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A NOTE ON THE SUFFICIENCY OF THE PONTRYAGIN CONDITIONS FOR OPTIMAL CONTROL WHEN THE TIME HORIZON IS FREE*

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1. INTRODUCTION

The "Maximum Principle" of optimal control theory states the necessary conditions which a solution must satisfy (see Pontryagin *et al* [6], or Lee and Markus [4]). It was shown by Mangasarian [5] that these conditions are sufficient when the maximized Hamiltonian is concave in the state variables. However, Mangasarian explicitly considers only the case with a fixed time interval. (Arrow and Kurz [1], who provide a simple direct proof of the sufficiency theorem, also assume a fixed time.) There are, however, a number of applications of optimal control in economics where the final "time", say, is free. (As, for example, in central place urban models where the radius of the city can be chosen. See Dixit [2]]. For a model of a variable apocalypse with depletable resources, see Koopmans [3][1].) It is the purpose of this note to show that, given a weak condition for "time" dependence in addition to concavity of the maximized Hamiltonian, sufficiency still obtains. The proof proceeds by reducing the variable time case by a simple "trick" to the known fixed time situation.

2. PROBLEM FORMULATION

The basic problem is taken to be to maximize the objective functional

$$J = \int_{t_o}^{t_1} f^o(x,u,t)\,dt$$

subject to the "equations of motion" of the state variables [2]

$$\dot{x}_i(t) = f^i(x,u,t) \qquad i = 1,\ldots,n$$

over choice of the control vector function $\{u(t)\}$. It is assumed that for all time t,

$$u(t) \in \Omega \subset R^m$$

where Ω is the control set. The boundary conditions for the state variables are taken to be [3]

$$x_i(t_o) = x_i^o \qquad i = 1,\ldots,n$$

$$x_i(t_1) = x_i^f \qquad i = 1,\ldots,p$$

$$x_i(t_1) \text{ freely chosen} \quad i = p+1,\ldots,n$$

Finally, although the initial time t_o is taken to be fixed, the terminal time, t_1, will be allowed to vary.

As is shown in Pontryagin [6], for $\{u^*(t)\}$, t_1^*, and $\{x^*(t)\}$ to be an optimal admissible[4] control, terminal time, and associated path of the state variables respectively, it is necessary that for some "costate variables" ψ_i, $i=1,\ldots,n$.[5]

$$\mathcal{H}(x,u,\psi,t) = f^o(x,u,t) + \sum_{i=1}^{n} \psi_i f^i(x,u,t)$$

should be maximized by u^* over choice of $u \in \Omega$. The costate variables satisfy the adjoint equations

$$\dot{\psi}_i = -\frac{\partial \mathcal{H}}{\partial x_i} \qquad i = 1,\ldots,n$$

with

$$\psi_i(t_1^*) = 0 \qquad i = p+1,\ldots,n$$

corresponding to the free choice of the associated state variables. Finally, corresponding to the free choice of terminal time it is necessary that

$$\mathcal{H}(x^*,u^*,\psi,t_1^*) = 0$$

3. THEOREM

Suppose that the maximized Hamiltonian

$$M(x,\psi,t) = \underset{u\in\Omega}{\text{Max}}\ \mathcal{H}(x,u,\psi,t)$$

satisfies the following assumptions

1. $M(x,\psi,t)$ is concave in x
2. $M(x^*(t),\psi(t),t) \geqslant 0 \quad t \in [t_o,t_1^*]$
   $M(x^*(t_1^*),\psi(t_1^*),t) \leqslant 0 \quad t \geqslant t_1^*$. [6]

then the above Pontryagin necessary conditions are also sufficient.

4. PROOF

Consider the following new problem with fixed terminal time T. The objective functional is

$$J = \int_{t_o}^{T} v\ f^o(x,u,t)dt$$

which is to be maximized subject to the equations of motion

$$\dot{x}_i = v\ f^i(x,u,t) \qquad i = 1,\ldots,n$$

over choice of controls

$$(u,v) \in \Omega \times [0,1] \subset R^{m+1}$$

(Note that v has been introduced as a new control variable. Intuitively, v acts as a "switch" for the entire problem, so that a fixed terminal time may be formally considered.) The boundary conditions are now taken to be

$$x_i(t_o) = x_i^o \qquad i = 1,\ldots,n$$

$$x_i(T) = x_i^f \qquad i = 1,\ldots,p$$

$$x_i(T) \text{ freely chosen} \quad i = p+1,\ldots,n$$

Necessary conditions for $\{u^*(t),v^*(t)\}$ and $\{x^*(t)\}$ to be any optimal admissible control and associated path of the state variables are that, for some costate variables $\{\psi_i(t)\}$ $i = 1,\ldots n$,

$$\bar{\mathcal{H}}(x,u,v,\psi,t) = v\,\{f^o(x,u,t) + \sum_{i=1}^{n} \psi_i f^i(x,u,t)\}$$

$$= v\,\mathcal{H}(x,u,\psi,t)$$

should be maximized by (u^*,v^*). The adjoint equations are

$$\dot{\psi}_i = -\frac{\partial\bar{\mathcal{H}}}{\partial x_i} \qquad i = 1,\ldots,n$$

with

$$\psi_i(T) = 0 \qquad i = p+1,\ldots,n$$

Consider now any solution of the necessary conditions of the original problem. Suppose $\{u^*(t)\}$, t_1^*, $\{x^*(t)\}$, and $\{\psi(t)\}$ are the admissible control, terminal time, state variables, and costate variables respectively. (It is desired to show these are optimal). In fact, then, where a prime is used to denote the extension of a function to the larger interval, the controls

$$u^{*\prime}(t) = u^*(t) \qquad t \in [t_o,t_1^*]$$

$$= u^*(t_1^*) \qquad t \in [t_1^*,T]$$

$$v^* = 1 \qquad t \in [t_o,t_1^*]$$

$$= 0 \qquad t \in [t_1^*,T]$$

the state variables

$$x^{*\prime}(t) = x^*(t) \qquad t \in [t_o, t_1^*]$$
$$= x^*(t_1^*) \qquad t \in [t_1^*, T]$$

and the costate variables

$$\psi'(t) = \psi(t) \qquad t \in [t_o, t_1^*]$$
$$= \psi(t_1^*) \qquad t \in [t_1^*, T]$$

satisfy the necessary conditions for the transformed problem for any $T \geqslant t_1^*$. Indeed, consider firstly $t \in [t_o, t_1^*]$. Since $v \geqslant 0$, $u^{*\prime}(t) \in \Omega$ maximizes $\bar{\mathcal{H}}(x^{*\prime}, u, v, \psi, t)$ because $u^*(t) \in \Omega$ maximizes $\mathcal{H}(x^*, u, \psi, t)$. Then using the condition that

$$\mathcal{H}(x^*, u^*, \psi, t^*) = 0$$

and Assumption 2, it can be shown that $\{v^*(t)\}$ is, as asserted an "extremal" control for all $t \in [t_o, T]$. (An "extremal" control satisfies the Pontryagin necessary conditions). Then for $t \in [t^*, T]$ the choice of $u^{*\prime}(t)$ is irrelevant, so that $u^{*\prime}(t)$ is also extremal for all $t \in [t_o, T]$. The new variables clearly satisfy the appropriate boundary conditions. Finally, it is readily seen that the costate variables ψ_i', $i = 1, \ldots, n$ satisfy the adjoint equations and boundary conditions of the new problem.

Now the maximized value of the new Hamiltonian is, say,

$$\bar{M}(x, \psi', t) = M(x, \psi, t) \quad t \in [t_o, t_1^*]$$
$$= 0 \qquad t \in [t_1^*, T]$$

which is concave in x by Assumption 1. Hence the usual sufficiency theorem may be invoked (see Arrow and Kurz [1]), and $\{u^{*\prime}(t)\}$, $\{v^*(t)\}$ and $\{x^{*\prime}(t)\}$ are optimal for the new problem. Suppose now that $\{u(t)\}$, t_1 and $\{x(t)\}$ are any other admissible control, terminal time, and associated state variables for the original problem. Then, for any

$T \geqslant \max [t_1^*, t_1]$, it must be true that

$$\int_{t_o}^{T} v\, f^o(x', u', t)\,dt \leqslant \int_{t_o}^{T} v^* f^o(x^{*\prime}, u^{*\prime}, t)\,dt$$

where

$$v(t) = 1 \qquad t \in [t_o, t_1]$$
$$= 0 \qquad t \in [t_1, T]$$

and $\{x'(t)\}$ and $\{u'(t)\}$ extend $\{x(t)\}$ and $\{u(t)\}$ to $[t_o, T]$ precisely as $\{x^{*\prime}(t)\}$ and $\{u^{*\prime}(t)\}$ extend $\{x^*(t)\}$ and $\{u^*(t)\}$. In other words,

$$\int_{t_o}^{t_1} f^o(x,u,t)\,dt \leqslant \int_{t_o}^{t_1^*} f^o(x^*,u^*,t)\,dt$$

so that the necessary conditions of the original problem are also sufficient,as was to be proven.

5. AN APPLICATION

As an example of the use of this theorem, consider the following simple optimal urban model. Suppose that there are N identical individuals, each with utility function

$$u(s,z)$$

where s is space, and z is a composite consumption good. These individuals are to be allocated in a ring-shaped residential region with fixed inner radius, I, and variable outer radius, R. The entire arc can be used, so that

$$n(t)s = 2\pi t$$

where $n(t)$ is the number of individuals locating between t and t+dt. Also,

$$N = \int_I^R n(t)\,dt$$

There is available a certain fixed amount of the composite commodity, $\bar{Z}$.

This is used for individuals' consumption, but also to rent land at a fixed rate, $\bar{r}$ (corresponding to agricultural use), and to pay for transportation, which costs $\alpha \cdot t$ for each person living at distance t. Hence

$$\bar{z} = \int_I^R \{n(t)z + 2\pi t\bar{r} + n(t)\alpha t\}dt$$

The welfare function is taken to be

$$W = \int_I^R u(s,z)n(t)dt$$

which must be maximized subject to the above constraints. The appropriate Hamiltonian is

$$\mathcal{H} = u\left(\frac{2\pi t}{n}, z\right) n - \psi n - \phi(nz+n\alpha t+2\pi t\bar{r})$$

where ϕ and ψ are constants, in this case. Hence, assuming an interior solution (which can be guaranteed under certain conditions)

$$\frac{\partial \mathcal{H}}{\partial z} = 0 \quad \text{or} \quad u_z = \phi$$

and

$$\frac{\partial \mathcal{H}}{\partial n} = 0 \quad \text{or} \quad u = u_s s + u_z z + u_z \alpha t + \psi .$$

Using these first-order conditions, it can be shown that

$$M = 2\pi t u_z(r(t) - \bar{r})$$

where $r(t) = \frac{u_s}{u_z}$ is the shadow rent. The Hamiltonian is Ricardian rent, which is a result not dependent on these particular assumptions. In this case, it can be shown that

$$\frac{dr}{dt}(t) = -\frac{\alpha}{s} < 0$$

The condition corresponding to free choice of R is

$$M\Big|_R = 0 \quad \text{or} \quad r(t) = \bar{r}$$

Hence Assumption 2 of the theorem is clearly satisfied.[7] Assumption 1 is satisfied trivially because the "state variables" do not enter the Hamiltonian. Hence the above necessary conditions are also sufficient.

FOOTNOTES

1 Koopmans does not use the Maximum Principle but offers a direct sufficiency proof in the particular case he considers.

2 It must be assumed that $f^i(x,u,t)$ $i=0,\dots,n$ and the partial derivatives $\frac{\partial f^i}{\partial x_j}, \frac{\partial f^i}{\partial t}$ $i=0,\dots n;\ j=1,\dots,n$ exist and are continuous.

3 More general boundary conditions can be treated when an initial r_o-dimensional smooth manifold S_o and a final r_1-dimensional smooth manifold S_1 must be attained. That is,

$$(x(t_o),t_o) \in S_o \subset R^{n+1}$$

$$(x(t_1),t_1) \in S_1 \subset R^{n+1}$$

In order for the sufficiency theorem to be proven, additional hypotheses will be needed concerning the convexity of S_o and S_1. (For fixed-time and time-optimal sufficiency theorems with otherwise quite general initial and terminal manifolds, see Lee and Markus [4]).

4 Here "admissible" is taken to mean piecewise continuous, for the sake of simplicity. The directly analogous result holds for bounded measurable controls, however.

5 Strictly a costate variable $\psi_o \geqslant 0$ should be introduced. If $\psi_o > 0$, there is no loss of generality. The degenerate case where $\psi_o = 0$ is simply disregarded.

6 This can be guaranteed if

$$\frac{\partial \mathcal{H}}{\partial t}(x,u,\psi,t) \leqslant 0$$

which holds if, for example,

$$\frac{\partial f^o}{\partial t}(x,u,t) \leqslant 0 \quad \text{and} \quad \frac{\partial f^i}{\partial t}(x,u,t) = 0 \quad i=1,\dots,n$$

For then, the first part of Assumption 2 follows from

$$\frac{dM}{dt}(x^*,\psi,t) = \frac{\partial \mathcal{H}}{\partial t}(x^*,u^*,\psi,t) \leqslant 0 \qquad t \in [t_o,t_1^*]$$

Also

$$\mathcal{H}(x^*(t_1^*),u,\psi(t_1^*),t) \leqslant \mathcal{H}(x^*(\psi^*),u,\psi(t_1^*),t_1^*)$$

$$\leqslant M(x^*(t_1^*),\psi(t_1^*),t_1^*) = 0$$

for all $u \in \Omega$, $t \geqslant t_1^*$, so that the second part of Assumption 2 follows.

7 This example shows incidentally that the generality of Assumption 2 may be needed in applications. For it is not necessarily true here that $\frac{dM}{dt} \leqslant 0$ everywhere and hence it is not true *a fortiori* that $\frac{\partial \mathcal{H}}{\partial t} \leqslant 0$ everywhere.

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