1982

Conclusion, Competition, and Conjectures

John McMillan

20065

ISSN: 0318-725X

ISBN: 0-7714-0360-7

RESEARCH REPORT 8215

COLLUSION, COMPETITION, AND CONJECTURES*

by

John McMillan

Revised: June 1982

In Augustin Cournot's model of oligopoly, it is assumed that each firm believes that its rivals' actions are constant and unrelated to its own. Irving Fisher was the first of many to reject this assumption.

> But, as a matter of fact, no businessman assumes either that his rival's output or price will remain constant any more than a chess player assumes that his opponent will not interfere with his effort to capture a knight. On the contrary, his whole thought is to forecast what move the rival will make in response to one of his own. He may lower his price to steal his rival's business temporarily or with the hope of driving him out of business entirely. He may take great care to preserve the *modus vivendi* so as not to break the market and provoke a rate war. He may raise his price, if ruinously low, in hopes that his rival, who is in the same difficulty, may welcome the change, and follow suit. The whole study is a "dynamic" one, and far more complex than Cournot makes it out to be (pp. 126-27).

To model formally a process such as Fisher described, it would be necessary to take account of the levels of all past prices and outputs, as well as the firms' subjective beliefs about the information these variables contain about their rivals' likely future actions.[1] Moreover, to make an optimal decision, each firm would have to predict not only its rivals' outputs, but also its rivals' predictions of its output, its rivals' predictions of its prediction of their outputs, and so on in infinite regress. This paper adopts a simpler way of modelling this process of observation and anticipation. Suppose that, on the basis of past observations of its rivals' behavior, each firm estimates an apparent correlation between changes in its own actions and changes in its rivals' actions, and uses this apparent correlation in making its current output decision. These apparent correlations are the firms' conjectural variations. Unlike the standard conjectural-variations model[2] in which conjectures are given exogenously so that Fisher's point cannot be addressed, in the model of this paper the conjectures are endogenous. Given these conjectures at any point in time, the firm's perceived demand function, and therefore its optimization problem, are well-defined; the firm equates perceived marginal revenue to marginal cost.

As Fisher noted, introducing rivals' responses into Cournot's model implicitly introduces dynamic considerations. Interpreted literally, Cournot's model should not be criticized for ignoring reactions, because it is formally static: in a model in which each firm makes only one output decision, and the decisions are simultaneous so that no firm is able to revise its decision after observing its rivals' actions, Fisher's critique is irrelevant. Realistically, however, firms sell their outputs on more than one occasion. Moreover, Cournot himself introduced dynamic considerations by discussing the stability of equilibrium.[3] Studies of the stability of Cournot equilibrium are especially susceptible to Fisher's criticism: these studies assume that firms believe their rivals will produce the same quantity as they produced in the previous period, even though in each period during the transition to equilibrium this belief is seen to be wrong. This paper attempts to address Fisher's point by introducing endogenous conjectural variations into a model which is in other ways similar in its dynamic structure to the model used in Cournot stability studies.

Like the Cournot stability studies, the standard conjectural-variations model has firms believing something that is literally false. Each firm makes its profit-maximizing output decision under the conjecture that there is a simple functional relationship between its rivals' outputs and the output it chooses to produce; whereas in fact the rivals are themselves solving maximization problems. This paper proposes that the way to make sense of this conjectural-variations functional relationship is to interpret it not as a perceived causal relationship, but instead as an apparent correlation based on past experience. The model to be presented has each firm, on the basis of its observations of its rivals' outputs, gradually modifying its conjectures about the relationship between its own outputs and its rivals' outputs. The paper develops an equilibrium concept based on disequilibrium behavior: the nature of the equilibrium depends upon what the firms observe during the disequilibrium adjustments.

The standard model of conjectural variations has a continuum of equilibria. In the present model, the fact that conjectures are endogenous greatly limits the possible equilibria: the only endpoints of the adjustment process are implicit collusion and extreme rivalry. The assumption in the standard model that conjectures are fixed could be justified if such conjectures could be reached as equilibrium conjectures; however, in the model of this paper the process of revision of conjectures is unstable.

I. Endogenous Conjectures

Consider an oligopolistic industry producing a homogeneous commodity. Firms make their output decisions simultaneously, once in each time period. The market inverse demand function is f:

$$p_t = f\left(\sum_{i=1}^{n} x_i^t\right) \tag{1}$$

where p_t represents the price in period t and $x_i^t \geq 0$, $i=1,\ldots,n$, is the output of firm i in period t. Assume $f \geq 0$, and $f'(x) < 0$ whenever $f(x) > 0$. All firms face the same decreasing-returns-to-scale cost function c: $c' > 0$, $c'' > 0$. The cost function and the demand function do not change over time. In order to concentrate on the firms' interactions with each other rather than with their environment, assume each firm has perfect knowledge about its own cost function, the market demand function, and the price that ruled at any time in the past. It is not necessary that each firm be able to observe its rivals' past outputs directly because each firm's profit depends on the sum of the rivals' outputs, not the individual outputs; this sum can be deduced from knowledge of the demand function, the market price, and the firm's own output. Nor is it necessary that the firm know its rivals' cost functions. On the basis of their knowledge about their rivals' past actions, the firms must somehow predict what their rivals will do in the current period. No binding agreements among the firms are possible.

Define conjectural variations as follows. Denote the current period by t. Suppose that whenever firm i plans to change its output from one period to the next, by $x_i^t - x_i^{t-1}$, it predicts that its rivals' outputs will change proportionately. Denote the sum of i's rivals' actual outputs in period $t-1$ by x_{-i}^{t-1}, and i's belief about the sum of its rivals' outputs in period t by ξ_{-i}^t. Then i's beliefs about the correlation between its own actions and its rivals' actions are represented by the conjectural-variations term b_i^t, where

$$(2) \qquad b_i^t = \frac{\xi_{-i}^t - x_{-i}^{t-1}}{x_i^t - x_i^{t-1}} , \qquad i=1,\ldots,n;$$

that is

(3) $\xi_{-i}^{t} = b_i^t(x_i^t - x_i^{t-1}) + x_{-i}^{t-1}$, i=1,...,n.

The time superscript on the conjectural-variations term b_i^t is necessary because this term will change over time during the transition to a conjectural equilibrium.[4]

Suppose that, as in the Cournot stability studies, the oligopoly game is played once in each time period, but that firms seek in each period to maximize only current profits. Unlike the Cournot stability studies, however, at each point in time t during the transition to equilibrium, the ith firm has a point estimate b_i^t of the relationship between its own output changes and the sum of its rivals' output changes, as in (2).

In the demand function (1), substitute firm i's prediction of the other firms' total output, ξ_{-i}^t, for their actual total output, x_{-i}^t. This yields firm i's perceived demand function. Firm i's perceived demand function has as arguments only firm i's own current output, the parameters of the industry demand function, and the previous period's output levels; it does not depend on the (unknown) current output decisions of firm i's rivals. Given its beliefs about its rivals' behavior as summarized in (3), firm i predicts that its profit in period t will be

(4) $\pi_i^t = x_i^t \, f([1+b_i^t]x_i^t - b_i^t x_i^{t-1} + x_{-i}^{t-1}) - c(x_i^t)$, i=1,...,n.

Maximizing this single-period profit with respect to x_i^t (assuming $x_i^t \geq 0$),

(5) $f([1+b_i^t]x_i^t - b_i^t x_i^{t-1} + x_{-i}^{t-1}) + (1+b_i^t)x_i^t f'([1+b_i^t]x_i^t - b_i^t x_i^{t-1} + x_{-i}^{t-1}) - c'(x_i^t) = 0,$

i=1,...,n.

Equation (5) implicitly defines the i^{th} firm's profit-maximizing output as a function of its own and its rivals' outputs in the previous period and of its conjecture (represented by b_i^t) about its rivals' actions in the current period.

The second-order condition is

$$(6) \qquad 2(1+b_i^t)f'(\cdot) + (1+b_i^t)^2 x_i^t f''(\cdot) - c''(x_i^t) < 0, \; i=1,\ldots,n, \text{ for all } x_i^t \geq 0.$$

Each firm's output decision is based on its point estimate b_i^t of the conjectural-variations term. This point estimate is based on past observations of actual actions. Each firm makes its output decision as in (5). At the end of period t each firm can compare its rivals' actual total output, x_{-i}^t, with what it had predicted, ξ_{-i}^t. Thus at the end of period t, the firm receives new information about this apparent relationship; the firm observes the actual ratio of its opponents' output change to its own output change,

$$(7) \qquad \beta_i^t = \frac{x_{-i}^t - x_{-i}^{t-1}}{x_i^t - x_i^{t-1}} \quad , \qquad i=1,\ldots,n \; .$$

(Contrast with equation (2).) With this new information, the firm revises for period $t+1$ its point estimate b_i^{t+1} of the conjectural-variations term; suppose there is an updating operator B_i^t which uses the new information β_i^t to generate a new point-estimate of the conjectural-variations parameter b_i^{t+1}:

$$(8) \qquad b_i^{t+1} = B_i^t(b_i^t, \beta_i^t) \; .$$

The function B_i^t will be assumed to be continuously differentiable, with $\partial B_i^t/\partial \beta_i^t > 0$ so that observations do affect conjectures and the updating process is not trivial. For example, conjectures may be updated in adaptive or error-learning fashion.

In order to behave as described here, the firms need minimal information. They need to know their own cost functions, the market demand function, and past prices. They need not know their rivals' costs, how many rivals there are (though they must know there exists at least one rival), or the identity of the rivals. (The model assumes for the sake of simplicity that the firms face identical costs, but the firms themselves need not know this.) Thus the firms may have much less information than they would need in order to compute their Cournot-Nash (rational-expectations) strategies; behaving as in this model may be the best they can do.

II. Equilibrium

Once equilibrium has been reached, no firm will change its output. Thus β_i^t, being a ratio of output changes (equation (7)), is undefined. It seems reasonable, since no firm receives any new information in this case, to postulate that the updating operator B_i^t has the property that the conjecture b_i^t is left unchanged when the observation β_i^t is a ratio of zeros. Assume, moreover, that an observation β_i^t which is defined and different from the *ex ante* belief b_i^t results in a changed *ex post* belief $b_i^{t+1} \neq b_i^t$. These properties of the updating operator are summarized in the following assumption: for all firms $i=1,\ldots,n$,

$$(9)\quad \begin{aligned} &(a)\quad b_i^{t+1} = b_i^t \quad \text{if} \quad x_i^t = x_i^{t-1} \quad \text{and} \quad x_{-i}^t = x_{-i}^{t-1} \\ &(b)\quad b_i^{t+1} = b_i^t \quad \text{only if either } b_i^t = \beta_i^t \\ &\qquad\qquad\qquad \text{or } x_i^t = x_i^{t-1} \text{ and } x_{-i}^t = x_{-i}^{t-1}. \end{aligned}$$

(9b) says that *ex ante* beliefs are left unchanged *ex post* only if they are confirmed by observation. Note that (9a) leaves open the possibility that, if outputs do change so that β_i^t is not a ratio of zeros, then a confirmation of beliefs does not necessarily mean that the beliefs remain unchanged (that is, it may happen that $b_i^{t+1} \neq b_i^t$ even though $\beta_i^t = b_i^t$). Clearly these requirements would be satisfied for example if beliefs were updated adaptively.

Define an equilibrium of this trial-and-error learning process to be a vector of outputs and conjectures $(x_1^*,\ldots,x_n^*;\ b_1^*,\ldots,b_n^*)$ such that each firm's output x_i^* maximizes its profit given its beliefs b_i^* about its rivals' reactions and given that the equilibrium vector of outputs $(x_1^*,\ldots,x_n^*)$ was also produced in the previous period. From (9a), this definition of equilibrium implies that conjectures b_i^* do not change once equilibrium has been reached.

It follows from this definition that equilibrium points can be found simply by putting $x_i^{t-1} = x_i^t$ for all i=1,...,n in equation (5), for given values of b_i^t, i=1,...,n, and solving for x_i^t. In the standard conjectural-variations model (with the conjectures exogenously fixed) there are infinitely many equilibria. A similar phenomenon occurs in this model, in the sense that, for a wide range of initial beliefs b_i^1, i=1,...,n, there is a corresponding set of initial output levels x_i^o, i=1,...,n, such that no firm has any incentive to change its output. For example, suppose firstly that $b_i^1 = 0$, i=1,...,n. Then, from (5), $x_i^1 = x_i^o$ if x_i^o is the Cournot level of output. Thus, from (9a), no revision of beliefs will occur; $b_i^2 = b_i^1 = 0$. In other words, if all firms start with Cournot conjectures and producing Cournot outputs, they will stay at the Cournot equilibrium forever. Secondly, suppose that, each firm's initial conjecture is $b_i^1 = -1$. Then $x_i^1 = x_i^o$ if $f(x_i^o + x_{-i}^o) = c'(x_i^o)$; that is, price equals marginal cost. Thus if initial output levels sum to the perfectly competitive output level, and if each firm's initial conjecture is $b_i^1 = -1$, the system will stay at perfect competition. Thirdly, suppose all firms initially have the conjecture $b_i^1 = n - 1$. If the initial outputs are X^m/n, where X^m represents the joint-profit maximizing level of industry output, then (5) says industry marginal revenue equals marginal cost; total industry profit is maximized. No firm has any incentive to revise its beliefs and the system stays at the joint-profit maximizing equilibrium.

Thus there are infinitely many conjectural equilibria. By varying the initial conjectures b_i^1, any outcome between perfect competition and the joint-profit maximum can be maintained as an equilibrium for some particular allocation of initial outputs. It was assumed, however, in discussing these equilibria that the system started at one of the equilibria. It will now be shown that if disequilibrium output adjustments do occur, the set of possible equilibria is much smaller.

III. Collusion and Competition

Suppose that the system does not start with outputs and conjectures such that no firm wants to change its output; suppose instead that some transition does take place. Outputs change over time, so that β_i^t is defined. Note that from (7),

$$(10) \qquad 1 + \beta_i^t = \frac{\sum_{j=1}^{n} x_j^t - \sum_{j=1}^{n} x_j^{t-1}}{x_i^t - x_i^{t-1}} \quad .$$

Thus the following relationship must hold among the observations β_i^t:

$$(11) \qquad \begin{aligned} &(a) \quad \text{either } \beta_i^t = -1 \qquad \text{for all } i=1,\ldots,n \\ &(b) \quad \text{or } \sum_{i=1}^{n} \frac{1}{1+\beta_i^t} = 1 \ . \end{aligned}$$

(Note that (11a) occurs when total industry output does not change from period t-1 to period t; (11b) when it does change.)

If a non-trivial transition does take place, so that the beliefs b_i^t are altered in the light of the observations β_i^t, then assumption (9b) ensures that the property (11) of the observations is inherited by any steady-state beliefs b_i^*:

$$(12)\quad \begin{array}{ll} \text{(a) either } b_i^* = -1 & \text{for all } i=1,\ldots,n \\ \text{(b) or } \displaystyle\sum_{i=1}^{n} \frac{1}{1+b_i^*} = 1. & \end{array}$$

This is because, if β_i^t is not a ratio of zeros and if $\beta_i^t \neq b_i^t$ for some t, b_i^t cannot be a steady-state set of conjectures because (9b) would imply some revision of conjectures so that $b_i^{t+1} \neq b_i^t$. (Throughout, asterisks will be used to denote equilibrium values of variables.)

Condition (12) limits the outcomes which can be reached as endpoints of this learning process. In particular, (12) is not satisfied by the Cournot conjectures ($b_i^* = 0$, $i=1,\ldots,n$); it is, however, satisfied by perfectly competitive conjectures ($b_i^* = -1$, $i=1,\ldots,n$) and by joint-profit maximizing conjectures ($b_i^* = n-1$, $i=1,\ldots,n$).

More generally, as is shown in the appendix, at the equilibria associated with the conjectures (12), it is as if the market maximizes the weighted sum of profits $\Sigma_{i=1}^{n}\alpha_i\pi_i$ for some $\alpha_1,\ldots,\alpha_n$ (with some α_i nonzero but not necessarily positive).

If $\Sigma_{i=1}^{n}\alpha_i\pi_i$ is maximized for some $\alpha_i \geq 0$, $i=1,\ldots,n$, then the equilibrium lies on the profit-possibility frontier. (Such outcomes will be referred to in this paper as implicitly collusive.) These equilibria occur when (12b) holds and $b_i^* \geq 0$ for all $i=1,\ldots,n$. To prove this, rewrite (5) as

$$(13) \qquad \frac{f}{(1+b_i^*)} + x_i^* f' - \frac{c_i'(x_i^*)}{(1+b_i^*)} = 0 \ .$$

Sum over all firms, using $\Sigma_{i=1}^n 1/(1+b_i^*) = 1$:

$$(14) \qquad f + Xf' = \sum_{i=1}^{n} \frac{c_i'(x_i^*)}{(1+b_i^*)} ,$$

where X denotes total industry output. With $b_i^* \geq 0$, $0 \leq 1/(1+b_i^*) \leq 1$. Thus (14) says that industry marginal revenue equals a weighted average of each firm's marginal cost. Clearly this is a necessary condition for $Xf(X) - \Sigma_{i=1}^n c(X/(1+b_i^*))$ to be maximized; that is, for total industry profit to be maximized subject to the constraint that the i^{th} firm has a market share of $1/(1+b_i^*)$. Varying the b_i^*'s (with $b_i^* \geq 0$) traces out the profit-possibility frontier. Note one extreme case which satisfies (12b) with $b_i^* \geq 0$, and therefore is on the profit-possibility frontier: suppose for one i, $b_i^* = 0$, while $b_j^* = \infty$, j=1,...,i-1, i+1,...,n. In an equilibrium with such conjectures, the firms j, for $j \neq i$, produce nothing while firm i produces as a monopolist.

The alternative possibility, also consistent with (12), is that the equilibrium maximizes $\Sigma_{j=1}^n \alpha_j \pi_j$ with $\alpha_j < 0$ for at least one j=1,...,n. This occurs when $b_i^* < 0$ for at least one i=1,...,n. (To see this, note that (A2) shows that $b_i^* < 0$ if and only if $\pi_{ii} < 0$. Since $\pi_{ik} < 0$ for $i \neq k$, (A12) then holds if and only if some α_j's are strictly positive and some α_j's are strictly negative.) Thus, in the weighted sum of profits being maximized at equilibrium, some firms have negative weight. At equilibrium, given the conjectures, no firm will want to change its output; however, with $\pi_{ii} < 0$, the direct effect of a decrease in firm i's output would be an increase in all firms' profits (thus this outcome is clearly not on the profit-possibility frontier). From (12b), if there is a $b_i^* < 0$, then there is a $b_k^* \leq -1$ (because for $b_i^* < 0$, either $b_i^* \leq -1$ or $0 > b_i^* > -1$; in the latter case (12) can only hold if there is a $b_k^* < -1$). If $b_k^* \leq -1$ then (5) implies that for firm k, price is less than or equal to marginal cost at equilibrium. For these reasons, the outcome in which one or more of the b_i^*'s (or equivalently one or more of the α_j's) is negative will be referred to as cutthroat competition. A particular example of this is perfect competition ($b_i^* = -1$ for all i=1,...,n).

To illustrate these equilibria, consider duopoly. Condition (12) reduces to the simpler statement $b_1^* = 1/b_2^*$. Either both b_1^* and b_2^* are positive, in which case the outcome is implicitly collusive; or both b_1^* and b_2^* are negative, in which case the outcome is one of cutthroat competition. In the former case, total industry profit is maximized subject to market shares being $1/(1+b_1^*)$ and $b_1^*/(1+b_1^*)$. In the latter case, (A2) implies $\pi_{11} < 0$ and $\pi_{22} < 0$. Thus, from (A12), one of the α's is positive and the other negative: at the equilibrium, a weighted difference of the firms' profits is maximized. Since both b_i^*'s are negative the firms produce more in total than they would at the Cournot equilibrium.[5]

Thus, because of the existence of the algebraic relationship (11) among the different firms' observations, and because this relationship must be inherited by any equilibrium conjectures b_i^* so that (12) holds, the only possible outcomes that can be reached as the endpoints of an adjustment process are implicit collusion (of which joint-profit maximization is a special case) and cutthroat competition (of which perfect competition is a special case). The unsatisfactory property of the standard conjectural-variations model, the fact that almost any outcome can be an equilibrium, is evidently ameliorated when we make conjectures endogenous.

IV. Disequilibrium Dynamics: Convergence

To illustrate the implications for conjectural equilibrium of observations of apparent correlations between output changes, suppose the system is initially in one of the conjectural equilibria described above. Now suppose that at time t an exogenous change in each firm's costs or in market demand disturbs the system. Suppose each firm has perfect information about this exogenous disturbance. Then each firm will change its output according to (5); $x_i^t \neq x_i^{t-1}$, i=1,...,n. Each firm will observe an apparent correlation β_i^t, and the relationship (10) among these β_i^t's will hold. Thus if the original conjectures b_i^t did not satisfy (12) (that is, the original equilibrium was neither one of cutthroat competition nor on the profit-possibility frontier), some or all of the firms will receive disconfirming information about their conjectures: $b_i^t \neq \beta_i^t$. By (9b), therefore, revision of conjectures is called for so that $b_i^{t+1} \neq b_i^t$ and the original equilibrium is upset by the exogenous cost or demand change. Suppose, on the other hand, that the pre-disturbance conjectures did satisfy (12); in particular, suppose initially $b_i^t = -1$ so that the system was initially at the perfectly competitive equilibrium. By symmetry, the initial outputs are the same for each firm; $x_i^t = x_j^t$, i,j=1,...,n. Moreover, from (5), each firm will react to the disturbance symmetrically, so that $x_i^{t+1} = x_j^{t+1}$, i=1,...,n. From (7), since $x_i^t = x_j^t$, i,j=1,...,n, the i^{th} firm's observation $\beta_i^t = (n-1) \neq -1$; the original conjectures $b_i^t = -1$ are refuted. Suppose instead that initially the system is at the joint-profit maximizing equilibrium, with $b_i^t = n-1$, i=1,...,n. Again, by symmetry, $x_i^{t+1} = x_j^{t+1}$, i,m=1,...,n. Thus, since $x_i^t = x_j^t$, i,j=1,...,n, $\beta_i^t = n-1$; therefore $\beta_i^t = b_i^t$, i=1,...,n and the original conjectures are confirmed. The joint-profit maximizing conjectural equilibrium is not disturbed by the observations of reactions following an exogenous change.[6]

Thus, although this model has a continuum of equilibria, the conjectures upon which most of these equilibria are based are destroyed by the new information about rivals' reactions that results from exogenous changes. (Note, however, that only the first step in the adjustment was examined; the possibility that conjectures might eventually return to their original values has not been ruled out here.)

The state of the system at time t is described by the vector of outputs and beliefs $y^t = (x_1^t,\ldots,x_n^t,b_1^{t+1},\ldots,b_n^{t+1})$. The transition to y^{t+1} is defined by (5), (7), and (8). Equations (5), (7), and (8) implicitly define a system of nonlinear first-order difference equations in y^t. Is this difference-equation system stable? We now demonstrate convergence from some particular initial outputs and conjectures; in the next section we consider the question of convergence from arbitrary initial conditions.

Consider first the special case in which the initial outputs sum to the perfectly competitive level of industry output, $x_{-i}^o + x_i^o = X^c$; and the initial conjectures are $b_i^1 = -1$, $i=1,\ldots,n$. Then from (5), in period 1 each firm will produce $x_i^1 = X^c/n$. Thus each firm will observe $\beta_i^1 = \dfrac{(n-1)X^c/n - x_{-i}^o}{X^c/n - x_i^o} = -1$. Thus $b_i^2 = -1$ and the system reaches the perfectly competitive equilibrium after one adjustment.

Suppose instead that the initial variables are symmetric: $x_i^o = x_j^o$ and $b_i^1 = b_j^1$ for all $i,j=1,\ldots,n$, but that the initial outputs are not profit maximizing given the initial beliefs so that the outputs will change ($x_i^o \neq x_i^1$,

$i=1,\ldots,n$) and the firms will receive observations β_i^1, $i=1,\ldots,n$. Then, from the symmetry, at each stage of the process each firm will observe $\beta_i^t = (n-1)$, $i=1,\ldots,n$, $t=1,2,3,\ldots$. Since each firm repeatedly observes $\beta_i^t = (n-1)$, if the updating operators are the same for each firm and are contractions then eventually the beliefs b_i^t must converge upon $(n-1)$. Once this happens, no further convergence of the b_i^t's will occur; the outputs x_i^t will still change, however. A result due to Sato and Nagatani now becomes relevant. Sato and Nagatani investigated the stability of a conjectural-variations model in which conjectures are fixed, and showed that a sufficient condition for convergence of outputs with fixed conjectures is, in the notation of this paper,

$$\text{(15)}\quad \begin{aligned} &\text{(a)}\quad -\infty < b_i a_i - 1 < 0 \\ &\text{and} \\ &\text{(b)}\quad |b_i a_i| + \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_j| < 1, \quad i=1,\ldots,n, \end{aligned}$$

where

$$\text{(16)}\quad a_i \equiv \frac{f' + (1+b_i)x_i^t f''}{2(1+b_i)f' + (1+b_i)^2 x_i^t f'' - c''(x_i)} .$$

In the special case currently under consideration, $b_i = n-1$, $i=1,\ldots,n$, and $x_i^t = x_j^t$, $i,j=1,\ldots,n$. Moreover, from the second-order condition (6) the denominator of a_i is strictly negative. Thus the first inequality in (15a) is satisfied. The second inequality in (15a) follows from (15b). Given the symmetry, (15b) reduces to

$$\text{(17)}\quad 2(n-1)\,|a_i| < 1.$$

Elementary manipulations, using (6), show that (17) is satisfied in the problem under consideration if

$$\text{(18)} \quad \frac{2f' - c''}{n(n-2)x_i^t} < f'' < \frac{-2(2n-1)f' + c''}{(3n-2)nx_i^t} \quad ;$$

that is, the demand function is neither too concave nor too convex. Thus if all firms start with identical initial outputs and conjectures, all update their conjectures in the same way, and the updating operators are contractions, then the system converges on the joint-profit maximizing outcome provided (18) is satisfied (unless $x^o, b^o) = (x^1, b^1)$).

V. Disequilibrium Dynamics: Instability

The special examples in the previous section showed that there exist initial conditions from which the system converges upon the joint-profit maximum or perfect competition. Convergence is not, however, typical; rather, it is a knife-edge property. It will now be shown that, if this learning process starts from arbitrary initial outputs and conjectures, it is unstable. The reason for this instability is straightforward. The firms are observing a ratio of changes $\beta_i^t = (x_{-i}^t - x_{-i}^{t-1})/(x_i^t - x_i^{t-1})$; if the system were to approach a steady state, both denominator and numerator will become small. Thus small changes in one of the outputs would generate arbitrarily erratic changes in the observation β_i^t.

An equilibrium is stable in Liapunov's definition if nearby solutions stay nearby for all future time. Conversely, and more precisely, an equilibrium $y^* = (x_1^*, \ldots, x_n^*;\ b_1^*, \ldots, b_n^*)$ is unstable if there is a neighborhood U of y^* such that for every neighborhood U_1 of y^* in U, there is at least one path y^t, starting at $y^o \varepsilon U_1$, which does not lie entirely in U (Morris Hirsch and Stephen Smale, pp. 185-87). Suppose the initial state is $y^o = (x_1^* + \epsilon_1, \ldots, x_n^* + \epsilon_n;\ b_1^*, \ldots, b_n^*)$ for some nonzero $\epsilon_1, \ldots, \epsilon_n$. (That is, initially conjectures are at their equilibrium values while outputs are near, but not at, their equilibrium values.) Denote the subsequent state by $y^1 = (x_1^1, \ldots, x_n^1;\ b_1^1, \ldots, b_n^1)$. Taking a Taylor-series expansion around y^*, using the first-order condition (5) and the implicit-function theorem (denoting $\Sigma_{j \neq i}\ \epsilon_j$ by ϵ_{-i}),

$$(19) \qquad x_i^1 - x_1^* \approx \frac{(b_i^* f' + b_i^*(1 + b_i^*)x_i^* f'')\,\epsilon_i - (f' + (1 + b_i^*)x_i^* f'')\,\epsilon_{-i}}{2(1 + b_i^*)f' + (1 + b_i^*)^2 x_i^* f'' - c''} \quad .$$

Assume that

$$(20) \qquad f' + (1 + b_i^*)x_i^* f'' \neq 0, \quad \text{for some } i=1,\ldots,n;$$

the alternative case will be considered below. From (19), the change in the i^{th} firm's output from period 0 to period 1 is

$$(21) \qquad x_i^1 - (x_i^* + \epsilon_i) \approx \frac{(-(2+b_i^*)f' - (1+b_i^*)x_i^* f'' + c'')\,\epsilon_i - (f' + (1+b_i^*)x_i^* f'')\,\epsilon_{-i}}{2(1+b_i^*)f' + (1+b_i^*)^2 x_i^* f'' - c''} \quad .$$

The denominator of the right-hand side of (19) and (21) is nonzero because of the second-order condition (6). Since some output changes will occur, it can be presumed that the b_i^*'s satisfy (12). By (12) it cannot be the case that all the b_i^*'s are infinite: assume $b_1^* < \infty$ and that (20) holds for i=1. The initial outputs are arbitrary: choose ϵ_1, ϵ_{-1} so that (for $-(2+b_1^*)f' - (1+b_1^*)x_1^* f'' + c'' \neq 0$):

$$(22) \qquad \epsilon_1 = \frac{(f' + (1+b_1^*)x_1^* f'')\,\epsilon_{-1}}{-(2+b_1^*)f' - (1+b_1^*)x_1^* f'' + c''} \quad .$$

Then, from (21), the change in firm 1's output from period 1 to period 2 is approximately zero:

(23) $x_1^1 - (x_1^* + \epsilon_1) \approx 0$.

Now choose $\epsilon_2, \ldots, \epsilon_n$ such that their sum satisfies (22) and such that, using (21), the total change in the other firms' output is nonzero:

(24) $\sum_{j=2}^{n} x_j^1 - (x_j^* + \epsilon_j) \neq 0$.

(Clearly there are enough degrees of freedom so that this can be done: the point y^o was not an equilibrium by construction, so that at least one firm j, j=2,...,n, will change its output.) Then from the definition of β (7), the first firm observes an arbitrarily large β_1^1. Thus the updating rule (8) (given the continuity of B_i^t and the fact that $\partial B_i^t/\partial \beta_i^t > 0$) prescribes an arbitrarily large change in the conjectural-variations parameter b_i^2. Thus for every neighborhood of an equilibrium point there is a path which leaves the neighborhood, provided (20) holds.

Return now to consider the case in which (20) does not hold:

(25) $f' + (1 + b_i^*)x_i^* f'' = 0 \quad$ for all i=1,...,n.

Note incidentally that this implies

(26) $\sum_{i=1}^{n} \frac{f'}{(1 + b_i^*)} + \sum_{i=1}^{n} x_i^* f'' = 0$

or, using X to denote total industry output

(27) $f' + Xf'' = 0$

because of (12b). (It cannot be the case that both (25) and (12a) hold for some i.) Equation (27) is satisfied by the particular demand function

(28) $$f(X) = k_1 - k_2 \ln X ,$$

where k_1 and k_2 are strictly positive constants. If (25) holds, then it can be seen from (19) that, if the system starts with equilibrium conjectures and close-to-equilibrium outputs, it moves immediately to within a first-order approximation of the equilibrium point. However, suppose (25) holds and the system starts at an arbitrary point in the neighborhood of equilibrium $y^{0/} = (x_1^* + \epsilon_1, \ldots, x_n^* + \epsilon_n;\ b_1^* + \delta_1, \ldots, b_n^* + \delta_n)$, with $\epsilon_i \neq 0,\ \delta_i \neq 0,\ i=1,\ldots,n$. Taking a Taylor-series expansion around the equilibrium point y^*,

(29) $$x_i^1 - x_i^* \approx \frac{(b_i^* f' + b_i^*(1+b_i^*)x_i^* f'')\epsilon_i - (f' + (1+b_i^*)x_i^* f'')\epsilon_{-i} - x_i^* f' \delta_i}{2(1+b_i^*)f' + (1+b_i^*)^2 x_i^* f'' - c''}$$

Using (25), this reduces to

(29) $$x_i^1 - x_i^* \approx \frac{-x_i^* f' \delta_i}{2(1+b_i^*)f' + (1+b_i^*)^2 x_i^* f'' - c''} .$$

Thus the change in the i^{th} firm's output from period 0 to period 1 is

(30) $$x_i^1 - (x_i^* + \epsilon_i) \approx \frac{[-x_i^* \delta_i - 2\epsilon_i(1+b_i^*)]f' - \epsilon_i(1+b_i^*)^2 x_i^* f'' + \epsilon_i c''}{2(1+b_i^*)f' + (1+b_i^*)^2 x_i f'' - c''} .$$

If δ_1 and ϵ_1 are chosen so that

(31) $$\delta_1 = \frac{-2\epsilon_1(1+b_1^*)f' - \epsilon_1(1+b_1^*)^2 x_1^* f'' + \epsilon_1 c''}{x_1^*} ,$$

then the change in the first firm's output is approximately zero. If δ_j and ϵ_j, for some $j=2,\ldots,n$, are chosen so that the equivalent to (31) does not hold for j, then once again the first firm will observe an arbitrarily large β_1^1. Thus a path starting from a point satisfying (31) will leave any neighborhood of y^*.

Hence the system of first-order nonlinear difference equations defined implicitly by equations (5), (7) and (8) is locally unstable. Since it is locally unstable, it is clearly globally unstable.

VI. Concluding Comments

Modelling firms as conjecturing that there is a relationship between their own outputs and their rivals' outputs is a short-cut way of modelling oligopolistic interdependencies. The conjectures are literally false: each firm conjectures that its rivals' output decisions simply follow a rule making their outputs functions of its own output, when in fact the rivals are solving maximization problems. Presumably the firm believes that such a relationship exists because of its past experiences. This paper modelled the development of such beliefs.

In order to behave as in this model, the firms need very little information. They must know their own characteristics, the market demand function, and past market prices. They need not know anything about the other firms beyond the fact that there exists at least one rival firm. Thus, while Cournot-Nash behavior may in some sense be more rational than the behavior described here, the firms may not have enough information to compute their Cournot-Nash strategies.

In the model of this paper, if no disequilibrium adjustments take place there is a continuum of equilibria (as in the standard conjectural-variations model). Most of these equilibria, however, are eliminated if the firms observe actual output changes of their rivals; the new information firms obtain causes them to change their conjectures. The only possible endpoints of a non-trivial adjustment process are implicit collusion and cutthroat competition. If the firms are initially at a symmetric conjectural equilibrium and there is an exogenous change in costs or demand, the subsequent adjustment will cause the firms to alter their conjectures unless the initial equilibrium was joint-profit maximizing. Although there exist initial outputs and conjectures from which the system converges to the joint-profit maximizing solution or the perfectly competitive solution, if the system starts from arbitrary initial conjectures and outputs then the process of revision of conjectures is unstable.

If it is reasonable, as this paper has proposed, to model oligopolists as forming conjectures about rivals' reactions on the basis of experience, then two conclusions may tentatively be drawn from the foregoing analysis. Firstly, it is unlikely that an oligopolistic market will reach a conjectural equilibrium. (Thus empirical attempts to measure conjectural variations which presume conjectures are fixed--for example, that of Gyoichi Iwata--may be misdirected.) Secondly, if the market does reach an equilibrium, it is more likely to be at one of the extremes of implicit collusion or cutthroat competition than at an intermediate outcome such as the Cournot point. (Thus the model provides some very limited justification for the rejection of Cournot's equilibrium concept by Joseph Bertrand, Edward Chamberlin, and others on the grounds that rational oligopolists will recognize their mutual dependence and somehow find their way to a tacitly collusive solution. This is only limited justification because the implicitly collusive equilibria are unstable and because cutthroat competition is also a possibility.) These conclusions should, of course, be accepted with some caution because the model of this paper is very special.

There are two directions of generalization (which may help to remove the model's instability). Firstly, the firm's beliefs about its rivals' reactions are completely described in this model by a single number, b_i^t. It might be more satisfactory to represent the firm's conjectures by a subjective probability distribution over possible reactions of its rivals. After each new observation, the firm would update its prior probability distribution. These probability distributions over time would follow a Markov process, the convergence properties of which could be studied. Secondly, this model has a shortcoming which it shares with most conjectural-variations models[8] and all Cournot stability studies: although the model is dynamic, the firms myopically maximize only current profits. In a fully dynamic model, a firm would consider the effects of its current actions on its future profits. In such a dynamic model there would be a trade-off between current profits and information: the firm might vary its current output levels, sacrificing some current profits, in order to obtain more accurate information about its rivals' reactions and therefore higher profits in the future.

APPENDIX

In this appendix, it is proved that an equilibrium with conjectures satisfying (12) maximizes the weighted sum of profits $\Sigma_{i=1}^{n} \alpha_i \pi_i$, with $\alpha_i \neq 0$ for some i.

Note firstly that in equilibrium $x_i^t = x_i^{t-1}$, so that the first-order condition (5) can be written as (denoting $\partial\pi_i/\partial x_j$ by π_{ij})

(A1) $$\pi_{ii} + b_i^* \pi_{ij} = 0 , \quad i=1,\ldots,n$$

where $\pi_{ij} = \pi_{ik}$ for all j,k≠i. Since $\pi_{ij} < 0$,

(A2) $$\text{sign}(\pi_{ii}) = \text{sign}\ (b_i)$$

Rewrite (12) as

(A3) $$\prod_{i=1}^{n} (1+b_i^*) - \sum_{k=1}^{n} \prod_{\substack{i=1 \\ i\neq k}}^{n} (1+b_i^*) = 0 .$$

Equation (A3) holds if and only if

(A4) $$D \equiv \begin{vmatrix} -b_1^* & 1 & \ldots & 1 \\ 1 & -b_2^* & & 1 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 1 & 1 & \ldots & -b_n^* \end{vmatrix} = 0 .$$

To prove this, rearrange this determinant as

$$\text{(A5)}\quad D = \begin{vmatrix} -b_1^* & 1 & 1 & \cdots & & 1 \\ 1+b_1^* & -(1+b_2^*) & 0 & & & 0 \\ 0 & 1+b_2^* & -(1+b_3^*) & & & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1+b_{n-1}^* & -(1+b_n^*) \end{vmatrix}$$

$$= \prod_{i=1}^{n-1}(1+b_i^*) \begin{vmatrix} -b_1^* & 1 & 1 & \cdots & 1 \\ 1 & -\frac{1+b_2^*}{1+b_1^*} & 0 & & 0 \\ 0 & \frac{1+b_2^*}{1+b_1^*} & -\frac{1+b_3^*}{1+b_1^*} & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1+b_n^*}{1+b_{n-1}^*} \end{vmatrix} .$$

Adopt the following abbreviation: $\gamma_1 \equiv b_1^*$, $\gamma_2 \equiv (1+b_2^*)/(1+b_1^*),\ldots,$ $\gamma_n \equiv (1+b_n^*)/(1+b_{n-1}^*)$. Then

$$\text{(A6)}\quad \begin{vmatrix} -\gamma_1 & 1 & 1 & \cdots & 1 \\ 1 & -\gamma_2 & 0 & & 0 \\ 0 & 1 & -\gamma_3 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & -\gamma_n \end{vmatrix} = (-1)^n\gamma_1\gamma_2\ldots\gamma_n - \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -\gamma_3 & 0 & & 0 \\ 0 & 1 & -\gamma_4 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & -\gamma_n \end{vmatrix}$$

$$= (-1)^n\gamma_1\gamma_2\cdots\gamma_n - (-1)^{n-2}\gamma_3\gamma_4\cdots\gamma_n + \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & -\gamma_4 & & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \cdots & -\gamma_n \end{vmatrix}$$

$$= (-1)^n\gamma_1\gamma_2\cdots\gamma_n + (-1)^{n-1}\gamma_3\gamma_4\cdots\gamma_n + (-1)^{n-3}\gamma_4\cdots\gamma_n + \cdots$$

$$+ (-1)^k(-1)^{n-k-1}\prod_{i=k}^{n}\gamma_i + \cdots + (-1)^{n-1}\gamma_n + (-1)^{n+1}$$

Thus

$$\text{(A7)}\qquad D = \prod_{i=1}^{n}(1+b_i^*)[(-1)^n\gamma_1\gamma_2\cdots\gamma_n + (-1)^{n-1}(\gamma_3\gamma_4\cdots\gamma_n + \gamma_4\gamma_5\cdots\gamma_n + \cdots + \gamma_n + 1)]$$

$$= (-1)^n[b_1^*\prod_{i=2}^{n}(1+b_i^*) - \sum_{k=2}^{n}\prod_{\substack{i=1\\ i\neq k}}^{n}(1+b_i^*)]$$

$$= (-1)^n[\prod_{i=1}^{n}(1+b_i^*) - \sum_{k=1}^{n}\prod_{\substack{i=1\\ i\neq k}}^{n}(1+b_i^*)].$$

From (A7), (A4) holds if and only if (A3) holds.

From (A1), $b_i = -\pi_{ii}/\pi_{ij}$ for all $j\neq i$. Substituting this in (A4) (using the k in π_{ik} to denote any $k\neq i$):

$$\text{(A8)}\qquad \begin{vmatrix} \frac{\pi_{11}}{\pi_{1k}} & 1 & \cdots & 1 \\ 1 & \frac{\pi_{22}}{\pi_{2k}} & & 1 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 1 & 1 & \cdots & \frac{\pi_{nn}}{\pi_{nk}} \end{vmatrix} = 0\ .$$

Notice that

$$\begin{bmatrix} \frac{\pi_{11}}{\pi_{1k}} & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \frac{\pi_{22}}{\pi_{2k}} & & & & 1 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \frac{\pi_{nn}}{\pi_{nk}} \end{bmatrix} \begin{bmatrix} \pi_{1k} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \pi_{2k} & & & & 0 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \pi_{nk} \end{bmatrix}$$

$$= \begin{bmatrix} \pi_{11} & \pi_{2k} & \cdot & \cdot & \cdot & \pi_{nk} \\ \pi_{1k} & \pi_{22} & & & & \pi_{nk} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \pi_{1k} & \pi_{2k} & \cdot & \cdot & \cdot & \pi_{nn} \end{bmatrix} .$$

From (A9) since $\pi_{ik} = \pi_{ij}$ for all $k,j \neq i$, since the determinant of a product is the product of the determinants, and since the second matrix on the left-hand side of (A9) is nonsingular, (A8) holds if and only if

$$\text{(A10)} \qquad \begin{vmatrix} \pi_{11} & \pi_{21} & \cdot & \cdot & \cdot & \pi_{n1} \\ \pi_{12} & \pi_{22} & & & & \pi_{n2} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \pi_{1n} & \pi_{2n} & \cdot & \cdot & \cdot & \pi_{nn} \end{vmatrix} = 0 .$$

Condition (A10) holds if and only if there exist $\alpha_1,\ldots,\alpha_n$, with $\alpha_j \neq 0$ for some j, such that

$$\text{(A12)} \quad \sum_{j=1}^{n} \alpha_j \pi_{ji} = 0 , \quad i=1,\ldots,n.$$

That is, (A10) is a necessary condition for $\Sigma_{j=1}^{n} \alpha_j \pi_j$ (for some $\alpha_1,\ldots,\alpha_n$ not all zero but not necessarily positive) to be maximized.

REFERENCES

Bertrand, Joseph, Review of Cournot's Recherches sur les Principes Mathématiques de la Théorie des Richesses, Journal des Savants, September 1883, 499-508.

Bishop, Robert L., "Duopoly: Collusion or Warfare?" American Economic Review, December 1960, 50, 933-961.

Bushaw, D. W. and Clower, R. W., Introduction to Mathematical Economics, Homewood, Ill.: Irwin, 1957.

Chamberlin, Edward, The Theory of Monopolistic Competition, Cambridge, Mass.: Harvard University Press, 1933.

Cournot, Augustin, Researches into the Mathematical Principles of the Theory of Wealth, trans. Nathaniel T. Bacon, New York: Macmillan, 1897.

Fisher, Irving J., "Cournot and Mathematical Economics," Quarterly Journal of Economics, January 1898, 12, 119-138.

Hirsch, Morris W., and Smale, Stephen, Differential Equations, Dynamical Systems, and Linear Algebra, New York: Academic Press, 1974.

Iwata, Gyoichi, "Measurement of Conjectural Variations in Oligopoly," Econometrica, September 1974, 42, 947-966.

Kamien, Morton I., and Schwartz, Nancy, "Conjectural Variations," Discussion Paper No. 466S, Northwestern University, March 1981.

Okuguchi, Koji, Expectations and Stability in Oligopoly Models, New York: Springer-Verlag, 1976.

Sato, R. and Nagatani, K., "The Stability of Oligopoly with Conjectural Variations," Review of Economic Studies, October 1967, 34, 409-416.

Seade, Jesus, "The Stability of Cournot Revisited," Journal of Economic Theory, August 1980, 23, 15-27.

FOOTNOTES

*I am grateful to John Chilton, Peter Howitt, Glenn MacDonald, Peter Morgan, Alan Slivinski, and Victoria Zinde-Walsh for comments and advice, and to the Social Sciences and Humanities Research Council of Canada for research support.

[1]For a discussion of a firm's estimation of its current sales prospects based on its observations of its rivals' past actions, and the idea of the perceived demand function as a summary of this estimate, see D. W. Bushaw and R. W. Clower, pp. 179-185.

[2]See, for example, Morton Kamien and Nancy Schwartz, and the references therein.

[3]See Cournot (pp. 81-82). For recent studies of the stability of Cournot equilibrium, see for example Koji Okuguchi and Jesus Seade.

[4]The conjectural-variations models of R. Sato and K. Nagatani and Seade (pp. 24-25) have a dynamic structure similar to this model's, except that the conjectural-variations term is fixed exogenously. Note that the linearity of (3) rules out the possibility that conjectures might be of the kinked-demand type.

[5]These two kinds of outcomes are reminiscent of Robert Bishop's "collusion" and "warfare" solutions. They are different, however, because Bishop made the unorthodox assumption that each firm seeks to maximize a weighted sum of its own and its rivals' profits. In this model, each firm maximizes its own profit given its conjectures; the effect of this, however, is that at equilibrium it is <u>as</u> <u>if</u> the market maximizes a weighted sum of profits.

[6]It is easy to find examples of non-symmetric (that is, non-joint-profit-maximizing) equilibria on the profit-possibility frontier which are upset by new information of this sort. This does not rule out the possibility that some nonsymmetric profit-possibility-frontier equilibria (or nonsymmetric cutthroat-competition equilibria) may not be disturbed by such exogenous changes.

[7]On the other hand, it is easy to show that the sufficient condition for instability of this output-adjustment process due to Seade (p. 24) is never satisfied in this special case because it contradicts the second-order condition (6).

[8]The standard conjectural-variations model, because it has firms responding to other firms' actions, must be implicitly if not explicitly dynamic. Such a model can be formally static only if it is assumed that responses occur instantaneously.