations of $T(R, Z)$ in order to retain some spatial information. The average radial profile, $\overline{T}(R)$, which is the density-weighted average temperature of each vertical column, describes how the temperature of the disk changes with radius. It is also useful to examine some temperature profile average in the z direction because gravity darkening breaks the spherical symmetry of the star. We developed, $\overline{T}(u)$ as the density-weighted average temperature on each scale height, u, as defined by Equation 4.15.

Together the $\overline{T}(R)$ and $\overline{T}(u)$ diagnostics show the spacial structure of the
disk temperatures in a fair amount of detail, but are compact enough to be shown for many models (see Figures 4.10 through 4.15 and also Figure 4.18, and Figures 5.5 through 5.12).

Another way of illustrating disk temperature structure is with histograms of the disk temperatures, allowing the divergence of the disk temperatures with gravitational darkening to be seen. As demonstrated in Chapter 5 rapid rotation increases both extremes of disk temperatures (see Figures 5.14 and 5.15).

7.2 Gravitational Darkening in Disk Models

Gravitational darkening causes a divergence of the stellar surface temperatures with increasing rotation; the bulk of the surface becomes cooler, while the polar regions increase in temperature. Although in these models the total luminosity of the star remains unchanged, the energy in the stellar radiation field around the equatorial region of the star is reduced which in turn affects the disk. As a result, cooling occurs in much of the disk, with the density-weighted temperature averages decreasing by $\approx 15\text{-}20\%$. Heating over the larger disk is also seen in the increase in the volume-weighted average temperature at rotational speeds above 80\% of the critical rate. Such changes cannot be produced by simply lowering the effective temperature of the star with rotation.

The cool region in the disk, near the star and in the equatorial plane, generally increases in both radius and height as the rotation rate is increased. The hot sheaths, above and below the cool region, generally decrease in temperature and move to larger scale heights with increased rotation.
7.3 Spherical Gravity Darkening

Some of the models created for this work used the spherical approximation of gravitational darkening (SGD), which ignores the distortion of the stellar surface. SGD does a reasonable job of approximating full gravitational darkening for rotation rates less than 80% of critical. For the B2V models at 80% of critical, the difference between SGD and FGD density-weighted temperature average is only 7.5 K or 0.09%, with a good match in the global temperature parameters below these speeds. There is also similarity in the vertically-averaged and radially-averaged disk temperature distributions for FGD and SGD models, indicating that SGD approximates the bulk temperature structure of the circumstellar disk below 80% of critical. Above these rates, there are noticeable departures, and by critical rotation, the difference in the density-weighted averages has reached 745.5 K or 9%. In addition, the vertical and radial average temperature no longer show close agreement.

7.4 Combined Effects of Density and Rotation

Chapter 5 systematically examines the effects of rotation for a range of disk densities and spectral types, and characterizes the effects of rotation on our disk models. Below 0.20 $v_{\text{frac}}$, the effects of rotation on the thermal structure of disks are not noticeable. For moderate rotation, 0.20 $v_{\text{crit}}$ to 0.60 $v_{\text{crit}}$, the changes in the disk temperatures are small. The effects become larger from 0.60 $v_{\text{crit}}$ to 0.80 $v_{\text{crit}}$ as the equator cools, and by 0.80 $v_{\text{crit}}$ to 0.99 $v_{\text{crit}}$, the equatorial region becomes even cooler and the stellar pole is hot enough to influence the disk.

Considering the combined effects of rotation and density is a necessary
step as it provides a sense of scale to the impact of rotation on the disk temperatures by allowing comparison to the effects of density. Disk temperatures are strongly affected by both density and rotation as the cooling of the density-weighted average temperatures indicate (see Figures 5.1 through 5.4 and Table 5.3). As expected, density has a larger effect on the temperatures of these disks than rotation because density increases the optical depths throughout the disk and reduces the amount of photoionizing radiation able to penetrate the disk by a large factor. Rotation, by contrast, does not change the luminosity of the star, but only redirects it away from the mid-plane and the effects on the local radiation field are smaller. However, while smaller than the effects of density, the impact of rotation is still about the same order of magnitude. Increasing rotation from zero to $0.80 \frac{v}{v_{\text{frac}}}$ can have the same effect on the density-weighted average temperatures as increasing the density by factors of 1.5 to 5, depending on the model. The effects become even larger closer to critical rotation. Increasing rotation from zero to $0.99 \frac{v}{v_{\text{frac}}}$ can have the same effect on the density-weighted average temperatures as increasing the density by 2.5 to 7.5 times, also depending on the model. So while not as important as density in determining disk temperatures, the impact of rotation can be important for moderate to strong rotation and should not be neglected.

Increasing either rotation or density causes Be star disks to become less isothermal. Figures 5.14 and 5.15 show an increase in the amount of very cool gas as rotation increases, and at high rotation rates, the amount of hot gas as well. Increasing the density of these disks increases the amount of cool gas, but does not significantly increase the fraction of hot temperatures. Therefore, only rapid rotation causes a noticeable increase in the temperatures of the upper disk due to the hotter stellar pole.
7.5 Hydrostatically Converged Models

When gravitational darkening is included in calculations of hydrostatically converged models, very thin disk models are produced for rapidly rotating stars because the pressure support in the vertical direction is reduced due to the larger, cool, equatorial region. These models predict very small scale heights for the densest disks, with $H/R$ reaching as low as 0.027 at 3.75 stellar radii. This may be unrealistically thin since these disks must still be thick enough for disk building around edge-on systems to obscure the star enough to produce the noticeable drop in V magnitude (Sigut & Patel, 2013). The models considered in § 5.3.2 are essentially the smallest and largest rotation rates possible, zero and 0.99 $v_{\text{crit}}$. It is likely that rotation rates are somewhat lower than 0.99 $v_{\text{crit}}$ in real systems with corresponding larger scale heights.

7.6 Alternative Forms of Gravity Darkening

One of the more interesting points considered in Chapter 5 was the different, weaker forms of gravity darkening suggested by interferometric observations (van Belle, 2010). These new forms have lower temperature differences between the pole and the equator. Weaker forms of gravitational darkening reduce the effects. Models of gravity darkening which lower the parameter $\beta$ decrease the effects of rotation on both the volume-weighted average temperatures and the density-weighted average temperatures in the disk. Using the formulation of Espinosa Lara & Rieutord (2011), instead of the standard gravitational darkening, does not produce a significant change over the density-weighted average temperatures, but it does increase the minimum in the volume-weighted temperature average at $v_{\text{frac}} = 0.80$ as well as decrease
the maximum at $v_{frac} = 0.99$, reducing the range of the volume-weighted temperature average, suggesting stronger effects on the heating of the upper disk due to the stellar pole.

7.6.1 The Continuum Energy Distributions

At low inclinations ($i = 0^\circ$ and $30^\circ$) the continuum brightens and all calculated photometric bands, (U, B, V, R, I, J, H, K) decrease in magnitude with rotation. This increase in brightness is largest for $i = 0^\circ$, with similar, though smaller, effects for $i = 30^\circ$. In the SEDs, this occurs as an increase in brightness in the UV/optical and a decrease in brightness in the far IR. This switchover from brightening to dimming depends on density and occurs at shorter wavelengths for denser disks and varies somewhat with spectral type and inclination angle. In systems without disks, brightening occurs with rotation at all wavelengths for low inclinations. At the highest densities, all disk emission becomes smaller with rotation at all wavelengths, but shortward of $\approx \log(\lambda) = 4.3$ the star dominates and the system brightens. At lower densities the disk emission from the UV to the near IR is dimmest in models with $v_{frac} = 0.80$; models rotating at $v_{frac} = 0.95$ or $v_{frac} = 0.99$ are brightest depending on wavelength. At longer wavelengths the disk emission decreases with rotation and the size of this decrease, is larger for denser disks. There is more complexity and variation across spectral type and wavelength at larger inclinations ($i = 60$ and $90^\circ$). Changes in the UV and optical are small are varied, while the disk becomes dimmer with increasing rotation at longer wavelengths. The decrease in brightness at long wavelengths is caused by the cooling of the circumstellar disk with rotation. Changes in the UV and optical are primarily due free-bound processes.
Despite similar behaviour in most adjacent photometric bands, there still can be colour differences due to variations in strength of the brightening/dimming of the continuum with wavelength. In the UV/optical, changes in colours correspond primarily to changes in the slopes of the Paschen and Brackett continuum. These colour changes are dictated by differences in disk free-bound emission and absorption. Differences in colour with rotation are typically fairly small for low inclinations in the UV/optical. At higher inclinations, density has a smaller effect on colour than rotation. Typically the disk becomes redder with rotation. This reddening is quite large by $i = 90^\circ$. In the IR, the colours are more strongly affected by density changes. The effects of density are smaller for rapidly rotating models, which causes the rapidly rotating models to have less scatter on an HR diagram. Since rapidly rotating models tend to be the brighter and redder, they appear similar to the non-rotating giant sequence.

7.6.2 The H$\alpha$ Line

The H$\alpha$ line is the primary diagnostic of classical Be stars. Its presence in emission defines the Be stars as a class. The H$\alpha$ line typically has large, normalized line heights, $(F_{\text{max}} - F_{\text{cont}})/F_{\text{cont}}$, (for example see Silaj et al. (2010) which reports values as high as 14, though 5-10 is more typical of an active Be star). The temperature structure of a disk around a rotating star resembles a denser disk as it is cooler overall. However, the H$\alpha$ line from a disk that surrounds a rapidly rotating star has a smaller normalized line height, similar to those from a less dense disk. The bulk of this change in height is due to an increase in the continuum flux. For an optically thick line, there is only a very small drop in the maximum flux due to rotation, of around 30% at $i = 0^\circ$, 

which may reflect the somewhat cooler disk. The increase in the continuum flux due to the rapidly rotating star is much larger, 140% at $i = 0^\circ$. The increase of the continuum is strongest for lower inclinations because of the stellar orientation and for denser disks, possibly due to enhanced reflection. Combined, these produce a drop in the height of the line above the continuum and a strong decrease in the equivalent width with increasing stellar rotation. The percent reduction in equivalent width is 60% for the densest model at $i = 0^\circ$, and 70% for the least dense disk at $i = 0^\circ$. The size of the drop is only 50-60% at $i = 30^\circ$ and 7-18% $i = 60^\circ$. There is no noticeable drop in maximum flux and normalized line height for $i = 90^\circ$ at B2V, however for B0V the normalized line height increases by 10% and the equivalent width increase by 30-50%. The optical depths are much larger through the disk at $i = 90^\circ$. It is also possible that the larger inclination increases the impact of the hotter regions.

The shape of the line is primarily a result of inclination and the height is controlled by a combination of rotation and density. However there are subtle effects, mostly around line center, caused by rotation and density. Increasing disk density enhances the sharpness of the line features, increases the height and width of the shoulder, narrows the peaks spacing slightly, and decreases the depth of the central absorption. Increasing rotation also decreases the depth of the central absorption.

7.7 Final Remarks

Stellar rotation and the resulting gravity darkening has significant effects on the disk temperatures, photometric colours and H$\alpha$ equivalent widths of our disk models. The effects, while generally not as strong as changes in disk
density, do become important with increasing rotation. The differences between Be star models with and without rotational effects are of sufficient size to impact the disk parameters found for individual stars using codes such as Bedisk. Models of Be star systems which do not include gravity darkening are incomplete and, if based primary on the height of the Hα line above the continuum, are likely to underestimate disk densities. For \( i = 0 \) the reduction in the height of the normalized Hα line with rotation for the models with \( \rho_o = 5.0 \times 10^{-11} \text{g/cm}^3 \), produces a line with height between the non-rotating models with densities \( 1.0 \times 10^{-11} \) and \( 2.5 \times 10^{-11} \text{g/cm}^3 \). The corresponding drop in equivalent width is \( \approx 2/3 \). The \( \rho_o = 5.0 \times 10^{-11} \text{g/cm}^3 \) model at \( i = 0 \) shows the largest changes in the Hα line. Rotation and inclination together have a strong affects on the continuum flux level, photometric magnitude and colour. This is an important aspects of gravity darkening that would not occur for iso-thermal stellar models, even if the effective temperature is chosen in a way to attempt to account for the effects of gravity darkening. Isothermal models of rapidly rotating Be stars are incomplete and do not include all phenomena associated with rotation. Many Classical Be stars rotate faster than \( 0.80 v_{\text{frac}} \) (Cranmer, 2005), and therefore gravitational darkening should be included in models of classical Be stars disks. Hence the rotation rate of the central B star should be regarded as an additional parameter required to model the spectra of these complex objects.
Bibliography


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Appendix A

Additional Calculations

The effective gravitational potential is given by:

\[ \Phi(R, \theta, \omega) = -\frac{GM_*}{R} - \frac{1}{2}\omega^2 R^2 \sin^2 \theta, \quad (A.1) \]

The potential is constant across the surface of the star;

\[ \Phi(R(\theta = 0), \omega) = \Phi(R(\theta = 90), \omega). \quad (A.2) \]

This gives

\[ -\frac{GM_*}{R_{\text{polar}}} = -\frac{GM_*}{R_{\theta}} + \frac{1}{2}\omega^2 R_{\theta}^2 \sin^2 \theta. \quad (A.3) \]

After rearranging the equation is

\[ 0 = \frac{GM_*}{R_{\text{polar}}} - \frac{GM_*}{R_{\theta}} + \frac{1}{2}\omega^2 R_{\theta}^2 \sin^2 \theta. \quad (A.4) \]

Multipling by \( R_{\theta} \), it follows that

\[ 0 = \frac{GM_* R_{\theta}}{R_{\text{polar}}} - GM_* + \frac{1}{2}\omega^2 R_{\theta}^3 \sin^2 \theta, \quad (A.5) \]
which after re-ordering the terms, becomes

\[ 0 = \frac{1}{2} \omega^2 R_\theta^3 \sin^2 \theta + \frac{G M_r}{R_{\text{polar}}} R_\theta - G M_r. \]  

(A.6)

In standard form the equation is

\[ 0 = R_\theta^3 + \frac{2 G M_r}{\omega^2 \sin^2 \theta R_{\text{polar}}} R_\theta - \frac{2 G M_r}{\omega^2 \sin^2 \theta}. \]  

(A.7)

Solutions to cubics of this form are found in Chapter 5 of Press et al. (1986).

\[ R_{\theta, \omega_{\text{frac}}} = \left( \frac{-3 R_p}{\omega_{\text{frac}} \sin \theta} \right) \cos \left( \frac{\arccos (\omega_{\text{frac}} \sin \theta) + 4\pi}{3} \right). \]  

(A.8)

This is often expressed using the scaled variable \( x(\theta, \omega_{\text{frac}}) = R_{\theta, \omega_{\text{frac}}} / R_p \) which is given by:

\[ x_{\theta, \omega_{\text{frac}}} = \left( \frac{-3}{\omega_{\text{frac}} \sin \theta} \right) \cos \left( \frac{\arccos (\omega_{\text{frac}} \sin \theta) + 4\pi}{3} \right). \]  

(A.9)

The equation is ill-conditioned if \( \omega_{\text{frac}} \sin \theta \geq 1 \). At \( \omega_{\text{frac}} \sin \theta = 1 \), \( x \) is 1.5.

**A.1 Conversion of \( v_{\text{frac}} \) and \( \omega_{\text{frac}} \)**

The theoretical equations of rotation are best written in terms of the fractional angular rotation speed, \( \omega_{\text{frac}} = \frac{\omega}{\omega_{\text{crit}}} \) yet the observations of rotation are measured in velocity and considered in terms of \( v_{\text{frac}} = \frac{v}{v_{\text{crit}}} \). The conversion is non-linear and given by \( v_{\text{frac}} = \frac{x(\omega_{\text{frac}})}{1.5} \omega_{\text{frac}} \), where \( x(\omega_{\text{frac}}) \) is given by Equation A.9 with \( \theta = \pi/2 \). This is because \( v_{r_{\text{meq}}} = r_{r_{\text{meq}}} \omega \) and the increase in \( r_{r_{\text{meq}}} \) with rotation is non-linear (see in Equation A.8). Figure A.1 shows the relationship between \( \omega_{\text{frac}} \) and \( v_{\text{frac}} \), as well as angular and equatorial speeds.
Figure A.1: Lower and left axes show $\omega_{frac}$ versus $v_{frac}$ for a rotating star, while the upper and right axes indicate the corresponding angular and equatorial speed for the B3V model listed in Table 6.2.

for the B3V model listed in Table 6.2. The relationship is initially linear but the slope begins to increase sharply as the function approaches (1.00, 1.00). This corresponds to $445 \text{ km/s}$ and $\approx 1.2 \text{ rev/day}$ for the B3V model.

A.2 Surface Orientation

Once we have calculated the shape of the star, we must deal with the orientation of the surface. This is necessary to calculate the region of the star that is visible from a point in the disk and the intensity of the light reaching that point from the each piece of the stellar surface that is visible. The surface vector is defined to be outwards from center so a piece of the stellar surface is only visible if it has positive component in the direction of the point in the disk. The local stellar surface is flat with respect to local gravity, so the direction of the surface normal is a unit vector opposite to the local gravity. In a non-rotating star this is straight out from the center of the star, no more, because $g$ now has a component in $\hat{\theta}$, recall:
\[ \ddot{g} (\theta) = \left( \omega^2 R_\theta \sin^2 \theta - \frac{GM_{\star}}{R_\theta^2} \right) \hat{r} + \left( \omega^2 R_\theta \sin \theta \cos \theta \right) \hat{\theta} \]  
\[ (A.10) \]

\[ |g (\theta)| = \sqrt{\left( \frac{GM_{\star}}{R_\theta} - \omega^2 R_\theta \sin^2 \theta \right)^2 + \omega^4 R_\theta^2 \sin^2 \theta \cos^2 \theta} \]  
\[ (A.11) \]

\[ \hat{n} (\theta) = \frac{-\ddot{g}}{|\ddot{g}|} \]  
\[ (A.12) \]

If we take \( \vec{R} (\theta, \phi, \omega) \) to be the vector from the origin located at the center of the star to a point on the stellar surface of length \( R_\theta \) with angular coordinates \( \theta \) and \( \phi \) and \( \vec{r}_s \) from the origin to an observing location in the disk, the orientation of the surface is found by taking the dot product of the unit vector that points from the stellar surface to the observing location \( \hat{n}_{\text{pointer}} = \frac{\vec{R} (\theta, \phi, \omega) - \vec{r}_s}{|\vec{R} - \vec{r}_s|} \) and the surface normal \( \hat{n}_{\text{surf}} = \frac{\vec{g}}{|\vec{g}|} \). This produced the cosine of the surface orientation.

### A.3 Change of Visible Sector

One more spatial factor affects the disk. In the vicinity of a sphere only a sector of the sphere is visible from a given point. From a location outside the star, an observer can only see the points on the stellar surface from which the cosine of surface orientation is positive:

\[ \frac{\vec{R} (\theta, \phi, \omega) - \vec{r}_s}{|\vec{R} - \vec{r}_s|} \cdot \frac{-\ddot{g}}{|\ddot{g}|} > 0 \]  
\[ (A.13) \]

This is the horizon effect. The further away the more you can see. As the
distance increases towards infinity, exactly half of the sphere becomes visible. The set of point on the edge of what is viewable from the observation point are on lines that connect the observation point to the surface point and are tangential to the surface, which mean this cosine is zero. For a sphere this is simple to find. The visible region is a spherical cap subtended by and angle \( \theta_{\text{edge}} \) where \( \theta_{\text{edge}} = \arccos \left( \frac{z_{\text{point}}}{r_{\text{point}}} \right) \) if the point is considered in cylindrical coordinates, \((r, z, 0)\). For a sphere the visible region is the same shape, a spherical cap for all points and has the same area for all points the same height above the sphere regardless of \( \theta \) or \( \phi \). This is no-longer true for a more complex shape like that of an rotating star. The terms in the dot produce are now fairly complex functions of \( \theta \), and is this constraint which must be employed to find the bounding edge of visible region. In order to find this we will need to take the dot product of \( n_{\text{surf}} = \frac{\vec{\mathbf{g}}}{|\vec{\mathbf{g}}|} \) and \( n_{\text{pointer}} = \frac{\vec{R}(\theta, \phi) - \vec{r}_s}{|\vec{R} - \vec{r}_s|} \) in Cartesian coordinates as functions of \( \theta \) and \( \phi \).

\[
\vec{r}_s = (r, z, 0)_{r,z,\phi} = (r, 0, z)_{x,y,z} \quad (A.14)
\]

\[
\vec{R} = (R_\theta, \theta, \phi)_{r,\theta,\phi} = (R_\theta \sin \theta \cos \phi, R_\theta \sin \theta \sin \phi, R_\theta \cos \theta)_{x,y,z} \quad (A.15)
\]

The value \( R_\theta \) is the result of solving the cubic to find the \( x \) as a function of \( \theta \) and multiplying by the polar radius \( R_p \) to get the stellar radius.

\[
\vartheta = \arccos \left( \frac{\omega_{\text{frac}} \sin \theta}{\omega_{\text{frac}} \sin \theta} \right) \quad (A.16)
\]

\[
R_\vartheta = \left( \frac{-3R_p}{\omega_{\text{frac}} \sin \theta} \right) \arccos \left( \frac{\vartheta + 4\pi}{3} \right) \quad (A.17)
\]

The vector from the observing location to the surface point is now restricted
to be a tangent, \( r_{\text{pointer}} = \vec{R} - \vec{r}_s = \vec{V}_{\text{tan}} \). (this is simpler, but equivalent to demanding that \( \hat{n}_{\text{pointer}} \) is tangent) For a point on the boundary of the visible region, \( \vec{g} \cdot \vec{V}_{\text{tan}} = 0 \)

\[
\vec{V}_{\text{tan}} = \vec{R} - \vec{r}_s = (R_\theta \sin \theta \cos \phi - r, R_\theta \sin \theta \sin \phi, R_\theta \cos \theta - z) \quad (A.18)
\]

We now need \( \vec{g} \) in Cartesian coordinates but still as a function of \( \theta \) and \( \phi \). We know the potential \( \Phi \) in Cartesian coordinates.

\[
\Phi = -\frac{GM}{\sqrt{x^2 + y^2 + z^2}} - \frac{\omega^2}{2} \left( x^2 + y^2 \right) \quad (A.19)
\]

taking the negation gradient gives:

\[
\vec{g}(x, y, z, \omega) = -\frac{GM}{(x^2 + y^2 + z^2)^{3/2}} (x, y, z) + \omega^2 (x, y, 0) \quad (A.20)
\]

\[
x = R \sin \theta \cos \phi \quad (A.21)
\]

\[
y = R \sin \theta \sin \phi \quad (A.22)
\]

\[
z = R \cos \theta \quad (A.23)
\]

\[
R = x^2 + y^2 + z^2 \quad (A.24)
\]

Since \( R \) is confined to be on the surface of the star \( R = R_\theta \), which produces:

\[
\vec{g} = -\frac{GM}{R_\theta^2} \left( \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta \right) + \omega^2 R_\theta \left( \sin \theta \cos \phi, \sin \theta \sin \phi, 0 \right) \quad (A.25)
\]
Expanding, $\omega^2 = \omega_{\text{frac}}^2 \times \omega_{\text{crit}}^2 = \omega_{\text{frac}}^2 \frac{8}{27} \frac{GM}{R_p^3}$, we have

$$\ddot{g} = \frac{-GM}{R_\theta^2} \left( \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta \right) + \frac{8}{27} \frac{\omega_{\text{frac}}^2}{R_p^3} \frac{GM}{R_\theta^2} \left( \sin \theta \cos \phi, \sin \theta \sin \phi, 0 \right),$$

(A.26)

or

$$\ddot{g} = \left( \frac{-GM}{R_\theta^2} \right) \left( \sin \theta \cos \phi \left( 1 - \frac{8}{27} \frac{\omega_{\text{frac}}^2}{R_p^3} \right), \sin \theta \sin \phi \left( 1 - \frac{8}{27} \frac{\omega_{\text{frac}}^2}{R_p^3} \right), \cos \theta \right).$$

(A.27)

If we define $\kappa_\theta = \frac{8}{27} \frac{\omega_{\text{frac}}^2}{R_p^3}$, we have

$$\ddot{g} = \left( \frac{-GM}{R_\theta^2} \right) \left( \sin \theta \cos \phi \left( 1 - \kappa_\theta \right), \sin \theta \sin \phi \left( 1 - \kappa_\theta \right), \cos \theta \right),$$

(A.28)

or

$$\ddot{g} = \left( \frac{-GM}{R_\theta^2} \right) \left( \sin \theta \cos \phi \left( 1 - \kappa_\theta \right), \sin \theta \sin \phi \left( 1 - \kappa_\theta \right), \cos \theta \right),$$

(A.29)

with

$$\vec{V}_{\text{tan}} = \vec{R} - \vec{r}_s = \left( R_{\theta} \sin \theta \cos \phi - r, R_{\theta} \sin \theta \sin \phi, R_{\theta} \cos \theta - z \right).$$

(A.30)

We need to take the dot product of this with $\vec{V}_{\text{tan}}$ and set it to zero in order to define the boundary. This gives

$$\ddot{g} \cdot \vec{V}_{\text{tan}} = 0 =$$

$$\left( \frac{-GM}{R_\theta^2} \right) \left( \sin \theta \cos \phi \left( 1 - \kappa_\theta \right), \sin \theta \sin \phi \left( 1 - \kappa_\theta \right), \cos \theta \right) \cdot \left( R_{\theta} \sin \theta \cos \phi - r, R_{\theta} \sin \theta \sin \phi, R_{\theta} \cos \theta - z \right),$$

(A.31)
or

\[ 0 = \left( -\frac{GM}{R_g^2} \right) \left[ \sin \theta \cos \phi (1 - \kappa_{\theta}) (R_{\theta} \sin \theta \cos \phi - r) + \sin \theta \sin \phi (1 - \kappa_{\phi}) (R_{\theta} \sin \theta \sin \phi) + \cos \theta (R_{\theta} \cos \theta - z) \right]. \tag{A.32} \]

Cancelling \( \left( -\frac{GM}{R_g^2} \right) \), it follows that

\[
\vec{g} \cdot \vec{V}_{tan} = 0 = \\
\sin \theta \cos \phi (1 - \kappa_{\theta}) (R_{\theta} \sin \theta \cos \phi - r) + \sin \theta \sin \phi (1 - \kappa_{\phi}) (R_{\theta} \sin \theta \sin \phi) + \cos \theta (R_{\theta} \cos \theta - z). \tag{A.33} 
\]

This needs some manipulation to get into a simpler form.

Continuing to rearrange one finds:

\[ 0 = \sin \theta \cos \phi (1 - \kappa_{\theta}) (R_{\theta} \sin \theta \cos \phi) - r (\sin \theta \cos \phi (1 - \kappa_{\theta})) \]
\[ + \sin \theta \sin \phi (R_{\theta} \sin \theta \sin \phi) - \sin \theta \sin \phi R_{\theta} \sin \theta \sin \phi + (R_{\theta} \cos \theta \cos \theta - z \cos \theta); \tag{A.34} \]

\[ 0 = R_{\theta} \sin^2 \theta \cos^2 \phi (1 - \kappa_{\theta}) - r \sin \theta \cos \phi - \kappa_{\theta} r \sin \theta \cos \phi \]
\[ + (R_{\theta} \sin^2 \theta \sin^2 \phi) - \kappa_{\theta} R_{\theta} \sin^2 \theta \sin^2 \phi + R_{\theta} \cos^2 \theta - z \cos \theta; \tag{A.35} \]
\[0 = R_\theta \sin^2 \theta \cos^2 \phi - \kappa_\theta R_\theta \sin^2 \theta \cos^2 \phi - r \sin \theta \cos \phi - \kappa_\theta r \sin \theta \cos \phi + (R_\theta \sin^2 \theta \sin^2 \phi) - \kappa_\theta R_\theta \sin^2 \theta \sin^2 \phi + R_\theta \cos^2 \theta - z \cos \theta; \tag{A.36}\]

\[0 = R_\theta \sin^2 \theta \cos^2 \phi + (R_\theta \sin^2 \theta \sin^2 \phi) - r \sin \theta \cos \phi - \kappa_\theta r \sin \theta \cos \phi - \kappa_\theta R_\theta \sin^2 \theta \sin^2 \phi + R_\theta \cos^2 \theta - z \cos \theta; \tag{A.37}\]

\[0 = R_\theta \sin^2 \theta - r \sin \theta \cos \phi - \kappa_\theta r \sin \theta \cos \phi - \kappa_\theta R_\theta \sin^2 \theta + R_\theta \cos^2 \theta - z \cos \theta. \tag{A.38}\]

Finally we have this form,

\[0 = R_\theta \sin^2 \theta - \cos \phi (r \sin \theta - \kappa_\theta r \sin \theta) - \kappa_\theta R_\theta \sin^2 \theta + R_\theta \cos^2 \theta - z \cos \theta, \tag{A.39}\]

which has only one \( \phi \) term.
\[ \phi = 0 \] defines is the R-z plane. In this plane the restriction is given by,

\[
\begin{align*}
0 &= R_\theta \sin^2 \theta \\
&\quad - (r \sin \theta - \kappa_\theta r \sin \theta) \\
&\quad - \kappa_\theta R_\theta \sin^2 \theta \\
&\quad + R_\theta \cos^2 \theta - z \cos \theta .
\end{align*}
\] (A.40)

This non-linear equation has two roots, \( \theta^1 \) and \( \theta^2 \), the extreme ranges of the \( \theta \) values within the visible region.

To define the whole region, \( \phi \) must be isolated. Returning to the \( \phi \neq 0 \) and one has

\[
\begin{align*}
\cos \phi (r \sin \theta - \kappa_\theta r \sin \theta) &= R_\theta \sin^2 \theta \\
&\quad - \kappa_\theta R_\theta \sin^2 \theta \\
&\quad + R_\theta \cos^2 \theta - z \cos \theta .
\end{align*}
\] (A.41)

Continuing to rearrange one finds:

\[
\begin{align*}
\cos \phi (r \sin \theta - \kappa_\theta r \sin \theta) &= R_\theta \sin^2 \theta - \kappa_\theta R_\theta \sin^2 \theta \\
&\quad + R_\theta \cos^2 \theta - z \cos \theta ;
\end{align*}
\] (A.42)

\[
\begin{align*}
\cos \phi (r \sin \theta - \kappa_\theta r \sin \theta) &= R_\theta \sin^2 \theta(1 - \kappa_\theta) + R_\theta \cos^2 \theta - z \cos \theta .
\end{align*}
\] (A.43)

Isolating \( \cos \phi \) gives

\[
\cos \phi = \frac{R_\theta \sin^2 \theta(1 - \kappa_\theta) + R_\theta \cos^2 \theta - z \cos \theta}{(r \sin \theta - \kappa_\theta r \sin \theta)} .
\] (A.44)

The equation for \( \phi \) is given by
\[
\phi = \arccos \frac{R_\theta \sin^2 \theta (1 - \kappa_\theta) + R_\theta \cos^2 \theta - z \cos \theta}{(r \sin \theta - \kappa_\theta r \sin \theta)}, \tag{A.45}
\]
which only has real values between \(\theta^1\) and \(\theta^2\).

This defines \(\phi\) around the edge of the visible region, between \(\theta^1\) and \(\theta^2\). This equation is ill-conditioned at a few points, if \(r = 0\) or \(\sin \theta = 0\), which corresponds to the center of the star, and either pole. It is also infinite when \(\kappa_\theta = 1\) which occurs at the equator of a critically rotating star and is quite physical, as a critically rotating star has no true surface at the equator. Considering \(r = 0\), the above only make sense at \(r=0\) if we are considering the surface of a star with radius 0, which means, a singularity not a star, and this needs no solution. However a valid function is required at the poles of the star, so two patches are required.

### A.3.1 First Patch: The top of the star, \(R_\theta = R_p\), \(\sin(\theta) = 0\) and \(\cos(\theta) = 1\)

The value of \(\vec{g} \cdot \vec{V}_{\text{tan}}\) and \(\vec{g} \cdot \vec{v}_{\text{tan}} = 0\) must be defined for the ill conditioned points individually.

\[
\vec{g} = \left( \frac{-GM}{R_p^2} \right)(0, 0, 1) \tag{A.46}
\]

\[
\vec{V}_{\text{tan}} = (0 - r, 0, R_p - z) \tag{A.47}
\]

\[
\vec{g} \cdot \vec{V}_{\text{tan}} = \left( \frac{-GM}{R_p^2} \right)(0, 0, 1) \cdot (0 - r, 0, R_p - z) \tag{A.48}
\]

\[
\vec{g} \cdot \vec{v}_{\text{tan}} = \left( \frac{-GM}{R_p^2} \right)(R_p - z) \tag{A.49}
\]
This equals zero when \(z = R_p\).
A.3.2 Second Patch: The bottom of the star, $R_\theta = R_p$, 
\[
\sin(\theta) = 0 \text{ and } \cos(\theta) = -1
\]

\[
\vec{g} = \left( \frac{-GM}{R_p^2} \right) (0,0,-1) \quad (A.50)
\]

\[
\vec{V}_{tan} = (0-r,0,-R_p-z) \quad (A.51)
\]

\[
\vec{g} \cdot \vec{V}_{tan} = \left( \frac{-GM}{R_p^2} \right) (0,0,-1) \cdot (0-r,0,-R_p-z) \quad (A.52)
\]

\[
\vec{g} \cdot \vec{V}_{tan} = \left( \frac{-GM}{R_p^2} \right) (-1) (-R_p-z) \quad (A.53)
\]

\[
\vec{g} \cdot \vec{V}_{tan} = \left( \frac{GM}{R_\theta^2} \right) (-R_p-z) \quad (A.54)
\]

This equals zero when $z = -R_p$. We now have a function valid at all points across the surface of a rotating star which allows the surface orientation to be defined.

A.4 Details of Von Zeipel’s Theorem

Von Zeipel’s theorem (von Zeipel , 1924) states that the radiative flux is proportional to the surface gravity,

\[
\vec{F} = C_1 \vec{g}. \quad (A.55)
\]

This is important for a rotating star on which the gravity decreases with increasing stellar co-latitude. The flux is related to the effective temperature
using the Stephan-Boltzmann Law,

$$|\bar{F}| = \sigma T^4_{\text{eff}}.$$  \hspace{1cm} (A.56)

This leaves us with the standard form of Von Zeipel’s theorem,

$$T = (C |\bar{g}|)^{1/4}. \hspace{1cm} (A.57)$$

In the standard treatment, $C$, is a constant.

Before we begin, the assumptions within this model should be clearly laid out.

1. Energy Transported by Radiation,

$$\bar{F} = -\frac{4acT^3}{3\kappa \rho} \nabla T.$$  \hspace{1cm} (A.58)

2. Hydrostatic Equilibrium,

$$\nabla P(\vec{s}) = \bar{g}_{\text{eff}}(\vec{s}) \rho(\vec{s}). \hspace{1cm} (A.59)$$

3. To prove $C$ is a constant we need to assume solid body rotation, uniform and constant rotation so all points within the star rotate together $\omega(r, z) = \omega$ (or cylindrical rotation, $\omega(r, z, \phi) = \omega(r)$) because this allows a potential to be defined and fulfills a requirement of the Wavre-Poincaré theorem (§4.3 Tassoul (1978)).

4. To find the constant $C$ we use the Roche Model which treats the gravitational potential due to the stellar mass as though it is unaffected by
any shape distortion so that \( \phi_{\text{mass}} = \frac{GM}{R} \).

5. To calculate a value for \( C \) for a specific rotating star we need to know how the luminosity is affected by rotation. It is common to assume that the stellar luminosity is not affected by rotation, \( L_\omega = L_0 \), where \( L_0 \) is the luminosity of the equivalent non-rotating star.

A.4.1 Derivation of Von Zeipel’s Theorem

The goal is to show that \( T_{\text{eff}} = (Cg)^{1/4} \).

One begins with the expression for purely radiative energy transport,

\[
F = -\frac{4acT^3}{3\kappa\rho} \nabla T.
\]

Using the chain rule, it follows that

\[
F = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dP} \nabla P.
\]

Because of hydrostatic equilibrium,

\[
\nabla P = \rho \bar{g}_{\text{eff}},
\]

which when substituted into the expression for energy transport gives,

\[
\tilde{F} = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dP} \rho \bar{g}_{\text{eff}}.
\]
Cancelling the ρ's we have,

\[ \tilde{F} = -\frac{4acT^3}{3\kappa} \frac{dT}{dP} \tilde{g}_{eff}. \]

If we define

\[ C_1 = \frac{4acT^3}{3\kappa} \frac{dT}{dP}, \]  

(A.60)

we have

\[ \tilde{F} = -C_1(T, \kappa, \frac{dT}{dP}) \tilde{g}_{eff}. \]  

(A.61)

Using the Stefan-Boltzmann law, we have

\[ \sigma T_{eff}^4 = |\tilde{F}| = C_1|\tilde{g}_{eff}|, \]

producing

\[ T_{eff} = \left( \frac{C_1}{\sigma|\tilde{g}_{eff}|} \right)^{\frac{1}{4}}. \]

The final form of this expression is

\[ T_{eff} = \left( C|\tilde{g}_{eff}| \right)^{\frac{1}{4}}. \]

(A.62)

with

\[ C = \frac{4acT^3}{3\kappa\sigma} \frac{dT}{dP}. \]  

(A.63)

It is worth noting that despite \( C \) being a function of \( T, \kappa \) and \( \frac{dT}{dP} \), it may still be considered to be constant across an equi-potential surface. This is because if we take the rotation as solid body rotation (or cylindrical if we wish to consider differential rotation) by the Wavre-Poincaré theorem, then the
surfaces of constant density and constant pressure coincide and these are also the equi-potential surfaces (Tassoul, 1978). With the density and pressure identical across a surface temperature must also be constant and thus the opacities and \( \frac{dT}{dP} \) would be constant as well.

The Wavre-Poincaré theorem is based on the principal that for cylindrical rotation the gravity can be written in the form of a potential. We return to the assumption of hydrostatic equilibrium. The pressure gradient of the star at position \( \vec{s} \) is proportional to the local gravity, \( \nabla P(\vec{s}) = \vec{g}_{eff}(\vec{s})\rho(\vec{s}) \), but for solid body rotation the local gravity is just the gradient of the potential, \( \vec{g}_{eff}(\vec{s}) = \nabla \Phi(\vec{s}) \). We have \( \nabla P(\vec{s}) = \rho(\vec{s})\nabla \Phi(\vec{s}) \). Since \( \rho \) is a scalar, by \( dP = \rho d\Phi \), the equi-potential surfaces must also be surfaces of equal pressure and the pressure is only a function of \( \Phi \). Since \( dP/d\Phi = \rho \), \( \rho \) must also be constant across these surfaces and a function of \( \Phi \). If the chemical composition, \( \mu \), is taken to be constant on an equi-potential, then temperature must be constant as well (because all other state variables are constant). This can also be justified if we assume \( \mu \) is a function of \( T \) and \( \rho \) if the atmosphere is in local thermodynamic equilibrium and \( \rho \) is already constant across an equi-potential by Wavre-Poincaré theorem. This means that \( T \) is a function of \( \Phi \). Because temperature and pressure are both constant across an equi-potential, \( \frac{dT}{dP} \) cannot vary across an equi-potential. It is clear that with hydrostatic equilibrium and solid body rotation \( C_1 \) and \( C \) are constant on an equi-potential surface (Collins (1989)).

At first the argument which defines gravitational darkening may seem odd. We prove that the local gravity is the only varying term in the equation for \( T_{eff} \) by proving that the temperature is constant across the surface of a rotating star. \( T_{eff} \) varies across a rotating star, yet \( T \) is constant. This is because \( T_{eff} \) and \( T \) are different. \( T \) is the local thermodynamic temperature and state
variable of the gas, while $T_{\text{eff}}$ is derived from the radiative flux and reflects the spacings of the equipotentials in depth through the surface. Thus the flux varies across the stellar surface while the temperature is constant. This is of course an idealism: the variation in flux should effect the matter of the star.

### A.4.2 Using Von Zeipel’s Theorem

$C = 4a c T^3 / 3 \kappa \sigma |dT/dP|$ is not a particularly useful expression because $|dT/dP|$ is difficult to calculate. Maeder (2009) defines $C$ as:

$$C = \left( \frac{L}{4 \pi \sigma G M (1 - \frac{\omega^2}{2 \pi G \rho_M})} \right)^{1/4}, \quad (A.64)$$

where $\rho_M$ is the average stellar density, and is a function of rotation rate. The most straight forward method of evaluating $C$ is presented in Collins (1965) and begins with the definition of the stellar luminosity as the surface integral of the local flux,

$$L = \sigma \int_{A} T_{\text{eff}}^4 dA = \sigma \int_{A} C|g_{\text{eff}}|dA = \sigma C \int_{A} |g_{\text{eff}}|dA.$$

Rearranging we have,

$$C = \frac{L}{\sigma} \left( \int_{A} |g_{\text{eff}}|dA \right)^{-1}. \quad (A.65)$$

To solve this, we need to describe the stellar gravity. Using the Roche model, the gravity is given by, $|g| = \frac{G M}{R_p^2} f(x, \omega_{\text{frac}})$, where $x = R_\theta / R_p$ and $\omega_{\text{frac}} = \omega / \omega_{\text{crit}}$ and $f$ is defined by:

$$f(x, \omega_{\text{frac}}) = \left( \left( \frac{8}{27} \omega_{\text{frac}}^2 x \sin^2 \theta - 1 / x^2 \right)^2 + \left( \frac{8}{27} \omega_{\text{frac}}^2 x \sin \theta \cos \theta \right)^2 \right)^{1/2}. \quad (A.66)$$
Substituting this into the expression for $C$, we have,

$$C = \frac{L}{\sigma} \left( \int_A g_{eff} dA \right)^{-1} = \frac{L}{\sigma} \left( GM \int_A f(x, \omega_{frac}) \frac{dA}{R_p^2} \right)^{-1}.$$

Since $dA = R^2 d\Omega$, where $d\Omega$ is the solid angle which subtends the surface element, $\frac{dA}{R_p^2} = x^2 d\Omega$, leaving us with,

$$C = \frac{L}{\sigma} \left( GM \int_\Omega f(x, \omega_{frac}) x^2 d\Omega \right)^{-1}. \quad (A.67)$$

The integrand in this expression consists only of scaled variables across the solid angle of the star, $\omega_{frac}$, $x$, and $\theta$, since the shape of a rotating star is determined only by $\omega_{frac}$, this integral will only be a function of $\omega_{frac}$. We have,

$$C(\omega_{frac}) = \frac{L}{MF(\omega_{frac})} \quad (A.68)$$

with

$$F(\omega_{frac}) = \left( G\sigma \int_\Omega f(x, \omega_{frac}) x^2 d\Omega \right). \quad (A.69)$$

Once $C(\omega_{frac})$ has been found for one star by calculating $F(\omega_{frac})$, it can be rescaled for all other stars rotating at the same $\omega_{frac}$.

To calculate $C$, the stellar luminosity is needed. One final assumption is often made, that the internal energy generation is not effected by rotation and the luminosity remains constant when the rotation rate changes. Alternatively, the luminosity as a function of rotation is required.

The equations used to describe gravitational darkening in this work are the
following:

\[ T_{\text{eff}} = (C_{g_{\text{eff}}})^{\frac{1}{4}} \]  
\[ C_{\omega} = \frac{L}{MF(\omega_{\text{frac}})} \]

\[ F(\omega_{\text{frac}}) = G\sigma \left( \int_A f(x, \omega_{\text{frac}})x^2d\chi \right) \]  
\[ \frac{C_{\text{star}_1}(\omega)}{C_{\text{star}_2}(\omega)} = \frac{L_{\text{star}_1}(\omega)}{L_{\text{star}_2}(\omega)} \frac{M_{\text{star}_2}(\omega)}{M_{\text{star}_1}(\omega)}. \]

A.4.3 Weaknesses in Von Zeipel’s Theorem

After finishing our derivation we should consider any weaknesses in the theorem before moving on. Beyond the odd separation of \( T_{\text{eff}} \) and \( T_{\text{gas}} \), there is another contradiction within von Zeipel’s theorem. When energy is transported by radiative diffusion, \( F = F_{\text{rad}} = -\frac{4acT^3}{3\kappa\rho} \nabla T \), and energy is not created in the gas, which is the case throughout the radiative envelope of a star, we have \( \nabla F_{\text{rad}} = 0 \). Unfortunately, the gradient of the flux through the envelope of a gravity darkened star does not equal zero. Since the gradient of the total flux must be zero outside the core, if von Zeipel’s theorem holds there must be a flow to compensate for the non-zero gradient of the radiative flux. The flow is called the meridional circulation. This contradicts the assumption of pure radiative transport. This is what is known as von Zeipel’s paradox. The difficulties result from the fact that the assumptions which build von Zeipel’s theorem, solid body rotation, hydrostatic equilibrium and purely radiative energy transport, are contradictory and this shows itself in the fact that \( F_{\text{rad}} \neq 0 \).

However there are strengths in von Zeipel’s theorem. Pure radiative energy transport and hydrostatic equilibrium are idealizations that are not far from
the reality of a star, with energy transport close to ideal radiative diffusion (Claret, 2012). Even with circulations or convection, a star cannot be far from hydrostatic equilibrium without large scale instabilities which would be noticeable and stellar pulsations are only small periodic permutations in the hydrostatic equilibrium expression (Maeder (2009)).

Claret (2012) uses model atmospheres to improve the treatment of the radiative transport and remove the simple assumption of the diffusion equation and finds deviations from von Zeipel becoming important for optical depths of order of 1, where the surface becomes optically thin. This suggests, for the radiation field under the atmosphere, the diffusion approximation and von Zeipel’s theorem are valid. Because of this, $F \propto |g|$ is likely to be fairly accurate. The use of the blackbody temperature is an approximation, as stars are not true blackbodies, but Equation (A.56) is used to define an effective temperature as a means of characterizing a flux, like a brightness temperature, rather than the local thermodynamic temperature.

The next step to consider in a critique of von Zeipel’s theorem is the question of whether the equipotentials must be isobaric surfaces and all other terms in the equation can be treated as constant over the surface of a rotating star. This leads to the near contradiction: the gas temperature is taken as a constant across the surface of a rotating star, while the effective temperature decreases with increasing stellar co-latitude. Although this is not truly a contradiction, it would be reasonable for a higher radiative flux at the poles to produce higher local temperatures. The assumption or baroclinity is likely incorrect. Detailed stellar structure calculations suggest this. This allows the pressure, temperature and $\frac{\partial T}{\partial P}$ to vary across the surface and $C_\omega$ would no longer be constant.

The second assumption known to be an approximation is solid body ro-
tation, which except in the case of strong magnetic fields, does not occur in real stars. In particular, Maeder & Meynet (2008) indicate that the internal rotation profile varies on evolutionary timescales due to angular momentum transfer within these stars. Stellar structure calculations indicate that hot stars rotate differentially with depth and across the surface. But rotation rates may not be that different across the surface of a rapidly rotating star (Espinosa Lara & Rieutord, 2012; Zorec et al., 2011). Despite these imperfections, we can expect the gravity darkening law(s) found in rotating stars to be a departure from von Zeipel, but not something altogether different. Below the atmosphere, radiative diffusion is a good approximation.

A.5 Lucy’s Law

Von Zeipel’s theorem is based on the radiative energy transport. In regions of the star where convection occurs \( \vec{F} = -\frac{4\pi c T^3}{3 k \rho} \nabla T \) does not apply. When considering rotation in stars like the sun, with convection occurring in the outer layers, we need another approach. Lucy (1967) outlines a method of evaluating the relationship between the local temperature and \( g_{\text{eff}} \) from the adiabatic constant. Adiabatic constants are constants in relationships between two state variables (the temperature \( T_e \), the pressure \( P \), the volume \( V \) or the density \( \rho \)) during an adiabatic process, for example:

\[
K_1 = \rho^{1-\gamma} T_e \\
K_2 = P^{1-\gamma} T_e^\gamma \\
K_3 = V^{\gamma-1} T_e
\]
These can be used within a fully convective zone because the gradient of the specific entropy is zero, so the state variables of the gas through these zones follow adiabatic curves. Lucy (1967) argues that an adiabatic constant in a stellar atmosphere can be taken as a function of only \( g_{\text{eff}} \) and \( T_e \), defining:

\[
K(\log g, \log T_e) = K_0. \tag{A.74}
\]

This equation can be solved for the relationship between the temperature and gravity. Lucy (1967) uses the ad hoc relationship,

\[
T_e \propto g^\beta, \tag{A.75}
\]

between \( T_e \) and \( g \) (valid if the gravity variations over the surface of the star are small) to define \( \frac{\partial T_e}{\partial g} \). Since \( K \) is a constant over an adiabatic, \( \frac{\partial \log K}{\partial \log g} \) is zero through the convection zone, giving the expression,

\[
0 = \frac{\partial \log K}{\partial \log g} + \beta \frac{\partial \log K}{\partial \log T_e}. \tag{A.76}
\]

\( \beta \) can be evaluated from this equation using a convective stellar envelope model. Lucy (1967) uses tables from Baker (1963) to calculate \( \beta \) for a limited number of stellar masses, radii, luminosities, compositions and mixing lengths parameters. The masses considered were between 1 to 1.2 solar masses. They found that \( \beta \) ranges between 0.069 - 0.088 and choose \( \beta = 0.08 \) as a representative value. The relationship between the flux and gravity found using the blackbody function and gives:

\[
F \propto g^{0.32}. \tag{A.77}
\]
It is worth noting that in this case the relationship between the temperature and gravity relates to the thermodynamic temperature, not the effective temperature, because it is based on adiabatic temperature profiles, however the equation that relates the flux to the gravity now assumes the star can be treated as a blackbody.

Since the value of $\beta$ is dependent on a model rather than a theory it should be considered somewhat flexible, and unknown if models appropriate to the star being considered don’t exist. New calculations predict a very wide range of $\beta$’s for the convective regions of stars from 0.5 to 1.5 (Claret, 2012; Espinosa Lara & Rieutord, 2012). The $\beta$ values which occur for convection are less than those for radiative energy transport. This means that temperature differences across the surface of smaller rotating stars with convective envelopes is much smaller than that seen in stars with radiative envelopes. Late B-type stars and early A type stars have radiative envelopes but for rapid rotation rates the equators become cool enough for convection and the rotation law changes across the star. This effect becomes negligible by around B5V.
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