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A First Graduate Course in Economic Theory and Game Theory

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A FIRST GRADUATE COURSE IN ECONOMIC THEORY AND GAME THEORY

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<u>Contents</u>

Preface

The pages following contain suggested readings, some problems and a collection of notes. This material was assembled over the last few years in the course of instructing first year M.A. and Ph.D students in Economics at the University of Western Ontario. I circulate it in hopes that someone may find it useful.

This is not a text, and so makes no claim to include all topics or to be comprehensive. Rather, its intent is simply to provide information sufficient to give the student familiarity with the main concepts found in the best general economics journals. Readings are those I found helpful in achieving this goal.

The problems range from easy to quite difficult, from purely theoretical to highly applied, and cover a wide range of topics. Many allow the student to discover some useful idea or method of attack.

The notes themselves are little more than a list of notation, definitions, theorems and proofs. They are intended to allow students to write less and concentrate more in class. They contain no motivation, intuition or explanations and so serve only as a skeleton.

NOTES ON CONSUMER THEORY

1. Commodities

The *definition of a commodity* must specify its physical characteristics, the date and place of availability, and whether delivery is contingent on any as yet unobserved events. Assume a finite list of commodities, indexed by $h = 1,\dots,l$; $l < \infty$. The *quantity of commodity h* is $x_h \in \mathbb{R}$. A *commodity bundle* is a vector $x = (x_1,\dots,x_l)' \in \mathbb{R}^l$. Let $\underline{0} = (0,\dots,0)' \in \mathbb{R}^l_+$.

2. Prices

All transactions occur at the outset in an initial exchange. The *price of commodity h* is $p_h \in \mathbb{R}$. Assuming "free disposal" implies $p_h \in \mathbb{R}_+$; this will be strengthened to $p_l \in \mathbb{R}_{++}$. A *price system* is a vector $p = (p_1,\dots,p_l)' \in \mathbb{R}^l_{++}$. The *value of any commodity bundle x* at prices p is $p'x$.

3. Consumers

Consumers choose a commodity bundle from those that are feasible and affordable. The *consumption set* $X = \mathbb{R}^l_+$ describes feasible x. (In general $X \subset \mathbb{R}^l$ is closed, convex and bounded below.) Affordable x are described by the *budget correspondence*. The consumer is endowed with an income of $w \in \mathbb{R}_{++}$. The budget correspondence $\gamma: \mathbb{R}^{l+1}_{++} \to \mathbb{R}^l_+$ is defined pointwise by

$$\gamma(p,w) = \{x \mid p'x \le w\} \cap \mathbb{R}^l_+.$$

γ is convex– and compact–valued and continuous (for $X = \mathbb{R}^l_+$).

4. Preferences

Preferences over commodity bundles are described by a binary relation $\succeq$, defined on $\mathbb{R}^l_+ \times \mathbb{R}^l_+$, with "$x \succeq y$" interpreted as "$x$ is at least as preferred as y". Assume:

A1: (Completeness): $\forall(x,y) \in \mathbb{R}_+^l \times \mathbb{R}_+^l$, either $x \succsim y$ or $y \succsim x$,

A2: (Transitivity): $\forall(x,y) \in \mathbb{R}_+^l \times \mathbb{R}_+^l$ and $(y,z) \in \mathbb{R}_+^l \times \mathbb{R}_+^l$ such that $x \succsim y$ and $y \succsim z$, $x \succsim z$.

Since A1 implies $\succsim$ is reflexive ($x \succsim x$), $\succsim$ is a complete preordering. Also impose

A3: (Continuity): $\forall x \in \mathbb{R}_+^l$, both $\{y \in \mathbb{R}_+^l | y \succsim x\}$ and $\{y \in \mathbb{R}_+^l | x \succsim y\}$ are closed.

Two other binary relations are implied: i) Indifference "$\sim$": $x \sim y \Leftrightarrow (x \succsim y$ and $y \succsim x)$; ii) Strict preference "$\succ$": $x \succ y \Leftrightarrow (x \succsim y$ and not $y \succsim x)$.

5. Representation of $\succsim$ by a Utility Function

Definition: Let $\succsim$ be a complete preordering on $\mathbb{R}_+^l \times \mathbb{R}_+^l$. The function $u:\mathbb{R}_+^l \to \mathbb{R}$ *represents* $\succsim \Leftrightarrow (\forall(x,y) \in \mathbb{R}_+^l \times \mathbb{R}_+^l$ with $x \succsim y$, $u(x) \geq u(y))$.

Theorem (Debreu): *Let $\succsim$ satisfy A1-A3. Then there exists a continuous function u representing $\succsim$.*

While the result is correct as stated, the proof to follow adds *monotonicity* to the list of hypotheses. (Monotonicity: $\forall(x,y) \in \mathbb{R}_+^l \times \mathbb{R}_+^l$ satisfying $x \neq y$ and $x_h \geq y_h$ for all h, $x \succ y$.)

PROOF: Let ι be the unit vector in $\mathbb{R}^l$. Let $x \in \mathbb{R}_+^l$. By strong monotonicity, there exists a scalar $u_x \in \mathbb{R}_+$ such that $u_x \iota \sim x$. Define $u:\mathbb{R}_+^l \to \mathbb{R}_+$ by $u(x) = u_x$.

To verify that u represents $\succsim$, suppose $x \succsim y$. Then $u_x \cdot \iota \sim x$ and $u_y \cdot \iota \sim y$. If $u_x < u_y$, strong monotonicity and A2 give $y \succ x$. Therefore $u_x \geq u_y$ and hence $u(x) \geq u(y)$. Conversely, suppose $u(x) \geq u(y)$. Then $u_x \geq u_y$ and $u_x \cdot \iota \succsim u_y \cdot \iota$, implying $x \succsim y$.

Intervals $I \subset [0,\infty)$ of the form $I = (\underline{u}, \bar{u})$ or $I = [0,\bar{u})$ are a basis for the order topology of $\mathbb{R}_+$. That is, every open set in $\mathbb{R}_+$ can be written as the union of such intervals, say $\cup_\alpha I_\alpha$.

Since $u^{-1}(\bigcup_{\alpha} I_{\alpha}) = \bigcup_{\alpha} u^{-1}(I_{\alpha})$, to demonstrate continuity of u it suffices to check that for all such I, $u^{-1}(I)$ is open.

$$\begin{aligned} u^{-1}((\underline{u}, \bar{u})) &= \{x \in \mathbb{R}^l_+ \,|\, \underline{u} < u(x) < \bar{u}\} \\ &= \{\{x \in \mathbb{R}^l_+ \,|\, u(x) \leq \underline{u}\} \cup \{x \in \mathbb{R}^l_+ \,|\, u(x) \geq \bar{u}\}\}^c \\ &= \{x \in \mathbb{R}^l_+ \,|\, u(x) \leq \underline{u}\}^c \cap \{x \in \mathbb{R}^l_+ \,|\, u(x) \geq \bar{u}\}^c \\ &= \{x \in \mathbb{R}^l_+ \,|\, \underline{u}\mathfrak{1} \succsim x\}^c \cap \{x \in \mathbb{R}^l_+ \,|\, x \succsim \bar{u}\mathfrak{1}\}^c \end{aligned}$$

which is open by A3. $u^{-1}([0,\bar{u}))$ is analysed similarly. ||

Theorem: Let $f{:}\mathbb{R} \to \mathbb{R}$ be continuous and strictly increasing and u a continuous representation of $\succsim$. Then $v = f o u$ is a continuous representation of $\succsim$.

PROOF: i) $v{:}\mathbb{R}^l_+ \to \mathbb{R}$, in which case v it is possible for v to represent $\succsim$; ii) Since u represents $\succsim$, $x \succsim y \Leftrightarrow u(x) \geq u(y)$. But by strict monotonicity of f, $u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y)$. Thus $x \succsim y \Leftrightarrow v(x) \geq v(y)$. Also, f and u are continuous, so $f \circ u$ is, as well. ||

6. The Consumer Problem

The consumer is assumed to select a commodity bundle that is feasible and affordable, and that is as preferred as any other feasible and affordable bundle. The *demand correspondence* φ: $\mathbb{R}^{l+1}_{++} \to \mathbb{R}^l_+$ is defined pointwise by

$$\varphi(p,w) = \{x \in \gamma(p,w) \,|\, x \succsim y,\ y \in \gamma(p,w)\}.$$

Observe that since u represents $\succsim$, $x \in \varphi(p,w) \Leftrightarrow x \in \underset{y\in\gamma(p,w)}{argmax}\ u(y)$. Since u is continuous on $\mathbb{R}^l_+$, and for each (p,w) $\gamma(p,w)$ is a compact subset of $\mathbb{R}^l_+$, u achieves a maximum on $\gamma(p,w)$ i.e. φ exists.

7. Properties of φ

a) $\forall\lambda \in \mathbb{R}_{++}$, $\forall(p,w) \in \mathbb{R}^{l+1}_{++}$, $\varphi(\lambda p, \lambda w) = \varphi(p,w)$;

b) If for some $(p,w) \in \mathbb{R}^{l+1}_{++}$, both $x \in \varphi(p,w)$ and $y \in \varphi(p,w)$, then $x \sim y$;

c) φ is upper–hemi continuous.

8. <u>Other Restrictions on $\succsim$</u>

For any $z \in \mathbb{R}^l$, let $\|z\| = (z'z)^{\frac{1}{2}}$. Also, $x = (x^1, x^2)$ is a partition of x into subvectors x^1 and x^2.

a) *Local Nonsatiation*: $\succsim$ is locally nonsatiated if $\forall x \in \mathbb{R}^l_+$ and $\varepsilon \in \mathbb{R}_{++}$, there exists $y \in \mathbb{R}^l_+$ with $\|x-y\| < \varepsilon$ and $y \succ x$.

b) *Monotonicity*: (Definition given above, p. 2.)

c) *Convexity*: $\succsim$ is convex if for all $x \in \mathbb{R}^l_+$, $\{y \in \mathbb{R}^l_+ | y \succsim x\}$ is convex.

d) *Strict Convexity*: $\succsim$ is strongly convex if for all $x \in \mathbb{R}^l_+$, $x \sim y$ and $x \neq y$ imply $\bar{x} \in \text{Int}\{\tilde{x} \in \mathbb{R}^l_+ | \tilde{x} \succsim x\}$, where $\bar{x} = \lambda x + (1-\lambda)y$ and $\lambda \in (0,1)$.

e) *Differentiability*: see Debreu, "Smooth Preferences," *Econometrica*, 40 (1972), 603–15.

f) *Separability*: $\succsim$ is (weakly) separable if for all $x \in \mathbb{R}^l_+$ and $y \in \mathbb{R}^l_+$, a) $(x^1, x^2) \succsim (y^1, x^2) \Leftrightarrow (x^1, y^2) \succsim (y^1, y^2)$; and b) $(x^1, x^2) \succsim (x^1, y^2) \Leftrightarrow (y^1, x^2) \succsim (y^1, y^2)$.

9. <u>More Predictions</u>

a) Let $\succsim$ be locally nonsatiated. Then $\forall (p,w) \in \mathbb{R}^{l+1}_{++}$, $x \in \varphi(p,w) \Rightarrow p'x = w$.

PROOF: Suppose $x \in \varphi(p,w)$ and $p'x < w$. Choose $\varepsilon > 0$ so that $\|x-y\| \leq \varepsilon$ implies $p'y < w$. By local nonsatiation there exists $y_\varepsilon \succ x$ with $\|x-y_\varepsilon\| \leq \varepsilon$. Thus $x \notin \varphi(p,w)$, a contradiction. ||

b) Consider $n \geq 2$ commodity bundles $x^1, x^2, \ldots, x^n$ and price vectors $p^1, p^2, \ldots, p^n$ arrayed in a vector $(x^1, p^1, x^2, p^2, \ldots, x^n, p^n) = (x^i, p^i)_{i=1}^{i=n}$. x^i is ***directly revealed preferred*** to x^j, written $x^i R^0 x^j$ if $p^{i\prime} x^i \geq p^{i\prime} x^j$. If the inequality is strict x^i is *strictly* directly preferred to x^j, denoted x^iPx^j. Let $\{i_1,\ldots,i_m\}$ be a subset of $\{1,\ldots,n\}$ having m elements. If for some such subset $x^iR^0x^{i_1}, x^{i_1}R^0x^{i_2},\ldots,x^{i_m}R^0x^j$ hold, x^i is *revealed preferred* to x^j, written x^iRx^j.

Definition: $(x^i,p^i)_{i=1}^{i=n}$ satisfies the *Generalized Axiom of Revealed Preference* (GARP) if x^iRx^j implies not x^jPx^i.

Definition: Let $\succsim$ satisfy A1–A3, be locally nonsatiated, and represented by the function u: $\mathbb{R}^l_+ \to \mathbb{R}$. *u rationalizes* $(x^i, p^i)_{i=1}^{i=n}$, if $\forall\ i \in \{1,\ldots,n\}$, $x^i \in \varphi(p^i, p^{i\prime} x^i)$.

Theorem (Afriat–Varian): There exists an increasing function u rationalizing $(x^i, p^i)_{i=1}^{i=n}$ if and only if $(x^i, p^i)_{i=1}^{i=n}$ satisfy GARP.

PROOF: Varian, Hal R. "The Nonparametric Approach to Demand Theory" *Econometrica*, 50 (July 1982), 945–73.

Theorem: Let $x^0 \in \varphi(p^0,w)$ and $x^1 \in \varphi(p^1, p^{1\prime}x^0)$. Then $(p^1 - p^0)'(x^1 - x^0) \leq 0$.

PROOF: Since $x^1 \in \varphi(p^1, p^{1\prime} x^0)$,

$$p^{1\prime}x^1 \leq p^{1\prime}x^0, \tag{9.1}$$

in particular $x^1R^0x^0$. Thus, since GARP must be satisfied, not x^0Px^1 must hold, or

$$p^{0\prime}x^0 \leq p^{0\prime}x^1. \tag{9.2}$$

Adding (9.1) to (9.2) gives the result. ||

c) If $\succsim$ is convex, $\forall(p,w) \in \mathbb{R}^{l+1}_{++}$, $\varphi(p,w)$ is convex.

PROOF: For some $(p,w) \in \mathbb{R}^{l+1}_{++}$, let $x^0 \in \varphi(p,w)$ and $x^1 \in \varphi(p,w)$; $x^0 \sim x^1$. For $\lambda \in [0,1]$, define $\bar{x} = \lambda x^0 + (1-\lambda)x^1$. Then $p'\bar{x} \leq w$ and by convexity of $\succsim$, $\bar{x} \succsim x^0$. Since $x^0 \succsim y$ for $y \in \gamma(p,w)$, $\bar{x} \in \varphi(p,w)$. ||

d) If $\succsim$ is strictly convex and locally nonsatiated, φ is single–valued and continuous.

PROOF: For some $(p,w) \in \mathbb{R}^{l+1}_{++}$, let $x^0 \in \varphi(p,w)$ and $x^1 \in \varphi(p,w)$, and assume $x^0 \neq x^1$. Then $\bar{x} = \frac{1}{2}(x^0 + x^1)$ satisfies $p'\bar{x} \leq w$ and $\bar{x} \in \text{Int}\{\tilde{x} \in \mathbb{R}^l_+ | \tilde{x} \succsim x\}$, by strict convexity. Thus $\exists\, \hat{x} \in \{\tilde{x} \in \mathbb{R}^l_+ |\ \hat{x} \succsim x\}$ with $p'\hat{x} < w$. Therefore $x^0 \notin \varphi(p,w)$, a contradiction. Therefore $x^0 = x^1$. Since φ is single–valued and upper hemi–continuous it is continuous. ||

Whenever φ is single–valued, it will be referred to as the Marshallian (or Ordinary) *demand function* $f\colon \mathbb{R}^{l+1}_{++} \to \mathbb{R}^l_+$.

e) Slutsky Conditions. Let $\succsim$ be locally nonsatiated and strictly convex. Assume $\succsim$ can be represented by $u \in C^2(\mathbb{R}^l_+)$. Let $\Omega \subset \mathbb{R}^{l+1}_{++}$ be open and such that $(p,w) \in \Omega$ implies $f(p,w) \in \mathbb{R}^l_{++}$. Then if $\Omega \neq \emptyset, f \in C^1(\Omega)$ and the $l \times l$ matrix $\nabla_p f + \nabla_w f \cdot f(p,w)'$ is negative semidefinite, symmetric and singular; in particular, $(\nabla_p f + \nabla_w f f')\, p = \underset{\sim}{0}$. ($\nabla_p f \equiv (\partial f_h/\partial p_k)$ $h,k = 1,\ldots,l$ and $\nabla_w f \equiv (\partial f_h/\partial w)$ $h = 1,\ldots l$.)

PROOF: Section 13.

10. Some Properties of $u \in C^2(\mathbb{R}^l_+)$

Suppose that $u \in C^2(\mathbb{R}^l_+)$ is strictly quasi concave. Denoting $\partial u/\partial x_h$ by u_h and $\partial^2 u/\partial x_h \partial x_k$ by u_{hk},

$$\nabla u = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_\ell \end{bmatrix} \text{ and } \nabla^2 u = \begin{bmatrix} u_{11},\dots,u_{1\ell} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_{\ell 1},\dots,u_{\ell\ell} \end{bmatrix}.$$

The relevant properties are:

a). $\nabla^2 u = (\nabla^2 u)'$;

b). For $x^0 \in \mathbb{R}^l_{++}$, if $\nabla u(x^0) \neq \underset{\sim}{0}$, define $O(x^0) = \{x \in \mathbb{R}^l_+ \,|\, \nabla u(x^0)' x = 0\}$. Then $\forall x \in O(x^0)$, $x' \nabla^2 u(x^0)\, x \leq 0$ with strict inequality almost everywhere in $\mathbb{R}^l_{++}$. Moreover, the matrix $\begin{bmatrix} \nabla^2 u(x^0) & \nabla u(x^0) \\ \nabla u(x^0)' & 0 \end{bmatrix}$ is nonsingular almost everywhere in $\mathbb{R}^l_{++}$;

c). Let $\bar{u}$ be the image of x under u; i.e. $\bar{u} = u(x)$. For x such that $u_h \neq 0$, the equation $u(x) - \bar{u} = 0$ may be solved for $x_h = q^h(x_{-h}, \bar{u})$, where $x_{-h} = (x_1,\dots,x_{h-1}, x_{h+1},\dots,x_l)$. q^h is twice continuously differentiable on the interior of its domain. For $u_k > 0$, all k, $-\partial q^h/\partial x_k \geq 0$ is called the Marginal Rate of Substitution (MRS_{hk}) of x_k for x_h; $\partial q^h/\partial x_k = -u_k/u_h$. Also $\partial^2 q^h/\partial x_k^2 \geq 0$, often referred to as "diminishing MRS."

11. Slutsky–Hicks Approach

Under the conditions given in 9 (e), $f(p,w)$ can be characterized as the interior solution to the programming problem

$$\max_x u(x) \text{ subject to } p'x = w.$$

The Lagrange function for this problem is

$$L(x,\lambda;p,w) = u(x) + \lambda(w - p'x),$$

where $\lambda \geq 0$ is an "undetermined multiplier." Assuming x^* is the maximal value of x, and λ^* the associated value of λ, implies

$$\nabla_x L(x^*, \lambda^*; p,w) = 0, \quad (\nabla u(x^*) - \lambda^* p = 0), \tag{11.1}$$

$$\nabla_\lambda L(x^*, \lambda^*; p,w) = 0, \quad (w - p'x^* = 0), \tag{11.2}$$

and

$$\forall x \text{ such that } p'x = 0,\ x'\nabla_x^2 L(x^*, \lambda^*; p,w)x \leq 0, \tag{11.3}$$

where $\nabla_x^2 L(\cdot)$ is the matrix of second partial derivatives of L with respect to x.

Properties to note are:

a). (11.1) implies $\lambda^* > 0$, since some $u_h(x^*) > 0$, by local nonsatiation (i.e. $p'x = w$ and $u_h(x^*) \leq 0$, all h, are incompatible). Then $\nabla_h u(x^*) > 0$ for all h, again from (11.1);

b). (11.3) is redundant. That is, since $\succeq$ is strongly convex, u is strictly quasi concave and property 10(b) holds at x^*. Now consider (11.3). Let x be such that $p'x = 0$. By (11.1), $\nabla u(x^*)'x = 0$, implying $x'\nabla^2 u(x^*)x \leq 0$. But $\nabla^2 u(x^*) = \nabla_x^2 L(x^*, \lambda^*; p,w)$, so (11.3) is satisfied.

12. Derivation of Slutsky Matrix

The predictions in 9(e) refer to the matrix $\nabla_p f + \nabla_w f f'$. This matrix is referred to as the Slutsky matrix and is obtained as follows.

Recall that the first order necessary conditions (11.1) and (11.2) can be written

$$\nabla u(x^*) - \lambda^* p = 0$$

and

$$w - p'x^* = 0.$$

Observe the Jacobian matrix of this system equals the matrix in 10(b), nonsingular almost everywhere. Also, expressions on the left hand side of each equation have continuous derivatives with respect to x, λ, p and w. Thus, by the Implicit Function Theorem there exists functions $f\colon \Omega \to \mathbb{R}^l_{++}$ and $\theta\colon \Omega \to \mathbb{R}_{++}$, with $f \in C^1(\Omega)$ and $\theta \in C^1(\Omega)$, such that

$$\nabla u[f(p,w)] - \theta(p,w)p = 0 \tag{12.1}$$

and

$$w - p'f(p,w) = 0. \tag{12.2}$$

Now since f and θ have continuous partial derivatives, differentials dx^* and $d\lambda^*$, dp and dw satisfy

$$\begin{bmatrix} dx^* \\ d\lambda^* \end{bmatrix} = \begin{bmatrix} \nabla_p f & \nabla_w f \\ \nabla_p \theta' & \nabla_w \theta \end{bmatrix} \begin{bmatrix} dp \\ dw \end{bmatrix}.$$

Also, given the continuous differentiability of the expressions in (12.1) and (12.2), it is also true that

$$\begin{bmatrix} dx^* \\ d\lambda^* \end{bmatrix} = \begin{bmatrix} \nabla^2 u & -p \\ -p' & 0 \end{bmatrix}^{-1} \begin{bmatrix} \lambda^* I_l & \underset{\sim}{0} \\ x^{*\prime} & -1 \end{bmatrix} \begin{bmatrix} dp \\ dw \end{bmatrix},$$

where I_l is the $l \times l$ identity matrix, and the inverse exists almost everywhere. It follows that since $x^* = f(p,w)$

$$\begin{bmatrix} \nabla_p f & \nabla_w f \\ \nabla_p \theta' & \nabla_w \theta' \end{bmatrix} = \begin{bmatrix} \nabla^2 u & -p \\ -p' & 0 \end{bmatrix}^{-1} \begin{bmatrix} \lambda^* I_l & \underset{\sim}{0} \\ f(p,w)' & -1 \end{bmatrix} \tag{12.3}$$

Define Z, z and ζ by

$$\begin{bmatrix} Z & z \\ z' & \zeta \end{bmatrix} \equiv \begin{bmatrix} \nabla^2 u & -p \\ -p' & 0 \end{bmatrix}^{-1}, \tag{12.4}$$

where Z is $l \times l$. Then the relationship between upper left hand submatrices in (12.3) is

$$\nabla_p f = \lambda^* Z + zf'. \tag{12.5}$$

Also, the relationship between the upper right hand submatrices is $\nabla_w f = -z$. Thus (12.5) becomes

$$\nabla_p f = \lambda^* Z - \nabla_w f f'. \tag{12.6}$$

Define the Slutsky matrix by $S \equiv \lambda^* Z$. Then (12.6) gives

$$S = \nabla_p f + \nabla_w f f'. \tag{12.7}$$

(12.7) is the *Slutsky Equation.*

13. <u>Proof of Slutsky Conditions</u> 9(e)

(a) $S = S'$

PROOF: S is a principal submatrix of a symmetric matrix. ||

(b) $Sp = \underset{\sim}{0}$

PROOF: Postmultiplying (12.4) by $\begin{bmatrix} \nabla^2 u & -p \\ -p & 0 \end{bmatrix}$ gives a matrix whose upper right submatrix, Zp equals $\underset{\sim}{0}$, implying $Sp = \underset{\sim}{0}$. ||

Note that since $p \in \mathbb{R}^l_{++}$ and $\lambda^* \in \mathbb{R}_{++}$, $Sp = 0$ implies Rank $(S) \leq l-1$.

c) $\forall y \in \mathbb{R}^l$, $y'Sy \leq 0$.

PROOF: Since $\lambda^* > 0$, $y'Sy \leq 0 \Leftrightarrow y'Zy \leq 0$.

From (12.4), premultiplied as in part (b), the upper left hand submatrices satisfy $Z\nabla^2 u - zp' = I_l$. Post multiplication by Z gives

$$Z\nabla^2 uZ = Z - zp'Z = Z. \tag{13.1}$$

Next, for any $y \in \mathbb{R}^l$, define $q \equiv Zy$. Note that $q'\nabla u(x^*) = \lambda^{-1}y'Zp = 0$, from (12.1) and (b). Thus $q \in O(x^*)$, in which case $q'\nabla^2 u(x^*)q \leq 0$. But $q'\nabla^2 u(x^*)q = y'Z\nabla^2 uZy = y'Zy$, from the definition of q and (13.1). ||

(a) – (c) are inherited by the matrix $\nabla_p f + \nabla_w f f'$; see (12.7).

14. <u>Miscellany</u>

a) Interpretation of S

Suppose p and w are allowed varied by dp and dw provided $x^*dp = dw$; i.e. w is adjusted so that x^* is affordable when the price system is $p + dp$. (This adjustment is called "Slutsky compensation".) Define $V^* = u[f(p,w)]$. Since u and f are continuously differentiable, differentials dV^*, dp and dw satisfy

$$dV^* = \nabla u'(\nabla_p f dp + \nabla_w f dw)$$
$$= \nabla u'(\nabla_p f + \nabla_w f f')dp \text{ (Slutsky compensation)}$$
$$= \lambda^* p' S dp (12.1 \text{ and definition of } S)$$
$$= 0. \quad (13(b))$$

The second equality demonstrates that S is the matrix of changes in x^* in response to (small) Slutsky compensated changes in p. That $dV^* = 0$ as a result of such variations in p implies that S^* is also the matrix of changes in x^* in response to (small) changes in p when w is adjusted to hold V^* constant ("Hicks compensation"). S is often called the matrix of *substitution effects* and the Slutsky equation written

$$\nabla_p f = S - \nabla_w f f';$$

i.e. the effect of p on f is the sum of the substitution (S) and *income effects* $(-\nabla_w f f')$. The diagonal elements of this matrix are

$$\frac{\partial f_h}{\partial p_h} = S_{hh} - x_h \frac{\partial f_h}{\partial w}$$

and $S_{hh} \leq 0$.

b) Interpretation of λ^*

For $dp = \underset{\sim}{0}$

$$dV^* = \nabla u' \nabla_w f dw$$
$$= \lambda^* p' \nabla_w f dw$$
$$= \lambda^* dw,$$

since $z = -\nabla_w f$ and $-p'z = 1$ (12.3, 12.4). ||

Thus, λ^* is the *marginal utility of income.*

c) The condition $p'\nabla_w f = 1$ is sometimes called "Engel Aggregation". Also, $p'S = 0 \Leftrightarrow p'(\nabla_p f + \nabla_w f f') = 0$, implying $p'\nabla_p f + f' = \underset{\sim}{0}'$; this expression is sometimes called "Cournot Aggregation".

15. Duality

a) Assuming only A1 – A3, it is possible to define the functions G: $\mathbb{R}^{l+1}_{++} \longrightarrow \mathbb{R}$ and E: $\mathbb{R}^{l}_{++} \times \mathbb{R} \rightarrow \mathbb{R}_{++}$ pointwise by

$$G(p,w) \equiv \max_{x \in \mathbb{R}^{\ell}_{+}} \{u(x) \mid p'x \leq w\}$$

and

$$E(p,u) \equiv \min_{x \in \mathbb{R}^{\ell}_{+}} \{p'x \mid u(x) \geq u\}.$$

G is referred to as the *indirect utility function* and E as the *expenditure function.*

b) When, in addition, $\succsim$ is locally nonsatiated, prediction 9(a) holds and G is i) strictly increasing and continuous in w; ii) nonincreasing and quasi–convex in p; and iii) homogeneous of degree 0 in (p,w). Also, E is *i*) strictly increasing and continuous in u; and ii) nondecreasing, linear homogeneous and concave in p. Moreover the following relationships between G and E hold:

$$\forall (p,u) \in \mathbb{R}^{l}_{++} \times \mathbb{R},\ G[p, E(p,u)] = u; \tag{15.1}$$

and

$$\forall (p,w) \in \mathbb{R}^{l+1}_{++},\ E[p, G(p,w)] = w. \tag{15.2}$$

c) When $\succsim$ is also strictly convex the demand correspondence becomes the (Marshallian, or Ordinary) demand function $f(p,w)$; *9(d)* again. Also, define h: $\mathbb{R}^{l}_{++} \times \mathbb{R} \rightarrow \mathbb{R}^{l}_{+}$ pointwise by

$$h(p,u) = \text{argmin}\{p'x \mid u(x) \geq u\}.$$

h is called the *Hicksian (or Compensated) demand function.* f and h are related by

$$\forall(p,w) \in \mathbb{R}_{++}^{l+1}, f(p,w) = h[p, G(p,w)] \tag{15.3}$$

and

$$\forall(p,u) \in \mathbb{R}_{+}^{l} \times \mathbb{R}, h(p,u) = f[p, E(p,u)]. \tag{15.4}$$

d) If the further condition $u \in C^2(\mathbb{R}_{+}^{l})$ is included, it may be shown that $G \in C^2(\Omega)$ and $E \in C^2(\Omega')$ where Ω' is defined for given $u(\cdot)$ by: $(p,w) \in \Omega \Longrightarrow (p, G(p,w)) \in \Omega'$. Three further results are

i) Shepherd's Lemma: $\forall(p,u) \in \Omega'$, $h(p,u)' = \nabla_p E(p,u)$ and $\nabla_p h(p,u) = \nabla_p^2 E(p,u)$;

PROOF: i) For $u \in C^2(\mathbb{R}_{++}^{l})$ and $(p,w) \in \Omega$, $h(p,u)$ may be characterized as an interior solution of

$$\min_x \; p'x \quad \text{s.t. } u(x) = u,$$

yielding first order necessary conditions

$$p - \mu^* \nabla u(x^*) = 0 \tag{*}$$

and $\quad u(x^*) - u = 0;$ (**)

x^* is the minimal commodity bundle and $\mu^* \neq 0$ a Lagrange multiplier. Since $E(p,u) = p'h(p,u)$,

$$\nabla_p E(p,u) = h(p,u) + p'\nabla_p h(p,u).$$

Differentiating (**) with respect to p gives

$$\nabla u(x^*)'\nabla_p h(p,u) = \underset{\sim}{0},$$

implying

ii) $\quad \nabla_p h = \nabla_p^2 E$ since $h = \nabla_p E$ is $\forall(p,u) \in \Omega'$. ‖

$p'\nabla_p h(p,u) = \underset{\sim}{0}$.

ii) Roy's Identity: $f(p,w) = \nabla_p G'(\partial G/\partial w)^{-1}$;

and iii) $\quad \nabla_p h = S$.

PROOF: Differentiating (15.3) with respect to p gives

$$\nabla_p h(p,u) = \nabla_p f[p, E(p,u)] + \nabla_w f[p,E(p,u)]\nabla_p E(p,u)'$$

$$= \nabla_p f[p,E(p,u)] + \nabla_w f[p,E(p,u)]h(p,u)' \qquad \text{(Shepherd)}$$

$$= \nabla_p f[p,E(p,u)] + \nabla_w f[p,E(p,u)]f[p,E(p,w)]' \qquad \text{(from 15.3)}$$

$$= S \qquad \text{(Slutsky eqn.)} \; \|$$

e) *Theorem: Let* $g\colon \mathbb{R}^{l+1}_{++} \to \mathbb{R}$ *be homogeneous of degree 0, nonincreasing and quasiconvex in its first l arguments, strictly increasing in its last argument, and twice continuously differentiable. Then there exists a binary relation* $\succsim$ *satisfying* A1–A3 *that is locally nonsatiated and strongly convex, and such that* $u \in C^2(\mathbb{R}^l_+)$*, for which*

$$\forall (p,w) \in \mathbb{R}^{l+1}_{++},\ g(p,w) = \max_{x \in \mathbb{R}^l_+} \{u(x) \,|\, p'x \leq w\},$$

This result demonstrates that every function g satisfying the conditions given is an indirect utility for some $\succsim$ having the stated characteristics. There is an analogous result for expenditure functions.

f) Can $u(\cdot)$ be obtained from f?

If E is known, $u(\cdot)$ for which E is the expenditure function can be constructed as follows. For each $u \in \mathbb{R}$ let $I_u = \{x \in \mathbb{R}^l_+ \,|\, x = \nabla_p E(p,u) \text{ for some } p \in \mathbb{R}^l_{++}\}$. Let $I = \{(u,x) \,|\, u \in \mathbb{R}, x \in I_u\}$. I is the graph of $u(\cdot)$, equivalent to $u(\cdot)$ itself.

If $f \in C^1(\mathbb{R}^{l+1}_{++})$, E may be obtained as follows. Select some (p^0,w^0) and u^0, and require

$$E(p^0,u^0) = w^0. \qquad (15.6)$$

From Shepherd's lemma

$$\nabla_p E(p,u^0) = h(p,u^0). \qquad (15.7)$$

(15.7) is a system of l partial differential equations, for which (15.6) serves as an initial condition, given u^0. The Frobenius theorem implies that (15.6) and (15.7) have a solution $E(p,u^0)$ if and only if $\nabla_p h$ is symmetric. But from 15(d)(iii), $\nabla_p h = S$, which is symmetric.

Thus, given $h(\cdot)$, $E(p,u^0)$ may be constructed for each $u^0 \in \mathbb{R}$.

Next, from (15.4), (15.7) is equivalent to

$$\nabla_p E(p,u) = f[p, E(p,u^0)], \tag{15.8}$$

which therefore, in conjunction with (15.6), has a solution. Altogether, given f, u may be obtained by solving (15.6) and (15.8) to obtain E, and proceding as above.

16. Uncertainty

Throughout this Section, $\succsim$ will be assumed to satisfy A1–A3 and strong monotonicity.

a) General Setup

Random events may occur. Assume there are $M < \infty$ distinct events, called ***states of nature*** and indexed by $i \in \{1,...,M\}$. Let x^i be a vector of commodities consumed only if state i occurs, and define $x = \begin{bmatrix} x^1 \\ \vdots \\ x^M \end{bmatrix}$. Such commodities are called *contingent claims*. With p the associated price system, the analysis may proceed as above without modification, with $l = \sum_{i=1}^{M} l\,(x^i)$. ($l: \mathbb{R}^\infty \to$ N is the number of elements in any vector, where N is the natural numbers.)

b) Expected Utility

It is often convenient to restrict preferences so that u may be written

$$u(x) = \sum_{i=1}^{M} \pi^i v(x^i), \tag{16.1}$$

where π^i is the probability with which state i occurs; let $\pi = \begin{bmatrix} \pi^1 \\ \vdots \\ \pi^M \end{bmatrix}$ and $\Pi = \{z \in \mathbb{R}^M_+ | 2 \le M < \infty, z'\iota = 1\}$. Note that v is independent of i. (Thus all x^i must have the same number of components.) (16.1) is called the *Expected Utility Hypothesis* and v the ***subutility*** function. One set of restrictions on u that yields (16.1) is additive separability across states, implying

$$u(x) = \sum_{1}^{M} v^i(x^i),$$

coupled with the restriction $v^i(x^i) = \rho^i v(x^i)$ for all x^i and some parameter $\rho^i \in (0,1)$, yielding

$$u(x) = \sum_{1}^{M} \rho^i v(x^i),$$

and finally, the assumption $\rho^i = \pi^i$, all i.

Observe that for any continuous monotonic transformation f: $\mathbb{R} \to \mathbb{R}$, $f \circ u = \sum_{1}^{M} \pi^i (f \circ v)$ if and only if f is of the form $f(z) = \alpha + \beta z$; $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_{++}$.

(16.1) will be assumed in the rest of this Section.

c) Risk Aversion

Definition: Given (16.1), $\succsim$ displays *risk aversion* if $\forall \pi \in \Pi$ and $\forall x \in \mathbb{R}^l_+$,

$$\sum_{i=1}^{M} \pi^i v(x^i) \leq v(\sum_{i=1}^{M} \pi^i x^i).$$

Theorem: $\succsim$ *displays risk aversion* $\Leftrightarrow$ *v is concave.*

PROOF: (1) Suppose v is concave. Then

$$\sum_{i=1}^{M} \pi^i v(x^i) = \sum_{i=1}^{M-2} \pi^i v(x^i) + (\pi^{M-1} + \pi^M) \left[\frac{\pi^{M-1} v(x^{M-1})}{\pi^{M-1} + \pi^M} + \frac{\pi^M v(x^M)}{\pi^{M-1} + \pi^M} \right]$$

$$\leq \sum_{i=1}^{M-2} \pi^i v(x^i) + (\pi^{M-1} + \pi^M) v \left[\frac{\pi^{M-1} x^{M-1} + \pi^M x^M}{\pi^{M-1} + \pi^M} \right] \quad \text{(concavity of } v\text{)}$$

$$\leq \sum_{i=1}^{M-3} \pi^i v(x^i) + (\pi^{M-2} + \pi^{M-1} + \pi^M) \cdot \quad v \left[\frac{\pi^{M-2} x^{M-2} + \pi^{M-1} x^{M-1} + \pi^M x^M}{\pi^{M-2} + \pi^{M-1} + \pi^M} \right]$$

$$\vdots$$

$$\leq v \left[\sum_{i=1}^{M} \pi^i x^i \right].$$

(ii) Suppose v is not concave. Let $m = l/M$. Then for some $x^1 \in \mathbb{R}^m_+$ and $x^2 \in \mathbb{R}^m_+$, and some $\lambda \in (0,1)$,

$$\lambda v(x^1) + (1-\lambda)v(x^2) > v[\lambda x^1 + (1-\lambda)x^2]. \qquad (*)$$

Let $\pi = (\lambda, 1-\lambda)'$ and $x = (x^1, x^2)'$. Then (*) violates the definition of risk aversion for such π and x. ||

For the rest of this Section, $M = l$ ($l(x^i) = 1$, all i).

d) Increasing Risk

Order states so that $x^i \leq x^{i+1}$, $i \in \{1,...,M-1\}$. Given π, define $P:\mathbb{R} \to [0,1]$ pointwise by $P(\tilde{x}) = \sum_{i=1}^{M} \pi^i I_{[x^i,\infty)}(\tilde{x})$, where $I:\mathbb{R} \to \{0,1\}$ is the indicator function. $P \in \mathcal{P}$, where $\mathcal{P}$ is the set of all distribution functions having support $\{x^1,...,x^M\}$.

Let $\{i_1, i_2, i_3, i_4\} \subset \{1,...,M\}$ with $i_1 < i_2 \leq i_3 < i_4$ ($M \geq 3$ is required).

Definition: $P' \in \mathcal{P}$ is a *simple mean preserving spread* of $P \in \mathcal{P}$ if i) $\pi' = (\pi^1,...,\pi^{i_1} + \underline{\varepsilon}, \pi^{i_1+1},...,\pi^{i_2} - \underline{\varepsilon}, \pi^{i_2+1},...,\pi^{i_3} - \bar{\varepsilon}, \pi^{i_4} + \bar{\varepsilon},...,\pi^M)$, with $\underline{\varepsilon} > 0$ and $\bar{\varepsilon} > 0$, and ii) $\int \tilde{x}dP = \int \tilde{x}dP'$.

Definition: A distribution function P' is a *mean preserving spread* (m.p.s) of $P \in \mathcal{P}$ if i) $P' \in \mathcal{P}$ and ii) there exists a sequence $\{P^j\}_{j=0}^{j=\infty}$, $P^j \in \mathcal{P}$, with $P^0 = P$, P^{j+1} a simple mean preserving spread of P^j, and $P^j \to P'$. (Here convergence is defined by $P^j \to P \Leftrightarrow \pi_j \equiv (\pi^1_j,...,\pi^M_j) \to \pi$.)

Theorem (*Rothschild/Stiglitz, J.E.T. 1970*): *P′ is a m.p.s. of P if and only if*

i) $\forall \tilde{x} \in \mathbb{R}, \int_{(-\infty,\tilde{x})} [P'(v) - P(v)]dv \geq 0;$

and

ii) $\int_{\mathbb{R}} [P'(v) - P(v)]dv = 0.$

e) Risk Aversion and Increasing Risk

Let $\mathcal{V}$ be the set of all increasing concave functions v: $\mathbb{R}_+ \to \mathbb{R}$. Observe that given π, and (16.1), $\int_{[0,\infty]} v(\tilde{x})dP = u(x^1,...,x^M)$.

Theorem: *Let P′ be an m.p.s. of P. Then* $\forall v \in \mathcal{V} \int_{\mathbb{R}_+} v(\tilde{x})dP \geq \int_{\mathbb{R}_+} v(\tilde{x})dP'$.

e) Indices of Risk Aversion

Let $\mu = \int \tilde{x}dP$ and $\sigma^2 = \int (\tilde{x}-\mu)^2 dP$. Assuming $\succsim$ displays risk aversion, the equation

$$v(\mu - r) = \int_{[0,\infty)} v(\tilde{x})dP \tag{16.2}$$

may be solved for $r \in [0,\infty)$ (i.e. $v(0) < \int v(\tilde{x})dP \leq v(\mu)$ and v is continuous. Thus the Intermediate Value Theorem applies.)

Definition: r is called the ***Risk Premium***

Theorem: *If* $v \in C^2(\mathbb{R}_+)$, $r \equiv v'(\mu)/v''(\mu) + \{[v'(\mu)/v''(\mu)]^2 + \sigma^2]\}^{\frac{1}{2}}$.

PROOF: Write (16.1) as

$$0 = \int_{[0,\infty)} [v(\mu - r) - v(\tilde{x})]dP.$$

Expanding $v(\mu - r)$ and $v(\tilde{x})$ in a second order Taylor series around μ gives

$$\begin{aligned} 0 &\cong \int \{[v(\mu) - rv'(\mu) + \tfrac{r^2}{2}v''(\mu)] \\ &\quad -[v(\mu) + v'(\mu)(\tilde{x}-\mu) + \tfrac{1}{2}v''(\mu)(\tilde{x}-\mu)^2]\}dP \\ &= -rv'(\mu) + \tfrac{r^2}{2}v''(\mu) - \tfrac{1}{2}v''(\mu)\sigma^2. \end{aligned} \qquad (*)$$

(*) is a quadratic in r. Solving, and taking the positive root gives the desired expression. ||

Let $R_A \equiv -v''(\mu)/v'(\mu)$. Then

$$r \cong -R_A^{-1} + [R_A^{-2} + \sigma^2]^{\frac{1}{2}}. \qquad (16.3)$$

The derivative of the right hand side of (16.3) with respect to R_A is

$$R_A^{-2}[1 - (1 + R_A^2\sigma^2)^{-\frac{1}{2}}] > 0 \qquad (16.4)$$

and, with respect to σ^2,

$$\tfrac{1}{2}[R_A^{-2} + \sigma^2]^{-\frac{1}{2}} > 0.$$

(16.4) is the motivation for

Definition: $R_A(\tilde{x}) \equiv -v''(\tilde{x})/v'(\tilde{x})$ is the *Coefficient of Absolute Risk Aversion.*

Similarly,

Definition: $R_R(\tilde{x}) \equiv -v''(\tilde{x})\tilde{x}/v'(\tilde{x})$ is the *Coefficient of Relative Risk Aversion.*

NOTES ON PRODUCER THEORY

1. Producers

A *production plan* is a vector $y \in \mathbb{R}^l$; $y_h \geq 0$ is interpreted as an *output* and $y_h < 0$ as an *input*. The value of a production plan y given a price system p is "profit" $p'y$. The producer is assumed to select a technologically feasible production plan that is profit maximizing.

2. Technology

Let $Y \subset \mathbb{R}^l$ be the *production set*. Y will be interpreted as including all technologically feasible production plans. Assume

B1: Y is closed;

B2: (Free disposal) $\mathbb{R}^l_- \subset Y$;

B3: (Irreversibility) If $y \in Y$ and $y \neq \underset{\sim}{0}$, $-y \notin Y$;

B4: $\forall p \in \mathbb{R}^l_{++}$, $\{y \in Y | p'y \geq 0\}$ is compact.

Observe that B2 and B3 imply $\mathbb{R}^l_+ \cap Y = \underset{\sim}{0}$, (the "No land of Cockaigne" assumption) and that B2 implies $\underset{\sim}{0} \in Y$ (the "possibility of inaction").

3. Representation of *Y* by a Production Function

Since Y is closed its boundary may be represented by a continuous implicit function: $F(y) = 0 \Leftrightarrow y \in Bndy\ Y$. Moreover, F may be required to satisfy $F(y) \leq 0 \Leftrightarrow y \in Y$.

This may be proved by observing that Y may be interpreted as a "weakly worse" set of a complete, transitive and continuous binary relation $\precsim$ defined on $\mathbb{R}^l \times \mathbb{R}^l$; "$y \precsim y'$" is read "$y$ is less productive than y'". Debreu's Theorem gives continuous F such that $y \precsim y' \Leftrightarrow F(y) \leq F(y')$. A monotonic transformation allows $F(\underset{\sim}{0}) = 0$, and $y \in Y \Leftrightarrow F(y) \leq 0$.

4. <u>Net Output Correspondence</u>

Given the producer's objective, and technological feasibility as described by Y, the *net output correspondence* ξ: $\mathbb{R}^l_{++} \to \mathbb{R}^l$ is defined pointwise by

$$\forall\, p \in \mathbb{R}^l_{++},\ \xi(p) \equiv \{y \in Y \mid p'y \geq p'\tilde{y},\ \tilde{y} \in Y\}.$$

Given $p \in \mathbb{R}^l_{++}$, let $\hat{Y}_p = \{y \in Y \mid p'y \geq 0\}$; $\hat{Y}_p$ is nonempty ($\underline{0} \in \hat{Y}_p$) and compact (B4). Thus $\hat{\xi}$: $\mathbb{R}^l_{++} \to \mathbb{R}^l$, defined pointwise by

$$\forall\, p \in \mathbb{R}^l_{++},\ \hat{\xi}(p) \equiv \{y \in \hat{Y}_p \mid p'y \geq p'\tilde{y},\ \tilde{y} \in \hat{Y}_p\},$$

exists, since the function $p'y$ is continuous on $\hat{Y}_p$.

Now, since $Y - \hat{Y}_p = \{y \in Y \mid p'y < 0\}$, $y \in \hat{\xi}(p)$ implies $p'y \geq p'\tilde{y}$, $\tilde{y} \in Y$; thus $\hat{\xi}(p) \subset \xi(p)$. Also, if ξ exists, $y \in \xi(p)$ implies $p'y \geq p'\tilde{y}$, $\tilde{y} \in Y_p$; thus $\xi(p) \subset \hat{\xi}(p)$. Therefore $\xi = \hat{\xi}$ and since $\hat{\xi}$ exists, so does ξ.

5. <u>Properties of ξ</u>

a) $\forall\, \lambda \in \mathbb{R}_{++}$, $\forall\, p \in \mathbb{R}^l_{++}$, $\xi(\lambda p) = \xi(p)$;

b) If for some $p \in \mathbb{R}^l_{++}$, $y^0 \in \xi(p)$ and $y^1 \in \xi(p)$, then $p'y^0 = p'y^1$;

c) $\xi(p)$ is upper–hemi continuous;

d) Consider $n \geq 2$ production plans $y^1,\ldots,y^n$ and price vectors $p^1,\ldots,p^n$ arrayed in a vector $(y^i, p^i)_{i=1}^{i=n}$.

Definition: $(y^i, p^i)_{i=1}^{i=n}$ satisfies the *Weak Axiom of Profit Maximization* (WAPM) if $\forall$ $i=1,\ldots,n$ and $j=1,\ldots,n$, $p^{i\prime} y^i \geq p^{i\prime} y^j$.

Definition: A production set Y *p–rationalizes* $(y^i, p^i)_{i=1}^{i=n}$ if $\forall$ $i=1,\ldots,n$, i) $y^i \in Y$; and ii) $p^{i\prime} y^i \geq p^{i\prime} y$, all $y \in Y$.

Theorem: There exists a production set p–rationalizing $(y^i, p^i)_{i=1}^{i=n}$ if and only if $(y^i, p^i)_{i=1}^{i=n}$ satisfies WAPM.

Theorem: Let $p^0 \in \mathbb{R}^l_{++}$, $p^1 \in \mathbb{R}^l_{++}$ $y^0 \in \xi(p^0)$ and $y^1 \in \xi(p^1)$. Then $(p^1 - p^0)'(y^1 - y^0) \geq 0$.

Proof: $y^0 \in \xi(p^0) \Rightarrow p^{0\prime} y^0 \geq p^{0\prime} y^1$. $y^1 \in \xi(p^1) \Rightarrow p^{1\prime} y^1 \geq p^{1\prime} y^0$. Adding the inequalities gives the desired result. ||

Corollary: If $p^0 \neq p^1$ and $p^0_h = p^1_h$ $h \neq k$, then $(p^1_k - p^0_k)(y^1_k - y^0_k) \geq 0$.

6. <u>Other Restrictions on Y and/or F</u>

Let $(y^1 y^2)$ be a partition of y into subvectors y^1 and y^2.

a) *Convexity*: Y is convex

b) *Strict Convexity*: Y is strictly convex if $y^0 \in Y$ and $y^1 \in Y$ imply $\bar{y} \in \text{Int } Y$, for $\bar{y} = \lambda y^0 + (1-\lambda)y^1$, $\lambda \in (0,1)$.

c) *Monotonicity*: Y is monotonic if $y^0 \in Y$ and $y^1 \in \mathbb{R}^{\ell}_{-}$ imply $y^0 + y^1 \in Y$.

d) *Differentiability of F*: $F \in C^2(\mathbb{R}^l)$.

e) *Linear Homogeneity*: $\forall \lambda \in \mathbb{R}_+$, $y \in Y \rightarrow \lambda y \in Y$.

f) *Separability of F*: F is separable if it may be written $F(y) = f[v^1(y^1), v^2(y^2)]$, where $F: \mathbb{R}^2 \rightarrow \mathbb{R}$, $v^1: \mathbb{R}^{l^1} \rightarrow \mathbb{R}$, $v^2: \mathbb{R}^{l^2} \rightarrow \mathbb{R}$ and $l^1 + l^2 = l$.

7. <u>More Properties of ξ</u>

(a) If Y is convex, then $\forall p \in \mathbb{R}^l_{++}$, $\xi(p)$ is convex.

PROOF: Let $y^0 \in \xi(p)$, $y^1 \in \xi(p)$. Then $\forall \lambda \in [0,1]$, $\bar{y} \in Y$ for $\bar{y} \equiv \lambda y^0 + (1-\lambda)y^1$. Also, $p'\bar{y} = p'y^0 \geq p'y$, $\forall y \in Y$. Thus $\bar{y} \in \xi(p)$. ||

(b) If Y is strictly convex, ξ is single valued and continuous.

PROOF: Let $y^0 \in \xi(p)$, $y^1 \in \xi(p)$ and $y^0 \neq y^1$. Then $\bar{y} \equiv \frac{1}{2}(y^0 + y^1)$ satisfies $\bar{y} \in \text{Int}\, Y$ by strict convexity, and $p'\bar{y} = p'y^0$. Thus $\exists\, y' \in Y$ with $p'y' > p'y^0$; i.e. $y^0 \notin \xi(p)$. It follows that $y^0 = y^1$ must hold.

Since ξ is single–valued and upper hemi–continuous, it is continuous. ||

When ξ is single–valued, it will be referred to as the net output function z.

8. Some Properties of $F \in C^2(\mathbb{R}^l)$.

Let Y be strictly convex and monotonic, and $F \in C^2(\mathbb{R}^l)$. Denote $\partial F/\partial y_h$ by F_h and $\partial^2 F/\partial y_h \partial y_k$ by F_{hk}.

a) $\nabla^2 F = (\nabla^2 F)'$

b) For $y^0 \in \mathbb{R}^l$, if $\nabla F(y^0) \neq \underset{\sim}{0}$, define $O(y^0) = \{y \in \mathbb{R}^l \,|\, \nabla F(y^0)'y = 0\}$. Then $\forall y \in O(y^0)$, $y'\nabla F(y^0)y \geq 0$ with strict inequality almost everywhere. Moreover, the matrix $\begin{bmatrix} \nabla^2 F(y^0) & \nabla F(y^0) \\ \nabla F(y^0)' & 0 \end{bmatrix}$ is nonsingular almost everywhere in $\mathbb{R}^l$.

c) For y such that $F_h \neq 0$, $F(y) = 0$ may be solved for $y_h = g^h(y_{-h})$, where $y_{-h} = (y_1,\ldots,y_{h-1}, y_{h+1},\ldots,y_l)$. g^h is strictly concave and twice continuously differentiable on the interior of its domain. For $h \neq k$, $\partial q^h/\partial y_k = -F_k/F_h < 0$ and $\partial^2 q^h/\partial y_k^2 < 0$.

9. Hicks–Samuelson Approach

Let Y be strictly convex, monotonic and such that $F \in C^2(\mathbb{R}^l)$. Then $\forall p \in \mathbb{R}^l_{++}$, *the matrix* ∇z *is symmetric, positive semidefinite and singular* ($\nabla zp = 0$, *in particular*).

Under the given conditions, z may be characterized as the interior solution to

$$\max_y p'y \text{ subject to } F(y) = 0.$$

The Lagrange function for this problem is

$$L(y,\eta;p) = p'y - \eta F$$

where $\eta \geq 0$ is an undetermined multiplier. Assuming y^* is the maximal net output vector and η^* the associated value of η implies

$$\nabla_y L(y^*, \eta^*; p) = 0, \qquad (p - \eta^* \nabla F(y^*) = 0) \tag{9.1}$$

$$\nabla_\eta L(y^*, \eta^*; p) = 0, \qquad (F(y^*) = 0) \tag{9.2}$$

and

$$\forall y \text{ such that } y'\nabla F(y^*) = 0,\ y'\nabla_y L(y^*,\eta^*;p)y \leq 0. \tag{9.3}$$

Observe that using (9.1), $\eta^* > 0$ and $F_h(y^*) > 0$ for all h. Also, the assumed quasi–convexity of F implies (9.3) is satisfied.

The proposition given above may be demonstrated as follows. First, an expression for ∇z is needed. Writing (9.1) and (9.2) as

$$p - \eta^*\nabla F(y^*) = 0 \tag{9.4}$$

and $\quad F(y^*) = 0, \qquad (9.5)$

note that the Jacobian of this system is nonsingular almost everywhere, by 8(b). Moreover, derivatives of the left–hand–side with respect to y, η and p are continuous. Thus the Implicit Function Theorem yields existence of functions z: $\mathbb{R}^l_{++} \to \mathbb{R}^l$ and ρ: $\mathbb{R}^l_{++} \to \mathbb{R}_+$ with $y^* = z(p)$, $\eta^* = \rho(p)$, $z \in \mathscr{C}^2(\mathbb{R}^l_{++})$ and $\rho \in \mathscr{C}^2(\mathbb{R}^l_{++})$. Therefore differentials dy^* and $d\eta^*$ exist such that

$$dy^* = \nabla z(p)dp \tag{9.6}$$

and $\quad d\eta^* = \nabla\rho(p)dp. \qquad (9.7)$

Also, since the left–hand–side of (9.4) – (9.5) is continuously differentiable,

$$\begin{bmatrix} \eta^*\nabla^2 F(y^*) & \nabla F(y^*) \\ \nabla F(y^*)' & 0 \end{bmatrix} \begin{bmatrix} dy^* \\ d\eta^* \end{bmatrix} = \begin{bmatrix} dp \\ 0 \end{bmatrix} \tag{9.8}$$

Define the matrix $\begin{bmatrix} A & a \\ a' & \alpha \end{bmatrix}$ to be the inverse of the matrix on the left–hand–side of (9.8). Then, in particular $\eta^* A\nabla^2 F(y^*) + a\nabla F(y^*)' = I_l$ and $A\nabla F(y^*) = \underline{0}$. Solving (9.8) for dy^* and $d\eta^*$ gives

$$\begin{bmatrix} dy^* \\ d\eta^* \end{bmatrix} = \begin{bmatrix} A & a \\ a' & \alpha \end{bmatrix} \begin{bmatrix} dp \\ 0 \end{bmatrix},$$

in which case $dy^* = Adp$. Thus, from (9.6),

$$\nabla z = A. \tag{9.9}$$

a) $\nabla z = (\nabla z)'$

PROOF: A is a principal submatrix of the inverse of a symmetric matrix. ||

b) $\nabla zp = \underset{\sim}{0}$

PROOF: $A\nabla F(y^*) = \underset{\sim}{0}$ implies $\nabla z\nabla F(y^*) = \underset{\sim}{0}$, and (9.4) gives $\nabla F(y^*) = (1/\eta^*)p$, $\eta^* > 0$. ||

c) $\forall q \in \mathbb{R}^l$, $q'\nabla z(p^*)q \geq 0$.

PROOF: $\eta^* A\nabla^2 F(y^*) + a\nabla F(y^*)' = I_l$

$$\Rightarrow \eta^* A\nabla^2 F(y^*)A + a\nabla F(y^*)'A = A$$

$$\Rightarrow A'\nabla^2 F(y^*)A = A, \tag{9.10}$$

since A is symmetric and $\nabla F(y^*)'A = \underset{\sim}{0}$.

Let $q \in \mathbb{R}^l$. Then

$$q'\nabla zq = q'Aq = \eta^* q'A'\nabla^2 F(y^*)Aq, \tag{9.11}$$

from (9.10). Let $r \equiv Aq$. Then,

$$\nabla F(y^*)'r = \nabla F(y^*)'Aq = 0,$$

in which case

$$r'\nabla^2 F(y^*)r \geq 0, \tag{9.12}$$

by quasiconvexity of F. But

$$r'\nabla^2 F(y^*)r = q'A'\nabla^2 F(y^*)Aq = (1/\eta^*)q'\nabla z(p)q,$$

by (9.11), so (9.12) gives the result. ||

9. The Le Chatelier Principle

Let $v\colon \mathbb{R}^l \to \mathbb{R}^n$ $(1 \le n < \infty)$, and denote the i^{th} coordinate of $v(y)$ by $v^i(y)$, $i \in \{1,\dots,n\}$. For any set $I \subset \{1,\dots,n\}$, let $V^I = \{y \in \mathbb{R}^l \mid v^i(y) \le 0,\ i \in I\}$. Assume $\mathrm{Int}(V^I \cap Y) \ne \emptyset$ for $I = \{1,\dots,n\}$, and V^I strictly convex for all I.

Now, for each I define: $z^I\colon \mathbb{R}^l_+ \to \mathbb{R}^l$ pointwise by

$$z^I(p) = \underset{y}{\operatorname{argmax}}\{p'y \mid y \in V^I \cap Y\}.$$

Theorem: Assume Y is strictly convex, monotonic and such that $F \in C^2(\mathbb{R}^l)$. Then $\forall I \subset \{1,\dots,n\}$, if $I' \subset I$ and $I' \ne I$, and if for some $\bar{p}$, $\bar{p}'z^I(\bar{p}) = \bar{p}'z^{I'}(\bar{p})$, the matrix $\nabla z^{I'}(\bar{p}) - \nabla z^I(\bar{p})$ is positive semi-definite when the derivatives exist and are continuous.

PROOF: Given p, let $\pi^I(p) = \max\limits_{y}\{p'y \mid y \in V^I \cap Y\}$.

Then, assuming ∇z^I exists and is continuous, $\nabla\pi^I(p) = z^I(p)$ and $\nabla^2\pi^I(p) = \nabla z^I(p)$. Let $\Gamma(p) = \pi^{I'}(p) - \pi^I(p)$. Then $\Gamma(p) \ge 0$ and $\Gamma(\bar{p}) = 0$, and therefore $\nabla\Gamma(\bar{p}) = 0$ and $\nabla^2\Gamma(\bar{p})$ is positive semidefinite. But $\nabla^2\Gamma(\bar{p}) = \nabla z^{I'}(\bar{p}) - \nabla z^I(\bar{p})$. ||

10. Duality

a) Assuming B1–B4, the *profit function* $\pi\colon \mathbb{R}^l_{++} \to \mathbb{R}_+$ may be defined pointwise by

$$\forall p \in \mathbb{R}^l_{++},\ \pi(p) \equiv \max_{y}\{p'y \mid y \in Y\}.$$

$\pi(p)$ is a linear homogeneous and convex function. That is, $\forall \gamma \in \mathbb{R}_{++}$

$$\begin{aligned}\pi(\gamma p) &= \max_{y}\{(\gamma p)'y \mid y \in Y\}\\ &= \gamma \max_{y}\{p'y \mid y \in Y\}\\ &= \gamma\pi(p).\end{aligned}$$

Also, for price vectors p^0 and p^1 let $\bar{p} = \gamma p^0 + (1-\gamma)p^1$. Then

$$\begin{aligned}\pi(\bar{p}) &= \max_y \{[\gamma p^0 + (1-\gamma)p^1]'y \mid y \in Y\} \\ &\leq \max_y \{\gamma p^{0'} y \mid y \in Y\} + \max_y \{(1-\gamma)p^{1'} y \mid y \in Y\} \\ &= \gamma\pi(p^0) + (1-\gamma)\pi(p^1),\end{aligned}$$

the latter by linear homogeneity.

b) Imposing the additional condition that Y is strictly convex and monotonic it follows that $\forall p \in \mathbb{R}^l_{++}$, $\pi(p) = p'z(p)$. Moreover, when $F \in C^2(\mathbb{R}^l)$

Theorem (*Hotelling*): $\nabla\pi = z$.

PROOF: For any p, $\pi(p) = p'z(p)$. Thus $\nabla\pi(p) = z(p) + p'\nabla z(p) = z(p)$, since $p'\nabla z(p) = \underset{\sim}{0}$. ||

c) Restricted Profit (or "Cost") Functions. Let $y = (y^1, y^2)$ and $p = (p^1, p^2)$; $y^1 \in \mathbb{R}^{l^1}$, $y^2 \in \mathbb{R}^{l^2}$, $l^1 + l^2 = l$. Fix y^2 at $\bar{y}^2$ and define π^R: $\mathbb{R}^l_{++} \times R^{l^2} \to \mathbb{R}$ pointwise by

$$\forall (p, \bar{y}^2) \in \mathbb{R}^l_{++} \times \mathbb{R}^{l^2},\ \pi^R(p, \bar{y}^2) = \max_{y^1}\{p'y \mid (y^1, \bar{y}^2) \in Y\}.$$

Like π, π^R is linear homogeneous and convex in p^1 and satisfies Hotelling's result with the obvious modifications. Moreover, π^R is strictly concave in $\bar{y}^2$.

In the particular case where given $\bar{y}^2 \geq 0$, and the function z^1: $\mathbb{R}^l_{++} \times \mathbb{R}^{l^2} \to \mathbb{R}^{l^1}$ defined pointwise by $z^1(p^1, \bar{y}^2) \equiv \underset{y^1}{\operatorname{argmax}}\{p^{1'} y \mid (y^1, \bar{y}^2) \in Y\}$ satisfies $z^1_h(p^1, \bar{y}^2) \leq 0$, all h, define the *cost function* c: $\mathbb{R}^{l^1}_{++} \times \mathbb{R}^{l^2} \to \mathbb{R}_+$ pointwise by $c(p^1, \bar{y}^2) \equiv -p^{1'} z^1(p^1, \bar{y}^2)$. Then c is nonnegative, linear homogeneous and concave in p^1, and convex in $\bar{y}^2$.

NOTES ON COMPETITIVE EQUILIBRIUM

1. Market Demand

If each consumer's preference preordering is locally nonsatiated, strictly convex and representable by $u \in C^2(\mathbb{R}^l_+)$, commodity demand may be summarized by the ordinary demand function $f \in C^1(\Omega)$ with the symmetric, negative semi–definite matrix $S = \Delta_p f + \Delta_w f f'$ satisfying $Sp = 0$.

Assume there are n consumers, with $1 \leq n < \infty$, and let $i \in \{1,\ldots,n\}$ index them along with their demand functions and incomes; that is, $f^i(p,w^i)$ is consumer i's commodity demand. Let $w \equiv (w^1,\ldots,w^n)' \in \mathbb{R}^n_{++}$.

Market Demand D: $\mathbb{R}^l_{++} \times \mathbb{R}^n_{++} \to \mathbb{R}^l_+$ is defined pointwise by

$$\forall (p,w) \in \mathbb{R}^l_{++} \times \mathbb{R}^n_{++},\ D(p,w) = \sum_{i=1}^{n} f^i(p,w^i). \tag{1.1}$$

It is immediate that $D \in C^1(\Omega)$, for Ω defined by $(p,w) \in \Omega \Leftrightarrow \forall i,\ (p,w^i) \in \Omega^i$. For such (p,w),

$$\nabla_p D(p,w) = \sum_{i=1}^{n} \nabla_p f^i(p,w^i) = \sum_{i=1}^{n} [S^i - \nabla_w f^i(p,w^i) f^i(p,w^i)'],$$

implying

$$\nabla_p D(p,w) + \sum_{i=1}^{n} \nabla_w f^i(p,w^i) f^i(p,w^i)' = \sum_{i=1}^{n} S^i. \tag{1.2}$$

Since $S \equiv \sum_{i=1}^{n} S^i$ is symmetric, negative semi–definite and yields $Sp = \underset{\sim}{0}$, the matrix on the left–hand side of (1.2) shares these properties.

Theorem: $\forall (p,w) \in \Omega$, *the matrix* $S \equiv \nabla_p D(p,w) + \sum_{i=1}^{n} \nabla_w f^i(p,w^i) f^i(p,w^i)'$ *is symmetric and negative semi–definite with* $Sp = \underset{\sim}{0}$.

2. Representative Consumer and Aggregation

Testing of the Theorem in Section 1 would require information on individual income and consumption. Under what conditions would aggregate information suffice? One such condition is the *Representative Consumer* restriction:

$$\forall\, i \in \{1,\dots,n\},\ \forall i' \in \{1,\dots,n\},\ w^i = w^{i'} \text{ and } \succsim^i = \succsim^{i'}.$$

Under this condition $f^i = f^{i'}$ and the left–hand side of (1.2) becomes

$$n[\nabla_p f^1(p,\bar{w}) + \nabla_w f^1(p,\bar{w})\, f^1(p,\bar{w}')],$$

for $\bar{w} \equiv (1/n)\,(\sum_{i=1}^{n} w^i)$. Testing would require data on $n, p, \bar{w}$ and aggregate consumption of each commodity. Observe that $\succsim^i$ is not restricted.

Another approach permits consumers to differ in terms of $\succsim^i$ and w^i, but restricts $\succsim^i$.

Definition: If $\forall (p,w^i) \in \Omega^i$ the indirect utility $G^i(p,w^i)$ can be written $G^i(p,w^i) = \alpha^i(p) + \beta(p)w^i$, G^i has the *Gorman Polar Form*; α^i: $\mathbf{R}^l_{++} \to \mathbf{R}$ and β: $\mathbf{R}^l_{++} \to \mathbf{R}_{++}$.

Observe that the requirement that G^i be an indirect utility restricts α^i and β. It can be shown that α^i and β satisfying the conditions do exist.

When $\succsim^i$ is such that G^i is of the Gorman Polar Form, $(p,w^i) \in \Omega^i$ implies

$$f^i(p,w^i) = [\nabla_w G^i(p,w^i)]^{-1} \nabla_p G^i(p,w^i) \qquad \text{(Roy)}$$

$$= [\nabla\, \alpha^i(p) + \nabla\beta(p) w^i] \div \nabla\beta(p)$$

and

$$\nabla_w f^i(p,w)^i = \nabla\beta(p)/\beta(p);$$

Note that $\nabla_w f^i(p,w^i)$ is independent of i and w^i, and that if all G^i are of the Gorman Polar Form, $(p,w) \in \Omega$ implies

$$D(p,w) = [\sum_{i=1}^{n} \nabla\alpha^i(p) + \nabla\beta(p) \sum_{i=1}^{n} w^i] + \beta(p),$$

which depends only on the market level information n,p and $\bar{w}$. Indeed, D is the demand behavior of a single consumer having income $\tilde{w} \equiv n\bar{w}$ and indirect utility $G(p,\tilde{w}) = \sum_{i=1}^{n} \alpha^i(p) + \beta(p)\tilde{w}$.

Referring back to (1.2), for $(p,w) \in \Omega$

$$S = \nabla_p D(p,\tilde{w}) + \beta(p)^{-1}\nabla\beta(p)\, D(p,\tilde{w})',$$

in which case the restrictions on S may also be tested with information on aggregate consumption, $p,\tilde{w}$ and n. The key is that $\nabla_w f^i(p,w^i)$ does not depend on i or w^i. The Gorman Polar Form is sufficient for this feature; it is also necessary.

Theorem (Gorman): $G^i(p,w^i)$ is of the Gorman Polar Form $\Leftrightarrow$ $\forall(p,w^i) \in \Omega^i, \forall(p,w'^i) \in \Omega^i, \nabla_w f^i(p,w^i) = \nabla_w f^i\,(p,w'^i)$.

3. <u>Market Net Output</u>

If each producer's production set is strictly convex and monotonic with boundary representable by $F \in C^2(\mathbb{R}^l)$, net output behavior for any producer may be summarized by $z \in C^1(\mathbb{R}^l_{++})$ and the symmetric positive semidefinite matrix ∇z satisfies $\nabla zp = \underset{\sim}{0}$.

Assume there are m producers, $1 \leq m < \infty$, and index them by j; $j \in \{1,\ldots,m\}$. Define market net output Z: $\mathbb{R}^l_{++} \to \mathbb{R}^l$ pointwise by

$$\forall p \in \mathbb{R}^l_{++}, Z(p) = \sum_{j=1}^{m} z^j(p). \tag{3.1}$$

It is immediate that ∇Z is symmetric and positive semidefinite with $\nabla Zp = \underset{\sim}{0}$, and that only the market–level data m, p and aggregate net output are needed to test these restrictions.

4. <u>Competitive Equilibrium</u>

To consider equilibrium it is necessary to specify commodity endowments and shares of producer profit. For consumer i, let $\bar{x}^i \in \mathbb{R}^l_+$ be an endowed commodity bundle and

$s^i \equiv (s_1^i,...,s_m^i)' \in [0,1]^m$ be a vector of profit shares; $\forall j, \sum_{i=1}^{m} s_j^i = 1$. Letting $\pi = (\pi^1,...,\pi^m)'$ be the vector of producer profits, w^i is replaced by $p'\bar{x}^i + s^{i\prime}\pi$. Demand behavior can be analysed as before, with the budget correspondence being $\gamma^i: \mathbb{R}^l_{++} \times \mathbb{R}^l_+ \times [0,1]^m \times \mathbb{R}^m_+ \to \mathbb{R}^l_+$ defined pointwise by $\forall(p,\bar{x}^i,s^i,\pi) \in \mathbb{R}^l_{++} \times \mathbb{R}^l_+ \times [0,1]^m \times \mathbb{R}^m_+$,

$$\gamma^i(p;\bar{x}^i,s^i,\pi) \in \{x \in \mathbb{R}^l_+ \,|\, p'x \le p'\bar{x} + s^{i\prime}\pi\}.$$

The demand correspondence is $\bar{\varphi}_i: \mathbb{R}^l_{++} \times \mathbb{R}^l_+ \times [0,1]^m \times \mathbb{R}^m_+ \to \mathbb{R}^l_+$, defined pointwise by

$$\forall(p,\bar{x}^i,s^i,\pi) \in \mathbb{R}^l_{++} \times \mathbb{R}^l_+ \times [0,1]^m \times \mathbb{R}^m_+,$$

$$\bar{\varphi}(p;\bar{x}^i,s^i,\pi) \in \{x \in \gamma^i(p;\bar{x}^i,s^i,\pi) \,|\, x \succeq y, \text{ all } y \in \gamma^i(p;\bar{x}^i,s^i,\pi)\}.$$

Let $\bar{f}^i$ be the associated demand function.

Market demand is then $\bar{D}: \mathbb{R}^l_{++} \times \mathbb{R}^{nl}_+ \times [0,1]^{nm} \times \mathbb{R}^m_+ \to \mathbb{R}^l_+$ defined pointwise by

$$\forall(p,\bar{x},s,\pi) \in \mathbb{R}^l_{++} \times \mathbb{R}^{nl} \times [0,1]^{nm} \times \mathbb{R}^m_+,$$

$$\bar{D}(p;\bar{x},s,\pi) = \sum_{i=1}^{n} [\bar{f}^i(p;\bar{x}^i,s^i,\pi) - \bar{x}^i],$$

where $\bar{x} = (\bar{x}^{1\prime},...,\bar{x}^{n\prime})'$ and $s = (s_1^1,...,s_m^n)$. Where needed, $\bar{D}$ will be written $(\bar{D}_1,...\bar{D}_l)$.

Definition: A *competitive equilibrium* is a price system p^* such that

$$\bar{D}[p;\bar{x},s,p^{*\prime}Z(p^*)] = Z(p^*).$$

The function $E \equiv \bar{D} - Z$ is called *excess demand*. Observe that $\forall p \in \mathbb{R}^l_{++}, p'E = p'(\sum f^i - \bar{x}^i) - p'Z = 0$. $p'E \equiv 0$ is sometimes called *Walras' law*.

4. Sonnenschein–Mantel–Debreu Theorem

Consider a pure exchange economy (either there are no producers or $Y^j = \mathbb{R}^l_-$, all j). Such a economy is completely described by $\bar{x}$ and $\succeq^i$, $i \in \{1,...,n\}$.

Equilibrium in an exchange economy requires $\bar{D} = \underset{\sim}{0}$. By Walras' law, if $\bar{D}_h = 0$ for $h = 1,...,l-1$, $\bar{D}_l = 0$ must hold. It is thus sufficient to consider whether p is such that $\bar{D}_l = 0$ for $h = 1,...,l-1$. Call the $l-1$ vector of excess demands $\bar{D}(p;\bar{x})$.

Note that if p is an equilibrium price system, then so is λp, $\forall\lambda \in (0,\infty)$. Thus p may be "normalized" by requiring $p'\iota = 1$; i.e. set $\lambda = (p'\iota)^{-1}$. Let $P = \{p \in \mathbb{R}^l_+ | p'\iota = 1\}$. Then, for fixed $\bar{x}$, $\tilde{D}$: $P \to \mathbb{R}^{l-1}_+$.

Theorem: *Let* Φ: $P \to \mathbb{R}^{l-1}_+$ *be continuous and* $\tilde{P} \subset P$ *be compact. Then* $\exists$ *an economy (endowments* $\bar{x}^i$ *and preferences* $\succsim^i$*) with* $n \geq l$ *such that* $\forall p \in \tilde{P}$, $\tilde{D} = \Phi$.

NOTES ON GAME THEORY

1. Definition of a Game in Extensive Form

A *game tree* Γ is a finite set V (with typical element v) and a reflexive partial order $\prec$ on $V \times V$ such that i) $\forall v \in V$, $\{\tilde{v} \in V | \tilde{v} \prec v\}$ is completely ordered by $\prec$; and ii) $\exists v_0 \in V$ satisfying $\forall v \in V$, $v_0 \prec v$.

Elements of V are referred to as *vertices*. v_0 is the *distinguished* vertex; v is a *terminal vertex* if $\{\tilde{v} \in V | \tilde{v} \neq v, v \prec \tilde{v}\} = \phi$; v' is a *follower* of v if $v' \neq v$ and $v \prec v'$; v' is an *immediate* follower of v if $v' \neq v$, $v \prec v'$ and $\{\tilde{v} \in V | \tilde{v} \neq v, \tilde{v} \neq v', v \prec \tilde{v} \prec v'\} = \phi$.

Let T be the set of terminal vertices. The set $\{\tilde{v} \in V | \tilde{v} \prec v, v \in T\}$ is called a *play* of the game. Also, let $S \equiv V \setminus T$ be the set of nonterminal vertices, $F: V \to 2^V$ be a correspondence such that $\forall v \in V$, $F(v) = \{\tilde{v} \in V | \tilde{v}$ is an immediate follower of $v\}$; and $\#: 2^V \to \{0,1,...\}$ be the function $\#(V')$ = number of vertices in V', $V' \subset V$.

Let $n \in \{1,2...\}$ and $i \in \{1,...,n\} \equiv N$; N is the *player set*. An *n–player game in extensive form*, G, is:

i) a game tree Γ;

ii) a function $P: T \to \mathbb{R}^n$

iii) a partition $\{S_0, S_1, ..., S_n\}$ of S;

iv) $\forall v \in S_0$, a probability distribution on $F(v)$ such that all elements of $F(v)$ occur with positive probability;

v) $\forall i \in N$, a partition $\{S_i^1, ..., S_i^{m_i}\} \equiv \{S_i^j\}_{j=1}^{m_i}$ of S_i, where m_i is the number of sets in the partition, satisfying a) $\forall j \in \{1,...,m_i\}$, $\forall v \in V, \{\tilde{v} \in V | \tilde{v} \prec v\} \cap S_i^j \subset \{v'\}$, $v' \in V$; and b) $\forall v \in S_i^j$, $\forall v' \in S_i^j$, $\#[F(v)] = \#[F(v')] \equiv f_i^j$;

vi) $\forall i \in N$, $\forall j \in \{1,...,m_i\}$, a partition $\left\{C_{i1}^j, ..., C_{if_i^j}^j\right\}$ of $\cup_{v \in S_i^j} F(v)$ such that $\forall v \in S_i^j$, $\forall c \in \left\{1,...,f_i^j\right\}$, $\#\left[F(v) \cap C_{ic}^j\right] = 1$.

In all that follows "game" will mean "n–player game in extensive form".

P is called the *payoff function*; $\{S_k\}_0^n$ is the *player partition*; S_i^j are the *information sets*, and each $\left\{C_{i1}^j,\ldots,C_{if_i^j}^j\right\}$ is a *choice partition.*

Player i is said to have *perfect information* if $\forall j \in \{1,\ldots,m_i\}$, $\#(S_i^j) = 1$. G is a *game of perfect information* if $\forall i \in N$, player i has perfect information.

2. Strategies

Let $C_i = \left\{C_{ic}^j\right\}_{\substack{j \in \{1,\ldots,m_i\} \\ c \in \{1,\ldots,f_i^j\}}}$. A *strategy for player i* is a function σ_i: $\left\{S_i^1,\ldots,S_i^{m_i}\right\} \to C_i$ such that $\forall j \in \{1,\ldots,m_i\}$, $\sigma_i(S_i^j) \in \left\{C_{i1}^j,\ldots,C_{if_i^j}^j\right\}$. Let the set of such functions be Σ_i; $\sigma_i \in \Sigma_i$. Define $\sigma = (\sigma_1,\ldots,\sigma_n)$ and $\Sigma = \underset{i \in N}{\times} \Sigma_i$; $\sigma \in \Sigma$. Where convenient, and with slight abuse of notation, σ will be written $\sigma = (\sigma_i, \sigma_{-i})$, with $\sigma_{-i} \equiv (\sigma_1,\ldots,\sigma_{i-1}, \sigma_{i+1},\ldots,\sigma_n)$; $\sigma_{-i} \in \Sigma_{-i} \equiv \underset{\substack{j \in N \\ j \neq i}}{\times} \Sigma_j$.

3. The Normal Form

Given any strategy vector $\sigma \in \Sigma$ the probability distribution over $F(v)$, $v \in S_0$, induces a probability distribution over vertices in T: τ: $T \times \Sigma \to [0,1]$ with $\forall\, \sigma \in \Sigma$, $\sum_{v \in T} \tau(v;\sigma) = 1$. Define the function p: $\Sigma \to \mathbb{R}^n$ pointwise by $p(\sigma) = \sum_{v \in T} \tau(v)P(v)$. p is called the *normal form* of the game. The i^{th} component of τ is written $\tau_i(\sigma)$ and referred to as *player i's expected payoff.*

A strategy $\sigma_i \in \Sigma_i$ is said to be *dominated* by $\bar{\sigma}_i \in \Sigma_i$ if: $\forall\, \sigma_{-i} \in \Sigma_{-i}$, $p(\bar{\sigma}_i, \sigma_{-i}) \geq p(\sigma_i, \sigma_{-i})$ with strict inequality for some σ_{-i}.

4. Mixed and Behavioral Strategies

A *mixed strategy* for player i is a function μ_i: $\Sigma_i \to [0,1]$ with $\sum_{\sigma_i \in \Sigma_i} \mu(\sigma_i) = 1$. If $\forall\, \sigma_i \in \Sigma_i$, $\mu_i(\sigma_i) \in (0,1)$, μ_i is said to be *completely mixed.* To distinguish between μ_i and σ_i,

σ_i is often called a *pure strategy*. Let M_i be the set of mixed strategies for i, $\mu = (\mu_1,...,\mu_n)$ (sometimes $\mu = (\mu_i, \mu_{-i})$) and $M \equiv \times_{i\in N} M_i$; $\mu \in M$.

A *behavioral strategy* for player i is a function $\tilde{\mu}_i$: $C_i \rightarrow [0,1]$ with $\forall\, j \in \{1,...,m_i\}$ $\sum_{c=1}^{f_i^j} \tilde{\mu}_i(C_{ic}^j) = 1$. The restriction of $\tilde{\mu}_i$ to $\{C_{i1}^j,...,C_{if_i^j}^j\}$ is called $\tilde{\mu}_i^j$.† If $\forall\, C_{ic}^j \in C_i$, $\tilde{\mu}_i(C_{ic}^j) \in (0,1)$, $\tilde{\mu}_i$ is said to be *completely mixed*. Let $\tilde{M}_i$ be the set of behavioral strategies for i, $\tilde{\mu} = (\tilde{\mu}_1,...,\tilde{\mu}_n)$ (sometimes $\tilde{\mu} = (\tilde{\mu}_i, \tilde{\mu}_{-i})$) and $\tilde{M} = \times_{i\in N} \tilde{M}_i$; $\tilde{\mu} \in \tilde{M}$.

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†If f: $X \rightarrow Y$ is a function and $X^0 \subset X$, the *restriction* of f to X^0 is the function f^0: $X^0 \rightarrow Y$ defined pointwise by $\forall x \in X^0, f^0(x) = f(x)$.

5. Kuhn's Theorem

For $v \in S_i^j$, let $A_i^j(v) = \{C_{ic}^{j'} \mid C_{ic}^{j'} \cap \{\tilde{v} \in V \mid \tilde{v} < v\} \neq \phi, j' \in \{1,...,m_i\}, c \in \{1,...,f_i^{j'}\}\}$. Player i is said to have *perfect recall* if $\forall\, j \in \{1,...,m_i\}$, $\forall\, v \in S_i^j$, $\forall\, v' \in S_i^j$, $A_i^j(v) = A_i^j(v')$. If all players have perfect recall, G is said to have perfect recall.

Any vector of mixed strategies μ, in conjunction with the distribution over $F(v)$ for $v \in S_0$, induces a distribution over T — ρ:$T \times M \rightarrow [0,1]$ with $\sum_{v\in T} \rho(v;\mu) = 1$. Likewise any vector of behavioral strategies $\tilde{\mu}$ induces $\tilde{\rho}$: $T \times \tilde{M} \rightarrow [0,1]$ with $\sum_{v\in T} \tilde{\rho}(v;\tilde{\mu}) = 1$.

It is not difficult to verify that given $\tilde{\mu} \in M$ and the implied $\tilde{\rho}$, there exists $\mu \in M$ such that $\forall\, v \in T$, $\tilde{\rho}(v;\tilde{\mu}) = \rho(v;\mu)$; ie. that any distribution over T induced by some $\tilde{\mu}$ can be induced by some μ. The contrary does not always hold.

Theorem (*Kuhn, 1953*). *If G has perfect recall, then* $\forall\, \mu \in M$ *there exists* $\tilde{\mu} \in \tilde{M}$ *such that* $\forall\, v \in T$, $\rho(v;\mu) = \tilde{\rho}(v;\tilde{\mu})$.

In what follows G will *always* be assumed to have perfect recall.

5. Nash Equilibrium

Definition: $\sigma^* \in \Sigma$ is an *equilibrium of G* if $\forall\, i \in N, \forall\, \sigma_i \in \Sigma_i, p_i(\sigma^*_i, \sigma^*_{-i}) \geq p_i(\sigma_i, \sigma^*_{-i})$.

Theorem: An equilibrium exists if G has perfect information.

Define π: $M \to \mathbb{R}^n$ pointwise by $\forall\, \mu \in M, \pi(\mu) = \sum_{v \in T} \rho(v;\mu)P(v)$, and $\tilde{\pi}$: $\tilde{M} \to \mathbb{R}^n$ pointwise by $\forall\, \tilde{\mu} \in \tilde{M}, \tilde{\pi}(\tilde{\mu}) = \sum_{v \in T} \tilde{\rho}(v;\tilde{\mu})P(v)$.

Definition: $\mu^* \in M$ is an *equilibrium in mixed strategies* if $\forall\, i \in N, \forall\, \mu_i \in M_i$, $\pi_i(\mu^*_i, \mu^*_{-i}) \geq \pi_i(\mu_i, \mu^*_{-i})$.

The definition of an *equilibrium in behavioral strategies* substitutes $\tilde{\mu}^*$, $\tilde{M}$ and $\tilde{\pi}$, for π, μ and M in the above definition.

Theorem: G has an equilibrium in mixed and behavioral strategies.

6. Subgame Perfect Equilibrium (Selten)

For any $v \in V$, let $\Gamma_v = \{\tilde{v} \in V \mid v \prec \tilde{v}\}$. With $\prec$ restricted to $\Gamma_v \times \Gamma_v$, Γ_v and $\prec$ form a *subtree*. Observe that if $v' \in \Gamma_v$, $F(v') \subset \Gamma_v$.

The *restriction* of the game G to Γ_v is

i) the subtree Γ_v;

ii) the restriction of P to $T \cap \Gamma_v$.

iii) the partition $\{S_0 \cap \Gamma_v, \ldots, S_n \cap \Gamma_v\}$ of $S \cap \Gamma_v$;

iv) $\forall v \in S_0 \cap \Gamma_v$, the probability distribution on $F(v)$;

v) $\forall i \in N$, the partition $\{S^1_i \cap \Gamma_v, \ldots, S^{m_i}_i \cap \Gamma_v\}$ of $S_i \cap \Gamma_v$;

vi) $\forall i \in N, \forall j \in \{1, \ldots, m_i\}$, the partition $\{C^j_{ic} \cap \Gamma_v\}_{c \in \{1,\ldots,f^j_i\}}$ of $\bigcup_{v \in S^j \cap \Gamma_v} F(v)$.

If $\forall i \in N, \forall j \in \{1, \ldots, m_i\}$, either $S^j_i \cap \Gamma_v = \phi$ or $S^j_i \cap \Gamma_v = S^j_i$, the restriction of G to Γ_v is called a *subgame* of G, G_v.

Observe that for any subgame G_v, $\forall i$, $\forall j$, $\forall c$ either $C^j_{ic} \cap \Gamma_v = \phi$ or $C^j_{ic} \cap \Gamma_v = C^j_{ic}$. It follows that the restriction of the behavioral strategy for player i to $\{C^j_{ic}\}_{C^j_{ic} \subset \Gamma_v}$ is well defined; call this function $\tilde{\mu}_{vi}$, the set of possible restrictions for player i $\tilde{M}_{vi}$, and the vector of such functions $\tilde{\mu}_v$; $\tilde{\mu}_v \in \tilde{M}_v \equiv \times_{i \in N} \tilde{M}_{vi}$.

Given a subgame G_v and restriction $\tilde{\mu}_v$ obtained from $\tilde{\mu}$ (in conjunction with probability distributions over $F(v')$, $v' \in S_0$) a probability distribution $\tilde{\rho}_v$: $(T \cap \Gamma_v) \times \tilde{M}_v \to [0,1]$, with $\sum_{v' \in T \cap \Gamma_v} \tilde{\rho}_v(v';\tilde{\mu}) = 1$, is implied. $\tilde{\rho}_v$ is the distribution over terminal vertices in $T \cap \Gamma_v$ given play begins at vertex v. $\tilde{\rho}$ (defined in Sec. 5) is thus exactly $\tilde{\rho}_{v_0}$.

Define $\tilde{\pi}_v$: $\tilde{M}_v \to \mathbb{R}^n$ pointwise by $\tilde{\pi}_v(\tilde{\mu}) = \sum_{v' \in T} \tilde{\rho}_v\,(v';\,\tilde{\mu})\,P(v')$, and an equilibrium of G_v in behavioral strategies analogously to the definition for G itself.

Definition: $\tilde{\mu}^* \in \tilde{M}$ is a *subgame perfect equilibrium* (SPE) of G if for all subgames G_v, $\tilde{\mu}^*_v$ is an equilibrium of G_v in behavioral strategies.

Theorem: IF $\tilde{\mu}^*$ is a SPE of G, $\tilde{\mu}^*$ is a NE.

Theorem: G has a SPE.

7. Sequential Equilibrium (Kreps–Wilson)

A *system of beliefs* is a function ξ: $S \backslash S_0 \to [0,1]$ with $\forall i \in N$, $\forall j \in \{1,\ldots,m_i\}$ $\sum_{v \in S^j_i} \xi(v) = 1$. Let $\xi^j_i(\cdot)$ be the restriction of ξ to S^j_i. An *assessment* is a pair $(\tilde{\mu}, \xi)$.

If $\tilde{\mu}$ is completely mixed, it is possible to calculate an associated system of beliefs using Bayes' rule and $\tilde{\mu}$. Denote such a system of beliefs by $\xi(\cdot\,|\tilde{\mu})$.

Definition: The assessment $(\tilde{\mu}, \xi)$ is *consistent* if there is a sequence of completely mixed behavioral strategies $\{\tilde{\mu}_n\}$ such that $\tilde{\mu}_n \to \tilde{\mu}$ implies $\xi(\cdot\,|\tilde{\mu}_n) \to \xi$.

For any $v \in S$ and associated subtree Γ_v, let $\tilde{\rho}_v(\cdot\,;\,\tilde{\mu})$ be the induced probability distribution over T, and $\tilde{\pi}_v(\tilde{\mu}) \equiv \sum_{v' \in T} \tilde{\rho}_v(v';\,\tilde{\mu})P(v')$.†

Definition: Given an assessment $(\tilde{\mu}, \xi)$, $\tilde{\mu}_i$ is *sequentially rational* for player i if $\forall j \in \{1,\ldots,m_i\}$, $\forall \tilde{\mu}'_i \in \tilde{M}_i$, $\sum_{v \in S_i^j} \xi_i^j(v)\tilde{\pi}_{vi}(\tilde{\mu}_i, \tilde{\mu}_{-i}) \geq \sum_{v \in S_i^j} \xi_i^j(v)\tilde{\pi}_{vi}(\tilde{\mu}'_i, \tilde{\mu}_{-i})$. $\tilde{\mu}$ is sequentially rational if $\tilde{\mu}_i$ is sequentially rational for player i, all i.

Definition: A *sequential equilibrium* (SE) is a consistent assessment $(\tilde{\mu}^*, \xi^*)$ where $\tilde{\mu}^*$ is sequentially rational.

Theorem: G has a SE.

Proof (Sketch): G has a SPE. The argument demonstrates that there is at least one SPE from which a consistent assessment $(\tilde{\mu}, \xi)$ can be constructed and where $\tilde{\mu}$ is sequentially rational.||

Theorem: If $(\tilde{\mu}^*, \xi^*)$ is an SE, $\tilde{\mu}^*$ is a SPE.

Proof: Let $(\tilde{\mu}^*, \xi^*)$ be a SE. If $\tilde{\mu}^*$ is not a SPE, there exists $v \in S$, with $S_i^j = \{v\}$ for some i and j, and a subgame G_v such that $\tilde{\mu}_v^*$ is not an equilibrium of G_v, implying $\tilde{\pi}_{vi'}(\tilde{\mu}^*_{i'}, \tilde{\mu}^*_{-i'}) <$ $\tilde{\pi}_{vi'}(\tilde{\mu}_{i'}, \tilde{\mu}^*_{-i'})$ for some $i \in N$ and $\tilde{\mu}_i \in M_i$. In particular, $\tilde{\mu}^*_{i'}$ must not be a best reply at some information set $S_{i'}^{j'}$ containing vertices occurring with positive probability given $\tilde{\mu}_v^*$; otherwise, the inequality would not be strict. But consistency requires $\xi_{i'}^{j'}$ to be computed using $\tilde{\mu}^*$ and Bayes' rule, in which case if $\tilde{\mu}^*_{i'}$ is not a best reply to $\tilde{\mu}^*_{-i'}$ it cannot be sequentially rational given such $\xi_{i'}^{j'}$. ||

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† v may not be such that the restriction of G to Γ_v is a subgame, in which case there is the minor technical issue that the restriction of $\tilde{\mu}$ to Γ_v is not well defined. $\tilde{\rho}_v$ must be constructed from $\tilde{\mu}$ directly.

8. Perfect Sequential Equilibrium (Grossman–Perry)

A prototypical *signalling game* has the following special structure:

i) n=2; ii) $S_0 = \{v_0\}$; iii) $m_1 = \#[F(v_0)]$ and $\forall j$, $S_1^j \subset F(v_0)$;

iv) $\forall j, \forall j', f_i^j = f_1^{j'} \equiv f_1$; v) $m_2 = f_1$; $\forall j$, $S_2^j = \bigcup_{j'=1}^{f_1} C_{1j'}^{j'}$.

Let $(\tilde{\mu}^*, \xi^*)$ be a sequential equilibrium of any game G and $T^* = \{v \in T \mid \rho_{v_0}(v; \tilde{\mu}^*) > 0\}$. Information set S_i^j is *off the equilibrium path* if $\forall v \in T^*$, $\{\tilde{v} \in V \mid \tilde{v} \prec v\} \cap S_i^j = \phi$. If S_i^j is off the equilibrium path, ξ_i^{j*} is referred to as an *off equilibrium path belief*. In the signalling game, only S_2^j and ξ_2^{j*} can be off the equilibrium path.

A perfect sequential equilibrium requires that off equilibrium path beliefs satisfy a condition in addition to consistency—*credibility*—developed as follows.

Given a sequential equilibrium $(\tilde{\mu}^*, \xi^*)$, let $S_2^{j'}$ be an off equilibrium path information set, and $\tilde{\mu}_1'$ be a behavioral strategy for player 1 such that for some $v \in T$ with $\rho_{v_0}(v \mid \tilde{\mu}_1', \tilde{\mu}_2^*) > 0$, $S_2^{j'} \cap \{\tilde{v} \in v \mid \tilde{v} \prec v\} \neq \phi$ holds; $\tilde{\mu}_1'$ is called a *signalling deviation*. Let $\xi_2^{j'}: S_2^{j'} \rightarrow [0,1]$ be the p obability distribution over vertices in $S_2^{j'}$ that is consistent with $(\tilde{\mu}_1', \tilde{\mu}_2^*)$, and define a system of beliefs ξ' pointwise by $\xi(v) = \xi^*(v)$, $v \notin S_2^{j'}$, $\xi'(v) = \xi_2^{j'}(v)$, $v \in S_2^{j'}$. Also, let $\tilde{\mu}_2'$ be sequentially rational for player 2 given beliefs ξ', and equal to $\tilde{\mu}_2^*$ for $C_{ic}^j \neq C_{ic}^{j'}$.

Definition: If $\forall v \in S_1^j$, $\tilde{\pi}_{v1}(\tilde{\mu}_1', \tilde{\mu}_2') \geq \tilde{\pi}_{v1}(\tilde{\mu}^*)$, with strict inequality for some v, $\tilde{\mu}_1'$ is called a *successful signalling deviation* for $S_i^{j'}$.

Definition: Given a sequential equilibrium assessment (μ^*, ξ^*), ξ^* is *credible* if there is no off equilibrium information set $S_i^{j'}$ for which a successful signalling deviation exists.

Definition: A sequential equilibrium assessment (μ^*, ξ^*) is a *perfect sequential equilibrium* (PSE) if ξ^* is credible.

Observe that if $\tilde{\mu}^*$ is completely mixed $(\tilde{\mu}^*, \xi^*)$ is a PSE. Also, there are examples for which no PSE exist.

9. Sufficiency of the Normal Form

Theorem: If μ^* $(\tilde{\mu}^*)$ is a mixed (behavioral) strategy equilibrium of G, it is also a mixed (behavioral) strategy equilibrium of every game having the game normal form as G.

Proof: $\pi(\mu) = \sum_{v \in T} \rho(v; \mu) P(v)$

Now $\rho(v;\mu) = \sum_{\sigma\in\Sigma} [\tau(v;\sigma) \prod_{i\in N} \mu_i(\sigma_i)]$.

Thus, $\pi(\mu) = \sum_{v\in T} \sum_{\sigma\in\Sigma} [\tau(v;\sigma) \prod_{i\in N} \mu_i(\sigma_i)]P(v)$

$$= \sum_{\sigma\in\Sigma} \prod_{i\in N} \mu_i(\sigma_i)[\sum_{v\in T} \tau(v;\sigma)\, P(v)]$$

$$= \sum_{\sigma\in\Sigma} \prod_{i\in N} \mu_i(\sigma_i)p(\sigma).$$

Thus whether μ^* is an equilibrium in mixed strategies depends only on $p(\sigma)$. Then, whether $\bar{\mu}^*$ is an equilibrium also depends only on $p(\sigma)$ (by Kuhn's Theorem). ||

10. Normal Form as a Primitive

In Section 5 the normal form p was treated as a derived object. In some instances it is convenient to consider it to be a primitive, as follows. A strategy for player i is a vector $(\sigma_i^1,\ldots,\sigma_i^{m_i}) \in \Sigma_i \subset \mathbb{R}^{m_i}$. Let $m = \sum_i m_i$, $\Sigma \equiv \times_{i\in N} \Sigma_i$, $p: \Sigma \to \mathbb{R}^n$, and write $p(\sigma) = (p_1(\sigma),\ldots,p_m(\sigma))$.

Treated as a primitive then, a normal form game g is a triple (N, Σ, p) and $\sigma^* = (\sigma_1^*,\ldots,\sigma_n^*)$ is an equilibrium if $\forall i \in N$, $\sigma_i^* = \operatorname{argmax}_{\sigma_i\in\Sigma_i} p_i(\sigma_i, \sigma_{-i}^*)$.

11. Normal Form with a Continuum of Choices

Theorem: Let (N, Σ, p) be a normal form game with

i) $\forall i$, $\Sigma_i \subset \mathbb{R}^m$ *is compact and convex;;*

ii) p *continuous;*

and iii) $\forall i$, $\forall \sigma_{-i}$, p_i *is strictly quasi concave in* σ_i.

Then $(N, \Sigma\, p)$ has an equilibrium.

Proof: Define the mapping r_i: $\Sigma_{-i} \to \Sigma_i$ pointwise by: $\forall \sigma_{-i} \in \Sigma_{-i}$, $r_i(\sigma_{-i}) = \operatorname{argmax}_{\sigma_i\in\Sigma_i} p_i(\sigma_i, \sigma_{-i})$.

r_i exists because Σ_i is compact and p_i is continuous. Moreover, a simple extension of Berge's maximum theorem† implies r_i is upper–hemi continuous, and strict quasiconcavity of p_i, in conjunction with convexity of Σ_i, implies that r_i is a function; thus r_i is continuous.

Define r: $\Sigma \to \Sigma$ pointwise by $\forall\, \sigma \in \Sigma, r(\sigma) = (r_1(\sigma_{-1}),...,r_n(\sigma_{-n}))$. Observe that Σ is compact and convex, and σ^* is an equilibrium if and only if $\sigma^* = r(\sigma^*)$.

r is a continuous function from a compact convex subset of $\mathbb{R}^m$ to itself. By Brouwer's fixed point theorem r has fixed point, σ^*. ||

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†Harris, *Dynamic Economic Analysis*, Theorem 1.1.

12. Repeated Games

Let $g \equiv (N, \Sigma, p)$ be a normal form game with $\Sigma \subset \mathbb{R}^m$ compact and p continuous; σ is a generic element of Σ. For any i, let $\underline{u}_i = \min_{\sigma_{-i} \in \Sigma_{-i}} \max_{\sigma_i \in \Sigma_i} p_i(\sigma)$, and define

$U \equiv \{u \in p(\Sigma) \,|\, u_i > \underline{u}_i\}$. $\bar{U}$ is the convex hull of U.

In a repeated game g^D, g, is played at each date $t \in D$, where without loss of generality $D = \{0,1,...,\bar{D}\}$; $\bar{D} = \infty$ is permitted. Information available to players at the outset of t is the *history* of play prior to t. Define, for $t \geq 1$, $H_t \equiv \times_{t'=0}^{t-1} \Sigma$ and $H_0 = \{\phi\}$. H_t is the set of possible histories prior to t, with generic element h_t.

A *strategy* $\hat{\sigma}_i$ for player i in the repeated game is a sequence of functions $\{\hat{\sigma}_{it}\}_{t \in D}$, where $\hat{\sigma}_{it}$: $H_t \to \Sigma_i$. Let $\hat{\Sigma}_i$ be the set of feasible strategies for i; $\hat{\Sigma} \equiv \times_{i \in N} \hat{\Sigma}_i$; $\hat{\sigma} \equiv (\hat{\sigma}_1,...,\hat{\sigma}_n)$, sometimes written $(\hat{\sigma}_i,\hat{\sigma}_{-i})$. Let $\sigma_t \in \Sigma$ denote actions taken at t, given $\hat{\sigma}$ and some h_t; i.e. $\sigma_t = \hat{\sigma}(h_t)$.

The *payoff function* for g^D is $\hat{p}$: $\hat{\Sigma} \to \mathbb{R}^n$ defined pointwise by:

$$\forall \hat{\sigma} \in \hat{\Sigma}, \;\; \hat{p}(\hat{\sigma}) = \sum_{t \in D} \delta^t p(\sigma_t),$$

where $\delta \in (0,1)$, $\sigma_t = \hat{\sigma}(h_t)$, $h_0 = \phi$, and $h_t = (\sigma_0,...,\sigma_{t-1})$.

A *repeated game* g^D is the normal form $(N, \hat{\Sigma}, \hat{p})$.

"Folk" Theorem: *Let $\bar{D} = \infty$. Then $\forall u \in U$, $\exists\, \hat{\sigma}^* \in \hat{\Sigma}$ and $\delta^u \in (0,1)$ such that $\delta \geq \delta^u$ implies i) $\hat{\sigma}^*$ is a NE of g^D; and ii) $\hat{p}(\hat{\sigma}^*) = u/(1-\delta)$.*

Proof: Let $\bar{\sigma} \in p^{-1}(u)$; $\bar{\sigma}$ exists since $u \in U$. Also let $\varphi^i \in \arg\left\{\min_{\sigma_{-i} \in \Sigma_{-i}} \max_{\sigma_i \in \Sigma_i} p_i(\sigma)\right\}$. Then $p_i(\varphi^i) = \underline{u}_i$.

Let $\bar{h}_t$ be the history describing $\bar{\sigma}$ being played at all dates prior to t. For any $h_t \neq \bar{h}_t$, define $i(h_t)$ to be the index i of the player who first failed to play $\bar{\sigma}_i$ in the history h_t when all other players i' chose $\bar{\sigma}_{i'}$, if there is such an i, and $i(h_t)$ equals 0 otherwise.

Consider the strategy $\hat{\sigma}^*_i$ defined pointwise by

$$\hat{\sigma}^*_i(h_t) = \begin{cases} \varphi_i^{i'} & \text{if } t \in D\backslash\{0\},\ i(h_t) = i' \in N, \\ \bar{\sigma}_i & \text{otherwise.} \end{cases}$$

Observe that i's payoff from equilibrium play by all players is $\hat{p}_i(\hat{\sigma}^*) = p_i(\bar{\sigma})/(1-\delta) = u_i/(1-\delta)$. Any deviation by i at t yields i a payoff of at most

$$\sum_{t'=0}^{t-1} \delta^{t'} u_i + \delta^t \max_{\sigma_i \in \Sigma_i} p_i(\sigma_i, \bar{\sigma}_{-i}) + \sum_{t'=t+1}^{\infty} \delta^{t'} \underline{u}_i.$$

Thus $\hat{\sigma}^*$ is a NE if, $\forall t \in D$

$$\begin{aligned} u_i/(1-\delta) &\geq \sum_{t'=0}^{t-1} \delta^{t'} u_i + \delta^t \max_{\sigma_i \in \Sigma_i} p_i(\sigma_i, \bar{\sigma}_{-i}) + \sum_{t'=t+1}^{\infty} \delta^{t'} \underline{u}_i \\ &= [u_i/(1-\delta)](1-\delta^t) + \delta^t \max_{\sigma_i \in \Sigma_i} p_i(\sigma_i, \bar{\sigma}_{-i}) + \delta^{t+1}\underline{u}_i/(1-\delta), \end{aligned}$$

or

$$(u_i - \delta \underline{u}_i)/(1-\delta) \geq \max_{\sigma_i \in \Sigma_i} p_i(\sigma_i, \bar{\sigma}_{-i}). \qquad (*)$$

Since $u_i > \underline{u}_i$, either (*) is satisfied for any δ or there is $\bar{\delta} \in (0,1)$ for which (*) is an equality. Define $\delta^u = \max\{0, \bar{\delta}\}$. ||

Theorem: If $\dim \bar{U} = n$, *SPE replaces NE in the Folk Theorem.*†

Theorem: If g *has an unique equilibrium,* σ^*, *and* $\bar{D} < \infty$, g^D *has a unique equilibrium,* $\hat{\sigma}^*$, *in which* $\forall i, \forall t, \forall h_t$, $\hat{\sigma}^*_{it}(h_t) = \sigma^*_i$.

Proof: For any $\hat{\sigma}_i \in \hat{\Sigma}_i$ and any sequence of histories $\{h_t\}_0^{\bar{D}}$,

$$\hat{p}_i(\hat{\sigma}_i, \hat{\sigma}^*_{-i}) = \sum_{t\in D} \delta^t p_i[\sigma_{it}, \sigma^*_{-it}]$$
$$\leq \sum_{t\in D} \delta^t \max_{\sigma_i \in \Sigma_i} p_i(\sigma_i, \sigma^*_{-it})$$
$$= \sum_{t\in D} \delta^t p_i(\sigma^*),$$

establishing that $\hat{\sigma}^*$ is a NE.

Suppose $\hat{\sigma}' \neq \hat{\sigma}^*$ is a *NE* of g^D. Then $\forall i$, $\forall \hat{\sigma}_i \in \hat{\Sigma}_i$, $\hat{p}_i(\hat{\sigma}') \geq \hat{p}(\hat{\sigma}_i, \hat{\sigma}'_{-i})$;

This must hold, in particular, for $\hat{\sigma}_i = \hat{\sigma}'_i$ for all histories h_t with $t < \bar{D}$, in which case, $\forall_i$, $\forall \sigma_i \in \Sigma_i$

$$p_i(\sigma'_{i\bar{D}}, \sigma'_{i\bar{D}}) \geq p_i(\sigma_i, \sigma'_{-i\bar{D}}). \qquad (*)$$

But by uniqueness of equilibrium in g, σ^* is the only vector for which (*) may obtain for all i. Thus $\sigma'_{\bar{D}} = \sigma^*$. Given this condition, $\sigma'_{\bar{D}-1} = \sigma^*$ may be demonstrated, and so on, yielding $\hat{\sigma}'(h_t) = \sigma^*$ all h_t; i.e. $\hat{\sigma}' = \hat{\sigma}^*$, a contradiction. ||

Theorem: Assume i) $\bar{D} < \infty$; ii) dim $\bar{U} = n$; and iii) g has two equilibria σ_A and σ_B with $\forall i \in N, p_i(\sigma_A) \neq p_i(\sigma_B)$. Then $\forall u \in U, \forall \varepsilon > 0, \exists D_0 < \infty$ and $\delta^u \in (0,1)$ such that $\bar{D} \geq D_0$ and $\delta \in (\delta^u, 1)$ imply g^D has a SPE, $\hat{\sigma}^*$, in which $|\hat{p}_i(\hat{\sigma}^*) - u\frac{1-\delta^{\bar{D}+1}}{1-\delta}| < \varepsilon$.

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†For any $X \subset \mathbb{R}^k$, dim X is the minimal number of linearly independent vectors needed to span X.

13. Anonymous Games (Mas–Collel)

Let $\Sigma \subset \mathbb{R}^m$ be compact and α a Borel probability measure on Σ.# Let $\mathcal{M}$ be the set of such measures, endowed with the topology of weak convergence.†

A player's *payoff* is a function p: $\Sigma \times \mathcal{M} \to \mathbb{R}$, *written* $p(\sigma, \alpha)$ where $\sigma \in \Sigma$ is the player's action and $\alpha \in \mathcal{M}$ is the probability measure of others' actions; that α is all that matters about what others do is what is meant by *anonymity*. Similarly, the game does not identify players apart from the features of this payoff function.

p is assumed continuous for each player. Let $\mathcal{P}$ be the set of such payoffs. An *anonymous game* is a probability measure on $\mathcal{P}$, v.

Let γ be a probability measure on $\mathcal{P} \times \Sigma$ and define i) $\gamma_{\mathcal{P}}$ by $\forall P \subset \mathcal{B}^{\mathcal{P}}, \gamma_{\mathcal{P}}(P) = \gamma(P \times \Sigma)$; and ii) γ_{Σ} by $\forall s \subset \mathcal{B}^{\Sigma}, \gamma_{\Sigma}(s) = \gamma(s \times \mathcal{P})$.

Definition: γ^* is an *equilibrium* of the game v if i) $\gamma^*_{\mathcal{P}} = \text{v}$; and ii) $\gamma^*\left[\left[\{(p,\sigma) \mid p(\sigma,\gamma^*_{\Sigma}) \geq p(\Sigma\, \gamma^*_{\Sigma})\}\right]\right] = 1$. Note that it is implicit that each player has $\gamma^*_{\mathcal{P}}$ measure 0.

Theorem: v has an equilibrium, γ^*.

In general, γ^* will not be symmetric, in the sense that some players having identical payoffs will take different actions in equilibrium.

#Let $\mathcal{A}$ be a collection of subsets of a set X. $\mathcal{A}$ is a σ algebra if i) $X \in \mathcal{A}$; ii) $A \in \mathcal{A} \Longrightarrow A^C \in \mathcal{A}$; and iii) $A_1 \in \mathcal{A}, A_2 \in \mathcal{A}, \ldots \Longrightarrow \left[\left(\bigcup_{n=1}^{\infty} A_n\right)\right] \in \mathcal{A}$. Given a topology on X, let $\mathcal{O}$ be the collection of open sets in X. There is a minimal σ algebra $\mathcal{B}^X$ containing $\mathcal{O}$; that is, $\mathcal{B}^X$ is a subset of every σ algebra containing $\mathcal{O}$. $\mathcal{B}^X$ is called the *Borel σ algebra* of subsets of X. Let μ: $\mathcal{B}^X \longrightarrow \mathbb{R}^*$ (extended reals). μ is a (Borel) *probability measure* if i) $\forall A \in \mathcal{B}^X, \mu(A) \geq 0$; ii) $\mu(X) = 1$; and iii) $A_1 \in \mathcal{B}^X, A_2 \in \mathcal{B}^X, \ldots$, and $A_n \cup A_{n'} = \phi$ unless $n=n'$, imply $\mu\left[\bigcup_{n=1}^{\infty} A_n\right] = \sum_{n=1}^{\infty} \mu(A_n)$.

†The topology of weak convergence is often called the weak* topology; see Hildenbrand, *Core and Equilibria of Large Economy*, p. 48 ff. The relevant fact here is that the sequence $\{\mu_n\}$ of measures in $\mathcal{M}$ converges to μ – $\mu_n \to \mu$ – if and only if for every bounded and continuous function f: $\Sigma \to \mathbb{R}$, the sequence of real numbers $\{\int f d\mu_n\}$ converges to $\int f d\mu$.

14. Anonymous Sequential Games (Jovanovic–Rosenthal)

An anonymous sequential game allows an anonymous game to be played at each date $t \in D = \{0,1,...\}$. It requires that differences in payoffs across players can be described by distinct values of an individual "state variable" θ, where $\theta \in \Theta$ and $\Theta \subset \mathbb{R}^m$ is compact. θ is permitted to evolve over time for each player.

Let $\mathcal{M}_{\Theta\Sigma}$ be the set of Borel probability measures on $\Theta \times \Sigma$, endowed with topology of weak convergence, and having generic element γ. Then define i) γ_Θ by $\forall A \subset \mathcal{B}^\Theta$, $\gamma_\Theta(A) = \gamma(A \times \Sigma)$; and ii) γ_Σ by $\forall s \in \mathcal{B}^\Sigma$, $\gamma_\Sigma(s) = \gamma(\Theta \times s)$.†

A *player's return at date t* is $p: D \times \Theta \times \Sigma \times \mathcal{M}_{\Theta\Sigma} \to \mathbb{R}$, written $p_t(\theta, \sigma, \gamma)$; assume $\forall t$, $\forall \theta$, $\forall \sigma$, $\forall \gamma$, $p_t(\theta, \sigma, \gamma) < \bar{p} < \infty$. A *given* sequence of individual states, actions and measures, $\{\theta_t, \sigma_t, \gamma_t\}_0^\infty$ yields a player a *payoff* of

$$\sum_{t \in D} \beta^t p_t(\theta_t, \sigma_t, \gamma_t),$$

where $\beta \in (0,1)$ is a fixed discount factor.

Individual state variables θ evolve as follows. There is an *initial distribution of* θ, summarized by a Borel probability measure on Θ, ν_0. Subsequently, given θ_t, σ_t and γ_t, the distribution of individual states at $t+1$ is given by a continuous probability measure on Θ, $F_t(\cdot\,;\theta_t,\sigma_t,\gamma_t)$.

Part of the definition of equilibrium will require a player to select some action $\sigma \in \Sigma$ given t, θ and the past and anticipated future behavior of other players as summarized by $\tilde{\gamma} \equiv \{\gamma_t\}_0^\infty$; $\tilde{\gamma} \in \tilde{\mathcal{M}}_{\Theta\Sigma} \equiv \mathcal{M}_{\Theta\Sigma} \times \mathcal{M}_{\Theta\Sigma} \times ...$ This problem can be characterized by a function $V: D \times \Theta \times \tilde{\mathcal{M}}_{\Theta\Sigma} \to \mathbb{R}$, where $\forall (t, \theta, \tilde{\gamma})$,

$$V_t(\theta,\tilde{\gamma}) = \max_{\sigma \in \Sigma} \left\{ p_t(\theta,\sigma,\gamma_t) + \beta \int_\Theta V_{t+1}(\theta', \tilde{\gamma}) F_t(d\theta'; \theta, \sigma, \gamma_t) \right\}.$$

Define $\tilde{\sigma}_t(\theta,\tilde{\gamma})$ to be the argmax of the expression in braces.

Definition: $\tilde{\gamma}^*$ is an *equilibrium* if

i) $\gamma^*_{\Theta_0} = \nu_0$ and $\forall t \in D$, $\gamma^*_{\Theta_{t+1}} = \int_{\Theta \times \Sigma} F_t(\cdot\,; \theta, \sigma, \gamma^*_t)\gamma^*_t\,(d\theta \times d\sigma)$;

ii) $\forall t \in D$, $\gamma^*_t \left[\left\{ (\theta,\sigma) \in \Theta \times \Sigma \,|\, \sigma = \tilde{\sigma}_t(\theta, \gamma^*_t) \right\} \right] = 1$;

iii) V_t and $\tilde{\sigma}_t$ are as defined above.

Theorem: Every anonymous sequential game has an equilibrium.

The game is *stationary* if neither p_t nor F_t depend on *t*. An equilibrium $\tilde{\gamma}^*$ is *stationary* if $\forall t \in D$, $\gamma_t^* = \gamma$ for some $\gamma \in \mathcal{M}_{\theta\Sigma}$

Theorem: For any stationary anonymous sequential game, there exists ν_0 *such that* $\tilde{\gamma}^*$ *is stationary.*

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†In general, feasible actions for any player can be allowed to depend on θ and γ_θ.

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1. Let consumer preferences $\succsim$ (weak preference) be defined on the consumption set $\mathbb{R}^l_+$, having typical element x. Denote the budget correspondence—the set of affordable x in $\mathbb{R}^l_+$—by $\gamma(p,w)$. Define strict preference by $x \succ y \Leftrightarrow (x \succsim y$, and not $y \succsim x)$, and the demand set by $\varphi \equiv \{x \in \gamma(p,w) \mid$ if $y \in \mathbb{R}^l_+$ and $y \succ x$, then $y \notin \gamma\}$.

   (i) Among other things, $\succsim$ is often assumed complete. Suppose instead that $\succsim$ is very incomplete, in the sense that for all x and y (with $x \in \mathbb{R}^l_+$, $y \in \mathbb{R}^l_+$ and $x \neq y$) neither $x \succsim y$ nor $y \succsim x$. Show that $\varphi = \gamma$, and explain why. What is the empirical content of the theory so specified?

   (ii) Now require $\succsim$ to be complete, but such that for all x and y, with $x \in \mathbb{R}^l_+$ and $y \in \mathbb{R}^l_+$, both $x \succsim y$ and $y \succsim x$. What is φ, and why? Compare the resulting demand set φ to that obtained in (i), and explain the relationship between them.

2. Let the consumption set be $[0,k]$, for some $k \in \mathbb{R}$. Let the preference relation $\succsim$ be defined by: $\forall\, x, y \in [0,k]$, $x \succsim y \Leftrightarrow x \geq y$.

   i). Show that $\succsim$ is continuous, complete and transitive.

   ii). For each $x \in [0,k]$, let $F(x) = \{y \in [0,k] \mid y \succsim x\}$. The mapping F is called the "weak preference correspondence". Draw the graph of F, and show that F is both u.h.c. and l.h.c.

iii). Let the unit price of the good be p, and income w. Let $\omega = w/p < k$. Interpret ω. Show that the demand correspondence

$\varphi(\omega) = \{x \in [0,\omega] \mid (y \succsim x \text{ and not } x \succsim y) \Rightarrow y \notin [0,\omega]\}$

is well defined (i.e. $\forall\, x \in [0,k]$, $x \in \varphi$ or $x \notin \varphi$) and $\varphi \neq \phi$.

iv). Suppose the preference ordering is as above except that exactly k units of the good is strictly worse than any other number. Reconsider parts (i) – (iii).

3. Let the consumption set be $\mathbb{R}_+^l$ and the preference preordering be represented by

$$u(x) = \prod_{h=1}^{l} x_h^{\alpha_h}, \text{ with } 0 < \alpha_h < \infty.$$

(a) Check that the preferences represented by u(x) are complete, transitive, continuous, locally nonsatiated, monotonic, strictly convex, and weakly separable on $\mathbb{R}_{++}^l$.

(b) Derive the demand function f(p,w) and verify that the Slutsky matrix has rank $l-1$ and is symmetric negative semidefinite.

(c) Suppose $l=2$ and that you do not know $u(x_1,x_2)$. However, you do know that the data are described by $k(x_2/x_1) = p_1/p_2$, for some $k \in \mathbb{R}_{++}$. Can such data be "rationalized"? Use these data to construct $u(x_1,x_2)$.

4. Consider a two–good consumer problem where preferences may be represented by $u(x) = x_1^{\alpha} x_2^{1-\alpha}$; $\alpha \in (0,1)$ and $x_h > 0$. The price system is $p = (p_1,p_2)'$, exogenous income is w, and each consumer has an endowment of x_1 equal to $\bar{x}_1 > 0$. There is no endowment of x_2.

(a) Let $y = x_1 - \bar{x}_1$, $y < 0$ (> 0) will be called sales (purchases). For all p, w, α and $\bar{x}_1$, what is the utility maximizing y? Call it y*. Suppose zero degree homogeneity of y* in (p,w) is a maintained hypothesis, and let $\omega \equiv w/p_1$. Compute $\partial y^*/\partial\omega$. Call this result the "prediction".

(b) Suppose that you, the investigator, assume preferences to be precisely as specified, but do not know the particular values of α and $\bar{x}_1$. However, you are willing to structure the differences among consumers in the following manner: α is the same for all consumers and $\bar{x}_1$ varies across consumers. $\bar{x}_1$ is a realization of a random variable X_1, where realizations are independent across consumers, and the density of X_1 is denoted $\phi(\bar{x}_1)$. $E(X_1) = \mu < \infty$ and $Var(X_1) < \infty$, where ϕ is known to you. Further, $\phi(\bar{x}_1) > 0$ implies $\bar{x}_1 > 0$. Let $\Delta \equiv X_1 - \mu$ and rewrite y* in terms of Δ. Calling the density of Δ $\psi(\delta)$, what fraction of the population chooses $y^* > 0$? < 0? What is the expected value of y* given ω, $E(y^*|\omega)$? For given α and μ, graph $E(y^*|\omega)$ in the (ω,y^*) plane and represent what might be various (ω,y^*) pairs in the picture (a "scatter diagram"). What is $\partial E(y^*|\omega)/\partial\omega$, and will a simple regression of y* on ω reveal $\partial E(y^*|\omega)/\partial\omega$ if the sample is large?

(c) Suppose the procedure which supplies data on y* and ω is such that observations are taken only if $y^* > 0$ (called "selection on y*"). For example, sales and purchases might be carried out at different locations, with sampling at just one. A large sample of such data will permit estimation of $E(y^*|y^* > 0, \omega)$. Explain, first using the scatter diagram, then formally, why $\partial E(y^*|y^* > 0, \omega)/\partial\omega < \partial E(y^*|\omega)/\partial\omega$. This inequality is one type of "selection bias". In a similar way, explain why selecting on $\omega > \bar{\omega}$ (for some $\bar{\omega}$) for example, generates $\partial E(y^*|\omega > \bar{\omega})/\partial\omega = \partial E(y^*|\partial\omega)/\partial\omega$. What is the critical economic distinction between y* and ω that causes selection on y* to be disastrous, and selection on ω innocuous?

(d) The best solution to sample selection problems is to construct the sample correctly in the first place. Such is not always possible, but even when it is investigators sometimes intentionally "oversample"; that is, they choose a non–random sample based on some characteristic of the data. Suppose that the fixed fraction γ of the sample is taken from consumers for whom $y^* > 0$, and $1-\gamma$ from those for whom $y^* \leq 0$. Estimates based on such data estimate

$$E(y^*|\gamma,\omega) \equiv \gamma E(y^*|\omega,y^* > 0) + (1-\gamma)E(y^*|\omega, y^* \leq 0).$$

Show that $\partial E(y^*|\gamma,\omega)/\partial\omega = \partial E(y^*|\omega)/\partial\omega$ if and only if γ equals exactly the fraction of the population that chooses $y^* > 0$. Procedures of this kind, and that referred to in (c), are examples of the general procedure called "choice based sampling". Assess the impact of choice based sampling on the process of testing theories.

(e) A famous technique for handling sample selection bias is the use of "Heckman's λ". In this case, λ is a an estimate of $E(\Delta|\Delta < \frac{\alpha\omega}{1-\alpha} - \mu)$, which is included as a regressor, along with ω, in a regression having y as dependent variable, where y is only observed if $y^* > 0$. The coefficient of ω is then used as an estimate of $\partial E(y^*|\omega)/\partial\omega$. This procedure works. Why?

5. Suppose there are l=2 goods, and all consumers have preferences characterizable by the utility function $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$; $\alpha_h > 0$, ($x_h > 0\ \forall h$)

(i) Assume $\alpha_1 + \alpha_2 = 1$, and so write $u(x) = x_1^{\alpha} x_2^{1-\alpha}$. Why does this restriction involve no additional loss of generality?

(ii) For a consumer facing prices $p = (p_1,p_2)'$ and income w, derive the Marshallian demand functions $f_h(p,w;\alpha)$ and Hicksian demands $h_h(p,u;\alpha)$. Check the properties of the Slutsky matrix, and show that for these preferences Marshallian demand is always more price elastic than Hicksian.

(iii) Derive the indirect utility function G(p,w) and expenditure function E(p,u) and check their properties.

(iv) Suppose you, the analyst, do not know α or w for any particular agent, but are willing to assume:

(a) for each agent, α and w are realizations of random variables A and W which are stochastically independent and have known densities $\pi(\alpha)$ ($> 0 \Leftrightarrow \alpha \in (0,1)$) and $\rho(w)$ ($> 0 \Leftrightarrow y \in (0,\infty)$), where E(A) and E(W) exist;

and

(b) estimation will provide a very good picture of <u>average</u> Marshallian demand

$$\bar{f}_h(p) = \int_0^1 \int_0^\infty f_h(p,w;\alpha)\pi(\alpha)\rho(w)\, dw d\alpha.$$

Given (a) and (b), what predictions can you make about the structure of average demand? Would an "average" consumer—i.e. one for whom α = E(A) and w = E(W)—choose average demand $\bar{f}_h(p)$? Why?

6. Consider a very simple decision problem wherein an individual chooses some variable y, and the realized value of that choice is X = y + A, where A is a random variable taking on values α and $-\alpha$, $\alpha \geq 0$, equiprobably. Suppose that for any given X=x, utility is f(x). (Here I have "substituted" in a constraint, so that a standard constrained optimization looks like an unconstrained one.) f(y) is twice continuously differentiable, strictly concave, and has a maximum at $\bar{y}$.

(i) What is the expression for expected utility given y?

(ii) Assume expected utility is maximized by some $y = y^* < \infty$. What are the first and second–order necessary conditions characterizing this choice?

(iii) Let V^* be the expected utility generated by y^*. Show that $dV^*/d\alpha = 0$ for $\alpha = 0$. What does this result say, in terms of what the individual would be willing to pay to eliminate risk A when α is small, and why is this so?

(iv) Does the result in (iii) continue to hold if $X = y + A + Z$, where Z is some random variable distributed independently of A?

7. Consider the infinite horizon consumer problem

$$\sup_{\{c_t, x_t\}_0^\infty} \sum_0^\infty \beta^t c_t,$$

$\beta \in (0,1)$,

$x_0 \in \mathbb{R}$ given,

$x_{t+1} \in (-\infty, x_t/\beta]$

and $\quad c_t \leq x_t - \beta x_{t+1}$. (*)

Let $V(x_0)$ denote the supremum given x_0.

1) Show that for all x_0, $V(x_0) = +\infty$.

2) Explain why the following argument is incorrect. "Consider the collection of constraints (*):

$c_0 \leq x_0 - \beta x_1$

$c_1 \leq x_1 - \beta x_2$,

.
.
.

Multiplying the t^{th} constraint by β^t gives

$c_0 \leq x_0 - \beta x_1$

$\beta c_1 \leq \beta x_1 - \beta^2 x_2$

.
.
.

Summing these inequalities yields

$$\sum_{0}^{\infty} \beta^t c_t \le x_0. \quad (**)$$

Therefore any feasible $\{c_t, x_t\}_0^\infty$ sequence satisfying (*) satisfies (**), in which case $V(x_0) \le x_0$."

3) Show that if the setup is modified to require $x_t \ge 0$, all t, then $V(x_0) = x_0$. Explain.

8. Consider a two person household whose collective preferences can be represented by $u(z) = {z^1}^{\beta} {z^2}^{1-\beta}$; $\beta \in (0,1)$. The home produced goods z^i are constructed according to

$$z^1 = {x^1}^{\alpha} {t^1}^{1-\alpha} \qquad \alpha \in (0,1)$$

and $$z^2 = {x^2}^{\gamma} {t^2}^{1-\gamma} \qquad \gamma \in (0,1)$$

where t^i is the time spent by person i (i=1,2) on home production, and x^i is the input of the sole purchasable commodity, x, which has price p. Person i produces good i and/or sells time in the market at price s^i. The household has w in exogenous income. Each agent has total time T available, and both pool their resources and seek to maximize u(z).

i) Set up the household's problem.

ii) Find the optimal z^i, x^i and t^i.

iii) Determine the effects of changes in s^i, p and w on optimal z^i, x^i, t^i.

iv) Explain why the t^i and z^i obtained in (ii) are the same as those which would follow from solving

Max u(z)

S.T. $\sum_i \bar{s}^i t^i = T(s^1 + s^2) + w$

$z^1 = \alpha^1 t^1$

and $z^2 = \alpha^2 t^2$

where $\alpha^1 = \left[\frac{s^1}{p} \frac{\alpha}{1-\alpha}\right]^{\alpha}$, and $\alpha^2 = \left[\frac{s^2}{p} \frac{\gamma}{1-\gamma}\right]^{\gamma}$, $\bar{s}^1 = \frac{s^1}{1-\alpha}$ and $\bar{s}^2 = \frac{s^2}{1-\gamma}$.

v) Examine optimal t^1/t^2 and explain the nature of its dependence on α, β, γ and s^1/s^2.

vi) Show that the elasticities of optimal t^i, x^i and z^i with respect to changes in w are all less than unity, while the same elasticities with respect to "full income" equal unity. What does this say about observed income elasticities?

9. Suppose the subutility v(x) displays constant absolute risk aversion:

$$\forall x, \; -v''(x)/v'(x) = \alpha,$$

for some constant $\alpha > 0$.

i) Show that v(x) can be written as $-e^{-\alpha x}$.

ii) Let consumption be the random variable $\tilde{X}$, and assume $\tilde{X} \sim N(\mu,\sigma^2)$. Suppose two situations are compared, where the difference between them is the magnitude of σ^2: $\sigma_1^2 > \sigma_0^2$. Show that increasing σ^2 from σ_0^2 to σ_1^2 is a Rothschild–Stiglitz mean–preserving spread.

(iii) For $\tilde{X}$ as in (ii), what is $E[v(\tilde{X})]$? Check that $\frac{\partial}{\partial\sigma^2}E[v(\tilde{X})] < 0$.

iv) Show that the "risk premium" k, defined by

$$v(\mu-k) = E[v(\tilde{x})],$$

is equal to $\frac{\alpha\sigma^2}{2}$.

10. Suppose that at each date $t \in [0,T]$, there is just one commodity, x(t). The utility associated with any consumption path is

$$\int_0^T v[x(t)]e^{-\rho t}dt,$$

where $\rho > 0$ is a constant, and

$$v[x(t)] \equiv \frac{x(t)^{1-\xi}}{1-\xi}$$

where $\xi > 0$ a constant. ξ=1 implies v[ ] = ℓn[ ]. Let r be the instantaneous rate of interest, and w an exogenous income at each date.

i) Explain why the budget constraint is

$$\int_0^T x(t)e^{-rt}dt = \int_0^T we^{-rt}dt.$$

ii) Find the utility maximizing consumption path $x^*(t)$, and explain how it varies with changes in w and ρ.

11. i) Suppose there is a single output y_1, and that $y_1 = g^1(y_2,...,y_l)$ can be represented by the function (α_h and β are exogenous parameters)

$$y_1 = \left\{ \sum_{h=2}^{l} a_h(-y_h)^{-\beta} \right\}^{-1/\beta}, \beta > -1, \sum_{h=2}^{l} \alpha_h = 1.$$

Define $\sigma = 1/(1+\beta)$. What is σ? Why?

ii) What are the marginal products, in terms of y_1, of $y_2,...,y_l$?

iii) For a given level of y_1, what is the input share

$$s_h = p_h y_h / \sum_{k=2}^{l} p_k y_k \qquad h = 2,...,l$$

in terms of marginal products.

iv) Show that the relative input shares s_h/s_k(h, k = 2,...,l) vary with relative input prices in a particular way, depending on σ.

v) Suppose we are interested in evaluating the statement that "relative input prices affect relative input shares". Strictly speaking, in the case analyzed in parts (ii) – (v), is this statement a valid claim?

vi) The α_h and β are exogenous. This means their values are determined outside the model. Suppose Mother Nature determines these values from an atomless distribution $\nu(\beta,\alpha_2,...,\alpha_l)$, with support being the closure of

$$\Gamma = \{(x_1,...,x_l) \in \mathbb{R}^l | x_1 > -1; 0 \le x_h \le 1 \text{ for } h = 2,..,l; \text{ and } \sum_{h=2}^{l} x_h = 1\}.$$

What kind of claim can you make about the statement in (vi)?

vii) Generating restrictions requires that conditions be placed on exogenous entities. (vii) illustrated one method of doing so. Would you find the behaviour ascribed to Mother Nature et al. in (vii) a reasonable model? How do you decide when a model is reasonable, and how might the one determining the α_h and β in (vii) be improved?

12. Suppose the demand side of an economy comprises n consumers, all of whom behave so as to satisfy GARP. Let x^{ij} be the vector of commodities demanded by consumer j when the price vector is p^i; i = 1,...,M indexes observations. Let $x^i = \sum_{j=1}^{n} x^{ij}$.

i) Suppose it is possible to renumber observations so that for all consumers x^{ij} is revealled preferred to $x^{i+1,j}$. Show that the market will behave so as to appear to satisfy GARP at the aggregate level:

$$p^{i\prime}x^i \geq p^{i\prime}x^k \Rightarrow \text{not } (p^{k\prime}x^i < p^{k\prime}x^k);\ i, k = 1,\ldots,M.$$

ii) Show that in the absence of the restrictions in part (i), even if all consumers behave so as to satisfy GARP, the aggregate consumptions need not do so.

iii) Explain what (i) and (ii) have to do with the existence of a representative consumer?

13. Consider the competitive firm using one input (x) to produce one output (q) according to the production function $q = x^{1/2}$. Output is sold at price p, and x purchased at price r. The firm faces a fixed cost of F.

(a) What is the firm's variable cost function? Total cost function? Call the latter $C(\cdot)$.

(b) What is the firm's supply curve?

(c) Assume demand is zero elastic at quantity Q. What is the equilibrium price (p^*), quantity of output per firm (q^*), and number of firms (N^*)?

(d) The aggregate cost of production is $N \cdot C$. Set up the problem of finding the N and q which minimize aggregate cost given that total output is Q. Call these $\bar{N}$ and $\bar{q}$. Show that $\bar{N} = N^*$ and $\bar{q} = q^*$. Also show that the Lagrange multiplier (on the constraint that Q be produced) equals p^*. Explain these conclusions.

(e) Does the result in (d) depend on the inelastic demand side?

(f) Let market demand for Q be $Q = D(p)$, with $D(p)$ twice continuously differentiable and $D' < 0$. Can the equilibrium outcomes p^*, q^*, N^* be derived by assuming the market solves a programming problem? If so, display the problem and show that it does what is required.

14. Suppose work involves accidents. To simplify, assume a "standard accident" involving no "pain and suffering" such that if it occurs, the worker loses a fraction k of his pay. All workers are homogeneous, but jobs are not: some jobs, although otherwise identical have larger probabilities of accidents than others, i.e., some kinds of work are inherently riskier than others (all have the same value k though). Index jobs by p, where p is the probability of an accident, defined over the interval [0,1]. Since jobs are different, and assuming all jobs get done in equilibrium, they each pay a different wage summarized by a function w(p). Assume workers are risk averse and utility depends only on consumption of a composite good. Each worker applies for the job offering the "optimal" value of p for him.

(a) Suppose no accident insurance is available. Derive conditions characterizing equilibrium job choice (be sure to state what, if any, restrictions this imposes on w(p)).

(b) Now suppose a system of workman's compensation is instituted, in which the worker buys actuarially fair insurance at net (i.e. the premium is paid only if there is no accident) premium $\pi = p/1-p$ per dollar purchased. Now characterize equilibrium job choice. What restrictions does this institution place on observed values of w(p)? What is special about the relationship between π and p?

(c) The restrictions on w(p) obtained in case (a) are rather different than in case (b). What is the fundamental economic reason for this? Also, what change in the assumptions for case (b) would make its implied restrictions similar to case (a)? Why?

15. Ideas lead to new ideas. Denote time by $t \in \{0,1,...\}$ and suppose that there is a list of ideas $\{\theta_0,\theta_1,...\}$ such that θ_{t+1} may be discovered if and only if θ_t has; θ_0 is primitive and given. Suppose that learning depends fundamentally on heterogeneity in knowledge as follows. There is a continuum of agents at each t. Let $s_t \in [0,1]$ be the fraction of the population knowing θ_t; thus $1 - s_t$ know other ideas in $\{\theta_0,...,\theta_{t-1}\}$. The fraction knowing θ_{t+1} at t+1 is

$$s_{t+1} = \lambda s_t(1 - s_t) \qquad (*)$$

(Thinking of s_t as a binomial random variable $s_t(1 - s_t)$ is its variance; hence the interpretation in terms of heterogeneity of knowledge.) $s_0 \in (0,1)$ is a given constant and $\lambda \in (0,4)$ ensures $s_{t+1} \in (0,1)$.

i) A general version of (*) is

$$s_{t+1} = f(s),$$

with f continuous. For $k \in \{1,2,...\}$ define the "iterate" function $f_k(s)$ by

$$f_k(s) \equiv f[f_{k-1}(s)] \qquad k = 2,3,...$$

and

$$f_1(s) \equiv f(s).$$

Show that a) if s^* is such that $f(s^*) = s^*$, $f_k(s^*)/=s^*$ for any k; b) if f is differentiable and if s^* is such that $f(s^*) = s^*$, then

$$f'_k(s^*) = f'(s^*)^k$$

holds for any k; and c) there is an s^* satisfying $s^* = f(s^*)$.

ii) The sequence $\{s_t\}_0^\infty$ is said to converge to s^* (written $s_t \rightarrow s^*$) if for any $\varepsilon > 0$ there exists T such that $t \geq T$ implies $|s_t - s^*| < \varepsilon$. Using (*) show that $\lambda \in (0,3)$ implies that $s_t \rightarrow s^*$ with s^* being the positive solution to $s^* = f(s^*)$, if there is one.

iii) Verify that for $\lambda \in (3,3.3)$, the equation $s^* = f_2(s^*)$ has exactly three solutions, say s_0^*, s_1^* and s_2^*, with $s_0^* < s_1^* < s_2^*$. Then, prove that $s_0 \neq s_1^*$ implies a) $s_t \longrightarrow s_1^*$ for any s_0; and b) the subsequences $\{s_0, s_2, s_4,...\}$ and $\{s_1,s_3,...\}$ converge to distinct limits, either s_0^* or s_2^*.

iv) Assume that $\{s_t\}_0^\infty$ may be obtained from

$$s_t = f_t(s_0) \qquad t \geq 1 \text{ so given,}$$

where f_t is f_k (above) with k = t. a) Let $d_t \equiv \partial s_t/\partial s_0$; $d_0 \equiv 1$. Can you say anything about $\{d_t\}_0^\infty$? For example, is $d_{t+1} \leq d_t$? Illustrate your answer by computing $\{s_t\}_0^{50}$, varying s_0 from $s_0 = .010$ to $s_0 = .011$ and computing d_t as the difference in the computed s_t divided by .001. Do this calculation for $\lambda = 2$, 3.25 and 3.98.

16. The people of Arva (called Arvans), having little else to do, knit socks. As anyone who has been there knows, the Arvans are distributed uniformly along a line of finite length.* Describe this neighborhood by the uniform density on the unit interval [0,1], so each person has an address a, $a \in [0,1]$. The people of Arva are a bit unsociable, and so knit by themselves at home. Moreover, they are creatures of habit, so each knits t(a) hours per day, taken to be exogenous and differentiable. But Arvans living closer to London are distracted by its many delights, and so knit less: $t'(a) < 0$. A firm offers (and we assume that Arvans accept) to supply each person living at a with wool, w(a), for all a. The firm buys wool at price p_w. In return the firm receives the socks, which it sells at Galleria for price p. All Arvans produce socks S according to the technology $S - g(w,t) = 0$, where this technology is twice continuously differentiable and has all the other properties usually given to Y (except w and t are measured positive here).

i) For any <u>given</u> allocation of wool by address (the function w(a)) how many socks are produced and what is the firm's profit?

ii) What is the condition determining the profit maximizing allocation w(a)?

iii) What condition ensures $w(a) > 0$ for all $a \in [0,1]$?

iv) What determines whether those who live close to London get more or less wool?

v) How does a change in p_w affect the optimal w(a), total production of socks, and profits of the firm?

vi) Let $g(w,t) = Aw^{\alpha}t^{\beta}$; $A > 0$, $\alpha \in (0,1)$ and $\alpha + \beta < 1$. Show that w(a) is predicted to be a log linear function of t(a) and p_w. What simple data generation process would make running a regression using data on wool allocation (by address), the price of wool, and time spent knitting an appropriate method for testing this theory?

*If you have not been there, little will be missed by assuming Arva to be isomorphic to Ottawa.

17. Some jobs are more pleasant because they are cleaner than others. Individuals have utility functions given by

$$U(C,D) = C - \rho D$$

where C is consumption of goods (off the job) and D is the amount of dirt encountered on the job. Individuals differ in their distaste for dirt and ρ is distributed uniformly on [0,1]. (Each individual's ρ is known to all.) Workers have no nonlabor income, all jobs involve identical effort and all workers work (which is why effort etc. is ignored in U). Firms, on the other hand, find dirt useful. Each firm hires only one worker and produces output with value 1 in a dirty environment; that is, one with $D = 1$. Each firm can, alternatively, provide a clean environment, that is set $D = 0$, but in this case gets output worth only α. Firms differ in the costs of clean–up and this difference is described by α being distributed uniformly on [0,1].

Suppose finally that w^d is the wage paid for a worker in a dirty environment and w^c is that paid in a clean environment. There are N workers and N firms.

(a) What determines each worker's choice of job?

(b) Given $w^d - w^c$, how many workers choose a dirty job?

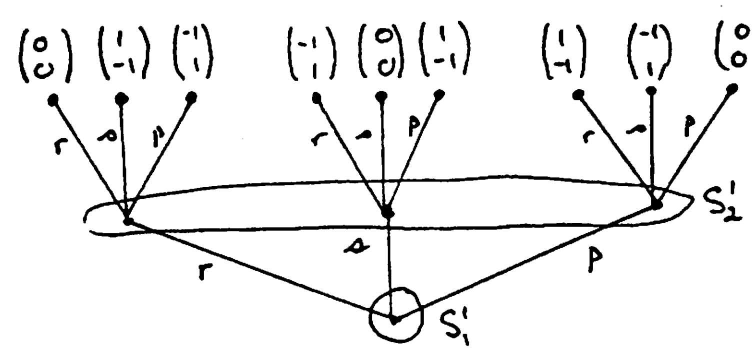
(c) What determines whether a given firm cleans up or not?

(d) Given $w^d - w^c$, how many firms do not clean up?

(e) Find the equilibrium $w^d - w^c$ and the equilibrium number of dirty jobs.

(f) What individuals end up in the dirty jobs and to which firms are they assigned? How does utility vary with ρ and how do profits vary with α? Explain.

18. Cor.sider the Rock–Scissors–Paper game depicted below.



(a) What is the set of feasible (pure) strategies, Σ_i for $i \in \{1,2\}$. Let σ_i^j be the j^{th} element of Σ_i.

(b) Let μ_i be a mixed strategy for player i, with $\mu_{ij} \equiv \text{Prob}\ \{\sigma_i^j\}$. What is $\pi_i(\mu_1,\mu_2)$ for i = 1,2.

(c) Why is there no equilibrium 2–tuple (μ_1^*,μ_2^*) with both $\mu_{1j}^* = 1$ and $\mu_{2j'}^* = 1$ for some j and j′?

(d) Show that there is no equilibrium in which any $\mu_{ij}^* = 1$ or 0. That is $\mu_{ij}^* \in (0,1)\ \forall\ i,j$ must hold.

(e) If μ_1^* is a best reply to μ_2^*, and μ_1^* is completely mixed, μ_1^* must be an <u>interior</u> solution to

$$\max_{\mu_1} \pi_1(\mu_1,\mu_2^*) \text{ subject to } \sum_j \mu_{1j} = 1.$$

An analogous requirement exists for μ_2^*.

Set up the appropriate Lagrangeans and obtain the first order conditions. Interpret them in terms of player i's payoff to following strategy σ_i^j.

(f) Show that the conditions obtained in (e) have an unique solution. Explain.

19. Consider the following situation. A firm has a product which it offers for sale to two customers. The product costs \$c (fixed) per unit to produce. The firm knows that one consumer would be willing to pay at most \$u for a single unit of the good, and the other would be willing to pay at most \$v, with u > v > c > 0. Assume prices, u, v and c are all integer multiples of 1c, price above u+1c are not allowed, and that no consumer will ever buy more than one unit under any circumstances. The firm does <u>not</u> know which consumer is willing to pay u, and which would pay only v. Trade proceeds as follows. The firm makes consumers an offer. This offer consists of two prices, at <u>each</u> of which the firm will sell exactly one or zero units. (If the firm wishes to sell 2 units at the same price, p, its offer is (p,p).) Given this offer, consumers either buy or leave the market, and the game ends. The firm is interested in maximizing profit (equal to 0 if no sales occur). Consumers maximize net "utility": u–p or v–p if p is the price paid. (Consumers may pay different prices here.) No purchase yields zero utility, and each consumer does not directly observe the other's purchase.

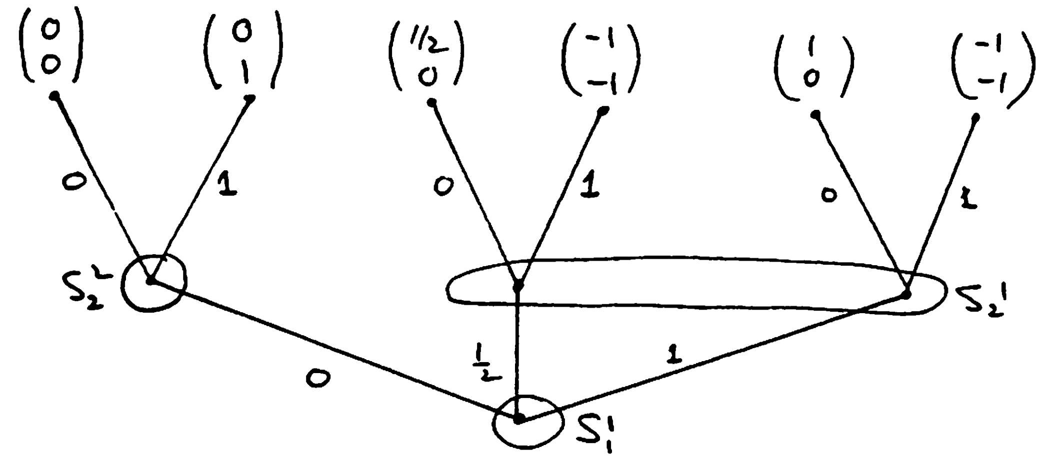
(a) Set this situation up as a 3–person game in extensive form. (For purposes of displaying the same tree, let u, v and c be small numbers, say u=3, v=2, c=1.)

(b) Show that the firm's offering (u,v) and both consumers purchasing if and only if they can do so for a price not greater than their willingness to pay, is an equilibrium of this game. What are the equilibrium payoffs?

(c) Show that the offer (c,c) with consumers purchasing if and only if they can do so at price c or less is also an equilibrium. What are the equilibrium payoffs?

(d) Why does the firm fare so poorly in part (c) despite the fact that it does the offering, and consumer's best alternative is valued at 0?

20. Consider the following game



i) Find all the equilibrium (NE) behavioural strategy vectors μ^*.

ii) Find all the Subgame Perfect Equilibria (SPE). Display an NE that is not SPE. Explain.

iii) Find all the sequential equilibrium assessments (μ^*, ρ).

21. Two teams, A and B are engaged in a "best 2 of 3" tournament. The winner gets $P > 0$ and the loser earns 0. In any game the outcome depends on team effort λ_i ($i = A,B$) as follows: $\Pr(A \text{ wins } |\lambda_A, \lambda_B) = \Phi(\lambda_A - \lambda_B)$ with $\Phi(x) + \Phi(-x) = 1$, $\Phi' > 0$ and $\Phi'' < 0$ for $x > 0$ ($x \equiv \lambda_A - \lambda_B$). The cost of effort to any team, in any game, is $c(\lambda_i)$ with $c(0) = 0$, $c'(0) = 0$, $c' > 0$ for $\lambda_i \neq 0$, $c'' > 0$. Assume teams are risk neutral.

i) Show that there is an unique symmetric SPE.

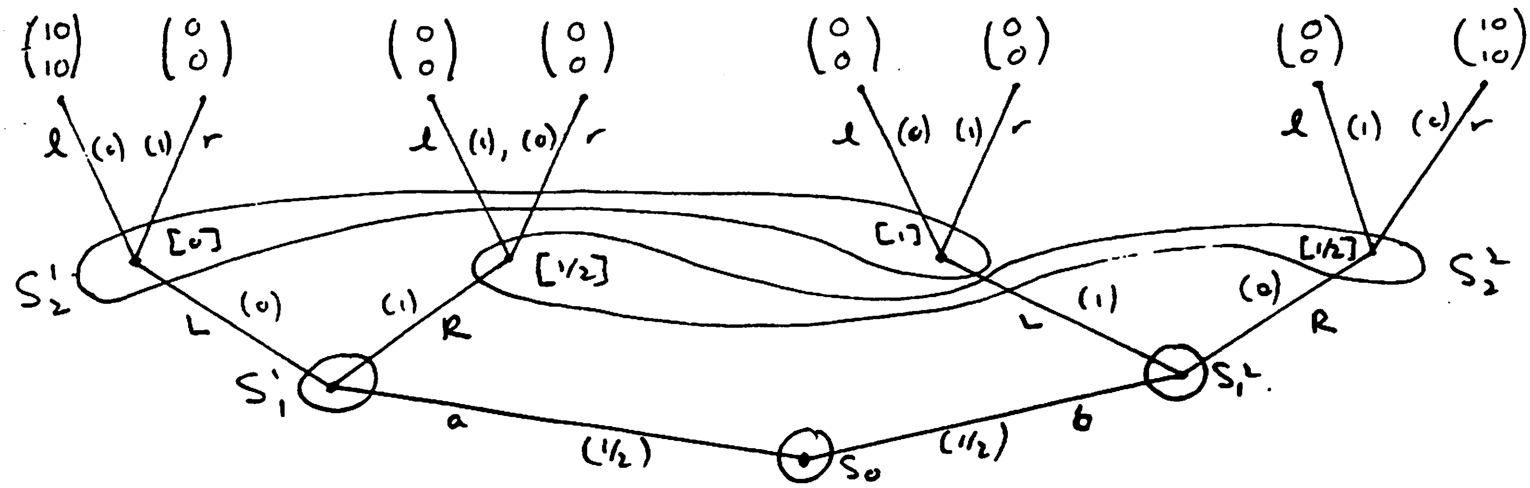
ii) Prove that in this SPE winning and losing are equally likely for team A in the first and third (if it occurs) games, but in the second, team A is more (less) likely to win if it won (lost) the first game. Explain.

22. In the game displayed below, the belief system is displayed in square brackets, and the behavioural strategies are in round brackets. The vertices immediately following the initial vertex are chosen with equal probability.

i) Verify that the assessment depicted is a sequential equilibrium. What payoffs occur with positive probability in equilibrium?

ii) If nature chooses the left vertex, it looks as if player 1 should earn a higher payoff than he in fact does. Why does he do so poorly?

iii) Show that the SE displayed is not a PSE, and display a PSE.



23. Consider a duopoly. Consumers are passive, their behavior being summarized by the demand function $Q = 1-p$ where Q is aggregate quantity produced and p is its price. The active players in the game are firms. Firm i ($i = 1,2$) has constant marginal cost, normalized to zero, and no other costs. Firm i may choose a price π_i, which must be chosen from the set $P = \{1/N, 2/N, ..., (N-1)/N, 1\}$, where $2 < N < \infty$. Let $p_k = k/N$ be the k^{th} element of P; $k = 1,...,N$. The firm charging the lowest price sells to the whole market. If both choose the same price, the quantity demanded at that price is divided between the firms equally.

i) What are the firms' strategy sets and payoff functions?

ii) Show that when firms announce prices simultaneously and noncooperatively, there is exactly one equilibrium, and display it.

iii) Suppose firm 1 must announce and commit to his price before firm 2 does. What are the equilibrium prices? Compare them to those you derived in (ii) and explain.

iv) Suppose that, along with the changes made in part (iii), firm 2's costs remain as specified but firm 1's constant marginal cost is equal to 1/N. Does this change alter your answer to (iii)?

24. Two agents seek to divide a pie of size 1. If they arrive at an agreement, they obtain the agreed upon shares; otherwise, both receive 0. The bargainers take turns making proposals. At date t=0 player A suggests he receives a share $s_0 \in [0,1]$. Player B immediately replies "OK" or "No". If he says "OK" the game ends, player A gets s_0 and player B obtains $1-s_0$. Otherwise, at date t=1, player B makes a proposal $1-s_1$, to which player A immediately responds "OK" or "No". If "OK" player A obtains s_1 and B gets $1-s_1$. And so on, for t=2,...

Player A's payoff is his share multiplied by δ_A^t where t is the date of agreement; similarly for player B, with δ_B^t multiplying his share: $\delta_A, \delta_B \in (0,1)$. A strategy for each player specifies, for each date t and history of previous proposals and replies (i.e. under perfect information "history" can replace "information set"), some proposal or reply, depending on whether the player is offering or responding to t.

i) Show that any division of the pie can be the NE outcome of this game.

ii) Now consider SPE. Let M be the largest share (not payoff $M\delta_A^t$) player A could obtain in any SPE of the game. Consider the subgame beginning at t=2. Why is the largest share player A could obtain in any SPE of that subgame also equal to M?

iii) Consider player B's offer at t=1. Why must player B receive a share of at least $1-\delta_A M$ in the SPE of any subgame beginning at t=1?

iv) Consider player A's offer at t=0. Why must player A receive a share at most equal to $1-\delta_B[1-\delta_A M]$, and why does $M = 1-\delta_B[1-\delta_A M]$ hold?

v) Modify the above argument ((ii) – (iv)) to show that M is also the least player A can obtain.

vi) Show that the strategy in which player A demands M whenever it is his turn to offer, and accepts any offer at least as large as M whenever it is his turn to accept or reject, with the corresponding strategy for B, is a SPE of this game. Is there any other SPE?

vii) Solve for M. Why does player 1's payoff rise with an increase in his discount factor δ_A?