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Rigid Body Attitude Estimation: An Overview and Comparative Study

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A thesis submitted in partial fulfillment of the requirements for the degree in Master of Engineering Science

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RIGID BODY ATTITUDE ESTIMATION: AN OVERVIEW AND
COMPARATIVE STUDY
(Thesis format: Monograph)

by

Nojan Madinehi

Graduate Program in Electrical and Computer Engineering

A thesis submitted in partial fulfillment
of the requirements for the degree of
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Abstract

The attitude estimation of rigid body systems has attracted the attention of many researchers over the years. The development of efficient estimation algorithms that can accurately estimate the orientation of a rigid body is a crucial step towards a reliable implementation of control schemes for underwater and flying vehicles.

The primary focus of this thesis consists in investigating various attitude estimation techniques and their applications.

Two major classes are discussed. The first class consists of the earliest static attitude determination techniques relying solely on a set of body vector measurements of known vectors in the inertial frame. The second class consists of dynamic attitude estimation and filtering techniques, relying on body vector measurements as well other measurements, and using the dynamical equations of the system under consideration.

Various attitude estimation algorithms, including the latest nonlinear attitude observers, are presented and discussed, providing a survey that covers the evolution and structural differences of these estimation methods.

Simulation results have been carried out for a selected number of such attitude estimators. Their performance in the presence of noisy measurements, as well as their advantages and disadvantages are discussed.

Keywords: Rigid Body, Attitude Estimation, Inertial Measurement Units, Kalman Filtering, Observer Design
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Chapter 1

General Introduction

The aerial robotics field has seen a growing interest during the last few decades because of its remarkable achievements in providing flying vehicles that can assist humans in a variety of difficult and hazardous tasks. Some examples of where these robotic systems may be employed include outer space (such as Earth-orbiting satellites), surveying and inspecting structures (such as tall skyscrapers and huge dams), investigation of hazardous or toxic environments, and traffic congestion and security applications. Such operations require aerial vehicles with a certain level of autonomy and manoeuvrability.

Unmanned Aerial Vehicles (UAVs) have shown great potentials in many indoor and outdoor applications. These vehicles can either be very large or relatively small in size depending on the applications they are intended to. From the heavy weight military drones to small vertical take-off and landing (VTOL) UAVs, the control of all these flying vehicles relies on some crucial sensors that provide the necessary flight information.

The UAV position, orientation and velocities are crucial states that need to be measured or estimated for the implementation of a successful motion control strategy. The position and linear velocity can be obtained using a Global Positioning System (GPS) for instance, while the angular velocity can be obtained using a body-attached gyroscope. As for the orientation (attitude), there is no sensor that measures it directly. However, the orientation is usually obtained using some attitude estimation algorithms relying on gyroscopic and
body vector measurements.

The earliest attempts to estimate the attitude of flying vehicles may go back to the time when mechanical gyroscopes, which provide measurements of angular velocity, were used in an integration process in which the knowledge of an initial attitude would be sufficient in finding the attitude in any other time. However, since these devices were primitive and usually had many problems with pressure, heat, etc., their ultimate performance was not satisfying and frequent restarting of the estimation process was required. With the advances in electronic devices, microelectromechanical systems (MEMS) have replaced those previous measurement devices. These components have provided low cost and lightweight Inertial Measurement Units (IMUs) for both industrial and research applications. The use of body vector measurements sensors such as accelerometers and magnetometers have allowed researchers to design better attitude estimation algorithms.

Probably one of the first and yet most influential works in the attitude estimation field was a mathematical problem proposed by Wahba in [Wahba, 1965]. The problem consists in finding the optimal attitude rotation matrix provided that a number of vectorial measurements are available. Several attempts to solve this problem resulted in the development of fast estimation methods, such as the Singular Value Decomposition (SVD) in [Markley, 1988], Quaternion Estimation (QUEST) in [Shuster and Oh, 1981], and Filter QUEST in [Shuster, 1989b], which were used in some of the NASA projects in 1980’s. Various solutions to the Wahba’s problem are categorized as a class of attitude estimators known as deterministic attitude estimators.

The emergence of Kalman filtering theory and its subsequent advantages in the estimation field led to a broad class of attitude estimators that revolutionized the real-time estimation of aerial system states and parameters. The nonlinear forms of such filters, known as Extended Kalman Filters (EKFs), were successful in accurately estimating attitude and other system states such as position and velocity of the flying object. The most popular EKFs were developed as Multiplicative EKF ([Markley, 2003]), and Additive EKF ([Bar-Itzhack and Oshman, 1985]). These two methods have been applied to many attitude
estimation problems, such as the IBM Space Precision Attitude Reference System (SPARS) ([Toda et al., 1969b]).

Despite the popularity of conventional EKFs, their dependence on the linearization of system equations was a disadvantage that could result in filter divergence. Therefore, some researchers tried to propose alternative estimation methods that did not require the linearization process. The Unscented Kalman Filtering (UKF) technique developed in [Julier and Uhlmann, 2004] relies on a set of points known as sigma points to estimate the mean of states and their covariance matrix. The method was specifically applied to attitude estimation applications in [Crassidis and Markley, 2003] and [VanDyke et al., 2004]. Another approach which consists in estimating the Probability Density Function (PDF) of states, known as Particle Filtering (PF), has been investigated in [Cheng and Crassidis, 2004], [Liu et al., 2007] and [Carmi and Oshman, 2009b] for attitude estimation purposes.

In the last decade, the emergence of a new and powerful class of attitude estimation techniques, relying on nonlinear observers, has brought new hopes for more reliable and stable attitude estimators. Rigorous stability proofs and strong mathematical arguments for the performance of nonlinear attitude observers are regarded as their important advantages over conventional filtering techniques.

Attitude filters with nonlinear structures such as the ones proposed in [Salcudean, 1991], and [Thienel and Sanner, 2003] inspired a number of subsequent works that led to a variety of nonlinear attitude filters and observers (e.g., [Mahony et al., 2008], [Tayebi et al., 2011]). These observers can estimate not only the attitude, but other system states and unknown parameters and have been shown to have good performance under noisy measurement conditions. Various solutions to the problems associated with real-time applications of these observers, in accelerated flights for instance, have been developed in [Hua, 2010], [Roberts and Tayebi, 2011b], and [Grip et al., 2012b].

Studies on the special structure of rigid body rotational and translational dynamics have also resulted in powerful observers known as Invariant Observers. Studies on systems possessing symmetries conducted in [Bonnabel et al., 2008], and [Lageman et al., 2010],
provided researchers with remarkable achievements in the attitude estimation field.

Other types of nonlinear attitude observers include those that provide simultaneous estimates of the rotational and translational states and are best suited for navigation purposes. The algorithms presented in [Rehbinder and Ghosh, 2003] and [Baldwin et al., 2007] rely on vision systems, and are shown to be efficient in indoor applications.

One of the objectives of this thesis is to provide a survey of the latest developments in the field of attitude estimation with an emphasis on the nonlinear observer techniques. Although there have been a number of attitude estimations surveys, such as [Lefferts et al., 1982] and [Crassidis et al., 2007], to the extent of the author’s knowledge, there has been no surveys on the newly developed nonlinear attitude observers. The importance of this work is that a thorough study of this class of attitude estimation techniques may pave the way for other researchers to not only get familiar with nonlinear attitude observers, but develop new tools with better performance. Nonlinear observers have great potentials for further research and this survey tries to enlighten this by providing information on the evolution and achievements of these tools.

This thesis is organized as follows: Chapter 2 provides a review of the basic definitions of rigid body attitude and mathematical preliminaries required to understand the dynamics of a flying vehicle. In Chapter 3, earliest attitude estimation techniques known as deterministic solutions will be discussed. Chapter 4 proceeds with the introduction of modern estimation methods including the complementary filters, Extended Kalman Filter and some of its variants such as Unscented Kalman Filters. A great part of this chapter will also be dedicated to reviewing the latest achievements in the field of nonlinear attitude observers that have brought significant new results in the attitude estimation field. This chapter, also, investigates the performance of some of the attitude estimation techniques under real-time conditions and provides simulations that shed light on the advantages and disadvantages of each technique. Chapter 5 brings the work in this thesis to a conclusion and discusses ideas for future developments.
Chapter 2

Attitude Representation and Mathematical Preliminaries

2.1 Introduction

In aerospace engineering, it is often needed to know the orientation of rigid bodies in space with respect to a reference frame attached to the Earth, the Sun, or the stars. The aim of this chapter is to review some of the commonly used attitude parameterizations used to describe a spacecraft’s orientation. The properties of these attitude parameterizations are discussed with their relative advantages and disadvantages in section (2.2). For a more complete and comprehensive discussion on different ways of representing the attitude, readers are referred to [Stuelpnagel, 1964a], [Shuster, 1993], and [Hughes, 1986].

The dynamic equations of motion for a rigid body with a brief introduction to the kinematic equations are presented in section (2.3). Their introduction helps define the special groups that represent the rotational and translational motion. Section (2.3.1) is dedicated to this aim.

Since the application of sensor measurements in attitude estimation problems is numerously discussed through this survey, section (2.4) will be including a brief introduction to the sensors pertinent to the attitude estimation problem.
2.1.1 Coordinate Frames

Let $I$ denote an inertial (fixed) frame and let $B$ denote the body-attached frame. The orientation (attitude) of a rigid body is defined as the orientation of frame $B$ with respect to frame $I$.

The attitude of a rigid body can be expressed by a variety of mathematical parameterizations, which can be either constrained with redundant elements, or unconstrained with minimal elements. The rotation matrix and the unit-quaternion are examples of constrained parameterizations and Euler angles, the Rodrigues parameters and the modified Rodrigues parameters (MRPs) are examples of minimal parameterizations.

In this section, we briefly review the most common attitude representations which are the rotation matrix, unit-quaternion and Euler angles. These parameterizations constitute the bases of most attitude estimation techniques and their properties and group structures are of great importance for the study of the evolution of attitude filters and observers.

2.2 Attitude Parameterizations

2.2.1 Direction Cosine Matrix

The Direction Cosine Matrix (DCM), commonly known as rotation matrix, is a widely used attitude representation and an element of the Lie group $SO(3)$, that is the special orthogonal group of dimension 3,

$$SO(3) := \{ R \in \mathbb{R}^{3\times3} | R^T R = R R^T = I_{3\times3}, \det(R) = 1 \}.$$  \hspace{1cm} (2.1)

This definition points to the orthogonal basis of columns in a rotation matrix, making it an orthogonal matrix itself. In this case, the inverse of matrix is equal to its transpose, \textit{i.e.} $R^T = R^{-1}$.

There are two definitions of rotation matrix: one that relates the orientation from the body-fixed frame $B$ to the inertial reference frame $I$, and the other that carries $I$ into the $B$. Although in most of the published papers (and in the present thesis as well) the first
representation is used, both definitions represent the same concept and can be easily transformed into each other. Let \( R \in SO(3) \) be the rotation matrix describing the orientation of a rigid body, and let \( \omega := (\omega_1, \omega_2, \omega_3)^T \) be the angular velocity of the rigid body expressed in the body frame. Then, the rigid body kinematics equation is given by

\[
\dot{R} = RS(\omega),
\]

(2.2)

where \( S(x) \) denotes the skew-symmetric matrix associated to \( x \). The skew symmetric matrix \( S(x) \) satisfies \( S(x)y = x \times y, \forall x, y \in \mathbb{R}^3 \), where \( (\times) \) denotes the vector cross product, and is given by

\[
S(\omega) = \begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix}.
\]

(2.3)

For a rotation matrix from the inertial frame to the body frame, the kinematics equation equivalent to (2.2) is expressed as

\[
\dot{R} = -S(\omega)R.
\]

(2.4)

The rotation matrix is a global and unique representation of orientation. Using this matrix, related vectors in reference coordinates of inertial and body frames can be mapped to each other. For example, let \( a_I \) be a vector expressed in the inertial frame \( I \) and \( a_B \) be the vector coordinates of \( a_I \) expresses in the body frame \( B \). Then,

\[
a_B = R^T a_I.
\]

(2.5)

In many estimation techniques, such as Kalman filters, the presence of measurement noise and uncertainties in the system parameters leads to non-orthogonal estimated rotation matrices. In these cases, numerical orthogonalization methods based on projection and reflection, such as Gram-Schmidt process and Householder transformation, have to be used to produce \( SO(3) \)-belonging rotation matrices.
2.2.2 Euler Angles

Euler angles are Euclidean parameterizations that lie in $\mathbb{R}^3$. Among all the different threedimensional parameterizations, Euler angles are the most widely used. A number of three-dimensional attitude parameterizations are presented and discussed in [Stuelpnagel, 1964b]. It is shown that no such parameterization can be both nonsingular and unique, which is also the case for Euler angles. The components of the vector of Euler angles $\Theta = [\phi, \theta, \psi]^T$, are referred to as the roll, pitch and yaw of the rigid body.

Given two coordinate systems $xyz$ and $x''''y''''z''''$, the process by which the three Euler angles transform the first coordinate system into the second system can be summarized as follows:

1. A positive rotation by an angle $\psi$ about the $z$ axis, leading to $x'y'z'$ where $z \equiv z'$.

2. A positive rotation by an angle $\theta$ about the $x'$ axis, leading to $x''y''z''$ where $x'' \equiv x'$.

3. A positive rotation by an angle $\phi$ about the $y''$ axis, leading to $x''''y''''z''''$ where $y'''' \equiv y''$.

The rotation matrix can be obtained as a product of three different rotation matrices, each corresponding to a rotation about three axes of the body frame $\mathcal{B}$. This is given by

$$ R = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}, $$

(2.6)

where $s$ and $c$ denote the sine and cosines operators. Direct computation of the time deriva-
tive of the Euler angles in light of (2.2) reads

\[\begin{align*}
\dot{\phi} &= \omega_1 + s\phi \tan \theta \omega_2 + c\phi \tan \theta \omega_3, \\
\dot{\theta} &= c\phi \omega_2 - s\phi \omega_3, \\
\dot{\psi} &= s\phi (c\theta)^{-1} \omega_2 + c\phi (c\theta)^{-1} \omega_3.
\end{align*}\]  

(2.7)

The Euler angles parameterization is not global and angles \(\phi\) and \(\psi\) along with their derivatives are not well-defined for \(\theta = \pm \pi/2\). This problem is not unique for the Euler angles parameterization and all other three-dimensional parameterizations have a similar singularity problem.

### 2.2.3 Unit Quaternion

Unit quaternion is a four-dimensional parameterization of attitude that allows avoiding singularities associated with the three-dimensional parameterizations. Euler was the first scientist to discover the abilities of this formulation and found out that an axis can be assigned to every rotation in three-dimensional space. This can be stated as follows:

**Euler’s Theorem:** Consider an element \(R\) of the Special Orthogonal group. For any rotation matrix \(R \in SO(3)\), a non-zero vector \(x\) exists that satisfies \(Rx = x\).

The existence of vector \(x\) that remains unchanged under a transformation of rotation matrix multiplication implies that any attitude can be specified in terms of a rotation by some angle about some fixed axis. Therefore, combination of a vector with a scalar can make a basis for attitude parameterization. The equality \(Rx = \lambda x\) also indicates that any rotation matrix has an eigenvalue equal to one. A new proof to the Euler’s theorem is recently presented in [Palais and Palais, 2007].

There are a number of four-element parameterizations of the three-dimensional space such as the Euler parameters, the Rodrigues parameters, and the Cayley-Klein parameters. Their definition and application along with transformations that connect them together can be found in the survey [Shuster, 1993]. The unit quaternion parameterization is another four-element attitude representation that has gained great attention among scientific com-
munity.

The unit quaternion is denoted by \( Q = (q_0, q) \in \mathbb{Q} \), where \( q_0 \in \mathbb{R} \) is its scalar part and \( q = (q_1, q_2, q_3) \in \mathbb{R}^3 \) its vector part. The non-Euclidean set of unit quaternions is defined by

\[
\mathbb{Q} := \{ Q \in \mathbb{R} \times \mathbb{R}^3 | |Q|^2 = 1 \}.
\]

As a result of the norm constraint in their definition, it can be seen that unit quaternion forms a unit sphere in \( \mathbb{R}^4 \). As previously discussed, it is known from the Euler’s theorem that the attitude of a rigid body can be described in terms of a rotation by some angle \( \theta \) along an axis \( \hat{k} \). This gives another definition of quaternion

\[
Q = (q_0, q) = (\cos(\theta/2), \sin(\theta/2)\hat{k}),
\]

where \( \theta \) is the angle of rotation and \( \hat{k} \) is a unit length rotation axis.

There is a simple transformation of a unit quaternion into its corresponding rotation matrix which is known as the Rodrigues formula and is given by

\[
R(Q) = I_3 + 2S(q)^2 - 2q_0S(q),
\]

\[
= \begin{bmatrix}
q_0^2 + q_1^2 - q_2^2 - q_3^2 & -2q_0q_3 + 2q_1q_2 & 2q_0q_2 + 2q_1q_3 \\
2q_0q_3 + 2q_1q_2 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & -2q_0q_1 + 2q_2q_3 \\
-2q_0q_2 + 2q_1q_3 & 2q_0q_1 + 2q_2q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2
\end{bmatrix}.
\]

Apart from quaternion, rotation matrix can also be constructed using the rotation angle \( \theta \) and rotation axis \( \hat{k} \) by the following transformation

\[
R(\theta, \hat{k}) = I_3 - \sin(\theta)S(\hat{k}) + (1 - \cos(\theta))S(\hat{k})^2.
\]

While the rotation space \( SO(3) \) has three-elements, quaternion representation has four elements which make it an over-parameterization of this space. This results in the transformation from \( \mathbb{Q} \rightarrow SO(3) \) to form a two-to-one map. In fact, it is obvious from (2.10) that both quaternions \( Q \) and \(-Q\) correspond to the same rotation matrix, i.e. \( R(Q) = R(-Q) \). Technically speaking, the two orientations are physically the same and point to a single
orientation in space. Despite this redundancy, the unit-quaternion representation is a non-singular (global) representation of the attitude.

Given two unit quaternions $Q_x = (q_{0,x}, q_x)$ and $Q_y = (q_{0,y}, q_y)$, the quaternion product denoted by $Q_z = (q_{0,z}, q_z)$ is defined as

$$Q_z = Q_x \otimes Q_y = \begin{pmatrix} q_{0,x}q_{0,y} - q_x^T q_y \\ q_{0,y}q_x + q_{0,x}q_y + q_x \times q_y \end{pmatrix},$$

where $(\otimes)$ denotes the quaternion multiplication and $(\times)$ denotes the cross product. It should be noted that the quaternion multiplication is not commutative and $Q_1 \otimes Q_2 \neq Q_2 \otimes Q_1$. The inverse of a quaternion denoted by $Q^{-1}$ is defined as $Q^{-1} = (q_0, -q)$, where

$$Q \otimes Q^{-1} = Q^{-1} \otimes Q = (1, 0).$$

The kinematics of unit quaternion attitude representation is given by the following equation

$$\dot{Q} = \frac{1}{2} Q \otimes (0, \omega) = \frac{1}{2} \begin{bmatrix} -q^T \\ q_0 I + S(q) \end{bmatrix} \omega.$$  \hspace{1cm} (2.14)

The unit quaternion has some advantages over the other attitude parameterizations. One advantage is that it works with a $4 \times 1$ vector rather than a $3 \times 3$ (9 elements) attitude matrix. This not only makes work and computations easy, it also helps in applications where normalization of attitude representation is needed because of perturbations involved in its estimation process; normalizing a unit quaternion by simply dividing by its norm is much easier than the non-trivial process of preserving the orthogonality of a rotation matrix.

On the other hand, this representation lacks uniqueness. As previously discussed, two quaternions $\pm Q \in \mathbb{Q}$ point to the same rotation matrix $R \in SO(3)$, i.e. an equivalent physical orientation. This is the cause of a well known problem in quaternion based attitude estimation and control algorithms. In fact, multiple equilibria (that represent the same physical orientation), having different stability properties, may be generated.
Chapter 2. Attitude Representation and Mathematical Preliminaries

2.3 Rigid Body Kinematics and Dynamics

The dynamic model of a rigid body or a flying vehicle consists of its rotational and translational motion. Based on the chosen attitude parameterization method, the rotational kinematics equation takes different forms which were presented for rotation matrix (2.2), Euler angles (2.7), and unit quaternions (2.14). Recalling $\omega \in \mathbb{R}^3$ as the body-measured angular velocity expressed in $\mathcal{B}$, the complete rotational dynamic equations of a flying rigid body using the rotation matrix representation is given by

$$\dot{R} = RS(\omega),$$  
$$I_b\dot{\omega} = -\omega \times I_b\omega + u,$$  \hspace{1cm} (2.15)

where $I_b \in \mathbb{R}^{3x3}$ denotes the inertia matrix of rigid body, and $u \in \mathbb{R}^3$ is the control torque input applied to the rigid body. In practice, the torque is computed according to the desired control strategy and is a function of estimated system parameters and states.

Let us denote the position and linear velocity of the rigid body with respect to the earth-fixed frame, by $p$ and $v$ respectively. The simplified translational dynamics of a flying rigid body can be expressed as follows

$$\dot{p} = v,$$  
$$\dot{v} = ge_3 + R^Ta,$$  \hspace{1cm} (2.16)

where $e_3 = [0, 0, 1]^T$ is the body-referenced $z$-axis and $a$ is the specific acceleration vector that is the sum of all the non-gravitational forces divided by the body mass $m$. $g$ is the Earth gravitational acceleration given by $g \approx 9.8 m/s^2$. Note that these equations are ideal and external forces and torques were not included. In practice, these forces play an important role in the process of designing controllers for aerial vehicles, such as small aircrafts. More complex system models that take the disturbance forces and torques into account are found in the works of [Roberts and Tayebi, 2011a] and [Pflimlin et al., 2007].

Although the translational dynamics may not seem to give information about the attitude, they are useful in designing filters where the measurements obtained from a GPS
and an IMU are available. These estimators provide estimates of not only the attitude of a flying vehicle, but its position and velocity.

### 2.3.1 SO(3) and SE(3)

In the study of rotational and translational dynamics of flying objects, it is always useful to refer to some special groups on which these dynamics are defined. The two groups are known as Special Orthogonal group $SO(3)$ and Special Euclidean group $SE(3)$. Because of the importance of the properties of these groups when dealing with nonlinear observers of special structures, a brief explanation of their properties is presented in this section. Detailed information on the structure of these two special groups can be found in [Belta and Kumar, 2002].

Let $GL(n)$ be the general linear group of dimension $n$. This group is a subset of $\mathbb{R}^{n\times n}$ and by definition, matrix operations of multiplication and inversion are smooth. Therefore, $GL(n)$ is a Lie group. The Special Orthogonal group $SO(n)$ is defined as a subgroup of this general linear group given by

$$SO(n) = \{ R \mid R \in GL(n), RR^T = R^T R = I_{n\times n}, \det R = 1 \}. \quad (2.17)$$

The $SO(n)$ describes the rotation group on $\mathbb{R}^n$. The affine group $GA(n)$ is defined as $GA(n) = GL(n) \times \mathbb{R}^n$, and the set of all rigid displacements in $\mathbb{R}^n$ is $SE(n) = SO(n) \times \mathbb{R}^n$. In the special case of three-dimensional space $\mathbb{R}^3$, $SO(3)$ and $SE(3)$ refer to the rotation group, and the group that includes both translations and rotations, respectively.

In order to explain the Special Euclidean group $SE(3)$, one should consider a rigid body moving in free space. Assume that the inertial reference frame $\mathcal{I}$ is fixed in space and attached to Earth, and the body-fixed frame $\mathcal{B}$ is rigidly attached to the rigid body’s centre of mass at point $O'$, as shown in Fig. (2.1). The pose (position and orientation) of the rigid body, $(R, p)$, can be described by a homogeneous transformation matrix, $T$, that corresponds to the displacement from the inertial frame $\mathcal{I}$ to the body frame $\mathcal{B}$. Then, the special Euclidean group $SE(3)$ can be defined as the set of all possible rigid body transformations.
in three dimensions

\[ SE(3) = \{ T \in \mathbb{R}^{4 \times 4} | T = \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix}, R \in \text{SO}(3), p \in \mathbb{R}^3 \} \].

(2.18)

This representation of an element of \( SE(3) \) is commonly known as homogeneous coordinates. The group is a closed subset of \( \text{GA}(3) \) and a Lie group. The inverse element associated with \( T \) is

\[ T^{-1} = \begin{pmatrix} R^T & -R^T p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R^T & P \\ 0 & 1 \end{pmatrix}, \]

(2.19)

where \( P = -R^T p \) is the position of the vehicle in the body frame. Letting \( V = R^T v \) denote the translational velocity of the vehicle in \( \mathcal{B} \), the standard expressions for the kinematics in inertial frame and body frame are given by

\[
\begin{cases}
\dot{p} = v, \\
\dot{P} = S(\omega)P - V.
\end{cases}
\]

(2.20)

Using the kinematics equations of rigid body, the kinematics of \( SE(3) \) element \( T \) are given by

\[ \dot{T} = TA, \]

(2.21)
where the matrix \( A \in \mathbb{R}^{4 \times 4} \) denotes the body-fixed frame velocity of the system

\[
A = \begin{pmatrix}
S(\omega) & V \\
0 & 0
\end{pmatrix}.
\] (2.22)

### 2.4 Sensor Measurements

Most of the available estimators of attitude and pose, such as the nonlinear observers and Kalman filters, use the knowledge of system inputs and measured outputs to predict the system states. Once the system states are predicted, the estimated outputs are obtained and compared to the existing outputs of the real system to correct the next prediction. For a dynamical system in a form of a rigid-body in space, the inputs are the applied torques and forces and the outputs are attitude and position of the system.

The most common and widely available sensor package that is used for estimation in aerial vehicles, specially the small-scale UAVs, are Inertial Measurement Units (IMUs), which provide the measurements of angular velocity, the magnetic field intensity in the surrounding environment, and the linear acceleration of the rigid body. These units are generally very cheap and small, with a total weight less than 50 grams. Although the readings of these sensors are prone to measurement noise, their availability and small size have made them perfect choices for estimation and control purposes in Aerial and Aeronautics applications. For a comprehensive study on the available IMU sensors, readers are referred to [Chao et al., 2010].

Other commercially available sensors that are used in attitude estimation applications are Star trackers, Sun sensors and Earth horizon scanners. The Star trackers and Sun sensors determine the attitude of flying aircraft by matching an observed star field to an \textit{a priori known} star catalog [Yadid-Pecht et al., 1997]. Horizon scanners rely on visual spectrum, photoelectric and optical brightness of the sky to find the horizon position. This allows to determine the attitude of the flying vehicle in daytime conditions [Taylor et al., 2003]. Although the mentioned sensors are mostly highly effective, their cost and size is a barrier towards using them in low-cost estimation applications and their use remains a matter of
interest for industrial purposes. As evident from most scientific publications, the application of IMUs are far more common than visual tracking sensors. Therefore, this survey provides only a brief description of performance of the common IMUs used in attitude observers and filters design.

2.4.1 Rate Gyros

The Gyroscopes or Rate Gyros provide measurements of the angular velocity of the body-fixed frame $B$ relative to the inertial frame $I$, expressed in $B$. Let $\omega \in \mathbb{R}^3$ be the system’s actual angular velocity, and $\omega_y \in \mathbb{R}^3$ be the measured output given by

$$\omega_y = \omega + \omega_b + n_\omega, \quad (2.23)$$

where $n_\omega$ denotes the measurement noise and $\omega_b$ denotes the existing bias in readings. In practice, the gyro bias is constant or slowly time-varying. Therefore, in many applications the dynamics of this bias is taken as

$$\dot{\omega}_b \approx 0. \quad (2.24)$$

2.4.2 Magnetometers

The magnetometers measure the magnetic field in the surrounding environment. Let $m_I \in \mathbb{R}^3$ be the constant, known magnetic field in an area and let $m_B \in \mathbb{R}^3$ denote the body-expressed vector associated with $m_I$. The magnetometer reading $m_y$, expressed in $B$, is then given by

$$m_y = m_B + n_m, \quad (2.25)$$

$$= R^T m_I + n_m,$$

where $n_m$ denotes the magnetometer’s measurement noise.

In practice, however, the magnetic disturbance can be non-negligible, specially when the vehicle is operating indoors, or the on-board magnetometer is strapped down to a flying
vehicle with electric motors. Although most research works have not considered the variations of magnetic field in a typical short-distance flight, the work in [Vissiere et al., 2007] have investigated the magnetic field changes according to Maxwell’s laws in environments where GPS data is not available and proposed a Kalman-based attitude estimation technique.

### 2.4.3 Accelerometers

These sensors measure the instantaneous linear acceleration of the body frame with respect to the inertial frame minus the gravitational acceleration field. Let the vehicle’s linear acceleration, expressed in $\mathcal{I}$ be denoted as $\dot{v}$, and let $\mathbf{g} = g e_3$ be the vector of gravity that points to centre of Earth in a NED coordinate. Then the output of a set of accelerometers, denoted as $a$, is

\begin{equation}
    a = R^T (\dot{v} - \mathbf{g}) + n_a,
\end{equation}

with $n_a$ representing the measurement noise. In many cases where the rigid body is not having an accelerated motion, the norm of gravity field vector ($|\mathbf{g}| \approx 9.8$) dominates other terms and can be assumed to be the only measured value. In this case, the following approximation holds

\begin{equation}
    a \approx -R^T \mathbf{g}.
\end{equation}

This is an estimate of the $z$ axis in $\mathcal{I}$, that is measured and expressed in $\mathcal{B}$. It will be shown in section (4.5) that such low-frequency estimate of a fixed vector in local frame is a fundamental requirement in the design of some filters and observers, notably nonlinear complementary filters.

On the other hand, if the body acceleration is not neglected, (2.26) can be used to represent the translational dynamics of the rigid body, with $a$ as a known input. Filters and observers that take advantage of this feature and try to estimate the attitude, provided that measurements of velocity are available, are known as velocity-aided attitude estimators.
Chapter 3

Static Attitude Determination

3.1 Introduction

Static Attitude Determination is probably the oldest systematic trend of estimating the attitude of a flying vehicle with acceptable accuracy. This class of attitude estimation techniques takes advantage of the body vector observations to numerically determine the attitude without necessarily considering its kinematics. In this way, the attitude is merely regarded as a matrix (or quaternion) that transforms a vector \( x \in \mathbb{R}^3 \) in one frame to a vector \( y \in \mathbb{R}^3 \) in another frame and as a result, can be obtained by mathematical optimization techniques. Therefore, the information of the original system's dynamics is disregarded and attitude is found on an optimization basis.

The method, also known as deterministic solution, is characterized by finding the attitude estimate in a single point in time when observations of some known vectors in the inertial frame are available in the body frame. It has a simple estimation process with relatively small computational cost. However, this comes with a lower accuracy than the other methods that rely on additional information of the system dynamics.

Although the earliest deterministic solution techniques relied only on body vector measurements and literally put the system equations aside, the emergence of recursive techniques that considered system dynamics for propagation of states in late 1990s provided
more reliable methods with remarkable resemblance to the Kalman filters.

Major development of these methods, also known as batch attitude determination algorithms, started with the early optimization methods proposed to solve the Wahba’s problem [Wahba, 1965]. In this problem, with the assumption that two sets of simultaneously observed unit vectors \( \hat{V}_1, ..., \hat{V}_N \) and \( \hat{W}_1, ..., \hat{W}_N \) are respectively known in the inertial frame (i.e. the reference coordinate system) and the body frame, orthogonal matrix \( R \), representing the rotation matrix, is numerically found by minimizing the loss function

\[
L(R) = \frac{1}{2} \sum_{i=1}^{N} a_i |\hat{W}_i - R\hat{V}_i|^2,
\]  

(3.1)

where the \( a_i, i = 1, ..., N \) are non-negative weights and \( N \) is the number of measurements. By normalizing these weights to have \( \sum_{i=1}^{N} a_i = 1 \), it is straightforward to show that

\[
L(R) = 1 - \sum_{i=1}^{N} a_i \hat{W}_i^T R \hat{V}_i = 1 - \text{tr}(RB^T),
\]  

(3.2)

where \( \text{tr} \) denotes the trace operator and matrix \( B \) is defined as

\[
B = \sum_{i=1}^{N} a_i \hat{W}_i \hat{V}_i^T.
\]  

(3.3)

Equation (3.2) reduces the problem to finding the appropriate matrix \( R \) that maximizes the term \( \text{tr}(RB^T) \). It should be noted that Wahba’s problem addresses the attitude determination in a closed-form reconstruction manner. For generalizations of this problem readers are referred to [Shuster, 2006] and [Psiaki, 2010].

Earlier solutions to the Wahba’s least squares problem included a method using polar decomposition of the matrix \( B \) proposed in [Farrell and Stuelpnagel, 1966], and other algorithms in [Wessner, 1966], [Velman, 1966], and [Brock, 1966]. Introduction of the Q-method [Keat, 1977], along with these algorithms divided the efforts of finding the optimal matrix \( R_{opt} \) into two classes of solutions where the first class directly computes matrix \( R_{opt} \) and the second tries to find the optimal quaternion associated with the orientation matrix.

Structural differences in numerous proposed algorithms belonging to each class result in different computational costs and execution times.
3.2 TRIAD

The earliest attitude reconstruction method, known as TRIAD [Lerner, 1978], was designed to work with only two non-collinear unit reference vectors $\hat{V}_1, \hat{V}_2$ in inertial frame and their corresponding unit observation vectors $\hat{W}_1, \hat{W}_2$ in body frame to construct new orthonormal reference with bases $(\hat{r}_1, \hat{r}_2, \hat{r}_3)$ and observation vectors $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$:

\begin{align*}
\hat{r}_1 &= \hat{V}_1, \\
\hat{r}_2 &= (\hat{V}_1 \times \hat{V}_2)/|\hat{V}_1 \times \hat{V}_2|, \\
\hat{r}_3 &= (\hat{V}_1 \times (\hat{V}_1 \times \hat{V}_2))/|\hat{V}_1 \times \hat{V}_2| \\
\hat{b}_1 &= \hat{W}_1, \\
\hat{b}_2 &= (\hat{W}_1 \times \hat{W}_2)/|\hat{W}_1 \times \hat{W}_2|, \\
\hat{b}_3 &= (\hat{W}_1 \times (\hat{W}_1 \times \hat{W}_2))/|\hat{W}_1 \times \hat{W}_2|
\end{align*}

from which the attitude matrix can be simply found by

\[ R = \sum_{i=1}^{3} \hat{b}_i \hat{r}_i^T. \]

Although this method seems to be very simple, in practice it suffers from the fact that parts of measurements are discarded. Therefore, the optimal attitude reconstruction methods were given more attention since they do not eliminate any parts of the observed vectors.

3.3 SVD and FOAM

A descendant of the method proposed in [Farrell and Stuelpnagel, 1966], Singular Value Decomposition (SVD) method is a point-by-point algorithm to determine the optimal attitude matrix in the Wahba problem framework [Markley, 1988]. In this approach, similar to the other deterministic techniques, only sensor measurements are used and information about the system model is disregarded. The method consists of a direct “singular value” decomposition [Golub and Loan, 1983] of the matrix $B$ that gives

\[ B = USV^T, \]

where $U$ and $V$ are orthogonal matrices and $S$ is a singular value diagonal matrix of the form

\[ S = \text{diag}(s_1, s_2, s_3), \]
with the singular values $s_i, i = 1, 2, 3$, obeying the inequalities $s_1 \geq s_2 \geq s_3 \geq 0$. Proper orthogonal matrices of $U_+$ and $V_+$ along with the diagonal matrix $S'$ are defined as

$$U_+ = U[\text{diag}(1, 1, \det U)], \quad (3.9)$$

$$V_+ = V[\text{diag}(1, 1, \det V)], \quad (3.10)$$

$$S' = \text{diag}(s_1, s_2, s_3(\det U)(\det V)), \quad (3.11)$$

where $\det$ denotes the determinant of a matrix and $(\det U)(\det V) = \pm 1$. Then, the matrix $B$ is decomposed into the following form

$$B = U_+ S' V_+^T, \quad (3.12)$$

and the optimal matrix $R_{\text{opt}}$, which minimizes the cost function (3.1), is found to be

$$R_{\text{opt}} = U_+ V_+^T = U[\text{diag}(1, 1, (\det U)(\det V))]V^T. \quad (3.13)$$

Another version of this method, known as Fast Optimal Attitude Matrix (FOAM) [Markley, 1993], uses the properties of the matrix $B$ to rewrite the optimal rotation matrix (3.13) as

$$R_{\text{opt}} = [(\kappa + \|B\|^2)B + \lambda \text{adj} B^T - BB^T B] / \xi, \quad (3.14)$$

where adj denotes the adjoint matrix and

$$\|B\|^2 = s_1^2 + s_2^2 + s_3^2, \quad (3.15)$$

and the scalar coefficients $\kappa$, $\lambda$, and $\xi$ are defined as

$$\kappa = s_2 s_3 + s_3 s_2 + s_1 s_2,$$

$$\lambda = s_1 + s_2 + s_3, \quad (3.16)$$

$$\xi = (s_2 + s_3)(s_3 + s_1)(s_1 + s_2).$$

Since the values of these coefficients depend on the SVD, FOAM takes advantage of an iterative computation strategy to avoid finding $s_1, s_2, s_3$ and instead, directly computing the
Chapter 3. Static Attitude Determination

three scalar coefficients. The coefficients $\kappa$ and $\xi$ can be expressed in terms of $\lambda$ and $B$ as

$$
\kappa = \frac{1}{2}(\lambda^2 - \|B\|^2),
$$

(3.17)

$$
\xi = \kappa \lambda - \det B.
$$

Using (3.14) and the fact that $\lambda = \text{tr}(R_{\text{opt}}B^T)$, $\lambda$ can be found by solving the following equation

$$
(\lambda^2 - \|B\|^2)^2 - 8\lambda \det B - 4\|\text{adj } B\|^2 = 0.
$$

(3.18)

Once this equation is recursively solved to find $\lambda$, all the other scalar coefficients can be computed. These will determine the optimal rotation matrix from (3.14).

In comparison to other methods, the FOAM algorithm is significantly higher in speed and is shown to be the most robust algorithm among the other deterministic attitude estimation methods. It also does not have problems in dealing with the special case of a 180 degrees rotation [Markley and Mortari, 2000], [Markley and Mortari, 1999]. The SVD and FOAM do not adopt quaternion parameterization and work entirely with a rotation matrix. This enables them to work without the requirement of computing eigenvalues and eigenvectors and save some computational time.

### 3.4 Q-Method and the QUEST

Since the four component quaternion representation and the rotation matrix are related to each other by simple relations, it can be shown that a search for an optimal matrix $R_{\text{opt}}$ in Wahba’s problem leads to the computation of an optimal quaternion corresponding to that rotation matrix [Keat, 1977]. The method, known in literature as the Q-method, simplifies the previous optimization techniques by using the $4 \times 1$ quaternion vector instead of $3 \times 3$ rotation matrix.

Given the observation pairs of $(\hat{V}_i, \hat{W}_i)$ and the positive coefficients $a_i$, let us define the following $3 \times 3$ matrix, $3 \times 1$ vector $\varepsilon$ and scalar $\sigma$ as

$$
S := B + B^T = \sum_{i=1}^{N} a_i \hat{W}_i \hat{V}_i^T + \hat{V}_i \hat{W}_i^T,
$$

(3.19)
Chapter 3. Static Attitude Determination

\[ Z := \sum_{i=1}^{N} a_i \hat{W}_i \times \hat{V}_i, \quad (3.20) \]

\[ \sigma := \text{tr}(B) = \sum_{i=1}^{N} a_i \hat{V}_i^T \hat{W}_i. \quad (3.21) \]

Defining the \(4 \times 4\) symmetric matrix \(K\) as

\[
K = \begin{bmatrix}
S - \sigma I_3 & Z \\
Z^T & \sigma
\end{bmatrix},
\]

results in (3.1) to be written into the quadratic quaternion function

\[ 1 - L(R) = g(Q) = Q^T K Q. \quad (3.23) \]

It is then clear that the minimization of \(L(R)\) is equivalent to finding the maximum value of the function \(g(Q)\). It is also easy to show that the optimal quaternion that maximizes (3.23) is the eigenvector associated with the largest positive eigenvalue of the matrix \(K\). In other words,

\[ KQ_{opt} = \lambda_{max} Q_{opt}. \quad (3.24) \]

Substituting (3.24) into (3.23) and applying the quaternion norm constraint gives the following expression for the optimized loss function

\[ L(R_{opt}) = 1 - \lambda_{max}. \quad (3.25) \]

Finding the largest eigenvalue of the matrix \(K\) and its corresponding optimal quaternion have been the target of many works, including two versions of ESOQ algorithm (EStimation of Optimal Quaternion) [Mortari, 1997], [Mortari, 2000], [Markley and Mortari, 2000], and most importantly, Shuster’s algorithm QUEST (QUaternion ESTimation) presented in [Shuster and Oh, 1981]. The latter is a popular algorithm for finding the optimal quaternion \(Q_{opt}\) and since it does not require the minimization of a cost function, it has been a fast attitude determination technique for real-time applications.

The QUEST relies on applying the Cayley-Hamilton theorem on matrix \(S\), which yields

\[ S^3 = \text{tr}(S)S^2 - \text{tr(adj S)}S + \det(S)I_3. \quad (3.26) \]
This characteristic equation is used to find an optimal vector $Y_{opt}$ defined as

$$Y_{opt} = X/\gamma,$$  \hspace{1cm} (3.27)

where

$$X = (\alpha I + (\lambda_{max} - \frac{1}{2} \text{tr } S)S + S^2)Z,$$ \hspace{1cm} (3.28)

and

$$\gamma = (\lambda_{max} + \frac{1}{2} \text{tr } S)\alpha - \det S,$$ \hspace{1cm} (3.29)

with

$$\alpha = \lambda_{max}^2 - \frac{1}{2} \text{tr } S)^2 + \text{tr(adj } S).$$ \hspace{1cm} (3.30)

By relating the definition of vector $Y_{opt}$ to the optimal quaternion, one can find $Q_{opt}$ as follows

$$Q_{opt} = \frac{1}{\sqrt{\gamma^2 + |X|^2}} \begin{pmatrix} \gamma \\ X \end{pmatrix}.$$ \hspace{1cm} (3.31)

Although the definition of $Y_{opt}$ is similar to the definition of a Gibbs vector [Shuster, 1993], the author avoided explicitly using this vector in the definition of unit quaternion since the Gibbs vector becomes infinite when the rotation angle passes $\pm \pi$. It can be seen from (3.28-3.30) that in this method, the computation of the optimal quaternion requires the value of $\lambda_{max}$ to be known. This value is provided by solving the following forth-order characteristic equation

$$\lambda^4 - (a + b)\lambda^2 - c\lambda + (ab + \frac{c}{2} \text{tr } S - d),$$ \hspace{1cm} (3.32)

where

$$a = (\frac{1}{2} \text{tr } S)^2 - \text{tr(adj } S), \quad b = (\frac{1}{2} \text{tr } S)^2 + Z^T Z,$$

$$c = \det S + Z^T S Z, \quad d = Z^T S^2 Z.$$ \hspace{1cm} (3.33)

Numerical algorithms, such as Newton-Raphson method, can be applied on the characteristic (3.32) with $\lambda = 1$ as a starting point. Based on (3.25) and assuming that $\lambda_{max}$ is close to unity, its value is easily computed by a very few number of iterations (generally
just a single iteration) with desirable accuracy. However, because of the problems associated with the reliability of these numerical methods, it is commonly believed that QUEST is less robust than the other point-to-point methods.

The ESOQ (Estimator of the Optimal Quaternion) [Mortari, 1997] is another approach that has the same structure as QUEST for obtaining $\lambda_{\text{max}}$, but a different method for finding $Q_{\text{opt}}$. It is based on defining the matrix

$$H = K - \lambda_{\text{max}} I,$$

where in light of (3.24), it is clear that the optimal quaternion is orthogonal to all the columns of $H$. Now, by representing $K$ in terms of its eigenvectors and eigenvalues (among which one is $\lambda_{\text{max}}$) and using the orthonormality of eigenvectors, the following equation can be derived

$$\text{adj}(K - \lambda I) = \sum_{j=1}^{4} \left( \prod_{i \neq j} (\lambda_i - \lambda) \right) Q_j Q_j^T,$$

in which choosing $\lambda = \lambda_{\text{max}} = \lambda_1$ gives

$$\text{adj}(H) = (\lambda_2 - \lambda_{\text{max}})(\lambda_3 - \lambda_{\text{max}})(\lambda_4 - \lambda_{\text{max}})Q_{\text{opt}} Q_{\text{opt}}^T.$$

Hence, the optimal quaternion can be computed easily by normalizing any non-zero column of the matrix $\text{adj}(H)$. Defining $G$ as the symmetric $3 \times 3$ matrix obtained from deleting the $k$-th row and $k$-th column of $H$, and $g$ as the $3 \times 1$ column vector obtained after deleting the $k$-th element of the $k$-th column of $H$, the elements of the optimal quaternion can be found as

$$Q_{\text{opt}}_k = -c \det(G),$$

$$Q_{\text{opt}}_{1, ..., k-1, k+1, ..., 4} = c \text{adj}(G) g.$$

The coefficient $c$ is determined by taking into account the quaternion norm constraint.

In comparison to other static attitude reconstruction algorithms such as SVD, Q-method, FOAM, ESOQ, and ESOQ2, the QUEST method is shown to be faster with regards to the execution time [Markley and Mortari, 2000], [Cheng and Shuster, 2007]. The application
of all these point-to-point algorithms is attractive for some researchers since there is no need to initialize them. Moreover, their estimations do not rely on linearization of rigid body equations, something that may result in inaccuracies and instability in the estimation method.

### 3.5 Recursive QUEST Algorithms

Since the QUEST is a single time point method, which means simultaneous reading of at least two measured vectors are needed to compute the orientation at a single time step, it falls short of using the valuable data obtained in the previous measurements. Filter QUEST and REQUEST (REcursive QUEST) algorithms take advantage of this data [Shuster, 1989b], [Bar-Itzhack, 1996].

In Filter QUEST, the propagation of the matrix $B$ in each observation is formulated by

$$B_k = \sum_{i=1}^{k} \frac{1}{\sigma_i^2} \hat{W}_i \hat{V}_i^T,$$  \hspace{1cm} (3.38)

where $\sigma_i^2, i = 1, ..., N$ are variance parameters [Shuster, 1989a]. By assigning an initial value for $B_0$, which can be assumed to be zero, the following sequence can be derived for computation of $B_k$ in each step

$$B_k = B_{k-1} + \Delta B_k, \hspace{0.5cm} k = 1, ..., N,$$  \hspace{1cm} (3.39)

where

$$\Delta B_k = \frac{1}{\sigma_k^2} \hat{W}_k \hat{V}_k^T, \hspace{0.5cm} k = 1, ..., N.$$  \hspace{1cm} (3.40)

Since the attitude constantly changes with time, it is assumed that the rotation matrix $R$ is related to its previous value via the following relation

$$R_k = \Phi_{k-1} R_{k-1}, \hspace{0.5cm} k = 1, ..., N,$$  \hspace{1cm} (3.41)

with $\Phi_k$ being known as the state transition matrix for $R$. This attitude matrix update equation is derived from a sequentialization of the rigid body attitude dynamics of (2.4).
Since the matrix $B$ is directly constructed from vector observations, sequentialization of $\hat{W}_k$ using (3.41) gives an expression of $B_k$ that changes in the same way as $R_k$

$$B_{k|k-1} = \alpha \Phi_{k-1} B_{k-1|k-1},$$  \hspace{1cm} (3.42)

where $0 < \alpha < 1$ is a forgetting factor and is placed in the equation to give exponentially less weights to older measurements. Substituting (3.40) and (3.42) into (3.39) yields the Filter QUEST’s propagation phase for $B$

$$B_{k|k} = \alpha \Phi_{k-1} B_{k-1|k-1} + \frac{1}{\sigma_k^2} \hat{W}_k \hat{V}_k^T.$$  \hspace{1cm} (3.43)

The updated matrix $B_k$ is then used to construct an updated $K$ matrix and the rest of estimation technique continues in the same manner as the previously discussed QUEST algorithm.

While the Filter QUEST emphasizes on the propagation of the matrix $B_k$ using its previous value $B_{k-1}$ and the vector measurements in the $k$-th step (i.e. $\Delta B_k$), the REQUEST algorithm is based on the propagation of Davenport’s $K$ matrix with the following update process

$$K_{k|k} = \alpha \Phi_{k-1} K_{k-1|k-1} \Phi_{k-1}^T + \frac{1}{\sigma_k^2} K_k,$$  \hspace{1cm} (3.44)

with

$$K_k = \begin{bmatrix} W_k V_k^T + V_k W_k^T - (W_k^T V_k) I_{3\times3} & W_k \times V_k \\ (W_k \times V_k)^T & W_k^T V_k \end{bmatrix}.$$  \hspace{1cm} (3.45)

The quaternion transition matrix at step $k$ can be computed by assuming a constant angular velocity $\omega_k$ in the time increment $\Delta t_k$ between two consecutive time steps and is expressed by

$$\Phi_k = \exp(\Omega_k \Delta t_k),$$  \hspace{1cm} (3.46)

where $\Omega_k$ is a skew-symmetric matrix defined as

$$\Omega_k = \frac{1}{2} \begin{bmatrix} -S(\omega_k) & \omega_k \\ -\omega_k^T & 0 \end{bmatrix}.$$  \hspace{1cm} (3.47)

Once the $K_k$ is obtained in each step, the algorithm continues with a search for optimal quaternion $Q_{opt}$ in the same eigenvalue-based procedure of QUEST. Although the
procedure needed to compute the $K$ matrix in each step is computationally more expensive in comparison to the Filter QUEST, the two methods are mathematically equivalent [Shuster, 2009], [Crassidis et al., 2007]. For better estimations, smoothers have been provided for both methods. Optimal and adaptive versions of REQUEST are also presented in [Choukroun et al., 2004], [Choukroun, 2007].

Although the Filter QUEST is more computationally effective than the REQUEST, it has not been used in practice because of its high estimation error in comparison with the standard Kalman filter. Furthermore, it is not able to estimate other system parameters such as rate gyro bias, which is a typical problem with gyro readings in practice and results in an incorrect measurement of angular velocity. The search for solutions to this problem in such attitude estimation techniques led to the development of new sequential methods capable of handling models with arbitrary dynamics.

### 3.6 Extended QUEST

The original Wahba loss function (3.1) is only minimized when an optimal attitude matrix $A$ is found, given that the set of inertial vectors and body frame measurements are known. However, in real-time applications there are other important unknown system parameters that have to be estimated. This cannot be done by a conventional form of Wahba’s loss function unless those parameters are integrated into the optimization problem. Therefore, the original problem has to be extended to include additional parameters other than the attitude.

In [Markley, 1989] and [Markley, 1991], a new extended form of the Wahba loss function is proposed

$$ L(R) = \frac{1}{2} \sum_{i=1}^{N} a_i |\hat{W}_i - R\hat{V}|^2 + \frac{1}{2}(\beta - \beta^-)^T W(\beta - \beta^-), $$

where $\beta$ is the vector of additional parameters, $\beta^-$ is an *a priori* estimate of $\beta$, and $W$ is a symmetric positive-semidefinite matrix of weights. The algorithm is based on the
assumption that $Q_{opt}$ is available and is used to find the optimal error vector $\delta \beta_{opt}$. Then with the *a priori* estimate $\beta^-$, updated vector of additional parameters along with new state matrix $\Phi$ are found. The algorithm repeats the same process until $\delta \beta_{opt}$ becomes small enough.

Similar to this method, Extended QUEST algorithm developed in [Psiaki, 2000] is an example of the class of optimal attitude searching methods that are able to extend their estimation to other system parameters and states. The work is based on the development of an iterative QUEST-based method using the square-root information filtering techniques along with linearization of the system dynamics.

The proposed method includes a vector $x_k$, which holds the auxiliary filter states, and $w_k$, the process noise vector. These vectors along with the quaternion $Q_k$ are used to reformulate Wahba’s loss function as

$$ J = \frac{1}{2} \sum_{i=1}^{N} \frac{1}{\sigma_i^2} |\hat{W}_i| - R[Q_k] |\hat{V}_i| |^2 + \frac{1}{2} |R_{QQ(k-1)}(Q_{k-1} - \hat{Q}_{k-1})|^2 $$

$$ + \frac{1}{2} |R_{ww(k-1)} w_{k-1}|^2 + \frac{1}{2} R_{xQ(k-1)} (Q_{k-1} - \hat{Q}_{k-1}) \cdot R_{xx(k-1)} (x_{k-1} - \hat{x}_{k-1}) |^2, $$

where vectors $\hat{Q}_{k-1}$ and $\hat{x}_{k-1}$ are *a posteriori* or best estimates of quaternion $Q$ and auxiliary state vector $x$ at step $k-1$. The matrix $R_{ww(k-1)}$ is the square root of the *a priori* information matrix for random noise $w_{k-1}$, and $R_{xQ(k-1)}$, $R_{QQ(k-1)}$, $R_{xx(k-1)}$ are weight matrices in the loss function. Apart from the quaternion norm constraint, the optimization problem is constrained by the attitude kinematics and transition equations of the form

$$ Q_k = \Phi[t_k, t_{k-1}; Q_{k-1}, x_{k-1}, w_{k-1}] Q_{k-1}, $$

$$ x_k = h_x[t_k, t_{k-1}; Q_{k-1}, x_{k-1}, w_{k-1}], $$

where $h_x[.]$ is the transition function of the auxiliary states. Defining $\hat{Q}_k$ and $\hat{x}_k$ as *a priori* estimates of $Q_k$ and $x_k$, the propagation phase is given by

$$ \hat{Q}_k = \Phi[t_k, t_{k-1}; \hat{x}_{k-1}, 0] \hat{Q}_{k-1}, $$

$$ \hat{x}_k = h_x[t_k, t_{k-1}; \hat{Q}_{k-1}, \hat{x}_{k-1}, 0], $$
with the mean value of process noise being considered as zero. The method then proceeds
with a linearization of the dynamic model about \textit{a posteriori} estimates of the \((k - 1)\)-th
step and \textit{QR} factorization of the propagated information matrix with the upper triangular
matrix \(R\) (not to be mistaken by the rotation matrix) given by
\[
R = \begin{bmatrix}
\tilde{R}_{Q(k)} & 0 & 0 \\
\tilde{R}_{xQ(k)} & \tilde{R}_{xx(k)} & 0 \\
\tilde{R}_{wQ(k)} & \tilde{R}_{wx(k)} & \tilde{R}_{ww(k-1)}
\end{bmatrix},
\]
where the matrices \(\tilde{R}_{Q(k)}, \tilde{R}_{xQ(k)},\) and \(\tilde{R}_{ww(k-1)}\) are square matrices. Reformulating the loss
function (3.49) with the \textit{a priori} estimates yields
\[
J = \frac{1}{2} \sum_{i=1}^{N} \frac{1}{\sigma_{\tilde{W}(i)}^2} |\tilde{W}(i) - R[Q_k] \tilde{V}(i)|^2 + \frac{1}{2} |\tilde{R}_{Q(k)}(Q_k - \tilde{Q}_k)|^2 \\
+ \frac{1}{2} |\tilde{R}_{xQ(k)}(Q_k - \tilde{Q}_k) + \tilde{R}_{xx(k)}(x_k - \tilde{x}_k)|^2.
\]
(3.55)

Since the auxiliary state vector \(x_k\) has no effect on the quaternion norm constraint, the
loss function can be easily optimized with respect to this vector by setting \(\partial J/\partial x_k\) equal to
zero. This results in the following optimal value of the auxiliary state vector
\[
x_{(k)}_{opt} = \tilde{x}_k - \tilde{R}_{xx(k)}^{-1} \tilde{R}_{xQ(k)}(Q_k - \tilde{Q}_k).
\]
(3.56)

Substituting this optimal vector into the loss function (3.55) results in obtaining a
quaternion-based loss function that is more general than that of QUEST’s
\[
J = \frac{1}{2} Q_k^T \left( \sum_{i=1}^{N} \frac{1}{\sigma_{\tilde{W}(i)}^2} K_i Q_k \right) + \frac{1}{2} |\tilde{R}_{Q(k)}(Q_k - \tilde{Q}_k)|^2,
\]
subject to the quaternion norm constraint. This quadratic loss function is then minimized
and the solution gives \textit{a posteriori} estimate of the quaternion \(\hat{Q}_k\). By substituting this into
(3.56), one can obtain the best estimate for auxiliary state vector at step \(k\) as
\[
\hat{x}_k = \tilde{x}_k - \tilde{R}_{xx(k)}^{-1} \tilde{R}_{xQ(k)}(\hat{Q}_k - \tilde{Q}_k).
\]
(3.58)

The Extended QUEST algorithm is a generalization of the recursive QUEST meth-
ods and can be reduced to previous methods under appropriate conditions [Psiaki, 2000].
Its structure, however, is more complex than the commonly used Extended Kalman Filters (EKF), since it combines the standard Kalman filtering technique with quadratically-constrained programming methods. It shows robustness to large initial attitude uncertainties and in practice, is insensitive to changes in covariance tuning which gives it a broader convergence domain with regards to initial conditions. The ability of the Extended QUEST filter in working with poor or no initial estimates in each update stage is another advantage of this work.

### 3.7 Sequential Optimal Attitude Recursion Filter

Sequential Optimal Attitude Recursion (SOAR) filter is a newly proposed static attitude estimation algorithm [Christian and Lightsey, 2010]. In this method, like most of the Kalman-like recursive attitude estimators, *a priori* information is used to compute the state estimations in each step and are then propagated to the next phase.

In SOAR, the attitude error is described by the following small error quaternion

\[
\delta Q = Q \otimes \hat{Q}^{-1},
\]

(3.59)

where it is interpreted as the quaternion that rotates the best estimated attitude \( \hat{Q} \) to the true attitude \( Q \). Assuming small angles for this rotation gives

\[
\delta Q = \begin{bmatrix} \delta q_0 \\ \delta q \end{bmatrix} \approx \begin{bmatrix} 1 \\ \delta \theta/2 \end{bmatrix},
\]

(3.60)

where the three-dimensional \( \delta \theta = \delta \theta e_{\delta \theta} \) represents a small rotation with magnitude \( \delta \theta \) about the axis of rotation’s unit vector \( e_{\delta \theta} \). The complete state vector for this filter is defined as

\[
x = \begin{bmatrix} \theta \\ \beta \end{bmatrix},
\]

(3.61)

where \( \beta \) is the vector of unknown parameters.

The filter is developed based on the relation between the attitude profile matrix and the covariance matrix. According to Cramer-Rao inequality [Sorenson, 1980], the covariance
Chapter 3. Static Attitude Determination

Matrix $P_{xx}$ of state vector $x$ obeys

$$P_{xx} = E[(x - \hat{x})(x - \hat{x})^T] \geq (F_{xx})^{-1}, \quad (3.62)$$

where $F_{xx}$ is the Fisher information matrix and is defined as

$$F_{xx} = E \left[ \frac{\partial^2 J(x)}{\partial x \partial x} \right]. \quad (3.63)$$

The Fisher information matrix is equal to the inverse of covariance matrix under certain conditions. It also approaches $P_{xx}^{-1}$ as the number of measured vectors, $N$, becomes infinity [Shuster, 1989a]:

$$P_{xx}^{-1} = \lim_{N \to \infty} F_{xx}, \quad (3.64)$$

Assuming that the inverse of the covariance matrix is approximately equal to the Fisher matrix, the former can be expressed as a partitioned form of the latter given by

$$P^{-1} \approx F = \begin{bmatrix} F_{\theta \theta} & F_{\theta \beta} \\ F_{\beta \theta} & F_{\beta \beta} \end{bmatrix}. \quad (3.65)$$

On the other hand, the Wahba’s cost function can be evaluated at the system’s true attitude and be rewritten as a function of $\delta \theta$ to give the new loss function in the two following forms

$$J(\delta \theta) = 1 - \text{tr} \left[ (I_{3\times 3} - S(\delta \theta) + \frac{1}{2} S(\delta \theta)^2)RB^T \right]$$

$$= 1 - \dot{Q}^T K \dot{Q} + \frac{1}{2} \delta \theta^T F_{\theta \theta} \delta \theta. \quad (3.66)$$

From (3.66), the Fisher information matrix can be computed by taking the partial derivative of $J(\delta \theta)$ twice with respect to $\delta \theta$:

$$F_{\theta \theta} = \frac{\partial^2 J}{\partial \theta \partial \theta} = \text{tr}[RB^T]I_{3\times 3} - RB^T. \quad (3.67)$$

Here, assuming that a priori attitude estimate $\hat{R}$ and covariance matrix $P_{\theta \theta}^{-1} \approx F_{\theta \theta}$ are known, the attitude profile matrix $B$ can be computed as

$$B = \left[ \frac{1}{2} \text{tr}(F_{\theta \theta})I_{3\times 3} - F_{\theta \theta} \right] \hat{R}. \quad (3.68)$$
Equation (3.68) can be used to find the a priori estimate $B^-$ of the profile matrix, provided that a priori estimated quaternion $\hat{Q}^-$ and covariance matrix $P_{\hat{\theta}\theta}$ are available. The attitude matrix $\hat{R}$ can also be computed from the transformation of the a priori quaternion into the matrix form. The attitude profile matrix $B^-$ constructs an a priori Davenport’s $K$ matrix denoted as $K^-$. This matrix is added to the matrix $K^m$ constructed from the new measurements, and gives the modified Davenport matrix $K^+ = K^m + K^-$. A posteriori optimal attitude estimate is then computed by solving the familiar equation of

$$K^+ \hat{Q}^+ = \lambda \hat{Q}^+,$$

for $\hat{Q}^+ = (\hat{q}_0^+, \hat{q}^+_n)$, using any appropriate solution method to the normal Wahba problem. Once this optimal estimate is found, the update to the auxiliary parameters vector $\delta \beta$ is computed as

$$\delta \hat{\beta}^+ = \delta \beta^- - 2(F_{\hat{\beta}\theta})^{-1} F_{\hat{\beta}\theta} \Psi(\hat{Q}^-) \hat{Q}^+,$$

with

$$\Psi(\hat{Q}^-) = \begin{bmatrix} \hat{q}_0^+ I_3 & -\hat{q}^- \end{bmatrix}.$$

The remaining steps are to compute the updated covariance matrix and its corresponding propagation along with the propagated state vector to the next step. Details of this procedure can be found in [Christian and Lightsey, 2010].

The SOAR filter is a relatively computationally expensive method because it relies on the calculation of some inverse matrices during the recursion. However, since it is similar in structure to a Multiplicative Extended Kalman Filter (MEKF), it behaves in a similar way when the errors are small and linearization assumptions hold.

### 3.8 Simulations

This section deals with the performance of the QUEST algorithm under different measurement conditions that vary from completely ideal conditions to more realistic cases.
In the ideal case, all the measurements and filter inputs are assumed to be noise-free. This means that, for instance, for the case of magnetic field vector, the value of this vector in the surrounding environment is constant and known in the inertial frame and the value in the body frame is available without any added measurement noise. Also, it is assumed that there are no magnetic field generators (such as powerful electric motors) in the environment. For the case of accelerometers, the ideal measurement is defined as the measurement of the Earth gravity vector the body frame.

### 3.8.1 Parameters and Conditions

In this experiment, all the simulations are performed in MATLAB and Simulink. It is assumed that the real system works in a non-stop trend and the estimators obtain their inputs from IMU sensors on-board the flying vehicle.

The trajectory of the angular velocity, known in the body frame, is chosen as

$$\omega(t) = \begin{bmatrix} 0.2 \sin(0.1t), 0.5 \sin(0.5t), 0.4 \sin(0.8t + \frac{\pi}{3}) \end{bmatrix}^T \text{rad/s}. \quad (3.72)$$

The inertial-referenced vector of gravity $\mathbf{g}$ and the Earth magnetic field vector $\mathbf{m}$ in the environment are taken as

$$\mathbf{g} = \begin{bmatrix} 0, 0, -9.8 \end{bmatrix}^T \text{m/s}^2 \in I,$$

$$\mathbf{m} = \begin{bmatrix} 1, 0, 1 \end{bmatrix}^T \text{(normalized)} \in I.$$

All the measurement noises added to the ideal sensor outputs are taken as independent normally distributed random 3-dimensional vectors with zero mean. The variance of noise vectors are taken as follows

Accelerometer noise = 0.01 m/s$^2$

Magnetometer noise = 0.01 (normalized measure).

The measurements noise vectors were added to the ideal body-measured vector of each sensor output.
3.8.2 Error Definitions

In order to make a basis for comparison between various attitude estimators, several error definitions can be used. These include the Euclidean norm of the difference between the identity matrix and the error rotation matrix, the 4-element error quaternion between the actual and estimated quaternions, and the error between the actual and estimated Euler angles.

The Euclidean norm is useful in comparisons where some of the matrices involved in the estimation technique do not belong to the Special orthogonal group $SO(3)$ and their norm is required to converge to 1. In this way, the error is defined as

$$\text{Error norm} = \|I_{3\times3} - \tilde{R}\|, \quad (3.73)$$

where $\tilde{R} = R^T\hat{R}$, with $R$ being the rotation matrix representing the actual orientation of the rigid body and $\hat{R}$ being the estimated rotation matrix. In MATLAB, the Euclidean norm function $\text{norm}(A)$ returns the largest singular value of the matrix $A$.

In case of estimators/observers in which the quaternion representation is used, an error quaternion vector of the form

$$\tilde{Q} = Q \otimes \dot{Q}^{-1} \quad \text{or} \quad \tilde{Q} = \dot{Q} \otimes Q^{-1}, \quad (3.74)$$

can be used. In this case, the desired error quaternion should converge to $\tilde{Q} = (\tilde{q}_0, \tilde{q})^T = (1, 0)$.

Another error definition consists of discrepancy between the actual and estimated Euler angles. The error Euler angles $\tilde{\phi}$, $\tilde{\theta}$ and $\tilde{\psi}$ are derived from the error rotation matrix $\tilde{R}$.

3.8.3 QUEST

As discussed, the QUEST algorithm and many other subsequent static attitude estimation techniques rely on the availability of at least two pairs of vector observations $(\hat{V}_i, \hat{W}_i), i = 1, \ldots, N$. In the case where only two vector measurements are used, the largest eigenvalue
of matrix $K$ can be simply computed as

$$\lambda_{max} = \sqrt{a_1^2 + 2a_1a_2 \cos(\theta_V - \theta_W) + a_2^2},$$

(3.75)

where

$$\cos(\theta_V - \theta_W) = (\hat{V}_1 \cdot \hat{V}_2)(\hat{W}_1 \cdot \hat{W}_2) + |\hat{V}_1 \times \hat{V}_2||\hat{W}_1 \times \hat{W}_2|.$$

(3.76)

The optimal unit quaternion of (3.31) associated with the optimal rotation matrix in (3.1) was found with the same weights given to both measurements

$$a_i = 0.5, \text{ for } i = 1, 2.$$

![Figure 3.1: Error Euler angles of the QUEST algorithm with ideal noise-free IMU sensor measurements.](image)

The results of simulations are shown in Figures (3.1-3.2). Figure (3.1) shows the error Euler angles versus time and indicates the satisfactory performance of the QUEST under ideal conditions. It can be seen that the algorithm is successful in obtaining optimized quaternions/rotation matrices from the very beginning. This is due to the fact that QUEST does not rely on attitude kinematics and the optimization process is performed in each time step.

Figure (3.2) shows the effect of measurement noise. In this case, although the mean error remains small, the ultimate performance shows sensitivity to the noise magnitude.
This is due to the fact that some parameters used in this algorithm (such as $S$ and $Z$) are obtained from the vector multiplications of noisy vectors to each other. The same happens for the computation of $\lambda_{max}$ using (3.75).

### 3.8.4 Filter QUEST

The recursive QUEST algorithms include the propagation of either $B$ or $K$ through the rotational dynamic equations. As discussed before, the two known recursive QUEST algorithms Filter QUEST and REQUEST are mathematically equivalent. However, the REQUEST is more computationally demanding since it requires the propagation of the matrix $K$.

Figure (3.3) shows the performance of the Filter QUEST algorithm under noisy measurements. In case of ideal noise-free measurements, the filter shows the same performance of QUEST.
3.9 Discussion

The static attitude reconstruction algorithms, specially the various QUEST-based methods discussed in this thesis, are aimed to find the optimal attitude given a set of vector observations at a given time. Although the initial solutions to Wahba problem did not consider the attitude kinematics, recursive forms of QUEST algorithm try to propagate the estimated attitude and covariance matrix over time, which results in better estimations.

In that sense, the recursive static algorithms are similar to Kalman filters and previous studies have confirmed this similarity in the performance of the two distinct filters. While the propagation phase is a key part in the process of both filters, the most important difference resides in the way each method finds the optimal attitude.

The recursive methods, such as the Filter QUEST presented in the simulations, provide better estimations with regards to measurement noise. The propagation of profile matrix when observations are not simultaneously obtained, filters the available data and enhances the algorithm performance.

In the practical sense, arbitrary number of observation pairs used in the process can be obtained from star tracking sensors. The star sensors are widely used in spacecraft attitude estimation and provide the body-measured coordinates of a known vector in the local frame pointing to a known star whose coordinates do not change with time. Here, the
only degrading factor is the sensor measurement noise. For small-scale UAVs, however, the
IMU set of magnetometers along with accelerometers might not result in good estimations
by using this method due to the linear acceleration of the rigid body.

In comparison to other attitude estimation algorithms, static determination methods
are more computationally demanding due to their optimization-based nature. This has
made the study of modern algorithms, such as Kalman filters and attitude observers, more
appealing to the scientific community.
Chapter 4

Dynamic Attitude Filtering and Estimation

4.1 Introduction

The need for fast attitude estimation techniques that are able to adapt themselves to changes in system states and give more reliable estimations led to the development of dynamic filtering methods and observers. In these techniques, not only vectorial observations are used, but the system dynamics are exploited in the design strategy to capture and predict the behavior of the system.

As discussed in the previous chapter, the main disadvantage of the static attitude estimation methods is their inability to take the full system dynamics into account. In these methods, the attitude is estimated regardless of the nonlinear structure of system and only vectorial measurements were used. However, dynamic methods incorporate the system equations in estimation process and therefore, have the potential of providing better estimation results.

Probably the most popular dynamic attitude estimators are Kalman filters and its variants such as the Extended Kalman Filter (EKF). These filters have the advantage of being specifically designed to work under noisy measurements conditions. The EKFs are nonlin-
ear versions of the original Kalman filter and have been applied to many aerospace applications during the last decades. A description of this type of filters is presented in section (4.2) along with their earliest applications in the forms of Multiplicative Extended Kalman Filters and Additive Extended Kalman Filters. More modern Kalman filtering techniques such as Unscented Kalman Filtering and Invariant Kalman Filters will also be covered in section (4.2). These will help the reader to become familiar with the use of these techniques for the attitude estimation problem.

The second main approach in dynamic attitude filtering is the complementary filtering. Along with Kalman filters, the complementary filters have been successful in providing reliable attitude estimations under real-time conditions. As their name states, these filters use various sensor measurements to “complement” each other in obtaining a better estimation. Section (4.3) will be dedicated to a brief presentation of such filters.

The nonlinear observers are another major group of dynamic attitude estimators. In these estimation tools, ideal vectorial, position and velocity measurements are assumed.

The study of nonlinear observers takes a considerable portion of this survey. These attitude estimation tools are relatively new and, to some extent, were neglected in the latest attitude estimation survey in [Crassidis et al., 2007]. While it is generally considered that these tools are in their infancy, many different types of nonlinear observers have started to gain huge attention from the scientific community. In section (4.4), the latest theoretical studies on the application of special properties of the rotational dynamics in designing nonlinear observers will be discussed. Section (4.5) will discuss the nonlinear complementary filters and their importance in the evolution of nonlinear observers. In sections (4.6) and (4.7), special solutions to some of the problems associated with real-time applications of such observers are presented and the inclusion of system’s position and velocity is discussed. This paves the way for section (4.8), in which the aim is to observe the Pose (Position and Orientation) of rigid body systems by simultaneously using both rotational and translational dynamics of the system. The chapter ends with simulations performed for some of the dynamic methods and a discussion on their performance under different
4.2 Extended Kalman Filters

The Kalman Filtering (KF) techniques in aerospace applications have been the subject of extensive research in the past few decades and the field has experienced huge progress from its original development in 1960’s [Grewal and Andrews, 2010]. From the original filter designed for linear systems to the forms compatible for nonlinear systems, the Kalman filters have been adapted to many estimation problems and have been applied to various actual missions.

While the basic Kalman filter was developed for linear systems, the technique can also be used for nonlinear systems provided that a linearization of the system in each step is performed about the best estimate state obtained in the previous step. The approach is called Extended Kalman Filtering (EKF) and has been the most widely used technique in real-time attitude estimation [Crassidis et al., 2007].

During the last decades, EKF has been successfully applied to many Aeronautics applications [Toda et al., 1969b], [Farrell, 1967], [Garcfa-Velo and Walker, 1997]. The existing EKF approaches for the attitude determination differ in the parametrization of attitude.

The earliest applications used the Euler angles parameterization [Farrell, 1970]. However, since three-dimensional parameterizations of the attitude face topological obstructions, Euler angles and other similar parameterizations cannot be both global and non-singular [Stuelpnagel, 1964b]. However, the four-dimensional quaternion parametrization represents the attitude with only one redundant parameter. This representation is global and can be easily transformed into a rotation matrix. Another advantage of the quaternion is that system kinematics equation can be expressed in a bilinear form of the attitude and the angular velocity vector. These advantages have all contributed to the popularity of quaternion-based Kalman filters among researchers [Markley, 2003], [Bar-Itzhack and Oshman, 1985].

In order to maintain orthogonality in the estimated attitude and avoid singularity of the
covariance matrix, the quaternion norm constraint should be taken into account by normalizing the quaternion estimate vector. It is known that in case of a slowly time-varying attitude, normalization can result in faster convergence and when attitude changes fast, it is necessary to do this in order to avoid divergence [Deutschmann et al., 1992]. The most simple quaternion normalization consists in dividing the estimated quaternion after each update stage by its Euclidean norm [Bar-Itzhack, 1971]. Although this act, known as the “brute force” normalization, is outside the filter’s algorithm and its estimation process, it is shown that normalizing the estimated quaternion does not affect the propagation of the covariance matrix. In [Bar-Itzhack et al., 1991], several quaternion normalization algorithms are compared and new methods are introduced. The authors in [Lefferts et al., 1982] discussed the problem with three different approaches.

The author in [Shuster, 2003a], [Shuster, 2003b] examined both cases of constrained and unconstrained quaternion estimations. It was shown, with examples, that the unconstrained quaternion estimates may lead to different estimation results that can depend on the choice of measurement sensitivity matrix. Therefore, it is advised to use constrained techniques to avoid problems such as obtaining singular inverse covariance matrices. The authors in [Zanetti et al., 2009] have recently showed that constrained estimation is mathematically equivalent to the unconstrained estimation when a brute force normalization is applied.

In general, the Kalman filtering approach has shown better performance than many other attitude estimation approaches. In [Marques et al., 2000], a comparison between the dynamic EKF and the deterministic method of SVD can be found. The experiment, which uses Sun sensor measurements, shows the superiority of EKF over SVD with regards to the accuracy of estimations.

The Basics of Extended Kalman Filtering

The Extended Kalman Filter simply relies on the model linearization, and uses the obtained Jacobian matrices in the Kalman filter [Jazwinski, 1970]. Consider the following nonlinear
discrete-time system

\[
x_k = f(x_{k-1}) + w_{k-1},
\]
\[
y_k = h(x_k) + v_k,
\]

where \(x_k\) is the system state with an initial value of \(x_0\), \(f(.)\) is the process nonlinear vector function, and \(h(.)\) is the observation nonlinear vector function. \(w_k\) is the process noise vector and \(v_k\) is the vector of measurement noise. The covariance matrices associated with process and measurement noises are \(Q_k\) and \(R_k\), respectively. Both noises are assumed to have zero means and are uncorrelated with initial state. \(y_k\) is the measurements output vector that is related to state vector through the function \(h(x)\).

It is assumed that initial estimates of the state vector \(x_0\) and its mean, \(\mu_0\), are available. From these, the initial optimal estimate, \(x_{a0}\), and error covariance, \(P_{0}\), are derived. Now assuming that an optimal estimate for state \(x_{a(k-1)}\) along with covariance \(P_{k-1}\) at step \(k - 1\) are available, it is desired to make a prediction for the state in step \(k\) and then correct this prediction using the data taken from measurements.

The prediction phase (or model forecast step) is provided by

\[
x_k^f \approx f(x_{a(k-1)})
\]
\[
P_k^f = J_f(x_{a(k-1)})P_{k-1}J_f^T(x_{a(k-1)}) + Q_{k-1}
\]

where \(J_f(.)\) is the Jacobian of \(f(.)\) derived in the process of linear approximation of this nonlinear function around the best estimate \(x_{a(k-1)}\), and is defined as

\[
J_f = \left| \frac{\partial f(x)}{\partial x} \right|_{x_{a(k-1)}}.
\]

This forecast is done without any knowledge about the true system outputs at step \(k\) and therefore, the measurements data should be assimilated with the prediction in order to give the best unbiased estimate. A way to approximate this best estimate is to assume that it’s a linear combination of predicted values \(x_k\) and outputs \(y_k\), that is

\[
x_k^a = a + K_k y_k,
\]
where $a$ can be computed using the unbiasedness condition and is given by

$$a = x^f_k - K_k h(x^f_k).$$

(4.5)

Hence, the next step, which is known as the correction phase, is formulated by

$$x^a_k \approx x^f_k + K_k(y_k - h(x^f_k)),$$

$$K_k = P^f_k J_h^T(x^f_k)(J_h(x^f_k)P^f_k J_h^T(x^f_k) + R_k)^{-1},$$

$$P_k = (I - K_k J_h(x^f_k))P^f_k,$$

(4.6)

where $K_k$ is the Kalman gain matrix, and $J_h(x^f_k)$ is the Jacobian of the nonlinear function $h(.)$ in point $x^f_k$ defined by

$$J_h = \frac{\partial h(x)}{\partial x} \bigg|_{x^f_k}.$$  

(4.7)

When the new state estimate and covariance matrix at step $k$ are obtained, they will then be used as available a priori data for the next step.

It can be seen that analytical propagation of state distribution was made possible in EKF when a first-order linearization is performed on the nonlinear system equations. In this process, only the first two terms of the truncated Taylor expansion series were used. There are also some highly accurate second-order versions of this method, known as second-order Extended Kalman Filter (EKF2) [Vathsal, 1987], [Roth and Gustafsson, 2011]. These methods may not be preferable in practice where execution time and possibly high dimensions of system pose a barrier towards the usage of algorithms with high computational complexity. However, advantages of these methods are their usefulness in systems with extreme nonlinearities and their ability to increase the convergence domain. Global convergence of EKF is not guaranteed and high errors in initial estimates may lead to divergence or poor estimation [Song and Grizzle, 1995], [Crassidis et al., 2007], [Psiaki, 2005].

**Multiplicative EKF**

The Multiplicative Extended Kalman Filtering (MEKF) approaches [Lefferts et al., 1982], [Markley, 2003] rely on the quaternion attitude representation. As discussed before, the
unit quaternion is a form of attitude representation with widespread application in aerospace and control engineering and it is useful to develop an EKF which gives quaternion estimates. However, the geometry of the quaternion space needs to avoid using linear terms like $Q - \hat{Q}$, in which the quaternion norm constraint is not respected during the standard linear correction phase. Therefore, studies have been done in order to change the EKF to a compatible form for quaternion use where the error is defined as a quaternion multiplication between unit quaternion and its estimate. The MEKF represents the attitude error as the quaternion product

$$\delta Q(a) = Q \otimes \hat{Q}^{-1},$$

where $\hat{Q}$ is the estimated unit quaternion, and $a$ is a three-dimensional vector by which the $\delta Q(a)$ is parameterized. Assuming that rotations are small, this error quaternion becomes

$$\delta Q(a) = \bar{1} + \frac{1}{2} \bar{a} + H.O.T,$$

where $\bar{1}$ is the identity quaternion and $\bar{a}$ is the quaternion with $a$ as its vector part and 0 as the scalar part (i.e. $\bar{a} = (0, a)$). Based on the choice of $a$, there exist several ways of parameterizing $\delta Q(a)$, which are discussed in detail in [Markley, 2003]. An alternative method [Gray, 2001] defines $a$ as the attitude error in the inertial frame and thus reverses the order of multiplication in (4.8).

The method is based on the idea of having an unconstrained estimate of $a$ and using the globally nonsingular $\hat{Q}$ to represent attitude in such a way that the estimation for true attitude quaternion be $\delta Q(\hat{a}) \otimes \hat{Q}$. In this sense, the estimate of $a$ is chosen as $\hat{a} := E[a]$, and normalized $\hat{Q}$ is then chosen in a way that vector estimate $\hat{a}$ becomes identically zero. Hence according to (4.9), $\delta Q(0)$ is an identity quaternion and $\hat{Q}$ becomes the estimate of true attitude quaternion [Markley, 2004b].

Let $\hat{\omega}$ be the angular velocity used in the kinematics equation of the estimated system, the dynamics of MEKF attitude error in (4.8) can be computed as

$$\delta \dot{Q} = \frac{1}{2} \begin{pmatrix} 0 \\ \omega \end{pmatrix} \otimes \delta Q - \frac{1}{2} \delta Q \otimes \begin{pmatrix} 0 \\ \hat{\omega} \end{pmatrix},$$

(4.10)
If (4.10) is substituted in the derivative of (4.9) and then linearized, it gives the derivative of $a$ as

$$\dot{a} = \omega_{\text{meas}} - \dot{\omega} - \frac{1}{2}S(\omega_{\text{meas}} + \dot{\omega})a + n_\omega,$$

(4.11)

where the gyro noise is denoted as $n_\omega$. In the measurement phase, knowing the vector $v_I$ in the inertial frame and it’s corresponding observation $v_B$ in the body frame, one can predict the latter using the estimated attitude

$$\dot{v}_B = R^T(\hat{\mathbf{Q}})v_I.$$

(4.12)

Recalling the correction phase in an EKF, the derivative of the attitude error estimate $\dot{a}$ can be computed with $a = 0$:

$$\dot{a} = \omega_{\text{meas}} - \dot{\omega} + \sigma_z^{-2}P_aS(v_B)\hat{v}_B,$$

(4.13)

where $\sigma_z^2$ is the covariance of the zero-mean Gaussian measurement noise $n_z$. $P_a$ denotes the covariance of $a$ and is propagated using the approximation $\dot{\omega} \approx \omega_{\text{meas}}$, neglecting higher order terms

$$\dot{P}_a = P_aS(\omega_{\text{meas}}) - S(\omega_{\text{meas}})P_a + \sigma_z^2P_a + \sigma_z^{-2}P_aS(\hat{v}_B)^2P_a.$$

(4.14)

Using (4.13), provided that the derivative of the estimated error vector is identically zero, the estimation equation for the angular velocity can also be found as

$$\dot{\omega} = \omega_{\text{meas}} + \sigma_z^{-2}P_aS(v_B)\hat{v}_B.$$

(4.15)

One of the advantages of MEKF method is its ability to preserve the quaternion unit norm, since the product of quaternion estimation error and the $a$ priori quaternion estimate is a quaternion itself. Extensions of MEKF can be found in [Bonnabel et al., 2009b] and [Martin and Salaun, 2010], where the velocity-aided navigation system uses the GPS data to extend the state vector to include the rigid body position and velocity. These so called “Attitude and Heading Reference Systems” (AHRS) methods mostly use extended versions of MEKF to take advantage of inertial sensors together with position and velocity.
Chapter 4. Dynamic Attitude Filtering and Estimation

sensors, [Bijker and Steyn, 2008], [Bauer and Bokor, 2011]. This method has been repeatedly used in many aerospace missions including the Space Precision Attitude Reference System (SPARS) in 1969 [Toda et al., 1969b], [Toda et al., 1969a], and since then has been a widely used method in practice, [Ernandes et al., 2007], [Bijker and Steyn, 2008].

Additive EKF

Additive Extended Kalman Filter (AEKF) [Bar-Itzhack and Oshman, 1985] is designed based on the classic EKF measurement update stage of adding a correction term to the \textit{a posteriori} estimate and then normalizing this estimate to preserve the unit-norm property of the quaternion. The method uses the classical EKF error between the true quaternion and its estimate

\[
\Delta Q_k = Q_k - \hat{Q}_k,
\]

with its time derivative given by

\[
\dot{\Delta Q}_k = F_Q \Delta Q_k + G_Q \Omega \omega,
\]

with

\[
F_Q = \frac{1}{2} \begin{bmatrix}
-S(\omega) & \omega \\
-\omega^T & 0
\end{bmatrix}, \quad G_Q = \frac{1}{2} \begin{bmatrix}
\hat{q}_0 I + S(\hat{q}) \\
-\hat{q}^T
\end{bmatrix}.
\]

Unlike the MEKF where the attitude error vector \(a\) is used to parameterize the attitude, AEKF uses the quaternion representation to reconstruct \(R(Q)\). However, if the obtained quaternion does not have a unit norm, an alternative solution is to use the normalized quaternion \(Q/|Q|\) to compute the rotation matrix

\[
R_n(Q) = R(Q/|Q|) = |Q|^{-2} R(Q).
\]

The rotation matrix \(R_n(Q)\), which is known as ray representation AEKF, works only with quaternion and needs no other parameterizations as in MEKF, hence it is theoretically simpler than the multiplicative approach. This representation has been used as an attitude estimator in ALEXIS and CAPER spacecraft [Psiaki et al., 1997], [Psiaki et al., 2002].
A presentation of different methods for normalizing quaternion estimates in AEKF and MEKF can be found in [Deutschmann et al., 1992]. Markley has compared these two methods and has provided an answer to the question of whether to use AEKF or MEKF, in favor of the latter [Markley, 2004b], [Markley, 2004a]. The comparison argues that while the AEKF appears to have a resemblance to linear Kalman filters, since the process noise and any dynamic parameters enter the quaternion kinematics equation multiplicatively rather than additively, as per (4.17), the resemblance is deceiving. Also, MEKF is computationally more efficient than the AEKF.

Both of the mentioned approaches use linearization of the rigid body nonlinear dynamics to make use of the Extended Kalman filtering. While this approach can result in good filter performance provided that the linearized model is a good approximation of the nonlinear model, it may lead to poor performance or divergence in the estimation if the approximation is not adequate. This problem along with the problems associated with existence of the Jacobian matrices and their calculation have resulted in the development of new methods that bypass the conventional EKF linearization.

### 4.2.1 Unscented Kalman Filters

The Unscented Kalman Filter (UKF) has been proposed as an alternative to the widely adopted EKF [Julier and Uhlmann, 2004], [Crassidis et al., 2007], [Crassidis, 2006]. The method is based on parameterizing the state estimate in Euclidean spaces and then numerically approximating its mean and covariance. In fact, the main assumption is that approximating a Gaussian distribution is easier than approximating an arbitrary nonlinear model. The motivation behind the development of the UKF is mainly related to the inaccuracies of the EKF that are inherited from its linearization process. As discussed before, those methods that try to overcome the inaccuracy and possible divergence problems by increasing the order of the filter to involve second-order terms, result in strong computational burdens.

The method is a deterministic sampling approach and approximates the optimal gain and prediction terms in the Kalman filter framework. Instead of approximating the non-
linear system in a linearization process, the UKF uses the true nonlinear model and rather approximates the distribution of the state random variable. It considers the nonlinear system equations of

\[ x_{k+1} = f(x_k, u_k) + w_k \]
\[ y_k = h(x_k, u_k) + v_k \]

(4.20)

where \( u \) is a known input, \( w_k \) and \( v_k \) are system and measurement noises, respectively. The algorithm starts with generating a set of points called *sigma points*. These points are propagated through the true nonlinear system equation. Assuming an \( N_x \times N_x \) state covariance matrix \( P \), the columns of matrices \( \pm \sqrt{N_x}P \) can be used to generate a symmetric set of \( 2N_x \) points with desired mean and covariance [Julier and Uhlmann, 2004]. Hence, with an *a priori* given matrix \( P^*_{k} \), process noise covariance matrix \( Q_k \), and the best estimate \( \hat{x}^+_k \) of the state vector in the \( k \)-th step, the set of sigma points is computed as

\[ x^{(i)}_k = \hat{x}^+_k \pm \gamma \sqrt{P^*_{k} + Q_k}, \text{ for } i = 1, \ldots, 2N_x + 1 \]

(4.21)

where \( \gamma \) is a scaling parameter affecting the spread of the sigma points [Crassidis, 2006]. Assuming no process noise, each of the sigma points is then transformed via (4.20) to provide

\[ x^{(i)}_{k+1} = f(x^{(i)}_k, u_k), \]

(4.22)

with the predicted weighted average \( \hat{\mu}_{k+1} \) and covariance \( \hat{K}_{k+1} \) defined by

\[ \hat{\mu}_{k+1} = \sum_{i=0}^{p} W^{(i)} x^{(i)}_{k+1}, \]
\[ \hat{K}_{k+1} = \sum_{i=0}^{p} W^{(i)} \{ x^{(i)}_{k+1} - \hat{\mu}_{k+1} \} \{ x^{(i)}_{k+1} - \hat{\mu}_{k+1} \}^T, \]

(4.23)

with \( W^{(i)} \) being a weighting matrix and \( p = 2N_x + 1 \). The next step is to instantiate the predicted points through the observation model (4.20). This gives a set of outputs \( y^{(i)}_{k+1} \) that
is then used to find the predicted observation $y_{k+1}$ and innovation covariance $\hat{S}_{k+1}$:

$$y_{k+1}^{(i)} = h(x_{k+1}^{(i)}, u_{k+1}),$$

$$\hat{y}_{k+1} = \sum_{i=0}^{p} W^{(i)} y_{k+1}^{(i)}, \quad (4.24)$$

$$\hat{S}_{k+1} = \sum_{i=0}^{p} W^{(i)} [\hat{y}_{k+1}^{(i)} - \hat{\mu}_{k+1}] [\hat{y}_{k+1}^{(i)} - \hat{\mu}_{k+1}]^{T}. \quad (4.25)$$

The cross covariance matrix of the internal states and the outputs is also determined by

$$\hat{K}_{xy}^{k+1} = \sum_{i=0}^{p} W^{(i)} [x_{k+1}^{(i)} - \hat{\mu}_{k+1}] [y_{k+1}^{(i)} - \hat{y}_{k+1}]^{T}. \quad (4.26)$$

Using these obtained equations, the method then proceeds with an update phase as in a normal Kalman filter. Therefore, the Kalman gain matrix can be written as

$$W_{k+1} = \hat{K}_{xy}^{k+1} \hat{S}_{k+1}^{-1}, \quad (4.27)$$

by which the state update can be found using the linear EKF rule of

$$\hat{x}_{k+1}^{s} = \hat{\mu}_{k+1} + W_{k+1} (y_{k+1} - \hat{y}_{k+1}), \quad (4.28)$$

and the updated covariance matrix is given by

$$P_{k+1}^{+} = \hat{K}_{k+1} - W_{k+1} \hat{S}_{k+1} W_{k+1}^{T}. \quad (4.29)$$

The UKF is preferable to the traditional EKF for its features like improved accuracy, robustness and simplicity of implementation, and they all come with computational complexity comparable to that of EKF [Norgaard et al., 2000], [Wan et al., 2004]. Moreover, unlike the EKF, unscented filtering does not need to compute Jacobian matrices and is suitable for systems where the computation of the Jacobian matrix is hard [Wan et al., 2000].

Since the UKF uses a computation process based on addition to construct the predicted estimate, it shares with the EKF the same problem of the unit quaternion norm constraint. Authors in [Crassidis and Markley, 2003] addressed this problem by using a Generalized Rodrigues Parameters (GRPs) error vector to propagate and update a nominal quaternion.
They show that the UKF algorithm has better performance in comparison to the conventional EKF especially when initial estimates have large errors.

Another approach introduced in [VanDyke et al., 2004] takes advantage of the system’s rotational dynamics equations to estimate both the attitude and the angular velocity. The state vector used in this unscented filter is defined as

\[ x = \begin{bmatrix} \delta q \\ \omega \end{bmatrix}, \quad (4.29) \]

where \( \delta q \) is the vector part of the error quaternion defined by

\[ \delta Q = Q \otimes \hat{Q}^{-1}. \quad (4.30) \]

The state in step \( k \) is initialized by assuming that the error quaternion is zero

\[ \hat{x}_k = [\delta q_k^T \omega_k^T]^T = [0 0 0 \omega_k^T]^T. \quad (4.31) \]

As in (4.21), \( \hat{x}_k = \hat{x}_k^c \) is used to compute the sigma points

\[ \hat{x}_k^{(i)} = [\delta q_k^{(i)} \omega_k^{(i)}]^T \quad \text{for} \quad i = 1, ..., 2N_x + 1. \quad (4.32) \]

The \( \delta q_k^{(i)} \) error vector in each sigma point is then transformed to its associated quaternion using the unit norm constraint

\[ \delta Q_k^{(i)} = [\delta q_k^{(i)}]^T \sqrt{1 - \delta q_k^{(i)} \delta q_k^{(i)}^T} \quad \text{for} \quad i = 1, ..., 2N_x + 1. \quad (4.33) \]

Substituting these quaternions into (4.30) gives the four-element sigma point quaternions. The quaternion and angular velocity sigma points are then propagated using system dynamic equation

\[ \begin{cases} \dot{Q} = \frac{1}{2} Q \otimes \omega, \\ \dot{\omega} = I_b^{-1}(-\omega \times I_b \omega + u), \end{cases} \quad (4.34) \]

to give \( Q_{k+1}^{(i)} \) and \( \omega_{k+1}^{(i)} \). Using the quaternion-rotation matrix transformation, the measurements vector with two vectorial measurements is subsequently defined as

\[ y_{k+1}^{(i)} = \begin{bmatrix} R^T(Q_{k+1}^{(i)})v_1^{(i)} \\ R^T(Q_{k+1}^{(i)})v_2^{(i)} \\ \omega_{k+1}^{(i)} \end{bmatrix}. \quad (4.35) \]
where $v_1$ and $v_2$ are known vectors in the inertial reference frame. The method then proceeds in the same manner as the UKF as discussed before. Once the updated state vector

$$\hat{x}_{k+1} = [\delta\hat{q}_{k+1}^T \omega_{k+1}^T]^T$$

is found, it is easy to find the new quaternion error estimate $\delta\hat{Q}_{k+1}$ using the same approach as (4.33), and then obtaining the estimated quaternion in step $k + 1$ using

$$\hat{Q}_{k+1} = \delta\hat{Q}_{k+1} \otimes \hat{Q}_k.$$  \hspace{1cm} (4.36)

The authors show that their algorithm outperforms the EKF in the presence of noisy measurements and poor initial estimates [Sekhavat et al., 2007]. On the other hand, one disadvantage is that as evident from (4.34), the rigid body’s inertia matrix must be exactly known in order to find the propagated angular velocity points. This may not be preferable in practice where the lack of accurate knowledge of this matrix may result in poor performance of the filter in its propagation phase.

The recently developed Particle Filter (PF) algorithm is a generalization of the UKF based on random sample (or particle) representations of Probability Density Function (PDF) of the states [Gordon et al., 1993]. Instead of having only $2N_x + 1$ particles, as in UKF, unlimited number of particles in PF allows reconstructing the states PDF within a sampling process. The advantage of this strategy is better filter performance when the system is strongly nonlinear or the measurements are contaminated with non-Gaussian noises. Albeit, this comes with high computational expense [Crassidis et al., 2007].

The idea has also been used for attitude estimation purposes. The authors in [Cheng and Crassidis, 2004] have presented a PF that utilizes Modified Rodrigues Parameters (MRPs) for its estimation of attitude and gyro bias. In [Liu et al., 2007], a separation of the nonlinear dynamics, like the orientation, and the linear dynamics, like the gyro bias, in the system has led to a less computationally expensive algorithm. Other PFs aimed for attitude estimation can be found in [Carmi and Oshman, 2009b], [Carmi and Oshman, 2009a].
4.2.2 Invariant Kalman Filters

Invariant Extended Kalman Filter (IEKF) [Bonnabel et al., 2009b], is a newly-proposed filter for AHRS systems, based on the symmetry-preserving observers design approach [Bonnabel et al., 2008], [Bonnabel et al., 2009a]. The approach simply exploits the natural symmetries in the rigid body system dynamics and uses this property to design filters and observers that remain invariant by body-fixed rotations and linear translations.

The method starts with proposing a pre-observer for the original system, with the correction terms having the same invariant properties. This results in an error system whose trajectory does not depend on the original system trajectory and input. The choice of strategy to obtain the correction term gains can either be on a Kalman filtering base or an observer design trend.

In [Bonnabel et al., 2009b], the authors use the following system dynamic model

\[
\begin{aligned}
\dot{Q} &= \frac{1}{2} Q \otimes (\omega_{meas} - \omega_b), \\
\dot{v} &= \mathbf{g} + \frac{1}{a_s} Q \otimes a_{meas} \otimes Q^{-1}, \\
\dot{\omega}_b &= 0, \\
\dot{a}_s &= 0,
\end{aligned}
\]  

(4.37)

where \(a_s > 0\) is a constant scaling factor for the accelerometer reading, which measures \(a_{meas} = a_s a\), with \(a\) being the specific acceleration. It will be shown in section (4.4) that the model remains invariant under a constant rotation of \(Q_g\) and linear body-fixed translation \(V_g\).

The task of the Invariant EKF is to use the IMU data along with GPS to estimate attitude, velocity, gyro bias, and accelerometer unknown scaling factor. For this, a pre-
observer is given as follows:

\[
\dot{\hat{Q}} = \frac{1}{2} \hat{Q} \otimes (\omega_{\text{meas}} - \hat{\omega}_b) + \hat{Q} \otimes (K_Q E),
\]

\[
\dot{\hat{v}} = g e_3 + \frac{1}{\hat{a}_s} \hat{Q} \otimes a_{\text{meas}} \otimes \hat{Q}^{-1} + \hat{Q} \otimes (K_v E) \otimes \hat{Q}^{-1},
\]

\[
\hat{\omega}_b = K_\omega E,
\]

\[
\hat{a}_s = \hat{a}_s K_a E,
\]

with \(K_Q, K_v, K_\omega, K_a\) being the filter gains. The invariant output error is given by

\[
E = \begin{pmatrix}
\hat{Q}^{-1} \otimes (\hat{v} - y_v) \otimes \hat{Q} \\
\hat{Q}^{-1} \otimes B \otimes \hat{Q} - y_B
\end{pmatrix},
\]

where \(y_v\) is the GPS-obtained linear velocity of the rigid body, \(B\) is the Earth magnetic field in NED coordinates known in inertial frame, and \(y_B\) is the noise-contaminated magnetometer reading in the body frame. The next step consists in defining the state error vector

\[
\eta := \begin{pmatrix}
\mu \\
\nu \\
\beta \\
\alpha
\end{pmatrix} = \begin{pmatrix}
Q^{-1} \otimes \hat{Q} \\
\hat{Q}^{-1} \otimes B \otimes \hat{Q} - y_B
\end{pmatrix}.
\]

The Invariant EKF linearizes the invariant estimation error \(\eta\) dynamics (instead of the system dynamics \(f(x, u)\) as in the conventional EKF) about the latest estimated state and computes the optimal observer gains for the resulting system. The linearization of the dynamics of this vector gives an approximation equation of the form

\[
\begin{pmatrix}
\delta \mu \\
\delta \nu \\
\delta \beta \\
\delta \alpha
\end{pmatrix} = (A - KC) \begin{pmatrix}
\delta \mu \\
\delta \nu \\
\delta \beta \\
\delta \alpha
\end{pmatrix} - M \begin{pmatrix}
w_Q \\
w_v \\
w_\omega \\
w_a
\end{pmatrix} + KN \begin{pmatrix}
v_v \\
v_B
\end{pmatrix}.
\]

Once the \(A, C, M, K, N\) matrices are obtained, the gain matrix \(K = \text{diag}(K_Q, K_v, K_\omega, K_a)\) and the covariance matrix update can be found from EKF equations of

\[
K = PC^T(NN^T)^{-1}
\]

\[
P = AP + PA^T + MM^T - PC^T(NN^T)^{-1}CP
\]
Also, in order to make sure that the norm constraint of the estimated quaternion is respected, a correction term can be used in the following way

\[
\dot{\hat{Q}} = \frac{1}{2} \hat{Q} \otimes (\omega_{\text{meas}} - \hat{\omega}_b) + \hat{Q} \otimes K_\theta E + \lambda (1 - |\hat{Q}|^2) \hat{Q},
\]  

(4.41)

where \(\lambda\) is a constant scalar chosen arbitrarily. The authors have shown that the specific IEKF representing a generalized form of MEKF not only respects the unit norm constraint of the estimated quaternion, but results in time-invariant Jacobian matrices leading to a larger domain of convergence [Martin and Salaun, 2010].

### 4.3 Linear Complementary Filters

Complementary filters have been known for quite a long time as reliable solutions to various estimation problems and have provided researchers with valuable techniques in real-time states estimation [Li, 1997], [Campolo et al., 2009], [Tayebi and McGilvray, 2006].

In practical estimation applications, both the Kalman filters and the complementary filters have been successfully implemented. However, most forms of the EKF require linearizations of system equations and fail in respecting the nonlinear structure of the configuration space of problems they are involved with. However, complementary filters and nonlinear observers can better cope with the nonlinear nature of systems without breaking their natural structure of dynamics [Daum, 2005].

These filters provide a simple environment in which multiple noisy measurements of the same signal are fused together to complement each other in providing better estimates. They can even be generalized to fuse information deriving from sensors when the sensed variables are related by differential equations, for example position and velocity. In many real-time applications, the differential equations relating the sensed variables or states may be nonlinear and this is typically the case when the attitude is concerned.

As an example, consider having two measurements \(y_1(t) = x + w_1\) and \(y_2(t) = x + w_2\) of signal \(x(t)\), where \(w_1(t)\) is a high frequency noise and \(w_2(t)\) is a low frequency disturbance.
A complementary filter given by
\[
\hat{X}(s) = F_1(s)Y_1 + F_2(s)Y_2 = X(s) + F_1(s)W_1(s) + F_2(s)W_2(s),
\]
(4.42)
where \( F_1(s) + F_2(s) = 1 \), with \( F_1(s) \) being a low pass sub-filter and \( F_2(s) \) a high pass sub-filter, can be chosen in a way that the combination of two sub-filters \( F_1(s) \) and \( F_2(s) \) provides a less distorted output of the receiving signal. Therefore, while the overall complementary filter behaves as an all-pass for the desired signal, it filters the low frequency disturbances and high frequency noise.

For small UAVs, a set of magnetometers plus accelerometers can be used to construct the Euler angles. For a set of magnetometers, assume that \( r_m \) is the known magnetic field of the surrounding environment and \( b_m = R^T r_m \) is the magnetic field vector measured in the body frame. Using the Euler angles representation of the rotation matrix and decomposing the rotation matrix into three matrices \( R_\phi, R_\theta \) and \( R_\psi \), where each corresponds to a rotation about axes \( X, Y \) and \( Z \), respectively, the body-frame representation of the measured magnetic field can be expressed as
\[
b_m = R^T_\phi R^T_\theta R^T_\psi r_m + w_m,
\]
(4.43)
with \( w_m \) being the magnetometer measurement noise. Assuming that \( m_p := R_\gamma(\theta)R_\phi(\phi)b_m \) denotes the projection of the magnetometer reading on the \( x-y \) plane, an algebraic manipulation of (4.43) results in a derivation of the yaw angle
\[
\psi = \arctan2(r^y_m m^x_p - r^x_m m^y_p, r^x_m m^x_p + r^y_m m^y_p).
\]
(4.44)
As evident from the usage of vector \( m_p \) components in (4.44), knowledge of the pitch and roll is needed for the calculation of yaw. These angles can be obtained from an accelerometer set that works under a low linear acceleration of the rigid body and thus measures the inertial frame gravity vector \( g = ge_3 \) in the body frame
\[
b_a \approx -R^T_\phi R^T_\theta g = \begin{bmatrix} g \sin \theta \\ -g \cos \theta \sin \phi \\ -g \cos \theta \cos \phi \end{bmatrix}.
\]
Therefore, algebraic calculation of pitch and roll angles can be simply obtained from $b_a$ using the following equations [Vasconcelos et al., 2011]

$$\phi = \arctan2 (-b^y_a, -b^z_a)$$

$$\theta = \begin{cases} \arctan \left(-\frac{b^y_a \sin \phi}{b^z_a}\right), & \sin \phi \neq 0 \\ \arctan \left(-\frac{b^y_a \cos \phi}{b^z_a}\right), & \cos \phi \neq 0 \end{cases} \tag{4.45}$$

Once the $\theta$ and $\phi$ are calculated, they can be used to find the yaw angle $\psi$ from (4.44).

Another way to estimate the Euler angles is to use linear observers. One of the earliest observers of this kind was proposed in [Rehbinder and Hu, 2000]. The designed observer estimates the two Euler angles (pitch and roll) using a linear modeling of the outputs $y_1$ and $y_2$ of two inclinometers attached to the rigid body. The inclinometer, also known as a tilt-meter, measures the tilt angle with respect to the gravity field and provides a measurement of the Euler angles in low frequency domain. The inclinometers are assumed to have the following first order linear dynamics

$$\dot{y}_1 = \tau_1 (\theta - y_1),$$

$$\dot{y}_2 = \tau_2 (\phi - y_2),$$

where $\tau_1 = 1/T_1$ and $\tau_2 = 1/T_2$ are the inverse time constants of the inclinometers.

Using the Euler angles dynamics and taking the state vectors $x_1 = [\theta, \phi]^T$, $x_2 = [y_1, y_2]^T$, and vector $x = [x_1^T, x_2^T]^T$ as the full system state, the system dynamics with inclinometer outputs can be written as

$$\begin{cases} \dot{x}_1 = m(x_1)\omega, \\
\dot{x}_2 = \begin{bmatrix} \tau_1 & 0 \\
0 & \tau_2 \end{bmatrix} (x_1 - x_2), \\
y = Cx, \end{cases} \tag{4.46}$$

with

$$m(x_1) = \begin{bmatrix} 0 & \cos \phi & -\sin \phi \\
1 & \sin \phi \tan \theta & \cos \phi \tan \theta \end{bmatrix},$$
and

\[ C = \begin{bmatrix} 0 & I \end{bmatrix}. \]

Assuming that perfect angular velocity measurements are available, a Luenberger type observer [Luenberger, 1971] of the form

\[
\dot{\hat{x}} = f(\hat{x}, \omega) + L(y - C\hat{x}),
\]

(4.47)

is proposed for the system described in (4.46), with \( L \) being the observer gain matrix. It is shown that errors for the proposed observer have exponentially reducing bounds and a high gain choice for the observer also guarantees fast convergent estimates. The authors show that same performance can be achieved by using only one inclinometer.

Rehbinder and Hu’s observer use only a set of tilt-meters and a gyroscope. The observer stability and convergence properties were analyzed by conventional linear techniques for Luenberger observers. Their observer does not include the yaw angle (\( \psi \)) and thus fails to estimate the complete attitude. As seen before, a set of body-attached magnetometers may be used to estimate the yaw angle using the known Earth magnetic field in the inertial and the measured Earth magnetic field.

Authors in [Baerveldt and Klang, 1997] proposed one of the earliest complementary filters for attitude estimation. Their method is based on using a set of gyroscopes accompanied by a set of inclinometers.

The gyro measurements can be integrated over time to obtain a measurement of the Euler angles \( \Theta(t) = [\theta, \phi, \psi]^T \). Since the gyroscopes measure the angular velocity \( \omega \), it can be assumed that for small variations in system attitude and hence Euler angles, the angular velocity is approximately equal to the rate of change in angles, i.e. \( \dot{\Theta} \approx \omega \). Having the initial value \( \Theta_0 \) at \( t = t_0 \), the integration

\[
\Theta = \int_{t_0}^{t} \omega d\tau + \Theta_0,
\]

(4.48)

gives the Euler angles at any time.

The problem for the attitude obtained from integration of gyro readings is having a drift caused by the integration of constant values (such as gyro bias) over time. The
inclinometer-obtained attitude, on the other hand, has considerable phase losses. Therefore, both measurements alone are not suitable for having a reliable estimate of the attitude. However, a combination of these two measurements in a complementary filter can result in a better attitude estimation.

The filters $F_i(s)$ and $F_g(s)$ for the inclinometer and gyroscope outputs, respectively, were simply chosen by the authors as

$$
F_i(s) = \frac{2\tau s + 1}{(\tau s + 1)^2},
F_g(s) = \frac{\tau^2 s}{(\tau s + 1)^2},
$$

such that $F_i(s) + F_g(s) = 1$. Comparing the Bode plots of the contributions of each sensor measurement to the attitude estimate, the value of the parameter $\tau$ can be chosen. Experiments performed by the authors showed that the filters perfectly collaborated with each other in canceling the high frequency noise and low frequency offset error of the rate gyro.

A generalized form of the chosen transfer functions in Baerveldt and Klang’s work can be written as

$$
F_1(s) = \frac{2\xi \omega_n s + \omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2},
F_2(s) = \frac{s^2}{s^2 + 2\xi \omega_n s + \omega_n^2},
$$

where filter parameters $\omega_n$ and $\xi$ can be chosen according to designer needs.

In [Vasconcelos et al., 2011], the authors propose a discrete-time complementary filter that provides a useful basis for comparisons between the complementary filters and the EKF. In their work, the states to be estimated are the Euler angles $\Theta = [\psi, \theta, \phi]^T$, and the rate gyro bias $\omega_b$. Discretization of the Euler angles dynamics subject to sample-and-hold gives

$$
\Theta_{k+1} = \Theta_k + T Q(\Theta_k) \omega_k,
$$

with

$$
Q(\Theta) = \begin{bmatrix}
0 & \sin \phi \sec \theta & \cos \phi \sec \theta \\
0 & \cos \phi & -\sin \phi \\
1 & \sin \phi \tan \theta & \cos \phi \tan \theta
\end{bmatrix}.
$$
where $T$ denotes the sampling time. The gyro bias dynamics also reads $\omega_{b,k+1} = \omega_{b,k} + w_{\omega_b,k}$, where $w_{\omega_b,k}$ is the vector of zero-mean, Gaussian white noise with covariance matrix $\Xi_{\omega_b}$.

By taking $w_{\omega_b}$ as a Gaussian white noise vector for the angular velocity measurements with covariance $\Xi_{\omega_b}$, the dynamics of the discretized system with state vector $X_k = [\Theta_k, \omega_{b,k}]$ can be found

$$X_{k+1} = \begin{bmatrix} I & -TQ(\Theta_k) \\ 0 & I \end{bmatrix} X_k + \begin{bmatrix} TQ(\Theta_k) \\ 0 \end{bmatrix} \omega_k + \begin{bmatrix} -TQ(\Theta_k) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} w_{\omega_b,k} \\ w_{\omega_b,k} \end{bmatrix}. \quad (4.51)$$

The system measurements are the pitch and roll angles provided by two on-board inclinometers. The yaw angle is also computed from the Earth’s magnetic field measurements. Therefore, a vector $y_k = Q^{-1}(\Theta_{k-1})\Theta_k + v_k$ of the observed Euler angles that are transformed to the angular velocity plus a Gaussian white noise $v_k$ with covariance $\Xi_v$, is available.

The attitude filter is given by

$$\dot{\hat{X}}_{k+1} = \begin{bmatrix} I & -TQ(\Theta_k) \\ 0 & I \end{bmatrix} \dot{\hat{X}}_k + \begin{bmatrix} TQ(\Theta_k) \\ 0 \end{bmatrix} \omega_k + \begin{bmatrix} Q(\Theta_k)(K_1 - I) + Q(\Theta_{k-1})K_2 \\ \Theta_{k-1} \end{bmatrix} (y_k - \hat{y}_k), \quad (4.52)$$

$$\hat{y}_k = Q^{-1}(\Theta_{k-1})\hat{\Theta}_k, \quad (4.53)$$

where $K_1, K_2$ are gain matrices.

The gain matrices are chosen to be the Kalman gains for the auxiliary linear time-invariant system of the form

$$\begin{bmatrix} x_{\Theta,k+1} \\ x_{B,k+1} \end{bmatrix} = \begin{bmatrix} I & -TI \\ 0 & I \end{bmatrix} \begin{bmatrix} x_{\Theta,k} \\ x_{B,k} \end{bmatrix} + \begin{bmatrix} -TI & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} w_{\omega,b,k} \\ w_{B,k} \end{bmatrix}, \quad (4.54)$$

$$y_k = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_{\Theta,k} \\ x_{B,k} \end{bmatrix} + v_k.$$

The main idea is to prove that if $K_1, K_2$ are the steady-state Kalman gains for the time-invariant system of (4.54), then the complementary filter in (4.52) is Uniformly Asymptotically Stable (UAS). This is shown by writing the error system dynamics using (4.51) and (4.52). The proposed filter is UAS if the origin of the unperturbed estimation error
dynamics (i.e., without the state and measurement noises) is UAS. On the other hand, it is
straight-forward to show that the auxiliary system of (4.54) is UAS because of the system’s
detectability and stabilizability features. By defining a Lyapunov transformation that maps
the state vector of the auxiliary error system to the original error system state vector, it is
finally proved that the origin of the estimation error system is uniformly asymptotically
stable.

4.4 Symmetry-Preserving and Invariant Observers

The application of Invariant Observers in attitude estimation has recently been introduced
in a series of papers on symmetry-preserving observers [Bonnabel et al., 2008],
[Bonnabel et al., 2009a]. As discussed before, the theory of invariant observers design ex-

tploits the natural features of systems that possess symmetries and respect their geometrical
structures in the design. Prior to these works, authors in [Aghannan and Rouchon, 2003]
had developed a general framework for observer design for Lagrangian mechanical sys-
tems with position measurements. The symmetry-preserving observer they proposed led to
development of other intrinsic observers whose performances do not depend on the choice
of coordinates [Maithripala et al., 2005].

Authors in [Lageman et al., 2010] also presented a design technique for nonlinear ob-
servers based on the theory of Gradient Observers. The fundamentals of this theory is
similar to the symmetry-preserving observers but with a different manner of exploiting the
invariance properties. While the invariant observers have invariant structures, the gradient
observers take advantage of an invariant cost function for their development, but may not
necessarily be invariant themselves.

The dynamic equations of the rigid body considered in [Bonnabel et al., 2008] is

\[
\dot{Q} = \frac{1}{2} Q \otimes \omega, \\
\dot{V} = S(V) \omega + Q^{-1} \otimes g \otimes Q + b_u,
\]

(4.55)

where \( V \) denotes the body-frame vector of rigid body’s translational velocity, \( g \) is the vector
of gravity known in the inertial frame, and \( b_a \) is the specific acceleration vector. The output signals available are the velocity \( V \), and the body-frame magnetometer measurements \( y_B \) of the surrounding magnetic field. The complete output vector \( y \) is therefore defined as

\[
y := (y_V, y_B) = (V, Q^{-1} \otimes B \otimes Q).
\]

In order to see how these dynamic equations are invariant, let us define the group multiplication for an arbitrary \((Q_g, V_g) \in G\), with \( G \) denoting the underlying Lie group

\[
\phi_{(Q_g, V_g)}(Q, V) = \begin{pmatrix} Q \otimes Q_g \\ Q^{-1} \otimes V \otimes Q_g + V_g \end{pmatrix},
\]

and for known input \( u = (b_a^T, \omega^T)^T \), the output map is given by

\[
\psi_{(Q_g, V_g)}(b_a, \omega) = \begin{pmatrix} Q^{-1} \otimes b_a \otimes Q_g - V_g \times (Q^{-1} \otimes \omega \otimes Q_g) \\ Q^{-1} \otimes \omega \otimes Q_g \end{pmatrix}.
\]

Then, by setting \( \phi_{(Q_g, V_g)}(Q, V) = (Q_G, V_G) \), and \( \psi_{(Q_g, V_g)}(b_a, \omega) = (B_G, \omega_G) \), it can be shown that

\[
\dot{Q}_G = \frac{1}{2} Q \otimes \omega \otimes Q_g
\]

\[
= \frac{1}{2} Q \otimes Q_g \otimes Q^{-1}_g \otimes \omega \otimes Q_g,
\]

\[
= \frac{1}{2} Q_G \otimes \omega_G,
\]

and

\[
\dot{V}_G = Q^{-1}_g \otimes \dot{V} \otimes Q_g,
\]

\[
= Q^{-1}_g \otimes V \times \omega \otimes Q_g + Q^{-1}_g \otimes g \otimes Q_G + Q^{-1}_g \otimes b_a \otimes Q_g
\]

\[
- V_g \times (Q^{-1}_g \otimes \omega \otimes Q_g) + V_g \times (Q^{-1}_g \otimes \omega \otimes Q_g),
\]

\[
= Q^{-1}_g \otimes V \otimes Q_g \times Q^{-1}_g \otimes \omega \otimes Q_g + V_g \times (Q^{-1}_g \otimes \omega \otimes Q_g)
\]

\[
+ Q^{-1}_g \otimes g \otimes Q_G + B_G,
\]

\[
= V_g \times \omega_G + Q^{-1}_G \otimes g \otimes Q_G + B_G.
\]
This shows that the dynamics equations remain unchanged under constant rotations and translations. The invariance property of the output function can also be easily shown using the transformation group $(Q_g)_{g\in G}$ given by

$$\varrho(Q_g, V, y_b) = (Q_s^{-1} \otimes Q_s + V_g, Q_s^{-1} \otimes Q_s).$$  \hfill (4.60)

For the described system in (4.55), the complete set of invariants is given by

$$I(Q, V, b_a, \omega) = \psi_{\gamma(Q,V)} \begin{pmatrix} \omega \\ b_a \end{pmatrix} = \begin{pmatrix} Q \otimes \omega \otimes Q^{-1} \\ Q \otimes (b_a + S(V)\omega) \otimes Q^{-1} \end{pmatrix}.$$  \hfill (4.61)

Hence, the invariant output error can be taken as

$$E = \begin{pmatrix} E_V \\ E_B \end{pmatrix} = \begin{pmatrix} \hat{Q} \otimes (\hat{V} - V) \otimes \hat{Q}^{-1} \\ B - \hat{Q} \otimes y_B \otimes \hat{Q}^{-1} \end{pmatrix},$$  \hfill (4.62)

with $\hat{Q}$ and $\hat{V}$ being the estimated states with the invariant dynamics described by the following pre-observer

$$\begin{align*}
\dot{\hat{Q}} &= \frac{1}{2} \hat{Q} \otimes \omega + (L_{\hat{Q}}^0 E_V + L_{\hat{Q}}^0 E_B) \otimes \hat{Q}, \\
\dot{\hat{V}} &= S(\hat{V})\omega + \hat{Q}^{-1} \otimes \hat{g} \otimes \hat{Q} + b_a + \hat{Q}^{-1} \otimes (L_{\hat{V}}^V E_V + L_{\hat{B}}^V E_B) \otimes \hat{Q},
\end{align*}$$  \hfill (4.63)

where $L_{\hat{Q}}^0$, $L_{\hat{B}}^0$, $L_{\hat{V}}^V$, and $L_{\hat{B}}^V$ are the observer’s gain matrices. In order to choose a value for these gain matrices, first the dynamics of the equivalent state errors $\eta_Q = \hat{Q} \otimes Q^{-1}$ and $\eta_V = Q \otimes (\hat{V} - V) \otimes Q^{-1}$ are found using the dynamic models of the actual system (4.55) and the pre-observer (4.63). Assuming that estimated states are close enough to the actual states, the dynamics of the equivalent state errors are linearized to obtain a new set of dynamic equations with the gain matrices $L_{\hat{Q}}^0$, $L_{\hat{B}}^0$, $L_{\hat{V}}^V$, and $L_{\hat{B}}^V$ involved. Letting $\delta E_V = \delta \eta_V$ and $\delta E_B = 2S(B)\delta \eta_Q$ denoting the first order approximations of the error states, the new error dynamics is given by

$$\begin{align*}
\frac{d}{dt} \delta \eta_V &= 2S(g)\delta \eta_Q + L_{\hat{V}}^V \delta \eta_V + 2L_{\hat{B}}^V S(B)\delta \eta_Q, \\
\frac{d}{dt} \delta \eta_Q &= L_{\hat{Q}}^0 \delta \eta_V + 2L_{\hat{Q}}^0 S(B)\delta \eta_Q.
\end{align*}$$

Chapter 4. Dynamic Attitude Filtering and Estimation

Here, the constant matrices can be chosen in a way that by decomposing the subsystems representing the dynamics of the elements of $\delta \eta_V$ and $\delta \eta_Q$, some decoupled subsystems can be derived. Stabilizing these linear subsystems ensures the stability of the main observer. In fact, a locally asymptotically stable observer is used as a mean to find the main observer gain with the basic concept that it is always possible to turn an asymptotic observer with a local gain design into an invariant one with the same local behavior.

Authors in [Martin and Salaun, 2007] also developed an invariant observer for attitude and gyro bias estimation. They designed their gyro bias observer based on the low-acceleration assumption and therefore, no GPS signals were used. Their observer has the advantage of being able to successfully cope with the magnetic disturbances.

In general, the invariant observers respect the important geometric features of the nonlinear system and preserve its constraints and natural symmetries. These observers have local convergence around every trajectory of the system and since the error dynamics of the filter is independent of the system trajectory, the observer’s domain of convergence with respect to initial conditions becomes large. The disadvantage, however, may be related to the fact that is difficult to show global stability with this kind of observers.

4.4.1 Gradient Observers

In parallel with the works on the invariant observers design, the application of Gradient and Gradient-Like Observers in attitude estimation was presented in [Lageman et al., 2010]. The theory emphasizes on the invariance properties of nonlinear systems. The methodology is independent of Bonnabel’s work and follows similar strategies as those of the design of intrinsic observers studied by previous researchers. Instead of directly designing an invariant observer on the underlying group, the authors in [Lageman et al., 2010] take advantage of classical observers structure and investigate a novel approach in finding the innovation terms in the observer design.

Consider a left invariant nonlinear system of the form

$$\dot{X} = Xu,$$  \hspace{1cm} (4.64)
where $X$ is a member of a Lie group $G$ with identity element $e$, and $u$ denotes the system’s input signal. Then, assuming that an observer designed to estimate the nonlinear system states has an internal mode of the system, from which measurement $Y$ is available, the following observer is proposed

$$\dot{\hat{X}} = F_{\hat{X}}(\hat{X}, Y, u, t) = \hat{X}u + \alpha(\hat{X}, Y, u, t), \quad (4.65)$$

where $u$ denotes the measurements of the system input $u$, and $\alpha$ is a smooth function that plays the role of a correction term in the observer structure. From the perspective of the invariant observers, it can be seen that the addition of this term makes the whole observer non-left invariant since $\alpha$ is not necessarily left invariant. This is in contrast with the invariant pre-observers in the invariant observers design strategy where the correction term is itself invariant.

The authors’ approach in choosing an appropriate correction term is to use the gradient descent direction of a smooth, non-negative cost function $f : G \times G \rightarrow \mathbb{R}$ of the estimated states and the measured outputs. In this way, the proposed observer can be expressed as

$$\dot{\hat{X}} = \hat{X}u - \text{grad}_{\hat{X}} f(\hat{X}, Y). \quad (4.66)$$

Let us consider the following right invariant state error

$$E_r(\hat{X}, X) := \hat{X}X^{-1}. \quad (4.67)$$

Taking the time derivative of the error function and setting $v = u$ results in

$$\dot{E}_r = (\frac{d}{dt} \hat{X})X^{-1} + \hat{X}(\frac{d}{dt} X^{-1}),$$

$$= \hat{X}uX^{-1} - \text{grad}_{\hat{X}} f(\hat{X}, Y)X^{-1} - \hat{X}(X^{-1} XuX^{-1}),$$

$$= - \text{grad}_{\hat{X}} f(\hat{X}, Y)X^{-1}, \quad (4.68)$$

$$= - \text{grad}_{\hat{X}} f(\hat{X}X^{-1}, e),$$

$$= - \text{grad}_{\hat{X}} f(E_r, e),$$

The importance of this result is in that the error dynamics is only dependent on the error itself and not on the trajectories of the system and observer states. From this perspective,
the gradient and invariant observers share the same property. This feature gives the observer a large domain of attraction. From a convergence point of view, it is not hard to prove that the error dynamics of $E_r$ is convergent to the identity element $e$.

The function $f$ should be chosen as a Morse-Bott with non-degenerative Hessian in the normal direction. This results in a cost function with $e$ as its unique global minimum. Construction of such functions constitutes a section of the authors’ work with some useful suggestions given. The paper also considers the requirements of the invariance of the cost function and proposes general functions whose gradients can be used to design gradient-like observers in a similar manner to the one discussed here. It should be mentioned that analogous to a left invariant nonlinear system, gradient observers can be designed for right invariant systems with the same technique. Also, all the results on error dynamics features can be extended to the right invariant systems case.

**Application in Attitude Estimation:** The authors in [Lageman et al., 2010] provide an example of their observer’s application for the attitude estimation problem with the assumption that the measurable state of the system is the rotation matrix $Y = R_y$. The system input, taken as the angular velocity measurements, is $v = S(\omega_y)$. The right and left invariant errors can then be expressed as

$$E_r = \hat{R}R^T, \quad E_l = R^T\hat{R},$$

and the invariant cost function $f(\hat{X}, Y)$ can be chosen as the Frobenius norm of the difference between the estimated rotation matrix and the measured one

$$f(\hat{R}, R_y) = \frac{k_f}{2}\|\hat{R} - R_y\|_F^2,$$

with $k_f$ being a positive scalar. Taking into account the fact that the example uses the induced metric on the Special Orthogonal group, the gradient of function $f$ can be expressed as the orthogonal projection of the Euclidean gradient in $\mathbb{R}^{3 \times 3}$ to the tangent space of $SO(3)$. Hence, the following filter is given

$$\dot{\hat{R}} = \hat{R}v + k_f\hat{R}P_a(\hat{R}^T R_y),$$

(4.71)
where $\mathbb{P}_a$ is the operator that extracts the anti-symmetric part of a matrix, \textit{i.e.}, $\mathbb{P}_a(A) = (1/2)(A - A^T)$.

This observer is very similar to the nonlinear passive complementary filter in [Mahony et al., 2008]. In both works, a reconstruction of the rotation matrix is needed for generating the error terms required to find the observer’s correction terms. However, the strategies adopted to design the two estimators are different. Lageman’s work is similar in nature to a previous work in [Shimizu, 2000]. In this work, a same observer structure as (4.65) is proposed, but with a different choice of the correction term as the gradient of a performance function that consists of squared errors between the measured and estimated outputs.

A general framework for observers design on $SE(3)$ has recently been developed in [Hua et al., 2011]. In this work, an extension of the observer design problem for invariant dynamics addressed by Lageman for $SO(3)$ dynamics is discussed. The design strategy takes the basic concept of \textit{invariance} in dynamical systems on Lie group and extends the idea to the dynamics defined on the Special Euclidean group $SE(3)$.

Consider system dynamics for a rigid body’s pose $T \in SE(3)$ given by (2.21). The system is left invariant if the system structure is left unchanged under constant translations and constant rotation of the body-fixed frame $\mathcal{B}$. According to the gradient observer design methodology, for a left invariant system an observer can be designed by first finding a right invariant cost function. In this case, a cost function $f: SE(3) \times SE(3) \to \mathbb{R}$ is called right invariant if for all poses $\hat{T}, T, T_0 \in SE(3)$ with constant $T_0$, the following relationship exists

$$f(\hat{T}T_0, TT_0) = f(\hat{T}, T). \quad (4.72)$$

Following the structure of the left invariant observer in (4.66), the authors propose a $SE(3)$-suited observer given by

$$\dot{\hat{T}} = \hat{T}A - \text{grad}_T f(\hat{T}, T), \quad (4.73)$$

with initial value $T(t)|_{t=0} = \hat{T}_0 \in SE(3)$. 

The measurements are taken as the body-expressed position of some known landmarks in the local frame. Consider a set of known inertial landmarks \( z_i \) and the matrix \( T_y \) constructed from the measured angular and translational velocities. The cost function is chosen to be

\[
 f(\hat{T}, T_y) = \frac{1}{2} \sum_{i=1}^{N} k_i (\hat{T}^{-1} - T_y^{-1}) z_i^2,
\]

(4.74)

where \( k_i, i = 1, \ldots, N \), are some positive constant scalars. By calculating the gradient of this cost function with regards to the estimated \( \hat{T} \), one can verify that

\[
 \text{grad}_{\hat{T}} f(\hat{T}, T) = -\mathbb{P} \left( \sum_{i=1}^{N} k_i (\hat{T}^{-1} - T^{-1}) z_i z_i^T \right) \hat{T},
\]

(4.75)

where \( \mathbb{P}(X) \), for all \( X \in \mathbb{R}^{4 \times 4} \) denotes the orthogonal projection of \( \mathbb{R}^{4 \times 4} \) onto \( SE(3) \). The cost function (4.74) can be regarded as a Lyapunov function of the error \( E_r(\bar{T}, T) := \bar{T} - \hat{T} T^{-1} \) and right invariant with regards to its variable. In fact, the cost function can be expressed as follows

\[
 f(\bar{T}, T_y) = \mathcal{L}(\bar{T}) = \frac{1}{2} \sum_{i=1}^{N} k_i (\bar{T} - I) z_i^2.
\]

(4.76)

Therefore, the gradient of this invariant Lyapunov function with regards to the estimated pose can be used to give the nonlinear observer on \( SE(3) \) as

\[
 \begin{aligned}
 \dot{\hat{T}} &= \hat{T} (A_y - \alpha) \\
 \alpha &= \hat{T}^{-1} \text{grad}_{\hat{T}} f(\hat{T}, T).
\end{aligned}
\]

(4.77)

The convergence properties of the observer are analyzed using Lyapunov arguments. For this, the time derivative of the Lyapunov function (4.76) is obtained as

\[
 \dot{\mathcal{L}}(E_r) = \langle \mathbb{P} \left( \sum_{i=1}^{N} k_i (I_4 - E_r) z_i z_i^T \right), \sum_{i=1}^{N} k_i (E_r z_i - z_i) (E_r z_i)^T \rangle \\
 = \langle \mathbb{P} \left( \sum_{i=1}^{N} k_i (I_4 - E_r) z_i z_i^T \right), \mathbb{P} \left( \sum_{i=1}^{N} k_i (E_r z_i - z_i) (E_r z_i)^T \right) \rangle \\
 = \langle \mathbb{P} (\sum_{i=1}^{N} k_i (I_4 - E_r) z_i z_i^T), \mathbb{P} (\sum_{i=1}^{N} k_i (E_r - I_4) z_i z_i^T) \rangle \\
 = -\| \mathbb{P} (\sum_{i=1}^{N} k_i (I_4 - E_r) z_i z_i^T) \|^2,
\]

(4.78)
with $\langle \cdot, \cdot \rangle$ denoting the Euclidean inner product. Setting the negative semi-definite derivative of the Lyapunov function to zero and invoking LaSalle’s invariance principle, one can show the asymptotic stability of the equilibrium $E_r - I_4$. For a more comprehensive study of the stability, time-derivative of the error

$$E_r = \begin{bmatrix} R_e & p_e \\ 0 & 1 \end{bmatrix}, \quad (4.79)$$

is taken and by assuming that at least 3 non-collinear vectors $z_i$ are available, it is shown that the error dynamics has only four equilibrium points $(R_e, p_e) = (R_{e_i}^*, p_{e_i}^*), i = 1, ..., 4$. The locally exponentially stable equilibrium point is $(R_{e1}^*, p_{e1}^*) = (I_3, 0)$, and the other 3 equilibria are unstable. For instability proof of these equilibria, Chetaev-like arguments are provided.

This observer has the advantage of being designed directly on the Special Euclidean group. The authors provide an interesting study of the application of Lie-algebra studies in the design of observers for special groups and their work can be regarded as a continuity to the previous works on the invariant observers. However, it is evident that while the invariant observers are observers with an invariant structure, the gradient and gradient-like observers are observers with non-invariant structures, but an invariant cost function from which the correction terms of the observer are derived.

### 4.5 Nonlinear Complementary Filters

As discussed in section (4.3), linear complementary filters have been long known to be reliable tools for attitude estimations. Nonlinear complementary filters, however, are relatively new and have various differences with the linear filters of this kind including different structure.

In many of these filters, reconstructions of the attitude is required for the filter as inputs and is then “filtered” to give better results. In such filters, observer-like structures with Lyapunov-based arguments are used to guarantee an ultimate convergence of the filter...
output to the real attitude. By “measured attitude” we mean an estimation provided from numerical methods, such as QUEST, through vectorial measurements.

Nonlinear complementary filters are designed based on the nonlinear structure of the system and therefore, give better results than linear approximative filters. With strong Lyapunov theory arguments, the estimated states are guaranteed to be closer to the actual system states.

The earliest work in this field that captured the true nonlinear nature of the rotational dynamics was presented in [Salcudean, 1991]. The proposed filter has a nonlinear structure and stability proofs for nonlinear systems using Lyapunov analysis are provided. The work includes a globally convergent nonlinear filter for the attitude and angular velocity of a rigid body. The filter uses both the Inertia matrix $I_f$ and the vector of applied torque $\tau$ in the estimation law to estimate the angular velocity $\hat{\omega}$ and the orientation

$$I_f \hat{\omega} = \tau + \frac{1}{2} k_p I_f^{-1} \tilde{q} \text{sign}(\tilde{q}_0),$$

$$\dot{\hat{R}} = \left[ \hat{R}^T (\hat{\omega} + k_v I_f^{-1} \tilde{q} \text{sign}(\tilde{q}_0)) \right] \times \hat{R},$$

(4.80)

where $\tilde{Q} = (\tilde{q}_0, \tilde{q}) = Q \otimes \hat{Q}^{-1}$ is the quaternion error between the actual attitude quaternion and the estimated quaternion. $k_p$ and $k_v$ are positive coefficients and $[,]_x$ is the equivalent representation of the skew-symmetric matrix. For the proof of filter convergence, a Lyapunov function $V_1$ is defined as

$$V_1 = \mu^T \mu + k_p (1 - \tilde{q}_0 \text{sign}(\tilde{q}_0))^2 + \tilde{q}^T \tilde{q},$$

(4.81)

where $\mu := I_f \omega - I_f \hat{\omega}$. It is shown that the function is decreasing along the quaternion error trajectories and the quaternion error scalar $\tilde{q}_0$ converge to 1 as the time goes to infinity (i.e., $\lim_{t \to \infty} \tilde{q}_0(t) = 1$). In order to show the convergence of $\hat{\omega}$ to the actual angular velocity, a second Lyapunov function was chosen as

$$V_2 = z^T P z = z^T \begin{bmatrix} p_1 I & I \\ I & p_2 I \end{bmatrix} z,$$

(4.82)

where $z := [\mu, \tilde{q}]^T$ is the vector of states error, and the positive scalars $p_1$ and $p_2$ are chosen in a way that $p_1 p_2 > 1$ so that the matrix $P$ becomes positive definite. While the
function $V_2$ is a quadratic positive definite function of error vector $z$, it is shown that the derivative of Lyapunov function is negative semi-definite with regards to this vector. From the boundedness of the second derivative of the Lyapunov function, it is deduced that the norm of error vector $\|z\|$ converges to zero. Hence, the norm of vector $\mu$ converges to zero which results in $\lim_{t \to \infty} \|\omega - \hat{\omega}\| = 0$.

Authors in [Vik and Fossen, 2001] designed a quaternion-based filter for gyro bias, gyro scale factor, and gyro misalignment angles. They assume that the gyro bias $\omega_b$, gyro scale factor $s_g = [s_x, s_y, s_z]^T$, and gyro misalignment angles $\phi_g = [\phi_{xy}, \phi_{xz}, \phi_{yx}, \phi_{yz}, \phi_{zx}, \phi_{zy}]^T$, associated with the gyro measurements exponentially decay with time and have dynamics of the form

$$
\dot{\omega}_b = -T_1^{-1}\omega_b + w_1,
\dot{s}_g = -T_2^{-1}s_g + w_2,
\dot{\phi}_g = -T_3^{-1}\phi_g + w_3,
$$

(4.83)

where $T_i \in \mathbb{R}^{3 \times 3}, i = 1, 2, 3$ are diagonal matrices with constant positive components and $w_i, i = 1, 2, 3$ are the vectors of Gaussian measurement noises. Same assumptions were made for the accelerometer errors. A nonlinear observer fairly similar to their work and under the same exponentially decaying gyro bias assumption was also proposed in [Guerrero-Sanchez et al., 2009]. However, this was not followed by most other succeeding researchers, who rather worked with constant (or slowly time-varying) gyro bias assumption, disregarding redundant gyro scale factors and misalignment angles assumptions.

Using the aforementioned assumptions, the attitude observer with bias, scale factor, and
misalignment angles estimates is proposed as

\[
\begin{align*}
\dot{\hat{Q}} &= \frac{1}{2} \begin{bmatrix}
-\hat{q}^T \\
\hat{q}_0 I + S(\hat{q})
\end{bmatrix} \hat{R} \left[ (I + \hat{\Delta})\omega_{\text{meas}} + \hat{\omega}_b + K_1 \hat{q} \ \text{sign}(\hat{q}_0) \right] - \frac{1}{2} \begin{bmatrix}
-\hat{q}^T \\
\hat{q}_0 I - S(\hat{q})
\end{bmatrix} \omega_{\text{in}} \\
\dot{\hat{\omega}}_b &= -T_1^{-1} \hat{\omega}_b + \frac{1}{2} K_2 \hat{q} \ \text{sign}(\hat{q}_0) \\
\dot{\hat{s}}_g &= -T_1^{-1} \hat{s}_g + \frac{1}{2} K_3 \text{diag}(\hat{q}) \omega_{\text{meas}} \ \text{sign}(\hat{q}_0) \\
\dot{\hat{\phi}}_g &= -T_1^{-1} \hat{\phi}_g + \frac{1}{2} K_4 \Gamma(\hat{q}) \omega_{\text{meas}} \ \text{sign}(\hat{q}_0)
\end{align*}
\]

(4.84)

with

\[
\hat{\Delta} = \begin{bmatrix}
\hat{s}_x & \hat{\phi}_{xy} & \hat{\phi}_{xz} \\
\hat{\phi}_{yx} & \hat{s}_y & \hat{\phi}_{yz} \\
\hat{\phi}_{zx} & \hat{\phi}_{zy} & \hat{s}_z
\end{bmatrix}, \quad \Gamma^T(\hat{q}) = \begin{bmatrix}
0 & 0 & 0 & 0 & \hat{q}_0 \\
0 & \hat{q}_1 & 0 & 0 & 0 \\
0 & 0 & \hat{q}_1 & 0 & 0 \\
0 & 0 & 0 & \hat{q}_2 & 0 \\
0 & 0 & 0 & 0 & \hat{q}_3
\end{bmatrix},
\]

where \(\omega_{\text{meas}} = \omega_y\) is the IMU measurement of the rigid body angular velocity, \(\omega_{\text{in}} = \omega_{\text{ie}} + \omega_{\text{en}}\), where \(\omega_{\text{ie}}\) is the Earth Centred Inertial (ECI) angular velocity with respect to Earth Centred Earth Fixed (ECEF) frame and contains the Earth rotation rate, and \(\omega_{\text{en}}\) is the ECEF frame angular velocity described in North, East, Down (NED) coordinates. Matrices \(K_1,K_2,K_3,\) and \(K_4\) are positive gain matrices.

With a rigorous stability proof using a Lyapunov function that contains positive semi-definite bilinear functions of the error vectors, the origin of filter’s error system was proved to be globally exponentially stable. While the filter seems to be precise in its estimation of many unknown system parameters, its disadvantage is the unnecessary introduction of unimportant parameters such as gyro misalignment angles. This strategy was not followed by other researchers and the main reason behind this is that in practice, the most important unknown parameter to be estimated is the gyro bias and other error parameterizations can be neglected. Although the assumption of an exponentially decaying gyro bias was softened by subsequent researchers to have a time-constant variable, the same adaption law was used by others for estimating this unknown system parameter.

In [Boskovic et al., 2000], the authors proposed an observer with the sole task of esti-
mating the gyro bias using the rigid body angular velocity dynamics:
\[
\begin{align*}
I_f \ddot{\omega} &= S(I_f \omega_y) \omega_y + [S(\omega_y)I_f - S(I_f \omega_y) - S(\hat{\omega}_b)I_f] \hat{\omega}_b + \Lambda (\dot{\omega} + \dot{\omega}_b - \omega_y) + \tau, \\
\dot{\hat{\omega}}_b &= -M^T (\dot{\omega} + \dot{\omega}_b - \omega_y),
\end{align*}
\]  
(4.85)

where \( M := [S(\omega_y)I_f - S(I_f \omega_y) - S(\hat{B})I_f + S(I_f \tilde{B}) + \Lambda] \), and \( \Lambda \) is a negative definite matrix. It can be seen that the observer relies on knowing the input torque \( \tau \). For this, the authors propose an adaptive sliding control law for attitude and rotational velocity control in which attitude measurements are required. This may not be a suitable solution since an important goal in observer design is to have estimations of system states and parameters without interference in the control strategy. Another disadvantage of directly using the rigid body angular velocity dynamics in observer design is the requirement of spacecraft inertia matrix to be known.

An extension of the filter in (4.80) is proposed in [Thienel and Sanner, 2003], which includes the estimation of the constant gyro bias vector without estimating the angular velocity
\[
\begin{align*}
\dot{\tilde{Q}} &= \frac{1}{2} \begin{bmatrix}
-\tilde{q}^T \\
\tilde{q}_0 I_3 + S(\tilde{q})
\end{bmatrix} R^T(\tilde{Q})(\omega_y - \hat{\omega}_b + k_p \tilde{q} \text{sign}(\tilde{q}_0)), \\
\dot{\hat{\omega}}_b &= -\frac{1}{2} \tilde{q} \text{sign}(\tilde{q}_0),
\end{align*}
\]  
(4.86)

where \( \tilde{Q} = Q \otimes \tilde{Q}^{-1} \) is the quaternion error between the estimated quaternion and the actual attitude quaternion, and the term \( R^T(\tilde{Q}) \) is the error rotation matrix associated with this quaternion error. One can see the similarity of using the quaternion error term in bias estimator in their work and the one presented in (4.84). Other attitude filters with error quaternions or rotation matrices involved in their structure have more or less the same structure for their bias estimator.

The time derivative of the quaternion and bias error vectors can be found as
\[
\begin{align*}
\dot{\tilde{Q}} &= \frac{1}{2} \begin{bmatrix}
-\tilde{q}^T \\
\tilde{q}_0 I_3 + S(\tilde{q})
\end{bmatrix} [\tilde{\omega}_b + k_p \tilde{q} \text{sign}(\tilde{q}_0)], \\
\dot{\hat{\omega}}_b &= \frac{1}{2} \tilde{q} \text{sign}(\tilde{q}_0),
\end{align*}
\]  
(4.87)
where \( \tilde{\omega}_b = \omega_b - \hat{\omega}_b \).

The Lyapunov function used in their proof is quite like Salcudean’s Lyapunov function (4.81) with the difference in the presence of bias error vector

\[
\mathcal{V} = \frac{1}{2} [(1 - \tilde{q}_0 \text{sign}(\tilde{q}_0))^2 + \tilde{q}^T \tilde{q}] + \frac{1}{2} \tilde{\omega}_b^T \tilde{\omega}_b. \tag{4.88}
\]

Using (4.87) and (4.86), one finds the time derivative of the Lyapunov function as

\[
\dot{\mathcal{V}} = -\frac{k_p}{2} \tilde{q}^T \tilde{q}. \tag{4.89}
\]

This ensures that all error vectors are globally uniformly bounded. It can also be shown that the second derivative of Lyapunov function is bounded. Therefore, invoking Barbalat Lemma allows to conclude that \( \|\tilde{q}\| \to 0 \) as \( t \to \infty \).

Authors in [Thienel and Sanner, 2007] also designed a filter to estimate the rotation rates for Hubble Space Telescope. Their filter, although similar in structure to their previous filter, does not take into account the gyro bias term. However, it provides detailed analysis of the error sources in practical applications. They compared their filter to a Kalman filter and showed that when attitude measurement error is considered, the nonlinear filter not only has smaller rotation rate errors than the Kalman filter, but converges to the actual angular velocity with no oscillations while the Kalman filter persistently oscillates around this value.

A similar filter of this kind was also proposed in [Tayebi et al., 2007], where the quaternion representation is used instead of the rotation matrix. The observer does not directly use the term \( R^T(\dot{\tilde{Q}}) \), but involves the vector part of the quaternion error in the observer dynamics

\[
\begin{aligned}
\dot{\tilde{Q}} &= \frac{1}{2} \begin{bmatrix} -\tilde{q}^T \\ \tilde{q}_0 I_3 + S(\tilde{q}) \end{bmatrix} [\omega_y - \hat{\omega}_b + \Gamma_1 \tilde{q}], \\
\dot{\hat{\omega}}_b &= -\Gamma_2 \tilde{q},
\end{aligned} \tag{4.90}
\]

where \( \Gamma_1 \) and \( \Gamma_2 \) are symmetric positive-definite matrices. It is shown that the quaternion and gyro rate bias error vectors are globally bounded and for all initial conditions such that \( (\tilde{Q}(0), \tilde{\omega}_{b,0}) \neq (-1,0,0,0,0) \), the filter results in \( \lim_{t \to \infty} \tilde{q}_0(t) = 1 \) and \( \lim_{t \to \infty} \hat{\omega}_b(t) = \omega_b \).
4.5.1 Explicit Complementary Filter and Compatible Observers

One of the most influential works in the field of nonlinear observers design for the attitude estimation was presented in [Mahony et al., 2008]. In this work, the authors present three forms of nonlinear complementary filters named as Direct Complementary Filter, Passive Complementary Filter and Explicit Complementary Filter. The latter is often simply called “Nonlinear Complementary Filter” and is inspired by the traditional linear complementary filtering.

All these observers assume that at least two non-collinear known vectors in the inertial frame and their corresponding vector observations in the body frame are available. The first two observers use an instantaneous algebraic reconstruction of the rotation matrix in their structure. This matrix, denoted as $R_y$, is constructed from accelerometer and magnetometer readings. However, the problem with accelerometer and magnetometer outputs, where magnetic disturbances or high acceleration maneuvers negatively affects the sensor readings, makes the algebraic reconstruction non-reliable. Therefore, both direct and passive complementary filters were given less considerations than the Explicit Complementary Filter, where the estimated rotation matrix $\hat{R}_y$ and gyro bias $\hat{\omega}_b$, are explicitly found using only the vector observations.

The main idea is to assume that a set of $r_i, i = 1, ..., n$ and $b_i, i = 1, ..., n$ are available in inertial and body frames, respectively, and the vectors are related through $b_i = R^T r_i$. Based on this, the Explicit Complementary Filter with bias correction is given by

$$
\dot{\hat{R}} = \hat{R}S (\omega_y - \hat{\omega}_b + k_p \sigma), \\
\dot{\hat{\omega}}_b = -k_I \sigma, \\
\sigma = \sum_{i=1}^{n} k_i b_i \times \hat{b}_i,
$$

(4.91)

where $\hat{b}_i := \hat{R}^T r_i$ is the estimated vector of body-referenced vectors associated with a known inertial direction. $k_p$ and $k_I$ are positive scalar gains, and $k_i$ are positive coefficients chosen in a way that the matrix $M_0 = \sum_{i=1}^{n} k_i r_i r_i^T$ has three distinct eigenvalues $\lambda_i, i = 1, 2, 3$. The necessary condition of having at least two non-parallel measurement vectors results in the
matrix $M := R^T M_0 R$ to be positive semi-definite and plays an important role in the stability proof. Using this definition, the authors show that the observer has three unstable equilibria

$$(\hat{R}_i, \hat{\omega}_b, i) = (U_0 D_i U_0^T R, \omega_b), \quad i = 1, 2, 3$$

(4.92)

where $D_i$’s are diagonal matrices defined by

$$D_1 = \text{diag}(1, -1, -1), \quad D_2 = \text{diag}(-1, 1, -1), \quad D_3 = \text{diag}(-1, -1, 1),$$

and $M_0 = U_0 \text{diag}(\lambda_1, \lambda_2, \lambda_3) U_0^T$. For all initial conditions except for the given unstable equilibria, the error $(\tilde{R}, \tilde{\omega}_b) = (\hat{R}^T R, \omega_b - \hat{\omega}_b)$ exponentially converges to $(I, 0)$.

The Lyapunov function used for the analysis of the observer stability and convergence is

$$V = \sum_{i=1}^n k_i - \text{tr}(\tilde{R}M) + \frac{1}{k_i} |\tilde{\omega}_b|^2.$$  (4.93)

Taking the time derivative of this Lyapunov function results in

$$\dot{V} = -k V ||P_a(\tilde{R}^T M_0 R)||^2.$$  (4.94)

Since the derivative of the Lyapunov function is negative semi-definite, one can show that all the error signals, including the $\tilde{\omega}_b$, are bounded. Also, invoking Barbalat lemma ensures that $\dot{V}$ asymptotically tends to zero. The stability of equilibria can then be analyzed using the eigenvalue-eigenvector decomposition of the error matrix $\tilde{R}$. Instability of equilibria given in (4.92) can also be shown using Chetaev’s equilibrium instability arguments.

Mahony’s observer is similar to the Luenberger observer in the sense that while the observer maintains the overall structure of system dynamics of the actual attitude, the innovation term $\sigma$ corrects the estimation dynamics. Having at least two non-collinear vector measurements in this observer is necessary since with only one vector observation, the system states cannot be observed. However, it can be shown that under the “persistent excitation” condition, in which it is assumed that the direction of the observed vector or the orientation of the rigid body is permanently changing with time, a single vector observation is sufficient for estimating the attitude [Mahony et al., 2009].
A small modification of the observer’s bias estimation, proposed in [Grip et al., 2011], gives a projection-based bias estimation

\[ \hat{\omega}_b = \text{Proj}(\hat{\omega}_b, -k_I \sigma), \]

to ensure that bias estimate \( \hat{\omega}_b \) remains within a compact, convex set. The projection function is defined on the set \( \Omega_b := \{ \omega_b \in \mathbb{R}^3 | P(\omega_b) \leq 0 \} \), with \( P : \mathbb{R}^3 \rightarrow \mathbb{R} \) being a smooth, convex function with gradient \( \nabla P^T \) and interior \( \Omega_b^0 \). Another set \( \hat{\Omega}_b := \{ \hat{\omega}_b \in \mathbb{R}^3 | P(\hat{\omega}_b) \leq \sigma \} \) slightly larger than \( \Omega_b \) can also be defined. The projection function is defined as follows

\[ \text{Proj}(\hat{\omega}_b, -k_I \sigma) = p(\hat{\omega}_b, -k_I \sigma)(-k_I \sigma), \] (4.95)

where

\[ p(\hat{\omega}_b, -k_I \sigma) = \begin{cases} 
I_{3 \times 3} & \text{if } \hat{\omega}_b \in \Omega_b^0 \\
I_{3 \times 3} - \min\{1, \frac{P(\hat{\omega}_b)}{\sigma} \} \frac{\nabla P^T \hat{\omega}_b}{\|\nabla P\|} & \text{or } \nabla P^T \cdot (-k_I \sigma) \leq 0 \\
I_{3 \times 3} & \text{otherwise}
\end{cases} \] (4.96)

In [Grip et al., 2012a], the same authors expanded their idea to the situation where stationary reference vectors are no longer assumed and provided a second observer for biases in vectorial observations.

A generalization of the Mahony’s complementary filter is proposed in [Jensen, 2011]. The filter is basically the same nonlinear complementary filter in (4.91) with time-varying matrix gains. The filter is given by

\[ \dot{\hat{R}} = \hat{R} \mathcal{S} (\omega_y - \hat{\omega}_b + K_P \sigma), \]
\[ \dot{\hat{\omega}}_b = -K_I \sigma, \] (4.97)
\[ \sigma = \sum_{i=1}^{n} k_i b_i \times \hat{b}_i, \]

where constant positive scalar gains \( k_P \) and \( k_I \) are replaced by the positive-definite matrix gains \( K_P \) and \( K_I \). By some additional assumptions on the positive semi-definiteness of \( K_I \),
upper and lower boundedness of $K_P$ and $K_I$ by some positive constants, and boundedness of $\dot{K}_P$, $\dot{K}_I$, and $\ddot{K}_I$, the Lyapunov function

$$V = \sum_{i=1}^{n} k_i - \text{tr}(\tilde{R}M) + \frac{1}{2} \tilde{\omega}_b^T K_I^{-1} \tilde{\omega}_b,$$

(4.98)

is used for local asymptotic stability proof of the equilibrium $(\tilde{R}, \tilde{\omega}_b) = (I, 0)$ of the error dynamics. The filter has a greater tuning space than the explicit complementary filter with the same stability characteristics and it is shown that bias-free Multiplicative Extended Kalman Filter and constant-gain MEKF are special cases of the proposed filter.

The authors in [Tayebi et al., 2011] proposed an IMU-based dynamic attitude estimator similar to the explicit complementary filter, but with new proofs for filter convergence based on the quaternion attitude representation. The basic idea of their observer consists in maintaining the structure of the attitude kinematics in $S^3$ by incorporating a correction term in the angular velocity part of a kinematics equations similar to the original quaternion-based attitude kinematics

$$\dot{\hat{Q}} = \frac{1}{2} \hat{Q} \otimes \begin{pmatrix} 0 \\ \omega_y - z_y \end{pmatrix},$$

(4.99)

with

$$z_y = \sum_{i=1}^{n} \gamma_i S(\hat{b}_i) b_i,$$

(4.100)

where $\gamma_i > 0$, and $\hat{b}_i = R(\hat{Q})^T r_i$, with an arbitrary initial condition $\hat{Q}(0) \in S^3$. In this formulation, the term $z_y$ is similar in its correcting task to the $\sigma$ in (4.91). Defining $\tilde{Q} = Q \otimes \hat{Q}^{-1}$ as the error quaternion, it is shown that $\lim_{t \to \infty} \tilde{Q}(t) = (\text{sign}(\tilde{q}_0), 0)$ for almost any initial conditions except for a a set of Lebesgue measure zero described by $\Psi = \{\tilde{Q} = (\tilde{q}_0, \tilde{q}) \in S^3 | \tilde{q}_0 = 0\}$.

The authors also considered the case where the observer innovation term, based on the vector measurements, is pre-filtered. The dynamic observer with a low-pass filter is given
by
\[ \dot{\mathbf{Q}} = \frac{1}{2} \mathbf{Q} \otimes \begin{pmatrix} 0 \\ \omega_y - \alpha \psi \end{pmatrix}, \]
(4.101)
\[ \dot{\psi} = -\alpha \psi + \alpha z_y, \]
with \( \psi \) as an auxiliary parameter, and the Lyapunov function considered for stability proof was chosen as
\[ V = \frac{1}{2} \sum_{i=1}^{n} \gamma_i |(b_i - \hat{b}_i)|^2 + \frac{1}{2} |\psi|^2. \]
(4.102)
Also, the authors proposed a control law for the stabilization of the following velocity-free, attitude-free, torque input
\[ \tau = z_y - z_\rho, \]
(4.103)
with
\[ z_\rho = \sum_{i=1}^{n} \rho_i S(b_i) r_i, \]
(4.104)
where only inertial measurements are used without any information on the angular velocity.

The two observers in [Mahony et al., 2008] and [Tayebi et al., 2011] share the same advantage of using only the sensor measurements for attitude estimation and do not need to algebraically compute the orientation. This decreases the computational load and prevents errors from entering the estimations as a result of using a previously computed attitude. Their design is also simple and allows for implementation of such observers on embedded architectures running on low powers.

Other observers closely related to the mentioned nonlinear complementary filters were independently derived in [Campolo et al., 2006] and [Vasconcelos et al., 2008a].

Another type of nonlinear observers, known as Compatible Observers, have been proposed in [Vasconcelos et al., 2008b] and [Mahony et al., 2009]. In the first work, a set of GPS receivers positioned on the flying vehicle with a known order are assumed. The position of each receiver in the inertial frame is denoted as \( p_j, j = 1, ..., r \), and a number of GPS satellites are also assumed to be available with \( p_S, i = 1, ..., s \), as the position of each of those satellites known and expressed in \( I \). The GPS pseudorange measurements are found
by the distance from the GPS satellites to the receivers and a distance offset \( b_c \) due to the clock bias

\[
\rho_{ij} = \| \mathbf{p}_j - \mathbf{p}_S \| + b_c. \tag{4.105}
\]

One of the receivers can be placed at the origin of the body. This receiver is denoted as receiver 1 with position \( \mathbf{p}_1 \) and the positions of other receivers are denoted by \( \mathbf{p}_{i+1} \). Let

\[
\mathbf{x}_i := \mathbf{p}_{i+1} - \mathbf{p}_1, \tag{4.106}
\]

that is expressed and known in \( \mathcal{B} \). By taking the \( \mathbf{x}_i \) vectors as the columns of the matrix

\[
\mathbf{X} := [\mathbf{x}_1 \mathbf{x}_2 \ldots \mathbf{x}_{r-1}] \in \mathbb{R}^{3 \times (r-1)},
\]

a linear combination of the body vectors can be expressed as

\[
\mathbf{y}_i := \sum_{i=1}^{r-1} \mathbf{b}_{ij} \mathbf{x}_i \iff \mathbf{Y}_X = \mathbf{Xb}_X,
\]

where \( \mathbf{b}_X \in \mathbb{R}^{(r-1) \times (r-1)} \) is invertible by construction and \( \mathbf{Y}_X := [\mathbf{y}_1 \mathbf{y}_2 \ldots \mathbf{y}_{r-1}] \in \mathbb{R}^{3 \times (r-1)} \). The vector \( \mathbf{Y}_X \) can be transformed to the inertial frame coordinated by \( \bar{\mathbf{Y}}_X := R\mathbf{Y}_X \) and to the observer frame by \( \hat{\mathbf{Y}}_X := \hat{R}\mathbf{Y}_X \). The nonlinear observer with bias correction is given by

\[
\begin{align*}
\dot{\hat{R}} &= \hat{R} \hat{S}(\hat{\omega}), \\
\dot{\hat{\omega}} &= \hat{R}^T \hat{Y}_X \hat{Y}_X^T \dot{\hat{R}} (\hat{\omega}_y - \hat{\omega}_b) - k_{\omega} \sigma, \\
\dot{\hat{\omega}}_b &= k_{\omega} \sigma, \\
\sigma &= \hat{R}^T \sum_{i=1}^{n} (\hat{Y}_X \mathbf{e}_i) \times (\hat{Y}_X \mathbf{e}_i),
\end{align*}
\tag{4.108}
\]

with \( \mathbf{e}_i \) being the unit vector where \( e_j = 1 \), for \( j = i \). The observer inputs \( \hat{Y}_X = -[\mathbf{f}_p(\rho_2) - \mathbf{f}_p(\rho_1), \ldots, \mathbf{f}_p(\rho_r) - \mathbf{f}_p(\rho_1)]b_X \) and \( \hat{Y}_X = \hat{R}Xb_X \) can be calculated using the vectors \( \rho_j := [\rho_{1j} \ldots \rho_{mj}], j = 1, \ldots, r \), obtained from (4.105) with a constant range bias assumption and known coordinates of pseudo-satellites installed at ground level. \( \mathbf{f}_p(\rho_j), j = 1, \ldots, r \) are functions of the sensor measurements and observer estimates that includes matrices described by the pseudoranges measurements and satellite’s position. The definition and derivation of this function can be found in the original work.
Exponential stability of the origin of the error system for the proposed observer for both biased and unbiased velocity measurements was shown. The work was also extended to the design of a position and linear-velocity observer of the following form

\[ \dot{\hat{p}} = \hat{v} - k_p \hat{p}, \]
\[ \dot{\hat{v}} = \hat{R}b_a + g e_3 - k_v \hat{v}, \]  
(4.109)

where \( b_a \) is the accelerometer reading in the body frame. The same stability analysis was applied to the cascaded system for biased and unbiased velocity measurements.

It should be noted that the complete observer is in fact on \( SE(3) \) since the translational motion dynamics of the moving vehicle were considered in the observer design. However, this observer is discussed in this section since the structure of the attitude observer of (4.108) is on the Special Orthogonal group \( SO(3) \) and even without the position and velocity observer, which does not have any effects on the main attitude observer, the estimated attitude is obtained from a compatible observer on the rotation group. Moreover, the basic idea of designing such an attitude observer is based on the use of multiple GPS receivers installed onboard the flying vehicle, which results in a technique that relies only on GPS data for its estimation. The method, thus, has a fundamental difference in structure with most other \( SE(3) \) based observers that require position data obtained from a set of on-board cameras or image-based position/velocity estimations. The work is partially an extension to the authors’ previous work in [Vasconcelos et al., 2008a], where an attitude observer on \( SO(3) \) with biased angular velocity readings was designed.

In [Mahony et al., 2009], the authors provided a detailed examination of compatible observers and extended the results obtained in [Vasconcelos et al., 2008b] and [Mahony et al., 2008] to design an observer based on vectorial measurements in both inertial and body-fixed frames of known time-varying references. In fact, their design strategy combines the complementary filters and compatible observers providing a design paradigm for vector measurements based nonlinear observers.

In a compatible observer, a set of \( N \) known vectors \( b_{0i}, i = 1, ..., N \) are available in the body fixed frame and their corresponding vectors in the inertial frame are expressed as
\( d_i = Rb_{0i} \).

Although the definition of vector measurements in this case is different than the conventional definition used in complementary filters and other nonlinear observers, the authors propose an observer with a structure quite similar to that of a complementary filter. The following compatible observer

\[
\dot{\hat{R}} = \hat{R}S(\omega + \hat{R}^T \beta), \\
\beta = \sum_{i=1}^{N} k_i S(\hat{d}_i)d_i,
\]

with \( \hat{d}_i = \hat{R}b_{0i} \) is shown to yield a locally exponentially stable equilibrium \( \hat{R} = \hat{R}R^T = I_{3 \times 3} \). The region of attraction includes at least all initial conditions such that \( \text{tr}(I_{3 \times 3} - \hat{R}) < 4 \).

It should be noted that similar to a complementary filter, the assumption of having at least two non-collinear vectors among body-known directions \( b_{0i} \) still holds.

The combined observer assumes that inertial measurements \( d_i \) associated with \( N_1 \) body-fixed reference directions \( d_{0i} \), such that \( d_i = R d_{0i} \), and body measurements \( r_i \) associated with \( N_2 \) inertial reference directions \( r_{0i} \), such that \( r_i = R^T r_{0i} \), are available. In this case, the estimates vectors \( \hat{b}_i = \hat{R}b_{0i}, \ i = 1, ..., N_1 \), in the inertial frame and the body-frame estimates vectors \( \hat{a}_j = \hat{R}^T r_{0j}, \ j = 1, ..., N_2 \) can be used to form a unified observer of the form

\[
\dot{\hat{R}} = \hat{R}S(\omega + \hat{R}^T \beta + \alpha) \\
\beta = \sum_{i=1}^{N_1} k_i S(\hat{d}_i)d_i, \\
\alpha = -\sum_{j=1}^{N_2} k_j S(\hat{a}_j)r_j.
\]

The basic assumption for the combined observer design is to have at least one pair of non-collinear directions in \( \{Rb_{0i}\} \cup \{a_{0j}\} \). Some improvements have been brought to this combined attitude observer by [Jensen, 2011] through the inclusion of time-varying gains.
4.6 Global Attitude Estimators Evolving Outside SO(3)

As discussed before, a direct cosine matrix is a member of the Special Orthogonal group \( SO(3) \) and a unit quaternion is a member of the Quaternion group \( S^3 \). These two representations are the most commonly used attitude representations in the nonlinear observer design for the rigid body attitude.

As the previously discussed attitude estimation classes, nonlinear attitude observers design techniques also experienced an evolution of ideas according to the challenges and open problems encountered in this field. In Kalman filtering methods, a considerable attention was given to the conservation of the nature of generated estimates, \( \text{i.e.,} \) if the estimated state was a rotation matrix it was desired to be a member of the Special Orthogonal group, or if the quaternion representation was estimated, normalization methods were used to preserve the unit norm of the estimated quaternion. This is necessary since the Kalman filters essentially deal with sensor measurements.

In the case of nonlinear observers, the earliest observers all gave estimations that were members of the group in which the attitude representation was defined. In other words, the observer structure imposed restrictions on \( \hat{\mathbf{R}}(t) \in SO(3) \) or \( \hat{\mathbf{Q}}(t) \in S^3 \), so that the estimate of the orientation would necessarily become a member of rotation group itself. However, some recent papers have investigated the ways in which the estimates where allowed to evolve in an Euclidian space and converge asymptotically to a final value that belongs to \( SO(3) \). These observers provide the interesting ability to overcome the topological obstructions to the global asymptotic stability on the Euclidean spaces, discussed in [Bhat and Bernstein, 2000], [Chaturvedi et al., 2011].

There are just a few nonlinear observers in the literature, that do not preserve the rotation group structure. In the following sections, we review two of the most recent nonlinear observers that have such characteristics and give details on their difference with conventional group-preserving attitude observers.

The problem addressed in [Grip et al., 2012b], is to design an observer for an interconnected system where a nonlinear subsystem \( S_1 \) is connected to a linear subsystem \( S_2 \), with
connection signal $z$ being the output of the first system and the input of the second system $S_2$. The two subsystems are given by

$$S_1 : \begin{cases} \dot{x} = f(u, w, x) \\ z = h(u, w, x) \end{cases} \quad (4.112)$$

$$S_2 : \begin{cases} \dot{w} = Aw + Bu + B_z z \\ y = Cw + Du + D_z z \end{cases} \quad (4.113)$$

A schematic of the overall nonlinear system is shown in Fig. (4.1).

![Figure 4.1](image)

Figure 4.1: Block diagram of the interconnected nonlinear and linear systems, from [Grip et al., 2012b]

![Figure 4.2](image)

Figure 4.2: Block diagram of the rotational and transitional dynamics of a flying vehicle expressed in a cascaded structure.

The goal is to design an observer for the complete system to estimate the states of both nonlinear and linear subsystems. Defining $w_e = [w^T, z^T]^T$ as the extended state vector, the
extended system dynamics reads

\[
\dot{w}_e = A_e w_e + B_e u + B_d d(u, \dot{u}, x),
\]

\[
y = C_e w_e + D_a u,
\]

(4.114)

with

\[
d(u, \dot{u}, x) := \frac{\partial h}{\partial u}(u, x)\dot{u} + \frac{\partial h}{\partial x}(u, x)f(u, x),
\]

and

\[
A_e = \begin{bmatrix} A & B_z \\ 0 & 0 \end{bmatrix},
B_e = \begin{bmatrix} B_u \\ 0 \end{bmatrix},
B_d = \begin{bmatrix} 0 \\ I \end{bmatrix},
C_e = \begin{bmatrix} C & D_z \end{bmatrix}.
\]

In order to design an observer for the interconnected system, the methodology is to assume that an observer already exists for the nonlinear subsystem \(S_1\) and is given by

\[
\dot{\hat{x}} = f(u, \hat{x}) + g(u, \hat{x}, \hat{z}),
\]

(4.115)

leading to exponential stability results. Making some other assumptions on the detectability of the pair \((A, C)\) in the linear system \(S_2\) and the boundedness of functions \(g(\cdot)\) and \(d(\cdot)\), the observer for the linear subsystem state in the extended system is defined as

\[
\dot{\hat{w}} = A_e \hat{w} + B_e u + B_d d(u, \dot{u}, \hat{x}) + K_e (y - C_e \hat{w} - D_a u - D_z \hat{z}),
\]

\[
\hat{z} = h(u, \hat{x}) + \xi,
\]

\[
\dot{\xi} = -\frac{\partial h}{\partial x}(u, \hat{x})g(u, \hat{x}, \hat{z}) + K_z (y - C_e \hat{w} - D_a u - D_z \hat{z}),
\]

(4.116)

where \(K_u\) and \(K_z\) are matrices of observer gains and are determined in a way to assure that \(A - K\) is Hurwitz, with \(K = [K_u^T, K_z^T]^T\). The extended observer system is then expressed as

\[
\dot{\hat{w}}_e = A_e \hat{w}_e + B_e u + B_d d(u, \dot{u}, \hat{x}) + K (y - C_e \hat{w}_e - D_a u).
\]

(4.117)

The attitude estimation problem can also be formulated as the problem of estimating the states of two subsystems where the nonlinear subsystem \(S_1\) represents the rotational rigid body kinematics and the linear subsystem \(S_2\) is composed of translational dynamics

\[
S_1 : \begin{cases} \dot{\hat{R}} = R S(\omega) \\ \dot{r}_u = R b_u \end{cases},
\]

(4.118)
The apparent acceleration in the inertial frame \( r_a = R b_a \) is assumed to be known. The vector \( b_a \) is the apparent acceleration measured in the body frame and an input to the second subsystem. The known signals are the Earth magnetic field \( b_m = R^T r_m \), the apparent acceleration \( b_a = R^T r_a \) and the angular velocity \( \omega \), measured in the body frame, as well as the linear velocity \( v \) measured in the inertial frame using a GPS.

One should notice that based on (4.120), the estimated rotation matrix \( \hat{R} \) does not belong to \( SO(3) \).

The concept of using such a correction matrix is taken from TRIAD algorithm. Since \( J = \hat{R} A_B A_B^T \), a simple Lyapunov function of the form \( V = \frac{1}{2} \| \hat{R} \|^2 \) can be used to show the exponential convergence of estimated rotation matrix \( \hat{R} \) to the actual rotation matrix \( R \).

The complete observer, using an estimate of the inertial apparent acceleration, aided by the linear-velocity measurement, is given by

\[
\begin{align*}
\dot{\hat{R}} &= \hat{R} \hat{S}(\omega) + \Gamma J(r_m, b_m, b_a, \hat{R} b_a + \xi, \hat{R}), \\
\dot{\hat{v}} &= \hat{R} b_a + \xi + g e_3 + K_{vp} \hat{p} + K_{vo} \hat{v}, \\
\dot{\hat{p}} &= \hat{v} + K_{pp} \hat{p} + K_{pv} \hat{v}, \\
\dot{\xi} &= -\Gamma J(r_m, b_m, b_a, \hat{R} b_a + \xi, \hat{R}) b_a + K_{zp} \hat{p} + K_{zv} \hat{v},
\end{align*}
\]  

(4.121)

where \( \hat{v} = v - \hat{v} \), and \( K_{pp}, K_{pv}, K_{vp}, K_{vo}, K_{zp}, \) and \( K_{zv} \) are the observer gain matrices and are chosen to ensure stability of the error dynamics.
Grip’s method expands the observer system in an interesting way to include the linear equations of translational motion along with the rotational motion. It can be pointed that since the gyro bias estimation problem is not addressed in this work, it remains a topic for future research.

Another observer with elements out of the special group was recently presented in [Batista et al., 2012]. In this observer, it is assumed that at least two vector observation pairs are available \( (i.e., \text{small-acceleration assumption}) \). It has a cascade structure in which a first observer is designed to estimate the gyro bias and the second observer takes this estimate as an input to find an estimate of the rotation matrix.

Assuming that vectors \( b_i = R^T r_i \) are known in \( \mathcal{B} \) and the gyroscope measures the bias-contaminated angular velocity \( \omega_y = \omega + \omega_b \), one can find the time derivative of \( b_i \) as

\[
\dot{b}_i = -S(\omega_y)b_i - S(b_i)\omega_b, \quad i = 1, \ldots, N. \tag{4.122}
\]

The first observer is designed by taking the observation vectors and gyro bias as subsystem states. The gyro bias is considered to remain constant with time and has dynamics \( \dot{\omega}_b = 0 \).

The observer dynamics is then given by

\[
\begin{align*}
\dot{\hat{b}}_i &= -S(\omega_y)\hat{b}_i - S(b_i)\hat{\omega}_b + \alpha_i(b_i - \hat{b}_i), \quad i = 1, \ldots, N, \\
\dot{\hat{\omega}}_b &= \sum_{i=1}^{N} \beta_i S(\hat{b}_i)b_i
\end{align*} \tag{4.123}
\]

where \( \alpha_i, \beta_i, i = 1, \ldots, N \), are positive scalar constants. The Lyapunov function

\[
\mathcal{V}_1 = \frac{1}{2} \sum_{i=1}^{N} \beta_i |\hat{b}_i|^2 + \frac{1}{2} |\hat{\omega}_b|^2, \tag{4.124}
\]

with \( \hat{\omega}_b = \omega_b - \hat{\omega}_b \), is used to show the global exponential stability of the observer error dynamics. Note that the \( \hat{b}_i, i = 1, \ldots, N \) are directly estimated, independent of the estimate for the rotation matrix \( \hat{R} \). The advantage of this approach is that the bias is obtained without need for rotation matrix estimation.

In order to design an attitude observer, the authors consider estimating the rows of the
rotation matrix considered as states defined as

\[ x_2 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{R}^9, \]

where

\[ R = \begin{bmatrix} z_1^T \\ z_2^T \\ z_3^T \end{bmatrix}, \quad z_1, z_2, z_3 \in \mathbb{R}^3. \]

Using the rotational dynamics, one can show

\[ \dot{x}_2 = -S_3(\omega_y - \omega_b)x_2, \quad (4.125) \]

where

\[ S_3(x) := \text{diag}(S(x), S(x), S(x)) \in \mathbb{R}^{9 \times 9}. \]

The observer for state vector of this subsystem is given by

\[ \dot{\hat{x}}_2 = -S_3(\omega_y - \omega_b)\hat{x}_2 + C_2^T Q^{-1} C_2 (x - \hat{x}), \quad (4.126) \]

where \( Q \in \mathbb{R}^{3N \times 3N} \) is a positive definite symmetric matrix and \( C_2 \) is a \( 3N \times 9 \) matrix that relates the output vector \( b = [b_1^T, b_2^T, ..., b_N^T]^T \) to the rotation matrix rows

\[ b = C_2 x_2. \]

Subtracting (4.125) from (4.126) gives the error dynamics

\[ \dot{\tilde{x}}_2 = A_2 \tilde{x}_2, \quad (4.127) \]

where

\[ A_2 := -S_3(\omega_y - \omega_b) - C_2^T Q^{-1} C_2. \]

Under the assumption of having at least two non-collinear vectors \( r_j \) in the inertial frame, it can be shown that for a known gyro bias, the origin of (4.127) is globally exponentially stable.
The final observer system is then a cascade of the two observer subsystems where the estimated bias is fed into the second observer

\[
\dot{\hat{b}}_i = -S(\omega_y)\hat{b}_i - S(b_i)\hat{\omega}_b + \alpha_i(b_i - \hat{b}_i),
\]

\[
\dot{\hat{\omega}}_b = \sum_{i=1}^{N} \beta_i S(\hat{b}_i)b_i, \tag{4.128}
\]

\[
\dot{\hat{x}}_2 = -S_3(\omega_y - \hat{\omega}_b)x_2 + C_TQ^{-1}C_2(x_2 - \hat{x}_2).
\]

The proof for stability of the final nonlinear observer is given in a straight-forward manner. Since the first observer remains intact in the cascading process, its stability in the cascade system is guaranteed and \(\lim_{t \to \infty} \|S_3(\tilde{\omega}_b(t))\| = 0\). The stability of the second part is also analyzed by incorporating the error bias term \(\tilde{\omega}_b\) into (4.127). Since the origin of the unperturbed system (i.e., without the term \(\tilde{\omega}_b\)) is globally exponentially stable and the \(\tilde{\omega}_b\) converges exponentially to zero, the origin of the perturbed system is also globally exponentially stable.

It is obvious from (4.126) that the estimated rotation matrix has no restrictions on its bound. Therefore, it is another observer with estimations that do not evolve in \(SO(3)\). In this case, the approach has enabled the authors to achieve global asymptotic stability.

The observers designed in the two mentioned works are two of the few observers that have attitude estimations which are members of the non-Euclidean special group. The concept of designing observers that do not evolve in \(SO(3)\) may be interesting since in the absence of restrictions on the structure of the estimated matrix, one may have free hands for many different innovations. Moreover, these observers achieve global results, in contrast with the almost global results achieved with the observers evolving in \(SO(3)\). Although this may not be an important issue in practice as the noise prevent the system from remaining in the unstable equilibria, this achievement is theoretically important.


4.7 Velocity-Aided Attitude Estimation

The case of time-varying reference vectors has been studied by researchers interested in the situation where the known inertial vectors are no longer available. In fact, instead of having a set of \( N \) constant vectors \( r_{0i}, i = 1, ..., N \) in the inertial frame \( I \) and time-varying observations \( b_i(t), i = 1, ..., N \) in the body frame \( B \), available vectorial observations are time-varying pairs of \((r_i(t), b_i(t))\). This is the case when, for instance, an accelerometer is used in accelerated flights. The linear acceleration in the inertial frame is unknown and time-varying. It can be expected that the knowledge of vehicle’s linear velocity may help in better using the accelerometer readings for attitude estimation purposes. This strategy is known as Velocity-Aided Attitude Estimation.

As discussed in the second chapter, an accelerometer measures the apparent acceleration \( b_a \), which is the sum of observed gravity vector in body frame and the vehicle’s linear acceleration vector

\[
b_a = -R^T g + R^T \dot{v}.
\]  

For the majority of the existing nonlinear attitude observers based on accelerometers and magnetometers measurements, it is assumed the rigid body is subject to relatively small linear accelerations and hence, the accelerometer output is approximated by \( b_a \approx -R^T g \).

As discussed in section (4.3), using an accelerometer (under low acceleration assumption), one can obtain the roll \( \phi \) and pitch \( \theta \) angles. Although some filters have been designed based on this estimation of roll and pitch (e.g. in [Wang et al., 2004]), it should be noted that since the linear acceleration vector \( \dot{v} \) is unknown, the assumption that apparent acceleration corresponds to a known vector in inertial frame no longer holds for accelerating vehicles. This results in inaccurate attitude estimates when nonlinear complementary filters or other observers based on low-acceleration assumption are used.

The available solutions to this problem use GPS data in order to compensate for the lack of information on the linear acceleration vector in the inertial frame. The velocity and sometimes the position obtained from the GPS, are used in the observers pro-
posed in such works as [Hua, 2010], [Roberts and Tayebi, 2011b], [Mahony et al., 2011], [Grip et al., 2012b].

In [Hua, 2010], the author uses the rotation matrix representation and proposes an attitude observer similar to the nonlinear complementary filter with the difference that the inertial acceleration is assumed to be unknown. The proposed attitude observer, aided by the linear velocity measurement, is given by

\[ \dot{\hat{R}} = \hat{R}S(\omega + \sigma), \]
\[ \sigma = k_m S(b_m)\hat{b}_m + k_a S(b_a)\hat{R}^T(v - \hat{v}), \]
\[ \hat{v} = k_v (v - \hat{v}) + ge_3 + \hat{R}b_a, \]

with \( k_m, k_a, k_v \) positive constant gains. Defining \( (\hat{R}, \hat{v}) = (\hat{R}^TR, v - \hat{v}) \), it is shown that for all initial values except for the set

\[ \mathcal{U} := \{ R \in SO(3) \mid \text{tr}(R) = -1 \}, \]

the variables \( (\hat{R}, \hat{v}) \) exponentially converge to the equilibrium \( (I_3, 0) \). It should be noted that the eigenvalues of an orthogonal matrix are of the form

\[ \text{eig}(R) = \{ 1, \cos(\theta) + i \sin(\theta), \cos(\theta) - i \sin(\theta) \} \]

where \( \theta \) is the angle from the angle-axis representation and \( i \) denotes the imaginary unit. In the case that all eigenvalues are real, \( \theta \) has to be either 0 or \( \pm \pi \). The first possibility corresponds to the desired case where rigid body orientation is fixed and \( R = I \). The second case, however, corresponds to the undesired equilibria where \( \text{tr}(R) = -1 \).

The other approach the author takes consists in estimating the inertial vector \( r_a \) corresponding to the observed vector \( b_a \). For this, an auxiliary matrix \( A \) is defined to play the role of rotation matrix. It is proved that the observer

\[ \dot{\hat{v}} = k_v (v - \hat{v}) + ge_3 + Ab_a, \]
\[ \dot{A} = AS(\omega) + k_d (v - \hat{v})b_a^T, \]
with $k_v, k_A$ some positive constant gains, guarantee that the error $(r_a - Ab_a, \tilde{v})$ converges to zero. The Lyapunov function proposed to show this convergence is given by

$$ V = \frac{1}{2} \| \tilde{v} \|^2 + \frac{1}{2k_A} \| R - A \|^2, \quad (4.132) $$

where the time-derivative of the function is found to be $\dot{V} = -k_i \| \tilde{v} \|^2$. Applying Barbalat Lemma results in the conclusion that $\hat{v}$ converges to $v$. The convergence of vector $Ab_a$ to $r_a$ can also be shown by some assumptions on the boundedness of $\ddot{v}$ and the angular velocity $\omega$. These result in exponential stability proof of the equilibrium ($\tilde{v} = 0, r_a - Ab_a = 0$) to zero.

It should be noted that although the term $Ab_a$ ultimately converges to $r_a = Rb_a$, the auxiliary matrix $A$ is not a rotation matrix. This can be easily seen from (4.131), where the innovation term for dynamics of auxiliary matrix is directly added to the right-hand side of the equation and not to the angular velocity. The same can be seen from the choice of Lyapunov function in (4.132), where the conventional terms containing $\tilde{R} = \hat{R}^T R$ are replaced by $\| R - A \|^2$.

Combining the two observer systems of (4.130) and (4.131), the full observer system is obtained as

$$ \dot{\hat{R}} = \hat{R}S(\omega + \sigma), $$

$$ \sigma = k_m S(b_m) \hat{R}^T r_m + k_A S(b_a) \hat{R}^T (Ab_a + k_A (v - \hat{v})), $$

$$ \dot{\hat{v}} = k_v (v - \hat{v}) + ge_3 + Ab_a, $$

$$ \dot{A} = AS(\omega) + k_A (v - \hat{v}) b_a^T. \quad (4.133) $$

Unlike the rotation-matrix-based estimation approach in [Hua, 2010], the authors in [Roberts and Tayebi, 2011b] take a quaternion-based approach with a much simpler observer stability proof. Their observer is given by

$$ \dot{\hat{Q}} = \frac{1}{2} \hat{Q} \otimes \begin{pmatrix} 0 \\ \omega + \sigma \end{pmatrix}, $$

$$ \sigma = k_m S(b_m) \hat{R}^T r_m + k_A k_v S(b_a) \hat{R}^T (v - \hat{v}), $$

$$ \dot{\hat{v}} = k_v (v - \hat{v}) + ge_3 + \hat{R}b_a + \frac{1}{k_v} \hat{R}S(\sigma)b_a. \quad (4.134) $$
where almost global exponential convergence is proven for all initial states except for quaternion error $\hat{Q}$ characterized by $\hat{q}_0 = 0$. The Lyapunov function considered for the observer’s stability proof is

$$V = \frac{\gamma}{2} \hat{r}^T \hat{r} + \gamma_q (1 - \hat{q}_0^2),$$

with

$$\hat{r} = k_v \hat{v} - (I - \hat{R}) r_a.$$  (4.136)

The definition of the signal $\hat{r}$ helped the authors to fuse the accelerometer readings with the linear velocity obtained from GPS. In fact, the update law for $\hat{v}$ in (4.134) approaches the equation describing the relationship between accelerometer output and linear acceleration when velocity estimate error $\hat{v}$ goes to zero. It is reported that the observer performed well under assumptions of a relatively large linear acceleration of rigid body.

In applications where the GPS data is not available, such as indoor, or in environments such as forests where GPS data is weak and discontinuous, air pressure measurements can replace the velocity and position measurements in the observer design since they give a sense of vehicle’s linear velocity through the surrounding air. The authors in [Mahony et al., 2011] used this idea to estimate the attitude of a fixed-wing UAV in accelerated mode without GPS measurements. The filter, which in nature is quite similar to an explicit complementary filter, is a $SO(3)$ observer of the form

$$\dot{\hat{Q}} = \frac{1}{2} \hat{Q} \otimes \begin{pmatrix} 0 \\ \omega_{meas} + \delta \end{pmatrix},$$  (4.137)

where the correction term $\delta$ is a PI filter of an error vector $e$ given by

$$\delta = k_p e + k_i \int_0^t e d\tau,$$  (4.138)

$$e = \vec{v} \times \hat{v},$$

with $k_p$ and $k_i$ being the proportional and integral gains, respectively. In their notation, $\vec{v} := \hat{\mathbf{g}}/\|\hat{\mathbf{g}}\|$ is the low-frequency normalized estimate of the gravitational direction and $\hat{v} = \hat{R}^T e_3$ is the expected gravitational direction in the body-fixed frame. Vector $\hat{\mathbf{g}}$ is the
estimate of the gravitational direction given by
\[ \hat{g} = -(y_a - \hat{b}_a), \quad (4.139) \]
where \( y_a \) is the accelerometer measurement in the body frame and \( \hat{b}_a \) is the estimate of the rigid body’s acceleration with respect to the inertial frame expressed in body frame. Acceleration of the rigid body can be estimated as
\[ \hat{b}_a = \omega \times V_{\text{air}}, \quad (4.140) \]
where the body frame expressed airspeed \( V_{\text{air}} \) can be formulated as a function of the changing angle of attack \( \alpha \) and its magnitude \( |V_{\text{air}}| \) is measured from the calibrated dynamic pressure measurements. The angle of attack \( \alpha \) is also a function of airspeed magnitude and pitch rate \( \dot{\theta} \). For the sake of simplicity, the details of these air-related relations are not presented here and readers are encouraged to see the reference [Mahony et al., 2011] for more information. The block diagram of the overall filter can be seen in Fig. (4.3), with \( \omega_y = \omega_{\text{meas}} \).

![Figure 4.3: Block diagram of the complementary filter with airspeed measurements, from [Mahony et al., 2011]](image)

Although the authors give no stability proofs for the proposed filter, the experimental results for the filter behavior show a performance comparable to that of an Extended...
Kalman Filter with GPS measurements. The advantage of their strategy, however, is that the problem of GPS data requirement for velocity compensation was overcome by using an additional measurement of another variable (here, airspeed). This is a considerable improvement for the vehicles that fly in environments without having access to GPS data or with poor reception. An example of such indoor attitude observers is the extended Kalman filter designed in [Vissiere et al., 2007], where an orthogonal trihedron structure of four magnetometers is used to take advantage of the positional variations of surrounding magnetic field of a flying small UAV. The magnetic field is a function of position and comparing the four magnetometers readings results in an estimate of vehicles position. This compensates the lack of GPS data in indoor or covered spaces.

### 4.8 Nonlinear Observers on SE(3)

Extending the attitude estimation problem to pose estimation by using nonlinear observers has attracted the attention of the research community in the last decade. Some researchers have tried to design nonlinear observers using the GPS data fused with IMU measurements (see e.g., [Vik and Fossen, 2001], [Baldwin et al., 2007], [Vasconcelos et al., 2008b]), while others tried to develop techniques to take advantage of vision-based camera measurements.

Computer vision applications in the estimation of the pose have long been known and studied by many researchers, reported in the survey papers of [Huang and Netravali, 1994], and [Olensis, 2000]. In this section, we try to present the latest developments in the field of pose estimation with visual and vectorial measurements.

One of the pioneering works in the nonlinear observer design for pose estimation was proposed in [Rehbinder and Ghosh, 2003], where vision-based measurements along with inertial measurements were used to develop a locally convergent observer for attitude estimation. The observer evolves on the Special Orthogonal group $SO(3)$ and does not include position and velocity estimation, but provides a theoretical framework for the combination of IMU measurements and camera recordings to estimate translational motion.
Consider a set of fixed lines \( l_i, i = 1, ..., N \), known in the inertial frame. The lines are represented by

\[
l_i = \{ x_I \in \mathbb{R}^3 : x_I = \xi_{i,I} + d_{i,I}s, s \in \mathbb{R} \},
\]

where \( \xi_i \) is a point arbitrarily taken on \( l_i \), and \( d_i \) is its direction vector. The scalar \( s \) determines the length of vector in each direction. In the body frame, the line is represented by

\[
l_i = \{ x_B \in \mathbb{R}^3 : x_B = R^T \xi_{i,I} + d_{i,I}s - p, s \in \mathbb{R} \},
\]

with \( p \) denoting the rigid body position expressed in the inertial frame. The camera is assumed to be placed in the rigid body such that its focal point coincides with the origin of \( \mathcal{B} \). With such assumption, each line \( l_i \) is projected onto the image plane \( P_I \) as an intersection of this plane with the camera focal point \( P_I \). The normal vector \( \eta_i \) to \( P_I \) is given by

\[
\eta_i = R^T [d_i \times (\xi_i - p)].
\]

Since the camera focal plane never becomes perpendicular to the plane in which the line exists, i.e., \( d_i \) and \( \xi_i - p(t) \) are not parallel, it can be assumed that the norm of \( \eta_i \) never becomes zero. Therefore, the observations in the body frame can be taken as

\[
y_i = \mu_i \frac{\eta_i}{||\eta_i||},
\]

with \( \eta_i \in \{-1, 1\} \) being an unknown parameter for sign ambiguity. The existence of this sign parameter for the unknown line directions does not affect the structure of the observer.

The vectors \( y_i \) correspond to the normalized normal vectors of some known lines in the inertial frame projected onto the camera plane. Assuming that at least three non-collinear inertial lines are available, by intuition a reconstruction of the orientation is possible, since each observer normal vector can be assigned as a basis to generate orthogonal bases in \( \mathcal{B} \). In this way, the attitude observer with point and line observations is given by

\[
\begin{align*}
\dot{\hat{R}} &= \hat{R} S(\omega_c) + \sigma, \\
\sigma &= k_i \sum_{i=1}^N y_i^T (\hat{R}^T d_i) S(y_i \times \hat{R}^T d_i),
\end{align*}
\]

where \( \hat{R} \) is the estimated attitude matrix and \( \sigma \) is the estimation error.
where $k_i$ are positive scalars. In order to show the convergence of this observer, the following Frobenius norm Lyapunov function is considered
\[
\mathcal{V} = \frac{1}{2} \|X\|_F^2 = \frac{1}{2} \|I - \hat{R}^T R\|_F^2. \tag{4.146}
\]
The local representation of the error rotation matrix $\hat{R} = \hat{R}^T R$ is derived as a function of $\|X\|_F$ using the Rodrigues attitude representation
\[
\hat{R} = I + S(\hat{k}) \frac{\|X\|}{\sqrt{2}} + O(\|X\|^3), \tag{4.147}
\]
where $\hat{k} \in \mathbb{R}^3$ with $\|\hat{k}\| = 1$. The term $O(x^k)$ contains higher order terms with $x^k$ as its lowest degree term. Local exponential convergence of the estimated attitude is shown by taking the time derivative of the Lyapunov function (4.146)
\[
\dot{\mathcal{V}} = -(R^T \hat{k})^T \left( \sum_{i=1}^N k_i \frac{P_i p_i^T}{\|p_i\|^2} \right) (R^T \hat{k}) \|X\|^2 + O(\|X\|^3). \tag{4.148}
\]

In order for the observer to be locally convergent, a condition of trivial observability is investigated. Letting $p_i(t)$ be the shortest vector from the position $p(t)$ to the line $l_i$, i.e., $p_i(t) = p(t) - \xi_i(t)$ is chosen such that $d_i^T (p(t) - \xi) = 0$, and the information matrix be given by
\[
Q(t) = \sum_{i=1}^N \frac{p_i(t) p_i^T(t)}{\|p_i(t)\|^2}, \tag{4.149}
\]
the system is called *trivially observable* if the following condition is met
\[
\det Q(t) \geq q, \tag{4.150}
\]
for a non-negative $q$. Assuming that this observability condition holds, the Lyapunov function derivative of (4.148) leads to
\[
-\dot{\mathcal{V}} \geq q \min[k_i] (R^T \hat{k})^T (R^T \hat{k}) \|X\|^2 + O(\|X\|^3) \geq q \min[k_i] \|\hat{k}\|^2 \|X\|^2 + O(\|X\|^3) \tag{4.151}
\]
The proposed observer produced good results in simulations with both biased and non-biased angular velocity measurements. However, one disadvantage of such strategy is that
the direction of the lines should be known. This assumption can be avoided by developing observers working only with known points in the inertial frame, known as landmarks.

Authors in [Baldwin et al., 2007] designed a complementary filter on the Special Euclidean group $SE(3)$ for the pose estimation. In their work, they study two different complementary filter forms, direct filters and explicit filters. While the latter is simply named Complementary Filter on $SE(3)$, both filters are designed on methodologies similar to their corresponding filters in nonlinear complementary filtering on $SO(3)$, and require an estimate of the rotation matrix and the position in their structure. These estimates are obtained from classical estimation algorithms for pose reconstruction. The velocity is also obtained from direct integration of accelerometer readings.

Let $(R_y, P_y, V_y)$ be the noise free measurements of the system attitude, position and velocity in the body frame, the complementary filter structure is defined as

$$
\dot{\hat{T}} = \hat{T} \hat{A}, \tag{4.152}
$$

with the estimated angular and translational velocities in $\hat{A}$ given by

$$
\begin{align*}
\hat{\omega} &= (S(\omega_y) - k_R \text{Ad}_{R^T} P_d(\hat{R}R_y^T))_{\otimes} \\
\hat{V} &= V_y - S(\hat{\omega} - \omega_y)P + k_p(\hat{P} - P_y)
\end{align*} \tag{4.153}
$$

where $(.)_{\otimes}$ denotes the inverse of skew-symmetric transformation. $k_p$ and $k_R$ are positive gains that are selected based on the analysis of the sensors used. The $\text{Ad}_T$ denotes the adjoint operator defined as

$$
\text{Ad}_T Q = TQT^{-1}. \tag{4.154}
$$

Similar to the definition of the error rotation matrix, the error matrix element of $SE(3)$ is defined as

$$
\tilde{T} = \begin{pmatrix} \tilde{R} & \tilde{p} \\ 0 & 1 \end{pmatrix} = \hat{T}T^{-1}, \tag{4.155}
$$

with

$$
\tilde{R} = \hat{R}R^T, \quad \tilde{p} = \hat{p} - \hat{R}p. \tag{4.156}
$$
Using the definition of the error element in $SE(3)$ and the fact that translational and rotational dynamics can be regarded as separate components of that element, the following familiar Frobenius norm-based Lyapunov function, in $SE(3)$, is proposed

$$
\mathcal{L} = \frac{1}{2} \| I_{4 \times 4} - \hat{T} \|_F^2,
$$

(4.157)

where it can simply be shown that the Lyapunov function is the addition of two sub-functions $\mathcal{L}_R$ and $\mathcal{L}_p$ defined as

$$
\mathcal{L}_R = \text{tr}(I_{3 \times 3} - \tilde{R}),
$$

$$
\mathcal{L}_p = \frac{1}{2} \| \tilde{p} \|_2^2,
$$

(4.158)

Both terms demonstrate the basic structure of the underlying group with $SO(3)$ properties of the rotational dynamics.

The derivative of the Lyapunov sub-functions , in view of (4.153), are given by

$$
\dot{\mathcal{L}}_R = -k_R \| \mathcal{P}_o(\tilde{R}) \|_2^2,
$$

$$
\dot{\mathcal{L}}_p = -2k_P \| \tilde{p} \|_2^2.
$$

(4.159)

Therefore, it can be deduced that $(\tilde{R}, \tilde{p})$ converge exponentially to $(I, 0)$. In other words, the $SE(3)$ element $\hat{T}$ is exponentially convergent to the actual pose $T$. Simulation result show robustness of the filter to noise added to the measurements and high stability of the proposed technique in presence of disturbances. It is demonstrated that the effect of noise in angular velocity measurements is more prominent than other sensor measurements, and not only causes delayed convergence and oscillations in the estimated attitude, but in an offset error in the estimations of the position. This result can be expected from the fact that while rotational dynamics is independent of the translational dynamics, the latter is affected by the angular velocity of vehicle.

In [Baldwin et al., 2009], the authors also proposed an observer for pose estimation using bearing measurements without requirement of the algebraic construction of the rotation matrix to be included in the observer structure.
Omnidirectional cameras allow obtaining landmark bearings directly and easily. The bearing (azimuthal) locations are extracted without requiring assumptions on the smoothness or rigidity of the objects in the scene. Landmark bearings are important measurements, on which some map-building techniques such as the Simultaneous Localization and Mapping (SLAM) systems rely [Spero, 2005], and have also long been studied for tracking problems with EKF techniques [Aidala and Hammel, 1983].

Consider a set of landmark points \( z_i, i = 1, ..., N \) known in the inertial frame \( I \) and distributed in the space in such a way that non-collinear points are available. The body-expressed coordinates of these landmarks, denoted as \( Y_i \) in \( B \), can be found using the inverse of the matrix \( T \) as

\[
Y_i = h(T, z_i) = T^{-1}z_i.
\] (4.160)

The assumption that the position of landmarks do not change with time (i.e., stationary landmarks), gives the time derivative of the landmark positions in \( B \) as

\[
\dot{Y}_i = -S(\omega)Y_i - V.
\] (4.161)

In visual measurements, the bearing of the \( i \)-th landmark from the origin of \( B \) observed by the spherical camera can be shown as \( X_i \in S^2 \), which is a projection of the visible point \( Y_i \) onto the image surface

\[
X_i = \frac{Y_i}{\|Y_i\|}.
\] (4.162)

The kinematics of the projected point is derived by taking the time-derivative of \( X_i \) on the spherical surface, that is known as optical-flow equations:

\[
\dot{X}_i = -S(\omega)X_i - \frac{\pi_{X_i}}{\|Y_i\|} V,
\] (4.163)

where \( \pi_{X_i} = (I_{3 \times 3} - X_iX_i^T) \) is the projection \( \pi_X : \mathbb{R}^3 \rightarrow T_XS^2 \), which is the tangent space of the sphere \( S^2 \) at the point \( X \in S^2 \). For more information on the computation of optical flow in spherical coordinates, readers are referred to [Vassallo et al., 2002]. For an application of optical flow for UAV control and autonomous landing, see [Herisse et al., 2011].
Assuming that at least three non-collinear points $z_i$ are available with a bounded piece-wise continuous $A$ and trajectory $T$, the proposed local observer is given by

$$\begin{align*}
\dot{\hat{T}} &= \hat{T} \begin{bmatrix} A_y + \begin{pmatrix} S(\xi_\omega) & \xi_V \\ 0 & 0 \end{pmatrix} \\ \end{bmatrix}, \\
\xi_\omega &= -k_\omega \sum_{i=1}^N \hat{X}_i \times X_i, \\
\xi_V &= -k_V \sum_{i=1}^N \frac{\pi \hat{X}_i}{\|\hat{Y}_i\|},
\end{align*}$$

(4.164)

where $A_y$ denotes the matrix of the measured velocities in $B$ and the estimated vectors $\hat{X}_i$ and $\hat{Y}_i$ are defined as

$$\begin{align*}
\hat{X}_i &= \frac{\hat{Y}_i}{\|\hat{Y}_i\|}, \\
\hat{Y}_i &= \hat{T}^{-1}z_i.
\end{align*}$$

(4.165)

As for the definition of the error in the previous work, the error matrix is taken as $\tilde{T} = \hat{T}T^{-1}$. However, since there are no measured poses available, the Lyapunov function using difference between the measured and estimated bearings is given by

$$\mathcal{V} = \frac{1}{2} \sum_{i=1}^N \|\hat{X}_i - X_i\|^2 = \frac{1}{2} \sum_{i=1}^N \|\tilde{X}_i\|^2.$$ 

(4.166)

Since the observer structure respects the geometry of the underlying Special Euclidean group, a necessary and sufficient condition for $\hat{X}_i = X_i$ is for the group matrix error $\tilde{T}$ to become the identity matrix $I_{4 \times 4}$. First order local approximations of the estimated variables around the trajectory $\tilde{T} = I$ can be found as $\hat{X}_i = X_i + \alpha_i$ and $\|\hat{Y}_i\| = \|Y_i\| + \beta_i$, where $\alpha_i$ and $\beta_i$ are small perturbations. The stability analysis then continues by substituting these approximations into the time derivative of the Lyapunov function and by a choice of sufficiently large gain $k_V$, local asymptotic stability of the observer error dynamics around $I$ is proved.

The observer has the advantage of using only the bearing measurements for pose estimation and requires no IMU measurements. The linear velocity can be available from GPS data and the angular velocity, measured by gyroscopes, is assumed to give bias-free readings. Since the rotation matrix is not required to be known in this observer, it has an advantage over the $SE(3)$ complementary filter.
The application of landmark position measurements in pose estimation has also been the basis of a recent work in [Vasconcelos et al., 2010]. In this study, a set of landmarks, whose position is known in $I$, is assumed and the attitude is estimated without the use of IMUs. While it is assumed that the position of landmarks in $B$ is measured by onboard sensors such as cameras or Laser Radars (LADARs), the velocity of the flying vehicle is considered to be available from a Doppler-effect-based sensor.

Consider the set of points $p_i, i = 1, ..., N$ with coordinates $x_i, i = 1, ..., N$ in the local frame. The position of each point expressed in the body frame is $q_i = R^T x_i - p$, with $p$ being the position of the rigid body with respect to $I$ expressed in the body frame $B$. The landmarks centroid is taken as the origin of the inertial frame. The two known sets of landmark coordinates in local and body frames can be expressed in matrix form as $X = [x_1 ... x_N]$ and $Q = [q_1 ... q_N]$, with $X, Q \in M(2, N)$.

Taking the rotational dynamics and the rigid body kinematics equation (2.20) into account, the authors propose an observer of the following form

$$
\begin{align*}
\dot{\hat{R}} &= \hat{RS}(\hat{\omega}), \\
\dot{\hat{P}} &= -S(\hat{\omega})\hat{P} + \hat{V},
\end{align*}
$$

(4.167)

where the terms $\hat{\omega}$ and $\hat{V}$ are to be found from the Lyapunov stability analysis. The Lyapunov function is defined based on the error between the estimated landmark position and the measured values and is given by

$$
V = \frac{1}{2} \sum_{i=1}^{N} ||\hat{u}_i - u_i||^2,
$$

(4.168)

where the vectors $u_i$ are linear combinations of the landmark position measurements in $B$ defined as $u_j := \sum_{i=1}^{N-1} a_{ij}(q_{i+1} - q_i)$ and $u_N = -\frac{1}{N} \sum_{i=1}^{N} q_i$, with $a_{ij}$ being positive constant scalars. The estimated vectors $\hat{u}_i$ are given as $\hat{u}_i := \hat{R}^T u_i$. The concatenation of the new transformed vectors $u_i$ can be expressed in matrix form as $U = [u_1 ... u_{N-1}], with U \in M(3, N - 1)$.

Taking the last term of the Lyapunov function, that is associated with index $i = N$, the
function can be expressed by the matrices of transformed vectors

\[
\mathcal{V} = \frac{1}{2} \sum_{i=1}^{N-1} ||\hat{u}_i - u_i||^2 + \frac{1}{2} ||\hat{u}_N - u_N||^2,
\]

\[
= \frac{1}{2} ||(\hat{R}^T - R^T)U||^2 + \frac{1}{2} ||\hat{P} - P||^2,
\]

\[
= \frac{1}{2} ||(I - \tilde{R})U||^2 + \frac{1}{2} ||\tilde{P}||^2.
\]

(4.169)

Therefore, the Lyapunov function can be expressed as \( \mathcal{V} = \mathcal{V}_R + \mathcal{V}_P \).

The time-derivative of the Lyapunov function component associated with rotational motion allows to design \( \hat{\omega} \) as follows

\[
\dot{\hat{\omega}} = \omega - k_\omega R^T (UU^T \hat{R} - \tilde{R}^T UU^T) \otimes (4.170)
\]

where \( k_\omega \) is a positive scalar. It is then shown that the attitude error \( \tilde{R} = I \) of the error dynamics system is almost globally asymptotic stable (aGAS) and exponentially stable with the domain of attraction given by

\[
R_A = \{ \tilde{R} \in SO(3) : ||I - \tilde{R}||^2 < 8 \}. \quad (4.171)
\]

The time-derivative of the Lyapunov function component associated with the translational motion leads to the design of \( \hat{V} \) as follows

\[
\dot{\hat{V}} = V + (S(\omega) - k_V I) \dot{\hat{P}} + k_\omega S(\dot{\hat{P}}) R^T (UU^T \hat{R} - \tilde{R}^T UU^T) \otimes (4.172)
\]

by which the dynamics of the position error becomes \( \dot{\hat{P}} = -k_V \hat{P} \), with \( k_V \) being a positive scalar. Therefore, the origin of the position error dynamics is globally exponentially stable. It should be noted that the stability analysis of the observer for the rotational motion was considered independently from the translational motion. However, because of the involvement of the angular velocity in the position dynamics, the effect of the correction term (4.170) can be seen in the position observer. This strategy is similar to the authors previous work on the pose estimation using GPS data with IMU measurements [Vasconcelos et al., 2008b]. As discussed before, the work included an observer of cascaded structure, where the attitude observer works independently from the position ob-
server. The estimates of the rotation matrix are then fed to the second observer for position estimation.

The observer (4.167) with the correction terms (4.170) and (4.172) is designed based on ideal sensor readings. The authors expand their observer to include gyro bias and unknown time-constant bias in velocity measurements. Complementing the original Lyapunov function (4.168) with some additional terms associated with bias errors, the final observer is designed to include bias estimations for both angular and translational velocity measurements.

The paper has the advantage of using only position and velocity measurements along with gyro readings in estimating the attitude. From a theoretical point of view, the observer yields almost global asymptotic results and exponential convergence of measurement biases estimates. This is a considerable advantage over the mostly locally-established observers.

From an application perspective, the ability of the attitude observer to work without inertial measurements and sensors such as magnetometers and accelerometers and relying only on the position measurements can be regarded as a positive point of the estimation technique. However, in practice, some of their assumptions are difficult to satisfy. For example, the exact positioning of the landmark centroid as origin of the inertial frame may not be a rational assumption. In fact, the accuracy of the estimation is in a way dependent on the distance of the flying vehicle from the centre of the inertial frame. This “dependency” of the estimation method on a non-moving set of landmarks poses an obstacle in practical applications of such technique. On the other hand, the authors other previously discussed work [Vasconcelos et al., 2008b] on the attitude and position estimation gives a greater region of mobility for the vehicle’s pose observer to properly function.
4.9 Simulations

In this section, some of the attitude observers and filters discussed in this chapter are simulated. The known inertial-frame vectors of gravity and magnetic field are taken similar to those in section (3.8). For this part, the trajectory of the rigid body position in the inertial frame is specified as

\[ p(t) = [200 \sin(0.1t), 150 \cos(0.1t), 6t]^T \text{m}, \]  

\[ \dot{v}(t) = [-2 \sin(0.1t), -1.5 \cos(0.1t), 0]^T \text{m/s}^2. \]  

From the first and second derivatives of which the velocity \(\dot{v}\) and linear acceleration \(\ddot{v}\) of the rigid body are obtained

\[ v(t) = [20 \cos(0.1t), -15 \sin(0.1t), 6]^T \text{m/s}, \]  

Figure (4.4) illustrates the trajectory of the rigid body position obtained from (4.173) in a 100 seconds time interval.

Figure 4.4: The trajectory of the rigid body position in a 100 seconds time interval.
In these simulations, all the estimators are tested with the same initial conditions for orientation and velocity. The initial attitude of the actual system is taken as

\[ R(0) = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}, \]

and the initial conditions for estimators were chosen to be \( \hat{R}(0) = I_{3\times3} \) or equivalently \( \hat{Q}(0) = [1, 0, 0, 0]^T \). For simulations in which a constant gyro bias is assumed, the bias vector is chosen as \( \omega_b = [0.5, -0.5, 1]^T \text{deg/s} \). For velocity-aided estimators, the initial velocity was taken as \( \hat{v} = [0, 1, 0]^T \text{m/s} \).

While the previous noise assumptions for accelerometer and magnetometer measurements in section (3.8) hold here, the following noise variances were considered

- Gyroscope noise = 1\text{deg/s}
- GPS-velocity noise = 0.01\text{m/s}.

### 4.9.1 Invariant EKF

The Invariant Extended Kalman Filter in (4.38) is simulated. This filter is designed based on an accelerated motion assumption and takes the accelerometer, magnetometer and gyro measurements, in the body frame, well as the linear velocity, in the inertial frame, as the filter inputs. The constant accelerometer scaling factor is taken as

\[ a_s = 0.9. \]

Figure (4.5) shows the error Euler angles obtained from the simulation under the realistic conditions of noisy IMU measurements and accelerated motion. The filter shows satisfactory performance with small errors in the estimated angles. It also shows no sensitivity to the magnitude of the linear acceleration and performs well under low and high linear accelerations. Moreover, the gain matrices \( K_q, K_v, K_\omega \) and \( K_a \) converge to some constant matrices over time.
The IEKF has an invariant structure in its pre-observer. As discussed before, this results in a system of error dynamics that is independent of the original system trajectory and the estimated states. While the origin of the error system is stable, the added measurement noise prevents the ultimate error Euler angles from converging to zero. If the Kalman gains are properly chosen in a way that the filter does not diverge at the beginning, the error trajectory remains close to zero for all times. Therefore, this attitude estimation technique is only sensitive to measurement noise and as long as the characteristics of the noise is known, the filter can be tuned to work well and generate acceptable estimations.

4.9.2 Unscented EKF

For this part, an Unscented Extended Kalman filter for the attitude estimation designed in [Crassidis and Markley, 2003] is simulated. The filter uses two noisy vector measurements. The vector of states is taken as

\[
X = \begin{bmatrix} \delta q \\ \omega_b \end{bmatrix}, \tag{4.175}
\]
where the $\tilde{Q} = [\delta q_0, \delta q]^T = Q \odot \hat{Q}^{-1}$ is defined as the quaternion error between the estimated quaternion attitude and the real attitude. $\omega_b$ is also the gyro bias vector that was chosen to be $\omega_b = [0.5, -0.5, 1]^T \text{deg/s}$.

The algorithm was initiated with the following state vector and covariance matrix

$$X(0) = [0 0 0 1 1 1]^T, \quad P(0) = \begin{bmatrix} P_{X,0} & 0 \\ 0 & P_{\omega,0} \end{bmatrix},$$

with

$$P_{X,0} = 0.25I_{3 \times 3}, \quad P_{\omega,0} = 0.04I_{3 \times 3}.$$

Figure (4.6) shows the error Euler angles $\tilde{\phi}$, $\tilde{\theta}$ and $\tilde{\psi}$. As the IEKF, the filter is able to provide good estimations of the system Euler angles under noisy measurements conditions. The advantage of this filter is that it can easily be expanded to include other system states such as the linear velocity. In this case, the velocity vector is added to the state vector and the sigma points are also propagated through the linear velocity dynamics equation.

![Figure 4.6: Error Euler angles obtained from the Unscented Kalman Filter under noisy measurements condition.](image-url)
4.9.3 Nonlinear Complementary Filter

An explicit nonlinear complementary filter of the form given in (4.91) is studied with the following set of observer gains

\[ k_p = 3, \ k_I = 1.5, \ k_i = 1, \text{ for } i = 1, 2. \]

The simulations are performed under several assumptions that contain both ideal and realistic conditions. Figure (4.7) shows the performance of this observer in ideal conditions. The output error in the Euler angles properly converges to zero in a fast trend and so does the estimated rotation matrix to the actual rotation matrix. Fast convergence of the correction term \( \sigma \) to zero guarantees the ultimate convergence of the estimated rotation matrix to the actual attitude.

![Figure 4.7: Error Euler angles of the nonlinear complementary filter with ideal noise-free sensor measurements.](image)

In Fig. (4.8), the added measurement noise results in an imperfect attitude estimation. However, it can be seen that the estimation error remains bounded and close to 0.
The effect of a large linear acceleration on the performance of the complementary filter relying on IMU measurements can be seen in Fig. (4.9). In this simulation, instead of assuming that the accelerometer measures the gravity vector in the body frame, we assume that the accelerometer measures the apparent acceleration in the body frame, i.e., \( b_a = R^T(\dot{v} - g) \).

Figure (4.10) shows the performance of the filter under noisy measurements and large linear accelerations (i.e., the approximation \( b_a \approx -R^T g \) does not hold anymore). The figure depicts how the large linear acceleration of the rigid body dominates the measurements noise in making large estimation errors.

Consideration of the linear velocity and position dynamics in the observer design has solved this problem to a great extent. This will be seen and discussed in the velocity-aided attitude estimators simulations.
4.9.4 Velocity-aided Attitude Observer

In this section, simulations are performed on a velocity-aided attitude observer of the form given in [Roberts and Tayebi, 2011b] and introduced in (4.134). As discussed before, in this observer, the unavailability of a known IMU vector \( r_a \) in the inertial frame is compensated with the help of the linear velocity of the rigid body obtained from the GPS data. Unlike the observer in [Hua, 2010] (Eq. (4.133)), where an auxiliary matrix \( A \) evolving outside the \( SO(3) \) is used to transform the body acceleration vector into the inertial frame, all estimated rotation matrices and quaternions in this observer are constrained. In order to make a comparison between the two observers, the definition used for attitude error in this part was chosen to be the norm error between the identity matrix \( I_3 \) and the error matrix \( \tilde{R} \).

The observer gains for (4.134) are chosen as follows

\[
 k_m = 1.8, \quad k_I = 0.1, \quad k_v = 4.
\]

Figure (4.11) shows how the attitude error converges to zero with a noise-free measure-
Figure 4.10: Error Euler angles of the nonlinear complementary filter in accelerated mode with noisy measurements.

ments assumption. In Fig. (4.12), however, the effect of adding measurement noise to the sensor outputs is visible. While the observer tries to force the attitude error norm to zero, the noisy vectors $b_m$ and $b_a$ directly affect the correction term $\sigma$.

The performance of this observer under noisy measurement conditions is comparable to those of the IEKF and the nonlinear complementary filter with the same noise-contaminated vectorial measurements. The velocity-aided observer is able to compensate for the linear acceleration term in the apparent acceleration and provides reliable attitude estimations.

The velocity-aided observer of (4.133) with the following choice of gains was simulated

$$k_m = 4 \quad k_a = 0.2 \quad k_v = 4 \quad k_A = 2.4.$$  

As discussed before, in this observer, the apparent acceleration $r_a$ in the inertial frame is estimated using an auxiliary matrix $A$ that does not belong to $SO(3)$. However, the vector $Ab_a$ converges asymptotically to $r_a$, which leads to the convergence of $A$ to $R \in SO(3)$. Figure (4.13) shows the performance of this observer in ideal noise-free measurements.
Figure 4.11: Performance of the velocity-aided algorithm under noise-free measurements assumption (Accelerated mode)

conditions. The figure demonstrates how the norm of the auxiliary matrix converges to 1.

Figure (4.14) shows the performance of this observer under noisy sensor measurements conditions. In the first part of this figure, it can be seen that the measurement noise prevents the auxiliary matrix $A$ from converging to $SO(3)$, resulting in an incorrect estimation of the apparent acceleration vector $r_a$ and the linear velocity $\hat{v}$. This error is propagated through the cascade structure of the observer and leads to large attitude estimation errors.

4.9.5 Global Observer non-evolving on $SO(3)$

In this case, a global observer of the form (4.121) is implemented with a simpler structure that does not include position estimations. The gains of the new observer were chosen as

$$
K_{vv} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad K_{zv} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.
$$

The performance of this observer under ideal conditions is shown in Fig. (4.15), where one can see the convergence of the estimation error to zero. Also, Fig. (4.16) shows the ultimate
Figure 4.12: Performance of the velocity-aided algorithm with noisy measurements (Accelerated mode)

error obtained under noisy measurements condition. Both simulations are performed with an accelerated motion assumption.

It should be noted that the matrix $J$ defined as

$$J = (A_f - \hat{R}A_B)A_B^T,$$  \hfill (4.176)

is highly dependent on the matrix $A_B$ that is given by

$$A_B = [b_m, b_m \times b_a, b_m \times (b_m \times b_a)],$$ \hfill (4.177)

which shows high sensitivity to the measurement noise. Both vectors of magnetic field $b_m$ and apparent acceleration $b_a$ in the body frame are contaminated with noise and the cross product between these vectors makes the use of this matrix unreliable in generating an orthogonal basis.

On the other hand, because of the special structure of this observer, the estimated rotation matrix $\hat{R}$ may not start its trajectory inside $SO(3)$. This makes it vulnerable to measurements noise since it might not get the chance to overcome the initial gap and make its way into the special group. In this case, the correction matrix $J$ will never converge to zero and the estimated matrix $\hat{R}$ remains out of the $SO(3)$. Figure (4.16) shows how the choice
Figure 4.13: (a) Convergence of the auxiliary matrix $A$ to $R \in SO(3)$, (b) Convergence of the attitude error norm to zero.
Figure 4.14: Performance of the velocity-aided observer with auxiliary matrix not belonging to $SO(3)$ in noisy measurements condition.
of the gain matrix $\Gamma$ results in the estimated rotation matrix to even increase its distance from the special group rather than decreasing it. In this simulation, it was also observed that a large gain $\Gamma$ led to instability in the estimation system.

### 4.9.6 Invariant Observer

An invariant observer of the form given in (4.63) is simulated under accelerated motion assumption. The inputs of the filter are taken as the magnetic field vector measurement and rigid body velocity, both in $\mathcal{B}$.

The observer gains are chosen in a way that the linearized error system is stable. The chosen gains are

$$
\mathcal{L}_V^O = \begin{bmatrix} 0 & -1.6 & 0 \\ 1.6 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{L}_b^O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -8B(2) & 8B(1) & 0 \end{bmatrix},
$$

$$
\mathcal{L}_V^V = \begin{bmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad \mathcal{L}_b^V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

(4.178)
where $B(1)$ and $B(2)$ are the Earth magnetic field vector values in the directions $X$ and $Y$, respectively.

Figures (4.17) and (4.18) show the performance of this observer in noise-free and noisy IMU measurements, respectively.

In Fig. (4.18), the effect of measurement noise is visible. The invariant observer is also sensitive to noise and based on the noise intensity, its various gains have to be tuned to maintain the stability of the observer. The performance of this observer in estimating the Euler angles is quite comparable to that of the extended Kalman filters.

4.10 Discussion

In this chapter, some of the dynamic attitude estimation algorithms, including the latest nonlinear attitude observers, are discussed and simulation results were presented. The study of the structure of each filter and the provided simulations help in making comparisons between different techniques.

The Invariant EKF and Unscented EKF have been chosen for simulations. Since the
IEKF can be regarded as a generalization of the Multiplicative EKF and the probability-analysis-based Particle filters techniques can be simplified to the UKF, the two choices seem to provide a good representation of the class of attitude estimators of the Kalman-type.

The simulation results for the extended Kalman filters show good performance of in estimating the attitude with noisy measurements. This is due to the fact that these filters are specifically designed to compensate for the effect of noise. Once the noise characteristics of the sensor readings are determined, the filter can be simply tuned. It should be noted that real-time noise cannot be assumed to be white noise with zero mean. Non-Gaussian noise is long known to have effects on sensor measurements and its existence in the system measurements can degrade the filter output.

The initial value issue was also noticed during the simulations. The Kalman filters are derived through linearization processes and this makes large initial errors a potential threat to the performance of the filter. For the unscented filter, it was seen that large initial errors in the state vector and the covariance matrix led to a non-positive predicted covariance matrix. Unlike the Kalman filters, the studied nonlinear observers did not show vulnerability to initial state values.
For most of the nonlinear observers, the results are semi-global and for some even global. This means that the choice of the initial filter states did not have an effect on the ultimate performance and the filter remains stable under different initial conditions. The simulations with noise-free measurements show how perfectly these observers obtain the true rigid body attitude. By adding measurement noise to sensor readings, however, the performance was degraded to a relatively acceptable level.

However, this is not the case for all nonlinear attitude observers. In fact, the simulations for the nonlinear observers with global results, that do not evolve on \( SO(3) \), show how a relatively small noise can have an important negative impact on the estimated attitude.

In nonlinear observers with global results, one has to note the requirement of the attitude rotation matrix or the attitude quaternion to leave the rotation group. With this kind of observers, in the noise-free case it is guaranteed that the estimated rotation matrix, while not-constrained in \( SO(3) \), converges asymptotically to the actual rotation matrix. However, this convergence is not guaranteed anymore in the case of noisy measurements.
Let us take a closer look at the following observer

\[
\dot{\hat{R}} = \hat{R}S(\omega_B) - \Gamma J(m_B, m_I, a_B, a_I, \hat{R}).
\]  

(4.179)

In real-time conditions, the body-frame measurements \(\omega_B, m_B\) and \(a_B\) are contaminated with noise. Therefore, the measured vectors are given by

\[
\begin{align*}
\omega_{\text{meas}} &= \omega_B + \nu_\omega, \\
m_{\text{meas}} &= m_B + \nu_m, \\
a_{\text{meas}} &= a_B + \nu_a.
\end{align*}
\]  

(4.180)

Substituting these measurements into the matrix \(A_B\) in (4.177) and disregarding the noise associated with the accelerometer measurements for the sake of simplicity (i.e., \(a_{\text{meas}} \approx a_B\)) gives

\[
A_{B,\text{meas}} = [R^T m_I + \nu_m, S(R^T m_I)a_B + S(\nu_m)a_B, S^2(R^T m_I + \nu_m)a_B]
\approx [R^T m_I, S(R^T m_I)a_B, S^2(R^T m_I)a_B] + \\
[\nu_m, S(\nu_m)a_B, S(R^T m_I)S(\nu_m) + S(\nu_m)S(R^T m_I)a_B]
:= A_B + N,
\]  

(4.181)

Substituting \(A_{B,\text{meas}}\) into \(J\), one can simply show that

\[
J_{\text{meas}} \approx J + (A_I - \hat{R}^T A_B)N - \hat{R}^T NA_B
= J + F,
\]  

(4.182)

Finally, substituting the measured matrix \(J_{\text{meas}}\) into the observer (4.179) gives

\[
\dot{\hat{R}} = \hat{R}S(\omega_B) - \Gamma J + \hat{R}S(\nu_\omega) - \Gamma F.
\]  

(4.183)

The first two terms in the right hand side of (4.183) are ideal terms that ensure the convergence of the estimator. However, the term \(\Gamma F\) is a potentially destabilizing term since a high gain choice of \(\Gamma\) increases the effect of the noise-related matrix \(F\) and feeds the observer with highly noisy data. On the other hand, a very small choice of the gain matrix \(\Gamma\)
simply decreases the importance of the correction term in the observer and slows down the convergence rates.

Since for noisy measurements the correction matrix never converges to zero, it can be expected that the rotation matrix obtained from this method never converges to \(SO(3)\). Depending on the intensity of the noise and the choice of the gain matrix \(\Gamma\), the ultimate estimation can lead to highly unreliable values.

These results are in accordance with the previously discussed simulation results of the velocity-aided attitude observer. In that observer, since the auxiliary matrix evolving out of \(SO(3)\) cannot converge to the special orthogonal group, the appropriate estimation of the linear velocity and the apparent acceleration does not occur. Since these two estimations are crucial for the cascaded structure to properly function, the whole system fails to work and yields large attitude errors.

Let us consider the structure of the observer in [Hua, 2010] given by

\[
\dot{\hat{v}} = k_v(v - \hat{v}) + ge_3 + Ab_a, \\
\dot{\hat{A}} = AS(\omega) + k_A(v - \hat{v})b_a^T.
\]  

(4.184)

which was previously presented in (4.131). The second term in the dynamic equation of the auxiliary matrix \(A\) is similar in function to the \(\Gamma J\) term in the previous observer. The convergence of the estimated linear velocity to the actual linear velocity of the rigid body, guarantees that the right hand side of the dynamics of \(A\) converges to \(AS(\omega)\) which suggest that \(A\) converges to \(SO(3)\). However, the existence of noise in sensor measurements prevents the last term of the right hand side of the \(A\)-dynamics from vanishing, and hence forcing \(A\) to remain out of \(SO(3)\).

Let us calculate the derivative of the Lyapunov function (4.132) proposed to analyze the observer convergence. For the sake of simplicity, we consider only the accelerometer and gyroscope readings to be contaminated with noise. With the new noisy measurements,
the new derivative becomes
\[
\dot{V}_{\text{meas}} = -k_v |\tilde{v}|^2 + \tilde{v}^T (R - A)a_B - \tilde{v}^T A\nu_a + \text{tr} \left[ (AS(\omega) + AS(\nu_\omega) + k_A \tilde{v}a_B^T + k_A \tilde{v}^T - RS(\omega))(A - R)^T \right] = -k_v |\tilde{v}|^2 - \tilde{v}^T (A - R)a_B - \tilde{v}^T A\nu_a + \text{tr} ((A - R)^T (AS(\nu_\omega))R^T)
\]
In ideal conditions, a high gain choice for $k_A$ leads to fast convergence of the filter. However, the Lyapunov function derivative of (4.185) shows that this gain can highly affect the Lyapunov derivative and hamper the convergence results. A small choice of $k_A$ will also downgrade the effect of the correction term $k_A \tilde{v}a_B^T$ on the auxiliary matrix update equation.

These two examples demonstrate the serious obstacles towards the real-time implementation of the observers with unconstrained elements. The failure to keep the effects of the measurement noise limited might be the main reason behind this. In a constrained structure, the norm of the estimated rotation matrix is not affected by the measurement noise. However, the norm of the unconstrained estimated rotation matrices is highly affected by measurement noise. This leads to the estimated matrices to remain outside the special orthogonal group without having a chance to converge to it.

In section we examined the performance of various attitude estimators of Kalman-type and nonlinear observers under measurement noise conditions. In the performed simulations, we notice how the performance of a specific group of observers degrades in the existence of noise in system measurements. This group involves those observers with elements non-evolving on $SO(3)$.

In order to see why these observers fail to converge in such conditions, a structural analysis is performed. It is shown that the choice of specific gains may lead to an increased noise effect on the norm of the involved matrices, resulting in failure of these unconstrained elements to converge to the rotational group. This effect can be seen in two ways: one that is direct, consisting of the inability of the unconstrained estimated rotation matrix to converge to $SO(3)$, and the other consisting of a constrained rotation estimate fed with
highly unreliable values coming from an unconstrained estimation of some other system elements.
Chapter 5

General Conclusion

The attitude estimation of flying vehicles has been an interesting field of study for many researchers over the years. Efficient control strategies for flying vehicles rely heavily on a good estimation of the attitude. Moreover, in many applications such as surveillance, infrastructure inspection, and aerial photography the estimation of the aircraft position and velocity is important as well. Therefore, powerful and reliable estimation tools are required to generate good estimations under real-time conditions such as noisy sensor measurements and unknown system parameters.

For many decades, conventional techniques such as Extended Kalman Filters were the workhorses of the attitude estimation field. With the emergence of new methods, it is believed that new generations of estimation tools may be able to replace the existing methods due to their superior reliability, accuracy, and domain of convergence. Among these techniques, nonlinear attitude observers have gained huge attention among the scientific community during the last decade.

All the attitude estimation techniques require measurements provided by a range of sensors attached to the rigid body system. These provide information on the vehicle’s angular velocity and vectorial measurements in the body frame. For small scale and low-cost applications, gyroscopes, accelerometers and magnetometers are the most commonly used sensors that are commercially available.
A theoretical study of the different classes of attitude estimation techniques and algorithms is performed. The basics of static attitude determination and various methods of this kind are presented and discussed. The reasons behind the adoption of new attitude estimation methods that are dynamic are also discussed.

Various dynamic methods are briefly introduced and the latest publications in the fields of Kalman filtering and linear complementary filtering are presented with discussions on their relative advantages and disadvantages.

In this thesis, a closer look on the evolution and application of the nonlinear attitude observers is performed with a study on the latest developments in this new field of research. This is the first survey conducted on the many different types of such observers and provides readers with details on the design and convergence properties of nonlinear attitude observers.

While it is shown that all the attitude estimation techniques remain stable under appropriate initial conditions, the effect of noise and disturbances on the performance of various attitude estimators is investigated. Simulations have been carried out to illustrate the performance of the selected attitude estimation techniques and provide an idea on the positive and negative points of each technique.

We argue that constrained attitude observers yield better results than unconstrained methods in noisy measurement conditions. We observe and report how a large choice of gains in these observers result in high estimation errors. Although there is no doubt in the theoretical importance of unconstrained nonlinear attitude observers, their structures pose barriers towards their real-time implementation. This is further investigated by a structural analysis of these observers, where the sensitivity to measurement noise is discussed.

In general, it can be seen that a large number of studies have been devoted to the special groups \( SO(3) \) and \( SE(3) \) in nonlinear observers design, and have encouraged researchers to focus on the inherent properties of these groups. Further studies on these properties may enable researchers to design more reliable and efficient attitude observers for real-time applications. These include the estimation of gyro bias with a single vector measurement,
or estimation of the vehicle position and velocity in indoor applications with no access to GPS data.

Application of new forms of measurements in the design of nonlinear observers is also an open problem for research. Optical flow measurements obtained from on-board cameras may help in finding new ways of designing IMU-based pose observers in indoor applications. On-board cameras may play an important role in future attitude estimation techniques because of their ability to fill the lack of direct measurements of velocity and position. These would eventually enable the system to gain more autonomy in its estimation and function in different environments.
Bibliography


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