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# Bargaining with Interdependent Values

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**Abstract**

A seller and a buyer bargain over the terms of trade for an object. The seller receives a perfect signal determining the value of the object to both players, while the buyer remains uninformed. We analyze the infinite horizon bargaining game in which the buyer makes all the offers. When the static incentive constraints permit first-best efficiency, then under some regularity conditions the outcome of the sequential bargaining game becomes arbitrarily efficient as bargaining frictions vanish. When the static incentive constraints preclude first-best efficiency, the limiting bargaining outcome is not second-best efficient, and may even perform worse than the outcome from the one-period bargaining game. With frequent buyer offers, the outcome is then characterized by recurring bursts of high probability of agreement, followed by long periods of delay in which the probability of agreement is negligible.

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# 1 Introduction

One of the most vexing problems in economics is why rational parties have such a difficult time reaching mutually beneficial agreements. Even a casual glance at the evidence shows that bargaining inefficiencies abound. These inefficiencies take on many forms : failure to reach agreement when gains of trade exist (e.g., lawsuits that go to trial), delays in reaching agreement (e.g., labor disputes such as strikes or work slowdowns (Cramton and Tracy, 1992)), the build-up of significant expenses in brokering an agreement (e.g., lawyer fees), and settling on contractual terms that fail to fully realize all gains from trade.[1] Ever since Hicks (1932), economists have wondered why the bargaining parties do not simply avoid such inefficiencies by settling immediately at the terms they expect to eventually arrive at.

While some other theories have been advanced,[2] the most popular explanation for the existence of bargaining inefficiencies is that the parties lack information about an aspect critical to reaching agreement. If this information is privately held, active negotiation may be necessary in order to reveal the range of agreements that are acceptable to all parties. For example, when approaching a seller of a piece of real estate, a potential buyer may not know the minimal offer the seller would be willing to accept. The buyer can of course estimate the seller's reservation value, but may fail in his negotiations unless he makes an offer that even a seller with a high reservation value would accept. Such an offer is generally not optimal from the buyer's viewpoint, so some delay will necessarily result. From the viewpoint of the literature on incomplete information bargaining, this delay is not only a necessary evil, but acts as a useful device by which the parties can credibly convey the strength of their bargaining positions. A seller who repeatedly rejects offers and leaves his house on the market for a long time can credibly signal to the buyer that he is not eager to sell.

Unfortunately, this elegant explanation for bargaining inefficiencies has come under recent attack from the literature on the Coase Conjecture (Fudenberg, Levine and Tirole (1985), Gul, Sonnenschein and Wilson (1986)). The Coase Conjecture literature studies the simplest asymmetric information bargaining problem, in which a seller with known valuation makes repeated price offers

[1] Indeed, the screening literature explains distortions in contractual terms as an attempt to minimize informational rents.

[2] For example, Fernandez and Glazer (1991) develop a complete information bargaining model in which, following rejection of management's wage offer, workers can either strike or continue to work at the existing wage. Because management is not indifferent between these responses, there exists a multiplicity of equilibria involving immediate settlement, but differing in the agreed upon wage. This multiplicity in turn permits equilibria involving delay in agreement.

for the sale of a single unit of an indivisible asset to a buyer whose valuation for the asset is private information. According to the Coase Conjecture, if there is no restraint on the rate at which the seller can make price offers, then in the limit as the length of the time period between successive offers vanishes, the seller will offer the good for sale at the lowest possible buyer valuation.[3] In other words, bargaining inefficiencies can be explained only by exogenous limitations on the rate at which offers can be revised. While there are undoubtedly some practical limitations on the speed at which parties can formulate and interpret offers, these limitations are unlikely to be of a magnitude sufficient to explain significant bargaining failures, such as protracted strikes or the huge costs associated with major corporate lawsuits.[4] As a consequence, the Coase Conjecture questions the usefulness of asymmetric information as an explanation for bargaining inefficiencies.

If we are to retain asymmetric information as a foundation for the theory of bargaining, there appear to be two possible avenues. First, we can question the validity of the Coase Conjecture. One line of argument here maintains that concerns for reputation may guide player's behaviors. Papers in this vein are Ausubel and Deneckere (1989), who abandon the stationarity assumption driving the Coase Conjecture, and Myerson (1991, pp. 399-402) and Abreu and Gul (2000), who develop a psychological theory of bargaining. Another line of argument questions the validity of backward induction, either on the basis of experimental evidence (Neelin, Sonnenschein and Spiegel (1988)), or on theoretical grounds (Binmore (1988), Reny (1993), Rosenthal (1981), Samet (1996)). Secondly, we can question the appropriateness of the one-sided incomplete information private values model. For example, Myerson and Satterthwaite (1983) show that in the two-sided incomplete information private values model inefficiency necessarily occurs in any sequential equilibrium of any bargaining game, provided the supports of the distribution of buyer and seller type are not separated. Unfortunately, extensive models with two-sided incomplete information are very hard to analyze, and at present very little is known about their outcomes.

In the current paper, we instead investigate the consequences of allowing interdependencies in player's valuations, but retain the relative simplicity and elegance of the one-sided incomplete information model. We believe this is a fruitful area for investigation because many real world bargaining problems involve such interdependencies. For example, in lawsuits involving the health

[3]More precisely, this statement holds if the lowest possible buyer valuation strictly exceeds the seller's valuation, and a mild technical is satisfied (Analogous to assumptions (1) and (2) in the body of the current paper). The Coase Conjecture also holds without these assumptions, provided attention is restricted to stationary equilibria.

[4]For a particularly striking example of the latter, see Cutler and Summers (1988).

hazards of a manufacturer's product, or the environmental consequence of a production method, the manufacturer may have private information regarding the safety of his product or the risks associated with his production method that is relevant to the welfare of potential victims. Similarly, when negotiating the sale of an oil tract, the buyer may possess survey information regarding the richness of the underlying deposit that is relevant to the owner's willingness to sell. And in wage bargaining, the worker may have superior knowledge about his level of human capital or productivity. This level not only affects the worker's value to his current employer, but also his value to alternative employers, and hence his reservation value of staying with his current employment.[5]

We consider an environment in which there is a single seller bargaining over the terms at which to trade a single unit of an indivisible good. The seller receives a signal $q \in [0,1]$ determining his reservation value $c$. The signal also affects the buyer's valuation $v$, but the buyer is uninformed about the realization of the signal. A prototypical example of this situation arises in the market for used cars, where the seller may have information regarding the reliability of the car that is relevant to the buyer, but not easily verifiable. Ever since the pioneering work of Akerlof (1970), economists have been aware that such an environment can generate trading inefficiencies. Indeed, if trade is to be efficient, the buyer's expected value from trading must exceed the reservation value of the seller of the most reliable car, for otherwise there exists no price at which both are willing to trade. Assuming the seller's cost to be increasing in $q$, inefficiencies will therefore necessarily occur whenever this condition is violated, i.e. $E(v) < c(1)$.

We study the welfare performance of the infinite horizon bargaining game in which the uninformed party (the buyer) makes all the offers. The literature on the Coase Conjecture analyzes a special case of this model, in which the buyer's valuation does not depend upon the seller's signal. Note that in this so-called *private values* case, there always exists a single price mechanism that is both feasible and efficient (any price between the highest seller valuation and the buyer's valuation is a competitive equilibrium price). This observation raises a number of interesting questions. Suppose first that interdependencies in valuations are not too strong, so that the static incentive constraints still permit an efficient outcome to be attained, i.e. $E(v) \geq c(1)$. Do the same forces that lie behind the Coase Conjecture in the private values case then cause the outcome of sequential

[5] Alternatively, if the firm has superior information about the value of the worker's productivity, and the worker can capture some of this value in alternative employment situations, the reservation values of both players will be positively related.

bargaining to become efficient as bargaining frictions disappear? Next, suppose that the basic incentive constraints are such that every equilibrium outcome of every bargaining game must exhibit some inefficiency, i.e. $E(v) < c(1)$. Then where does the logic of the Coase Conjecture break down, and how does the limiting delay manifest itself? Can we characterize the limiting delay schedule? Does the uninformed party's incentive to accelerate trade still operate so as to select a limiting outcome that is second-best efficient?

Our paper brings both good and bad news. We prove that when the static incentive constraints permit first-best efficiency, and $v(\cdot)$ is increasing, then as in the private values case, the outcome of the sequential bargaining game becomes arbitrarily efficient when bargaining frictions are allowed to vanish. At the same time, we also show that whenever the static incentive constraints preclude first-best efficiency, the Coase Conjecture forces select a limiting outcome that does not maximize the expected gains from trade. Finally, we show that when the informed party can make frequent offers, the bargaining outcome is characterized by recurring bursts of high probability of agreement, followed by long periods of delay in which the probability of agreement is negligible.

The two papers most closely related to the present one are Evans (1989) and Vincent (1989). Evans considers a two-type example in which the buyer and seller differ only in their valuation for the high quality car, and studies the impact of relative discount factors on the bargaining outcome.[6] Vincent allows much more general interdependencies in valuations, and introduces an assumption guaranteeing existence of a Bayesian equilibrium. He also provides a two-type example demonstrating the possibility of limiting delay.[7] Neither of these papers, however, provides a characterization of the limiting bargaining outcome, or delineates necessary and sufficient conditions for delay to be present. They also do not explain how and why the Coase Conjecture forces operate during the bargaining process, or determine the key factors influencing the length of delay.

The remainder of the paper proceeds as follows. Section 2 presents the model and explains the notion of stationary equilibrium. Section 3 presents a simple two-type example that provides intuition for our main results. Section 4 proves general existence of stationary equilibrium and, under some (mild) regularity conditions, uniqueness of the supporting stationary triplet. Section

[6]Unfortunately, with equal discount factors Evans' model becomes rather degenerate : when the fraction of high quality cars falls below a critical threshold, every incentive compatible trading mechanism necessarily generates no surplus. Meanwhile, when the threshold fraction is exceeded, a single take-it-or-leave it offer already leads to an efficient outcome.

[7]However, Vincent's example satisfies the condition $E(v) < c(1)$, so that every Nash equilibrium of every bargaining game necessarily exhibits inefficiency.

5 provides a general characterization of the limiting equilibrium outcome as the discount factor between successive buyer offers approaches 1. Section 6 proves that this outcome is not second-best efficient when $E(v) < c(1)$, and briefly presents the analogous model in which the buyer is the informed party, with the seller making the offers. Section 7 concludes. All proofs are relegated to three appendices, unless otherwise noted.

# 2 The Model

A buyer and a seller bargain over the terms at which to trade a single unit of an indivisible good. The value of the good to each trader is determined by the realization of a random variable $q \in [0,1]$. More precisely, the signal $q$ respectively determines buyer and seller valuations through the functions $v(\cdot)$ and $c(\cdot)$ :

$$b = v(q) \quad s = c(q)$$

The functions $v(\cdot)$ and $c(\cdot)$ are required to be bounded and measurable.

We assume that one of the traders, the seller, is informed about the realization of the signal, while his bargaining partner, the buyer, only knows the distribution of the signal.[8] We say that the model has *private values* if $v(q)$ is constant, and that the model has *interdependent values*, otherwise. We will be primarily interested in the interdependent values case, but allow private values as a special case.

Because the functions $v(\cdot)$ and $c(\cdot)$ are general, we may without loss of generality assume that the distribution of the signal is uniform. If necessary, we then reorder the signals so that the function $c(\cdot)$ is increasing in $q$.[9] Note, however, that we do not similarly restrict the function $v(\cdot)$. We impose the regularity condition that $v(\cdot)$ and $c(\cdot)$ are left-continuous functions, that are right-continuous at $q = 0$. We also make an assumption of economic significance, namely that it is common knowledge amongst traders that the gains from trade are bounded away from zero :

**Assumption 1** *There exists $\Delta > 0$ such that $v(q) - c(q) \geq \Delta$ for all $q \in [0,1]$.*

[8] See Section 6 for the analogous model in which the seller is the uninformed party.

[9] More precisely, given any bounded measurable function $c' : [0,1] \rightarrow \mathbb{R}_+$, there always exists a measure preserving bijection $\phi$ on $[0,1]$ such that $c(q) = c'(\phi(q))$ is increasing in $q$.

Assumption 1 implies that the extreme form of inefficiency described by Akerlof (1970) never occurs.[10] However, unlike in the private values case, the existence of a "gap" (Assumption 1) no longer guarantees that first best efficiency is attainable. Specifically, we have :

**Lemma 1** *First best efficient trade is possible iff* $E(v(q)) = \int_0^1 v(q)dq \geq c(1)$

*Proof* : First best efficiency requires that all seller types $q$ trade with probability one. This implies that the expected transfer must be independent of the seller's type (otherwise any seller type would want to mimic the type that receives the highest expected transfer). Denoting this transfer by $t$, seller individual rationality for type $q = 1$ requires that $t \geq c(1)$. Since the buyer's expected utility from participating in the mechanism equals $E(v(q)) - t$, buyer individual rationality then implies $E(v(q)) \geq t \geq c(1)$. *Q.E.D.*

The bargaining protocol we wish to analyze in this paper is the infinite horizon bargaining game in which the uninformed party makes all the offers. In this game, there are an infinite number of time periods, indexed by $n = 0, 1, 2, ....$ In each period $n$ in which bargaining has not yet concluded, the buyer starts by offering the seller a price $p \in \mathbb{R}_+$ at which trade is to occur. Upon observing this offer, the seller can accept, in which case trade occurs at the proposed price and the game ends, or the seller can reject, in which case play moves to the next period. Note that each terminal node of the game can be identified with a pair $(p, n)$. We assume that the traders are impatient and discount surplus at the common rate $r > 0$. Let $\zeta$ be the length of the time interval between two successive buyer offers, and $\delta = e^{-r\zeta}$ the (common) discount factor. Then the terminal payoffs at node $(p, n)$ are $\delta^n(v(q) - p)$, for the buyer, and $\delta^n(p - c(q))$, for the seller.

In every period $n$, the information set of the buyer can be identified with a history of rejected offers, $(p_0, p_1, ..., p_{n-1})$. A pure behavioral strategy for the buyer therefore specifies, in every period $n$, her current offer as a function of the $n$-history of rejected prices. Similarly, in every period $n$, the information set of the seller can be identified with the same history concatenated with the current offer, $(p_0, p_1, ..., p_{n-1}, p_n)$. Let $A$ denote acceptance of an offer, and $R$ denote rejection of an offer. A pure behavioral strategy for the seller specifies for each period $n$ a decision in the set $\{A, R\}$, as a function of his type $q$, and as a function of the history $(p_0, p_1, ..., p_{n-1}, p_n)$.

[10]Assumption 1 implies that there always exists a feasible mechanim in which trade occurs with positive probability. Indeed, consider the mechanism in which all seller types in $[0, \varepsilon]$ trade at the price $c(\varepsilon)$, and types $q > \varepsilon$ do not trade. The buyer's expected utility in this mechanism equals $E[v(q) - c(\varepsilon)|q \leq \varepsilon]\varepsilon$. Since the first term in this expression converges to $v(0) - c(0) \geq \Delta > 0$, there exists $\varepsilon$ sufficiently small for which the above mechanism is incentive compatible and individually rational.

We are interested in the stationary equilibria of this bargaining game.[11] Formally, a *stationary equilibrium* is a sequential equilibrium in which the seller's acceptance decision is based only upon the current offer, and not on any other detail of the prior history. Thus, there exists a nondecreasing (left-continuous) function $P(q)$, such that seller type $q$ accepts the offer $p_n$ in period $n$ if and only if $p_n \geq P(q)$. Consequently, following any history (with no simultaneous seller deviations), the buyer's belief will always be a (left) truncation of the prior, i.e. a uniform distribution on an interval of the form $[q_n, 1]$. Furthermore, since in his acceptance decision the seller ignores all but the current offer, when the buyer formulates her offer the prior history of the game will not matter, except in so far as it is reflected in the cutoff level $q_n$. The cutoff level $q_n$ therefore acts as a state variable, so that stationary equilibria are Markovian.

In stationary equilibria, the acceptance function $P(\cdot)$ acts as a "static" supply curve to the buyer, who faces a tradeoff between screening more finely and delaying agreement. Let $G_q(z)$ denote the buyer's belief when the state is $q$ (the uniform distribution on $[q, 1]$), and let $g_q(z)$ denote the corresponding density. Also let $W(q)$ denote the buyer's maximized expected payoff when the state is $q$. The buyer's tradeoff is then captured by the dynamic programming equation :

$$W(q) = \max_{q' \geq q} \left\{ \int_q^{q'} (v(z) - P(q'))g_q(z)dz + \delta(1 - G_q(q'))W(q') \right\}. \tag{1}$$

To understand (1), observe that if the current state is $q$ and the buyer offers $P(q')$, thereby bringing the state to $q'$, all seller types in the interval $[q, q']$ accept.[12] Conditional on the offer being accepted, the buyer's net payoff from transacting with seller type $z \in [q, q']$ is $v(z) - P(q')$; the likelihood of this happening is $g_q(z) = 1/(1 - q)$. Integrating over all possible seller types in $[q, q']$ then yields the first term in (1). Rejection happens with probability $(1 - G_q(q'))$, moves the state to $q'$, and results in the seller receiving the expected payoff $W(q')$ with a one-period delay. Letting

[11] The reason for our interest in stationary equilibria is that in the private values case, the literature has established an intimate connection between stationarity of the informed party's acceptance behavior and the Coase Conjecture (Gul, Sonnenschein and Wilson, 1986). Furthermore, as we shall demonstrate in Section 4, under the assumption of a "gap", as far as equilibrium outcomes is concerned, there is no loss of generality in restricting attention to stationary equilibrium outcomes.

[12] Strictly speaking, this reasoning is only correct if $P(q)$ is strictly increasing in $q$ (as will be the case when $c(\cdot)$ is strictly increasing in $q$, see equation (3) below). If $P(\cdot)$ has a flat segment, and $q'$ is not the endpoint of this segment, then by charging $P(q')$ the buyer induces more acceptances than indicated in (1). However, in this case it is straightforward to show that the maximum in (1) is never attained on the interior of the flat segment (the buyer always prefers to induce the largest state consistent with the offer $P(q')$). The extra freedom allowed in (1), by letting the seller select the state rather than the price, is therefore without consequence.

$R(q) = (1-q)W(q)$ denote the buyer's *ex-ante* expected payoff from trading with seller types in the interval $[q, 1]$, equation (1) can be simplified to :

$$R(q) = \max_{q' \geq q} \left\{ \int_q^{q'} (v(z) - P(q'))dz + \delta R(q') \right\}. \tag{2}$$

Let $\Upsilon(q)$ denote the argmax correspondence in (2). By the Generalized Theorem of the Maximum (Ausubel and Deneckere, 1993) $\Upsilon$ is a nonempty- and compact-valued upper hemicontinuous correspondence, and the value function $R(\cdot)$ is continuous. Since the objective function in (2) has increasing differences in $(q', q)$, $\Upsilon$ is a nondecreasing correspondence, and hence single-valued at all but at most a countable set of $q$.

In equilibrium, the seller's acceptance decision must be optimal given the buyer's offer behavior, as described by (2). To see the implications of this requirement, define $t(q) = \min \Upsilon(q)$; then we must have[13] :

$$P(q) - c(q) = \delta(P(t(q)) - c(q)). \tag{3}$$

In other words, seller type $q$ is indifferent between accepting the price $P(q)$ and waiting one period for the (higher) offer $P(t(q))$.

The triplet $\{P(\cdot), R(\cdot), t(\cdot)\}$ determines a stationary equilibrium path in the following way. In the initial period, the buyer selects (possibly randomly) an offer $P(q)$, for some $q \in \Upsilon(0)$. Following this offer, all seller types in the interval $[0, q]$ accept, and all seller types in $(q, 1]$ reject. Since it is necessarily the case that $q > 0$, Equation (3) implies that following rejection of the offer $P(q)$ the seller must necessarily come back with the offer $P(t(q))$, even if $\Upsilon(q)$ is not single-valued. While the buyer may thus randomize in her initial offers, subsequent buyer offers are uniquely determined. Following the offer $P(t(q))$, all seller types in the interval $(q, t(q)]$ accept, and all seller types in the interval $(t(q), 1]$ reject. This process then continues : in case of rejection, the buyer raises her offer to $P(t^2(q))$, inducing all seller types in the interval $(t(q), t^2(q)]$ to accept, and so on, until the state

[13] Note that $t(q)$ is continuous at any point $q$ where $T(q)$ is single-valued. Now consider any point $q \in [0, 1]$ at which the functions $c(\cdot), P(\cdot)$ and $P(t(\cdot))$ are continuous; since all of these functions are increasing, this excludes at most a countable number of $q$. For any nonexcluded $q$, if the buyer induces the state $q$ by offering $P(q)$, then since $T$ is single-valued at $q$, the seller will necessarily offer $P(t(q))$ in the next period. For seller type $q$ to be willing to accept $P(q)$ it must therefore be the case that $P(q) - c(q) \geq \delta(P(t(q)) - c(q))$. But if we had strict inequality, and the seller offered a price slightly below $P(q)$, seller type $q$ would still strictly prefer to accept, contradicting the fact that $P(q)$ is a reservation price. Consequently, for any non-excluded point $q$ equation (3) must hold. Now if $q$ is an excluded point and $q > 0$ then there exists a sequence $\{q_n\}$ converging from below to $q$. Since each of the functions $c(\cdot), P(\cdot)$ and $t(\cdot)$ is left-continuous, it follows that (3) must in fact hold for all $q > 0$.

$q = 1$ is reached.[14]

## 3 A Two-Type Example

In this Section, we present a simple two-type example to provide intuition for how and when the Coase Conjecture forces operate to produce equilibrium limiting delay, when the discount factor converges to one. The example allows us to derive an explicit closed form solution for the equilibrium, thereby avoiding many of the technical intricacies present in the general model. Suppose the seller's cost and the buyer's valuation function are respectively given by:

$$c(q) = \begin{cases} 0 & \text{for } q \in [0, \hat{q}] \\ s & \text{for } q \in (\hat{q}, 1] \end{cases} \qquad v(q) = \begin{cases} \alpha & \text{for } q \in [0, \hat{q}] \\ s + \beta & \text{for } q \in (\hat{q}, 1] \end{cases}, \tag{4}$$

where $\alpha, \beta$, and $s$ are strictly positive (see Figure 1). As noted above, when the buyer's valuation function $v(\cdot)$ is constant, we obtain the private values model as a special case. In the present example, this translates to the condition that $\alpha = s + \beta$.

We will start by using backward induction to construct the stationary equilibrium, and then use the explicit solution to both analyze the extent of the equilibrium limiting delay, and the economic forces that underlie it. Our derivation proceeds at an intuitive level.[15]

Observe first that the buyer's final equilibrium offer must be equal to the highest possible seller cost, $c(1) = s$. Any lower offer would not be accepted by all remaining seller types, while any offer greater than $c(1)$ would be accepted with probability one, and hence dominated. Suppose now that in equilibrium there are $n$ periods of bargaining remaining before the game concludes. Since seller types with valuation $s$ do not accept until the final round, the offer in the current round, $p_n$, will have to keep seller types with valuation 0 indifferent between accepting in that round and waiting

[14]The triplet $\{P(\cdot), R(\cdot), t(\cdot)\}$ also describes the equilibrium continuation following nonequilibrium buyer offers $p$ : all seller types whose reservation price falls below $p$ accept, and all other types reject. If $q$ is the induced state, and the offer satisfies the equation $p = P(q)$, then following rejection the buyer raises her offer to $P(t(q))$. If $p$ is not in the range of the function $P(\cdot)$, so that we have $p > P(q)$, then following rejection of $p$ the buyer randomizes between the minimum and maximum elements of $P(T(q))$ so as to rationalize type $q$'s acceptance of the previous offer $p$. Note that the latter type of offer will never arise along the equilibrium path, for the buyer could have lowered her offer to $P(q)$, and still have induced the same acceptances.

[15]Deneckere (1992) uses a similar procedure to compute an explicit equilibrium for the two-type independent private values model.

$n$ more periods to receive the final offer $s$, i.e.

$$P(q) = s\delta^n, \text{ for } q \in (q_n, q_{n-1}], \tag{5}$$

where $q_{n-1}$ denotes the highest buyer type whose acceptance price is $p_n$, and where we use the convention that $q_{-1} = 1$ and $q_0 = \hat{q}$. To determine the sequence of cutoff levels $\{q_n\}$, we must consider the buyer's optimization problem. When the state is $q_1$, the buyer must be indifferent between offering $p_0 = s$, which all remaining seller types accept, and offering $p_1 = s\delta$, which all seller types in $(q_1, q_0]$ accept, and returning in the next period with the final offer $p_0$. Letting $m_i = q_{i-1} - q_i$ denote the period $n$ ex-ante probability of agreement, we therefore have :

$$R(q_1) = (\alpha - s\delta)m_1 + \delta\beta m_0 = (\alpha - s)m_1 + \beta m_0,$$

Solving this equation for $m_1$ yields $m_1 = \frac{\beta}{s} m_0$, where $m_0 = (1 - \hat{q})$.

Similarly, when $n > 1$, at the state $q_n$ the buyer must be indifferent between making the offer $p_n$, which will be accepted by all seller types in $(q_n, q_{n-1}]$, and making the next higher offer $p_{n-1}$, which will be accepted by all seller types in $(q_n, q_{n-2}]$, i.e.

$$R(q_n) = (\alpha - s\delta^n)m_n + \delta R(q_{n-1}) = (\alpha - s\delta^{n-1})(m_n + m_{n-1}) + \delta R(q_{n-2}). \tag{6}$$

Solving for $m_n$ from (6) yields $s\delta^{n-1}(1-\delta)m_n = (\alpha - s\delta^{n-1})m_{n-1} + \delta(R(q_{n-2}) - R(q_{n-1}))$. Also, using the middle expression in (6) for $R(q_{n-2})$ and the right-hand expression in (6) for $R(q_{n-1})$, yields $R(q_{n-2}) - R(q_{n-1}) = -(\alpha - s\delta^{n-2})m_{n-1}$. Combining the last two equations then produces a difference equation in $m_n$ :

$$m_n = \frac{\alpha}{s\delta^{n-1}} m_{n-1}. \tag{7}$$

Defining $\rho = \frac{\alpha}{s}$, we may solve this difference equation by forward recursion, using the boundary condition $m_1 = \frac{\beta}{s} m_0$ :

$$m_n = \rho^{n-1}\delta^{-\frac{n(n-1)}{2}} m_1 = \rho^{n-1}\delta^{-\frac{n(n-1)}{2}} \frac{\beta m_0}{s} \tag{8}$$

Let us write $m_n(\delta)$ to explicitly denote the dependence of the solution in (8) on $\delta$, and let $N(\delta) = \min\{n : \sum_{i=1}^{n} m_i(\delta) \geq 1\}$. For simplicity, assume that we are in the generic case where $\sum_{i=1}^{N} m_i(\delta) > 1$. We may then summarize the solution as follows :

**Proposition 1** *Let $v(\cdot)$ and $c(\cdot)$ be given by (4), let $0 < q_{N-1} < ... < q_0 = \hat{q}$ be defined recursively by (7), and let $p_n$ be defined by (5). Then for all $\delta < 1$ the unique stationary triplet is given by :*

$$
\begin{array}{ll}
P(q) = p_n & q \in [0, q_{n-1}],\ \textit{if}\ n = N \\
 & q \in (q_n, q_{n-1}],\ \textit{if}\ n = 0, 1, \cdots, N-1 \\[1ex]
t(q) = q_{n-2} & q \in [0, q_{n-1}],\ \textit{if}\ n = N \\
 & q \in (q_n, q_{n-1}],\ \textit{if}\ n = 2, 3, \ldots, N-1 \\
 & q \in (q_1, 1],\ \textit{if}\ n = 1 \\[1ex]
R(q) = p_{n-1}(q_{n-1} - q) + \delta R(q_{n-2}) & q \in [0, q_{N-1}],\ \textit{if}\ n = N \\
 & q \in (q_n, q_{n-1}],\ \textit{if}\ n = 2, 3, \ldots, N-1 \\
 & q \in (q_n, 1],\ \textit{if}\ n = 1
\end{array}
$$

According to Proposition 1, the buyer starts out by offering $p_{N-1}$, which all seller types in $[0, q_{N-2}]$ accept. Upon rejection, the buyer raises her offer to $p_{N-2}$, which is accepted by all seller types in $(q_{N-2}, q_{N-3}]$, and so on until the state $q_0$ is reached, at which point the seller makes her final offer $p_0 = s$. Bargaining therefore lasts for $N(\delta)$ periods.[16]

We are interested in the behavior of the above solution as $\delta$ converges to 1. To gain some insight into this question, let us first consider the case where $\rho \geq 1$. Note that this case includes the private values model, where $\rho = 1 + \frac{\beta}{s} > 1$. The economic significance of the inequality $\rho \geq 1$ is that it implies $\alpha > s\delta^n = p_n$ for all $n \geq 1$, so that at any point in the game the buyer always expects to earn a positive surplus if her offer $p_n$ is accepted. As we work backwards from the terminal state, the buyer's expected discounted surplus therefore grows, i.e. $R(q_n) - R(q_{n-1}) > 0$. Since the buyer trades off gains from increased price discrimination against delayed receipt of the continuation value, she will therefore become more reluctant to price discriminate as $n$ increases. Thus, the acceptance probability is higher in earlier stages of the bargaining process; formally this is reflected in the fact that $m_n > m_{n-1}$ for all $n > 1$. Note that this inequality immediately implies that the number of bargaining rounds $N(\delta)$ is finite, and uniformly bounded in $\delta$.[17] It follows that the Coase Conjecture holds, for if bargaining can last for at most $N$ rounds, the initial price is no lower than $\delta^{N-1}s$, and hence converges to $s = c(1)$ as $\delta \to 1$. We conclude that when $\rho \geq 1$ the solution behaves qualitatively exactly like in the private values case.

When $\rho < 1$, however, the equilibrium takes on a different character from the private values case. Indeed, with $\rho < 1$, it is always the case that when $\delta$ is sufficiently close to 1 there exists an

[16] In the nongeneric case where $\sum_{i=1}^{N} m_i(\delta) = 1$, the buyer may randomize between the initial offers $p_N$ and $p_{N-1}$, so that bargaining can last for one additional period.

[17] Indeed, $m_1 = \frac{\beta}{s}(1 - \hat{q})$ does not depend on $\delta$.

initial range of integers $n \geq 1$ for which $\alpha < s\delta^n$. For such $n$ the buyer expects to earn a negative surplus when the seller accepts her offer $p_n$. Working backwards from the state $q_0$, the buyer's expected profits are decreasing in $n$, as long as the inequality $\alpha < s\delta^{n-1}$ continues to hold. Since the buyer trades off gains from increased price discrimination against delayed receipt of the continuation value, she will therefore become more eager to price discriminate as $n$ increases. Formally, this is reflected in the fact that the ex-ante acceptance probability $m_n$ is decreasing in $n$. Importantly, observe that when $\delta$ approaches 1, the number of time periods over which the inequality $\alpha < s\delta^{n-1}$ holds increases without bound as the discount factor approaches 1. Consequently, unlike in the private values case, real delay may occur, even in the limit, as bargaining frictions are allowed to vanish.

To determine whether or not real delay occurs, let us calculate $a = \sum_{i=1}^{\infty} m_i(1)$. If $a > 1$,[18] we can define $\hat{N} = \min\{n : \sum_{i=1}^{n} m_i(1) \geq 1\}$. Bargaining then lasts no more than $\hat{N}$ periods, regardless of the discount factor $\delta$. As observed above, the Coase Conjecture then holds. This is true despite the fact that when $\rho < 1$, for large $\delta$ the acceptance probability $m_n$ is decreasing in $n$ $(1 < n \leq \hat{N})$. If $a = 1$, the number of bargaining rounds is finite for any $\delta < 1$, but increases without bound as $\delta \to 1$. Nevertheless, as Proposition 2 below shows, the Coase Conjecture holds for this case as well. Some simple computations show that the condition $a \geq 1$ is equivalent to the condition $E(v) \geq c(1)$.[19] We conclude that in the two-type model the Coase Conjecture holds if and only if the static incentive constraints permit an efficient outcome (see Lemma (1)).

When $a < 1$, then in the limit the backward construction "gets stuck" at the quantity $q^* = 1 - a$. The reason for this is straightforward. By the definition of $a$, for any $q > q^*$ there exists an $n < \infty$ such that for all $\delta < 1$ we have $\sum_{i=1}^{n} m_i(\delta) \geq 1 - q$. For any such state, it will take no more than $n$ periods before the buyer makes her final offer, independently of the discount factor $\delta$. Thus for any state $q > q^*$, the Coase Conjecture applies, yielding the buyer a limiting expected surplus $\widetilde{R}(q) = E[v(z) - c(1)|z \geq q]$. Now consider Figure 2 : the buyer makes an expected loss of $(s - \alpha)$ on every trade in $(q^*, \hat{q})$, and an expected gain of $\beta$ on every trade in $(\hat{q}, 1)$. The point $q^*$ is such that the buyer's loss on the interval $(q^*, \hat{q})$ is equal to the profit on the interval $(\hat{q}, 1)$. In other words, the buyer's limiting expected revenue is equal to zero at $q^*$, $\widetilde{R}(q^*) = 0$. Consequently, the buyer's incentive to accelerate trade vanishes at $q^*$.

[18] Note that when $\rho \geq 1$, we have $a = \infty$, so that in this case the inequality $a > 1$ always holds.

[19] Indeed, for $\rho < 1$ we have $a = m_0(1 + \frac{\beta}{s}\frac{1}{1-\rho}) = (1 - \hat{q})(1 + \frac{\beta}{s-\alpha})$. The inequality $a \geq 1$ is therefore equivalent to the condition $E(v) = \alpha\hat{q} + (\beta + s)(1 - \hat{q}) \geq s$.

In fact, real delay must necessarily occur at the state $q^*$. The buyer must expect to make a profit from trading with seller types $q < q^*$, i.e. trade at a price $p \le \alpha$. Those types must prefer accepting $p$ immediately to waiting to receive the final offer $c(1) = s$, necessitating real delay. One of the main breakthroughs in this paper is to figure out exactly how much delay there must be at the state $q^*$. To this end, define $\tau$ such that a seller with valuation 0 is indifferent between accepting the offer $\alpha$ and waiting a length of time $\tau$ to receive the offer $s$, i.e. $\alpha = e^{-r\tau}s$. Proposition 2 below shows that the limiting delay must equal $2\tau$. The reason for the doubling of $\tau$ is the symmetry of the situation. This symmetry can be most easily seen if for each $n$ we select $\delta$ such that $\rho\delta^{-n} = 1$, so that $p_n = \alpha$. It then follows from equation (7) that $m_{n+1} = m_n$, $m_{n+2} = \rho\delta^{-(n+1)}m_{n+1} = \delta^{-1}m_{n+1} = \delta^{-1}m_n = m_{n-1}$, etc. We conclude that it takes as much time to go from $q_n - \varepsilon$ to $q_n$ as it does to go from $q_n$ to $q_n + \varepsilon$. Observe that $\lim_{\delta\to 1} q_n = q^*$,[20] and hence that the limiting real delay to move from $q_n$ to $q_n + \varepsilon$ is equal to $\tau$. It follows that the limiting acceptance price of type $q^*$ must be $p = e^{-2r\tau}s = \rho^2 s$.

To complete the description of the limiting outcome when $a < 1$, observe that for $q < q^*$ the buyer's limiting continuation surplus is strictly positive (it is no lower than $\alpha - \rho^2 s$). As a consequence, the buyer has an incentive to accelerate trade, and the Coase Conjecture again applies : for any $\varepsilon > 0$ there exists $n < \infty$ such that regardless of the discount factor it takes no more than $n$ steps for the buyer to trade with type $q$. We therefore have :[21]

**Proposition 2** *The Coase Conjecture obtains iff $a \ge 1$. When $a < 1$, then as $\delta$ converges to 1, all seller types in $[0,\, 1-a)$ trade immediately at the price $s\rho^2$, and all types in $(1-a,\, 1]$ trade at the price $s$ after a delay of length $T$ discounted such that $e^{-rT} = \rho^2$, where $\rho = \frac{\alpha}{s}$.*

The two-type example shows us that it is the possibility of ex-post buyer regret (i.e., the buyer's expectation to earn a negative surplus should her offer be accepted) that slows down the bargaining. This slowdown may or may not be sufficiently strong to produce limiting delay. Whether the Coase Conjecture holds, and thus whether there is no limiting delay, depends on whether the condition $E(v) \ge c(1)$ is satisfied. In the latter case, the buyer's expected continuation surplus remains bounded away from zero for all states $q > 0$, so the incentive to speed up receipt of this continuation value dominates the buyer's incentive to price discriminate. When $E(v) < c(1)$ the Coase Conjecture forces still operate at all values of the state where the buyer's expected

[20]Since the acceptance price of $q_n$ is $\alpha$, we must have $q_n \le q^*$. But if the $\lim_{\delta\to 1} q_n$ were less than $q^*$, then by the definition of $q^*$ the buyer would earn negative expected surplus, a contradiction.

[21]Theorem 3 in Section 5 contains Proposition 2 as a special case, so we do not provide a formal proof here.

continuation value remains bounded away from zero. However, there exists a state $q^* = 1 - a$ for which the limiting continuation value converges to zero. Near this state, the incentive to price discriminate remains strong as the discount factor converges to one, allowing real delay to occur. As a consequence, when $\delta$ is near one, the bargaining outcome is characterized by two short time periods during which there is a high probability of agreement, interspersed with a long period of delay in which the probability of agreement is negligible. The length of the delay is increasing in $c(1) - c(q^*)$, decreasing in $v(q^*) - c(q^*)$, but does not depend on $v(1)$.

Intuitively, it would appear that many of these properties do not depend on there being just two types. However, there might now be more than one location in which there is limiting delay. It is also unclear whether the length of the limiting delay can still be determined for general cost and valuation functions, and whether it will depend more intricately on the global structure of those functions (rather than just $c(q^*)$, $v(q^*)$ and $c(1)$). We address these questions in the next two sections.

# 4 Existence and Uniqueness

For the special case of private values, Gul, Sonnenschein and Wilson (1986) demonstrate that there exists a unique stationary triplet, and that all sequential equilibrium outcomes are the outcomes of some stationary equilibrium, provided that Assumptions 1 holds and the seller's cost function satisfies a Lipschitz condition at $q = 1$:

**Assumption 2** *There exists $L < \infty$ such that $c(1) - c(q) \leq L(1 - q)$ for all $q \in [0, 1]$.*

Theorem 1 below generalizes the Gul, Sonnenschein and Wilson result to the case of interdependent values. The key step in establishing uniqueness of the stationary triplet is to show that there exists a critical value of the state $q_1 < 1$, such that in any sequential equilibrium, whenever the state exceeds $q_1$, the buyer must make an offer that all remaining seller types will accept (see Lemmas A-1 and A-2 in Appendix A). This uniquely pins down a stationary triplet $(R, t, P)$ on the interval $[q_1, 1]$. We then use backward induction on the state, employing the functional equations (2) and (3), to successively extend the triplet $(R, t, P)$ to the entire interval $[0, 1]$.

Our construction differs from the one in Gul, Sonnenschein and Wilson (1986) in two crucial ways. First, our extension is maximal in the sense that we construct a decreasing sequence of cutoff levels $\{q_n\}$ with the property that for each $n > 1$ there exists no state less than $q_{n+1}$ for which the buyer selects an offer acceptable to seller types in the interval $[q_n, 1]$. We use this property extensively in the proof of our characterization theorem in Section 5 (Theorem 3). Secondly, in the private values case, the buyer's expected surplus $R(q)$ is decreasing in $q$. Assumption 1 then guarantees that $R(q) > 0$ for all $q \in [0,1]$. This property allows the extension process to terminate in a finite number of steps (for a precise argument, see Ausubel and Deneckere, 1989, Lemma 2). When valuations are positively related, however, $R(q)$ may be increasing in $q$ over some interval, so it is quite conceivable that the extension procedure might never terminate. Our proof makes essential use the maximality of the extension to establish that this cannot happen : bargaining always ends in a finite number of rounds.[22]

**Theorem 1** *Suppose that Assumptions 1 and 2 hold. Then for any $\delta < 1$ there exists a unique stationary triplet $(R(q), t(q), P(q))$ on $q \in [0,1]$, and every sequential equilibrium outcome is the outcome of some stationary equilibrium. Furthermore, there exists $N(\delta) < \infty$ such that bargaining concludes with probability one in $N(\delta)$ periods.*

While Theorem 1 guarantees that bargaining will end in a finite number of periods, there is a big difference with the private values case. When values are private, the number of bargaining rounds remains bounded above as the discount factor approaches 1 (see Deneckere, 1992). With interdependent values, this property cannot generally hold. Otherwise, the Coase Conjecture would always apply, and according to Lemma 1 this is impossible whenever $E(v(q)) < c(1)$. Under the latter assumption, the number of bargaining rounds must necessarily increase without bound as the discount factor approaches 1.

Our second result establishes existence of stationary equilibrium under extremely weak conditions. We drop Assumption 2, and replace Assumption 1 with the much weaker condition :

**Assumption 3** *$v(q) \geq c(q)$, for every $q \in [0,1]$.*

Our technique of proof consists of approximating $v(\cdot)$ and $c(\cdot)$ by functions that satisfy Assumptions 1 and 2, and arguing that an appropriate chosen limit of the stationary equilibria of the approximating games is a stationary equilibrium of the limit game. This generalizes Ausubel and Deneckere (1989, Theorem 4.2) to the case of interdependent values.

**Theorem 2** *Suppose Assumption 3 holds. Then there exists a stationary equilibrium.*

[22] Vincent (1989, Theorem 1) adapts Gul, Sonnenschein and Wilson's (1986) arguments to establish existence under Assumptions 1 and 2, but his proof fails to demonstrate that the extension does not get stuck at some state $q > 0$. Our proof also dispenses with Vincent's requirement that $v(\cdot)$ be nondecreasing.

## 5 Characterizing the limiting equilibrium path

In this section, we will use the results of Section 4 to characterize the limiting delay whenever the buyer's and seller's valuation function are neoclassical,[23] i.e. :

**Assumption 4** *$v(\cdot)$ and $c(\cdot)$ are step functions, each having at most a finite number of steps.*

To describe the limiting revenue and acceptance functions, let us set $q_0^* = 1$, $p_0^* = c(1)$, and iteratively define

$$q_n^* = \max\left\{ q \in [0,\, q_{n-1}^*) : \int_q^{q_{n-1}^*} (v(z) - p_{n-1}^*)\mathrm{d}z \le 0 \right\}, \tag{9}$$

whenever the set in (9) is nonempty, and $q_n^* = 0$ otherwise, and

$$p_n^* = c(q_n^*) + \frac{(v(q_n^*) - c(q_n^*))^2}{p_{n-1}^* - c(q_n^*)}, \tag{10}$$

ending the process whenever $q_n^*$ reaches 0. Note that Assumptions 1 and 4 guarantee that this happens in a finite number of steps. Let us denote this number by $K$. In order to simplify the proofs, we also make the following nondegeneracy assumption.

**Assumption 5** *The functions $v(\cdot)$ and $c(\cdot)$ are continuous at $q_n^*$, for all $n \in \{1, ..., K\}$.*

For each $n \in \{1, ..., K-1\}$ let us also define $T_n$ as the solution to $e^{-rT_n} = \rho_n^2$, where

$$\rho_n = (v(q_n^*) - c(q_n^*)) \,/\, (p_{n-1}^* - c(q_n^*)). \tag{11}$$

The intuition behind the above construction is analogous to the intuition for the limiting solution in the two-step example of Section 3. Let $\tilde{R}(q)$ and $\tilde{P}(q)$ respectively denote the buyer's expected revenue and the seller's acceptance function, in the limit as the length of the time period between successive offers converges to zero (our proofs below show that these limits are well defined, with the possible exception of $\tilde{P}(q_n^*)$, which is defined by making $\tilde{P}$ left-continuous). Then we have the following generalization of Proposition 2:

**Theorem 3** *Suppose Assumptions 1, 4 and 5 hold. Let $q_n^*$, $p_n^*$ and $T_n$ be given by (9), (10) and (11), respectively. Then in the limit, as $\delta \to 1$, the seller's acceptance function and the buyer's*

[23] Note that arbitrary valuation functions can be arbitrarily closely approximated by neoclassical ones.

*revenue function converge to :*

$$\tilde{P}(q) = p_n^* \qquad q \in [0,\, q_n^*],\ if\ n = K-1$$
$$\qquad q \in (q_{n+1}^*,\, q_n^*],\ if\ n = 0, ..., K-2$$

$$\tilde{R}(q) = \int_q^{q_n^*} (v(z) - p_n^*)dz \qquad q \in [0,\, q_n^*],\ if\ n = K-1$$
$$\qquad q \in (q_{n+1}^*,\, q_n^*],\ if\ n = 0, ..., K-2$$

*Furthermore, the buyer successively offers* $\{p_{K-1}^*, ..., p_0^*\}$*, but delays trade for a length of time* $T_n$ *between the offers* $p_n^*$ *and* $p_{n-1}^*$*.*

To interpret Theorem 3, observe that when the state $q$ satisfies $q > q_1^*$, the buyer's expected revenue is strictly positive. The Coase Conjecture forces thus cause her to accelerate trade, and in the limit propose the concluding offer $c(1)$ instantaneously. As in the two type case, this also explains why the seller's acceptance function over the interval $(q_1^*, 1]$ is equal to $c(1)$. At $q = q_1^*$ the buyer's revenue becomes zero, and there no longer is any incentive to accelerate trade. As a consequence, there is real delay at $q = q_1^*$. The length of this delay, $T_1$, is determined so that the seller of type $q_1^*$ is indifferent between accepting the price $p_1^*$ and waiting for a length of time $T_1$ to receive the price $p_0^* = c(1)$, i.e. $p_1^* - c(q_1^*) = \rho_1^2(c(1) - c(q_1^*))$. The equation for $p_1^*$ is derived in an analogous fashion to the two type case.

Over the interval $(q_2^*, q_1^*]$ the buyer's expected revenue is again positive, so again the Coase Conjecture causes her to offer the price $p_1^*$ instantaneously. As a consequence the seller's acceptance function over the interval $(q_2^*, q_1^*]$ is equal to $p_1^*$, and so on.

For the special case where $K \leq 2$, the proof of Theorem 3 follows from Corollary 1 and Theorem 3′, below. The proof for the general case consists of a finite number of repetitions of the arguments for $K = 2$, because the acceptance price $p_n^*$ takes on an analogous role over the interval $(q_{n-1}^*, q_n^*]$ to that of the acceptance price $p_0^* = c(1)$ over the interval $(q_1^*, 1]$.

Suppose then that $K \leq 2$, so that either $q_1^* = 0$ or $q_2^* = 0$. Define

$$\tilde{q} = \inf\{q : \lim_{\delta \to 1} P(q) = c(1)\}. \tag{12}$$

Note that by Lemma A-2 we have $P(q) = c(1)$, for all $q \geq q_1$, regardless of the length of time between periods, so $\tilde{q}$ is well defined. Our next lemma shows that $\tilde{q} = q_1^*$, i.e. that the buyer's limiting revenue is equal to $\tilde{R}(q)$ for all $q \geq q_1^*$, and that the seller's limiting acceptance function is equal to $p_0^* = c(1)$ on the interval $(q_1^*, 1]$. Essentially, the proof of Lemma 2 consist of showing that the Coase Conjecture applies on the interval $(q_1^*, 1]$.

**Lemma 2** *Suppose Assumptions 1, 4 and 5 hold. Then* $\tilde{q} = q_1^*$.[24]

An immediate consequence of Lemma 2 is the following generalization of Proposition 2 :

**Corollary 1** *Suppose Assumptions 1 , 4 and 5 hold. Then the Coase Conjecture holds if and only if* $q_1^* = 0$ .[25]

Indeed, if $q_1^* > 0$ and Assumptions 1, 4 and 5 hold, then by Lemma 2 we necessarily have $\tilde{P}(0) < c(1)$. Thus, $q_1^* = 0$ is necessary for the Coase Conjecture to hold. Now if $q_1^* = 0$, then Lemma 2 implies that $\tilde{P}(q) = c(1)$ for all $q > 0$. What remains to be shown is that there cannot be any delay at $q = 0$, i.e. that $\tilde{P}(0) = c(1)$. This requires a more sophisticated proof, along the lines of the proof of the Coase Conjecture for the private values case when there is no "gap."

Corollary 1 reduces the question of whether or not the limiting bargaining outcome is efficient to the question of whether or not $q_1^* = 0$. Since the condition $q_1^* = 0$ is equivalent to the condition $E[v(z) - c(1)|\ z \geq q] \geq 0$ for all $q$, we may rephrase Corollary 1 as :

**Corollary 2** *Suppose Assumptions 1 , 4 and 5 hold. Then the Coase Conjecture holds if and only if* $E[v(z) - c(1)|\ z \geq q] \geq 0$ *for all* $q$.

Observe that in the two-type case, whenever Assumption 1 holds, the condition $E[v(z) - c(1)|\ z \geq q] \geq 0$ is equivalent to the condition $E(v) \geq c(1)$. As a consequence, we were able to conclude that the Coase Conjecture held if and only if the static incentive constraints admit an efficient outcome. In general, however, it is possible that $E(v) \geq c(1)$ but that $E[v(z) - c(1)|\ z \geq q] = 0$ for some $q \in (0,1)$, so that $q_1^* > 0$. Our next example illustrates this.

**Example** : Modify the example from Section 3 as follows. Pick $\varepsilon > 0$ and $s > 0$, and select $\alpha$ and so that $q_1^* > \varepsilon$. Now redefine $v$ on $[0, \varepsilon]$ so that $v(q) = \gamma$, and select $\gamma$ sufficiently large that $E(v) \geq c(1)$.

In the above example, it is feasible for trade to occur at the price c(1), but because $q_1^* > 0$ the limiting bargaining outcome exhibits real delay (see Corollary 1). In order for there not to be any limiting delay, the condition $E(v) \geq c(1)$ therefore generally needs to be strengthened. However, suppose the function $v(\cdot)$ is nondecreasing and Assumption 1 holds. Then the condition

[24] A careful perusal of the proof of Lemma 2 reveals that Assumptions 4 and 5 may be replaced by the weaker condition that $v(\cdot)$ is nondecreasing in a right neighbourhood of $q_1^*$.

[25] The proof of Corollary 1 actually shows that Assumption 1 and the condition $q_1^* = 0$ are sufficient to imply the Coase Conjecture.

$E(v) \geq c(1)$ is equivalent to the condition of Corollary 2, and hence both necessary and sufficient for the absence of delay.

Our remaining task is to determine the extent of the equilibrium delay when $q_1^* > 0$.[26] As we saw in Section 3, determining this delay is equivalent to finding the limiting acceptance price of type $q_1^*$.

**Theorem 3′** *Suppose that Assumptions 1, 4 and 5 hold, and suppose that $q_1^* > 0$. Then the acceptance price at $q_1^*$ converges to $p_1^*$ as $\delta$ tends to 1.*

The proof of Theorem 3′ is long and hard. The essential ideas are as follows. Assumption 4 implies that $c(\cdot)$ can have at most $M$ jumps ($M < \infty$), and hence that $P(\cdot)$ can have at most $M$ jumps in the interior of any interval $(q_n, q_{n-1}]$. We show that any such jumps can persist for at most an arbitrarily small amount of real time when $\delta$ is sufficiently large. Now because $q_1^* > 0$ there is real delay at $q_1^*$. There will therefore exist a neighborhood of $q_1^*$ such that whenever $\delta$ is sufficiently large there are no inner jumps in $P(\cdot)$. As a consequence, when the state is $q_{n+1}$ the buyer will be indifferent between inducing the state $q_n$ and the state $q_{n+1}$. It is this property which allows us to estimate the a relationship between $m_n$ and $m_{n+1}$, in a fashion analogous to the two-type case (see the derivations leading up to equation (7)), and therefore to calculate the limiting delay at $q_1^*$.

## 6 The efficiency of sequential bargaining

Suppose that the condition of Lemma 1 is violated, so that we necessarily have $q_1^* > 0$. We would like to know whether the limiting solution described in the previous section is then ex-ante efficient, i.e. maximizes the gains from trade over all incentive compatible and individually rational trading outcomes. To see that this cannot generally be the case, let us first consider the two-type example studied in Section 3 (see Proposition 2). Note that in the limiting outcome, all seller types in the interval $[0, \hat{q}]$ have the same valuation, and hence are indifferent between trading at time 0 at the price $s\rho^2$, and trading at time $T$ at the price $s$ (where $e^{-rT} = \rho^2$). However, in the limiting outcome, only types $q$ in the interval $[0, q_1^*]$ (where $q_1^* = 1 - a$) trade at time zero; the remainder trade at time $T$. Social welfare can thus be increased by having all types $q \in (q_1^*, \hat{q}]$ trade at time

[26]Note that whenever $E(v) < c(1)$, it is necessarily the case that $q_1^* > 0$.

zero at the price $s\rho^2$ instead. Indeed, all such seller types are indifferent between these two options, and the buyer pays the same discounted price, but gets to trade earlier.[27]

The two-type example gives a clear and unambiguous answer, but leaves open the possibility that for more general type distributions the limiting bargaining mechanism might sometimes achieve the constrained welfare optimum. We will now slightly alter the argument from the two-type case to demonstrate that this is *never* the case. Indeed, under Assumptions 4 and 5 there exists an $\varepsilon > 0$ such that $c(q_1^* + \varepsilon) = c(q_1^*)$. Just as in the two-type case, we can therefore improve welfare by letting all types in the interval $(q_1^*, q_1^* + \varepsilon)$ trade at the price $p_1^*$ at the same time as type $q_1^*$. Incentive compatibility is maintained, for each of these seller types is indifferent between this option and trading after a delay of length $T_1$ at a price $p_0^* = c(1)$. Hence we have shown :

**Theorem 4** *Suppose Assumptions 1, 4 and 5 hold, and suppose that $q_1^* > 0$. Then the limiting bargaining outcome is not ex-ante efficient, i.e. there exists an incentive compatible and individually rational mechanism that yields higher expected gains from trade.*

Theorem 4 shows that whenever $E(v) < c(1)$, there exist feasible mechanisms that yield higher welfare than in the frictionless bargaining outcome of our model. Consequently, when values are strongly interdependent, many of the lessons we have learned from the private values model may be overturned . As an example of this, we demonstrate below that the relative performance of different bargaining institutions may depend significantly on the degree to which valuations are interdependent.

Ever since Ronald Coase's (1972) famous paper, a central tenet of bargaining theory has been that a player's inability to commit to walking away from the bargaining table may not only seriously undermine her bargaining power, but may also enhance the efficiency of the bargaining outcome. In other words, the welfare distortions are lower when the uninformed party lacks commitment power than when she has perfect commitment power. We claim that when values are interdependent, this conclusion may be reversed. To see this, let us again consider the two-type example studied in Section 3, and let us assume that the fraction of high valuation seller types is sufficiently small that $q_1^* > 0$, i.e. $\hat{q} > \beta/(s + \beta - \alpha)$. Amongst all incentive compatible mechanisms, the one

[27] In the resulting mechanism, the buyer will enjoy strictly positive expected surplus; this means we can increase the probability of trade on the interval $(\hat{q}, 1]$ above $\rho^2$, thereby further increasing welfare. We can maintain incentive compatibility by raising the price paid by seller types in $[0, \hat{q}]$ in such a way as to keep them indifferent between the two options. The ex-ante optimal mechanism obtains when the probability of trade over the interval $[0, \hat{q}]$ cannot be raised any further without making the buyer sustain losses.

most preferred by the buyer is the one in which she gets to make a single take-it-or-leave-it offer (see Samuelson (1984)). Since there are relatively few high valuation seller types, the buyer's optimal take-it-or-leave-it offer is equal to zero. Under perfect commitment power social welfare therefore coincides with the buyer's expected revenue, i.e. equals $\alpha\hat{q}$. At the same time, the limiting bargaining outcome from Section 3 has trade occur immediately with seller types in $[0, q_1^*]$, and with delay discounted to $\rho^2$ with seller types in $(q_1^*, 1]$. In the absence of commitment power, welfare therefore equals $\alpha q_1^* + \rho^2[\alpha(\hat{q} - q_1^*) + \beta(1-\hat{q})] = \alpha\hat{q} - \rho\beta(1-\hat{q}) < \alpha\hat{q}$, i.e. falls short of welfare under perfect commitment! Intuitively, this can happen because with relatively few high valuation types the inefficiencies associated with ordinary monopsony power are smaller than the inefficiencies caused by the Coase Conjecture forces.

Our welfare analysis also applies to the reverse bargaining model in which the buyer is the informed party, and the seller makes all the offers. To see this, let us assume that types are ordered such that $v(\cdot)$ is a non-increasing function, so that it can be interpreted as a demand curve.[28] Assumption 2 then becomes : there exists $L < \infty$ such that $v(q) - v(1) \leq L(1-q)$ for all $q \in [0,1]$. In a stationary equilibrium, the buyer adopts a stationary acceptance strategy, accepting the offer $p$ when his signal is $q$ if and only if $p \leq P(q)$. The seller's value function $R(q)$ must then satisfy a dynamic programming equation analogous to (2):

$$R(q) = \max_{q' \geq q} \left\{ \int_q^{q'} (P(q') - c(z))\mathrm{d}z + \delta R(q') \right\} \tag{13}$$

Let $t(q)$ be the minimum element from the argmax correspondence associated with (13); the buyer's acceptance function must then satisfy the indifference equation :

$$v(q) - P(q) = \delta(v(q) - P(t(q))). \tag{14}$$

By analogy, we may define

$$q_1^* = \max \left\{ q \in [0,1) : \int_q^1 (v(1) - c(z))\mathrm{d}z \leq 0 \right\},$$
$$p_1^* = v\left(q_1^*\right) - \frac{\left(v(q_1^*) - c\left(q_1^*\right)\right)^2}{v(q_1^*) - v\left(1\right)},$$

and similarly for $\{q_n^*, p_n^*\}$ when $n > 1$. With these definitions, all of our results apply immediately. For example, under Assumptions 1, 4 and 5, there is limiting delay if and only if $q_1^* > 0$. In the latter

[28]This model is of independent interest, for it can be interpreted as a model of a durable goods monopoly, in which the seller is subject to learning-by-doing (in case $c(\cdot)$ is decreasing), or sells exhaustible resources (in case $c(\cdot)$ is increasing).

case, there is a delay for a length of time discounted to $\rho^2$, where $\rho = (v(q_1^*) - c(q_1^*))/(v(q_1^*) - v(1))$, and so on.

# 7 Conclusion

In this paper, we considered a bilateral trading situation, in which there is one-sided incomplete information. We analyzed the infinite-horizon bargaining game in which the uninformed party makes all the offers, but departed from the standard model by allowing valuations to be interdependent. We showed that the Coase conjecture forces are still very much operative, but that there may be limiting delay, at states for which the uninformed party's limiting expected payoff vanishes. Under these circumstances, when the discount factor is sufficiently large, bargaining will characterized by short periods with substantial likelihood of agreement, followed by long periods with very low probability of agreement. Such delay may occur even if the static incentive and individual rationality constraints permit an efficient outcome to be obtained, unless additional regularity conditions are imposed. We also demonstrated that when the static incentive constraints do not permit first best efficiency, then from a welfare viewpoint the limiting bargaining outcome displays "excessive delay."

The inability of the bargaining game in which the uninformed party makes all the offers to replicate the ex-ante efficient outcome opens up an interesting avenue for future research. Indeed, other institutions may then yield superior outcomes. This could help explain why parties sometimes resort to other mechanisms, such as arbitration. Even staying strictly within the framework of infinite horizon bargaining, interesting questions arise. For example, it is a well accepted wisdom that in order to promote efficiency in bargaining, the power should go the party that has the private information. Ausubel and Deneckere (1989b) lend some credibility to this belief, by showing that with private values, the bargaining game in which the informed party makes all the offers yields the efficient outcome (even when bargaining frictions are present). With interdependent values, first best efficiency cannot be attained when the static incentive constraints do not permit so, and as we saw above may not be attained when they do so. Thus, it remains an open question whether transferring bargaining power to the party that has superior information generally improves bargaining efficiency.

## Appendix A : Proof of Theorem 1

Consider first the one-period bargaining problem, starting at an arbitrary state $q < 1$. Since the buyer's offer is final, the seller's reservation price then coincides with his cost. Thus the buyer selects $y \in [q, 1]$ to maximize :

$$\pi(y;q) = \int_q^y (v(z) - c(y))\mathrm{d}z$$

Lemma A-1 establishes that if $q$ is sufficiently near 1, then in the one-period bargaining game the buyer will select an offer that all remaining seller types accept. Despite the presence of monopsony power, in this case there are no allocative distortions.

**Lemma A-1** *Suppose that Assumptions 1 and 2 hold. Then there exists $\overline{q} < 1$ such that for all $q > \overline{q}$ the unique maximizer of $\pi(y;q)$ is $y = 1$.*

*Proof* : Let $\varepsilon = \Delta/2$; by left-continuity of $v(\cdot)$ there exists $\overline{q}_1 < 1$ such that $v(q) \geq v(1) - \varepsilon$ for all $q > \overline{q}_1$. Let $\overline{q}_2 = 1 - \Delta/(2L)$, and define $\overline{q} = \max\{\overline{q}_1, \overline{q}_2\}$. Then we have $\pi(y;q) = \int_q^y (v(z) - c(y))\mathrm{d}z$ $\leq \int_q^y (v(z) - c(1) + L(1-y))\mathrm{d}z = \pi(1;q) - \int_y^1 (v(z) - c(1) - L(y-q))\mathrm{d}z$, where the first inequality follows from Assumption 2. Using the fact that $q > \overline{q}_1$ the term under the integrand can in turn be bounded below by $v(1) - \varepsilon - c(1) - L(y-q) \geq \Delta/2 - L(1-q) > 0$. Thus if $y < 1$ we have $\pi(y;q) < \pi(1;q)$, as was to be demonstrated. *Q.E.D.*

The intuition behind Lemma A-1 is straightforward. When the buyer raises output marginally from $y < 1$, the increased acceptance probability raises her expected payoff by at least $\Delta$. The cost associated with raising output is that all seller types who would have accepted previously will now be receiving a higher offer. When $q$ approaches 1, the number of such seller types becomes arbitrarily small, so raising output will be profitable unless the rate at which the offer must be increased is unbounded. Assumption 2 prevents this from happening.

Define $S(q) = \arg\max \pi(y;q)$ and let $q_1 = \inf\{q : 1 \in S(q)\}$; the previous result implies that $q_1 < 1$. Lemma A-2 shows that when the state exceeds $q_1$ and there is more one bargaining period remaining, the buyer will still "clear the market" by making the offer $p = c(1)$. This is not obvious, for in the infinite horizon model the acceptance price of any seller type $q \in [q_1, 1]$ will generally exceed $c(q)$.

**Lemma A-2** *Suppose that Assumptions 1 and 2 hold. Then in every sequential equilibrium, after any history in which it is the buyer's turn to move and the state $q > q_1$, the buyer makes the offer $c(1)$, and all remaining seller types accept.*

*Proof* : Since in equilibrium the seller can never expect an offer $p > c(1)$ (Fudenberg, Levine and Tirole, 1985, Lemma 2), the seller will accept any such price with probability one. This implies that the buyer's (ex-ante) equilibrium continuation payoff is bounded below by $\pi(1;q)$. At the same time, Samuelson (1984) has shown that the optimal static mechanism for the buyer involves a take-it-and-leave-if offer to the seller. Consequently, Lemma A-1 implies that in any sequential equilibrium, after any history in which it is her turn to move, the most the buyer can expect as a continuation payoff is $\pi(1;q)$. Combining both results, we see that the buyer's equilibrium continuation payoff equals $\pi(1;q)$, and is uniquely attained by offering $p = c(1)$, which the seller accepts. *Q.E.D.*

Lemma A-2 yields a unique candidate stationary triplet $(R, t, P)$ on the interval $[q_1, 1]$ :

$$\begin{cases} R(q) &= \max_{y \geq q} \int_q^y (v(z) - c(y))\mathrm{d}z \\ t(q) &= \min \arg\max_{y \geq q} \int_q^y (v(z) - c(y))\mathrm{d}z \\ P(q) &= (1-\delta)c(q) + (1-\delta)\delta c(t(q)) + \delta^2 c(1) \end{cases}, \tag{A-1}$$

This is the easiest to see when $q > q_1$, for then (A-1) reduces to $t(q) = 1$ and $R(q) = \pi(1;q)$, as required by Lemma A-2. Furthermore, if contrary to the equilibrium the seller is offered a price $p < c(1)$, he will expect the buyer to return with the counteroffer $c(1)$ in the next period, and therefore will accept $p$ iff $p \geq (1-\delta)c(q) + \delta c(1)$, as indicated in (A-1). The same argument applies at $q = q_1$ if $t(q_1) = 1$. Meanwhile, if $t(q_1) < 1$, application of equation 3, and using the fact that $t(q_1) > q_1$, yields the stated formula for $P(q_1)$. That (A-1) is a stationary triplet is shown in the next lemma.

**Lemma A-3** *Consider the triplet defined in (A-1). Then* $R(q) = \max_{y \geq q}\{\int_q^y (v(z) - P(y))\,dz + \delta R(y)\} > 0$ *and* $t(q) = \min \arg\max_{y \geq q}\{\int_q^y (v(z) - P(y))\,dz + \delta R(y)\}$, *for all* $q \in [q_1, 1]$.

*Proof* : First, we prove that $R(q)$ as defined in (A-1) satisfies $R(q) > 0$ for all $q \in [q_1, 1]$. Let $q \in [q_1, 1]$, $w = \liminf_{\varepsilon \downarrow 0} v(q+\varepsilon)$, and $c = \lim_{\varepsilon \downarrow 0} c(q+\varepsilon)$. By Assumption 1, we have $w \geq c + \Delta$. Consequently, there exists $\varepsilon > 0$ such that $v(q') > c(q+\varepsilon)$ for all $q' \in (q, q+\varepsilon)$. The definition of $R(\cdot)$ then implies that $R(q) > 0$.

By the definition of $q_1$, we know that $t(q) = 1$ for all $q > q_1$, so that (A-1) yields $P(q) = (1-\delta)c(q) + \delta c(1)$. Furthermore, $R(q) > 0$ implies that we must have $t(q_1) > q_1$. Hence, for all

$q \in [q_1, 1]$, we have

$$\begin{aligned}
&\max_{y\geq q}\left\{\int_q^y (v(z)-P(y))\mathrm{d}z+\delta R(y)\right\}\\
&=\max_{y\geq q}\left\{\int_q^y (v(z)-(1-\delta)c(y)-\delta c(1))\mathrm{d}z+\delta\int_y^1 (v(z)-c(1))\mathrm{d}z\right\}\\
&=\max_{y\geq q}\left\{(1-\delta)\int_q^y (v(z)-c(y))\mathrm{d}z+\delta\int_q^1 (v(z)-c(1))\mathrm{d}z\right\}\\
&=\int_q^1 (v(z)-c(1))\,\mathrm{d}z=R(q).
\end{aligned}$$

The one before last equality follows from the fact that $\max_{y\geq q}\{\int_q^y (v(z)-c(y))\mathrm{d}z\} = \int_q^1 (v(z)-c(1))\mathrm{d}z$. Since $\int_q^1 (v(z)-c(1))\mathrm{d}z$ is constant, we also have $\arg\max_{y\geq q}\{\int_q^y (v(z)-P(y))\mathrm{d}z+\delta R(y)\} = \arg\max_{y\geq q}\{\int_q^y (v(z)-c(y))\mathrm{d}z\} = t(q)$. *Q.E.D.*

***Proof of Theorem 1 :*** Given a stationary triplet on the interval $[q_n, 1]$, we extend it in a unique way to a stationary triplet on a larger interval $[q_{n+1}, 1]$. For $q \in [0,1]$, let $R_1(q)$ be the buyer's profit when constrained to select the state in $[q_n, 1]$, and let $\Upsilon_1$ be the corresponding argmax correspondence:

$$\left\{\begin{array}{lcl} R_1(q) & = & \max_{y\in[q_n,1]}\int_q^y (v(z)-P(y))\mathrm{d}z+\delta R(y)\\ \Upsilon_1(q) & = & \arg\max_{y\in[q_n,1]}\int_q^y (v(z)-P(y))\mathrm{d}z+\delta R(y)\end{array}\right. .$$

Also, extend $P(q)$ to the entire interval $[0,1]$ by setting $P_1(q) = (1-\delta)c(q)+\delta P(t_1(q))$, where $t_1(q) = \min \Upsilon_1(q)$. Next, for $q \in [0, q_n]$ let $R_2(q)$ denote the buyer's profit when constrained to select the state in $[q, q_n]$ (using the extended acceptance function):

$$\left\{\begin{array}{lcl} R_2(q) & = & \max_{y\in[q,q_n]}\int_q^y (v(z)-P_1(y))\mathrm{d}z+\delta R_1(y)\\ \Upsilon_2(q) & = & \arg\max_{y\in[q,q_n]}\int_q^y (v(z)-P_1(y))\mathrm{d}z+\delta R_1(y)\end{array}\right. .$$

Finally, define $q_{n+1} = \max\{q \geq 0 : R_1(q) \leq R_2(q)\}$ whenever the latter set is nonempty, and $q_{n+1} = 0$ otherwise.

We now claim that $q_{n+1} < q_n$. To see this, note that $R_2(q_n) = \delta R_1(q_n) < R_1(q_n)$, and that by the theorem of the maximum $R_1$ and $R_2$ are continuous functions. For $q \in [q_{n+1}, 1]$ define $P(q) = P_1(q), R(q) = R_1(q)$, $t(q) = t_1(q)$ if $q > q_{n+1}$, and $t(q_{n+1}) = t_2(q_{n+1})$. We also claim that $(P, t, R)$ is a stationary triplet on $[q_{n+1}, 1]$. To show this, we need to establish that for any $q \in [q_{n+1}, q_n)$ it is the case that $R_1(q) = \max_{y\in[q,1]}\int_q^y (v(z)-P(y))\mathrm{d}z+\delta R_1(y)$. But if the maximizer $y \in [q, q_n)$, we would have $R_1(q) = R_2(q)$, contradicting the definition of $q_{n+1}$. Consequently, $y \in [q_n, 1]$ and the required equality holds by construction.

Next, we claim that $R(q_{n+1}) > 0$. Let $\bar{t}(q) = \max \Upsilon(q)$, and let $\varepsilon$ be such that $q+\varepsilon \in (q, \bar{t}(q))$. Since $\bar{t}(q)$ is feasible from the state $q+\varepsilon$, we have $R(q+\varepsilon) \geq \int_{q+\varepsilon}^{\bar{t}(q)} (v(z) - P(\bar{t}(q)))\,\mathrm{d}z$ $+\delta R(\bar{t}(q))$. It then follows from $R(q) = \int_q^{\bar{t}(q)} (v(z) - P(\bar{t}(q)))\,\mathrm{d}z + \delta R(\bar{t}(q))$ that

$$R(q+\varepsilon) \geq R(q) - \int_q^{q+\varepsilon} (v(z) - P(\bar{t}(q)))\mathrm{d}z \tag{A-2}$$

Since $q+\varepsilon$ is feasible from $q$, we also have

$$R(q) \geq \int_q^{q+\varepsilon} (v(z) - P(q+\varepsilon))\,\mathrm{d}z + \delta R(q+\varepsilon) \tag{A-3}$$

Plugging (A-2) into (A-3) yields

$$\begin{aligned}(1-\delta)R(q) &\geq \int_q^{q+\epsilon} \Big((1-\delta)v(z) - P(q+\epsilon) + \delta P(\bar{t}(q))\Big)\mathrm{d}z \\ &\geq \int_q^{q+\epsilon} \Big((1-\delta)\Big(v(z) - c(q+\epsilon)\Big) - \delta\Big(P(t(q+\epsilon) - P(\bar{t}(q))\Big)\Big)\,\mathrm{d}z \end{aligned} \tag{A-4}$$

As shown in the proof of Lemma A-3, the first term under the integrand is bounded away from 0 for sufficiently small $\varepsilon$. We will have proven the claim if we can show that $\lim_{\varepsilon\downarrow 0}(P(t(q+\epsilon)-P(\bar{t}(q)) = 0$. Since $\Upsilon(q)$ is a nondecreasing continuous correspondence, we have $\lim_{\varepsilon\downarrow 0} t(q+\epsilon) = \bar{t}(q)$. The above equality can therefore fail only if $P(\cdot)$ has a discontinuity at $\bar{t}(q)$. However, if this is the case, we necessarily have $t(q+\varepsilon) = \bar{t}(q)$ when $\varepsilon$ is sufficiently small. Otherwise the buyer could select $\bar{t}(q)$ when the state is $q+\varepsilon$, thereby lowering the price discontinuously, and increasing his profit.

Finally, we will show that it takes only a finite number of extensions before $q_{n+1}$ reaches 0. Suppose to the contrary that $\lim_{n\to\infty} q_n = q_\infty > 0$. Since $P(q_n) = (1-\delta)\sum_{j=0}^{n}$ $(\delta^{n-j} c(t^{n-j}(q))) + \delta^{n+1} c(1)$, it must be the case that $\lim_{n\to\infty} P(q_n) = c_+(q_\infty)$, where $c_+(q)$ denotes the right hand limit of $c(q)$. By Assumption 1 there exists $n_0 > 0$ such that $\forall n > n_0$ and all $q \in (q_\infty, q_{n_0}]$ we have $0 < v(q) - P(q_{n_0}) < \bar{v}$, where $\bar{v} = \sup\{v(q) : q \in [0,1]\}$. Let $\epsilon = \frac{(1-\delta)R(q_{n_0})}{\bar{v}}$. We will show that $q_n - q_{n+1} \geq \epsilon$ for all $n > n_0$, contradicting the assumption that $\{q_n\}$ converges to $q_\infty$. To see this, observe that

$$\begin{aligned} R(q_{n+1}) &= \int_{q_{n+1}}^{t(q_{n+1})} (v(z) - P(t(q_{n+1})))\mathrm{d}z + \delta R(t(q_{n+1})) \\ &\leq \bar{v}(q_n - q_{n+1}) + \delta R(t(q_{n+1})) \\ &\leq \bar{v}(q_n - q_{n+1}) + \delta R(q_{n+1}) \end{aligned}$$

The last inequality follows because $v(q) - P(q_{n_0}) > 0$ for $q \in [q_\infty, q_{n_0}]$ implies that the function $R(q)$ is decreasing in $q$ for $q \in [q_\infty, q_{n_0}]$. Indeed, if $q' > q$ then $R(q) \geq \int_q^{t(q')} (v(z) - P(t(q')))\,\mathrm{d}z$

$+\delta R(t(q')) = R(q') + \int_q^{q'} (v(z) - P(t(q)))\,\mathrm{d}z > R(q')$. Hence we have,

$$q_n - q_{n+1} \geq \frac{(1-\delta)R(q_{n+1})}{\bar{v}} \geq \frac{(1-\delta)R(q_{n_0})}{\bar{v}} = \epsilon,$$

where the last inequality follows again from the monotonicity of $R(\cdot)$ on $[q_\infty, q_{n_0}]$. *Q.E.D.*

## Appendix B : Proof of Theorem 2

Consider a sequence $\Delta_n > 0$ such that $\Delta_n \downarrow 0$, and a sequence $q_n < 1$ such that $q_n \uparrow 1$. For each $n$, define $v_n(q) = v(q) + \Delta_n$, and $c_n(q) = c(q)$ for $q \leq q_n$, and $c_n(q) = c(q_n) + [c(1) - c(q_n)]\frac{q-q_n}{1-q_n}$, for $q > q_n$. Observe that the game with valuation functions $\{v_n(\cdot), c_n(\cdot)\}$ satisfies the assumptions of Theorem 1, and so has a stationary triplet $\{R_n, P_n, t_n\}$.

Since $[0,1]$ is compact, it follows from Prohorov's Theorem (Billingsley, 1968, p.37) that the sequence $P_n$ has a weakly convergent subsequence (more precisely, Prohorov proves this for the right continuous versions of $P_n$, but the result is obviously true if we take left continuous versions instead). By taking this subsequence, and renumbering indices, we may without loss of generality assume that the original sequence $\{P_n\}$ is weakly convergent. Thus, $P_n$ converges "in distribution" to a left-continuous nondecreasing function $P(q)$, i.e. $P_n(q) \rightarrow P(q)$ at every point $q$ where $P(\cdot)$ is continuous.

Next, we claim that each $R_n$ is Lipschitz continuous, with Lipschitz constant $\bar{v} + c(1) + \Delta_n$. Indeed, we know that $R_n(\cdot)$ is continuous and that its left-hand derivative, $R_n^-(q) = \lim_{\epsilon \downarrow 0} \frac{R_n(q)-R_n(q-\epsilon)}{\epsilon} = -(v_n(q) - P_n(t_n(q)))$, exists and is bounded by $\bar{v}+c(1)+\Delta_n$. Using reasoning analogous to the proof of the mean value theorem, we obtain that for any two values $x_1, x_2$ in $[0,1]$,

$$|R_n(x_1) - R_n(x_2)| \leq (\bar{v} + \Delta_n + c(1))\,|x_1 - x_2|.$$

This in turn implies that $\{R_n\}$ is an equicontinuous family of functions, and hence has a subsequence which converges uniformly to a continuous limit $R$. Again, by taking a further subsequence if necessary, we may assume that the original sequence converges to $R$.

Let $J_n(q) = \max_{q' \geq q}\{\int_q^{q'}(v(z) - P_n(q'))dz + \delta R_n(q')\}$ and $J(q) = \max_{q' \geq q}\{\int_q^{q'}(v(z) - P(q'))dz + \delta R(q')\}$. Because $P_n$ converges "in distribution" to $P$, and $R_n$ converges uniformly to $R$, the hypotheses of the Generalized Theorem of the Maximum (Ausubel and Deneckere, 1989, p. 527) are satisfied. Since for each $n$ we have $J_n = R_n$, it follows that $J(q) = \lim_{n\rightarrow\infty} R_n(q) = R(q)$, i.e. (2) holds.

It remains to be shown that (3) is also satisfied. Consider any $q \in [0,1)$ where $t(\cdot)$, $P(\cdot)$, and $P(t(\cdot))$ are continuous. Since each of these functions is nondecreasing, at most countably many $q$ are excluded. We will first argue that for such $q$ equation (3) must hold. Observe first that since $q$ is a continuity point of $P(\cdot)$, we have $\lim_{n\rightarrow\infty} P_n(q) = P(q)$. Secondly, since $c_n(\cdot)$ converges uniformly to $c(\cdot)$, we have $\lim_{n\rightarrow\infty} c_n(q) = c(q)$. Thirdly, the Generalized Theorem of the Maximum implies that any cluster point of $\{t_n(q)\}$ belongs to $\Upsilon(q)$. Since $t(\cdot)$ is continuous at $q$, $\Upsilon(\cdot)$ is single-valued at $q$, and hence $\lim_{n\to\infty} t_n(q) = t(q)$. Finally, let $p$ be any accumulation point of the sequence $\{P_n(t_n(q))\}$. We claim that $p = P(t(q))$. First, let us show that $p \geq P(t(q))$. To this effect, let $r_k \uparrow t(q)$ and $s_k \downarrow t(q)$ be sequences of continuity points of $P(\cdot)$. Then for all $k$, there exists $N(k)$ such that for all $n \geq N(k)$ we have $t_n(q) \in (r_k, s_k)$. Consequently, $P_n(r_k) \leq P_n(t_n(q)) \leq P_n(s_k)$, for all $n \geq N(k)$. Since $r_k$ and $s_k$ are continuity points of $P(\cdot)$, it follows upon taking limits as $n \to \infty$ that $P(r_k) \leq p \leq P(s_k)$. Exploiting left-continuity of $P(\cdot)$, and again taking limits as $k \to \infty$, we then obtain $P(t(q)) \leq p \leq \lim_{s\downarrow t(q)} P(s)$. Secondly, suppose that contrary to the claim we have $p > P(t(q))$. We will show that for large $n$ this contradicts that $t_n(q)$ is a profit-maximizing choice for the buyer. Indeed, observe that $\int_q^{r_k}(v(z) - P_n(r_k))\mathrm{d}z + \delta R_n(r_k) - R_n(q) = \int_{t_n(q)}^{r_k}[v(z) - P_n(t_n(q))]\mathrm{d}z + [P_n(t_n(q)) - P_n(r_k)][r_k - q] + \delta[R_n(r_k) - R_n(t_n(q))]$. Now letting $n = N(k)$, and letting $k \to \infty$, we see that the above expression converges to $[p - P(t(q))][t(q) - q]$. Next, we claim that $t(q) > q$, showing that for large $k$ the choice $r_k$ dominates the choice $t_{N(k)}(q)$, the desired contradiction. Indeed, suppose to the contrary that $t(q) = q$. Then there must exist $\varepsilon > 0$ such that $t(q+\varepsilon) = t(q)$. For if we have $t(q+\varepsilon) > t(q)$ for all $\varepsilon > 0$ then $P(t(q+\varepsilon)) \geq p > P(t(q))$, contradicting continuity of the function $P(t(\cdot))$ at $q$. Now let $q' \in (q, q+\varepsilon)$ be a continuity point of $t(\cdot)$. Then since $t_n(q') \geq q'$ for every $n$, we would obtain the contradiction $q = t(q') = \lim_{n\to\infty} t_n(q') \geq q'$, so we must have $t(q) > q$. We conclude that $\lim_{n\to\infty} P_n(t_n(q)) = P(t(q))$.

Now observe that since $\{P_n, R_n, t_n\}$ is a stationary triplet, we have for each $n$ :

$$P_n(q) - c_n(q) = \delta(P_n(t_n(q)) - c_n(q)).$$

By taking limits as $n \to \infty$, we see that (3) must hold for all but at most a countable number of $q$. Finally, consider any of the excluded $q \in [0,1]$, and select as sequence of nonexcluded $q_k$ such that $q_k \uparrow q$. Since (3) holds for all $k$, and since the functions $P_n(\cdot), c_n(\cdot)$, and $P_n(t_n(\cdot))$ are all left-continuous, we see upon taking limits as $k \to \infty$ that (3) also holds at $q$. We conclude that $\{P(\cdot), R(\cdot), t(\cdot)\}$ is a stationary triplet for the game with valuations functions $v(\cdot)$ and $c(\cdot)$. *Q.E.D.*

## Appendix C : Characterization of the Limit

**Lemma C-1** *For any $\varepsilon > 0$, there exists a finite integer $N$ and a constant $\eta > 0$ such that $t(q) - q \geq \eta$ for all $q \in (q_1^* + \varepsilon, q_1)$ and $q_N \leq q_1^* + \varepsilon$, for all $\delta < 1$.*

*Proof* : Note that $R(q) - R(t(q)) = \int_q^{t(q)} (v(z) - P(t(z)))\mathrm{d}z$. Consequently, $\int_q^{t(q)} (v(z) - P(t(q)))\mathrm{d}z = R(q) - \delta R(t(q)) = (1-\delta)R(q) + \delta(R(q) - R(t(q)) = (1-\delta)R(q) + \delta \int_q^{t(q)} (v(z) - P(t(z)))\mathrm{d}z$. Combining the outer inequalities then yields :

$$(1-\delta)\int_q^{t(q)} v(z)\mathrm{d}z - \int_q^{t(q)} (P(t(q)) - \delta P(t(z)))\,\mathrm{d}z = (1-\delta)R(q) \tag{C-1}$$

Note that $P(t(q)) - \delta P(t(z)) \geq (1-\delta)c(t(q)) \geq 0$ , $v \leq \bar{v}$ and $R(q) \geq \tilde{R}(q)$. Substituting into (C-1) then yields $t(q) - q \geq \frac{\tilde{R}(q)}{\bar{v}}$. Since $\tilde{R}(q)$ is continuous and strictly positive over $[q_1^* + \varepsilon, 1]$, $d = \min_{q \in [q_1^* + \varepsilon,\, q_1]} \tilde{R}(q)$ exists and is strictly greater than 0. Hence, $t(q) - q$ is uniformly bounded from below by $\eta = \frac{d}{\bar{v}}$. Let $N$ be the smallest integer greater than $\frac{1-(q_1^*+\varepsilon)}{\eta}$; then $q_N \leq q_1^* + \varepsilon$. *Q.E.D.*

***Proof of Lemma 2*** : First, we claim that $\tilde{q} \geq q_1^*$. If $q_1^* = 0$, the inequality is trivial, so suppose that $q_1^* > 0$. Observe that since for all $q > \tilde{q}$ we have $\lim_{\delta \to 1} P(q) = c(1)$, it is the case that $\tilde{R}(q) = \phi(q)$ for all $q > \tilde{q}$, where $\phi(q) = \int_q^1 (v(z) - c(1))\mathrm{d}z$. Next, observe that $v(q_1^*) < c(1)$. Indeed, if we had $v(q_1^*) \geq c(1)$, then since (by Assumptions 4 and 5) $v(\cdot)$ must be constant in a neighborhood of $q_1^*$ , we would have $\phi'(q) \leq 0$ in a right neighborhood of $q_1^*$, contradicting the definition of $q_1^*$. Suppose now that contrary to the claim we had $q_1^* > \tilde{q}$. Since $\phi'(q_1^*) = v(q_1^*) - c(1) < 0$, this would imply that $\tilde{R}(q) = \phi(q) < 0$ in a left neighborhood of $q_1^*$. But this is impossible since for every $\delta < 1$ we have $R(q) > 0$, and hence $\tilde{R}(q) = \lim_{\delta \to 1} R(q) \geq 0$.

Next, we show that $\tilde{q} \leq q_1^*$. Suppose to the contrary we had $\tilde{q} > q_1^*$. Select any $q' \in (q_1^*, \tilde{q})$. From Lemma C-1, we know that it takes at most a finite periods for the buyer to offer price $c(1)$ regardless of $\delta$. This implies $\tilde{P}(q') = c(1)$, which contradicts to the definition of $\tilde{q}$. *Q.E.D.*

***Proof of Corollary 1*** : It remains to be shown that there can be no limiting delay at $q_1^* = 0$, i.e. that $\tilde{P}(0) = c(1)$. Suppose to the contrary that $\liminf_{\delta \to 1} P^\delta(0) = b < c(1)$. In the generic case where $\widetilde{R}(0) > 0$, select $\tilde{q}_1 \in (0,1)$ and define $q' = 0$. The second part of the proof of Lemma 2 now applies literally, with $\tilde{q}_1$ taking the place of $\tilde{q}$, yielding the desired contradiction.

Next, let $\widetilde{R}(0) = 0$. Define $N(\zeta)$ be the smallest index such that $q_{N(\zeta)} \leq 0$. We will show that $\lim_{\zeta \to 0} N(\zeta)\zeta = 0$, which then implies $\tilde{P}(0) = c(1)$. Observe that $q_1 > 0$, for $q_1 = 0$ would imply that $N(\zeta) = 1$, which completes the proof. It therefore follows from the definition of $q_1$ that if $y_q = \sup S(q) = \arg\max_{y' \in [q,1]} \pi(y';q)$ then $y_q < 1$ for all $q \in [0, q_1)$, so that $\int_q^{y_q}(v(z) - c(y_q))\mathrm{d}z > \int_q^1(v(z) - c(1))\mathrm{d}z \geq 0$. Select $x \in (0, q_1)$. By the Theorem of the Maximum $\int_q^{y_q}(v(z) - c(y_q))\mathrm{d}z$ is continuous in $q$. Hence $d = \min_{q \in [0,x]} \{\int_q^{y_q}(v(z) - c(y_q))\mathrm{d}z\}$ exists and is strictly greater than 0. Hence, applying the definition of $R$ and the inequality $P(q) \leq (1-\delta)c(q) + \delta c(1)$, we obtain

$$\begin{aligned} R(q) &\geq \int_q^{y_q}(v(z) - P(y_q))\mathrm{d}z + \delta \int_{y_q}^1 (v(z) - c(1))\mathrm{d}z \\ &\geq \delta \int_q^1 (v(z) - c(1))\mathrm{d}z + (1-\delta)\int_q^{y_q} (v(z) - c(y_q))\,\mathrm{d}z \\ &\geq (1-\delta)\int_q^{y_q} (v(z) - c(y_q))\,\mathrm{d}z \geq (1-\delta)\,d. \end{aligned}$$

Now from (C-1) we obtain $q - t(q) \geq \frac{R(q)}{\bar{v}} \geq \frac{(1-\delta)d}{\bar{v}}$. Hence for any $\varepsilon < x$, it takes at most $n(\delta)$ periods to move from state 0 to state $\varepsilon$, where $n(\delta)$ is the smallest integer greater than $\frac{\varepsilon\bar{v}}{(1-\delta)d}$. From Lemma C-1, there exists an $N_0$ such that $q_{N_0} \leq \varepsilon$. Hence $N(\zeta) \leq n(\delta) + N_0$. Since $\lim_{\zeta\to 0} \zeta N(\zeta) \leq \lim_{\zeta\to 0}\zeta(n(\delta) + N_0) = \lim_{\zeta\to 0}\frac{\zeta\varepsilon\bar{v}}{(1-e^{-r\zeta})d} = \frac{\varepsilon\bar{v}}{rd}$, by letting $\varepsilon$ go to zero we obtain $\lim_{\zeta\to 0}\zeta N(\zeta) = 0$ *Q.E.D.*

***Proof of Corollary 2 :*** From the first part of the proof of the Lemma 2, we know that $E[v(z) - c(1)\,|z \geq q] \geq 0$ for all $q$ implies $q_1^* = 0$. Otherwise we would have $E[v(z) - c(1)\,|z \geq q] < 0$ for $q$ in a left neighborhood of $q_1^*$. Conversely, the condition $q_1^* = 0$ and the definition of $q_1^*$ immediately imply that $E[v(z) - c(1)\,|z \geq q] \geq 0$ for all $q$. *Q.E.D.*

***Proof of Theorem 3' :*** By Assumptions 4 and 5, there exists $\varepsilon_0 > 0$ such that $v(q) = v(q_1^*)$ and $c(q) = c(q_1^*)$ for all $q \in [q_1^* - \varepsilon_0,\, q_1^* + \varepsilon_0]$. Let $\rho = \frac{v(q_1^*) - c(q_1^*)}{c(1) - c(q_1^*)}$. The proof then consists of four steps :

1. Show that for a length of real time approximately discounted to $\rho$ the state $q_n$ (constructed in Theorem 1) stays in a neighborhood of $q_1^*$. More precisely, for any $\varepsilon \in (0, \frac{c(1)-v(q_1^*)}{2(c(1)-c(q_1^*))})$ and any $\varepsilon_1 < \frac{\varepsilon_0}{6}$, there exists $\delta_0 < 1$ such that for all $\delta \geq \delta_0$, $n_1(\varepsilon,\delta)$ satisfying $\delta^{n_1} = 1 - \varepsilon$, and $n_2'(\delta)$ satisfying $\delta^{n_2'} = \rho$, we have $q_n \in (q_1^* - \varepsilon_1,\, q_1^* + \varepsilon_1)$ for all $n$ satisfying $n_1 \leq n \leq n_2' + 1$.

2. Show that for sufficiently large $\delta$, we have $t(q) = q_{n-1}$ for all $q \in (q_{n+1}, q_n]$, $n_1(\varepsilon,\delta) - 2 \leq n \leq \bar{n}(\delta)$, where $\bar{n}(\delta) = \max\{n : q_n \geq q_1^* - \varepsilon_0\}$. In other words, there are no inner jumps in the seller's acceptance function for all $q$ in the neighborhood $(q_{\bar{n}}, q_{n_1-2}]$ of $q_1^*$. Note that from Step 1 we have $\bar{n} > n_2' > n_1$ and that $\bar{n} - n_1$ becomes arbitrarily large as $\delta_0$ goes to 1.

3. Similarly to the two step case, because there are no inner jumps in $P(\cdot)$ within any interval $(q_{n+1}, q_n]$ for $n$ satisfying $n_1 - 2 \le n \le \bar{n} - 1$, we have $R(q_n) = (q_{n-1} - q_n)(v(q_1^*) - P(q_{n-1})) + \delta R(q_{n-1}) = (q_{n-2} - q_n)(v(q_1^*) - P(q_{n-2})) + \delta R(q_{n-2})$, for $n$ satisfying $n_1 \le n \le \bar{n}$. Letting $\rho(\varepsilon) = \frac{v(q_1^*) - c(q_1^*)}{P(q_{n_1}) - c(q_1^*)}$, we therefore get a similar relation between the discount factor $\delta$ and $m_n = q_{n-1} - q_n$, i.e. $\frac{m_{n+1}}{m_n} = \delta^{-(n-n_1-1)}\rho(\varepsilon)$. Define $\hat{n}(\varepsilon, \delta)$ to be the turning period, that is the solution to $\rho(\varepsilon)\delta^{-(\hat{n}-1)} \le 1 < \rho(\varepsilon)\delta^{-\hat{n}}$, or $\delta\rho(\varepsilon) \le \delta^{\hat{n}} < \rho(\varepsilon)$. We show that $q_{\hat{n}} < q_{n_1}$, that there exists a $q_{n_2}$ such that $q_{\hat{n}} > q_{n_2} \ge q_1^* - 5\varepsilon_1$, and that the length of the delay in $[q_{n_2}, q_{n_1}]$ must satisfy $e^{-rT} = (1-\varepsilon)^{-2}\rho^2(\varepsilon)$. By letting $\varepsilon \to 0$ and $\varepsilon_1 \to 0$, we further prove that $\tilde{P}(q_1^*) = p_1^*$.

4. Prove that $\tilde{P}(q) = p_1^*$ for $q \in (q_2^*, q_1^*]$.

**Proof of Step 1 :** First, given $\varepsilon > 0$ and $\delta^{n_1} = 1 - \varepsilon$, we have

$$\begin{aligned} P(q_{n_1}) &= (1-\delta)\sum_{j=0}^{n_1-1}\delta^j c(t^j(q_{n_1})) + \delta^{n_1}c(1) \le (1-\delta)\sum_{j=0}^{n_1-1}\delta^j c(q_1) + \delta^{n_1}c(1) \\ &= c(1) - \varepsilon(c(1) - c(q_1)) < c(1). \end{aligned}$$

The inequality $c(q_1) < c(1)$ follows from the definition of $q_1 = \inf\{q : t(q) = 1\}$. Otherwise, by the left continuity of $c(\cdot)$, and Assumption 4 we can find $q' < q_1$ such that $c(q) = c(1)$ for all $q \in (q', 1]$. This would imply $t(q') = 1$, yielding a contradiction. Since $\tilde{P}(q_1^* + \varepsilon_1) = c(1)$, there exists a $\delta_0$ such that for all $\delta > \delta_0$, $P(q_{n_1}) < P(q_1^* + \varepsilon_1)$. Hence, we obtain $q_{n_1} < q_1^* + \varepsilon_1$.

Secondly, from the proof of Theorem 1, we know that $R(q) > 0$ for all $q \in [0, 1]$. Hence,

$$\begin{aligned} 0 < R(q_1^* - \varepsilon_1) &= \int_{q_1^*-\varepsilon_1}^{q_1^*} (v(z) - P(t(z)))\,\mathrm{d}z + \int_{q_1^*}^{1} (v(z) - P(t(z)))\mathrm{d}z \\ &\le (v(q_1^*) - P(t(q_1^* - \varepsilon_1)))\varepsilon_1 + \int_{q_1^*}^{1} (v(z) - P(t(z)))\mathrm{d}z. \end{aligned} \tag{C-2}$$

The second inequality uses the fact that $v(z) = v(q_1^*)$ over the interval $[q_1^* - \varepsilon_1, q_1^*]$, and that $P(\cdot)$ is increasing. Lemma 2 shows that $\tilde{R}(q_1^*) = 0$ which implies that the second term in (C-2) converges to 0 as $\delta \to 1$. Hence for sufficiently large $\delta$, (C-2) implies $v(q_1^*) > P(t(q_1^* - \varepsilon_1)) > P(q_1^* - \varepsilon_1)$. Now observe that there exists $n'$ such that $q_1^* - \varepsilon_1 < q_{n'} \le t(q_1^* - \varepsilon_1)$. Since in equilibrium a seller with type $q_{n'}$ would accept a price $P(q_{n'})$ rather than a price $c(1)$ after $n'$ periods, we have $P(q_{n'}) - c(q_{n'}) \ge \delta^{n'}(c(1) - c(q_{n'}))$. This implies $\delta^{n'} \le \frac{P(q_{n'}) - c(q_1^*)}{c(1) - c(q_1^*)} < \frac{v(q_1^*) - c(q_1^*)}{c(1) - c(q_1^*)}$. Hence $n_2' < n'$ and $q_{n_2'+1} > q_1^* - \varepsilon_1$.

**Proof of Step 2 :** We first prove that if there are $M$ steps in $c(\cdot)$ over the interval $(q_n, 1] \subset (q_1^*, 1]$, then there are at most $M$ steps in $P(q)$ for $q \in (q_n, q_{n-1}] \subset (q_1^*, 1]$. For $n = 1$, since $P(q) = (1-\delta)\, c(q) + \delta c(1)$, $P$ and $c$ have the same number of steps for $q \in (q_1, 1]$. Suppose the statement is true for $n \geq 1$, i.e. $c(\cdot)$ has $M'$ steps in $(q_n, 1]$ and $P(\cdot)$ has at most $M'$ steps in $(q_n, q_{n-1}]$. Now suppose $c(\cdot)$ has $M$ steps for $q \in (q_{n+1}, 1]$. Then $c(\cdot)$ has $M - M'$ steps for $q \in (q_{n+1}, q_n]$. Note that $P(q) = (1-\delta)\, c(q) + \delta P(t(q))$ and $t(q) \in (q_n, q_{n-1}]$ for $q \in (q_{n+1}, q_n]$. Hence, if $q$ is a discontinuity point of $P$, then $q$ is either a discontinuity point of $c(\cdot)$ or of $P(t(\cdot))$, or of both. Since $t(\cdot)$ is increasing, $P(t(q))$ can have at most $M'$ discontinuity points for $q \in (q_{n+1}, q_n]$. Furthermore, since $c(\cdot)$ and $P(t(\cdot))$ are increasing, there are at most $(M - M') + M' = M$ steps in $P(\cdot)$ over the interval $(q_{n+1}, q_n]$.

Let $\bar{M}$ denote the number of discontinuity points of $c(\cdot)$ on $[q_1^*, 1]$. Since $c(\cdot)$ is constant over $[q_1^* - \varepsilon_0, q_1^* + \varepsilon_0]$, the number of the inner jumps in $P(\cdot)$ over $(q_{n+1}, q_n]$ is bounded by $\bar{M}$ and is nonincreasing in $n$ for $n$ strictly less than $\bar{n}$ and satisfying $q_n < q_1^* + \varepsilon_0$. We will show that there are no more inner jumps in $P(\cdot)$ by the time we reach $q_{n_1}$, i.e. $\delta^{n_1} = 1 - \varepsilon$. The argument goes as follows : First select $\varepsilon'$ such that $(1-\varepsilon')^{\bar{M}} > 1 - \varepsilon$. Let $n_0(\delta, \varepsilon')$ be the solution of $\delta^{n_0} = 1 - \varepsilon'$. From the argument of Step 1, we know $q_{n_0} \in (q_{n_1}, q_1^* + \varepsilon_0)$ and there are at most $\bar{M}$ inner jumps in $P$ over $(q_{n_0+1}, q_{n_0}]$. Secondly, suppose that after $n_0$ periods $m$ inner jumps persist. Let $n_0'(\delta, \varepsilon')$ be the solution of $\delta^{n_0' - n_0} = 1 - \varepsilon'$. We will show that after $n_0'$ periods at least one inner jump collapses. By applying this argument till all the inner jumps disappear, we know it takes at most $n_1'(\delta, \varepsilon')$ periods, where $n_1'(\delta, \varepsilon')$ is the solution of $\delta^{n_1'} = (1-\varepsilon')^{\bar{M}}$ . Since $\delta^{n_1'} > 1 - \varepsilon = \delta^{n_1}$, for sufficiently large $\delta$ we have $n_1' < n_1 - 2$. Therefore we can conclude that there are no inner jumps in the seller's acceptance function for all $q$ in $(q_{\bar{n}}, q_{n_1 - 2}]$. Hence, to complete the proof, we only need to show that if there are $m$ steps in $P(q)$ for $q \in (q_{n_0+1}, q_{n_0}]$, then there are at most $m - 1$ steps in $P(q)$ for $q \in (q_{n_0'+1}, q_{n_0'}]$.

Suppose two adjacent intervals $(q_{n+1}, q_n]$ and $(q_{n+2}, q_{n+1}]$ have the same number of discontinuity points in $P(\cdot)$, denoted by $m$. Let $(q_{n,1}, q_{n,2}, \ldots, q_{n,m})$ and $(q_{n+1,1}, q_{n+1,2}, \ldots, q_{n+1,m})$ denote those discontinuity points, and define $x_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,m})$, where $x_{n,i} = q_{n,i} - q_{n,i-1}$ for $i = 2, \cdots, m$, and $x_{n,1} = q_{n,1} - q_{n+1}$. We first show that there is a linear mapping $A_n$ such that $x_{n+1} = A_n(\delta, n_0) x_n$. (See figure 3.)

< Insert Figure 3 >

Since $c(\cdot)$ is constant over $(q_{n+2}, q_{n+1}]$, the jumps in $P$ requires the jumps in $t(\cdot)$. We therefore have $\Upsilon(q_{n+1,i}) = \{q_{n,i}, q_{n,i+1}\}$, $i = 1, \ldots, m-1$ and $\Upsilon(q_{n+1,m}) = \{q_{n,m}, q_{n-1,1}\}$, i.e., $R(q_{n+1,i}) = P(q_{n,i})(q_{n,i} - q_{n+1,i}) + \delta R(q_{n,i}) = P(q_{n,i+1})(q_{n,i+1} - q_{n+1,i}) + \delta R(q_{n+1,i})$ for $i = 1, \ldots, m-1$ and a similar equation for $i = m$. Using the same trick as in the two type case (leading up to Equation 7), we have $\frac{q_{n,i-1} - q_{n+1,i-1}}{x_{n,i}} = \frac{(1-\delta)(v(q_1^*) - c(q_1^*))}{\delta^{n-n_0}(P(q_{n_0,i+1}) - P(q_{n_0,i}))}$. Since $P(q_{n_0,i}) = (1-\delta)\sum_{k=0}^{n_0-1}\delta^k c(t^k(q_{n_0,i})) + \delta^{n_0} c(1)$, we have $P(q_{n_0,i+1}) - P(q_{n_0,i}) = (1-\delta)\, a_i(\delta, n_0)$, where

$$a_i(\delta, n_0) = \sum_{k=0}^{n_0-1} \delta^k \left(c(t^k(q_{n_0,i+1})) - c(t^k(q_{n_0,i}))\right). \tag{C-3}$$

Letting $b_{n,i}(\delta, n_0) = \frac{v(q_1^*) - c(q_1^*)}{\delta^{n-n_0} a_i(\delta, n_0)}$, we obtain a linear mapping $x_{n+1} = A_n(\delta, n_0) x_n$,

$$A_n(\delta, n_0) = \begin{pmatrix} b_{n,1}+1 & -b_{n,2} & 0 & \cdots & 0 & 0 \\ 0 & b_{n,2}+1 & -b_{n,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{n,m-1}+1 & -b_{n,m} \\ -1 & -1 & -1 & \cdots & -1 & b_{n,m} \end{pmatrix}. \tag{C-4}$$

Secondly, let $N(\delta)$ be the number such that $q_{N(\delta)} \le q_1^* + \varepsilon_0 < t(q_{N(\delta)})$. Lemma C-1 implies $N(\delta)$ is uniformly bounded in $\delta$. Using the fact that $c(\cdot)$ is constant over $(q_1^* - \varepsilon_0, q_1^* + \varepsilon_0)$ we know that $c(t^k(q_{n_0,i+1})) - c(t^k(q_{n_0,i})) = 0$ for $k = 0, 1, \ldots, n_0 - N$. Hence we may rewrite (C-3) as

$$a_i(\delta, n_0) = \sum_{k=0}^{N-1} \delta^{n_0-(N-k)} (c(t^{n_0-N+k}(q_{n_0,i+1})) - c(t^{n_0-N+k}(q_{n_0,i}))) \tag{C-5}$$

Thirdly, let $C_i = \{\sum_{k=0}^{N-1}(c(t^{n_0-N+k}(q_{n_0,i+1})) - c(t^{n_0-N+k}(q_{n_0,i}))) : \delta \in (0,1)\}$. Note that by definition both $n_0$ and $t(\cdot)$ are functions of $\delta$. Hence, potentially $C_i$ may have infinitely many elements. However, since $c(\cdot)$ has at most $\bar{M}$ different values in $(q_1^* + \varepsilon_0, 1]$, there are only finitely many elements in $C_i$.

Fourthly, let $a' \in C_1 \times C_2 \times \cdots \times C_m$ and $\Delta_m(a') = \{\delta \in (0,1) : \sum_{k=0}^{N-1}(c(t^{n_0-N+k}(q_{n_0,i+1})) - c(t^{n_0-N+k}(q_{n_0,i}))) = a_i', i = 1, \ldots, m\}$. We will show that there exists a $\delta_{a'} \in (0,1)$ such that for $\delta \in \Delta_m(a') \cap (\delta_{a'}, 1)$ and $n > n_0'(\delta, \varepsilon')$, at least one element in $x_n$ becomes negative, i.e., at least one inner jump collapses. Let $\delta_0 = \max\{\delta_{a'} : a' \in C_1 \times C_2 \times \cdots \times C_m\}$. Since there are only finitely possible $a'$, we know $\delta_0$ exists and is strictly less than 1. Applying the argument for all possible $a'$, we conclude that for $\delta \in (\delta_0, 1)$, at least one inner jump collapses for $n > n_0'$. This then completes the proof of Step 2. Henceforth, we therefore only consider the case where $a' \in C_1 \times C_2 \times \cdots \times C_m$ and $\delta \in \Delta_m(a')$.

Let $a_i(\varepsilon') = (1-\varepsilon')a_i'$. By (C-5) we have :

$$a_i(\delta, n_0) = (1 + O(1-\delta^N))a_i(\varepsilon') \tag{C-6}$$

Define $b_i(\varepsilon') = \lim_{\delta \to 1} b_{n_0,i}(\delta, n_0) = \frac{v(q_i^*) - c(q_i^*)}{a_i(\varepsilon')}$ and

$$A(\varepsilon') = \lim_{\delta \to 1} A_{n_0}(\delta, n_0). \tag{C-7}$$

Let $\lambda_1$, $\lambda_2$,..., $\lambda_m$ be the eigenvalues of $A(\varepsilon')$ such that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m|$, where $|\cdot|$ is the absolute value of a complex number i.e. $|\alpha + \beta i| = \sqrt{\alpha^2 + \beta^2}$. To simplify the argument we assume all the eigenvalues are real and $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_m|$.[29] From Claim 2(a) below, without loss of generality we have $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$. Let $v_i$ be the eigenvector for $A(\varepsilon')$ corresponding to $\lambda_i$. For any vector $x_n \in \mathbb{R}^m$, define

$$\beta(x_n) = \sum_{i=1}^{m} x_{n,i}$$

We select $v_i$ such that $\beta(v_i) = 1$ for all $i = 1, \ldots, m$. Now expand any vector $x_n \in \mathbb{R}^m_{++}$ on $\{v_1, \ldots, v_m\}$, the eigenspace of $A(\varepsilon')$ :

$$x_n = \sum_{i=1}^{m} r_{n,i} v_i, \text{ where } r_{n,i} \in \mathbb{R}. \tag{C-8}$$

Let $\kappa_n = \delta^{n-n_0}(1 + O(1-\delta^N))$, $e_i$ the $i$th unit vector in $\mathbb{R}^m$, and select $\alpha_i$ such that $e_m = \sum_{i=1}^{m} \alpha_i v_i$. Then using (C-4), (C-6) and (C-7) we have

$$\begin{aligned} x_{n+1} = A_n(\delta, n_0)x_n &= \frac{1}{\kappa_n}(A(\varepsilon') - (1-\kappa_n)I)x_n + (\frac{1-\kappa_n}{\kappa_n}\beta(x_n))e_m \\ &= \frac{1}{\kappa_n}\sum_{i=1}^{m}((\lambda_i - (1-\kappa_n))r_{n,i} + (1-\kappa_n)\beta(x_n)\alpha_i)v_i. \end{aligned} \tag{C-9}$$

From (C-8) and (C-9), we have

$$r_{n+1,i} = \frac{1}{\kappa_n}((\lambda_i - (1-\kappa_n))r_{n,i} + (1-\kappa_n)\beta(x_n)\alpha_i). \tag{C-10}$$

For any $k$ and any two vectors $x(\delta)$, $y(\delta) \in \mathbb{R}^k$, define $x \approx y$ if $\lim_{\delta \to 1} \frac{x_i}{y_i} = 1$ for $i = 1, \ldots, k$. Using (C-10), Claim 1 below establishes that $x_{n+1} \approx r_{n+1,1}v_1$ for some $n < n_0'(\delta, \varepsilon')$. Therefore from Claim 2(b) below, we obtain the desired result that for sufficiently large $\delta$ at least one element in $x_{n+1}$ becomes negative for some $n$ less than $n_0'$.

[29] By slightly modifying the proof, we get the same result when some eigenvalues have the same absolute values or when they are complex numbers.

**Proof of Step 3 :** We first prove that $n_1 < \hat{n} < n_2' + 1$, implying $q_1^* - \varepsilon_1 < q_{\hat{n}} < q_{n_1} \leq q_1^* + \varepsilon_1$ according to Step 1. It then follows that $q_{n_1} - q_{\hat{n}} < 2\varepsilon_1$, so that $q_{n_1} - 2(q_{n_1} - q_{\hat{n}}) > q_{n_1} - 4\varepsilon_1 > q_1^* - \varepsilon_0$. From Step 1 we know that $\delta^{n_2'+1} = \delta\rho < \delta\rho(\varepsilon) \leq \delta^{\hat{n}}$. Hence $\hat{n} < n_2' + 1$. Observe also that $\lim_{\varepsilon\to 0} P(q_{n_1}) = c(1)$, $\lim_{\varepsilon\to 0} \rho(\varepsilon) = \rho < 1$ and $\lim_{\varepsilon\to 0} \delta^{n_1} = 1$. Hence for sufficiently small $\varepsilon$ we have $\delta^{\hat{n}} < \delta^{n_1}$ or $\hat{n} > n_1$.

Secondly, we show that the $\tilde{P}(q_1^*) = p_1^*$. We do this in 3 steps :

**Step 3.1** : Show that $\exists\ \delta_1 \geq \delta_0$ and $n_2(\varepsilon, \delta)$ such that $n_2(\varepsilon, \delta) > \hat{n}(\varepsilon, \delta) > n_1(\varepsilon, \delta)$ and such that $\forall \delta > \delta_1$, we have $m_{n_2(\varepsilon,\delta)} \in ((1-\varepsilon)m_{n_1(\varepsilon,\delta)}, m_{n_1(\varepsilon,\delta)}]$. Show furthemore that this implies that the total delay in $[q_{n_2-1}, q_{n_1}]$, $(n_2(\varepsilon,\delta) - n_1(\varepsilon,\delta) - 2)\zeta$, converges to $T$, where $T$ satisfies $e^{-rT} = (1-\varepsilon)^{-2}\rho^2(\varepsilon)$.

From the analogue to Equation (8), we know that $m_{n_2}/m_{n_1} = (\rho(\varepsilon)\,\delta^{-\frac{(n_2+n_1-1)}{2}})^{n_2-n_1}$. Let $\psi(\zeta, n_2) = (\rho(\varepsilon)\, e^{\frac{r\zeta(n_2+n_1-1)}{2}})^{n_2-n_1}$ denote this ratio, and choose $\zeta_1$ and $n_2(\varepsilon,\delta) > \hat{n}(\varepsilon,\delta)$ such that $\forall \zeta < \zeta_1$ we have $\psi(\zeta, n_2(\varepsilon,\delta)) \in (1-\varepsilon,\ 1]$. This can be done because $\lim_{\zeta\to 0}\psi(\zeta, \hat{n}(\varepsilon,\delta)) = 0$, and because $\lim_{n_2\to\infty}\psi(\zeta, n_2; n_1) = \infty$. Observe now that (for sufficiently small $\zeta$) $e^{-r\zeta n_1} = \delta^{n_1} = 1-\varepsilon$, so we have $\lim_{\zeta\to 0}\rho(\varepsilon)\, e^{\frac{r\zeta(n_2-n_1-2)}{2}} = \lim_{\zeta\to 0}(1-\varepsilon)\,\rho(\varepsilon)\, e^{\frac{r\zeta(n_2(\zeta)+n_1-1)}{2}} = \lim_{\zeta\to 0}(1-\varepsilon)$ $[\psi(\zeta, n_2)]^{\frac{1}{n_2(\zeta)-n_1}} = 1-\varepsilon$, implying that $\lim_{\zeta\to 0}\zeta(n_2 - n_1 - 2) = T$.

**Step 3.2** : Show that $q_{n_2} \geq q_1^* - 5\varepsilon_1$.

(a) $m_{n_2-i} = (\rho(\varepsilon)\,\delta^{-\frac{(n_2+n_1-1)}{2}})^{n_2-n_1-2i} m_{n_1+i}$ for all $i$ such that $n_2 - n_1 - 2i > 0$.

(b) Since $\rho(\varepsilon)\,\delta^{-\frac{(n_2+n_1-1)}{2}} \leq 1 < \rho(\varepsilon)\,\delta^{-\hat{n}}$, we have $n_2 + n_1 - 1 \leq 2\hat{n}$. This implies that for all $i \leq n_2 - \hat{n} - 1$, we have $n_2 - n_1 - 2i > 0$ and $n_1 + i \leq \hat{n}$.

It follows from (a) and (b) that $\sum_{i=\hat{n}+1}^{n_2-1} m_i(\zeta) = \sum_{i=1}^{n_2-\hat{n}-1} m_{n_2-i}(\zeta) \leq \sum_{i=1}^{n_2-\hat{n}-1} m_{n_1+i}(\zeta) \leq \sum_{i=n_1+1}^{\hat{n}} m_i(\zeta) = q_{n_1} - q_{\hat{n}}$. Hence, $q_{n_2-1} = q_{n_1} - \sum_{i=n_1+1}^{n_2-1} m_i(\zeta) \geq q_{n_1} - 2(q_{n_1} - q_{\hat{n}}) \geq q_1^* - 5\varepsilon_1$.

**Step 3.3** : Show that $\tilde{P}(q_1^*) = p_1^*$.

In the proof of Step 2, we have shown that for $n = N, \ldots, n_1$, $x_{n+1} = A_n(\delta, n_0)x_n$, $\lim_{\delta\to 0} A_n(\delta, n_0) = A(\varepsilon')$ and that we need at most $\bar{M}$ pairs of $(n_0, \varepsilon')$ to kill off all inner jumps. For each pair of $(n_0, \varepsilon')$, Equations (C-5) and (C-6) imply that we can rewrite $A_n(\delta, n_0) = \delta^{-n}(1 + O(1-\delta^N))A$ for $n = N, \ldots, n_1$, where $A$ is one of finitely many possible linear mappings. Hence, there exists a sufficiently large $j \in \mathbb{Z}$, such that for sufficiently large $\delta$, we have $\frac{m_{n+1}}{m_n} \geq \delta^{-n}\rho(\varepsilon)^j$ for $n = N, \ldots, n_1$. Therefore $m_{n_1} \geq \rho(\varepsilon)^j\,\delta^{n_1-1}m_{n_1-1} \geq (\rho(\varepsilon)^j\,\delta^{-\frac{n_1+N-1}{2}})^{n_1-N}m_N$. Let $j' = j(n_1 - N)$. Suppose

$\lim_{\delta \to 1} q_{n_2+j'} \geq q_1^* - \varepsilon_0$. Analogously to Equation (8), we have

$$\begin{aligned}
\lim_{\delta \to 1} m_{n_2+j'} &= \lim_{\delta \to 1} (\rho(\varepsilon)\, \delta^{-\frac{n_2+j'+n_1-1}{2}})^{n_2+j'-n_1} m_{n_1} \\
&\geq \lim_{\delta \to 1} (\rho(\varepsilon)\, \delta^{-\frac{n_2+j'+n_1-1}{2}})^{n_2+j'-n_1} (\rho(\varepsilon)^j\, \delta^{-\frac{n_1+N-1}{2}})^{n_1-N} m_N \\
&= \lim_{\delta \to 1} \psi(\zeta, n_2)\, \rho(\varepsilon)^{2j'}\, \delta^{-\frac{j'((2n_2+j'-1)+\frac{1}{j}(n_1+N-1))}{2}} m_N \\
&\geq (1-\varepsilon) \lim_{\delta \to 1} \psi(\zeta, n_2)^{\frac{2j'}{n_2-n_1}}\, \delta^{-\frac{j'\left((j+\frac{1}{j}-2)n_1-(j-\frac{1}{j})N+1-\frac{1}{j}\right)}{2}} m_N \\
&= (1-\varepsilon) \lim_{\delta \to 1} \delta^{-\frac{j'\left((j+\frac{1}{j}-2)n_1-(j-\frac{1}{j})N+1-\frac{1}{j}\right)}{2}} m_N \\
&= (1-\varepsilon) \lim_{\delta \to 1} (1-\varepsilon)^{-\frac{j\left((j+\frac{1}{j}-2)n_1-(j-\frac{1}{j})N+1-\frac{1}{j}\right)}{2}} m_N
\end{aligned} \tag{C-11}$$

Note that for large $\delta$ we have $j\left((j+\frac{1}{j}-2)n_1-(j-\frac{1}{j})N+1-\frac{1}{j}\right) \geq 0$. Hence, Equation (C-11) implies $\lim_{\delta \to 1} m_{n_2+j'} \geq (1-\varepsilon) m_N$. Since $N$ is uniformly bounded in $\delta$ and $m_n$ is continuous in $\delta$, $\lim_{\delta \to 1} \frac{m_N}{m_{N-1}} = d'$ exists as well. Thus by $q_{N-1} \geq q_1^* + \varepsilon_0$ and Lemma C-1, we have $m_N \geq d' m_{N-1} \geq d'\eta$. We conclude that $\lim_{\delta \to 1} q_{n_2+j'} \leq \max\{q_1^* - \varepsilon_0, \lim_{\delta \to 1} q_{n_2} - (1-\varepsilon) m_N\} < q_1^*$. Let $t'$ be the delay from $n_2 + j'$ to $n_2$, i.e. $e^{-rt'} = (1-\varepsilon)^j$.

Letting $\varepsilon$ and $\varepsilon_1$ go to zero, Steps 3.1 and 3.2 yield $\tilde{P}(q) = p_1^*$ for $q \in (\lim_{\delta \to 1} q_{n_2+j'}, q_1^*]$. This completes the proof of Step 3.

**Proof of Step 4 :** Similarly to the proof that $\tilde{P}(q) = c(1)$ for $q \in (q_1^*, 1]$ , first define

$$\tilde{q} = \inf\{q : \lim_{\zeta \to 0} P(q) = p_1^*\}. \tag{C-12}$$

Note that by Step 3, we have $\tilde{P}(q_1^*) = p_1^*$. Hence, $\tilde{q}$ is well defined. Similarly to the argument in Lemma 2, we can show that $\tilde{q} = q_2^*$, i.e. that the buyers limiting revenue is equal to $\tilde{R}(q)$ for all $q \geq q_2^*$, and that the seller's limiting acceptance function is equal to $p_1^*$ on the interval $(q_2^*, q_1^*]$. *Q.E.D.*

### *Proof of* Claims 1-3

**Claim 1** *$x_{n+1} \approx r_{n+1,1} v_1$ for some $n_0 < n < n_0'(\delta, \varepsilon')$.*

*Proof* : Note that $\lambda_i$ and $v_i$ depend on $A(\varepsilon')$ only and hence remain constant as $\delta$ goes to 1. Since $\beta(x_{n_0}) = \sum_{i=1}^m x_{n_0,i} = \sum_{i=1}^m r_{n_0,i}$, we know $r_{n_0,i} = O(\beta(x_{n_0}))$. Let $N' = \frac{1}{\sqrt{\zeta}}$. Consider the following three cases :

**Case 1 :** $(1-\kappa_{n_0})\beta(x_{n_0}) = o(r_{n_0,1})$. Let $n(\varepsilon',\delta) = n_0 + N'$. Then $\lim_{\delta\to 1}\delta^{n-n_0} = \lim_{\zeta\to 0} e^{-r\sqrt{\zeta}} = 1$ and $\lim_{\delta\to 1}\kappa_n = 1$. Hence, from (C-10), we obtain $\frac{r_{i+1,1}}{r_{i,1}} = \lambda_1 + O(1-\delta)$, $i = n_0,\ldots,n$ and $x_{n+1} \approx \lambda_1^{n-n_0} r_{n_0,1} v_1 + \cdots + \lambda_m^{n-n_0} r_{n_0,m} v_m$. Since $\frac{\lambda_i}{\lambda_1} < 1$, we have $\lim_{\delta\to 1}(1-\delta^N)^{-1}(\frac{\lambda_i}{\lambda_1})^{n-n_0} = 0$. Combining this with $r_{n_0,i} = O(\ \beta(x_{n_0}))$, we obtain $r_{n,i} = o(r_{n,1})$ for $i = 2,\ldots,m$. Hence $x_{n+1} \approx \lambda_1^{n-n_0} r_{n_0,1} v_1 \approx r_{n+1,1} v_1$. Since $\lim_{\delta\to 1}\delta^{n-n_0} = 1$ and $\delta^{n_0'-n_0} = 1-\varepsilon'$, we obtain $n < n_0'(\delta,\varepsilon')$ when $\delta$ is sufficiently large.

**Case 2:** $r_{n_0,1} = o((1-\kappa_{n_0})\beta(x_{n_0}))$. From Claim 3, we know $\alpha_1 \neq 0$. Hence, $r_{n_0+1,1} \approx \alpha_1(1-\kappa_{n_0})\beta(x_{n_0})$. This implies that for all $n > n_0+1$, $r_{n,1}$ and $\alpha_1$ have the same sign. Therefore from (C-10) we have $\frac{r_{n+1,1}}{r_{n,1}} > \lambda_1 - (1-\kappa_n)$ for all $n > n_0+1$. Let $n'(\delta,\varepsilon',\varepsilon'')$ be such that $\delta^{n'-n_0} = 1-\varepsilon'' > 1-\varepsilon'$. Hence $1-\kappa_{n'} \approx \varepsilon''$ and $\beta(x_{n'}) = O(|r_{n',1}|)$. Select $\varepsilon''$ and $\zeta$ sufficiently small so $\lambda_1 - (1-\kappa_n)$ still dominates $\lambda_i - (1-\kappa_n)$ for all $i = 2,\ldots,m$. Letting $n = n' + N'$ and using the similar argument as in Case 1, we obtain $x_{n+1} \approx r_{n+1,1} v_1$ and $n < n_0'$ when $\delta$ is sufficiently large.

**Case 3:** $r_{n_0,1} = \gamma_{n_0}\alpha_1(1-\kappa_{n_0})\beta(x_{n_0})$, where $\gamma_{n_0}$ is a function of $\delta$ satisfying $\underline{\lim}_{\delta\to 1}|\gamma_{n_0}| > 0$ and $\overline{\lim}_{\delta\to 1}|\gamma_{n_0}| < \infty$. From (C-10) we obtain $\frac{r_{n_0+1,1}}{r_{n_0,1}} = \lambda_1 - (1-\kappa_{n_0}) + \frac{1}{\gamma_{n_0}}$. Hence, if $\gamma_{n_0} < 0$, it is possible to have $\frac{r_{n_0+1,1}}{r_{n_0,1}} < \frac{r_{n_0+1,2}}{r_{n_0,2}}$, a potential threat to our goal $x_n \approx r_{n,1} v_1$. Hence for this case, we need a better estimate of $r_{n,1}$. To simplify the argument, we only consider the worst case: $\gamma_{n_0} < 0$, and $\beta(x_{n_0}) = O(|r_{n_0,2}|)$. From Case 1, we obtain $x_{n+1} \approx \lambda_1^{n-n_0} r_{n_0,1} v_1 + \lambda_2^{n-n_0} r_{n_0,2} v_2$ for $n_0 + N' \le n \le n_0 + 2N'$. If there exists an $n \le n_0 + 2N'$ such that either $(1-\kappa_n)\beta(x_n) = o(r_{n,1})$ or $r_{n,1} = o((1-\kappa_n)\beta(x_n))$ holds then by applying Case 1 or Case 2, respectively, we obtain $x_{n'+1} \approx r_{n'+1,1} v_1$ for some $n' < n_0'$ , which completes the proof. Now consider the case that for every $n = n_0 + N',\ldots,n_0+2N'$, we have $r_{n,1} = \gamma_n\alpha_1(1-\kappa_n)\beta(x_n)$. This implies $x_{n+1} \approx \lambda_2^{n-n_0} r_{n_0,2} v_2$, $\beta(x_{n+1}) = \lambda_2^{n-n_0} r_{n_0,2} + O(1-\kappa_n)$ and $\frac{\beta(x_{n+1})}{\beta(x_n)} = \lambda_2 + O(1-\kappa_n)$. Since $\lim_{\delta\to 1}\kappa_n = 1$, using (C-10) recursively, we have

$$r_{n+1,1} \approx \alpha_1(1-\kappa_{n_0})\beta(x_{n_0})\left(\gamma_{n_0}\lambda_1^{n-n_0+1} + \lambda_1^{n-n_0} + \lambda_1^{n-n_0-1}\frac{\beta(x_{n_0+1})}{\beta(x_{n_0})} + \cdots + \frac{\beta(x_n)}{\beta(x_{n_0})}\right) \quad \text{(C-13)}$$

Letting $f_{n_0}(N') = \gamma_{n_0}\lambda_1 + 1 + \frac{\beta(x_{n_0+1})}{\lambda_1\beta(x_{n_0})} + \cdots + \frac{\beta(x_{n_0+N'})}{\lambda_1^{N'}\beta(x_{n_0})}$ and using $\frac{\beta(x_{n+1})}{\beta(x_n)} = \lambda_2 + O(1-\kappa_n)$, we can further rewrite (C-13) to

$$r_{n+1,1} \approx \alpha_1(1-\kappa_{n_0})\beta(x_{n_0})\left(\lambda_1^{n-n_0} f_{n_0}(N') + \sum_{k=N'+1}^{n-n_0}\lambda_1^{n-n_0-k}\lambda_2^k\right). \quad \text{(C-14)}$$

Using (C-14), $\beta(x_{n_0}) \approx r_{n_0,2}$ and $\beta(x_{n+1}) \approx \lambda_2^{n-n_0} r_{n_0,2}$, we obtain

$$\frac{r_{n+1,1}}{\beta(x_{n+1})} \approx \alpha_1(1-\kappa_{n_0})\left((\frac{\lambda_1}{\lambda_2})^{n-n_0} f_{n_0}(N') + \sum_{k=N'+1}^{n-n_0} (\frac{\lambda_1}{\lambda_2})^{n-n_0-k}\right)$$
$$= \alpha_1(1-\kappa_{n_0})\left(\frac{-\lambda_2}{\lambda_1-\lambda_2} + (\frac{\lambda_1}{\lambda_2})^{n-n_0-N'}\left((\frac{\lambda_1}{\lambda_2})^{N'} f_{n_0}(N') + \frac{\lambda_2}{\lambda_1-\lambda_2}\right)\right)$$

Since $r_{n,1} = \gamma_n\alpha_1(1-\kappa_n)\beta(x_n)$ for all $n = n_0+N', \ldots, n_0+2N'$ and $\frac{\lambda_1}{\lambda_2} > 1$, we have $(\frac{\lambda_1}{\lambda_2})^{N'} f_{n_0}(N') + \frac{\lambda_2}{\lambda_1-\lambda_2} = 0$. Hence $\gamma_{n_0+2N',1} \approx \frac{-\lambda_2}{\lambda_1-\lambda_2}$. Combining this with (C-10) we obtain $\gamma_{n+1} \leq \gamma_n$ for all $n > n_0 + 2N'$. Select $\varepsilon'' \in (0, \varepsilon')$ such that $\frac{\lambda_1-\varepsilon''}{\lambda_2-\varepsilon''} > 1$. Let $n'(\delta, \varepsilon', \varepsilon'')$ be the solution of $\delta^{n'-n_0} = \varepsilon'' < \varepsilon'$. Since $\lim_{\delta\to1}(1-\kappa_{n'}) = \varepsilon''$, by a similar argument as above (let $n'$ take the role of $n_0$ and replace $N'$ with 0) we have

$$\frac{r_{n+1,1}}{\beta(x_{n+1})} \approx \alpha_1(1-\kappa_{n'})\left(\frac{-\lambda_2+\varepsilon''}{\lambda_1-\lambda_2} + (\frac{\lambda_1-\varepsilon''}{\lambda_2-\varepsilon''})^{n-n'}\left(\gamma_{n'} + \frac{\lambda_2-\varepsilon''}{\lambda_1-\lambda_2}\right)\right).$$

Since $\gamma_{n'} + \frac{\lambda_2-\varepsilon''}{\lambda_1-\lambda_2} \leq \gamma_{n_0+2N'} + \frac{\lambda_2-\varepsilon''}{\lambda_1-\lambda_2} < -\frac{\varepsilon''}{\lambda_1-\lambda_2}$ and $\frac{\lambda_1-\varepsilon''}{\lambda_2-\varepsilon''} > 1$, for $n = n' + N'$ we have $(1-\kappa_n)\beta(x_n) = o(r_{n,1})$. Applying a similar argument as in Case 1, we have $x_{n+1} \approx r_{n+1,1}v_1$ and $n < n_0'$ when $\delta$ is sufficiently large. *Q.E.D.*

**Claim 2** *(a) $\lambda_i > 0$, for $i = 1, \ldots, m$ and (b) if $x_n \approx r_{n,1}v_1$ then for sufficiently large $\delta$ at least one element in $x_n$ is negative.*

*Proof* : Let $h(\lambda) = \det(A(\varepsilon') - \lambda I)$. It can be shown that $h(y) > 0$ for all $y \leq 0$, $h(b_1+1) < 0$ for all $j = 1, \cdots, m-1$ and $h(b_m) < 0$ if $b_m = \min\{b_1+1, \ldots, b_{m-1}+1, b_m\}$. Hence, $\lambda_i > 0$, for $i = 1, \ldots, m$ and there exists at least one eigenvalue in $(0, \min\{b_1+1, b_2+1, \cdots, b_{m-1}+1, b_m\})$. From $\sum_{j=1}^m \lambda_j = \sum_{j=1}^{m-1}(b_j+1) + b_m$, we know that $\lambda_1 > \min\{b_1+1, \ldots, b_{m-1}+1, b_m\} = \bar{b} > 0$. If $\bar{b} = b_j + 1$ for some $j < m$, then $(A - \lambda_1 I)v_1 = 0$ implies $(b_j + 1 - \lambda_1)v_j - b_{j+1}v_{j+1} = 0$. Hence, $\text{sign}(v_{1,j}) \neq \text{sign}(v_{1,j+1})$ or $v_{1,j} = v_{1,j+1} = 0$. However, $v_{1,j} = v_{1,j+1} = 0$ induces $v_1 = 0$ which yields a contradiction. We can therefore conclude that $\text{sign}(v_{1,j}) \neq \text{sign}(v_{1,j+1})$. A similar argument applies to the case where $\bar{b} = b_m$. *Q.E.D.*

**Claim 3** *$e_m = \sum_{i=1}^m \alpha_i v_i$, for some $\alpha \in \mathbb{R}^m$ and $\alpha_1 \neq 0$, i.e. $e_m$ cannot be generated by $\{v_2, \ldots, v_m\}$.*

*Proof* : Let $S$ be the space generated by $v_2, \ldots, v_m$ and $V = [v_1\ v_2 \ldots v_m]$. Suppose $e_m \in S$, i.e. $e_m = \sum_{i=2}^m \alpha_i v_i = V\alpha$ for some $\alpha \neq 0$ with $\alpha_1 = 0$. Then using $V^{-1}AV\alpha = \Lambda\alpha$, we have $V^{-1}Ae_m = (0, \lambda_2\alpha_2, \lambda_3\alpha_3, \ldots, \lambda_m\alpha_m)'$. Premultiplying by $V$ yields $(0, \ldots, 0, -b_m, b_m)' = V(0, \lambda_2\alpha_2, \lambda_3\alpha_3, \ldots, \lambda_m\alpha_m)' = \sum_{i=2}^{m} \lambda_i \alpha_i v_i \in S$. Since both $-b_m e_{m-1} + b_m e_m$ and $e_m$ are in $S$, and since $S$ is a vector space, we know that $e_{m-1} \in S$. By induction, we can similarly show that $e_m, e_{m-1}, \ldots, e_1 \in S$, yielding a contradiction to the fact that $S$ is of dimension $m-1$. *Q.E.D.*

*Q.E.D.*

## References


Abreu, Dilip and Faruk Gul (2000), "Bargaining and Reputation," Econometrica, 68: (1), pp. 5-117.

Akerlof, George A. (1970), "The Market for 'Lemons' : Quality Uncertainty and the Market Mechanism," *Quarterly Journal of Economics*, 84, pp. 488-500.

Ausubel, Lawrence and Raymond J. Deneckere (1989), "Reputation in Bargaining and Durable Goods Monopoly," *Econometrica*, 57(3), pp. 511-531.

Ausubel, Lawrence and Raymond J. Deneckere (1993), "A Generalized Theorem of the Maximum," *Economic Theory*, 3, pp. 99-107.

Billingsley, Patrick (1968), *Convergence of Probability Measures*, John Wiley & Sons, New York.

Binmore, Ken (1888), "Modelling Rational Players II," *Economics and Philosophy*, 4(1), pp. 9-55.

Coase, Ronald H. (1972), "Durability and Monopoly," *Journal of Law and Economics*, 3, pp. 1-44.

Cutler, David M. and Lawrence H. Summers (1988), "The Costs of Conflict Resolution and Financial Distress : Evidence from the Texaco-Pennzoil Litigation," *The Rand Journal of Economics*, 19(2), pp. 157-173.

Cramton, Peter and Joseph Tracy (1992), "Strikes and Holdouts in Wage Bargaining : Theory and Data," *American Economic Review*, 82, pp. 100-121.

Deneckere, Raymond J. (1992), "A Simple Proof of the Coase Conjecture," mimeo, Northwestern University.

Evans, Robert (1989), "Sequential Bargaining with Correlated Values," *Review of Economic Studies*, 56(4), pp. 499-510.

Fudenberg, Drew, David K. Levine and Jean Tirole (1985), "Infinite-Horizon Models of Bargaining with One-Sided Incomplete Information," in Alvin Roth (Ed.), *Game Theoretic Models of Bargaining*, Cambridge University Press, Cambridge, England.

Gul, Faruk, Hugo Sonnenschein and Robert Wilson (1986), "Foundations of Dynamic Monopoly and the Coase Conjecture," *Journal of Economic Theory*, 39(1), 155-190.

Hicks, John Richard (1932), *The Theory of Wages*, MacMillan, London, England.

Myerson, Roger B. and Mark Satterthwaite (1983), "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, 28, 265-281.

Myerson, Roger B. (1985),"Analysis of Two Bargaining Problems with Incomplete Information," in *Game Theoretic Models of Bargaining*, A. Roth (ed.), Cambridge University Press, London, pp. 115-147.

Myerson, Roger B. (1991), *Game Theory : Analysis of Conflict*, Harvard University Press, Cambridge, Massachusetts.

Neelin, Janet, Hugo Sonnenschein and Matthew Spiegel (1988), "A Further Test of Noncooperative Bargaining Theory : Comment," *American Economic Review*, 78(4), pp. 824-835.

Reny, Philip J. (1993), "Common Belief and the Theory of Games," *Journal of Economic Theory*, 59 (2), pp. 257-274.

Rosenthal, Robert W. (1981), "Games of Perfect Information, Predatory Pricing and the Chain-Store Paradox," *Journal of Economic Theory*, 25, pp. 92-100.

Royden, H.L. (1988), *Real Analysis*, Macmillan Publishing Company, New York.

Samuelson, William (1984),"Bargaining Under Asymmetric Information," *Econometrica*, 52, pp. 995-1005.

Samet, Dov (1996), "Hypothetical Knowledge and Games with Perfect Information," *Games and Economic Behavior*, 17(2), pp. 230-251.

Vincent, Daniel R. (1989), "Bargaining with Common Values," *Journal of Economic Theory*, 48(1), pp. 47-62.

Figure 1: “Bargaining with Interdependent Values” (Deneckere and Liang)

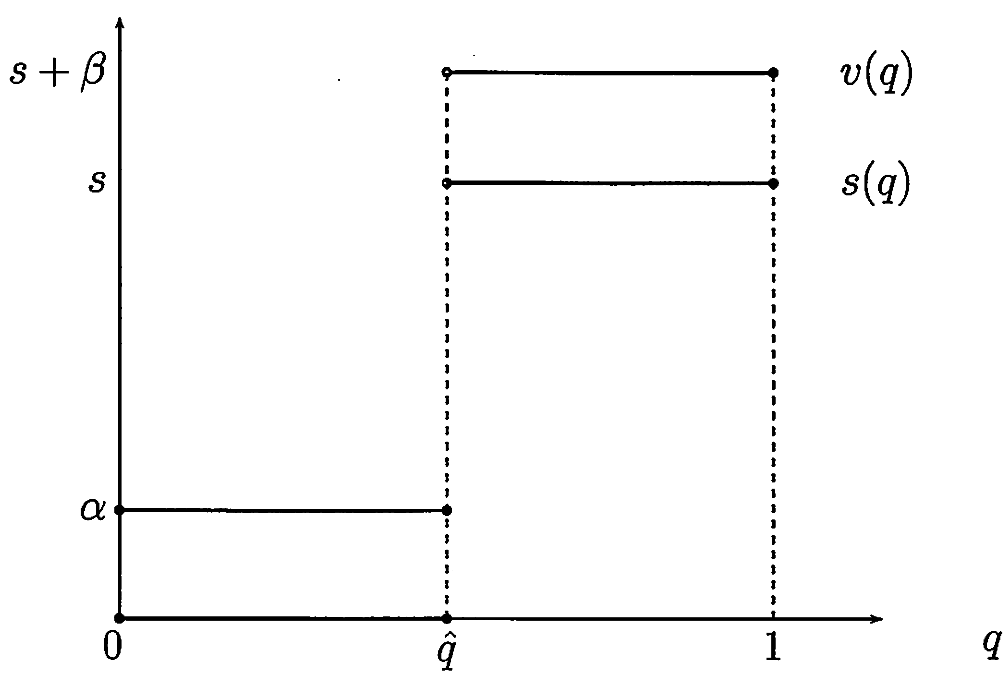

Figure 2: "Bargaining with Interdependent Values" (Deneckere and Liang)

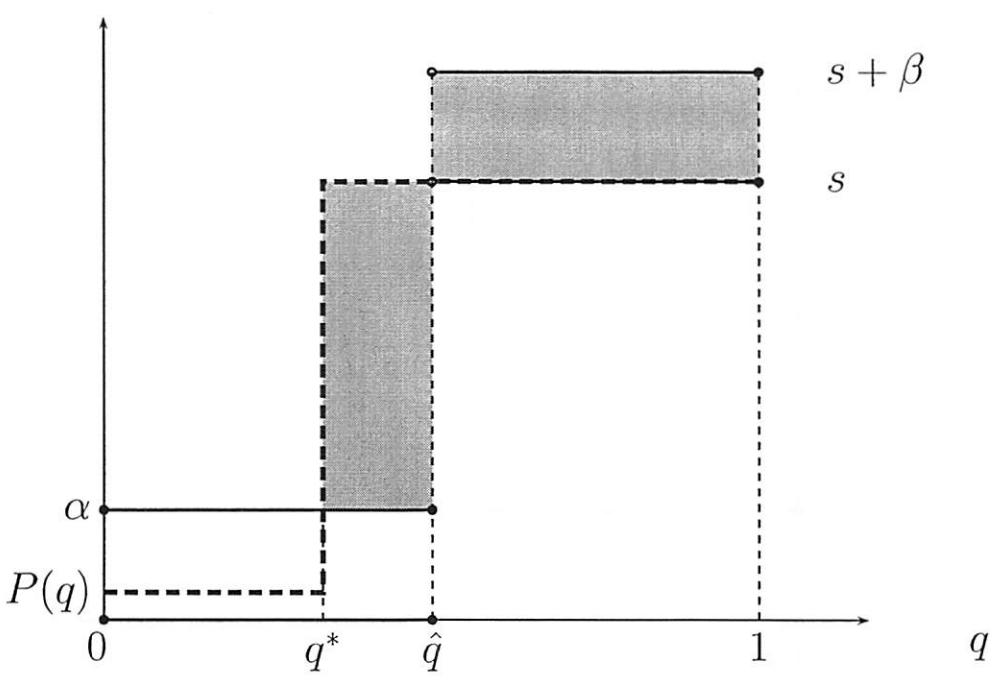

Figure 3: "Bargaining with Interdependent Values" (Deneckere and Liang)