1991

Dynamic Factor Demand and Value Function Methods

Stephen Gordon

ISSN:0318-725X
ISBN:0-7714-1344-0

RESEARCH REPORT 9114

DYNAMIC FACTOR DEMAND AND VALUE FUNCTION METHODS

by

Stephen Gordon

October 1991

Department of Economics

Social Science Centre

University of Western Ontario

London. Ontario. Canada

N6A 5C2

DYNAMIC FACTOR DEMAND AND VALUE FUNCTION METHODS

Stephen Gordon

October 1991

Department of Economics
University of Western Ontario
London, Ontario
Canada, N6A 5C2

ABSTRACT

This paper extends the applicability of Epstein's (1981) value function method of modelling dynamic factor demand to cases in which firms form non-static expectations over a finite planning period. If expected prices are not expected to change after the end of the planning horizon, then the duality between production functions and value functions--along with Bellman's Equation--can be used to transform the producer's problem from one with an infinite horizon to one with a finite horizon. This allows the analyst to provide exact solutions to the producer's problem without resorting to numerical approximations.

The paper also tests to see if the regularity conditions imposed by the duality theorem are consistent with data taken from the U.S. manufacturing sector. Exact posterior probabilities for the restrictions are provided by using Monte Carlo methods to integrate the posterior distribution over the region of the parameter space consistent with the regularity conditions.

1. INTRODUCTION

A fundamental objective of an empirical study of factor demand is to develop models that are capable of providing forecasts for levels of investment and labour demand. Furthermore, in order to avoid the criticism raised by Lucas (1976), these models should be consistent with the theory of a value-maximising firm. If inputs are costly to adjust, then the firm faces a dynamic programming problem. In order to predict firm behaviour, the analyst must estimate and solve the producer's problem.

The introduction of Euler equation methods has been a useful tool in the estimation of production functions and cost functions that reflect cost of adjustment (see, for example, Pindyck and Rotemberg (1983) and Shapiro (1986)). Under the assumption that value-maximising firms form rational expectations, Euler equation methods allow the estimation of the parameters of fairly flexible functional forms. However, since the producer's problem is not actually solved in the estimation (only the first-order conditions are assumed to hold; the transversality condition is not used), the analyst must also solve a dynamic programming problem in order to predict firm behaviour.

This is not a simple task. If the production function is quadratic and if prices are assumed to follow an autoregressive process, then the technique described by Epstein and Yatchew (1985) can be used to find closed-form solutions to the producer's problem. If a closed-form solution is not available, then numerical approximations must be used (see the survey in Taylor and Uhlig (1990)). While exercises of this sort are useful in analyzing the properties of real business cycle models and in evaluating their plausibility, they appear to have limited use in practical applications: no attempt to forecast real-world data have appeared in this literature.

A way to avoid the difficulties in solving the producer's problem is to use the stock market's assessment of the value of the firm. This is the approach taken by models of investment that use Tobin's (1969) Q. An attractive property of the Q models is that the analyst can generate an expression for investment that is a function of observed market data.

Unfortunately, these models have had only marginal success in explaining investment data - see, for example, Summers (1981) and Abel and Blanchard (1986). Moreover, these models cannot generate forecasts for investment without knowledge of the path of future stock market prices.

An alternative approach is simply to hypothesize a functional form for the firm's value function, which represents the maximized value of the firm and is a function of the firm's endowment of capital and of the future path of prices. This approach is due to Epstein (1981), and despite its attractiveness has received surprisingly little attention in the literature. Epstein (1981) is remarkable for two reasons: first, he establishes a duality between the form for the value function and that of the production function, and second, he provides decision rules for factor demand and output supply that are derived from the value function (i.e., a version of Hotelling's Lemma). Nonetheless, this approach has been used in only two empirical applications: Epstein and Denny (1983) and Bernstein and Nadiri (1989).

Although the value function approach appears to be the best available for analysts who wish to be able to provide forecasts, its lack of popularity can probably be attributed to the fact that Epstein (1981) relies on a constant discount rate and static expectations to derive his results. In his argument in favour of Euler equations methods, Shapiro (1986) is particularly critical of these restrictions. While it is possible to extend Epstein's (1981) results to include cases in which prices have a time trend, the restrictions on prices and on the discount rate are particularly troubling for those who wish to analyze the effects of temporary policy changes.

So far, applications of value function methods have relied on the version of Hotelling's Lemma provided by Epstein (1981). Epstein and Denny (1983) and Bernstein and Nadiri (1989) estimate decision rules derived under static expectations. This paper exploits an aspect of Epstein (1981) that has not received much attention, although it is central to the paper: the fact that there is a *duality* between the production function and the value function. Instead of requiring the assumption that current prices are expected to hold forever, the paper requires only that there be a certain point in the future after which relative prices are expected to remain constant.

Since Epstein (1981) provides us with a way of deriving the production function from the value function, Bellman's equation can be used to transform the original infinite-horizon problem to one with finite periods. The idea of a finite but shifting planning horizon is proposed by Nadiri and Prucha (1989). However, while Nadiri and Prucha (1989) assume that *output* is constant after the planning horizon, the method developed below assumes only that *relative prices* are expected to be constant.

The paper is divided into 5 sections. Section 2 adapts Epstein's (1981) duality theorem to discrete time models (Epstein's (1981) theorems are derived in continuous time). Section 3 uses the duality theorem to specify the producer's problem without the assumption of static expectations. Section 4 provides an example of a value function-production function pair and tests the restrictions imposed by the duality theorem, and Section 5 is the conclusion.

2. DUALITY IN DISCRETE TIME

This section adapts the theorems of Epstein (1981) to discrete-time models. The main difference is the treatment of the control variables. In continuous time, the firm controls the level of investment but must treat the level of the capital stock as fixed at any point in time. In discrete time, the firm can control the level of the capital stock, but if capital endowments are predetermined, the choice of the optimal capital stock defines optimal investment as well. In both models, the firm cannot choose the level of investment and the level of the capital stock separately at any point in time. The notation and proofs are taken from Epstein (1981): Ω^k is the open positive orthant in k-dimension Euclidean space; for a vector x, $x > 0$ means that $x \in \Omega^k$ and $x \geq 0$ means that $x \in \bar{\Omega}^k$, the closure of Ω^k; all vectors are column vectors, the superscript τ denotes transposition; 1 denotes the identity matrix of appropriate dimension; the diagonal matrix $\theta \cdot 1$ is denoted without ambiguity by θ, where θ is a scalar; if

$h(x) = h(x_1, ..., x_k)$ is a real valued function, h_{xx} is the matrix $(h_{x_i x_j})_{i,j}$; if $h(x) = (h^1(x)...h^k(x))$, then h_x is the matrix $(h^i_{x_j})_{i,j}$.

2.1 Duality Between Production and Value Functions

Price-taking firms are assumed to solve the infinite-horizon problem

$$J(K_0,p,w) \equiv \max_{\{(K_t, L_t)\}_1^\infty} \sum_{t=1}^{\infty}(1+r)^{-t+1}\left[F(L_t,K_t,I_t) - w^\tau L_t - p^\tau K_t\right] \tag{1}$$

where: $I_t = K_t - (1-\delta)K_{t-1}$, $K_0 > 0$, $F(L_t,K_t,I_t)$ is the production function, $L_t \in \bar{\Omega}^m$ is the perfectly variable factor, $K_t \in \Omega^n$ is the stock of the quasi-fixed factor (capital), $I_t \in \bar{\Omega}^n$ is gross investment, $w \in \Omega^m$ is the relative rental price of L_t (assumed to be constant over time), $p \in \Omega^n$ is the relative rental price of K_t (also constant), $r > 0$ is the rate of discount, δ is a diagonal $n \times n$ matrix of the depreciation rates of the capital stock, J is the value function, $\Theta \subset \Omega^{2n+m}$ is the domain of J and $\Phi \subset \Omega^{2n+m}$ is the domain of F.

Furthermore, define $\Phi(K_{t-1}) = \{(K_t,L_t) \mid (L_t,K_t,I_t) \equiv (L_t,K_t, K_t -(1-\delta)K_{t-1}) \in \Phi\}$ and $\Theta(K_{t-1}) \equiv \{(p,w) \mid (K_{t-1},p,w) \in \Theta\}$. Denote by $\hat{K}(K_0,p,w)$, $\hat{I}(K_0,p,w) \equiv \hat{K}(K_0,p,w) - (1-\delta) K_0$, $\hat{L}(K_0,p,w)$ and $\hat{y}(K_0,p,w)$ the optimal $t=1$ choices for inputs and outputs. $\hat{\lambda}(K_0,p,w)$ denotes the optimal shadow price associated with the capital endowment K_0. In order to distinguish cases where the previous period's capital stock is predetermined, denote $G(L_t,K_t;K_{t-1}) \equiv F(L_t,K_t,K_t - (1-\delta) K_{t-1}) \equiv F(L_t,K_t,I_t)$. Note that the domain of $G(\cdot,\cdot;K_{t-1})$ is simply $\Phi(K_{t-1})$. The f superscript indexes firms.

The regularity conditions imposed on technology are:

T.1: F maps Φ into $\bar{\Omega}^1$; F is twice differentiable.

T.2: $F_L, F_K > 0$, $F_I < 0$.

T.3: $G(L,K;K_0)$ is strongly concave in $\Phi(K_0)$ for each K_0.

T.4: For each $(K_0, p, w) \in \Theta$, a unique solution to the producer's problem exists. The policy functions $\hat{K}$, $\hat{L}$, $\hat{y}$ and the shadow price $\hat{\lambda}$ are at least once differentiable on Θ.

T.5: $1 - (1+r)^{-1}\hat{\lambda}_p(K_0, p, w)$ is nonsingular for each $(K_0, p, w) \in \Theta$.

T.6: For each $(L', K', K' - (1-\delta)K_0) \in \Phi$, there exists $(K_0, p', w') \in \Theta$ such that (L', K') is optimal in (1) at $t=1$ given endowment K_0 and prices p' and w'.

T.7: For each $(K_0, p, w) \in \Theta$, the problem (1) has a unique steady state capital stock $\bar{K}(p,w)$ that is independent of K_0 and is globally stable.

T.8: Each firm's production function F^f satisfies the restrictions necessary such that there exists an aggregate production function F'' that generates decision rules satisfying

$$\hat{K}''(\sum_f K_0^f, p, w) = \sum_f \hat{K}^f(K_0^f, p, w)$$

$$\hat{L}''(\sum_f K_0^f, p, w) = \sum_f \hat{L}^f(K_0^f, p, w)$$

$$\hat{y}''(\sum_f K_0^f, p, w) = \sum_f \hat{y}^f(K_0^f, p, w)$$

T.1 - *T.7* are adapted from Epstein (1981) and discussed there. *T.8* asserts the existence of a representative firm. Consistent aggregation is not necessary for the duality proof in continuous time, but it does simplify the proof in discrete time. In any case, consistent aggregation is a maintained assumption for any applied work that does not use firm-level data. Note also that adjustment costs are associated with gross investment, which is assumed to be non-negative. Adjustment costs could be associated with net investment, but only with an increase in the complexity of the model.

Suppose that F satisfies *T.1*-*T.8* and let J be defined by (1). Then J satisfies the

Bellman equation

$$J(K_0,p,w) = \max_{(L,K) \in \Phi(K_0)} \{F(L,K-(1-\delta)K_0) - w^{\tau}L - p^{\tau}K + (1+r)^{-1}J(K,p,w)\} \tag{2a}$$

or, equivalently,

$$J(K_0,p,w) = \max_{(L,K) \in \Phi(K_0)} \{G(L,K;K_0) - w^{\tau}L - p^{\tau}K + (1+r)^{-1}J(K,p,w)\} \tag{2b}$$

The value function method simply posits a functional form for $J(K_0,p,w)$ and exploits the 'inverse' to (2):

$$F^*(L,K,I) = \min_{(p,w) \in \Theta(K_0)} \{J([(1-\delta)^{-1}(K-I)], p,w) + w^{\tau}L + p^{\tau}K - (1+r)^{-1}J(K,p,w)\} \tag{3a}$$

or

$$G^*(L,K;K_0) = \min_{(p,w) \in \Theta(K_0)} \{J(K_0,p,w) + w^{\tau}L + p^{\tau}K - (1+r)^{-1}J(K,p,w)\} \tag{3b}$$

Equations (3) are to be interpreted as defining the equivalent production functions F^* and G^* given a function J that satisfies appropriate regularity conditions. The regularity conditions will involve the formulae:

$$\tilde{K}(K_0,p,w) \equiv \psi^{-1}(-J_p^{\tau}(K_0,p,w); p,w) \tag{4a}$$

where $\psi(K;p,w) \equiv K - (1+r)^{-1} J_p^{\tau}(K,p,w)$

$$\tilde{I}(K_0,p,w) \equiv \tilde{K}(K_0,p,w) - (1-\delta)\, K_0 \tag{4b}$$

$$\tilde{L}(K_0,p,w) \equiv (1+r)^{-1} J_w^{\tau}(\tilde{K},p,w) - J_w^{\tau}(K_0,p,w) \tag{4c}$$

$$\tilde{y}(K_0,p,w) \equiv J(K_0,p,w) + w^{\tau}\tilde{L} + p^{\tau}\tilde{K} - (1+r)^{-1} J(\tilde{K},p,w) \tag{4d}$$

It will be shown that (4) describes the optimal decision rules at $t=1$ in (1). The regularity conditions on J are:

V.1: J is a real-valued, bounded-from-below and twice differentiable function defined on θ.

V.2: (i) $(1-\delta)^{-1} J_K^{\tau}(K_0,p,w) + p - (1+r)^{-1} J_K^{\tau}(\tilde{K}(K_0,p,w), p,w) > 0$

(ii) $J_K(K_0,p,w) > 0$

V.3: For each $(K_0,p,w) \in \theta$, $\tilde{y} \geq 0$. For each K_0 such that $\theta(K_0)$ is nonempty, $(\tilde{L}(K_0,p,w), \tilde{K}(K_0,p,w))$ maps $\theta(K_0)$ onto $\Phi(K_0)$

V.4: The system $\tilde{K}_t(K_{t-1},p,w)$ defines a sequence $\{K_t\}$ such that $(K_t,p,w) \in \theta$ for all t and $\tilde{K}_t \to \bar{K}(p,w)$ a globally stable steady state, where $(\bar{K},p,w) \in \theta$

V.5: ψ_K is nonsingular

V.6: For $(K_0,p',w') \in \theta$, the minimum in (3) is attained at (p',w') if $(L,K,I) = (\tilde{L}(K_0,p',w')), \tilde{K}(K_0,p',w'), \tilde{I}(K_0,p',w'))$

V.7: The matrix $\begin{bmatrix} \tilde{L}_w & \tilde{L}_p \\ \tilde{K}_w & \tilde{L}_p \end{bmatrix}$ is nonsingular for $(K_0,p',w') \in \theta$.

V.8: $J^f(K_0^f,p,w)$ is of the Gorman polar form:

$$J^f(K_0^f,p,w) = A^f(p,w) + B^{\tau}(p,w)K_0^f$$

where $B(p,w)$ is common to all firms.

Theorems 1 and 2 are due to Epstein (1981):

Theorem 1

(a) *Let F satisfy (T) and define J by (1). Then J satisfies (V). If J is used to define* F^* *by (3) then* $F^* = F$.

(b) *Let J satisfy (V) and define F by (3). Then F satisfies (T). If F is used to define* J^* *by (1), then* $J^* = J$.

Theorem 2

Let F satisfy (T) and let J be the dual value function. The policy functions are given by $\hat{K} = \tilde{K}$, $\hat{I} = \tilde{I}$, $\hat{L} = \tilde{L}$, *and* $\hat{y} = \tilde{y}$.

Theorem 1 demonstrates the duality between F and J and Theorem 2 is the analogue of Hotelling's Lemma. The proofs of Theorems 1 and 2 are straightforward adaptations of the proofs provided in Epstein (1981) and are reproduced in Appendix 1.

For applied work, Theorems 1 and 2 provide a simple method of specifying decision rules that are consistent with the solution of (1). The analyst need only specify the form for J, and Theorem 2 provides decision rules that can be used as a basis for estimation of J and hence may be used of forecasting. If the form for F is desired, (3) is a straightforward method for its derivation.[1] Note that (3) is a much simpler problem than (1).

2.2 Technical Change

Theorems 1 and 2 can be readily adapted to models that incorporate technical change. the simplest case is to posit a rate of Hicks-neutral technical change g, so that

$$y_t = (1+g)^{t-1} F(L_t,K_t,I_t) = (1+g)^{t-1} G(L_t,K_t;K_{t-1}) \tag{5}$$

The natural analogue for value function methods is to suppose that the maximized value of the firm also grows over time:

$$J_t = (1+g)^{t-1} J(K_{t-1},p,w) \tag{6}$$

The question is whether or not Theorem 1 establishes a duality between (5) and (6).

[1]Of course, this would require the analyst to restrict attention to the class of value functions for which there is an analytical solution to (3).

The main complication is due to *T.7* and *V.4*, the assumptions that posit the existence of a globally stable steady state level of the capital stock. In a steady state, firms will not increase levels of inputs if the relative prices of those inputs are expected to increase at the rate of technical change. If this is the case, then (1) can be rewritten as

$$J_{t+1}(K_t,p,w;r,g) = \max_{\{(K_{t+j},L_{t+j})\}_1^\infty} \{ \sum_{j=1}^{\infty} (1+r)^{-j+1} [F(L_{t+j},K_{t+j},I_{t+j})(1+g)^{t+j-1}$$

$$- (1+g)^{t+j-1} w^\tau L_{t+j} - (1+g)^{t+j-1} p^\tau K_{t+j}]\} \qquad (7)$$

$$= (1+g)^t \max_{\{(K_{t+j},L_{t+j})\}_1^\infty} \{ \sum_{j=1}^{\infty} \left[\frac{1+r}{1+g}\right]^{-j+1} [F(L_{t+j},K_{t+j},I_{t+j})$$

$$- w^\tau L_{t+j} - p^\tau K_{t+j}]\} \qquad (8)$$

$$\equiv (1+g)^t J(K_t,p,w;r,g) \qquad (9)$$

Theorem 1 can now be used to establish the duality between the time-invariant functions J and F; all that needs to be done is modify the discount rate so that technical change is taken into account, as in the expression (8). The usual stability condition that $r > g$ is therefore required of this model. Given a functional form for J, we can determine the (unique) F by adapting (3):

$$F_t^*(L_t,K_t,I_t) = (1+g)^{t-1} \min_{(p,w) \in \Theta(K_0)} \{J([(1-\delta)^{-1}(K_t-I_t),p,w) - w^\tau L_t - p^\tau K_t$$

$$+ (1+\gamma)^{-1} J(K_t,p,w)]\} \qquad (10)$$

where $(1+\gamma) \equiv (1+r)/(1+g)$. Similarly, the decision rules based on J in equations (4) can be rewritten as

$$\tilde{K}(K_{t-1},p,w) \equiv \psi^{-1}(-J_p^{\tau}(K_{t-1},p,w);\ p,w) \tag{11a}$$

$$\text{where } \psi(K) \equiv K - (1+\gamma)^{-1}\ J_p^{\tau}(K,p,w)$$

$$\tilde{I}(K_{t-1},p,w) \equiv \tilde{K}(K_{t-1},p,w) - (1-\delta)K_{t-} \tag{11b}$$

$$\tilde{L}(K_{t-1},p,w) \equiv (1+\gamma)^{-1}\ J_w^{\tau}(\tilde{K}_t,p,w) - J_w^{\tau}(K_{t-1},p,w) \tag{11c}$$

$$\tilde{y}(K_{t-1},p,w) \equiv (1+g)^{t-1}[J(K_{t-1},p,w) + w^{\tau}\tilde{L}_t + p^{\tau}K_t - (1+\gamma)^{-1}\ J(\tilde{K}_t,p,w)] \tag{11d}$$

Note that in the steady state, technical change only affects the level of output.

2.3 Summary

Value function methods are a simple way of modelling dynamic factor demand, since the analyst need not solve the dynamic programming problem posed in (1) or (7). Moreover, value function methods provide a way of generating a rich variety of functional forms for value and production functions consistent with the producer's problem.

The main drawback to the use of value function methods is the maintained assumption of static expectations. The existing applications (Epstein and Denny (1983), Bernstein and Nadiri (1989)) estimate the decision rules that are derived from Theorem 2, which are derived under static expectations.

Static expectations are difficult to justify as either a theoretical proposition or as an empirical observation, and the lack of applied work based on Epstein (1981) can probably be attributed to this fact. However, existing applications have only used Theorem 2; the implications of Theorem 1 have not been fully exploited. The next section uses Theorem 1 to extend the use of value function methods to cases in which firms hold nonstatic expectations over a finite planning horizon.

3. NON-STATIC EXPECTATIONS

In this section, the producer's problem as specified in (1) will be generalized to allow for non-static expectations for a finite number of periods. This will mean abandoning Theorem 2, but Theorem 1 can still be used to transform the infinite-horizon problem (1) into a finite-horizon problem. In addition, short run fluctuations in the discount rate can also be accommodated.

3.1 Finite Horizon

Suppose that in the first s periods of the producer's problem, prices are expected to be (p_i,w_i), $i=1,2,\ldots,s$ and are expected to be (p,w) after period s. In addition, let r_i, $i=1,2,\ldots s$ be the discount rate in period i, and let the discount rate after period s be r. If this is the case, then the producer's problem is

$$J(K_0) = \max_{\{(K_t,L_t)\}_1^\infty} \left\{ \sum_{t=1}^{s} \beta_{t-1}[F(L_t,K_t,I_t) - w_t^\tau L_t - p_t^\tau K_t] \right.$$

$$\left. + \beta_s \sum_{t=s+1}^{\infty} (1+r)^{-t+s+1} [F(L_t,K_t,I_t) - w^\tau L_t - p^\tau K_t] \right\} \qquad (12)$$

where the discount rate β_t is defined by $1/(1+r_1)(1+r_2)\ldots(1+r_t)$. The relevant Bellman equation is

$$J(K_0) = \max_{\{K_t, L_t\}_1^s} \left\{ \sum_{t=1}^{s} \beta_{t-1}[F(L_t,K_t,I_t) - w_t^{\tau}L_t - p_t^{\tau}K_t] + \beta_s J(K_s,p,w) \right\} \qquad (13)$$

where $J(K_s,p,w)$ is simply the maximized value of an infinite-horizon problem identical to the one posed by (1). Since this is the case, we can use Theorem 1 to establish the duality between $J(K,p,w)$ and $F(L,K,I)$. Given a functional form for $J(K,p,w)$, we can derive $F(L,K,I)$ from (3) and use *both* expressions in (13).

Although the duality theorem is derived under the assumption of static expectations, its use is not restricted to models with that feature. If the analyst is willing to assume that after a point s periods in the future, relative prices are not expected to change, then Theorem 1 can be used to transform an infinite-horizon problem into one with a finite horizon.

Such an assumption is not implausible. If information is costly to obtain, then there will be a point in the future at which the cost of forming expectations for an additional future period will exceed the discounted gain. In other words, the difference between the solution to the producer's problem under 'rational' expectations and simply assuming that after period s prices will remain constant will not justify the expense of formulating price expectations for periods $s+1$ and onward.

Note that only expected relative prices are assumed to be constant after the end of the planning horizon. In Prucha and Nadiri (1989), the producer's problem is also transformed into one with a finite horizon, but with the assumption that after period s, the firm will be at steady-state levels of inputs and output. In the current specification, the firm will plan to converge to a steady state after period s according to the process described by (4).

3.2 Estimation and Solution

For applied work, the expression that describes $t=1$ behaviour is of most interest. In other applications of value function methods (Epstein and Denny (1983) and Bernstein and Nadiri (1989)), the assumption of static assumptions was maintained, so Theorem 2 could be used to provide the appropriate decision rule for estimation purposes. However, Theorem 2 cannot be used to provide an expression for $t=1$ behaviour for the firm facing (12).

If the length of the firm's planning horizon is known, then the appropriate decision rule can be derived from the solution to (12). This is the approach taken by Nadiri and Prucha (1989), who estimate the model with $s=4$ and $s=10$ years. However, if the analyst is unwilling to impose restrictions on s, then Euler equation methods are still valid. Since the evidence on the length of the planning horizon is sketchy, this paper uses Euler equations as the basis for estimation and hypothesis testing. The length of the planning horizon will be the subject of a subsequent paper.

4. AN EMPIRICAL APPLICATION

As indicated in the previous section, the current application of the value function method involves: (i) specifying a form for the value function J, (ii) solving (3) to find the form of the production function dual to J, and (iii) estimating the parameters of the production function from the Euler equations of the producer's problem. These are the steps taken in this section. In addition, the regularity conditions required for the duality theorem are tested.

4.1 Value and Production Functions

A largely unexploited advantage of value function methods is the wide variety of functional forms it offers for the analyst who wishes to model the behaviour of a representative firm. The popular assumption of constant returns to scale (see, for example, models that use Tobin's Q) is a sufficient condition for aggregation, but it is not necessary, as is noted by Blackorby and Schworm (1982). Existing applications of value functions methods (Epstein and Denny (1983), Bernstein and Nadiri (1989)) postulate quadratic value functions.

Suppose we have single quasi-fixed input (capital) and a single variable input (labour). Let the value function of the representative firm be:

$$J(K_0,p,w) = A\,p^{-\alpha}w^{-\beta} + \left[pB_{pK} + wB_{wK}\right]K_0 \tag{14}$$

Suppose further that the representative firm experiences a rate of Hicks-neutral technical change g, and that after period s, relative prices are expected to increase at the same rate (see the discussion in Section 2). If this is the case, then the form of the production function can be derived by solving the problem posed in (10). The dual production function is

$$F \equiv G \equiv c\,X_K^{\frac{\alpha}{\alpha+\beta+1}}\,X_L^{\frac{\beta}{\alpha+\beta+1}} \tag{15}$$

where:

$$c \equiv \left[\frac{\gamma A}{1+\gamma}\right]^{\frac{1}{\alpha+\beta+1}} \alpha^{\frac{-\alpha}{\alpha+\beta+1}}\,\beta^{\frac{-\beta}{\alpha+\beta-1}}\left[1 + \alpha^{\alpha+\beta+1} + \beta^{\alpha+\beta+1}\right] \tag{16a}$$

$$X_K \equiv B_{pK}\frac{(K-I)}{(1-\delta)} + K - \frac{B_{pK}K}{(1+\gamma)} \tag{16b}$$

$$\equiv B_{pK}K_0 + K - \frac{B_{pK}K}{1+\gamma} \tag{16c}$$

$$X_L \equiv \frac{B_{wK}(K-I)}{(1-\delta)} + L - \frac{B_{wK}K}{(1+\gamma)} \tag{16d}$$

$$\equiv B_{wK}K_0 + L - \frac{B_{wK}K}{(1+\gamma)} \tag{16e}$$

The form for the production function is similar to the Cobb-Douglas production function; this similarity is derived from the form of the first term of the value function (14). Equation (15) can be interpreted as $F(L,K,I)$ if the quasi-inputs X_K and X_L are represented by (16b) and (16d) or as $G(L,K;K_0)$ if they are represented by (16c) and (16e), respectively.

4.2 The Likelihood Function

As was noted in Section 3, Euler equation methods are appropriate for estimating the parameters of (15). The first-order conditions for the optimal choice of current-period inputs (i.e. the part of the solution to (12) that will be implemented) are

$$G_K(K_t,L_t;K_{t-1}) - p_t + (1+r_t)^{-1}\, G_{K_0}(K_{t+1},L_{t+1};K_t) = 0 \tag{17a}$$

$$G_K(K_t,L_t;K_{t-1}) - w_t = 0 \tag{17b}$$

Applying (17) to (15) generates equations that can be used as a basis for estimation. Define $\lambda = (\alpha, \beta, B_{pK}, B_{wK})^\tau$ the parameter vector, y_t = output in period t, $z_t = (K_t, L_t, y_t, K_{t-1}, K_{t+1}, r_t, p_t, w_t)^\tau$ the vector of observables for period t and $Z = (z_1,\ldots,z_N)^\tau$ the entire data set. Equations (18a) and (18b) form the basis of the likelihood function:

$$u_{K_t}(\lambda,\gamma,z_t) \equiv \left[\frac{\alpha}{\alpha+\beta+1}\right]\left[\frac{y_t}{X_{K_t}}\right]\left[1 - \frac{B_{pK}}{1+\gamma}\right] + \left[\frac{\beta}{\alpha+\beta+1}\right]\left[\frac{y_t}{X_{L_t}}\right]\left[\frac{-B_{wK}}{1+\gamma}\right] - p_t$$

$$+ (1+r_t)^{-1}\left[\frac{\alpha}{\alpha+\beta+1}\right]\left[\frac{y_{t+1}}{X_{K_{t+1}}}\right] B_{pK}$$

$$+ (1+r_t)^{-1}\left[\frac{\beta}{\alpha+\beta+1}\right]\left[\frac{y_{t+1}}{X_{L_{t+1}}}\right] B_{wK} \tag{18a}$$

$$u_{K_t}(\lambda,\gamma,z_t) \equiv \left[\frac{\beta}{\alpha+\beta+1}\right]\left[\frac{y_t}{X_{L_t}}\right] - w_t \tag{18b}$$

Define $u_t = (u_{K_t}, u_{L_t})^{\tau}$. With perfect foresight and no measurement error, u_t would be set equal to zero in every period to satisfy the first-order conditions (17). To account for these, the paper makes the weaker assumption that u_t is a random vector with mean zero. Moreover, let u_t be distributed as a bivariate normal with mean zero and covariance matrix Σ:

$$p(u_t | \lambda,\gamma,\Sigma,z_t) = (2\pi)^{-1} |\Sigma|^{-1/2} \, exp\,(-1/2 \, u_t^{\tau} \Sigma^{-1} u_t) \tag{19}$$

Furthermore, assume that errors are independent. Then the likelihood function given N observations is

$$L(\lambda,\gamma,\Sigma | Z) = (2\pi)^{-N} \, |\Sigma|^{-N/2} \, exp\{-(1/2) \sum_{t=1}^{N} u_t^{\tau} \Sigma^{-1} u_t\} \tag{20}$$

4.3 Data

The data are taken from the U.S. manufacturing sector. Output is represented by the total value added in the manufacturing sector, in billions of 1982 dollars. Labour inputs are measured in millions of full-time equivalent employees. Output and labour data are drawn from the National Income and Products Accounts tables. The output price is the deflator (1982=1), and wages per full-time equivalent employee (including employer contributions for social insurance) are measured in thousands of current dollars.

Capital stock and rental price data are provided by the Bureau of Labour Statistics. The capital measure used is Tornqvist's discrete-time version of the Divisia aggregate of the BLS series for structures and for equipment. The discount rate series is the BLS required rate of return.

The data are annual, and run from 1948 to 1987. However, since u_t depends on both a lead and a lag in K_t, the estimation period is 1949 to 1986, making the sample one of 38 observations.

4.4 Maximum Likelihood Estimation

The maximum likelihood estimates for γ are presented in Table 1. The likelihood function is virtually flat in γ, so Table 1 displays the *ML* estimates for λ for a range of values for γ. Varying γ does not significantly affect the estimates for the other parameters.

Before these estimates can be used to model the behaviour of firms, the regularity conditions required by Theorem 1 must be satisfied. One method of testing is to see if the *ML* estimates satisfy the regularity conditions at each data point (this is the approach taken by Epstein and Denny (1983)). The last column in Table 1 indicates whether or not the regularity conditions hold.[2] It suggests that for values of γ less than 0.6, the *ML* estimates do not satisfy T (the restriction of the first derivatives--*T.2*--is not satisfied).

However, such a test has its limitations. The difficulty is the fact that any conclusions about the restrictions must be conditioned by the data that were observed. Unfortunately, this is not consistent with classical statistics, which concerns itself with the behaviour of estimators across all possible data sets. One way around this problem is to restrict attention to functional forms in which the regularity conditions hold for all possible data sets, so long as the parameter vector lies within the appropriate region. If this is the case, then testing the regularity conditions can be done using classical techniques.

Even so, there is no compelling reason to restrict attention to these cases, since it is possible to make use of Bayesian techniques, where all inferences are conditioned by the data that were actually observed.

4.5 Bayesian Estimation

The first step in Bayesian estimation is to specify a prior distribution for the parameters (λ,γ,Σ). Following Box and Tiao (1973) pp. 421-428, assume that prior beliefs about λ,γ and Σ are independent, so that the prior can be represented by

$$p(\lambda,\gamma,\Sigma) = p(\lambda)p(\gamma)p(\Sigma) \tag{21a}$$

---

[2]See Appendix 2 for the application of T to (15).

Suppose further that we can represent a non-informative prior about λ so that

$$p(\lambda) \propto \text{constant} \tag{21b}$$

$$p(\Sigma) \propto |\Sigma|^{-3-2} \tag{21c}$$

Furthermore, let us restrict attention to the values of γ that are considered in Table 1, so that

$$p(\gamma)=p(\gamma_i),\ i = 0.04, 0.05,...,0.14 \tag{21d}$$

Give the prior described in (21) and the likelihood function (20), the posterior distribution for λ--after Σ is integrated out and conditional on each value of γ - is

$$p(\lambda|Z,\gamma) \propto |S(\lambda,\gamma)|^{-N/2} \tag{22}$$

where $S(\lambda,\gamma) = \sum_{t=1}^{N} u_t u_t^{\tau}$.

Define $T(Z)$ to be the region in the parameter space such that T is satisfied at each point in Z. Then the posterior probability that the production function satisfies the regularity conditions--denoted $p(T|Z,\gamma)$--is simply

$$p(T|Z,\gamma) = \int_{T(Z)} p(\lambda|Z,\gamma)d\lambda \tag{23}$$

Unfortunately, there is no closed-form solution for the integral in (23). Instead, Kloek and Van Dijk's (1978) method of Monte Carlo integration was used to evaluate the posterior probability $p(T|Z,\gamma)$ for each value of γ. Values for λ were drawn from a 4-variate normal distribution with means equal to the *ML* estimates and standard deviations equal to 2.5 times the estimated asymptotic standard errors reported in Table 1. Fifty thousand antithetic pairs were drawn from this distribution to evaluate the integral in (23).

For estimation purposes, T provides inequality constraints for the parameters. For $\gamma \geq 0.06$, the *ML* estimates satisfy T, so those estimates in Table 1 can also be interpreted as constrained *ML* estimates.

For Bayesian estimation, the inequality constraints mean that we need only concern ourselves with $T(Z)$--the region of the parameter space consistent with T. To minimize posterior expected quadratic loss, we calculate the posterior mean. As in Geweke (1986), this is calculated as

$$E(\lambda|Z,T,\gamma) = \frac{\int_{T(Z)} \lambda p(\lambda|Z,\gamma)d\lambda}{\int_{T(Z)} p(\lambda|Z,\gamma)d\lambda} \tag{24}$$

The posterior means and standard deviations of the parameters are presented in Table 2, along with the posterior probabilities associated with the regularity conditions. The probabilities associated with regularity conditions are slightly more consistent with larger values of γ. As for the point estimates, the *ML* and Bayes estimators for α and β are quite similar, and the estimates for the adjustment cost parameters B_{pK} and B_{WK} differ by about one half of their posterior standard deviations. Since the *ML* estimates in Table 1 can be seen as the mode of the posterior, the differences between Tables 1 and 2 can be interpreted as evidence on the skewness of the posterior in the region $T(Z)$.

So far, all inferences have been conditioned by the choice of the discount factor γ. Our belief about γ can be updated by applying Bayes' Theorem:

$$p(\gamma_i|Z) = \frac{\int_R p(Z|\lambda,\gamma_i)d\lambda \; p(\gamma_i)}{\sum_i \int_R p(Z|\lambda,\gamma_i)d\lambda \; p(\gamma_i)} \tag{25}$$

Table 3 offers two sets of posterior probabilities for two sets of prior probabilities. One set of prior beliefs posits that all 10 values of γ are equally likely (the "flat" prior); the other represents the prior beliefs of the author (the "informative prior"). Since the likelihood function is slightly inclined towards smaller values of γ, the posterior mean of γ is revised downward, but only slightly.

If we wish to add the regularity conditions to the conditioning set for our inferences, then the appropriate application of Bayes' Theorem is

$$p(\gamma_i|Z,T) = \frac{p(T|Z,\gamma_i)p(\gamma_i|Z)}{\sum_i p(T|Z,\gamma_i)p(\gamma_i|Z)} \tag{26}$$

Note that since larger values of γ are slightly more consistent with the regularity conditions, imposing them revises the posterior mean of γ partially upwards--almost exactly reversing the effect of conditioning on the data alone. Table 4 shows that the posterior probabilities for γ are virtually identical to the prior probabilities.

To remove the conditioning on γ, we can calculate the posterior probability $p(T|Z)$ by

$$p(T|Z) = \sum_i p(T|Z,\gamma_i)p(\gamma_i|Z) \tag{27}$$

and the posterior moments by

$$E(\lambda|Z,T) = \sum_i E(\lambda|Z,T,\gamma_i)p(\gamma_i|Z,T) \tag{28}$$

Table 5 reports these posterior probabilities and moments. Since the informative prior places more weight on large values of γ, the posterior probabilities associated with the regularity conditions is slightly higher than that generated with the flat prior, but the difference is not great. The probability associated with the regularity conditions is hardly greater than one half.

4.6 Partial Adjustment Coefficients

Note that J_{pK} for the value function in (14) does not depend on p or K_0. Since this is the case, Epstein's (1981) Theorem 3 says that an accelerator relationship can be used to describe behaviour after period s:

$$K_t - K_{t-1} = M(\bar{K}(p,w) - K_{t-1}) \tag{29}$$

where $\bar{K}$ is the steady state level of the capital stock and M is the (constant) partial adjustment coefficient:

$$M = 1 - \frac{B_{pK}}{1 - \frac{B_{pK}}{1+\gamma}} \tag{30}$$

Since M has an easily understood interpretation in determining how quickly firms respond to innovations, its values are presented in Table 6. Note that the Bayes estimates are slightly larger than the *ML* estimates, reflecting the negative skewness of the posterior for B_{pK}.

5. CONCLUSION

This paper suggest that the duality between production and value functions demonstrated by Epstein (1981) has empirical applications beyond continuous-time models with static expectations. In particular, the duality theorem can be used to transform an infinite-horizon producer's problem to one with a finite horizon. In addition, value function methods can be extended to accommodate technical change.

There are many extensions to be considered, some of which were mentioned above: disaggregating capital and labour inputs, incorporating non-Hicks-neutral technical change, modelling labour inputs as quasi-fixed and letting adjustment costs depend on net investment. In addition, it remains to be seen what length of the firm's planning horizon is most consistent with observed behaviour.

APPENDIX 1

Proof of Theorem 1 (Epstein (1981))

(a) Let F satisfy T and show J defined by (1) satisfies V.

V.1: J is well-defined on Θ by *T.4*. The boundedness of Φ and Θ implies that J is bounded below over Θ. The required differentiability is established by applying the envelope theorem, *T.1* and *T.4* to (3).

V.5: It is well-known that $J_K(K_0,p,w) = \hat{\lambda}(K_0,p,w)$, so *V.5* is a restatement of *T.5*.

V.6: Let $(K_0,p',w') \in \Theta$. $\hat{K}(K_0,p',w')$ and $\hat{L}(K_0,p',w')$ solve (2) given $(p,w) = (p',w')$. By the nature of problems (2) and (3), it follows that (p',w') is optimal in (3) given $(K,L) = (\hat{K}(K_0,p',w'), \hat{L}(K_0,p',w'))$. However, the first order conditions for an optimum in (3) yield $\hat{K} = \tilde{K}$ and $\hat{L} = \tilde{L}$ evaluated at (K_0,p',w') (This proves the major part of Theorem 2). There is now no need to distinguish between $(\hat{K},\hat{L})$ and $(\tilde{K},\tilde{L})$. Similarly, $\tilde{y} \equiv J(K_0,p,w) + w^{\tau}\tilde{L} + p^{\tau}\tilde{K} - (1+r)^{-1} J(\tilde{K},p,w) = \hat{y}$, where the equality follows from (3). *V.6* is dual to *T.6*.

V.2: Follows from *T.2* and the envelope theorem applied to (2).

V.3: Follows from $F \geq 0$ and *T.6*.

V.4: Follows from *T.4* and *T.7*.

V.7: $\tilde{L}$ and $\tilde{K}$ satisfy

$$G_L^{\tau}(\tilde{L},\tilde{K};K_0) = w$$
$$G_K^{\tau}(\tilde{L},\tilde{K};K_0) = p - (1+r)^{-1}\, \hat{\lambda}(\tilde{K},p,w) \tag{A.1}$$

Apply the strong concavity in K and L, the implicit function theorem, and the fact that $1 - (1+r)^{-1}\, \hat{\lambda}_p(K_0,p,w)$ is nonsingular (*T.5*).

V.8: See the proofs in Blackorby and Schworm (1982) or Epstein and Denny (1983).

Now use J to define F^* by (3). It must be shown that $F = F^*$ over their common domain. By $V.3$, it is enough to prove that $F = F^*$ for all arguments of the form $(\hat{L}(K_0,p,w), \hat{K}(K_0,p,w), \hat{I}(K_0,p,w))$, $(K_0,p,w) \in \Theta$. As in the proof that J satisfies $V.6$, $F^*(\hat{L}(K_0,p,w), \hat{K}(K_0,p,w), \hat{I}(K_0,p,w)) = J([(1-\delta)^{-1}(\hat{K}(K_0,p,w), - \hat{I}(K_0,p,w))], p,w) + w^{\tau}\hat{L}(K_0,p,w) + p^{\tau}\hat{K}(K_0,p,w) - (1+r)^{-1} J(\hat{K}(K_0,p,w),p,w) = \hat{y}(K_0,p,w) = F((\hat{L}(K_0,p,w), \hat{K}(K_0,p,w), \hat{I}(K_0,p,w))$.

(b) Let J satisfy V, define F by (3) and show F satisfies T. F is well-defined by $V.6$ and $V.3$.

$T.4$: Let $(K_0,p',w') \in \Theta$ and consider the corresponding version of (1). By the definition of F, $F(L,K,I) - w^{\tau}L - p^{\tau}K \leq J(K_0,p',w') - (1+r)^{-1} J(K,p,w)$ for all $(L,K) \in \Phi(K_0)$, and with equality is $(L,K) = (\tilde{L}(K_0,p',w'), \tilde{K}(K_0,p',w'))$. If this is the case, then (p',w') is optimal in (3) by $V.6$.

Therefore, for any finite $T \geq 1$

$$\sum_{t=1}^{T} (1+r)^{-t+1}[F(L_t,K_t,I_t) - w'^{\tau}L_t - p'^{\tau}K_t] \leq$$

$$\sum_{t=1}^{T} (1+r)^{-t+1}[J(K_{t-1},p',w') - (1+r)^{-1} J(K_t,p',w')]$$

$$= J(K_0,p',w') - (1+r)^{-T} J(K_T,p',w')$$

Therefore, the value of any feasible program is bounded above by $J(K_0,p',w')$. The inequality becomes an equality if $(L_t,K_t,I_t) = (\tilde{L}(K_{t-1},p',w'), \tilde{K}(K_{t-1},p',w'), \tilde{I}(K_{t-1},p',w'))$, yielding $\sum_{t=1}^{T} (1+r)^{-t+1}[F(\tilde{L}_t,\tilde{K}_t,\tilde{I}_t) - w'^{\tau}\tilde{L}_t - p'^{\tau}\tilde{K}_t] = J(K_0,p',w') - (1+r)^{-T} J(\tilde{K}_T,p',w')$.

By the stability of the steady state (*V.4*), $\bar{K}_T \to \bar{K}(p',w')$ as $T \to \infty$. Therefore, $(1+r)^{-T} J(\bar{K}_T,p',w') \to 0$ and $\sum_{t=1}^{\infty} (1+r)^{-t+1}[F(\bar{L}_t,\bar{K}_t,\bar{I}_t) - w'^{\tau}\bar{L}_t - p'^{\tau}\bar{K}_t] =$ $J(K_0,p',w')$. So J defines the value of programs corresponding to F. $\hat{\lambda}(K_0,p',w') = J_K(K_0,p',w')$, $\hat{K} = \bar{K}$, $\hat{L} = \bar{L}$, $\hat{I} = \bar{I}$ and $\hat{y} = \tilde{y}$. The required differentiability of $\hat{\lambda}$, $\hat{K}$, $\hat{L}$, $\hat{I}$ and $\hat{y}$ follows from (4) and *V.1*.

T.5: Restatement of *V.5*.

T.6: Let $(L',K',K' - (1-\delta)K_0) \in \Phi$ and let $(p',w') \in \Theta(K_0)$ be optimal in (3). That this price vector makes (L',K') optimal in (1) at $t=1$ was shown in the proof of *T.4*.

T.7: Restatement of *V.4*.

T.2: By *V.7* and the implicit function theorem, the functions $\hat{p}$ and $\hat{w}$ satisfying $(p',w') = (\hat{p}(L,K,I), \hat{w}(L,K,I))$ iff

$$(L,K,I) = (\bar{L}(K_0,p,w), \bar{K}(K_0,p,w), \bar{I}(K_0,p,w)) \qquad \text{(A.2)}$$

are well-defined and differentiable. By *V.6*, the minimum in (3) is attained at $(\hat{p}(L,K,I), \hat{w}(L,K,I))$. Apply (A.2), *V.2* and the envelope theorem to (3) to obtain

$$F_K^{\tau}(L,K,I) = (1-\delta)^{-1} J_K^{\tau}([(1-\delta)^{-1}(K-I)], \hat{p},\hat{w}) + \hat{p} - (1+r)^{-1} J_K^{\tau}(K,\hat{p},\hat{w}) > 0 \qquad \text{(A.3)}$$

$$F_I^{\tau}(L,K,I) = -(1-\delta)^{-1} J_K^{\tau}([(1-\delta)^{-1}(K-I)], \hat{p},\hat{w}) < 0$$

$$F_L^{\tau}(L,K,I) = \hat{w} > 0$$

(A.3) and the differentiability of J_K and $(\hat{p},\hat{w})$ yield the differentiability of F_K, F_I and F_L.

T.3: (A.3) can also be written as

$$G_K^{\tau}(L,K;K_{-1}) = \hat{p} - (1+r)^{-1} J_K^{\tau}(K_0,\hat{p},\hat{w})$$
$$G_L^{\tau}(L,K;K_{-1}) = \hat{w} \qquad \text{(A.3}'\text{)}$$

G is concave in (L,K) since it is the minimum of a family of linear and hence concave functions. Strong concavity requires that the appropriate Hessian be nonsingular. From (A.3′), it follows that

$$G_{KK} = \hat{p}_K - (1+r)^{-1} J_{Kp}(K,\hat{p},\hat{w})\hat{p}_K - (1+r)^{-1} J_{Kw}(K,\hat{p},\hat{w})\hat{w}_K - (1+r)^{-1} J_{KK}(K,\hat{p},\hat{w})$$
$$G_{KL} = \hat{p}_L - (1+r)^{-1} J_{Kp}(K,\hat{p},\hat{w})\hat{p}_L - (1+r)^{-1} J_{Kw}(K,\hat{p},\hat{w})\hat{w}_L$$
$$G_{LL} = \hat{w}_L$$
$$G_{LK} = \hat{p}_L$$

Note that $J_{KK} = 0$ by *V.8*. Therefore, we can write

$$\begin{bmatrix} G_{KK} & G_{KL} \\ G_{LK} & G_{LL} \end{bmatrix} = \begin{bmatrix} [1 - (1+r)^{-1} J_{Kp}(K,\hat{p},\hat{w})] & [-(1+r)^{-1} J_{Kw}(K,\hat{p}.\hat{w})] \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{p}_K & \hat{p}_L \\ \hat{w}_K & \hat{w}_L \end{bmatrix} \equiv RS$$

S is nonsingular from *V.7* and (A.2). R is nonsingular from *V.5*. Therefore RS is nonsingular, and G is strictly concave.

T.8: See Blackorby and Schworm (1982) or Epstein and Denny (1983).

Now use F to define J^* via (1). It was shown in the proof of *T.4* that $J^* = J$ □.

APPENDIX 2

Application of T to (15)

$T.1$: Satisfied by choice of functional form.

$T.2$: $$F_{L_t} = \frac{\beta}{\alpha+\beta+1}\frac{y_t}{X_{L_t}} > 0 \qquad \forall t$$

$$F_{K_t} = \frac{\alpha}{\alpha+\beta+1}\frac{y_t}{X_{K_t}}\left[\frac{B_{pK}}{1-\delta} - \frac{B_{pK}}{1+\gamma} + 1\right] + \frac{\beta}{\alpha+\beta+1}\frac{y_t}{X_{L_t}}\left[\frac{B_{wK}}{1-\delta} - \frac{B_{wK}}{1+\gamma}\right] > 0 \qquad \forall t$$

$$F_{I_t} = -\frac{\alpha\beta}{\alpha+\beta+1}\frac{y_t}{X_{K_t}}\frac{B_{pK}}{1-\delta} - \frac{\beta}{\alpha+\beta+1}\frac{y_t}{X_{L_t}}\frac{B_{wK}}{1-\delta} < 0 \qquad \forall t$$

$T.3$: $G_{KK} < 0,\ G_{LL} < 0,\ G_{KK}G_{LL} - (G_{KL})^2 > 0 \qquad \forall t$

$T.4$: Holds if $J(K_0,p,w)$ is strictly convex in (p,w). This implies

$$\alpha > -1,\ \beta > -1,\ \alpha\beta(\alpha+\beta+1) > 0$$

$T.5$: $1 - \frac{B_{pK}}{1+\gamma} \neq 0$

$T.6$: Same as $T.4$

$T.7$: $$\left|\frac{B_{pK}}{1 - \frac{B_{pK}}{1+\gamma}}\right| < 1$$

$T.8$: Satisfied by choice of functional form.

REFERENCES

Abel, A. B., and Blanchard, O. J., "The Present Value of Profits and Cyclical Movements in Investment" *Econometrica* 54 (1986) 249-273.

Bernstein, J. I., and Nadiri, M. I., "Research and Development and Intra-Industry Spillovers: An Empirical Application of Dynamic Duality" *Review of Economic Studies* 56 (1989) 249-267.

Blackorby, C., and Schworm, W., "Aggregate Investment and Consistent Intertemporal Technologies" *Review of Economic Studies* 44 (1982) 595-614.

Box, G. F. P., and Tiao, G. C., *Bayesian Inference in Statistical Analysis* (1973) Reading, Massachusetts: Addison-Wesley.

Epstein, L. G., "Duality Theory and Functional Forms for Dynamic Factor Demands" *Review of Economic Studies* 48 (1981) 81-95.

______________, and Denny, M. G. S., "The Multivariate Flexible Accelerator Model: Its Empirical Restrictions and an Application to U.S. Manufacturing" *Econometrica* 51(1983) 647-674.

______________, and Yatchew, A. J., "the Empirical Determination of Technology and Expectations: A Simplified Procedure" *Journal of Econometrics* 27 (1985) 235-258.

Geweke, J., "Exact Inference in the Inequality Constrained Normal Linear Regression Model" *Journal of Applied Econometrics* 1 (1986) 127-141.

Kloek, T., and van Dijk, H. K., "Bayesian Estimates of Equation System Parameters: An Application of Integration by Monte Carlo" *Econometrica* 46 (1978) 1-19.

Lucas, R. E., "Econometric Policy Evaluation: A Critique" in Karl Brunner and Allan Meltzer (eds.) *The Phillips Curve and labor Markets* (1976) Carnegie-Rochester Conference Series Vol. 1 Amsterdam: North-Holland 19-46.

Nadiri, M. I., and Prucha, I. R., "Dynamic Factor Demand Models, Productivity Measurement, and Rates of Return: Theory and an Empirical Application to the U.S. Bell System: Working Paper No. 89-26, University of Maryland.

Pindyck, R. S., and Rotemberg, J. J., "Dynamic Factor Demand Under Rational Expectations" *Scandinavian Journal of Economics* 85 (1983) 223-238.

Shapiro, M. D., "The Dynamic Demand for Capital and Labor" *Quarterly Journal of Economics* (1986) 513-542.

Summers, L. H., "Taxation and Corporate Investment: A Q-Theory Approach" *Brookings Papers on Economic Activity* 1 (1981) 67-140.

Taylor, J. B., and Uhlig H., "Solving Nonlinear Stochastic Growth Models: A Comparison of Alternate Solution Methods" *Journal of Business and Economic Statistics* 8 (1990) 1-18.

Tobin, J., "A General Equilibrium Approach to Monetary Theory" *Journal of Money, Credit and Banking* 1 (1969) 15-29.

Table 1
Maximum Likelihood Estimates

γ	α	β	B_{pK}	B_{wK}	Log L	T
0.05	1.4234	4.1526	0.32962	-0.002266	107.783	no
	(0.05)	(0.13)	(0.18)	(0.0026)		
0.06	1.4279	4.1626	0.30319	-0.001798	107.719	yes
	(0.05)	(0.13)	(0.18)	(0.0026)		
0.07	1.4314	4.1744	0.27686	-0.001333	107.670	yes
	(0.05)	(0.13)	(0.18)	(0.0025)		
0.08	1.4338	4.1865	0.25314	-0.000914	107.636	yes
	(0.05)	(0.14)	(0.187)	(0.0025)		
0.09	1.4354	4.1973	0.23459	-0.000581	107.614	yes
	(.05)	(0.14)	(0.17)	(0.0024)		
0.10	1.4363	4.2074	0.21820	-0.000294	107.603	yes
	(0.05)	(0.14)	(0.17)	(0.0023)		
0.11	1.4366	4.2161	0.20538	-0.000069	107.599	yes
	(0.05)	(0.14)	(0.17)	(0.0022)		
0.12	1.4365	4.2233	0.19554	0.000104	107.599	yes
	(0.04)	(0.14)	(0.17)	(0.0021)		
0.13	1.4363	4.2289	0.18907	0.000224	107.603	yes
	(0.04)	(0.14)	(0.17)	(0.0020)		
0.14	1.4358	4.2340	0.18318	0.000326	107.609	yes
	(0.04)	(0.14)	(0.16)	(0.0020)		

Table 2
Bayes Estimates

γ	α	β	B_{pK}	B_{wK}	(T|Z,γ)
0.05	1.4491 (0.05)	4.2530 (0.14)	0.19563 (0.18)	0.000452 (0.0029)	0.39858
0.06	1.4487 (0.04)	4.2569 (0.14)	0.18745 (0.18)	0.000579 (0.0028)	0.43702
0.07	1.4487 (0.05)	4.2621 (0.14)	0.18165 (0.19)	0.000686 (0.0027)	0.47139
0.08	1.4484 (0.05)	4.2676 (0.15)	0.17660 (0.19)	0.000797 (0.0027)	0.51740
0.9	1.4478 (0.05)	4.2741 (0.14)	0.16944 (0.19)	0.000931 (0.0026)	0.53694
0.10	1.4471 (0.05)	4.2791 (0.14)	0.16652 (0.19)	0.001004 (0.0025)	0.55634
0.11	1.4462 (0.05)	4.2824 (0.14)	0.16470 (0.18)	0.001043 (0.0024)	0.57134
0.12	1.4452 (0.04)	4.2848 (0.14)	0.16470 (0.19)	0.001062 (0.0023)	0.58500
0.13	1.4439 (0.04)	4.2861 (0.14)	0.16454 (0.18)	0.001062 (0.0022)	0.59917
0.14	1.4430 (0.05)	4.2884 (0.14)	0.16470 (0.17)	0.001067 (0.0021)	0.61022

Table 3
$p(\gamma|Z)$

γ	$p(\gamma)$	$p(\gamma\|Z)$	$p(\gamma)$	$p(\gamma\|Z)$
0.05	0.10	0.13030	0.03	0.03987
0.06	0.10	0.12051	0.05	0.06146
0.07	0.10	0.11176	0.07	0.07980
0.08	0.10	0.10494	0.15	0.16055
0.09	0.10	0.09856	0.20	0.20107
0.10	0.10	0.09371	0.20	0.19117
0.11	0.10	0.08968	0.15	0.13721
0.12	0.10	0.08624	0.07	0.06157
0.13	0.10	0.08339	0.05	0.04253
0.14	0.10	0.08092	0.03	0.02476
	$E(\gamma) = 0.095$		$E(\gamma) = 0.095$	
	$\sigma(\gamma) = 0.02872$		$\sigma(\gamma) = 0.02022$	
	$E(\gamma\|Z) = 0.09059$		$E(\gamma\|Z) = 0.09280$	
	$\sigma(\gamma\|Z) = 0.02882$		$\sigma(\gamma\|Z) = 0.02046$	

Table 4
$p(\gamma|Z,T)$

γ	$p(\gamma)$	$p(\gamma|ZT)$	$p(\gamma)$	$p(\gamma|Z,T)$
0.05	0.10	0.10032	0.03	0.02983
0.06	0.10	0.10174	0.05	0.05042
0.07	0.10	0.10177	0.07	0.07060
0.08	0.10	0.10488	0.15	0.15592
0.09	0.10	0.10223	0.20	0.20265
0.10	0.10	0.10071	0.20	0.19963
0.11	0.10	0.09898	0.15	0.14715
0.12	0.10	0.09745	0.07	0.06761
0.13	0.10	0.09652	0.05	0.04783
0.14	0.10	0.09539	0.03	0.02836

$E(\gamma) = 0.095$	$E(\gamma) = 0.095$
$\sigma(\gamma) = 0.02872$	$\sigma(\gamma) = 0.02022$
$E(\gamma|Z) = 0.09059$	$E(\gamma|Z) = 0.09280$
$\sigma(\gamma|Z) = 0.02882$	$\sigma(\gamma|Z) = 0.02046$
$E(\gamma|Z,T) = 0.09439$	$E(\gamma|Z,T) = 0.09462$
$\sigma(\gamma|Z,T) = 0.02853$	$\sigma(\gamma|Z,T) = 0.02006$

Table 5

	Flat Prior	Informative Prior
$P(T\|Z)$	0.51766	0.53276
$E(\alpha\|T,Z)$	1.4469	1.4472
$E(\beta\|T,Z)$	4.2732	4.2747
$E(B_{pK}\|T,Z)$	0.17383	0.17127
$E(B_{wK}\|T,Z)$	0.000865	0.000911

Table 6
Partial Adjustment Coefficients

γ	ML	Bayes
0.05	0.51956	0.70069
0.06	0.57535	0.71436
0.07	0.62650	0.72385
0.08	0.66936	0.73063
0.09	0.70108	0.74446
0.10	0.72781	0.75139
0.11	0.74799	0.75565
0.12	0.76310	0.75975
0.13	0.77294	0.76208
0.14	0.78175	0.76350

	Flat Prior	Informative Prior
E(M|T,Z)	0.74027	0.73347