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1 Introduction

A set of odds posted by a bookie is *coherent* if it is impossible to make a sure profit by betting against the bookie. DeFinetti [3, p. 63], among others, has argued that

> [it is] precisely this condition of coherence which constitutes the sole principle from which one can deduce the whole calculus of probability: this calculus then appears as a set of rules to which the subjective evaluation of probability of various events by the same individual ought to conform if there is not to be a fundamental contradiction among them.

The basis for this claim is the following theorem due originally to de-Finetti. *A set of odds is coherent if and only if they are derived from a probability measure.*[1] To illustrate, assume that a bookie posts odds of $\frac{1}{3}$ on A, $\frac{1}{3}$ on B, but $\frac{1}{2}$ on $A \cup B$. even though $A \cap B = \varnothing$. Then a smart bettor will sell the bookies a bet that pays \$1 if A happens, charging her $\frac{1}{3} - \varepsilon$, he will sell her another bet that pays \$1 if B happens for the same price, and pay her $\frac{1}{2} + \varepsilon$ for a bet that pays \$1 is $A \cup B$ happens. For $\varepsilon < \frac{1}{18}$, the bettor ends up with a sure gain of $\frac{1}{6} - 3\varepsilon > 0$.

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[1] A very general version of this result may be found in Heath and Sudderth [5].

But does the aforementioned theorem actually imply "a set of rules to which the subjective evaluation of probability ... ought to conform?" There are two interpretations we can make of this statement. One is that being in a betting environment forces a bookie to post odds in a way that makes her appear to have a subjective probability. The other is that she has a subjective probability and that placing her in a betting environment enables us to uncover her beliefs.

This second interpretation is clearly flawed, since in a betting environment, a risk averse bookie's odds will be posted so as to equate the supply and demand of bets, thus guaranteeing a sure payoff to the bookie. The supply and demand for bets measures the sentiments of the bettors, not of the bookie. For example, if the bookie believes that the probability of a certain event A is $\frac{1}{3}$, but bettors believe that the probability of this event is $\frac{1}{2}$, then her best strategy is to set the odds on A at $\frac{1}{2} - \varepsilon$. The bettors will thus bet on A. From the bookie's perspective, $\frac{1}{2} - \varepsilon$ dollars such played result in the lottery $(\frac{1}{2} - \varepsilon, \frac{2}{3}; \frac{1}{2} - \varepsilon - 1, \frac{1}{3})$. The expected value of this lottery is (almost) $\frac{1}{6}$ (see also Corollary 1 below).

In this paper we point out that the first interpretation suffers from a related flaw. It ignores the fact that the odds ratio posted by a bookie is merely a strategic decision in a game being played against the pool of bettors.[2] What we need to do is examine the equilibria of the underlying betting game in order to draw conclusions about the equilibrium odds. Below we construct an example in which this betting game has a subgame perfect equilibrium with incoherent odds, even though all players possess additive probabilities. It is true that these incoherent odds leave the bookie vulnerable to arbitrage—it's just that our particular collection of bettors does not find this arbitrage opportunity to be their most attractive collection of bets. If the bettors were only to concentrate on the sure gain, they would have to behave in a maximin fashion, behavior which most decision theorists would reject. Once the game theoretic nature of Dutch book interactions is recognized, all bets are off as to the kind of behavior we should expect to see.

Naturally, there are some unusual things about our example. We do not assume that the actual odds are common knowledge, or even commonly held. Indeed a difference of opinion is necessary for our example. The second thing that we need that is a bit unusual in a game theoretic setting is that

[2]Or against one bettor, about whose preferences and beliefs the bookie is uncertain.

our bettors are not expected utility maximizers. This does not bother us, since if it requires everyone else to be an expected utility maximizer to force any individual to behave in accordance with subjective probability, then the argument that everyone ought to behave as though they have a subjective probability loses most of its force. We do however use a risk averse expected utility maximizing bookie.

2 A game theoretic analysis

We consider the following simple situation. There are disjoint events A and B which exhaust the set of possible states. The bookie posts prices a and b for one dollar bets on A and B respectively. Bettors place bets after the prices are posted. A bettor may either buy or sell bets at the posted prices, so we will restrict prices to satisfy $0 \leqslant a, b \leqslant 1$. Coherence of course requires that $a+b=1$. For our purposes we assume that there is a single bettor, but that the bookie has incomplete information about his preferences and beliefs. The bettor is one of two types. To simplify the exposition, we treat this as if there are two bettors and that the bookie knows their preferences and beliefs, as, since we assume that the bookie is an expected utility maximizer, this does not change the decision problem for her.

We impose the following budget constraint on the bettors. Each bettor has only one dollar and is not permitted to buy on credit nor is he allowed to sell a bet (buy a negative quantity) unless he proves that he possesses sufficient funds to pay off in the event he loses.

Let x_i denote the amount that bettor i bets on event A, and y_i denote the amount on B. A negative value indicates a sale. As usual, for any number x, x^+ denotes $\max\{x, 0\}$ and x^- denotes $\max\{-x, 0\}$. Note that $x = x^+ - x^-$. We can write the budget constraint for a bettor facing prices a and b as

$$x^+ + y^+ + \frac{x^-}{a} + \frac{y^-}{b} \leqslant 1 + x^- + y^-.$$

Without loss of generality, we may restrict attention to prices satisfying $a + b \leqslant 1$. For suppose $a+b>1$. Then set $a' = 1-b$ and $b' = 1-a$, so $a'+b' \leqslant 1$. For bettor i, set $y_i' = -(1-a)x_i/a$, and $x_i' = -(1-b)y_i/b$. These new bets yield the same monetary payoffs in each event as the bets x_i and y_i. Furthermore, they satisfy the budget constraint $x'+y' \leqslant 1$. Formally then, the strategy set of the bookie is

$$S = \{(a,b) : a \geqslant 0, b \geqslant 0, a+b \leqslant 1\}.$$

Let the strategy set T of a bettor be the collection of all betting schemes satisfying his budget constraint:

$$T = \left\{ (\hat{x},\hat{y})\colon S \to \mathbb{R}^2 \;\middle|\; \begin{array}{l} \hat{x}^+(a,b) + \hat{y}^+(a,b) + \dfrac{\hat{x}^-(a,b)}{a} + \dfrac{\hat{y}^-(a,b)}{b} \\ \leqslant 1 + \hat{x}^-(a,b) + \hat{y}^-(a,b) \text{ for all } (a,b) \in S \end{array} \right\}.$$

To complete the description of the game we need to specify payoffs as a function of the strategies. Let $U(a,b,x_1,y_1,x_2,y_2)$ denote the bookie's payoff, and let $V_i(a,b,x_1,y_1,x_2,y_2)$ denote the payoff of bettor i. Since each bettor's payoff depends only on his own bets, for simplicity we will write $V_i(a,b,x_i,y_i)$. The bookie moves first, so the appropriate equilibrium concept is subgame perfect equilibrium.

Definition 1 *A* subgame perfect equilibrium, *or* equilibrium *for brevity, of the two bettor game is a vector $(a,b,\hat{x}_1,\hat{y}_1,\hat{x}_2,\hat{y}_2)$ in $S \times T \times T$ satisfying:*

1. *For each bettor i and for all (a,b) in S, $V_i(a,b,\hat{x}_i(a,b),\hat{y}_i(a,b)) \geqslant V_i(a,b,x,y))$ for all (x,y) satisfying $x^+ + y^+ + \frac{x^-}{a} + \frac{y^-}{b} \leqslant 1 + x^- + y^-$.*
2. *The bookie maximizes her payoff taking $\hat{x}$ and $\hat{y}$ as given functions. That is,*

$$U(a,b,\hat{x}_1(a,b),\hat{y}_1(a,b),\hat{x}_2(a,b),\hat{y}_2(a,b)) \geqslant$$
$$U(a',b',\hat{x}_1(a',b'),\hat{y}_1(a',b'),\hat{x}_2(a',b'),\hat{y}_2(a',b'))$$

for all $(a',b') \in S$.

For the remainder of our results, the players are assumed to evaluate lotteries using a *rank dependent* functional. That is, there is a utility function u and a continuous probability transformation function $g : [0,1] \to [0,1]$, strictly increasing and satisfying $g(0) = 0$ and $g(1) = 1$, which determine the value of a lottery, see [6]. The formula for the value of a lottery with (subjective) distribution function F is

$$V(F) = \int u(w)\, d(g \circ F)(w).$$

For a random variable taking on only two values, $v < w$, with probabilities q and $1-q$, the formula for the value reduces to

$$V(v,q;w,1-q) = u(v)g(q) + u(w)\,(1-g(q))\,. \tag{1}$$

When both u and g are concave (convex), the functional exhibits risk aversion (seeking), see [2]. If g is the identity function, then the rank dependent functional reduces to expected utility. We assume that the bettors' probability transformation functions are concave.

2.1 The bettors' decision problem

Let q and $1-q$ denote a bettor's subjective probability of A and B (for simplicity, we delete the index i). Given the prices a and b where $a+b \leqslant 1$, we may assume that $x, y \geqslant 0$, that is, the bettor does not sale bets on either A or B. Suppose, for example, that $y < 0$. then as before, the bettor is indifferent between selling y on B and betting $-(1-b)y/b$ on A at the rate $a' = 1-b$. Since $a \leqslant a'$, the bettor cannot be worse off by betting only on A. Assuming rank dependent preferences, the bettor's payoffs are given in Table 1.

	Bettor's Value
$\frac{x}{a} < \frac{y}{b}$	$g(q)u\left(\frac{x}{a}-x-y\right) + (1-g(q))\,u\left(\frac{y}{b}-x-y\right)$
$\frac{x}{a} > \frac{y}{b}$	$(1-g(1-q))\,u\left(\frac{x}{a}-x-y\right) + g(1-q)u\left(\frac{y}{b}-x-y\right)$
$\frac{x}{a} = \frac{y}{b}$	$u\left(\frac{x}{a}-x-y\right) = u\left(\frac{y}{b}-x-y\right)$

Table 1: Bettor's Payoffs from Purchases of Bets x on A and y on B

The next lemma simplifies the analysis of the bettors' best response behavior. It states that if $a+b \leqslant 1$, then a bettor will either *plunge* by betting everything on A or on B or else *hedge* by betting so as to receive the same payoff in either event. (Remember that if $a+b < 1$, then the payoff from hedging is strictly positive whichever event occurs.) The reason for this is that if $a + b \leqslant 1$ and the bettor chooses x and y to satisfy $x/a < y/b$, then he becomes better off by increasing y up to $y = 1$. Similarly if he chooses to set $x/a > y/b$, he should set $x = 1$, and if he chooses to set $x/a = y/b$, then he should set $x = a/(a+b)$ and $y = b/(a+b)$. The proof is a straightforward calculation.

Lemma 1 *If u is (weakly) convex and g is (weakly) concave, and if $a+b \leqslant 1$, then a bettor's optimal response is to Plunge on A, Plunge on B, or Hedge. The payoffs are given in Table 2.*

Action	Bettor's value
Plunge on A	$u(-1)g(1-q) + u\left(\frac{1-a}{a}\right)(1-g(1-q))$
Plunge on B	$u(-1)g(q) + u\left(\frac{1-b}{b}\right)(1-g(q))$
Hedge	$u\left(\frac{1-a-b}{a+b}\right)$

Table 2: Bettor's relevant payoffs.

The bettor's optimal strategy depends on which of these three options yields the highest utility. We start with the case of linear utility, $u(x) = x$. Define the parameters

$$\alpha = \frac{g(1-q)}{1-g(1-q)}$$

and

$$\beta = \frac{1-g(q)}{g(q)}.$$

Since g is concave, $\alpha \geqslant \beta$. Simple calculations prove the next lemma.

Proposition 1 *If a bettor's utility function u is linear and his probability transformation function g is concave, then his optimal strategies are:*

- *Plunge on A whenever $b/a \geqslant \alpha$;*

- *Plunge on B whenever $b/a \leqslant \beta$;*
- *Hedge whenever $\beta \leqslant b/a \leqslant \alpha$.*

These strategies are depicted in Figure 1.

Note that when g is linear, that is, when the bettor is risk neutral, $\alpha = \beta$ and he will buy either on A or on B, but not on both, unless $b/a = \alpha = \beta$, in which case he is indifferent between all three strategies.

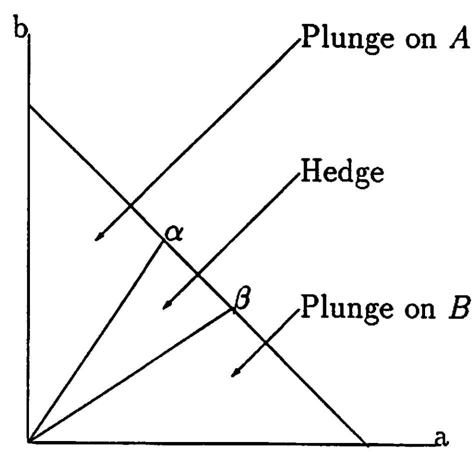

Figure 1: Bettor's optimal response (Linear utility)

It follows from Proposition 1 that if the bettor's utility function is linear, but his probability transformation function is concave, then his optimal strategy depends only on the ratio b/a. Since the bettor is buying bets, the bookie always prefers to raise both prices proportionately. We thus get the following result.

Theorem 1 *If all bettors maximize rank dependent functionals with linear utility functions and concave probability transformation functions, and the bookie's preferences are monotonic, then the bookie's equilibrium strategy satisfies $a + b = 1$.*

In the special case of identical expected value maximizing bettors, the bookie will set the prices equal to the bettors' subjective probabilities, so that they are indifferent among all bets. Otherwise, the bettors will bet everything on the event whose price is less than its subjective probability, and the bookie benefits by raising the price of this event. In this case, the bookie's prices are the bettor's subjective probabilities, not her own.

Corollary 1 *With only one type of bettor, if u and g are linear (i.e., the bettor is an expected value maximizer), then the equilibrium strategy of the bookie is to set $a = q$ and $b = 1 - q$ (q and $1 - q$ are the bettor's subjective probability of A and B).*

So far we have produced nothing counterintuitive. For that we need to analyze bettors with nonlinear convex utility functions. Specifically, consider a utility for a bettor of the following form.

$$u(x) = \begin{cases} e^{kx} & x \geqslant 0 \\ x + 1 & x \leqslant 0 \end{cases}$$

For $k \geqslant 0$ this is a (weakly) convex increasing function. Set

$$s = -ln[1 - g(1 - q)]$$

and

$$t = -ln[1 - g(q)].$$

The parameters s and t depend only on the bettor's belief q and his preferences through g. By choosing q and the concave function g carefully, we can choose s and t to be arbitrary positive numbers.

The equation of the set of (a, b) pairs making the bettor indifferent between plunging on A and plunging on B is

$$b = \frac{ak}{k - a(s - t)}. \tag{2}$$

This curve is labeled "A vs. B" in Figure 2. It is convex if $s > t$ and concave if $s < t$.

The equation of the (a, b) pairs for which plunging on A is indifferent to hedging is given by

$$b = \frac{a^2 s}{k - as}. \tag{3}$$

This curve is convex and intersects the line $a+b=1$ at the point $a=k/(s+k)$. It is labeled "A vs. H" in Figure 2. Finally, the locus of (a,b) pairs which the bettor indifferent between plunging on B and hedging is

$$a = \frac{b^2 t}{k - bt}. \tag{4}$$

This curve intersects the $a+b=1$ line at $a=t/(k+t)$. It is labeled "B vs. H" in Figure 2. Note that the transitivity of indifference guarantees that if two of these curves intersect, then all three of them intersect at the same point. Figure 2 depicts these loci for the case $s=1.8$, $t=1.5$, and $k=1$, although it is not drawn to scale. The regions are labeled with the bettor's preferences. That is, in the region marked "AHB," the bettor prefers plunging on A to hedging to plunging on B. The second drawing indicates the bettor's best responses.

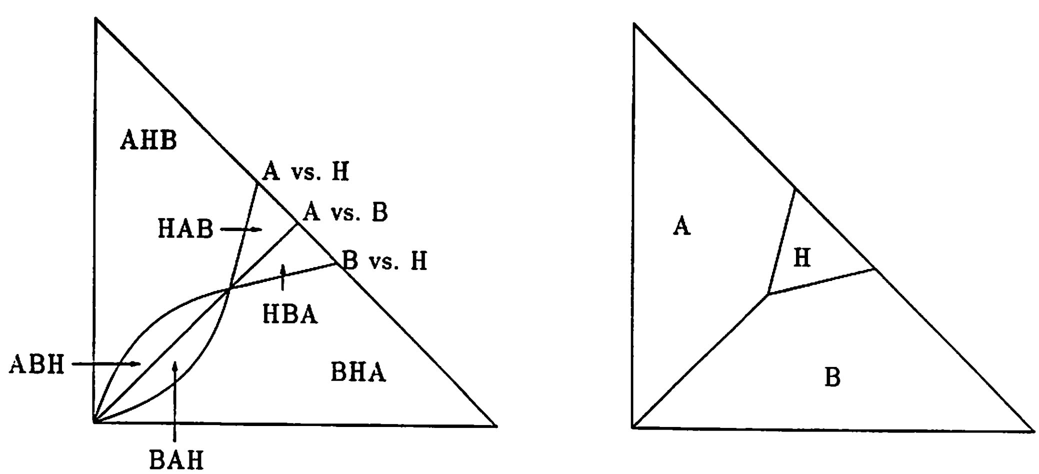

Figure 2: Bettor's choices

2.2 The bookie's optimal strategy

Suppose that when the bettor is indifferent between plunging and hedging, the bettor will plunge. (This will turn out to be the case in our equilibrium.) Even in this case, if there is only one bettor, the logic that when the bettor is plunging, the bookie wants to raise the price of the bettor's bet and when he's hedging, the bookie wants to raise both prices, drives the equilibrium prices to satisfy $a+b=1$. Of course, since the boundary lines are nonlinear, the bookie may have to change the price ratio while raising the prices.

It is this phenomenon that allows to construct an equilibrium with $a+b<1$. Since the set of directions that we can raise prices depends on the prices, if there are heterogeneous bettors, these sets of directions may not overlap. That is, it may be impossible to raise prices and keep both bettors making the same bets.

Suppose that the bookie is maximizing an expected utility functional and that there are two bettors, I and II, with optimal strategies as indicated in Figure 3. There are five points of special interest, labeled P, Q, R, S, and T. Point Q has the largest a for which both bettors will plunge on A, and point T has the largest b for which both bettors will plunge on B. At point R, Bettor I hedges while II plunges on A. At S, I plunges on B, while II hedges. The segment joining R and S has both bettors hedging. Finally, at point P, Bettor I is plunging on B and Bettor II is plunging on A. It is easy to see that the bookie's expected utility will be maximized at one of P, Q, T, or on the segment $\overline{RS}$. Letting p and $1-p$ denote the bookie's subjective probabilities of A and B, her expected utilities are given in Table 3.

P	$pu\left(2-\frac{1}{a}\right)+(1-p)u\left(2-\frac{1}{b}\right)$
Q	$pu\left(2-\frac{2}{a}\right)+(1-p)u(2)$
R	$pu\left(1-\frac{1}{a}\right)+(1-p)u(1)$
S	$pu(1)+(1-p)u\left(1-\frac{1}{b}\right)$
T	$pu(2)+(1-p)u\left(2-\frac{2}{b}\right)$
$\overline{RS}$	$u(0)$

Table 3: Bookie's candidate strategies.

It is possible to choose values for p, s_I, t_I, k_I, s_{II}, t_{II}, and k_{II}, and a concave increasing utility u for the bookie so that point P has the highest

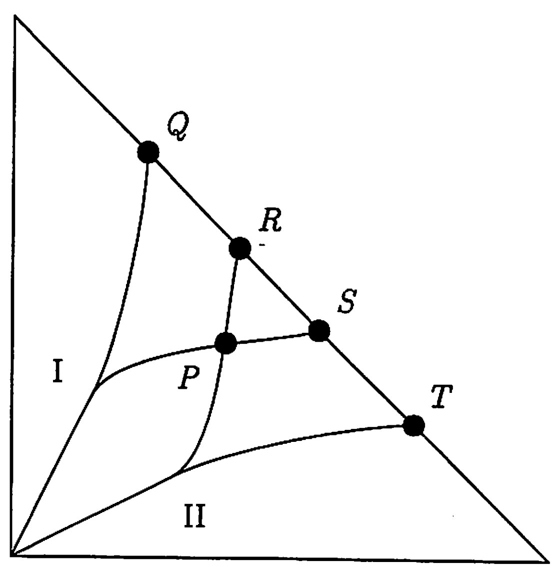

Figure 3: Two bettors

expected utility. For instance, choose $k_I = 2.857$, $k_{II} = 1$, $s_I = 28.57$, $t_I = 1$, $s_{II} = 3$, $t_{II} = 12$ (Figure 3 is based on these values, although it is not drawn to scale) and $p = .2$. For the bookie's utility choose

$$u(x) = \begin{cases} x & x \geqslant -2 \\ 4x + 6 & x \leqslant -2. \end{cases}$$

Then to three decimal places, the bookie's expected utilities are given in Table 4. In the equilibrium of this game the bookie chooses P and does not post additive prices.[3]

Point	(a, b)	Expected utility
P	$(.248, .727)$	0.080
Q	$(.091, .909)$	-13.200
R	$(.250, .750)$	-0.400
S	$(.259, .741)$	-0.215
T	$(.923, .077)$	-71.600
$\overline{RS}$		0.000

Table 4: Bookie's candidate strategies: Numerical example.

To be fair, there is another equilibrium, where the bettors hedge when indifferent, in which the bookie posts additive prices and her expected utility is zero. Our point is that there is at least one equilibrium (in fact the bookie's favorite) in which she sets nonadditive prices.

3 Conclusions

In a related paper [1] we pointed out that strategic behavior on the part of the bookie may eliminate the Dutch book argument against violations of the

[3]The utility function given above is not differentiable at $x = -2$, but it can be smoothed in a neighborhood of -2 without changing any of the relevant expected utilities. Thus we could specify differentiable utility with same equilibrium.

low of conditional probability. The analysis there too involves two bettors (with different beliefs). A major difference between the results of the current paper and the results obtained in [1] is that here, at least in one equilibrium situation, the bookie's optimal strategy *must* involve a violation of probability theory. In [1], on the other hand, the most we can get is a situation where posting non-multiplicative rates is as good as using multiplicative ones.

Several recent nonexpected utility models are based on the assumption that decision makers do not obey some of the basic rules of probability theory. Gilboa [4] and Schmeidler [7] present models of behavior with nonadditive probabilities. We do not claim that the reason for these violations is that people behave strategically. Nor do we want to suggest that the correct interpretation of the above mentioned models is game theoretic. However, we believe that these models and empirical evidence cannot be rejected as irrelevant on the grounds that violations of probability theory expose the decision maker to a Dutch book. All of these models analyze the behavior of a single agent. Dutch books must involve at least two agents, therefore the correct framework is game theoretic, and one must assume that agents behave strategically. Traditional analyses of Dutch books assume that the person offering choices to the subject is much more sophisticated than the subject. Our approach is more symmetric in that the subject bookie is at least as sophisticated as the bettors.

Our major claim is that when people behave strategically, it is wrong to interpret the betting rates they announce as their subjective probabilities of the different events. Instead, these rates should be understood as the prices at which subjects are willing to trade certain goods (simple lotteries tickets). If the market is noncompetitive—and the framework of deFinetti's Dutch book is basically noncompetitive—then the observed rates at which subjects are willing to exchange goods typically do not equal their true marginal rate of substitution between them.

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