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Generalized Gross Substitutes

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1. INTRODUCTION

In the study of the uniqueness and stability of competitive equilibrium the assumption of gross substitutability plays a key role [1]. Although it has little intuitive appeal it is one of the few assumptions known to imply both uniqueness and stability. However, the assumption has, up to now, been restricted to economies with single-valued excess demand functions.

The present paper defines a generalized concept of gross substitutability, applicable also to the case where excess demands must be expressed as multi-valued functions or correspondences, and demonstrates the uniqueness and stability of the competitive equilibrium under this generalized assumption. It makes use of the results reported in [2] on existence and stability of solutions to differential equations whose right hand sides are not single-valued. The paper also gives an example satisfying the assumption--namely, the case of a pure exchange economy in which all traders have linear utility functions.

Thus the paper represents not only an application of the results in [2] but also an extension of our understanding of the role of substitutability in rendering stable the tâtonnement adjustment process. It is well-known that stability is guaranteed if income-effects are absent from demand functions or if all goods are gross substitutes. It was also shown in [3] that local stability would obtain under certain conditions if one trader's indifference surfaces are flat enough. The present paper supplements the results of [3] by showing that global stability obtains if every trader's indifference surfaces are perfectly flat, the extreme case of perfect substitutability.

2. THE ASSUMPTION

First, some notation. The vector inequality $x' \geqq x$ means $x'_i \geqq x_i$ $\forall i$, $x' \geq x$ means $x' \geqq x$ and $x' \neq x$, and $x' > x$ means $x'_i > x_i$ $\forall i$. The set $\mathfrak{R}^n_+$ is defined as $\{x \in \mathfrak{R}^n: \ x > 0\}$. We use the Euclidean norm: $|x| \equiv (\Sigma x_i^2)^{1/2}$.

Let $\xi: \mathfrak{R}^n_+ \to \mathfrak{R}^n$ be the excess-demand correspondence. Assume that:

(C) $\forall p \in \mathfrak{R}^n_+$, ξ is upper hemi-continuous at p, and the set $\xi(p)$ is non-empty, compact, and convex,

(W) $p \cdot x = 0 \quad \forall x \in \xi(p)$, $p \in \mathfrak{R}^n_+$, and

(H) $\xi(\lambda p) = \xi(p) \quad \forall p \in \mathfrak{R}^n_+$, $\lambda \in \mathfrak{R}_+$.

An equilibrium is defined as a price vector, $\hat{p} \in \mathfrak{R}^n_+$ such that $0 \in \xi(\hat{p})$. We shall assume that:

(E) an equilibrium exists.

Consider the tâtonnement process, defined as the system of (possibly multi-valued) differential equations:

(T) $\frac{dp}{dt} \in \xi(p)$.

We say that the function $p: [0,\bar{t}] \to \mathfrak{R}^n_+$ is a solution to (T) on $[0,\bar{t}]$ starting at $p^o \in \mathfrak{R}^n_+$, iff

(S.1) $p(0) = p_0$,

(S.2) p is absolutely continuous[1] on $[0,\bar{t}]$, and

(S.3) $\frac{dp}{dt}$ exists and satisfies (T) a.e. on $[0,\bar{t}]$.

Likewise, we say that the function $p: [0,\infty) \to \mathfrak{R}^n_+$ is a solution to (T) on $[0,\infty)$ starting at $p^o \in \mathfrak{R}^n_+$, iff (S.1) holds, and if for every $\bar{t} > 0$ (S.2) and (S.3) hold; any such solution is referred to as a trajectory of (T).

Let D be a closed, convex subset of $\mathfrak{R}^n_+$. The system (T) is said to be quasi-stable on D iff (a) for every $p^o \in D$ a solution exists to (T) on $[0,\infty)$ starting at p^o, (b) every trajectory of (T) starting in D is bounded, and (c) every limit point of every such trajectory is an equilibrium. Likewise the system (T) is said to be asymptotically stable on D iff it is quasi-stable on D and every trajectory starting in D has only one limit point. Clearly if for every $p^o \in \mathfrak{R}^n_+$ there is a closed convex set D containing p^o such that (T) is quasi-stable (resp. asymptotically stable) on D then (T) is quasi-stable (resp. asymptotically stable) on $\mathfrak{R}^n_+$.

Let $I \equiv \{1,\ldots,n\}$ be the index set for commodities. In the case where ξ is describable as a single-valued function, $x: \mathfrak{R}^n_+ \to \mathfrak{R}^n$, the assumption of gross substitutability is [1,pp. 227-9]:

(GS)
(i) If $p,p' \in \mathfrak{R}^n_+$, $j \in I$, $p'_j > p_j$, and $p'_i = p_i$ $\forall_i \in I - \{j\}$, then $x_i(p') \geqq x_i(p)$ $\forall_i \in I - \{j\}$.

(ii) there is no partition $\{I_1,I_2\}$ of I and p', $p \in \mathfrak{R}^n_+$ such that $p'_i = p_i$ $\forall i \in I_1$, $p'_i > p_i$ $\forall i \in I_2$, and $x_i(p') = x_i(p)$ $\forall i \in I_1$.

It is well known [1, pp. 227-9, 288-9] that under (GS) the equilibrium is unique, in the sense that for some $\hat{p} \in \mathfrak{R}^n_+$ every equilibrium is of the form $p = \lambda\hat{p}$, $\lambda \in \mathfrak{R}_+$, and that (T) is asymptotically stable on $\mathfrak{R}^n_+$.

Our assumption of generalized gross substitutability is:

(GGS)
(i) $\forall p, p' \in \mathfrak{R}^n_+$, if there is a partition $\{I_1,I_2\}$ of I such that $p'_i = p_i$ $\forall i \in I_1$ and $p'_j > p_j$ $\forall j \in I_2$, then $\sum_{I_1} p_i x'_i \geqq \sum_{I_1} p_i x_i$ $\forall x \in \xi(p)$, $x' \in \xi(p')$, and

(ii) a strict inequality holds above if p is an equilibrium.

In the case where ξ is always single-valued, (GGS) is a slight generalization of (GS). What remains to be shown is that (GGS) also implies uniqueness and stability.

3. UNIQUENESS

Let $\hat{p} \in \mathbb{R}^n_+$ be any equilibrium. We want to show that if $p \in \mathbb{R}^n_+$ is also an equilibrium then $p = \lambda\hat{p}$ for some $\lambda \in \mathbb{R}_+$. First, we demonstrate:

Lemma 1: If $p \in \mathbb{R}^n_+$ and $p \neq \lambda\hat{p}$ for any $\lambda \in \mathbb{R}_+$, then $\forall x \in \xi(p)$, $\sum_{i \in M(p)} p_i x_i > 0$ and $\sum_{i \in S(p)} p_i x_i \leqq 0$, where $M(p) \equiv \{i \in I: p_i/\hat{p}_i \leqq p_j/\hat{p}_j \ \forall j \in I\}$ and $S(p) \equiv \{i \in I: p_i/\hat{p}_i \geqq p_j/\hat{p}_j \ \forall j \in I\}$.

Proof: We shall show first that $\sum_{i \in M(p)} p_i x_i > 0$. Take any such p. Define $\mu \equiv \min (p_i/\hat{p}_i)$. Then $p_i = \mu\hat{p}_i \ \forall i \in M(p) \neq \phi$, and $p_i > \mu\hat{p}_i$ $\forall i \in I - M(p) \neq \phi$. Therefore, by (GGS), $\sum_{i \in M(p)} p_i x_i \geqq \sum_{i \in M(p)} p_i \hat{x}_i$ $\forall x \in \xi(p)$, $\hat{x} \in \xi(\mu\hat{p})$. But, by (H), $\mu\hat{p}$ is an equilibrium; that is, $0 \in \xi(\mu\hat{p})$. Therefore $\sum_{i \in M(p)} p_i x_i > 0 \ \ \forall x \in \xi(p)$. To show that $\sum_{i \in S(p)} p_i x'_i \leqq 0$ we proceed in an analogous way, comparing p to the equilibrium $\sigma\hat{p}$, where $\sigma \equiv \max (p_i/\hat{p}_i)$. However, no strict inequality may be obtained because (GGS) does not require a strict inequality to hold in part (i) if p' is an equilibrium.

It follows immediately from Lemma 1 that the equilibrium is unique, for if $p \neq \lambda\hat{p}$ for any $\lambda \in \mathbb{R}_+$, then $\forall x \in \xi(p)$ $\exists i \in I$ such that $x_i > 0$; so that $0 \notin \xi(p)$.

4. STABILITY

Take any $p^o \in \Re^n_+$. We want to show first that a solution exists to (T) on $[0,\infty)$ starting at p^o. To do this we start by constructing an artificial dynamical system.

Define $C \equiv \{p \in \Re^n_+: \min p_i/\hat{p}_i \geq \min p^o_i/\hat{p}_i$, and $\max p_i/\hat{p}_i \leq \max p^o_i/\hat{p}_i\}$, and $C_2 \equiv \{p \in \Re^n_+: \min p_i/\hat{p}_i \geq 1/2 \min p^o_i/\hat{p}_i$ and $\max p^o_i/\hat{p}_i \leq 2\max p^o_i/\hat{p}_i\}$. Then C and C_2 are both compact, convex subsets of $\Re^n_+$, and $p^o \in C \subset \text{int } C_2$. For any $p \in \Re^n - C_2$, define $\tilde{p}(p)$ as the (unique) solution to $\min_{\{\tilde{p} \in C_2\}} |p-\tilde{p}|$. Then define the correspondence $\zeta: \Re^n \to \Re^n$ as:

$$\zeta(p) \equiv \begin{cases} \xi(p) & \text{if } p \in C_2 \\ \xi(\tilde{p}(p)) & \text{otherwise} \end{cases}$$

Consider the modified system of differential equations:

$$(T') \qquad \frac{dp}{dt} \in \zeta(p)$$

Solutions to (T′) are defined as functions $p: [0,\bar{t}] \to \Re^n$ or $p: [0,\infty) \to \Re^n$, which, like solutions to (T), satisfy (S.1) ~ (S.3).

It is easily verified that $\forall p \in \Re^n$, ζ is upper hemi-continuous at p, and the set $\zeta(p)$ is non-empty, compact and convex. Furthermore, since $\xi(C_2)$ is bounded, there exists a positive number α such that $\forall p \in \Re^n$, $\sup_{x \in \zeta(p)} |x| \leq \alpha(1+|p|)$. Therefore, by the existence theorem of Castaing-Valadier [2, pp. 291-2], there exists a $\hat{t} > 0$ such that

(V.1) $\forall \bar{p}^o \in \Re^n$, the set $S_{\hat{t}}(\bar{p}^o)$ of solutions to (T′) on $[0,\hat{t}]$ starting at $\bar{p}^o$ is non-empty and compact in $C_u([0,\hat{t}]; \Re^n)$, the space of continuous functions from $[0,\hat{t}]$ to $\Re^n$ endowed with the uniform convergence topology; and

(V.2) $\forall A \subset \mathfrak{R}^n$, A compact, the correspondence

$$S_{\hat{t}}: A \to C_u([0,\hat{t}]; \mathfrak{R}^n): \bar{p}^o \to S_{\hat{t}}(\bar{p}^o) \text{ is upper hemi-continuous.}$$

It follows immediately that for any $\bar{p}^o \in C$ a solution exists to (T′) on $[0,\infty)$ starting at $\bar{p}^o$, the projection of which onto $[0,\bar{t}]$ satisfies (V.1) and (V.2). We wish to show that this solution is also a solution to (T). To this end we have three Lemmas:

Lemma 2: If for all $i \in I$ $p_i(t)$ is an absolutely continuous function on the interval $[0,\bar{t}]$ then so are the functions $\mu(t) = \min(p_i(t)/\hat{p}_i)$ and $\sigma(t) = \max(p_i(t)/\hat{p}_i)$.

Proof: We shall prove the theorem for $\mu(t)$. The proof for $\sigma(t)$ is analogous. Take any $\varepsilon > 0$. Then, by the definition of absolute continuity there is a $\delta > 0$ such that:

(L.1) $\frac{1}{\hat{p}_i} \sum_{q\in Q} \left| p_i(t_q) - p_i(\tau_q) \right| < \varepsilon/n$ for all finite sets of pairwise disjoint sub-intervals $\{(t_q,\tau_q): q \in Q\}$ of $[0,\bar{t}]$ such that $\sum_{q\in Q} (\tau_q - t_q) < \delta$, and all $i \in I$.

Consider any such set. For each $q \in Q$ choose $i_q \in I$ such that

(L.2) $$\left|\lambda(t_q) - \lambda(\tau_q)\right| \leqq \left(\frac{1}{\hat{p}_{i_q}}\right)\left| p_{i_q}(t_q) - p_{i_q}(\tau_q) \right|$$

For each $i \in I$, define $Q_i \equiv \{q \in Q: i_q = i\}$. Then each set: $\{(t_q,\tau_q): q \in Q_i\}$ satisfies the premises of (L.1). Therefore, from (L.1) and (L.2):

$$\sum_{q\in Q} \left|\lambda(t_q) - \lambda(\tau_q)\right| \leqq \sum_{i\in I} \sum_{q\in Q_i} \left| p_i(t_q) - p_i(\tau_q) \right| < \sum_{i\in I} \varepsilon/n = \varepsilon \blacksquare$$

Lemma 3: If: (i) p is a solution to (T) on $[0,\bar{t}]$ starting at $\bar{p}^o \in \mathfrak{R}^n_+$, where $\bar{t} > 0$,

(ii) $p(t')$ is not an equilibrium, for some $t' \in (0,\bar{t})$,

and (iii) $\frac{d\mu}{dt}$, $\frac{d\sigma}{dt}$, and $\frac{dp}{dt}$ exist at t' and $\frac{dp}{dt}$ satisfies (T) at t' (where μ,σ are defined as in Lemma 2), then:

(iv) $\frac{d\mu}{dt} > 0$ and $\frac{d\sigma}{dt} \leqq 0$ at t'.

Proof: We shall prove the Lemma only for μ; the proof for σ follows analogously. Suppose (i) ~ (iii) are satisifed by p, $\bar{p}^o$, $\bar{t}$, t'.

Define $N \equiv \{i \in M(p(t')):$ $\frac{1}{\hat{p}_i}\frac{dp_i}{dt}\Big|_{t=t'} \geqq \frac{1}{\hat{p}_j}\frac{dp_j}{dt}\Big|_{t=t'}$ $\forall j \in M(p(t'))\}$. Then by (T) and Lemma 1,

(L.3) $\frac{dp_i}{dt}\Big|_{t=t'} > 0$ $\forall i \in N$.

Also, by continuity, there exists an $h \in \mathfrak{R}_+$ such that $\forall t \in (t'-h, t')$, $M(p(t)) \subset N$. Consider any sequence $\{t^q\}$ such that $t^q \uparrow t'$. Then $\exists\, k \in N$ such that $k \in M(p(t^q))$ for an infinite subsequence of $\{t^q\}$. For notational simplicity we shall suppose that $\{t^q\}$ is that subsequence. By the definition of $M(\,)$, $(p_k(t^q)/\hat{p}_k) = \lambda(t^q)$ $\forall q$. Also, by the definition of N, $p_k(t')/\hat{p}_k = \lambda(t')$. Therefore,

(L.4) $\frac{d\mu}{dt}\Big|_{t=t'} = \lim_{q\to\infty} \frac{\lambda(t^q) - \lambda(t')}{t^q - t'} = (\frac{1}{\hat{p}_k}) \lim_{q\to\infty} \frac{p_k(t^q) - p_k(t')}{t^q - t'} = (\frac{1}{\hat{p}_k}) \frac{dp_k}{dt}\Big|_{t=t'}$.

The Lemma follows from (L.3) and (L.4)∎

Lemma 4: If p is a solution to (T′) on $[0,\infty)$ starting at $\bar{p}^o \in C$, then the function $w(t) \equiv \sigma(t) - \mu(t)$ is non-negative, and strictly decreasing on any interval where $w > 0$ throughout the interval.

Proof: Non-negativity follows immediately from the definition of w. To prove the lemma we need only show that if $t' > 0$ and $w(t') > 0$ then there is an interval $Y \equiv (t' - h, t' + h)$ such that if $t_1, t_2 \in Y$, $t_1 < t_2$, then $w(t_1) > w(t_2)$. This will follow *a fortiori* if we can show that:

(L.5) $\sigma(t_1) \geqq \sigma(t_2)$ and $\mu(t_1) < \mu(t_2)$ if $t_1, t_2 \in Y$, $t_1 < t_2$.

To show this suppose first that p remains always in C_2. Take any $t' > 0$ such that $w(t') > 0$. By the definition of w and the uniqueness of equilibrium, $p(t')$ is not an equilibrium. Thus, by continuity, $\exists$ an interval $Y \equiv (t' - h, t' + h)$ such that:

(L.6) $p(t)$ is not an equilibrium if $t \in Y$.

Furthermore it follows from the absolute continuity of μ and σ on Y (Lemma 2) that [4, pp. 321, 337]:

(L.7) $\frac{d\sigma}{dt}$ and $\frac{d\mu}{dt}$ exist a.e. on Y,

and that [4, p. 340]:

(L.8) $\sigma(t_1) - \sigma(t_2) = \int_{t_2}^{t_1} \frac{d\sigma}{dt}\, dt$, and $\mu(t_1) - \mu(t_2) = \int_{t_2}^{t_1} \frac{d\mu}{dt}\, dt$, $\forall\, t_1, t_2 \in Y$.

Also, from Lemma 3, (L.6) ~ (L.8), and the fact that, since it never leaves C_2, p is a trajectory of (T):

(L.9) $\frac{d\mu}{dt} > 0$ and $\frac{d\sigma}{dt} \leqq 0$ a.e. on Y.

Thus if p never leaves C_2, (L.5) follows from (L.8) and (L.9). Furthermore the above reasoning also shows that p cannot leave C_2. For if it did then, by the definition of C_2, there would have to be some non-degenerate interval U over which p remained in C_2, p was not an equilibrium, and either σ was not decreasing or μ was not increasing. But we have just shown that to be impossible.

It follows that $\forall\ \bar{p}^o \in C$ a solution exists to (T) on $[0,\infty)$ starting at $\bar{p}^o$, the projection of which onto $[0,\bar{t}]$ satisfies (V.1) and (V.2). We have already seen that such solutions exist to (T'). By Lemma 4 all such solutions remain forever in $C \subset C_2$. Thus they are also solutions to (T).

We say that a function W: $\Re^n_+ \to \Re^n$ is a Lyapounov function for (T) on C iff:

(P.1) W is continuous on $\Re^n_+$,

(P.2) for every trajectory p of (T) starting in C the function $W(p(t))$ converges as $t \to \infty$, and

(P.3) if there exists $\tilde{t} \in \Re_+$ and a trajectory p of (T) starting in C such that $W(p(t))$ is constant on $[0,\tilde{t}]$, then $p(0)$ is an equilibrium.

According to Theorem 6.1 of [2], if a Lyapounov function exists for (T) on C, then (T) is quasi-stable on C.

The function $W(p) = \max p_i/\hat{p}_i - \min p_i/\hat{p}_i$ is a Lyapounov function. Property (P.1) is obvious, and properties (P.2) and (P.3) follow immediately from Lemma 4. Thus (T) is quasi-stable on C. Since the initial choice of p^o was arbitrary, therefore (T) is quasi-stable on $\Re^n_+$. Furthermore, we shall show that every trajectory of (T) has only one limit point. By (W) the norm $|p|$ is constant over any trajectory since $\frac{d}{dt}|p| = |p|^{-1}\, p \cdot x$ for some $x \in \xi(p)$ a.e., which must equal zero. Thus, by the uniqueness of equilibrium, each trajectory can have only one equilibrium as a limit point. Since each limit point is an equilibrium a trajectory can have only one limit point. Therefore (T) is asymptotically stable on $\Re^n_+$.

5. AN EXAMPLE

An example of an economic model in which excess demands satisfy (GGS) but not (GS) is given by the case of a pure exchange economy in which all agents have linear utility functions. Let $\xi^i(p)$ be the set of solutions to the problem: $\underset{\{x\in R^n\}}{\text{Max}}\ q^i \cdot x$ subject to $p \cdot x = 0$ and $x + w^i \geqq 0$, where $w^i, q^i \in \mathfrak{R}^n_+$; $i=1,\ldots,m$. The vector w^i can be thought of as trader i's endowment vector. Define $\xi(p) \equiv \sum_{i=1}^{m} \xi^i(p)$. Then ξ satisfies (W), (H) and (C). It is also easily shown using standard proofs [1] that (E) holds. The set $\xi(p)$ is not generally single-valued, so that (GS) does not hold. We wish to show that (GGS) holds.

To do this we begin by showing that ξ^ℓ satisfies part (i) of (GGS) $\forall \ell \in \{1,\ldots,m\}$. Consider any such ℓ. For any $p \in \mathfrak{R}^n_+$ define $\lambda^\ell(p) \equiv \underset{\{j\in I\}}{\text{Max}}\ (q^\ell_j/p_j)$, and $M^\ell(p) \equiv \{j \in I: q^\ell_j/p_j = \lambda^\ell(p)\}$. It can easily be shown that:

$$(T.1)\quad \xi^\ell(p) = \{x \in \mathfrak{R}^n: px = 0,\ x + w^\ell \geqq 0,\ x_j + w^\ell_j = 0\ \forall j \notin M^\ell(p)\}.$$

Now take any $p, p' \in \mathfrak{R}^n_+$ and a partition $\{I_1, I_2\}$ of I such that $p'_j = p_j\ \forall j \in I_1$ and $p'_j > p_j\ \forall j \in I_2$. Take any $x \in \xi^\ell(p)$, $x' \in \xi^\ell(p')$. We wish to show that:

$$(T.2)\quad \sum_{j\in I_1} p_j x'_j \geqq \sum_{j\in I_1} p_j x_j$$

We shall do this in three stages:

(i) Suppose that $M^\ell(p) \subset I_1$. Then, since $p'x' = 0$,

$$(T.3)\quad \sum_{j\in I_1} p_j x'_j = -\sum_{j\in I_2} p'_j x'_j.$$

Also, by the definitions of M^ℓ, $M^\ell(p') = M^\ell(p)$, so that $M^\ell(p') \cap I_2 = \phi$. Therefore, from (T.1),

$$(T.4)\quad \sum_{j\in I_2} p'_j \cdot x'_j = -\sum_{j\in I_2} p'_j w^\ell_j .$$

From (T.3) and (T.4),

$$(T.5)\quad \sum_{j\in I_1} p_j x_j' = \sum_{j\in I_2} p_j' w_j^{\ell}$$

By exactly analogous reasoning,

$$(T.6)\quad \sum_{j\in I_1} p_j x_j = \sum_{j\in I_2} p_j w_j^{\ell}$$

The conclusion (T.2) with strict inequality follows from (T.5) and (T.6), together with the assumptions that $w^{\ell} > 0$ and $p_j' > p_j \quad \forall j \in I_2$.

(ii) Next, suppose that $M^{\ell}(p) \subset I_2$. Then, from the definition of ξ^{ℓ},

$$\sum_{j\in I_1} p_j x_j' \geqq - \sum_{j\in I_1} p_j w_j^{\ell} = \sum_{j\in I_1} p_j x_j$$

(iii) Finally, suppose that $M^{\ell}(p)$ intersects both I_1 and I_2. Then $M^{\ell}(p') \subset I_1$, so that:

$$(T.7)\quad \sum_{j\in I_1} p_j x_j' = - \sum_{j\in I_2} p_j' x_j' = \sum_{j\in I_2} p_j' w_j$$

Likewise, by the definition of ξ,

$$(T.8)\quad \sum_{j\in I_1} p_j x_j = - \sum_{j\in I_2} p_j x_j \leqq \sum_{j\in I_2} p_j w_j^{\ell}$$

In this case (T.2) with a strict inequality follows from (T.7) and (T.8) as it does from (T.5) and (T.6).

It follows immediately from these results that part (i) of (GGS) holds for ξ. By the same reasoning it follows that part (ii) holds if it can be shown that whenever p is an equilibrium, case (ii) ($M^{\ell}(p) \subset I_2$) above cannot hold for all $\ell \in \{1,\ldots,m\}$. To see that this must be true suppose that $M^{\ell}(p) \subset I_2 \ \forall \ell \in \{1,\ldots,m\}$. Then, by the definition of each ξ^{ℓ}, if $x \in \xi(p)$ then $x_j = - \sum_{\ell=1}^{m} w_j^{\ell} \neq 0$, so that p cannot be an equilibrium.

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FOOTNOTES

[1] A function $x: R \to \mathfrak{R}^m$ (where R is any interval of the real line) is absolutely continuous on R iff $\forall \varepsilon > 0 \; \exists \; \delta > 0$ such that $\sum_{q \in Q} |x(t_q) - x(\tau_q)| < \varepsilon$ for all finite sets of pairwise disjoint subintervals $\{(t_q, \tau_q): q \in Q\}$ of R such that $\sum_{q \in Q} (\tau_q - t_q) < \delta$.