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CHOICE-SET FORMS ARE DUAL TO OUTCOME-SET FORMS

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Abstract. Alós-Ferrer and Ritzberger (2013) specify each node in a game tree as the set of outcomes that yet remain conceivable. In contrast, Streufert (2015a) specifies each node as the set of choices that have already been made. This symmetry suggests that the two formulations are "dual" in some sense.

In this paper I develop this duality. In particular, I define suitable conversion procedures, and show that there is a one-to-one correspondence between choice-set forms and outcome-set forms. The analysis encompasses discrete forms with finite or infinite horizons.

1. Introduction

1.1. Motivation

Von Neumann and Morgenstern (1944, Sections 9 and 10) specify each node in a game tree as a set of outcomes. Recently, this outcome-set formulation has been insightfully extended to the infinite horizon by the discrete extensive forms of Alós-Ferrer and Ritzberger (2013 henceforth AR). [The present paper always assumes discreteness. Still more general outcome-set formulations that do not satisfy discreteness are developed in Alós-Ferrer and Ritzberger (2005, 2008).]

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In contrast, Streufert (2015a henceforth S1) specifies each node in a game tree as a set of choices (i.e. agent-specific actions). These choice-set forms appear to be "dual" to the outcome-set forms of AR. Specifically, in a choice-set form, the initial node is the empty set. Thereafter every node is a superset of its predecessors. Symmetrically, in an outcome-set form, the initial node is the set of all conceivable outcomes. Thereafter every node is a subset of its predecessors.

This paper develops this duality. Specifically, I show how to convert a choice-set form into an outcome-set form. Then I show the senses in which the original choice-set form and its conversion are equivalent. Further, in the reverse direction, I show how to convert an outcome-set form into a choice-set form. Then I show the senses in which the original outcome-set form and its conversion are equivalent. Finally, I demonstrate that these two conversion processes are inverses, and consequently, that there is a meaningful one-to-one correspondence between choice-set forms and outcome-set forms. This duality is the main result.

AR show that outcome-set forms cannot accommodate absent-mindedness. Similarly, S1 shows that choice-set forms cannot accommodate absent-mindedness. These two facts accord with the above duality. In contrast, absent-mindedness can be accommodated by the extensive forms of Osborne and Rubinstein (1994 henceforth OR). Accordingly, I show that OR forms without absent-mindedness, choice-set forms, and outcome-set forms are all in one-to-one correspondence with one another. This triple equivalence follows easily from a theorem of S1 and the duality of this paper.

1.2. Overview

Section 2 sets the stage by recalling the definition of a choice-set form from S1, and by recalling the definition of a concise AR* outcome-set form from Streufert (2014b henceforth S2). S2's theorems show that concise AR* outcome-set forms and AR discrete extensive forms are equally general (though the latter can specify simultaneous moves without multiple information sets).

Section 3 defines two conversion procedures. The first converts a no-trivial-move choice-set form into a concise AR* outcome-set form. Essentially, the terminal choice-set nodes of the original form become the outcomes of the new outcome-set form. The second procedure converts a concise AR* outcome-set form into a no-trivial-move choice-set form. Essentially, the outcome-set choices in the original form become the choices in the new choice-set form. The well-definition and attractive properties of these two procedures are established in Theorems 1 and 2, respectively.

Section 4 shows that the two procedures are inverses of one another. The situation is complicated because both procedures agglomerate old objects to define new objects. For instance, the first procedure uses sets of choices to define outcomes, and the second procedure uses sets of outcomes to define choices. I compose the two procedures in order to prove that they are inverses. Accordingly, the first procedure followed by the second procedure uses collections of sets of old choices to define new choices. Yet, it turns out that this double procedure merely renames the choices of the old choice-set form. Accordingly, Theorem 3 defines as "equal" two choice-set forms that are equivalent by renaming choices. Similarly, it defines as "equal" two concise AR* outcome-set forms that are equivalent by renaming outcomes. Then it establishes a bijection, from the class of no-trivial-move choice-set forms, onto the class of concise AR* outcome-set forms. In this precise sense, choice-set forms are dual to outcome-set forms. This is the main result.

Section 5 recalls S1's result that the class of no-absent-minded OR forms is in one-to-one correspondence with the class of choice-set forms. That result and this paper's Theorem 3 are two links in a triple equivalence between no-absent-minded OR forms, choice-set forms, and AR* outcome-set forms. Corollary 5.1 combines the two to provide the third link. In particular, it shows that the class of no-trivial-move no-absent-minded OR forms is in one-to-one correspondence with the class of concise AR* outcome-set forms. The generality of the former relative to the latter is a secondary contribution of this paper.

2. Setup

Section 2.1 summarizes the relevant material from S1 about choice-set forms. Then Section 2.2 summarizes the relevant material from S2 about concise AR* outcome-set forms. Finally Section 2.3 discusses some of the similarities between the two.

2.1. Choice-set forms

Let C be an arbitrary set, and call a member c of the set C a *choice*. A *choice-set preform* (`S1` equation (8)) is a pair (C,N) such that

(1a) $$N \text{ is a nonempty collection of subsets of } C\ ,$$

(1b) $$C \subseteq \cup N\ ,$$

(1c) $$N \smallsetminus T = \{\ \cup T^* \mid T^* \text{ is an infinite chain in } T\ \}\ ,$$

(1d) $$(\forall t {\neq} \{\})(\exists ! c)\ c {\in} t \text{ and } t {\smallsetminus} \{c\} {\in} T\ , \text{ and}$$

(1e) $$(\forall t^1, t^2)\ F(t^1){=}F(t^2) \text{ or } F(t^1){\cap}F(t^2){=}\varnothing\ ,$$

where

(2) $$T := \{\ n \mid n \text{ is finite}\ \} \text{ and}$$

(3) $$F := \{\ (t,c) \mid c {\notin} t \text{ and } t{\cup}\{c\}{\in}T\ \}\ .$$

Call a member n of the set N a *node*, and call F the *feasibility correspondence*.

One node $n^\flat$ is said to *precede* another node n if $n^\flat \subset n$. Equivalently, n is said to *succeed* $n^\flat$. By `S1` Lemma B.5, $\{\}$ must be a node, and this node clearly precedes all other nodes. At the other extreme, define a *terminal* node to be a node with no successor. By `S1` Corollary 5.2(c), the collection of nonterminal nodes equals $F^{-1}(C)$. This result is displayed in the first two columns of the next-to-last row of Table 1. Finally, define

(4) $$p := \{\ (t, t{\smallsetminus}\{c\}) \mid c{\in}t \text{ and } t{\smallsetminus}\{c\}{\in}T\ \}\ .$$

By (1d), p is a function from $T \smallsetminus \{\{\}\}$. Call p the *immediate predecessor* function.

Let I be an arbitrary set, and call a member i of the set I a *player*. A *choice-set form* (`S1` equation (14)) is a pair $((C_i)_i, N)$ such that

(5a) $$(\cup_i C_i, N) \text{ is a choice-set preform (1)}\ ,$$

(5b) $$(\forall i{\neq}j)\ C_i{\cap}C_j = \varnothing\ , \text{ and}$$

(5c) $$(\forall i)(\forall t)\ F(t){\subseteq}C_i \text{ or } F(t){\cap}C_i{=}\varnothing\ .$$

A preform is a one-player form. Specifically, (C,N) is a choice-set preform iff $((C),N)$ is a choice-set form, where $(C_i)_i = (C)$ is taken to mean $I = \{1\}$ and $C_1 = C$. In this sense, definitions and results for forms also apply to preforms.

2.2. Concise AR* outcome-set forms

This subsection reviews S2's definition of a concise AR* outcome-set form. Such a form is essentially a reformulated discrete extensive form (AR Definition 6) that has been restricted to satisfy conciseness (S2 equation (13)). In particular, a concise AR* outcome-set form is less general than an AR discrete extensive form in the sense that the latter can specify simultaneous moves without multiple information sets. However, S2 Theorems 1 and 2 show that the former is as general as the latter in every other matter of interest to game theorists.

Let W be an arbitrary set, and call a member w of the set W an *outcome*. An AR* *outcome-set tree* (S2 equation (1)) is a pair $(W,\dot{N})$ such that

(6a) $\dot{N}$ is a collection of subsets of W containing W but not $\varnothing$,

(6b) $$(\forall \dot{n}^1{\neq}\dot{n}^2)\ \dot{n}^1{\supset}\dot{n}^2 \text{ or } \dot{n}^2{\supset}\dot{n}^1 \text{ or } \dot{n}^1{\cap}\dot{n}^2{=}\varnothing\ ,$$

(6c) $$\dot{N} \supseteq \{\{w\}|w\}\ ,$$

(6d) $$\dot{N} \supseteq \{\ \cap\dot{N}^* \mid \dot{N}^* \text{ is a nonempty chain in } N\ \}\ ,$$

(6e) $$\text{and } \dot{N} \subseteq \dot{T}{\cup}\{\{w\}|w\}\ ,$$

where $\dot{T}$ is defined by

(7) $$\dot{T} := \{\ \dot{n} \mid \{\dot{n}^\flat|\dot{n}^\flat{\supset}\dot{n}\} \text{ is finite }\}\ ,$$

and $\{\{w\}|w\}$ is the collection of singletons of the form $\{w\}$. A member $\dot{n}$ of the collection $\dot{N}$ is called a *node*.

One node $\dot{n}^\flat$ is said to *precede* another node $\dot{n}$ if $\dot{n}^\flat \supset \dot{n}$. Equivalently, $\dot{n}$ is said to *succeed* $\dot{n}^\flat$. Note that W precedes all other nodes. At the other extreme, define a *terminal* node to be a node with no successor. By (6a) and (6c), it is immediate that the collection of terminal nodes equals $\{\{w\}|w\}$. This immediate result is displayed by the appearance of $\{\{w\}|w\}$ in the last row of Table 1. Finally (by S2 Lemma A.1, or by AR equation (2) without the convention concerning W), define the *immediate predecessor* function $\dot{p}:\dot{T}{\smallsetminus}\{W\}{\rightarrow}\dot{T}$ by

(8) $$\dot{p}(\dot{t}) := \min\{\dot{t}^\flat|\dot{t}^\flat{\supset}\dot{t}\}\ .$$

An $\texttt{AR}^*$ *outcome-set preform* ($\texttt{S2}$ equation (8)) is a triple $(W,\dot{N},\dot{C})$ such that[1]

$$\text{(9a)} \quad (W,\dot{N}) \text{ is an } \texttt{AR}^* \text{ outcome-set tree (6) ,}$$

$$\text{(9b)} \quad \dot{C} \text{ is a collection of nonempty subsets of } W \text{ ,}$$

$$\text{(9c)} \quad (\forall\dot{t})\ \dot{p}^{-1}(\dot{t}) = \{\ \dot{t}\cap\dot{c} \mid \dot{c}\in\dot{F}(\dot{t})\ \} \text{ ,}$$

$$\text{(9d)} \quad (\forall\dot{t}) \text{ the members of } \dot{F}(\dot{t}) \text{ are disjoint , and}$$

$$\text{(9e)} \quad (\forall\dot{t}^1,\dot{t}^2)\ \dot{F}(\dot{t}^1){=}\dot{F}(\dot{t}^2) \text{ or } \dot{F}(\dot{t}^1)\cap\dot{F}(\dot{t}^2){=}\varnothing \text{ ,}$$

where $\dot{T}$ and $\dot{p}$ are derived from $(W,\dot{N})$ by (7) and (8), and where $\dot{F}$ is defined by

$$\text{(10)} \quad \dot{F} := \{\ (\dot{t},\dot{c}) \mid \dot{c}\not\supseteq\dot{t} \text{ and } (\exists\dot{t}^\sharp\in\dot{p}^{-1}(\dot{t}))\ \dot{c}\supseteq\dot{t}^\sharp\ \} \text{ .}$$

A member $\dot{c}$ of the collection $\dot{C}$ is called a *choice*.

$\dot{F}$ is called the *feasibility correspondence*. Its domain, denoted by $\dot{F}^{-1}(\dot{C})$, is equal to the collection $\dot{N}\smallsetminus\{\{w\}|w\}$ of nonterminal nodes by $\texttt{S2}$ Lemma A.4 and $\texttt{S2}$ equation (4). This result is displayed by the equality in the next-to-last row of Table 1. Finally, a preform $(W,\dot{N},\dot{C})$ is said to be *concise* ($\texttt{S2}$ equation (13)) if

$$\text{(11)} \quad (\forall\dot{c})\ \dot{c} \subseteq \cup\dot{F}^{-1}(\dot{c}) \text{ .}$$

Conciseness requires that every outcome in every choice is contained in at least one node from which the choice is feasible.

Let I be an arbitrary set, and call a member i of the set I a *player*. Then a *concise* $\texttt{AR}^*$ *outcome-set form* ($\texttt{S2}$ equation (20)) is a triple $(W,\dot{N},(\dot{C}_i)_i)$ such that[1]

$$\text{(12a)} \quad (W,\dot{N},\cup_i\dot{C}_i) \text{ is a concise (11) } \texttt{AR}^* \text{ outcome-set preform (9) ,}$$

$$\text{(12b)} \quad (\forall i{\neq}j)\ \dot{C}_i\cap\dot{C}_j = \varnothing \text{ , and}$$

$$\text{(12c)} \quad (\forall i)(\forall\dot{t})\ \dot{F}(\dot{t}){\subseteq}\dot{C}_i \text{ or } \dot{F}(\dot{t})\cap\dot{C}_i{=}\varnothing \text{ ,}$$

where $\dot{T}$ is defined by (7) and where $\dot{F}$ is derived from $(W,\dot{N},\cup_i\dot{C}_i)$ by (10). A preform is a one-player form. Specifically, $(W,\dot{N},\dot{C})$ is a concise $\texttt{AR}^*$ outcome-set preform iff $(W,\dot{N},(\dot{C}))$ is a concise $\texttt{AR}^*$ outcome-set

[1]The $\texttt{S2}$ counterparts of conditions (9c), (9d), and (12c) constrain the node $\dot{t}$ to lie within the collection $\dot{X}$ defined in $\texttt{S2}$ equation (4). This difference is inconsequential because [1] $\dot{t}\notin\dot{X}$ implies both $\dot{p}^{-1}(\dot{t}){=}\varnothing$ and $\dot{F}(\dot{t}){=}\varnothing$ by $\texttt{S2}$ Lemma A.3(a) and $\texttt{S2}$ Lemma A.4(b) and thus [2] the three conditions hold vacuously when $\dot{t}\notin\dot{X}$.

	choice-set form $((C_i)_i, N)$	concise AR* outcome-set form $(W, \dot{N}, (\dot{C}_i)_i)$
choice	$c \in C := \cup_i C_i$	$\dot{c} \in \dot{C} := \cup_i \dot{C}_i$
node	$n \in N$	$\dot{n} \in \dot{N}$
player	$i \in I$	$i \in I$
node with finitely many predecessors	$t \in T$	$\dot{t} \in \dot{T}$
feasibility correspondence	F	$\dot{F}$
immediate-predecessor function	p	$\dot{p}$
collection of nonterminal nodes	$F^{-1}(C)$	$\dot{N} \setminus \{\{w\}\|w\} = \dot{F}^{-1}(\dot{C})$
collection of terminal nodes	$N \setminus F^{-1}(C)$	$\{\{w\}\|w\} = \dot{N} \setminus \dot{F}^{-1}(\dot{C})$

Table 1. See the second paragraph of Section 2.3.

form, where $(\dot{C}_i)_i = (\dot{C})$ is taken to mean $I = \{1\}$ and $\dot{C}_1 = \dot{C}$. In this sense, definitions and results for forms also apply to preforms.

2.3. Discussion

Choice-set forms and concise AR* outcome-set forms are fundamentally different. The former specifies nodes and outcomes in terms of choices. The latter specifies nodes and choices in terms of outcomes. Nonetheless, there are many similarities.

These similarities are reflected in the notation of Table 1. Its first three rows concern the primitive objects. Its next three rows concern the derived objects. And finally, its last two rows summarize results about terminal and nonterminal nodes. In particular, in a choice-set form, $F^{-1}(C)$ is the set of nonterminal nodes by S1 Corollary 5.2(c). Meanwhile, in a concise AR* outcome-set form, [a] $\{\{w\}|w\}$ is the set of terminal nodes by (6a) and (6c), and [b] $\dot{F}^{-1}(\dot{C})$ is the set of nonterminal nodes by S2 Lemma A.4 and S2 equation (4).

Note that I conserve notation in two unconventional ways. [1] There is no symbol (such as "X" or "$\dot{X}$") for the collection of nonterminal nodes. Accordingly, there is no symbol (such as "Z" or "$\dot{Z}$") for the collection of terminal nodes. Rather, the last rows of Table 1 express these two collections using symbols that have already been defined. [I make just two exceptions. [a] Note 1 used the symbol "$\dot{X}$" to make connections with S2. [b] The next section will define the symbol "Z"

as part of a conversion procedure. The symbol will not be used in any other context.]

[2] Neither choice-set forms nor concise AR* outcome-set forms explicitly specify agents (i.e. information sets). Rather, both use their feasibility correspondence to implicitly specify agents as the inverse images of choices. In particular, S1 equation (12) derives agents in a choice-set form, and S2 equation (10) derives agents in a concise AR* outcome-set form. This innovation traces back to Alós-Ferrer and Ritzberger (2005, Definition 7(i), page 791).

3. Conversion procedures

3.1. Converting choice-set forms to outcome-set forms

Consider a choice-set form $((C_i)_i, N)$. Derive its $C{=}\cup_i C_i$, T (2), and F (3). Then $((C_i)_i, N)$ is said to have *no-trivial-moves* if

$$(\forall t)\ |F(t)| \neq 1\ . \tag{13}$$

The remainder of this paragraph interprets (13). By the definition of F, $F^{-1}(C) = \{\, t \,|\, |F(t)|{\geq}1 \,\}$. Further, by the next-to-last row of Table 1, $F^{-1}(C)$ is the set of nonterminal nodes. Thus (13) is equivalent to requiring that every nonterminal node has at least two feasible choices. Hence (13) may be regarded as a trivial restriction.

Again consider a choice-set form $((C_i)_i, N)$. Derive its $C = \cup_i C_i$ and F (3). By the last row of Table 1, the set of terminal nodes is $N \setminus F^{-1}(C)$. For notational convenience, let

$$Z := N \setminus F^{-1}(C)\ , \tag{14a}$$

and let z denote an element of Z. Next, for any node n, let

$$Z_n := \{\ z \mid n{\subseteq}z\ \} \tag{14b}$$

be the set of terminal nodes that equal or follow n. Lemma A.2(a) shows that every Z_n is nonempty. Note that $(Z_n)_n$ is a function from N into the power set of Z. Thus it maps a set of choices into a collection of sets of choices. Its range is $\{Z_n|n\}$. Further, for any choice c, let

$$Z_c := \{\ z \mid c{\in}z\ \} \tag{14c}$$

be the set of terminal nodes that contain c. Lemma A.3(a) shows that every Z_c is nonempty. Note that $(Z_c)_c$ is a function from C into the power set of Z. Thus it maps a choice into a collection of sets of choices. Its range is $\{Z_c|c\}$.

Part (a) of the following theorem shows how a no-trivial-move choice-set form can be converted into a concise `AR`* outcome-set form. The remaining parts of the theorem describe the sense in which the original choice-set form and the new outcome-set form are "equivalent" to one another.

Theorem 1. *Suppose* $((C_i)_i, N)$ *is a choice-set form (5) with no-trivial-moves (13). Derive* $C{=}\cup_i C_i$, T *(2),* F *(3), and* p *(4). To convert this form, let*

$$W := Z \ , \ \dot{N} := \{Z_n|n\} \ , \ (\dot{C}_i)_i := (\{Z_c|c{\in}C_i\})_i \ ,$$

where Z, $(Z_n)_n$, *and* $(Z_c)_c$ *are defined by (14). Then*

(a) $(W, \dot{N}, (\dot{C}_i)_i)$ *is a concise* `AR`* *outcome-set form (12).*

Further, derive $\dot{C}{=}\cup_i\dot{C}_i$, $\dot{T}$ *(7),* $\dot{p}$ *(8), and* $\dot{F}$ *(10). Then the following hold.*

(b) $(Z_n)_n$ *is a bijection from* N *onto* $\dot{N}$*. Further,* $Z_{\{\}} = W$*.*
(c) $(Z_c)_c$ *is a bijection from* C *onto* $\dot{C}$*.*
(d) $\dot{T} = \{ \ Z_t \mid t{\in}T \ \}$*.*
(e) $\dot{p} = \{ \ (Z_{t^\sharp}, Z_t) \mid (t^\sharp, t){\in}p \ \}$*.*
(f) $(\forall t, c, t^\sharp)$ $(c{\notin}t$ *and* $t{\cup}\{c\}{=}t^\sharp)$ *iff* $(Z_t{=}\dot{p}(Z_{t^\sharp})$ *and* $Z_t{\cap}Z_c{=}Z_{t^\sharp})$*.*
(g) $\dot{F} = \{ \ (Z_t, Z_c) \mid (t, c){\in}F \ \}$*. (Proof A.6.)*

This and several other results are illustrated by the examples of Figures 1 and 2. These examples show (a) how the nodes of Selten's horse game can be formulated in various ways as either choice sets or outcome sets, and (b) how these various formulations can be converted from one to another by means of this paper's theorems.

More specifically, in Figure 1, each node and choice is labelled with three rows. The top row corresponds to the one-player no-trivial-move choice-set form $((C), N)$ that is defined by

$$(15) \qquad \begin{aligned} C &= \{d_1, a_1, d_2, a_2, \ell, r\} \ \text{ and} \\ N &= \{\{\}, \{d_1\}, \{a_1\}, \{a_1, d_2\}, \\ &\qquad \{d_1{,}\ell\}, \{d_1{,}r\}, \{a_1{,}d_2{,}\ell\}, \{a_1{,}d_2{,}r\}, \{a_1{,}a_2\}\} \ . \end{aligned}$$

In accord with Theorem 1, this top row is then converted into the one-player concise `AR`* outcome-set form $(Z, \{Z_n|n\}, (\{Z_c|c\}))$ that appears in the middle row. For the sake of visual clarity, I have introduced the symbols z^1, z^2, z^3, z^4, and z^5 to abbreviate the five terminal nodes

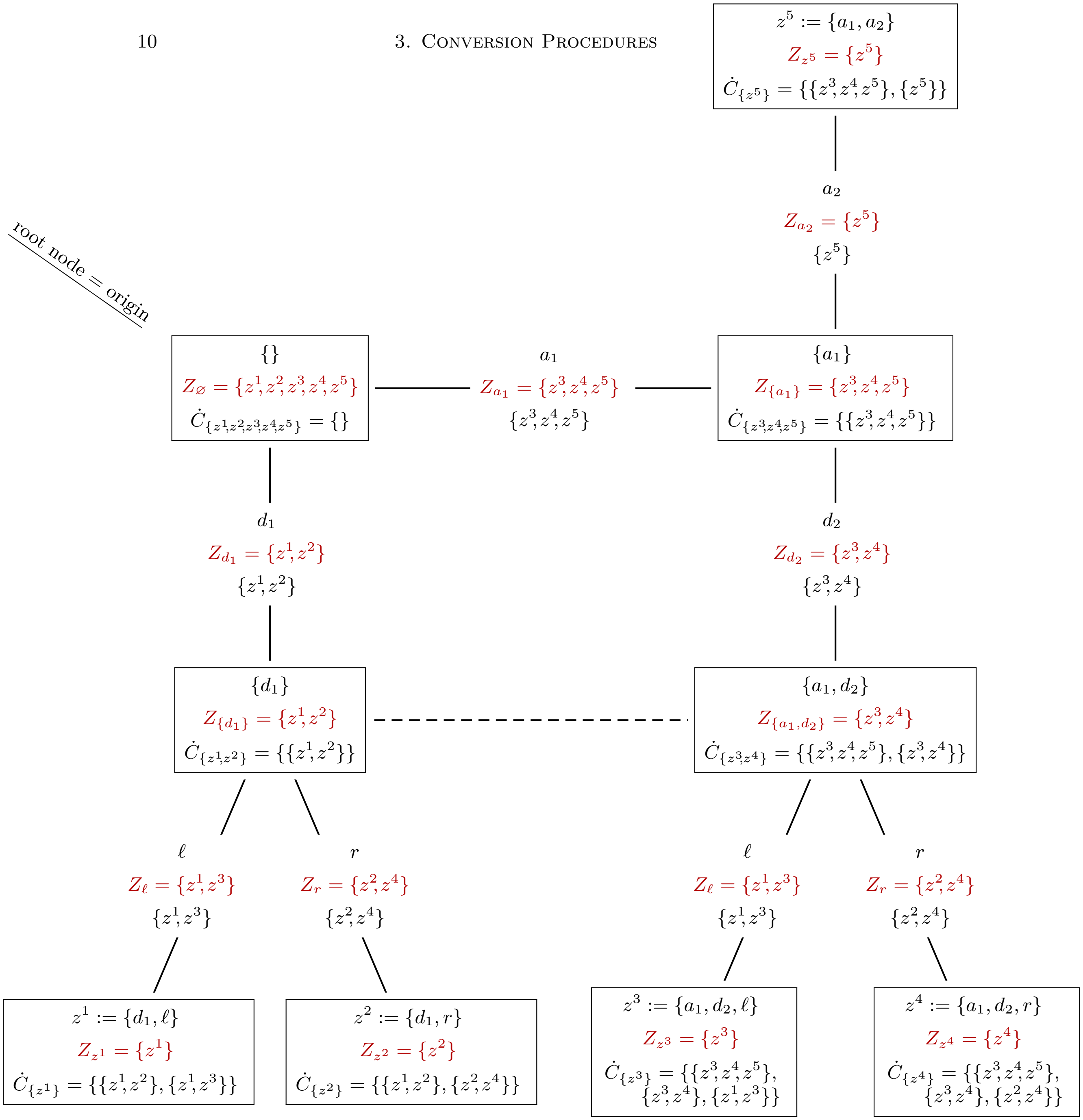

FIGURE 1. A choice-set form (top row), converted into an outcome-set form (middle row), and converted again into a new choice-set form (bottom row). The two choice-set forms are equivalent by renaming choices. (For readability, the symbols z^1, z^2, z^3, z^4, and z^5 abbreviate the terminal nodes of the original choice-set form.)

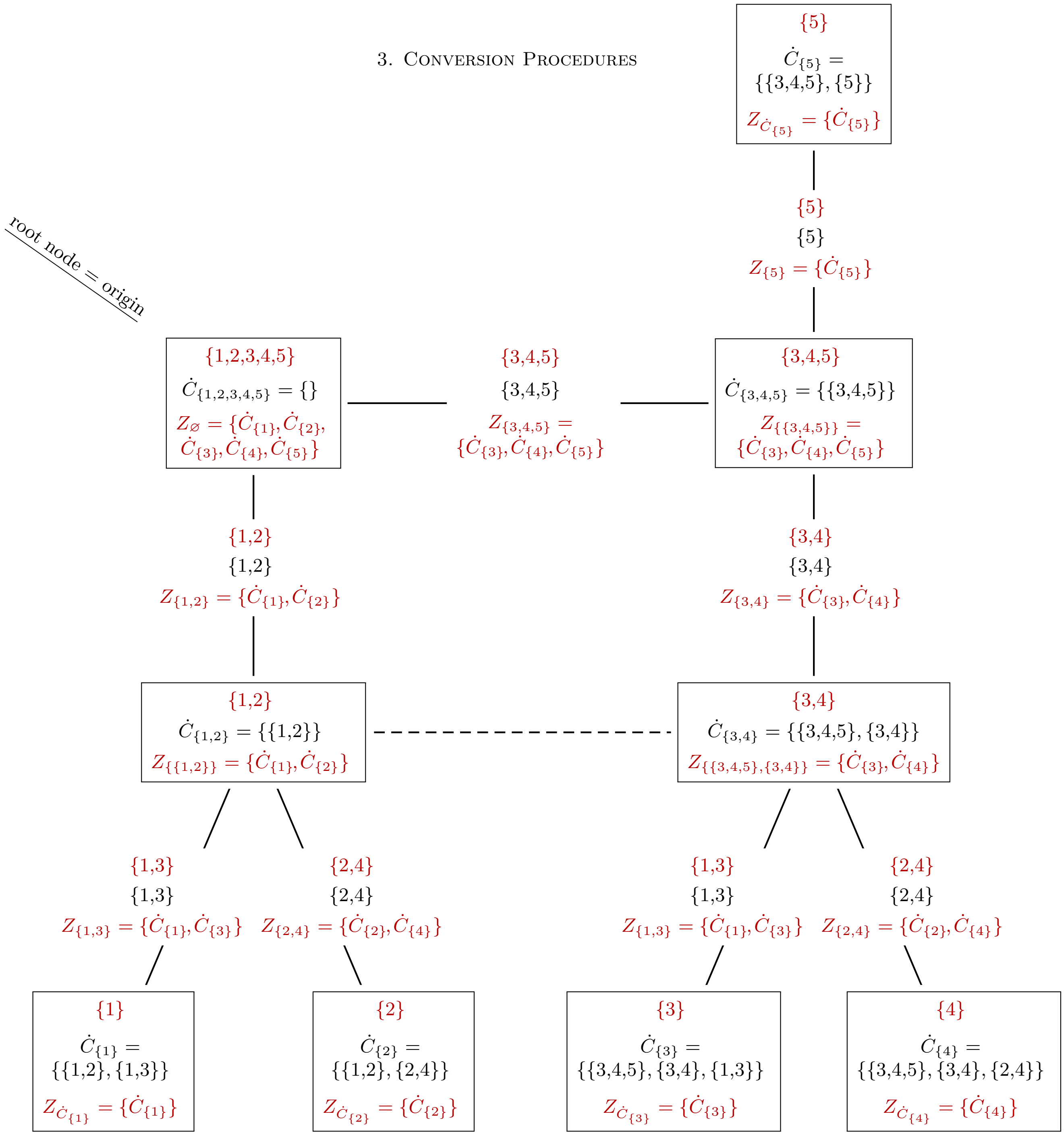

Figure 2. An outcome-set form (top row), converted into a choice-set form (middle row), and converted again into an outcome-set form (bottom row). The two outcome-set forms are equivalent by renaming outcomes.

of the original choice-set game. These five symbols are specific to this particular example. Note that $Z = \{z^1, z^2, z^3, z^4, z^5\}$.

Figure 2 illustrates a second, more difficult application of Theorem 1. There the choice-set form of the *middle* row is converted into the outcome-set form of the bottom row. This example will be defined and discussed later.

The conversion procedure of Theorem 1 fails if the original choice-set form has a trivial move. For example, consider the one-player choice-set form $((C), N)$ defined by $C = \{a\}$ and $N = \{\{\}, \{a\}\}$. Here the conversion procedure would set both $Z_{\{\}}$ and $Z_{\{a\}}$ equal to $\{\{a\}\}$, in contradiction to the bijection of Theorem 1(b). In general, the existence of a trivial move means that some node has exactly one immediate successor. In such a case, the node and its successor admit exactly the same set of outcomes. As a result, the two distinct choice-set nodes cannot be specified as two distinct sets of outcomes.

3.2. Converting outcome-set forms to choice-set forms

Consider a concise $\mathtt{AR}^*$ outcome-set form $(W, \dot{N}, (\dot{C}_i)_i)$ and let $\dot{C} = \cup_i C_i$. Then, for any node $\dot{n}$ let

$$\dot{C}_{\dot{n}} = \{ \ \dot{c} \mid \dot{c} \supseteq \dot{n} \ \} \ . \tag{16}$$

Thus $\dot{C}_{\dot{n}}$ is the set of choices on the way to the node $\dot{n}$. Note that $(\dot{C}_{\dot{n}})_{\dot{n}}$ is a function from $\dot{N}$ into the power set of $\dot{C}$. Thus it takes a set of outcomes into a collection of sets of outcomes. Its range is $\{\dot{C}_{\dot{n}}|\dot{n}\}$.

Part (a) of the following theorem shows how a concise $\mathtt{AR}^*$ outcome-set form can be converted into a no-trivial-move choice-set form. The remaining parts of the theorem show the sense in which the original outcome-set form and the derived choice-set form are "equivalent".

Theorem 2. *Suppose* $(W, \dot{N}, (\dot{C}_i)_i)$ *is a concise* $\mathtt{AR}^*$ *outcome-set form (12). Derive its* $\dot{C} = \cup_i \dot{C}_i$, $\dot{T}$ *(7),* $\dot{p}$ *(8), and* $\dot{F}$ *(10). To convert this form, let*

$$(C_i)_i := (\dot{C}_i)_i \text{ and } N := \{\dot{C}_{\dot{n}}|\dot{n}\} \ .$$

where $(\dot{C}_{\dot{n}})_{\dot{n}}$ *is defined by (16). Then*

(a) $((C_i)_i, N)$ *is a no-trivial-move (13) choice-set form (5).*

Further, derive $C = \cup_i C_i$, T *(2),* F *(3), and* p *(4). Then the following hold.*

(b) $(\dot{C}_{\dot{n}})_{\dot{n}}$ *is a bijection from* $\dot{N}$ *onto* N*. Further,* $\dot{C}_W = \{\}$*.*
(c) $T = \{\ \dot{C}_{\dot{t}} \mid \dot{t}{\in}\dot{T}\ \}$*.*
(d) $(\forall \dot{t}, \dot{c}, \dot{t}^\sharp)$ *(*$\dot{t}{=}\dot{p}(\dot{t}^\sharp)$ *and* $\dot{t}{\cap}\dot{c}{=}\dot{t}^\sharp$*) iff (*$\dot{c}{\notin}\dot{C}_{\dot{t}}$ *and* $\dot{C}_{\dot{t}}{\cup}\{\dot{c}\}{=}\dot{C}_{\dot{t}^\sharp}$*).*
(e) $F = \{\ (\dot{C}_{\dot{t}}, \dot{c}) \mid (\dot{t}, \dot{c}){\in}\dot{F}\ \}$*.*
(f) $p = \{\ (\dot{C}_{\dot{t}^\sharp}, \dot{C}_{\dot{t}}) \mid (\dot{t}^\sharp, \dot{t}){\in}\dot{p}\ \}$*. (Proof B.7.)*

For example, the top row of the labels in Figure 2 corresponds to the one-player concise $\mathtt{AR}^*$ outcome-set form $(W, \dot{N}, (\dot{C}))$ that is defined by

$$W = \{1,2,3,4,5\}\ , \tag{17}$$
$$\dot{N} = \{\, W,\ \{1,2\},\ \{3,4,5\},\ \{3,4\},\ \{1\},\ \{2\},\ \{3\},\ \{4\},\ \{5\}\,\}\ ,$$
$$\text{and } \dot{C} = \{\, \{1,2\},\ \{3,4,5\},\ \{3,4\},\ \{5\},\ \{1,3\},\ \{2,4\}\,\}\ .$$

In accord with Theorem 2, this top row is then converted into the one-player no-trivial-move choice-set form $((\dot{C}), \{\dot{C}_{\dot{n}}|\dot{n}\})$ that appears in the middle row. Note that the choices are unaltered.

Figure 1 provides another example. There the outcome-set form in the middle row is converted into the choice-set form of the bottom row. Again the choices are unaltered. (The middle row was obtained in the last subsection from the choice-set form in the top row.)

In general, Note 3 (on Lemma B.3) suggests that conciseness plays a deep role in the well-definition of this conversion procedure. This subtlety arises only in infinite-horizon forms.

4. The conversion procedures are inverses

4.1. Operator notation

Theorem 1(a) showed that any no-trivial-move choice-set form can be converted into a concise $\mathtt{AR}^*$ outcome-set form. Accordingly, we may define the operator $\widehat{\mathtt{Z}}$ that takes a no-trivial-move choice-set form to a concise $\mathtt{AR}^*$ outcome-set preform by the rule

$$\widehat{\mathtt{Z}}:\ ((C_i)_i, N)\ \mapsto\ (Z, \{Z_n|n\}, (\{Z_c|c{\in}C_i\})_i)\ , \tag{18}$$

where Z, $(Z_n)_n$, and $(Z_c)_c$ are defined by (14).

Conversely, Theorem 2(a) showed that any concise $\mathtt{AR}^*$ outcome-set form can be converted into a no-trivial-move choice-set form. Accordingly, we may define the operator $\widehat{\mathtt{C}}$ that takes a concise $\mathtt{AR}^*$ outcome-set form into a no-trivial-move choice-set form by the rule

(19) $$\widehat{\mathtt{C}}:\ (W,\dot{N},(\dot{C}_i)_i)\ \mapsto\ ((\dot{C}_i)_i,\{\dot{C}_{\dot{n}}|\dot{n}\})\ ,$$

where $(\dot{C}_{\dot{n}})_{\dot{n}}$ is defined by (16).

The remainder of this section develops the sense in which these two operators are inverses of one another.

4.2. Away from and back to choice-set forms

Consider two choice-set forms $((C_i)_i,N)$ and $((C'_i)_i,N')$. The two are said to be *equivalent by renaming choices* if there exists a bijection $\delta:\cup_i C_i\to\cup_i C'_i$ such that

(20a) $$(\forall i)\ \delta(C_i)=C'_i \text{ and}$$

(20b) $$\{\delta(n)|n\in N\}=N'\ .$$

Thus, two one-player choice-set forms $((C),N)$ and $((C'),N')$ are equivalent by renaming choices if there is a bijection $\delta{:}C\to C'$ satisfying (20b). This is illustrated by example in the next three paragraphs.

Consider Figure 1. There the operator $\widehat{\mathtt{Z}}$ takes the top row to the middle row, and the operator $\widehat{\mathtt{C}}$ takes the middle row to the bottom row. In particular, the top rows labelling the choices and nodes display the one-player choice-set form $((C),N)$ defined in (15). Next, the middle rows display $\widehat{\mathtt{Z}}[((C),N)]$. This is the one-player outcome-set form

$$(W,\dot{N},(\dot{C})):=(Z,\{Z_n|n\},(\{Z_c|c\}))\ ,$$

in which Z, $(Z_n)_n$, and $(Z_c)_c$ are derived from $((C),N)$ by (14). Finally, the bottom rows display $\widehat{\mathtt{C}}\circ\widehat{\mathtt{Z}}[((C),N)]$. This is the one-player choice-set form

$$((C'),N'):=((\dot{C}),\{\dot{C}_{\dot{n}}|\dot{n}\})\ ,$$

in which $(\dot{C}_{\dot{n}})_{\dot{n}}$ is derived from $(W,\dot{N},(\dot{C}))$ by (16).

The old (top-row) $((C),N)$ and new (bottom-row) $((C'),N')$ are equivalent by the choice-renaming function $\delta:=(Z_c)_c$. This δ can be specified exhaustively by[2]

(21a) $$\delta(d_1)=Z_{d_1}=\{z^1,z^2\}=\{\{d_1,\ell\},\{d_1,r\}\}$$

(21b) $$\delta(a_1)=Z_{a_1}=\{z^3,z^4,z^5\}=\{\{a_1,d_2,\ell\},\{a_1,d_2,r\},\{a_1,a_2\}\}$$

(21c) $$\delta(d_2)=Z_{d_2}=\{z^3,z^4\}=\{\{a_1,d_2,\ell\},\{a_1,d_2,r\}\}$$

[2]The symbols z^1, z^2, z^3, z^4, and z^5 are defined to abbreviate the old terminal nodes in this example (see the top row at each terminal node in Figure 1). Accordingly, the third equality in each part of (21) is purely definitional.

(21d) $\delta(a_2) = Z_{a_2} = \{z^5\} = \{\{a_1,a_2\}\}$

(21e) $\delta(\ell) = Z_\ell = \{z^1, z^3\} = \{\{d_1,\ell\}, \{a_1,d_2,\ell\}\}$

(21f) $\delta(r) = Z_r = \{z^2, z^4\} = \{\{d_1,r\}, \{a_1,d_2,r\}\}$.

Thus each new choice is the set of old terminal nodes that follow after the corresponding old choice. For example in (21a), the new choice $\{z^1, z^2\} = \{\{d_1,\ell\}, \{d_1,r\}\}$ is the set of old terminal nodes that follow after the old choice d_1.

Further, this δ is a bijection from C onto C'. Its surjectivity simply means that the right-hand sides of (21) correspond to the new choices of C'. Its injectivity is unsurprising because the values of δ are sets of sets of the arguments of δ. Although (20a) is vacuous because there is only one player, (20b) is tedious to verify. One must take every top-row node n and verify that its bottom-row node is $\delta(n)$. For instance, [1] consider the top-row node $\{a_1,d_2\}$, [2] note that (21b,c) imply

$$\delta(\{a_1,d_2\}) = \{\delta(a_1), \delta(d_2)\} = \{\ \{z^3,z^4,z^5\},\ \{z^3, z^4\}\ \}\ ,$$

and [3] note that the right-hand side is the corresponding bottom-row node. More enlightening is to see in Figure 1 that each new node is the set of new choices that lead to it. Similarly, each old node was the set of old choices that led to it. This preservation of structure is the essence of (20b).

The following lemma generalizes the example of Figure 1. It considers the general procedure of [1] converting away from a choice-set form and then [2] converting back to a choice-set form. The lemma shows that this general double procedure does nothing but rename choices.

Lemma 4.1. *Let* $\Phi := ((C_i)_i, N)$ *be a no-trivial-move (13) choice-set form (5). Define the conversion operators* $\widehat{\mathtt{Z}}$ *(18) and* $\widehat{\mathtt{C}}$ *(19). Then* Φ *and* $\widehat{\mathtt{C}}{\circ}\widehat{\mathtt{Z}}[\Phi]$ *are equivalent (20) by the choice-renaming function* $\delta := (Z_c)_c$*, where* $(Z_c)_c$ *is defined by (14c). (Proof C.1.)*

4.3. Away from and back to outcome-set forms

Consider two concise $\mathtt{AR}^*$ outcome-set forms $(W, \dot{N}, (\dot{C}_i)_i)$ and $(W', \dot{N}', (\dot{C}'_i)_i)$. The two are said to be *equivalent by renaming outcomes* if there exists a bijection $\theta{:}W{\rightarrow}W'$ such that

(22a) $\{\theta(\dot{n})|\dot{n}{\in}\dot{N}\} = \dot{N}'$ and

(22b) $(\forall i)\ \{\theta(\dot{c})|\dot{c}{\in}\dot{C}_i\} = \dot{C}'_i$.

Thus two one-player concise $\mathtt{AR}^*$ outcome-set forms $(W,\dot{N},(\dot{C}))$ and $(W',\dot{N}',(\dot{C}'))$ are equivalent by renaming outcomes if there is a bijection $\theta{:}W{\rightarrow}W'$ that satisfies (22a) and

$$\{\theta(\dot{c})|\dot{c}{\in}\dot{C}\} = \dot{C}' \ . \tag{23}$$

This is illustrated by example in the next three paragraphs.

Consider Figure 2. The operator $\widehat{\mathtt{C}}$ takes the top row to the middle row, and the operator $\widehat{\mathtt{Z}}$ takes the middle row to the bottom row. In particular, the top rows display the one-player outcome-set form $(W,\dot{N},(\dot{C}))$ defined in (17). Next, the middle rows display $\widehat{\mathtt{C}}[(W,\dot{N},(\dot{C}))]$. This is the one-player choice-set form

$$((C),N) := ((\dot{C}),\{\dot{C}_{\dot{n}}|\dot{n}\})$$

in which $(\dot{C}_{\dot{n}})_{\dot{n}}$ is derived from $(W,\dot{N},(\dot{C}))$ by (16). Finally, the bottom rows display $\widehat{\mathtt{Z}}{\circ}\widehat{\mathtt{C}}[(W,\dot{N},(\dot{C}))]$. This is the one-player outcome-set form

$$(W',\dot{N}',(\dot{C}')) := (Z,\{Z_n|n\},(\{Z_c|c\}))$$

in which Z, $(Z_n)_n$, and $(Z_c)_c$ are derived from $((C),N)$ by (14).

The old (top-row) $(W,\dot{N},(\dot{C}))$ is equivalent to the new (bottom-row) $(W',\dot{N}',(\dot{C}'))$ by the outcome-renaming function $\theta := (\dot{C}_{\{w\}})_w$. This θ can be specified exhaustively by

$$\theta(1) = \dot{C}_{\{1\}} = \{\{1{,}2\},\{1{,}3\}\} \tag{24a}$$

$$\theta(2) = \dot{C}_{\{2\}} = \{\{1{,}2\},\{2{,}4\}\} \tag{24b}$$

$$\theta(3) = \dot{C}_{\{3\}} = \{\{3{,}4{,}5\},\{3{,}4\},\{1{,}3\}\} \tag{24c}$$

$$\theta(4) = \dot{C}_{\{4\}} = \{\{3{,}4{,}5\},\{3{,}4\},\{2{,}4\}\} \tag{24d}$$

$$\theta(5) = \dot{C}_{\{5\}} = \{\{3{,}4{,}5\},\{5\}\} \ . \tag{24e}$$

Thus each new outcome is the set of old choices leading up to the corresponding old outcome. For example in (24a), the new outcome $\{\{1{,}2\},\{1{,}3\}\}$ is the set of old choices leading up to the old outcome 1.

This θ is a bijection from W onto W'. Its surjectivity simply means that the right-hand sides of (24) correspond to the new outcomes of W'. Its injectivity is unsurprising because the values of θ are sets of sets of the arguments of θ. To verify (22a), consider every node in the figure, let n be its first row, and verify that $\theta(n)$ is its third row. This is easily done via the definition of θ at each outcome, that is, via the first equality in each part of (24). To verify (23), consider every choice in the figure, let c be its first row, and verify that $\theta(c)$ is its third row. This too is easily done via the definition of θ at each outcome.

The following lemma generalizes the example of Figure 2. It considers the general procedure of [1] converting away from an outcome-set form and then [2] converting back to an outcome-set form. The lemma shows that this general double procedure does nothing but rename outcomes.

Lemma 4.2. *Let* $\dot{\Phi} := (W, \dot{N}, (\dot{C}_i)_i)$ *be a concise* $\mathtt{AR}^*$ *outcome-set form (12). Define the conversion operators* $\widehat{\mathtt{Z}}$ *(18) and* $\widehat{\mathtt{C}}$ *(19). Then* $\dot{\Phi}$ *and* $\widehat{\mathtt{Z}}\circ\widehat{\mathtt{C}}[\dot{\Phi}]$ *are equivalent (22) by the outcome-renaming function* $\theta := (\dot{C}_{\{w\}})_w$ *in which* $(\dot{C}_{\dot{n}})_{\dot{n}}$ *is defined by (16). (Proof C.2.)*

4.4. Theorem

Theorem 3. *For the purposes of this theorem, say that*

(i) two no-trivial-move (13) choice-set forms (5) are <u>*equal*</u> *if they are equivalent by choice renaming (20), and*

(ii) two concise $\mathtt{AR}^*$ *outcome-set forms (12) are* <u>*equal*</u> *if they are equivalent by outcome renaming (22).*

Define the conversion operators $\widehat{\mathtt{Z}}$ *(18) and* $\widehat{\mathtt{C}}$ *(19). Then the following hold.*

(a) $\widehat{\mathtt{Z}}$ *and* $\widehat{\mathtt{C}}$ *are well-defined given the above concepts of equality.*

(b) $\widehat{\mathtt{Z}}$ *is a bijection from the class of no-trivial-move choice-set forms onto the class of concise* $\mathtt{AR}^*$ *outcome-set forms.*

(c) $\widehat{\mathtt{Z}}^{-1} = \widehat{\mathtt{C}}$. *(Proof C.5.)*

Theorem 3 shows that there is a one-to-one correspondence between the class of no-trivial-move choice-set forms and the class of concise $\mathtt{AR}^*$ outcome-set forms. This, the natural results about $\widehat{\mathtt{Z}}$ in Theorem 1(b–g), and the natural results about $\widehat{\mathtt{C}}$ in Theorem 2(b–f), all substantiate the notion that choice-set forms are dual to outcome-set forms. This duality is the main contribution of this paper.

5. Corollary for $\mathtt{OR}^*$ choice-sequence forms

`S1` Theorem 2 shows that $\mathtt{OR}^*$ choice-sequence forms with no-absent-mindedness are in one-to-one correspondence with choice-set forms. An $\mathtt{OR}^*$ choice-sequence form (`S1` equation (6)) is a slight reformulation of an Osborne-Rubinstein (OR) extensive form, and the definition of no-absent-mindedness (S1 equation (16)) is taken directly from Piccione and Rubinstein (1997).

Essentially, S1 Theorem 2 and this paper's Theorem 3 together imply a triple equivalence between (a) no-trivial-move OR* choice-sequence forms with no-absent-mindedness, (b) no-trivial-move choice-set forms, and (c) concise AR* outcome-set forms. This triple equivalence is illustrated by Figure 3.

To be precise, S1 Theorem 2 shows that the operator $\widehat{\mathtt{R}}$ (S1 equation (17)) is a bijection from (a) the class of OR* choice-sequence forms with no-absent-mindedness onto (b) the class of choice-set forms. Meanwhile, this paper's Theorem 3 shows that $\widehat{\mathtt{Z}}$ is a bijection from (b') the class of no-trivial-move choice-set forms onto (c) the class of concise AR* outcome-set forms.

The composition of these two bijections is complicated by two minor issues. First, the range of $\widehat{\mathtt{R}}$ [(b) above] is a strict superset of the domain of $\widehat{\mathtt{Z}}$ [(b') above]. This happens because $\widehat{\mathtt{R}}$ can accommodate trivial moves while $\widehat{\mathtt{Z}}$ cannot. To address this issue, say that an OR*

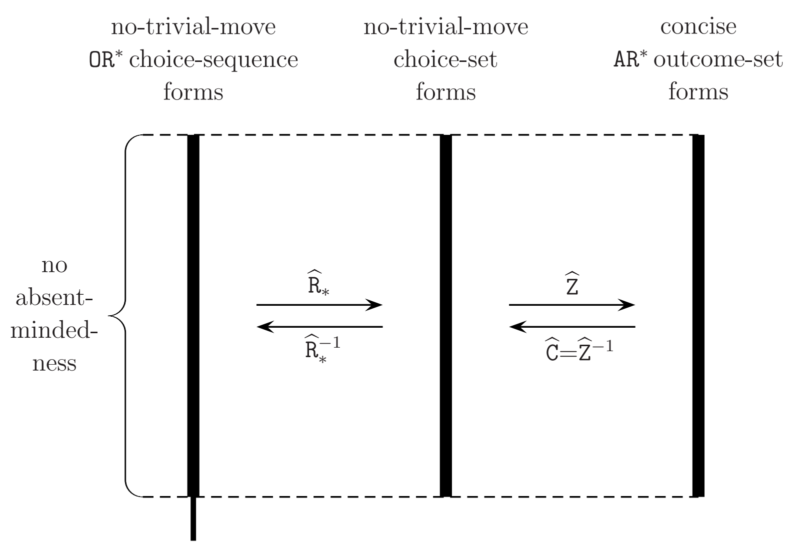

FIGURE 3. $\widehat{\mathtt{R}}_*$ is a restriction of the $\widehat{\mathtt{R}}$ from S1 Theorem 2. Theorem 1 concerns $\widehat{\mathtt{Z}}$, Theorem 2 concerns $\widehat{\mathtt{C}}$, and Theorem 3 shows $\widehat{\mathtt{C}}=\widehat{\mathtt{Z}}^{-1}$. Corollary 5.1 concerns $\widehat{\mathtt{Z}}\circ\widehat{\mathtt{R}}_*$.

choice-sequence form $((C_i)_i, \bar{N})$ has *no-trivial-moves* if

$$(\forall \bar{t})\ |\bar{F}(\bar{t})| \neq 1\ , \tag{25}$$

where $\bar{T}$ and $\bar{F}$ are derived by `S1` equation (2) and `S1` equation (3). (25) is directly comparable to (13) for choice-set forms. Corollary 5.1 will then define $\widehat{\mathtt{R}}_*$ to be the restriction of $\widehat{\mathtt{R}}$ to the forms in its domain with no-trivial-moves (thus the domain of $\widehat{\mathtt{R}}_*$ becomes the class of no-trivial-move $\mathtt{OR}^*$ choice-sequence forms with no-absent-mindedness). This is the $\widehat{\mathtt{R}}_*$ that appears in Figure 3.

The second minor complication is that $\widehat{\mathtt{Z}}$ is a bijection given Theorem 3's definitions of equality. In contrast, `S1` Theorem 2 shows that $\widehat{\mathtt{R}}$ is a bijection without such qualifications. To introduce such qualifications in the context of $\widehat{\mathtt{R}}$, say that two $\mathtt{OR}^*$ choice-sequence forms $((C_i)_i, \bar{N})$ and $((C'_i)_i, \bar{N}')$ are *equivalent by choice renaming* if there exists a bijection δ from $\cup_i C_i$ onto $\cup_i C'_i$ such that

$$(\forall i)\ \delta(C_i) = C'_i \text{ and} \tag{26a}$$

$$\{\ \delta\circ\bar{n} \mid \bar{n}\in\bar{N}\ \} = \bar{N}'\ . \tag{26b}$$

(26) is directly comparable to (20) for choice-set forms.

Corollary 5.1. *For the purposes of this corollary, say that*

(i) two no-trivial-move (25) $\mathtt{OR}^*$ *choice-sequence forms (*`S1` *equation (6)) are* <u>*equal*</u> *if they are equivalent by choice renaming (26),*

(ii) two no-trivial-move (13) choice-set forms (5) are <u>*equal*</u> *if they are equivalent by choice renaming (20), and*

(iii) two concise $\mathtt{AR}^*$ *outcome-set forms (12) are* <u>*equal*</u> *if they are equivalent by outcome renaming (22).*

Define the conversion operators $\widehat{\mathtt{R}}$ *(*`S1` *equation (17)) and* $\widehat{\mathtt{Z}}$ *(18). Further, let* $\widehat{\mathtt{R}}_*$ *be the restriction of* $\widehat{\mathtt{R}}$ *to the no-trivial-move (25) forms in its domain. Then the following hold.*

(a) $\widehat{\mathtt{R}}_*$ *is well-defined given the above concepts of equality.*

(b) $\widehat{\mathtt{R}}_*$ *is a bijection, from the class of no-trivial-move* $\mathtt{OR}^*$ *choice-sequence forms with no-absent-mindedness (*`S1` *equation (16)), onto the class of no-trivial-move choice-set forms.*

(c) $\widehat{\mathtt{Z}}\circ\widehat{\mathtt{R}}_*$ *is a bijection, from the class of no-trivial-move* $\mathtt{OR}^*$ *choice-sequence forms with no-absent-mindedness, onto the class of concise* $\mathtt{AR}^*$ *outcome-set forms.*

(d) $(\widehat{\mathtt{Z}}\circ\widehat{\mathtt{R}}_*)^{-1} = \widehat{\mathtt{R}}_*^{-1}\circ\widehat{\mathtt{Z}}^{-1}$. *(Proof C.9.)*

Corollary 5.1(c) shows that the class of no-trivial-move $\mathtt{OR}^*$ choice-sequence forms with no-absent-mindedness is in one-to-one correspondence with the class of concise $\mathtt{AR}^*$ outcome-set forms. This one-to-one correspondence is portrayed by Figure 3.

As mentioned earlier, $\mathtt{OR}^*$ choice-sequence forms are reformulations of OR extensive forms. Meanwhile, S2's theorems show that concise $\mathtt{AR}^*$ outcome-set forms are precisely as general as AR discrete extensive forms (assuming that simultaneous moves are specified by multiple information sets). Thus Corollary 5.1(c) implies that no-trivial-move no-absent-minded OR extensive forms are precisely as general as AR discrete extensive forms. The remainder of this section places this equivalence in the literature.

One direction of this equivalence is already known, namely, that no-trivial-move no-absent-minded OR forms are no more general than (discrete) AR forms. This was established by AR Example 10 and AR Proposition 6(a). Further, it is also known that this statement fails if either no-trivial-moves or no-absent-mindedness is relaxed, because both these restrictions are implied by the definition of (discrete) AR forms. The observation about no-trivial-moves was made in AR Example 10, and the observation about no-absent-mindedness follows from Alós-Ferrer and Ritzberger (2005, Proposition 13). But ultimately, these two restrictions are of little concern: No-trivial-moves is trivial, and no-absent-mindedness is minor.

Meanwhile, the reverse direction of the equivalence is new, namely, that no-trivial-move no-absent-minded OR forms are at least as general as (discrete) AR forms. This result is established by [1] Corollary 5.1(c)'s statement that $\widehat{\mathtt{Z}}\circ\widehat{\mathtt{R}}_*$ is surjective and [2] the first two sentences of the next-to-last paragraph. This result is a contribution to the literature because it had been previously understood (e.g. AR Example 10) that no-trivial-move no-absent-minded OR forms corresponded to a special case of (discrete) AR forms.

APPENDIX A. FOR THEOREM 1

Lemma A.1. *Suppose (C,N) is a choice-set preform (1) with its T (2). Further suppose a is a finite subset of n. Then $min\{t|a\subseteq t\subseteq n\}$ exists.*

Proof. On the one hand, suppose $n{\in}T$. Then by `S1` Corollary 5.1, $\{t|t{\subseteq}n\}$ is a finite chain. Thus $\min\{t|a{\subseteq}t{\subseteq}n\}$ is the smallest member of this chain that includes a. On the other hand, suppose $n{\notin}T$. Then by `S1` Corollary 5.3, $\{t|t{\subset}n\}$ is an infinite chain in T whose union is n. Thus $\min\{t|a{\subseteq}t{\subseteq}n\}$ is the smallest member of this chain that includes the finite set a. □

Lemma A.2. *Let (C,N) be a no-trivial-move (13) choice-set preform (1). Derive Z and $(Z_n)_n$ by (14a–b). Then the following hold.*
(a) $(\forall n)\ Z_n \neq \varnothing$.
(b) $Z_{n^1} \supseteq Z_{n^2}$ iff $n^1 \subseteq n^2$.
(c) $Z_{n^1}{\cap}Z_{n^2} = \varnothing$ iff ($n^1 \not\subseteq n^2$ and $n^2 \not\subseteq n^1$).
(d) $(\forall z)\ Z_z = \{z\}$.

Proof. Derive T (2) and F (3). Also note that

$$
\begin{aligned}
Z &= N \smallsetminus F^{-1}(C) \\
&= N \smallsetminus \{\ n \mid (\exists n^\sharp)\ n{\subset}n^\sharp\ \} \\
&= \{\ n \mid (\nexists n^\sharp)\ n{\subset}n^\sharp\ \}\ ,
\end{aligned} \tag{27}
$$

where the first equality is the definition of Z and the second follows from `S1` Corollary 5.2(c).

(a). Take any n. If $n \in Z$, then $n \in Z_n$. If not, (27) implies the existence of an n^1 such $n \subset n^1$, which starts the following iterative process.

1. If $n^1 \in Z$, then $n^1 \in Z_n$ because $n \subset n^1 \in Z$.
   If not, (27) implies an n^2 such that $n^1 \subset n^2$.
2. If $n^2 \in Z$, then $n^2 \in Z_n$ because $n \subset n^1 \subset n^2 \in Z$.
   If not, (27) implies an n^3 such that $n^2 \subset n^3$.

   $\ldots$.

This process will either terminate with some $n^k \in Z$ such that

$$n \subset n^1 \subset n^2\ \ldots \subset n^k\ ,$$

or generate an infinite sequence $(n^k)_{k\geq 1}$ such that

$$n \subset n^1 \subset n^2 \subset\ \ldots\ .$$

In the first contingency, $n^k \in Z_n$ because $n \subset n^k \in Z$. In the second contingency, we have an infinite chain $(n^k)_{k\geq 1}$ such that $n \subset \cup_k n^k$. Since each n^k has a successor, `S1` Corollary 5.2(b) implies that each

$n^k \in T$. Thus $(n^k)_{k\geq 1}$ is an infinite chain in T. Hence by (1c), $\cup_k n^k$ is an element of $N\smallsetminus T$. Since $\cup_k n^k \notin T$, `S1` Corollary 5.2(b) implies that $\cup_k n^k \notin \{n|(\exists n^\sharp)n{\subset}n^\sharp\}$. Thus by (27), $\cup_k n^k \in Z$. Therefore since $n \subset \cup_k n^k$, $\cup_k n^k \in Z_n$.

(b). Obviously

$$\begin{aligned} & n^1 \subseteq n^2 \\ \Rightarrow\ & \{z|n^1{\subseteq}z\} \supseteq \{z|n^2{\subseteq}z\} \\ \Rightarrow\ & Z_{n^1} \supseteq Z_{n^2}\ , \end{aligned}$$

where the second implication holds by the definition of $(Z_n)_n$. I will show the contrapositive of the converse. Accordingly, suppose that $n^1 \not\subseteq n^2$. Thus $n^1{\wedge}n^2 \subset n^1$, where $n^1{\wedge}n^2$ is a well-defined member of T by `S1` Corollary 5.4. So by `S1` Corollary 5.2(a) at $t{=}n^1{\wedge}n^2$ and $n^\sharp{=}n^1$, we may define $c^1 \in F(n^1{\wedge}n^2){\cap}n^1$. Two cases then arise: [1] $n^1{\wedge}n^2 \subset n^2$ or [2] $n^1{\wedge}n^2 = n^2$.

Case [1]. Suppose $n^1{\wedge}n^2 \subset n^2$. Then `S1` Corollary 5.2(a) at $t{=}n^1{\wedge}n^2$ and $n^\sharp = n^2$ implies the existence of a $c^2 \in F(n^1{\wedge}n^2){\cap}n^2$. Note that the definition of c^1 implies that $(n^1{\wedge}n^2){\cup}\{c^1\} \subseteq n^1$. Similarly, the definition of c^2 implies that $(n^1{\wedge}n^2){\cup}\{c^2\} \subseteq n^2$. Thus, if $c^2 = c^1$, we would have $(n^1{\wedge}n^2){\cup}\{c^1\} \subseteq n^1{\cap}n^2$, in contradiction to the definition of $\wedge$. Hence

$$c^2 \neq c^1\ . \tag{28}$$

In a different vein, part (a) allows us to take some $z^2 \in Z_{n^2}$. Note

$$c^2 \in n^2 \subseteq z^2\ , \tag{29}$$

where the set membership follows from the definition of c^2, and the set inclusion follows from the definition of Z_{n^2}.

`S1` Lemma B.6 implies that $|F(n^1{\wedge}n^2){\cap}z^2| \leq 1$. Note that both c^1 and c^2 belong to $F(n^1{\wedge}n^2)$ by their definitions. Thus the last two sentences, (29), and (28) together imply that $c^1 \notin z^2$. This implies $n^1 \not\subseteq z^2$ because $c^1 \in n^1$ by the definition of c^1. Hence $z^2 \notin Z_{n^1}$. So by the definition of z^2, $z^2 \in Z_{n^2}\smallsetminus Z_{n^1}$. Hence $Z_{n^2} \not\subseteq Z_{n^1}$.

Case [2]. Suppose $n^1{\wedge}n^2 = n^2$. Then the definition of c^1 implies that $c^1 \in F(n^2){\cap}n^1$. Because there are no-trivial-moves, there exists some $c^* \in F(n^2)$ such that

$$c^* \neq c^1\ . \tag{30}$$

Let $n^* = n^2 \cup \{c^*\}$. Then part (a) allows us to take some $z^* \in Z_{n^*}$. By the last two sentences

$$c^* \in n^* \subseteq z^* \ . \tag{31}$$

S1 Lemma B.6 implies that $|F(n^2) \cap z^*| \leq 1$. Note that both c^1 and c^* belong to $F(n^2)$ by early sentences in the last paragraph. Thus the last two sentences, (31), and (30) imply that $c^1 \notin z^*$. This implies $n^1 \not\subseteq z^*$ because $c^1 \in n^1$ by the definition of c^1. Hence $z^* \notin Z_{n^1}$. However, by the definitions of n^* and z^*, $n^2 \subseteq n^* \subseteq z^*$. Hence $z^* \in Z_{n^2}$. By the last and the third-to-last sentences, $z^* \in Z_{n^2} \setminus Z_{n^1}$. Hence $Z_{n^2} \not\subseteq Z_{n^1}$.

(c). Suppose $Z_{n^1} \cap Z_{n^2} = \varnothing$. Then since $Z_{n^1} \neq \varnothing$ by part (a), $Z_{n^1} \not\subseteq Z_{n^2}$. Thus by part (b), $n^2 \not\subseteq n^1$. A symmetric argument shows $n^1 \not\subseteq n^2$.

Conversely, suppose that $n^1 \not\subseteq n^2$ and $n^2 \not\subseteq n^1$. Because $n^1 \not\subseteq n^2$, $n^1 \wedge n^2 \subset n^1$, where $n^1 \wedge n^2$ is a well-defined member of T by S1 Corollary 5.4. Then S1 Corollary 5.2(a) at $t = n^1 \wedge n^2$ and $n^\sharp = n^1$ implies the existence of a $c^1 \in F(n^1 \wedge n^2) \cap n^1$. Thus

$$(n^1 \wedge n^2) \cup \{c^1\} \subseteq n^1 \ . \tag{32}$$

Similarly, because $n^2 \not\subseteq n^1$, $n^1 \wedge n^2 \subset n^2$. Then S1 Corollary 5.2(a) implies the existence of a $c^2 \in F(n^1 \wedge n^2) \cap n^2$. Thus

$$(n^1 \wedge n^2) \cup \{c^2\} \subseteq n^2 \ . \tag{33}$$

If $c^2 = c^1$, then (32) and (33) would imply $(n^1 \wedge n^2) \cup \{c^1\} \subseteq n^1 \cap n^2$, in contradiction to the definition of $\wedge$. Thus

$$c^2 \neq c^1 \ . \tag{34}$$

I must show that $Z_{n^1} \cap Z_{n^2} = \varnothing$. To do this, I will show that $z^2 \in Z_{n^2}$ implies $z^2 \notin Z_{n^1}$. Accordingly, take any $z^2 \in Z_{n^2}$. Then

$$c^2 \in n^2 \subseteq z^2 \ , \tag{35}$$

where the set membership holds by (33) and the set inclusion follows from $z^2 \in Z_{n^2}$. Meanwhile, S1 Lemma B.6 implies $|F(n^1 \wedge n^2) \cap z^2| \leq 1$. Note that both c^1 and c^2 belong to $F(n^1 \wedge n^2)$ by their definitions. Thus the last two sentences, (35), and (34) together imply that $c^1 \notin z^2$. This implies $n^1 \not\subseteq z^2$ because $c^1 \in n^1$ by (32). Hence $z^2 \notin Z_{n^1}$.

(d). Take any z. Note

$$\begin{aligned} Z_z &= \{\ z' \mid z \subseteq z'\ \} \\ &= \{z\} \cup \{\ z' \mid z \subset z'\ \} \end{aligned}$$

$$\subseteq \{z\} \cup \{\ n \mid z \subset n\ \}$$
$$= \{z\} \cup \varnothing\ ,$$

where the first equality is the definition of Z_z and the last equality follows from (27). Conversely, $\{z\} \subseteq \{\ z' \mid z \subseteq z'\ \} = Z_z$ by the definition of Z_z. By the last two sentences, $Z_z = \{z\}$. □

Lemma A.3. *Let (C, N) be a no-trivial-move (13) choice-set preform (1). Derive Z, $(Z_n)_n$, and $(Z_c)_c$ by (14). Then the following hold.*

(a) $(\forall c)\ Z_c \neq \varnothing$.
(b) $c^1 = c^2$ iff $Z_{c^1} = Z_{c^2}$.
(c) $c \in n$ iff $Z_c \supseteq Z_n$.

Proof. Derive T (2), F (3), and p (4).

(a). Take any c. By (1b), there exists n such that $c \in n$. Further, by Lemma A.2(a), there exists $z \in Z_n$. By the last two sentences, $c \in n \subseteq z$. Hence $z \in Z_c$.

(b). The $\Rightarrow$ direction is obvious. To prove the contrapositive of the $\Leftarrow$ direction, suppose that $c^1 \neq c^2$.

On the one hand, suppose that $(\nexists z)\ \{c^1, c^2\} \subseteq z$. Then $Z_{c^1} \cap Z_{c^2} = \varnothing$. Thus since $Z_{c^1} \neq \varnothing$ by part (a), $Z_{c^1} \neq Z_{c^2}$.

On the other hand, suppose that $(\exists z)\ \{c^1, c^2\} \in z$. Then Lemma A.1 at $a{=}\{c^1, c^2\}$ and $n{=}z$ allows us to define

$$t^{12} := \min\{\ t \mid \{c^1, c^2\} \subseteq t \subseteq z\ \}\ .$$

By the definition of t^{12}, $c^2 \notin p(t^{12})$ or $c^1 \notin p(t^{12})$. Without loss of generality, assume the former and define $t^1 = p(t^{12})$. Then

(36) $$c^1 \in t^1 \text{ and } c^2 \in F(t^1)\ ,$$

where the first half follows from the assumption $c^1 \neq c^2$. By no-trivial-moves (13), we may define c^* such that

(37) $$c^* \neq c^2 \text{ and } c^* \in F(t^1)\ .$$

So, by the definition of F we can construct the node $t^1 \cup \{c^*\}$, and further, by Lemma A.2(a), we may take $z^* \in Z_{t^1 \cup \{c^*\}}$. Note

(38) $$c^1 \in t^1 \subseteq z^*\ ,$$

where the set membership follows from the first half of (36) and the set inclusion follows from the definition of z^*. Meanwhile, by the second halves of (36) and (37), both c^2 and c^* belong to $F(t^1)$. By S1 Lemma B.6, $|F(t^1){\cap}z^*| \leq 1$. By the definition of z^*, $c^* \in z^*$. Thus, the last three sentences and the first half of (37) imply

$$c^2 \notin z^* \ . \tag{39}$$

(38) and (39) imply that $z^* \in Z_{c^1}{\smallsetminus}Z_{c^2}$. Hence $Z_{c^1} \neq Z_{c^2}$.

(c). The $\Rightarrow$ direction is straightforward: by the definitions of Z_c and Z_n,

$$c{\in}n \ \Rightarrow \ \{z|c{\in}z\}{\supseteq}\{z|n{\subseteq}z\} \ \Rightarrow \ Z_c{\supseteq}Z_n \ .$$

To prove the contrapositive of the $\Leftarrow$ direction, suppose that $c \notin n$. Two cases arise. On the one hand, suppose that $(\nexists z)\ n{\cup}\{c\}{\subseteq}z$. Then $Z_c{\cap}Z_n = \varnothing$. Thus, since $Z_n \neq \varnothing$ by Lemma A.2(a), $Z_n \not\subseteq Z_c$.

On the other hand, suppose that $(\exists z)\ n{\cup}\{c\}{\subseteq}z$. Take such a z. Since $c \notin n$ by assumption, $n \subset z$. Thus S1 Corollary 5.2(b) implies $n \in T$. Hence Lemma A.1, at its a equal to $n{\cup}\{c\}$ and its n equal to z, allows us to define

$$t^{nc} := \min\{\ t \mid n{\cup}\{c\}{\subseteq}t{\subseteq}z\ \} \ .$$

Since $c \notin n$ by assumption, $n \subset t^{nc}$. Therefore [1] $|n| < |t^{nc}|$ and [2] S1 Corollary 5.1(a) implies that $n = p^{|t^{nc}|-|n|}(t^{nc})$. These two facts imply that

$$n \subseteq p(t^{nc}) \ . \tag{40}$$

Hence the definition of t^{nc} implies that

$$t^{nc}{\smallsetminus}p(t^{nc}) = \{c\} \ . \tag{41}$$

By (41), $c \in F(p(t^{nc}))$. Further, by no-trivial-moves (13), we may take $c^* \neq c$ such that $c^* \in F(p(t^{nc}))$. Construct the node $p(t^{nc}){\cup}\{c^*\}$. By Lemma A.2(a), we may take $z^* \in Z_{p(t^{nc})\cup\{c^*\}}$. Note that

$$n \subseteq p(t^{nc}) \subseteq p(t^{nc}){\cup}\{c^*\} \subseteq z^* \ , \tag{42}$$

where the first set inclusion is (40) and the last set inclusion follows from the definition of z^*. Meanwhile, recall from the first two sentences of this paragraph that c and c^* are distinct members of $F(p(t^{nc}))$. By S1 Lemma B.6, $|F(p(t^{nc})){\cap}z^*| \leq 1$. By the definition of z^*, $c^* \in z^*$.

Thus by the last three sentences, $c \notin z^*$. This and (42) imply that $z^* \in Z_n \smallsetminus Z_c$. Hence $Z_n \not\subseteq Z_c$. □

Lemma A.4. *Let (C,N) be a no-trivial-move (13) choice-set preform (1), and derive its T (2), F (3), and p (4). Next, derive Z, $(Z_n)_n$, and $(Z_c)_c$ by (14). Further, let*

$$W := Z \ , \ \dot{N} := \{Z_n|n\} \ , \ \textit{and}\ \dot{C} := \{Z_c|c\} \ .$$

Then (a) $(W,\dot{N})$ satisfies (6a,b). So by `S2` *Lemma A.1, derive $\dot{T}$ (7), $\dot{p}$ (8), and $\dot{F}$ (10). Then the following hold.*

(b) $(Z_n)_n$ is a bijection from N onto $\dot{N}$. Further, $Z_{\{\}} = W$.
(c) $(Z_c)_c$ is a bijection from C onto $\dot{C}$.
(d) $\dot{T} = \{\ Z_t \mid t{\in}T\ \}$.
(e) $\dot{p} = \{\ (Z_{t^\sharp}, Z_t) \mid (t^\sharp, t){\in}p\ \}$.
(f) $(\forall t, c, t^\sharp)$ ($c{\notin}t$ and $t{\cup}\{c\}{=}t^\sharp$) iff ($Z_t{=}\dot{p}(Z_{t^\sharp})$ and $Z_t{\cap}Z_c{=}Z_{t^\sharp}$).
(g) $\dot{F} = \{\ (Z_t, Z_c) \mid (t,c){\in}F\ \}$.

Proof. (a). (6a). By definition, $W = Z$ and $\dot{N} = \{Z_n|n\}$. Thus it suffices to show that $\{Z_n|n\}$ is a collection of subsets of Z that contains Z but not $\varnothing$. By definition, each Z_n is a subset of Z. Further, since $\{\} \in N$ by `S1` Lemma B.5, $Z_{\{\}}$ is well-defined and $Z = \{z|\{\}{\subseteq}z\} = Z_{\{\}} \in \{Z_n|n\}$. Finally, since every $Z_n \neq \varnothing$ by Lemma A.2(a), we have $\varnothing \notin \{Z_n|n\}$.

(6b). Take any $\dot{n}^1$ and $\dot{n}^2$. By the definition of $\dot{N}$, there exist n^1 and n^2 such that $\dot{n}^1 = Z_{n^1}$ and $\dot{n}^2 = Z_{n^2}$. Thus there are four possibilities:

(43a) $$n^1 = n^2 \ , \text{ or}$$

(43b) $$n^1 \subset n^2 \ , \text{ or}$$

(43c) $$n^2 \subset n^1 \ , \text{ or}$$

(43d) $$(n^1 \not\subseteq n^2 \text{ and } n^2 \not\subseteq n^1) \ .$$

If (43a) holds, then the definition of n^1, $n^1{=}n^2$, and the definition of n^2 imply

(44a) $$\dot{n}^1 = Z_{n^1} = Z_{n^2} = \dot{n}^2 \ .$$

If (43b) holds, then the definition of n^1, Lemma A.2(b), and the definition of n^2 imply

(44b) $$\dot{n}^1 = Z_{n^1} \supset Z_{n^2} = \dot{n}^2 \ .$$

Similarly, if (43c) holds, then the definition of n^1, Lemma A.2(b), and the definition of n^2 imply

$$\dot{n}^1 = Z_{n^1} \subset Z_{n^2} = \dot{n}^2 \ . \tag{44c}$$

And finally, if (43d) holds, then the definitions of n^1 and n^2 and Lemma A.2(c) imply

$$\dot{n}^1 \cap \dot{n}^2 = Z_{n^1} \cap Z_{n^2} = \varnothing \ . \tag{44d}$$

Since (44a) or (44b) or (44c) or (44d) must hold, (6b) must hold.

(b). $(Z_n)_n$ is surjective by the definition of $\dot{N}$. $(Z_n)_n$ is injective by Lemma A.2(b). Further, $Z_{\{\}} = \{z|\{\}\subseteq z\} = Z = W$ by the definitions of $Z_{\{\}}$ and W.

(c). $(Z_c)_c$ is surjective by the definition of $\dot{C}$. $(Z_c)_c$ is injective by Lemma A.3(b).

(d). I argue

$$\begin{aligned} \dot{T} &= \{ \ \dot{n} \mid \{\dot{n}^\flat|\dot{n}^\flat\supset\dot{n}\} \text{ is finite } \} \\ &= \{ \ Z_n \mid \{Z_{n^\flat}|Z_{n^\flat}\supset Z_n\} \text{ is finite } \} \\ &= \{ \ Z_n \mid \{n^\flat|n^\flat\subset n\} \text{ is finite } \} \\ &= \{ \ Z_n \mid n \in T \ \} \ . \end{aligned}$$

The first equality holds by the definition of $\dot{T}$, the second holds by the first half of part (b), the third holds by Lemma A.2(b), and the last holds by the definition of T.

(e). I argue

$$\begin{aligned} \dot{p} &= \{ \ (\ \dot{t},\ \min\{\dot{t}^\flat|\dot{t}^\flat\supset\dot{t}\}\ ) \mid \dot{t}\neq W \ \} \\ &= \{ \ (\ Z_t,\ \min\{Z_{t^\flat}|Z_{t^\flat}\supset Z_t\}\ ) \mid Z_t\neq W \ \} \\ &= \{ \ (\ Z_t,\ \min\{Z_{t^\flat}|Z_{t^\flat}\supset Z_t\}\ ) \mid t\neq\{\} \ \} \\ &= \{ \ (\ Z_t,\ \min\{Z_{t^\flat}|t^\flat\subset t\}\ ) \mid t\neq\{\} \ \} \\ &= \{ \ (\ Z_t,\ Z_{\mathrm{argmin}\{Z_{t^\flat}|t^\flat\subset t\}}\ ) \mid t\neq\{\} \ \} \\ &= \{ \ (\ Z_t,\ Z_{\max\{t^\flat|t^\flat\subset t\}}\ ) \mid t\neq\{\} \ \} \\ &= \{ \ (\ Z_t,\ Z_{p(t)}\ ) \mid t\neq\{\} \ \} \\ &= \{ \ (\ Z_t,\ Z_{t^\flat}\ ) \mid (t,t^\flat)\in p \ \} \ . \end{aligned}$$

The first equality is the definition of $\dot{p}$, the second holds by the first half of part (b), and the third holds by both halves of part (b). The fourth equality holds by Lemma A.2(b), the fifth by manipulation, and the sixth by Lemma A.2(b) once again. The seventh equality holds by `S1` Corollary 5.1(b), and the last equality holds because the domain of p it $T\setminus\{\{\}\}$.

(f). Forward Direction. Take any t, c, and $t^\sharp$ such that [1] $c{\notin}t$ and [2] $t{\cup}\{c\}{=}t^\sharp$. By the definitions of Z_t and Z_c, by [2], and by the definition of $Z_{t^\sharp}$,

$$Z_t{\cap}Z_c = \{\ z \mid t{\cup}\{c\}{\subseteq}z\ \} = \{\ z \mid t^\sharp{\subseteq}z\ \} = Z_{t^\sharp}\ . \tag{45}$$

In a different vein, [1] and [2] together are equivalent to $c{\in}t^\sharp$ and $t{=}t^\sharp{\setminus}\{c\}$. Thus $t = p(t^\sharp)$ by the definition of p. Hence part (e) implies $Z_t = \dot{p}(Z_{t^\sharp})$. This and (45) were our goals.

Reverse direction. Take any t, c, and $t^\sharp$ such that [1] $Z_t = \dot{p}(Z_{t^\sharp})$ and [2] $Z_t{\cap}Z_c{=}Z_{t^\sharp}$. By part (e), [1] implies

$$t = p(t^\sharp)\ . \tag{46}$$

Further, since $Z_t{\cap}Z_c = \{\ z \mid t{\cup}\{c\}{\subseteq}z\ \}$ by the definitions of Z_t and Z_c, [2] implies

$$\{\ z \mid t{\cup}\{c\}{\subseteq}z\ \} = Z_{t^\sharp}\ . \tag{47}$$

[The left-hand side could be written as $Z_{t\cup\{c\}}$ if it had already been proved that $t{\cup}\{c\}$ is a node.]

This paragraph shows

$$c \notin t\ . \tag{48}$$

Suppose $c \in t$. Then $\{\ z \mid t{\cup}\{c\}{\subseteq}z\ \} = Z_t$. Hence (47) implies $Z_t = Z_{t^\sharp}$. Hence Lemma A.2(b) implies $t = t^\sharp$. This contradicts (46).

This paragraph defines and develops z^{tc}. Because $Z_{t^\sharp} \neq \varnothing$ by Lemma A.2(a), we may take $z^{tc} \in Z_{t^\sharp}$. By (47) we have

$$t{\cup}\{c\} \subseteq z^{tc}\ . \tag{49}$$

This and the next three paragraphs show that $t{\cup}\{c\}$ is a node. By (49) and by Lemma A.1 at $a{=}t{\cup}\{c\}$ and $n{=}z^{tc}$, we may let t^{tc} be $\min\{t^o|t{\cup}\{c\}{\subseteq}t^o{\subseteq}z^{tc}\}$. By (48), $t \subset t^{tc}$. Let $t' = p(t^{tc})$. Since both t and t' are predecessors of t^{tc} by the last two sentences, `S1` Corollary 5.1 implies that either [1] $t \subseteq t'$ or [2] $t' \subset t$. Note that [2], $t \subset t^{tc}$ (3 sentences ago), and $t' = p(t^{tc})$ (2 sentences ago) are contradictory. Thus [1] holds:

$$t \subseteq t' \; . \tag{50}$$

Further, (50), $t' = p(t^{tc})$, and the definition of t^{tc} imply

$$c \in F(t') \; . \tag{51}$$

By no-trivial-moves, there exists $c' \in F(t') \setminus \{c\}$. By Lemma A.2(a), we may take $z' \in Z_{t' \cup \{c'\}}$. The remainder of this paragraph shows that

$$z' \notin \{\; z \mid t \cup \{c\} \subseteq z \;\} \; . \tag{52}$$

Suppose it were. Then both $t' \cup \{c'\}$ and $t \cup \{c\}$ would be subsets of z'. Thus by (51) and the definition of c', c and c' would be distinct elements of $F(t') \cap z'$. This would contradict `S1` Lemma B.6.

This paragraph shows that $t \subset t'$ leads to a contradiction. Accordingly, assume $t \subset t'$. By (49), by the definition of z^{tc}, and by the definitions of t' and t^{tc}, we have that t, $t^\sharp$, and t' are all subsets of z^{tc}. Thus by `S1` Corollaries 5.1 and 5.3, $\{t, t^\sharp, t'\}$ is a chain. Thus $t = p(t^\sharp)$ (46) and $t \subset t'$ (this paragraph's assumption) imply $t^\sharp \subseteq t'$. Hence $t^\sharp \subseteq t' \subseteq t' \cup \{c'\} \subseteq z'$, where the last set inclusion holds by the definition of z'. Thus $z' \in Z_{t^\sharp}$. This and (52) contradict (47).

Since the previous paragraph shows $t \subset t'$ is false, (50) implies $t = t'$. So (51) shows $c \in F(t)$. So $t \cup \{c\}$ is a node. [If it were needed, it could be shown that $t \cup \{c\}$ equals the t^{tc} defined three paragraphs ago.]

Equation (49) and the definition of z^{tc} imply that t, $t \cup \{c\}$, and $t^\sharp$ are all subsets of z^{tc}. Thus, since $t \cup \{c\}$ is a node by the previous paragraph, `S1` Corollaries 5.1 and 5.3 imply that $\{t, t \cup \{c\}, t^\sharp\}$ is a chain. Thus (46) and (48) imply $t \cup \{c\} = t^\sharp$. This and (48) are our goals.

(g). I argue

$$\begin{aligned}
\dot{F} &= \{\; (\dot{t}, \dot{c}) \mid \dot{c} \not\supseteq \dot{t} \text{ and } (\exists \dot{t}^\sharp \in \dot{p}^{-1}(\dot{t}))\; \dot{c} \supseteq \dot{t}^\sharp \;\} \\
&= \{\; (Z_t, Z_c) \mid Z_c \not\supseteq Z_t \text{ and } (\exists Z_{t^\sharp} \in \dot{p}^{-1}(Z_t))\; Z_c \supseteq Z_{t^\sharp} \;\} \\
&= \{\; (Z_t, Z_c) \mid c \notin t \text{ and } (\exists Z_{t^\sharp} \in \dot{p}^{-1}(Z_t))\; c \in t^\sharp \;\} \\
&= \{\; (Z_t, Z_c) \mid c \notin t \text{ and } (\exists t^\sharp \in p^{-1}(t))\; c \in t^\sharp \;\} \\
&= \{\; (Z_t, Z_c) \mid (\exists t^\sharp)\; c \notin t,\; (t^\sharp, t) \in p,\; c \in t^\sharp \;\} \\
&= \{\; (Z_t, Z_c) \mid (\exists t^\sharp)\; c \notin t,\; t \cup \{c\} = t^\sharp \;\}
\end{aligned}$$

$$= \{ \ (Z_t, Z_c) \mid c{\notin}t, \ t{\cup}\{c\}{\in}T \ \}$$
$$= \{ \ (Z_t, Z_c) \mid (c,t){\in}F \ \} \ .$$

The first equality is the definition of $\dot{F}$, the second follows from parts (b) and (c), and the third follows from Lemma A.3(c). The fourth equality follows from part (e) and the sixth follows from the definition of p. The last equality follows follows from the definition of F. □

Lemma A.5. *Let $((C_i)_i, N)$ be a no-trivial-move (13) choice-set form (5). Next, let $C = \cup_i C_i$, and define Z, $(Z_n)_n$, and $(Z_c)_c$ by (14). Finally, let*

$$W := Z \ , \ \dot{N} := \{Z_n|n\} \ , \ \dot{C} := \{Z_c|c\} \ ,$$
$$\textit{and } (\forall i) \ \dot{C}_i := \{Z_c|c{\in}C_i\} \ .$$

Then the following hold.

(a) $(W, \dot{N})$ is an AR* *outcome-set tree (6).*

(b) $(W, \dot{N}, \dot{C})$ is a concise (11) AR* *outcome-set preform (9).*

(c) $(W, \dot{N}, (\dot{C}_i)_i)$ is a concise AR* *outcome-set form (12).*

Proof. Derive T (2), F (3), and p (4) from $((C_i)_i, N)$. Then, by Lemma A.4(a) and S2 Lemma A.1, derive $\dot{T}$ (7), $\dot{p}$ (8), and $\dot{F}$ (10) from $(W, \dot{N}, (\dot{C}_i)_i)$.

(a). Since (6a,b) was established by Lemma A.4(a), (6c–e) remain. I will prove these in the order (6c), (6e), (6d).

(6c). By definition, $W = Z$ and $\dot{N} = \{Z_n|n\}$. Thus it suffices to show that $\{\{z\}|z\} \subseteq \{Z_n|n\}$. Accordingly, take any z and note that $\{z\} = Z_z \in \{Z_{z'}|z'\} \subseteq \{Z_n|n\}$, where the equality holds by Lemma A.2(d), and where the set inclusion holds since $Z \subseteq N$ by the definition of Z.

(6e). I argue

$$\dot{N} = \{Z_n|n\}$$
$$= \{Z_n|(\exists n^\sharp)n{\subset}n^\sharp\} \cup \{Z_n|(\nexists n^\sharp)n{\subset}n^\sharp\}$$
$$\subseteq \{Z_t|t\} \cup \{Z_n|(\nexists n^\sharp)n{\subset}n^\sharp\}$$
$$= \dot{T} \cup \{Z_n|(\nexists n^\sharp)n{\subset}n^\sharp\}$$
$$= \dot{T} \cup \{Z_n|n{\in}N{\smallsetminus}F^{-1}(C)\}$$
$$= \dot{T} \cup \{Z_n|n{\in}Z\}$$

$$
\begin{aligned}
&= \dot{T} \cup \{Z_z|z\} \\
&= \dot{T} \cup \{\{z\}|z\} \\
&= \dot{T} \cup \{\{w\}|w\} \ .
\end{aligned}
$$

The first equality is the definition of $\dot{N}$, the set inclusion follows from `S1` Corollary 5.2(b), and the third equality follows from Lemma A.4(d). The fourth equality follows from `S1` Corollary 5.2(c), and the fifth equality follows from the definition of Z. The seventh equality follows from Lemma A.2(d), and the eighth equality follows from the definition of W.

(6d). Take any nonempty chain $\dot{N}^*$ in $\dot{N}$.

On the one hand, suppose there is a $\{w\} \in \dot{N}^*$. Note that $\varnothing \notin \dot{N}$ by (6a), which has already been proved. Thus, since $\dot{N}^*$ is a chain containing $\{w\}$, $\cap\dot{N}^* = \{w\}$. Hence $\cap\dot{N}^* \in \dot{N}$ by (6c), which has already been proved.

On the other hand, suppose there is no $\{w\} \in \dot{N}^*$. Then $\dot{N}^* \subseteq \dot{T}$ by (6e), which has already been proved. Thus by Lemma A.4(b,d), there exists a nonempty $T^* \subseteq T$ such that $\dot{N}^* = \{\ Z_t \mid t{\in}T^*\ \}$. Further, by Lemma A.2(b), T^* is a chain. On the one hand, if T^* is infinite, then (1c) implies $\cup T^* \in N$. On the other hand, if T^* is finite, then $\cup T^* \in T^* \subseteq N$. Thus in either case,

$$\cup T^* \in N \ . \tag{53}$$

I argue

$$
\begin{aligned}
\cap\dot{N}^* &= \cap\{\ Z_t \mid t{\in}T^*\ \} \\
&= \cap\{\ \{z|t{\subseteq}z\} \mid t{\in}T^*\ \} \\
&= \{\ z \mid (\forall t{\in}T^*)\ t{\subseteq}z\ \} \\
&= \{\ z \mid \cup T^*{\subseteq}z\ \} \\
&= Z_{\cup T^*} \\
&\in \dot{N} \ .
\end{aligned}
$$

The first equality follows from the definition of T^*, and the second equality follows from the definition of Z_t. The fifth equality follows from the definition of $Z_{\cup T^*}$, which is applicable by (53). Finally, the set membership follows from (53) and the definition of $\dot{N}$.

(b). Since (9a) was established in part (a), it suffices to prove (9b–e) and conciseness (11).

(9b). By definition, $\dot{C} = \{Z_c|c\}$. By Lemma A.3(a), each Z_c is nonempty. Further, each

$$Z_c = \{z|c{\in}z\} \subset Z = W \ ,$$

where the first equality is the definition of Z_c and the second equality is the definition of W.

(9c). Take any $\dot{t}^\flat$. By Lemma A.4(d), we may let $t^\flat$ be such that $\dot{t}^\flat = Z_{t^\flat}$. I argue

$$\begin{aligned}
\dot{p}^{-1}(\dot{t}^\flat) &= \{\ \dot{t}\ |\ (\dot{t},\dot{t}^\flat){\in}\dot{p}\ \} \\
&= \{\ \dot{t}\ |\ (\dot{t},Z_{t^\flat}){\in}\dot{p}\ \} \\
&= \{\ Z_t\ |\ (Z_t,Z_{t^\flat}){\in}\dot{p}\ \} \\
&= \{\ Z_t\ |\ (t,t^\flat){\in}p\ \} \\
&= \{\ Z_t\ |\ (\exists c)\ c{\in}t\ ,\ t{\smallsetminus}\{c\}{=}t^\flat\ \} \\
&= \{\ Z_t\ |\ (\exists c)\ c{\notin}t^\flat\ ,\ t^\flat{\cup}\{c\}{=}t\ \} \\
&= \{\ Z_t\ |\ (\exists c)\ c{\notin}t^\flat\ ,\ t^\flat{\cup}\{c\}{=}t\ ,\ Z_{t^\flat}{\cap}Z_c{=}Z_t\ \} \\
&= \{\ Z_{t^\flat}{\cap}Z_c\ |\ (\exists t)\ c{\notin}t^\flat\ ,\ t^\flat{\cup}\{c\}{=}t\ \} \\
&= \{\ Z_{t^\flat}{\cap}Z_c\ |\ c{\in}F(t^\flat)\ \} \\
&= \{\ Z_{t^\flat}{\cap}Z_c\ |\ Z_c{\in}\dot{F}(Z_{t^\flat})\ \} \\
&= \{\ Z_{t^\flat}{\cap}\dot{c}\ |\ \dot{c}{\in}\dot{F}(Z_{t^\flat})\ \} \\
&= \{\ \dot{t}^\flat{\cap}\dot{c}\ |\ \dot{c}{\in}\dot{F}(\dot{t}^\flat)\ \}\ .
\end{aligned}$$

The second equality holds by the definition of $t^\flat$, and the third holds by Lemma A.4(b,d). The fourth equality holds by Lemma A.4(e), and the fifth holds by the definition of p. The seventh equality in the $\subseteq$ direction holds by the $\Rightarrow$ direction of Lemma A.4(f). Meanwhile, the seventh equality in the $\supseteq$ direction is obvious. The eighth equality is a change of variable from t to c, and the ninth follows from the definition of F. The tenth equality follows from Lemma A.4(g), the eleventh follows from Lemma A.4(c), and the twelfth follows from the definition of $t^\flat$.

(9d). Suppose $\{\dot{c}^1, \dot{c}^2\} \subseteq \dot{F}(\dot{t})$ and $\dot{c}^1 \neq \dot{c}^2$. By $\dot{c}^1 \in \dot{F}(\dot{t})$ and by Lemma A.4(g), there exist c^1 and t^1 such that

(54a) $$\dot{t} = Z_{t^1} \ ,$$

(54b) $$\dot{c}^1 = Z_{c^1} \ ,$$

(54c) $$\text{and } c^1 \in F(t^1) \ .$$

Similarly, by $\dot{c}^2 \in \dot{F}(\dot{t})$ and by Lemma A.4(g), there exist c^2 and t^2 such that

(55a) $$\dot{t} = Z_{t^2} \ ,$$

(55b) $$\dot{c}^2 = Z_{c^2} \ ,$$

(55c) $$\text{and } c^2 \in F(t^2) \ .$$

(54a), (55a), and Lemma A.4(b) together imply $t^1 = t^2$. Thus by (54c) and (55c),

(56) $$\{c^1, c^2\} \subseteq F(t^1) \ .$$

Further, (54b), (55b), Lemma A.4(c), and $\dot{c}^1 \neq \dot{c}^2$ together imply

(57) $$c^1 \neq c^2 \ .$$

Now suppose that $\dot{c}^1 \cap \dot{c}^2 \neq \varnothing$. By (54b) and (55b), this is equivalent to $Z_{c^1} \cap Z_{c^2} \neq \varnothing$. Thus, there exists z such that $z \in Z_{c^1} \cap Z_{c^2}$. This is equivalent to

(58) $$\{c^1, c^2\} \subseteq z \ .$$

(56), (57), and (58) imply that $|F(t^1) \cap z| \geq 2$. This contradicts S1 Lemma B.6.

(9e). Before starting the main argument, I argue that

(59a) $$\begin{gathered}(\exists\, t^1, t^2) \\ F(t^1) \smallsetminus F(t^2) \neq \varnothing \text{ and } F(t^1) \cap F(t^2) \neq \varnothing \\[1em] \Leftrightarrow\ (\exists\, t^1, t^2, c^A, c^B) \\ c^A \in F(t^1) \smallsetminus F(t^2) \text{ and } c^B \in F(t^1) \cap F(t^2) \\[1em] \Leftrightarrow\ (\exists\, t^1, t^2, c^A, c^B) \\ \{(t^1, c^A), (t^1, c^B), (t^2, c^B)\} \subseteq F \text{ and } (t^2, c^A) \notin F \\[1.5em] \Leftrightarrow\ (\exists\, Z_{t^1}, Z_{t^2}, Z_{c^A}, Z_{c^B}) \\ \{(Z_{t^1}, Z_{c^A}), (Z_{t^1}, Z_{c^B}), (Z_{t^2}, Z_{c^B})\} \subseteq \dot{F} \text{ and } (Z_{t^2}, Z_{c^A}) \notin \dot{F}\end{gathered}$$

$$\Leftrightarrow\ (\exists\, \dot{t}^1, \dot{t}^2, \dot{c}^A, \dot{c}^B)$$
$$\{(\dot{t}^1, \dot{c}^A), (\dot{t}^1, \dot{c}^B), (\dot{t}^2, \dot{c}^B)\} \subseteq \dot{F} \text{ and } (\dot{t}^2, \dot{c}^A) \notin \dot{F}$$
$$\Leftrightarrow\ (\exists\, \dot{t}^1, \dot{t}^2, \dot{c}^A, \dot{c}^B)$$
$$\dot{c}^A \in \dot{F}(\dot{t}^1) \smallsetminus \dot{F}(\dot{t}^2) \text{ and } \dot{c}^B \in \dot{F}(\dot{t}^1) \cap \dot{F}(\dot{t}^2)$$
(59b)
$$\Leftrightarrow\ (\exists\, \dot{t}^1, \dot{t}^2)$$
$$\dot{F}(\dot{t}^1) \smallsetminus \dot{F}(\dot{t}^2) \neq \varnothing \text{ and } \dot{F}(\dot{t}^1) \cap \dot{F}(\dot{t}^2) \neq \varnothing\ .$$

The third equivalence follows from Lemma A.4(g). The fourth equivalence follows from Lemma A.4(b–d).

By assumption (1e), (59a) is false. Thus by the preceding paragraph, (59b) is false. Thus (9e) is true.

Conciseness (11). I will first show that

(60) $$(\forall c)\ Z_c \subseteq \cup\{\ Z_t \mid t{\in}F^{-1}(c)\ \}\ .$$

Accordingly, take any c and take any $z \in Z_c$. Since $\{c\}{\subseteq}z$, Lemma A.1 at $a{=}\{c\}$ and $n{=}z$ allows us to define

$$t^c := \min\{\ t \mid c{\in}t{\subseteq}z\ \}\ .$$

Since the definitions of p and t^c imply $p(t^c) \subset t^c \subseteq z$,

(61) $$z \in Z_{p(t^c)}\ .$$

Meanwhile, note that $c \in p(t^c)$ would contradict the minimization in the definition of t^c. Hence, $c \notin p(t^c)$ and $p(t^c){\cup}\{c\} = t^c$. By the definition of F, this implies $c \in F(p(t^c))$, which is equivalent to $p(t^c) \in F^{-1}(c)$. (61) and the conclusion of the last sentence imply

$$z \in Z_{p(t^c)} \subseteq \cup\{\ Z_t \mid t \in F^{-1}(c)\ \}\ .$$

Hence (60) has been established.

I will now prove (11). Accordingly, take any $\dot{c}$. By the definition of $\dot{C}$ there exists c such that $\dot{c} = Z_c$. I argue

$$\begin{aligned}\dot{c} &= Z_c \\ &\subseteq \cup\{\ Z_t \mid t \in F^{-1}(c)\ \} \\ &= \cup\{\ Z_t \mid Z_t \in \dot{F}^{-1}(Z_c)\ \} \\ &= \cup\{\ Z_t \mid Z_t \in \dot{F}^{-1}(\dot{c})\ \} \\ &= \cup\{\ \dot{t} \mid \dot{t} \in \dot{F}^{-1}(\dot{c})\ \} \\ &= \cup\dot{F}^{-1}(\dot{c})\ .\end{aligned}$$

The first equality is the definition of c. The set inclusion is (60) and the second equality follows from Lemma A.4(g). The third equality follows from the definition of c, and the fourth equality follows from Lemma A.4(b,d).

(c). Since (12a) has been established by part (b), it suffices to prove (12b) and (12c).

(12b). By assumption (5b), the members of $\{\, C_i \,|i\}$ are disjoint. Thus by Lemma A.4(c), the members of $\{\, \{Z_c|c{\in}C_i\}\,|i\}$ are disjoint. Thus by the definition of $(\dot{C}_i)_i$, the members of $\{\, \dot{C}_i \,|i\}$ are disjoint.

(12c). Before beginning the main argument, I argue that

$$
\begin{aligned}
&\text{(62a)} && (\exists\, i,t)\ F(t){\not\subseteq}C_i \text{ and } F(t){\cap}C_i{\neq}\varnothing \\
& &\Leftrightarrow\ & (\exists\, i,t,c^1,c^2)\ \{c^1,c^2\}{\in}F(t)\,,\ c^1{\notin}C_i\,,\ \text{and } c^2{\in}C_i \\
& &\Leftrightarrow\ & (\exists\, i,t,c^1,c^2)\ \{Z_{c^1},Z_{c^2}\}{\in}\dot{F}(Z_t)\,,\ c^1{\notin}C_i\,,\ \text{and } c^2{\in}C_i \\
& &\Leftrightarrow\ & (\exists\, i,t,c^1,c^2)\ \{Z_{c^1},Z_{c^2}\}{\in}\dot{F}(Z_t)\,,\ Z_{c^1}{\notin}\{Z_c|c{\in}C_i\}\,,\ \text{and} \\
& & & \quad Z_{c^2}{\in}\{Z_c|c{\in}C_i\} \\
& &\Leftrightarrow\ & (\exists\, i,t,c^1,c^2)\ \{Z_{c^1},Z_{c^2}\}{\in}\dot{F}(Z_t)\,,\ Z_{c^1}{\notin}\dot{C}_i\,,\ \text{and } Z_{c^2}{\in}\dot{C}_i \\
& &\Leftrightarrow\ & (\exists\, i,\dot{t},\dot{c}^1,\dot{c}^2)\ \{\dot{c}^1,\dot{c}^2\}{\in}\dot{F}(\dot{t})\,,\ \dot{c}^1{\notin}\dot{C}_i\,,\ \text{and } \dot{c}^2{\in}\dot{C}_i \\
&\text{(62b)} &\Leftrightarrow\ & (\exists\, i,\dot{t})\ \dot{F}(\dot{t}){\not\subseteq}\dot{C}_i \text{ and } \dot{F}(\dot{t}){\cap}\dot{C}_i{\neq}\varnothing\ .
\end{aligned}
$$

The second equivalence holds by Lemma A.4(g). The third equivalence holds by Lemma A.3(b) and the fourth equivalence holds by the definition of $(\dot{C}_i)_i$. The fifth equivalence holds by Lemma A.4(b–d).

By assumption (5c), (62a) is false. Thus by the preceding paragraph, (62b) is false. Thus (12c) is true. □

Proof A.6 (for Theorem 1)**.** Part (a) follows from Lemma A.5(c). The remaining parts follow from Lemma A.4(b–g). □

Appendix B. For Theorem 2

B.1. Lemmata concerning outcome-set preforms

Lemma B.1. *Suppose that $(W,\dot{N},\dot{C})$ is an* `AR`* *outcome-set preform (9) with its $\dot{T}$ (7), $\dot{p}$ (8), and $\dot{F}$ (10). Then the following hold.*

(a) $\{(\dot{t},\dot{c})|(\dot{t},\dot{t}{\cap}\dot{c}){\in}\dot{p}^{-1}\} = \dot{F}$.

(b) If $\dot{p}(\dot{t}){\cap}\dot{c} = \dot{t}$, *then* $\dot{c} \in \dot{F}(\dot{p}(\dot{t}))$.

Proof. (a). The statement of part (a) is equivalent to

$$(\forall \dot{t}, \dot{c}) \;\; (\dot{t}, \dot{t}{\cap}\dot{c}){\in}\dot{p}^{-1} \;\Leftrightarrow\; (\dot{t}, \dot{c}){\in}\dot{F} \;.$$

To prove the $\Leftarrow$ direction, assume $(\dot{t}, \dot{c}){\in}\dot{F}$. Then (9c) implies that $(\dot{t}, \dot{t}{\cap}\dot{c}){\in}\dot{p}^{-1}$. To prove the $\Rightarrow$ direction, assume $(\dot{t}, \dot{t}{\cap}\dot{c}){\in}\dot{p}^{-1}$. By the definition of $\dot{F}$, it suffices to prove

$$\dot{c} \not\supseteq \dot{t} \text{ and}$$
$$(\exists \dot{t}^\sharp{\in}\dot{p}^{-1}(\dot{t}))\; \dot{c} \supseteq \dot{t}^\sharp \;.$$

To see the first, suppose $\dot{c} \supseteq \dot{t}$. Then $\dot{t}{\cap}\dot{c} = \dot{t}$, in which case the assumption $(\dot{t}, \dot{t}{\cap}\dot{c}){\in}\dot{p}^{-1}$ implies $(\dot{t}, \dot{t}){\in}\dot{p}^{-1}$, in contradiction to the definition of $\dot{p}$. To see the second, note that $(\dot{t}{\cap}\dot{c}){\in}\dot{p}^{-1}(\dot{t})$ by assumption and that $\dot{c} \supseteq (\dot{t}{\cap}\dot{c})$.

(b). Suppose $\dot{p}(\dot{t}){\cap}\dot{c} = \dot{t}$. Then since $(\dot{p}(\dot{t}), \dot{t}) \in \dot{p}^{-1}$, we have

$$(\dot{p}(\dot{t}), \dot{p}(\dot{t}){\cap}\dot{c}) \in \dot{p}^{-1} \;.$$

By part (a) in the $\subseteq$ direction, this implies $(\dot{p}(\dot{t}), \dot{c}) \in \dot{F}$. □

Lemma B.2. *Let $(W, \dot{N}, \dot{C})$ be an* `AR`* *outcome-set preform (9) with its $\dot{T}$ (7) and $\dot{p}$ (8). Then the following hold.*

(a) If $\dot{t}{\neq}W$, there is exactly one $\dot{c}$ such that $\dot{p}(\dot{t}){\cap}\dot{c} = \dot{t}$.

(b) $(\forall \dot{t})$ $\{\dot{c}|\dot{c}{\supseteq}\dot{t}\}$ is finite.

Proof. Define $\dot{F}$ by (10).

(a). Take any $\dot{t} \neq W$.

This paragraph shows that there is at least one $\dot{c}$ that satisfies $\dot{p}(\dot{t}){\cap}\dot{c} = \dot{t}$. Since $\dot{t} \neq W$, $\dot{p}(\dot{t})$ exists. Note that

$$\dot{t} \in \dot{p}^{-1}(\dot{p}(\dot{t})) = \{\; \dot{p}(\dot{t}){\cap}\dot{c} \mid \dot{c}{\in}\dot{F}(\dot{p}(\dot{t})) \;\} \;,$$

where the set membership follows from the function-hood of $\dot{p}$ and the equality follows from (9c). Thus there exists a $\dot{c} \in \dot{F}(\dot{p}(\dot{t}))$ such that $\dot{t} = \dot{p}(\dot{t}){\cap}\dot{c}$ (the membership of $\dot{c}$ in $\dot{F}(\dot{p}(\dot{t}))$ is unnecessary in this proof).

This paragraph shows that no more than one $\dot{c}$ satisfies $\dot{p}(\dot{t}){\cap}\dot{c} = \dot{t}$. Accordingly, suppose that $\dot{c}^1$ and $\dot{c}^2$ satisfy $\dot{p}(\dot{t}){\cap}\dot{c}^1 = \dot{t}$ and $\dot{p}(\dot{t}){\cap}\dot{c}^2 = \dot{t}$. By Lemma B.1(b), both $\dot{c}^1$ and $\dot{c}^2$ belong to $\dot{F}(\dot{p}(\dot{t}))$. Further, since both $\dot{c}^1$ and $\dot{c}^2$ are supersets of $\dot{t}$, we have that $\dot{c}^1{\cap}\dot{c}^2 \neq \varnothing$. The last two sentences and (9d) imply $\dot{c}^1 = \dot{c}^2$.

(b). Take any $\dot{t}$. If $\dot{t} = W$, then $\{\dot{c}|\dot{c}\supseteq W\}$ is finite by (9b). Accordingly, assume $\dot{t} \neq W$. Then `S2` Lemma C.2(a) implies that there exists a unique integer $K \geq 1$ such that the sequence $(\dot{p}^k(\dot{t}))_{k=1}^K$ is well-defined and $W = \dot{p}^K(\dot{t})$. Further, by `S2` Lemma C.2(b), we may let λ be a function from $\{\dot{c}|W\supset\dot{c}\supseteq\dot{t}\}$ into $\{1, 2, ... K\}$ with the property that, for all $\dot{c}$ in its domain, $\dot{c} \in \dot{F}(\dot{p}^{\lambda(\dot{c})}(\dot{t}))$.

Now suppose $\{\dot{c}|\dot{c}\supseteq\dot{t}\}$ were infinite. Then by (9b), $\{\dot{c}|W\supset\dot{c}\supseteq\dot{t}\}$ must have more than K members. Thus by the definition of λ, $\{\dot{c}|W\supset\dot{c}\supseteq\dot{t}\}$ must have distinct members $\dot{c}^1$ and $\dot{c}^2$ such that $\lambda(\dot{c}^1) = \lambda(\dot{c}^2)$. Thus by the definition of λ again, both $\dot{c}^1$ and $\dot{c}^2$ belong to $\dot{F}(\dot{p}^{\lambda(\dot{c}^1)}(\dot{t})) = \dot{F}(\dot{p}^{\lambda(\dot{c}^2)}(\dot{t}))$. However, since both $\dot{c}^1$ and $\dot{c}^2$ belong to $\{\dot{c}|W\supset\dot{c}\supseteq\dot{t}\}$, $\dot{c}^1$ and $\dot{c}^2$ are not disjoint. The last two sentences and (9d) imply $\dot{c}^1 = \dot{c}^2$. This contradicts the distinctness in their definition. □

Lemma B.3. *Let $(W, \dot{N}, \dot{C})$ be a concise (11)* `AR*` *outcome-set preform (9) with its $\dot{T}$ (7). Then $\dot{c} \supseteq \dot{n}$ iff $(\exists\dot{t})$ $\dot{c} \supseteq \dot{t} \supseteq \dot{n}$.*[3]

Proof. The reverse direction is immediate. To prove the forward direction, assume $\dot{c} \supseteq \dot{n}$. If $\dot{n} \in \dot{T}$, the conclusion follows immediately by setting $\dot{t} = \dot{n}$. Thus we may assume $\dot{n} \notin \dot{T}$. By (6e), this implies the existence of a w such that $\dot{n} = \{w\}$. Thus we have

$$\dot{c} \supseteq \{w\} = \dot{n} \ .$$

Conciseness implies $\cup\dot{F}^{-1}(\dot{c}) \supseteq \dot{c}$, where $\dot{F}$ is defined by (10). Thus since $\dot{c} \ni w$, there is some $\dot{t}^w \in \dot{F}^{-1}(\dot{c})$ such that $\dot{t}^w \ni w$. Thus we have

$$\dot{c} \supseteq \dot{t}^w\cap\dot{c} \supseteq \{w\} = \dot{n} \ .$$

Since $\dot{t}^w \in \dot{F}^{-1}(\dot{c})$, (9c) implies $\dot{t}^w\cap\dot{c} \in \dot{T}$. Thus the last two sentences imply the lemma's conclusion by setting $\dot{t} = \dot{t}^w\cap\dot{c}$. □

[3] The forward direction can fail without conciseness. For instance, in Example 2 of `S2` Section 3.3, $\dot{c}$=$\{1\}$ is a superset of $\dot{n}$=$\{1\}$ and there is no $\dot{t}$ such that $\dot{c} \supseteq \dot{t} \supseteq \dot{n}$. Similarly, in Example 3 of `S2` Section 3.3, $\dot{c}$=`D`(22)$\cup\{.02\}$ is a superset of $\dot{n}$=$\{.02\}$ and there is no $\dot{t}$ such that $\dot{c} \supseteq \dot{t} \supseteq \dot{n}$. Conciseness and Lemma B.3 are used to justify the third equality in (78) below.

B.2. MAIN ARGUMENT FOR THEOREM 2

Lemma B.4. *Let* $(W,\dot{N},\dot{C})$ *be a concise (11)* AR* *outcome-set preform (9) with its* $(\dot{C}_{\dot{n}})_{\dot{n}}$ *(16). Then the following hold.*
(a) $\cap\dot{C}_{\dot{n}} = \dot{n}$ *(where* $\cap\varnothing$ *is defined to be* W*).*
(b) $\dot{n}^1{\supseteq}\dot{n}^2$ *iff* $\dot{C}_{\dot{n}^1}{\subseteq}\dot{C}_{\dot{n}^2}$*.*

Proof. (a). Take any $\dot{n}$. The $\supseteq$ direction follows immediately from the definition of $\dot{C}_{\dot{n}}$. The $\subseteq$ direction is equivalent to $W{\smallsetminus}\cap\dot{C}_{\dot{n}} \supseteq W{\smallsetminus}\dot{n}$. To prove this, take any $w \notin \dot{n}$. The next three paragraphs will show $w \notin \cap\dot{C}_{\dot{n}}$.

Let $\dot{T}^{wn} = \{\dot{t}|\dot{t}{\supseteq}\{w\}{\cup}\dot{n}\}$. Since $\dot{T}^{wn}$ is a chain by (6b), $\cap\dot{T}^{wn}$ is a member of $\dot{N}$ by (6d). Further,

$$\cap\dot{T}^{wn} \supset \{w\} \tag{63}$$

because [1] $\cap\dot{T}^{wn} \supseteq \{w\}{\cup}\dot{n}$ by the definition of $\dot{T}^{wn}$, [2] $w \notin \dot{n}$ by assumption, and [3] $\dot{n} \neq \varnothing$ by (6a). Since (63) implies that $\cap\dot{T}^{wn}$ is not a singleton, (6e) implies that $\cap\dot{T}^{wn}$ is a member of $\dot{T}$. Accordingly, we may use $\dot{t}^{wn}$ to denote $\cap\dot{T}^{wn}$. I argue

$$\dot{t}^{wn} \supset \{w\}\ , \tag{64a}$$

$$\dot{t}^{wn} \supset \dot{n}\ ,\ \text{and} \tag{64b}$$

$$(\nexists\dot{t})\ \dot{t}^{wn} \supset \dot{t} \supseteq \{w\}{\cap}\dot{n}\ . \tag{64c}$$

(64a) follows from (63) and the the definition of $\dot{t}^{wn}$. (64b) follows from the definitions of $\dot{t}^{wn}$ and $\dot{T}^{wn}$ and from the assumption that $w \notin \dot{n}$. (64c) follows immediately from the definitions of $\dot{t}^{wn}$ and $\dot{T}^{wn}$.

By (64a), S2 equation (4), and S2 Lemma A.3(b), we have that $\dot{p}^{-1}(\dot{t}^{wn})$ partitions $\dot{t}^{wn}$. Thus by (64b) there is some $\dot{t}^n \in \dot{p}^{-1}(\dot{t}^{wn})$ that intersects $\dot{n}$. By (6b), either $\dot{t}^n \supseteq \dot{n}$ or $\dot{n} \supset \dot{t}^n$. If the latter held, (64b) would imply $\dot{t}^{wn} \supset \dot{n} \supset \dot{t}^n$, which contradicts $\dot{t}^{wn} = \dot{p}(\dot{t}^n)$. Hence the former holds:

$$\dot{t}^n \supseteq \dot{n}\ . \tag{65}$$

Note the definition of $\dot{t}^n$ implies $\dot{t}^{wn} \supset \dot{t}^n$. So the last two sentences imply $\dot{t}^{wn} \supset \dot{t}^n \supseteq \dot{n}$. Thus by (64c), we must have

$$\dot{t}^n \not\ni w\ . \tag{66}$$

The definition of $\dot{t}^n$ and (9c) imply the existence of a $\dot{c}$ such that

$$\dot{t}^{wn}{\cap}\dot{c} = \dot{t}^n\ . \tag{67}$$

By (67) and (65), $\dot{c} \supseteq \dot{t}^n \supseteq \dot{n}$. Thus

$$\dot{c} \in \dot{C}_{\dot{n}} \ . \tag{68}$$

Further, by (67) and (66), $\dot{t}^{wn} \cap \dot{c} = \dot{t}^n \not\ni w$. Thus, since $\dot{t}^{wn} \ni w$ by (64a), $\dot{c} \not\ni w$. This and (68) imply that $\cap \dot{C}_{\dot{n}} \not\ni w$.

(b). Suppose $\dot{n}^1 \supseteq \dot{n}^2$. Then $\dot{C}_{\dot{n}^1} \subseteq \dot{C}_{\dot{n}^2}$ by the definition of $(\dot{C}_{\dot{n}})_{\dot{n}}$. Conversely, suppose $\dot{C}_{\dot{n}^1} \subseteq \dot{C}_{\dot{n}^2}$. Then $\cap \dot{C}_{\dot{n}^1} \supseteq \cap \dot{C}_{\dot{n}^2}$, which is equivalent to $\dot{n}^1 \supseteq \dot{n}^2$ by part (a). □

Lemma B.5. *Let $(W, \dot{N}, \dot{C})$ be a concise (11)* `AR`* *outcome-set pre-form (9) with its $\dot{T}$ (7), $\dot{p}$ (8), and $\dot{F}$ (10). Let*

$$C := \dot{C} \text{ and } N := \{\dot{C}_{\dot{n}} | \dot{n}\} \ ,$$

where $(\dot{C}_{\dot{n}})_{\dot{n}}$ is defined by (16). Finally, derive T (2), F (3), and p (4) from (C, N). Then the following hold.

(a) $(\dot{C}_{\dot{n}})_{\dot{n}}$ is a bijection from $\dot{N}$ onto N. Further, $\dot{C}_W = \{\}$.
(b) $T = \{ \ \dot{C}_{\dot{t}} \mid \dot{t} \in \dot{T} \ \}$.
(c) $(\forall \dot{t}, \dot{c}, \dot{t}^\sharp)$ $(\dot{t} = \dot{p}(\dot{t}^\sharp)$ and $\dot{t} \cap \dot{c} = \dot{t}^\sharp)$ iff $(\dot{c} \notin \dot{C}_{\dot{t}}$ and $\dot{C}_{\dot{t}} \cup \{\dot{c}\} = \dot{C}_{\dot{t}^\sharp})$.
(d) $F = \{ \ (\dot{C}_{\dot{t}}, \dot{c}) \mid (\dot{t}, \dot{c}) \in \dot{F} \ \}$.
(e) $p = \{ \ (\dot{C}_{\dot{t}^\sharp}, \dot{C}_t) \mid (\dot{t}^\sharp, \dot{t}) \in \dot{p} \ \}$.

Proof. (a). First sentence. $(\dot{C}_{\dot{n}})_{\dot{n}}$ is onto N by the definition of N. To show $(\dot{C}_{\dot{n}})_{\dot{n}}$ is injective, suppose $\dot{n}^1$ and $\dot{n}^2$ are such that $\dot{C}_{\dot{n}^1} = \dot{C}_{\dot{n}^2}$. Then by two applications of Lemma B.4(a), $\dot{n}^1 = \cap \dot{C}_{\dot{n}^1} = \cap \dot{C}_{\dot{n}^2} = \dot{n}^2$.

Second sentence. I argue $\dot{C}_W = \{\dot{c} | \dot{c} \supseteq W\} = \{\dot{c} | \dot{c} = W\} = \{\}$. The first equality is the definition of $\dot{C}_W$, and the second equality holds by (9b). The third equality follows from $W \notin \dot{C}$, which follows from conciseness via the straightforward paragraph following the statement of `S2` Lemma 3.2.

(b). I will argue that

$$\begin{aligned} T &= \{ \ n \mid n \text{ is finite } \} \\ &= \{ \ \dot{C}_{\dot{n}} \mid \dot{C}_{\dot{n}} \text{ is finite } \} \\ &= \{ \ \dot{C}_{\dot{n}} \mid \dot{n} \in \dot{T} \ \} \\ &= \{ \ \dot{C}_{\dot{t}} \mid \dot{t} \in \dot{T} \ \} \ . \end{aligned}$$

The first equality is the definition of T, and the second equality follows from part (a). Since the last equality is trivial, only the third equality remains. Accordingly, the next two paragraphs will argue that

$$\dot{C}_{\dot{n}} \text{ is finite iff } \dot{n}{\in}\dot{T} \ .$$

To prove the forward direction, assume $\dot{C}_{\dot{n}}$ is finite. Then the collection $\{\dot{C}_{\dot{n}^\flat}|\dot{C}_{\dot{n}^\flat}{\subseteq}\dot{C}_{\dot{n}}\}$ is finite simply because a finite set has a finite number of subsets. Hence $\{\dot{n}^\flat|\dot{C}_{\dot{n}^\flat}{\subseteq}\dot{C}_{\dot{n}}\}$ is finite by part (a). Hence $\{\, \dot{n}^\flat \,|\, \dot{n}^\flat{\supseteq}\dot{n} \,\}$ is finite by Lemma B.4(b). Hence $\dot{n}{\in}\dot{T}$ by the definition of $\dot{T}$.

To prove the reverse direction, assume $\dot{n}{\in}\dot{T}$. Then $\{\dot{c}|\dot{c}{\supseteq}\dot{n}\}$ is finite by Lemma B.2(b). Hence $\dot{C}_{\dot{n}}$ is finite by its definition.

(c). Forward Direction. Take any $\dot{t}$, $\dot{c}$, and $\dot{t}^\sharp$ such that $\dot{t} = \dot{p}(\dot{t}^\sharp)$ and $\dot{t}{\cap}\dot{c} = \dot{t}^\sharp$.

If $\dot{c}$ were an element of $\dot{C}_{\dot{t}}$, two applications of Lemma B.4(a) would result in $\dot{t} = \cap\dot{C}_{\dot{t}} = (\cap\dot{C}_{\dot{t}}) \cap \dot{c} = \dot{t}{\cap}\dot{c} = \dot{t}^\sharp$, which contradicts $\dot{t} = \dot{p}(\dot{t}^\sharp)$. Hence $\dot{c} \notin \dot{C}_{\dot{t}}$, which is the first of the two statements to be proved.

To prove the second, first note that $\dot{t} = \dot{p}(\dot{t}^\sharp)$ implies $\dot{t} \supseteq \dot{t}^\sharp$. This implies $\dot{C}_{\dot{t}} \subseteq \dot{C}_{\dot{t}^\sharp}$ by the definitions of $\dot{C}_{\dot{t}}$ and $\dot{C}_{\dot{t}^\sharp}$. Second note that $\dot{t}{\cap}\dot{c} = \dot{t}^\sharp$ implies $\dot{c} \in \dot{C}_{\dot{t}^\sharp}$ by the definition of $\dot{C}_{\dot{t}^\sharp}$. These two observations together imply $\dot{C}_{\dot{t}}{\cup}\{\dot{c}\} \subseteq \dot{C}_{\dot{t}^\sharp}$.

To show the converse, consider any $\dot{c}^+$ in $\dot{C}_{\dot{t}^\sharp}{\smallsetminus}\dot{C}_{\dot{t}}$. Then by the definition of $(\dot{C}_{\dot{n}})_{\dot{n}}$,

$$\dot{c}^+ \supseteq \dot{t}^\sharp \text{ and} \tag{69a}$$

$$\dot{c}^+ \not\supseteq \dot{t} \ . \tag{69b}$$

Hence by the assumption $\dot{t} = \dot{p}(\dot{t}^\sharp)$ and by the definition of $\dot{F}$,

$$\dot{c}^+ \in \dot{F}(\dot{t}) \ . \tag{70}$$

Meanwhile, the assumption $\dot{t}{\cap}\dot{c} = \dot{t}^\sharp$ implies

$$\dot{c} \supseteq \dot{t}^\sharp \ . \tag{71}$$

Further, the assumptions $\dot{t} = \dot{p}(\dot{t}^\sharp)$ and $\dot{t}{\cap}\dot{c} = \dot{t}^\sharp$ imply

$$\dot{c} \in \dot{F}(\dot{t}) \tag{72}$$

by Lemma B.1(b). (69a) and (71) imply that $\dot{c}^+{\cap}\dot{c} \neq \varnothing$. Thus, (70), (72), and (9d) imply that $\dot{c}^+ = \dot{c}$. Thus any $\dot{c}^+$ in $\dot{C}_{\dot{t}^\sharp}{\smallsetminus}\dot{C}_{\dot{t}}$ equals $\dot{c}$. Equivalently, $\dot{C}_{\dot{t}}{\cup}\{\dot{c}\} \supseteq \dot{C}_{\dot{t}^\sharp}$.

Reverse Direction. Take any $\dot{t}$, $\dot{c}$, and $\dot{t}^\sharp$ such that $\dot{c} \notin \dot{C}_{\dot{t}}$ and $\dot{C}_{\dot{t}} \cup \{\dot{c}\} = \dot{C}_{\dot{t}^\sharp}$. Note

$$(73) \qquad \dot{t} \cap \dot{c} = (\cap \dot{C}_{\dot{t}}) \cap \dot{c} = \cap(\dot{C}_{\dot{t}} \cup \{\dot{c}\}) = \cap(\dot{C}_{\dot{t}^\sharp}) = \dot{t}^\sharp ,$$

where the first equality holds by Lemma B.4(a), the second is tautological ($\dot{C}_{\dot{t}} \cup \{\dot{c}\}$ is a collection of choices), the third holds by $\dot{C}_{\dot{t}} \cup \{\dot{c}\} = \dot{C}_{\dot{t}^\sharp}$ (last sentence), and the last holds by Lemma B.4(a) (again). Also note

$$(74) \qquad \dot{t} \supset \dot{t} \cap \dot{c} = \dot{t}^\sharp ,$$

where the strict set inclusion holds because $\dot{t} \not\subseteq \dot{c}$ by $\dot{c} \notin \dot{C}_{\dot{t}}$ (two sentences ago), and where the equality holds by (73). Finally, the following four sentences argue

$$(75) \qquad (\nexists \dot{t}') \; \dot{t} \supset \dot{t}' \supset \dot{t}^\sharp .$$

Suppose there were such at $\dot{t}'$. Then by Lemma B.4(b), $\dot{C}_{\dot{t}} \subset \dot{C}_{\dot{t}'} \subset \dot{C}_{\dot{t}^\sharp}$. Thus by the assumption $\dot{C}_{\dot{t}} \cup \{\dot{c}\} = \dot{C}_{\dot{t}^\sharp}$, we have $\dot{C}_{\dot{t}} \subset \dot{C}_{\dot{t}'} \subset \dot{C}_{\dot{t}} \cap \{\dot{c}\}$. This is impossible simply because the same collection $\dot{C}_{\dot{t}}$ appears on both ends. (74) and (75) imply $\dot{t} = \dot{p}(\dot{t}^\sharp)$. This and (73) are our goals.

(d). I argue

$$\begin{aligned}
F &= \{ \ (t,c) \mid c \notin t \text{ and } t \cup \{c\} \in T \ \} \\
&= \{ \ (t,c) \mid (\exists t^\sharp) \ c \notin t \text{ and } t \cup \{c\} = t^\sharp \ \} \\
&= \{ \ (t,\dot{c}) \mid (\exists t^\sharp) \ \dot{c} \notin t \text{ and } t \cup \{\dot{c}\} = t^\sharp \ \} \\
&= \{ \ (\dot{C}_{\dot{t}},\dot{c}) \mid (\exists \dot{t}^\sharp) \ \dot{c} \notin \dot{C}_{\dot{t}} \text{ and } \dot{C}_{\dot{t}} \cup \{\dot{c}\} = \dot{C}_{\dot{t}^\sharp} \ \} \\
&= \{ \ (\dot{C}_{\dot{t}},\dot{c}) \mid (\exists \dot{t}^\sharp) \ \dot{t} = \dot{p}(\dot{t}^\sharp) \text{ and } \dot{t} \cap \dot{c} = \dot{t}^\sharp \ \} \\
&= \{ \ (\dot{C}_{\dot{t}},\dot{c}) \mid \dot{t} \cap \dot{c} \in \dot{p}^{-1}(\dot{t}) \ \} \\
&= \{ \ (\dot{C}_{\dot{t}},\dot{c}) \mid (\dot{t},\dot{c}) \in \dot{F} \ \} .
\end{aligned}$$

The first equality is the definition of F. The third equality follows from the definition of C. The fourth equality follows from parts (a) and (b). The fifth equality follows from part (c). The last equality follows from Lemma B.1(a).

(e). I argue

$$\begin{aligned}
p &= \{ \ (t^\sharp, t^\sharp \setminus \{c\}) \mid c \in t^\sharp \text{ and } t^\sharp \setminus \{c\} \in T \ \} \\
&= \{ \ (t^\sharp, t^\sharp \setminus \{c\}) \mid (\exists t) \ c \in t^\sharp \text{ and } t^\sharp \setminus \{c\} = t \ \}
\end{aligned}$$

$$= \{\ (t^\sharp, t^\sharp \smallsetminus \{\dot c\}) \mid (\exists t)\ \dot c {\in} t^\sharp \text{ and } t^\sharp \smallsetminus \{\dot c\} {=} t\ \}$$
$$= \{\ (\dot C_{\dot t^\sharp}, \dot C_{\dot t^\sharp} \smallsetminus \{\dot c\}) \mid (\exists \dot t)\ \dot c {\in} \dot C_{\dot t^\sharp} \text{ and } \dot C_{\dot t^\sharp} \smallsetminus \{\dot c\} = \dot C_{\dot t}\ \}$$
$$= \{\ (\dot C_{\dot t^\sharp}, \dot C_{\dot t}) \mid (\exists \dot c)\ \dot c {\in} \dot C_{\dot t^\sharp} \text{ and } \dot C_{\dot t^\sharp} \smallsetminus \{\dot c\} = \dot C_{\dot t}\ \}$$
$$= \{\ (\dot C_{\dot t^\sharp}, \dot C_{\dot t}) \mid (\exists \dot c)\ \dot c {\notin} \dot C_{\dot t} \text{ and } \dot C_{\dot t} \cup \{\dot c\} = \dot C_{\dot t^\sharp}\ \}$$
$$= \{\ (\dot C_{\dot t^\sharp}, \dot C_{\dot t}) \mid (\exists \dot c)\ \dot t {=} \dot p(\dot t^\sharp) \text{ and } \dot t {\cap} \dot c {=} \dot t^\sharp\ \}$$
$$= \{\ (\dot C_{\dot t^\sharp}, \dot C_{\dot t}) \mid \dot t {=} \dot p(\dot t^\sharp)\ \}$$

The first equality is the definition of p. The third equality holds by the definition of C. The fourth holds by parts (a) and (b). The fifth is a change of variables from $(\dot t^\sharp, \dot c)$ to $(\dot t^\sharp, \dot t)$. The seventh equality holds by part (c). The last equality is obvious in the $\subseteq$ direction, and the converse holds by (9c). □

Lemma B.6. *Suppose $(W, \dot N, (\dot C_i)_i)$ is a concise* AR* *outcome-set form (12). Let*

$$(C_i)_i := (\dot C_i)_i \text{ and } N := \{\dot C_{\dot n} | \dot n\}\ ,$$

where $(\dot C_{\dot n})_{\dot n}$ is defined by (16). Then the following hold.

(a) $(\cup_i C_i, N)$ is a choice-set preform (1) with no-trivial-moves (13).

(b) $((C_i)_i, N)$ is a choice-set form (5) with no-trivial-moves.

Proof. Note (12a) implies (9) and (6). Derive $\dot C := \cup_i \dot C_i$, $\dot T$ (7), $\dot p$ (8), and $\dot F$ (10) from $(W, \dot N, (\dot C_i)_i)$. Further, derive $C := \cup_i C_i$, T (2), F (3), and p (4) from $((C_i)_i, N)$. Note that, by the definitions of C, $(C_i)_i$ and $\dot C$,

(76) $$C = \cup_i C_i = \cup_i \dot C_i = \dot C\ .$$

(a). (1a). By the definitions of N and $(\dot C_{\dot n})_{\dot n}$, each member of N is a subset of $\dot C$. Thus by (76), each member of N is a subset of C.

Further, $\{\} \in N$ because $W \in \dot N$ by (6a) and because $\dot C_W = \{\}$ by Lemma B.5(a). Hence N is nonempty.

(1b). This paragraph proves

(77) $$(\forall \dot c)(\exists \dot n)\ \dot c \in \dot C_{\dot n}\ .$$

Accordingly, take any $\dot c$. By conciseness (11), $\dot F^{-1}(\dot c)$ is nonempty. Thus there is some $\dot t^\flat$ such that $\dot c \in \dot F(\dot t^\flat)$. Thus by (9c), $\dot t^\flat {\cap} \dot c \in \dot p^{-1}(\dot t^\flat)$, which easily implies that $\dot t^\flat {\cap} \dot c$ is a node. Thus, simply because $\dot c \supseteq \dot t^\flat {\cap} \dot c$, we have $\dot c \in \dot C_{\dot t^\flat \cap \dot c}$. Set $\dot n = \dot t^\flat {\cap} \dot c$.

To conclude, this paragraph shows $C \subseteq \cup N$. Accordingly, take any c. By (76), $c \in \dot{C}$. Thus by (77), there is some $\dot{n}$ such that $c \in \dot{C}_{\dot{n}}$. Hence $c \in \dot{C}_{\dot{n}} \subseteq \cup\{\dot{C}_{\dot{n}'}|\dot{n}'\} = \cup N$, where the last equality follows from the definition of N.

(1c). To show the $\subseteq$ half of (1c), take any $n{\in}N{\smallsetminus}T$. Then by the definition of N and Lemma B.5(a,b), $n{\in}\{\dot{C}_{\dot{n}}|\dot{n}\}{\smallsetminus}\{\dot{C}_{\dot{t}}|\dot{t}\}$. In other words, there exists an $\dot{n}{\notin}\dot{T}$ such that $n{=}\dot{C}_{\dot{n}}$.

By $\dot{n}{\notin}\dot{T}$ and the definition of $\dot{T}$, $\dot{n}$ has an infinite number of predecessors. By (6e), each of these predecessors is in $\dot{T}$. Hence $\{\dot{t}|\dot{t}{\supseteq}\dot{n}\}$ is an infinite subset of $\dot{T}$. Further, by (6b), $\{\dot{t}|\dot{t}{\supseteq}\dot{n}\}$ is an infinite chain in $\dot{T}$. Thus by Lemma B.4(b), $\{\dot{C}_{\dot{t}}|\dot{t}{\supseteq}\dot{n}\}$ is an infinite chain. Further, by Lemma B.5(b), $\{\dot{C}_{\dot{t}}|\dot{t}{\supseteq}\dot{n}\}$ is an infinite chain in T. Define T^* to be this infinite chain.

It remains to be shown that $n = \cup T^*$. I argue

(78)
$$
\begin{aligned}
n &= \dot{C}_{\dot{n}} \\
&= \{\ \dot{c} \mid \dot{c}{\supseteq}\dot{n}\ \} \\
&= \cup\{\ \{\dot{c}|\dot{c}{\supseteq}\dot{t}\} \mid \dot{t}{\supseteq}\dot{n}\ \} \\
&= \cup\{\ \dot{C}_{\dot{t}} \mid \dot{t}{\supseteq}\dot{n}\ \} \\
&= \cup T^*\ .
\end{aligned}
$$

The first equality follows from the definition of $\dot{n}$. The second equality is the definition of $\dot{C}_{\dot{n}}$. The $\supseteq$ half of the third equality is obvious. The converse follows from conciseness and Lemma B.3. The fourth equality follows from the definition of $\dot{C}_{\dot{t}}$. The fifth equality follows from the definition of T^*.

To show the $\supseteq$ half of (1c), suppose that T^* is an infinite chain in T. By Lemma B.5(a,b), there is an infinite $\dot{T}^*$ such that $T^* = \{\dot{C}_{\dot{t}}|\dot{t}{\in}\dot{T}^*\}$. Further, by Lemma B.4(b), $\dot{T}^*$ is a chain. Thus $\cap\dot{T}^* \in \dot{N}$ by (6d).

This paragraph argues

(79)
$$
\begin{aligned}
\cup T^* &= \cup\{\ \dot{C}_{\dot{t}} \mid \dot{t}{\in}\dot{T}^*\ \} \\
&= \cup\{\ \{\dot{c}|\dot{c}{\supseteq}\dot{t}\} \mid \dot{t}{\in}\dot{T}^*\ \} \\
&= \{\ \dot{c} \mid (\exists\dot{t}{\in}\dot{T}^*)\ \dot{c}{\supseteq}\dot{t}\ \} \\
&= \{\ \dot{c} \mid \dot{c}{\supseteq}{\cap}\dot{T}^*\ \} \\
&= \dot{C}_{\cap\dot{T}^*} \\
&\in N\ .
\end{aligned}
$$

The first equality follows from the definition of $\dot{T}^*$. The second equality follows from the definition of $\dot{C}_{\dot{t}}$. The third equality is a rearrangement. The $\subseteq$ half of the fourth equality is straightforward. The converse holds because $\dot{T}^*$ is a chain. The fifth equality is the definition of $\dot{C}_{\cap\dot{T}^*}$, which applies because $\cap\dot{T}^* \in \dot{N}$ by the conclusion of the previous paragraph. The set membership follows from the definition of N.

Finally, since T^* is an infinite chain, $\cup T^*$ must be an infinite set. Thus $\cup T^* \notin T$ by the definition of T. This and (79) yield $\cup T^* \in N \smallsetminus T$.

(1d). To start, note that

$$(80) \qquad (\forall \dot{t}, \dot{c}, \dot{t}^\sharp) \quad \dot{t}{=}\dot{p}(\dot{t}^\sharp) \text{ and } \dot{t}{\cap}\dot{c}{=}\dot{t}^\sharp$$
$$\Leftrightarrow \;\; \dot{c}{\notin}\dot{C}_{\dot{t}} \text{ and } \dot{C}_{\dot{t}}{\cap}\{\dot{c}\}{=}\dot{C}_{\dot{t}^\sharp}$$
$$\Leftrightarrow \;\; \dot{c}{\in}\dot{C}_{\dot{t}^\sharp} \text{ and } \dot{C}_{\dot{t}}{=}\dot{C}_{\dot{t}^\sharp}{\smallsetminus}\{\dot{c}\} \; .$$

The first equivalence is Lemma B.5(c). The second is a rearrangement.

Now fix any $t^\sharp \neq \{\}$. The next paragraph will show the existence of a $c{\in}t^\sharp$ such that $t^\sharp{\smallsetminus}\{c\}{\in}T$. The paragraph thereafter will show the uniqueness of such a c. To prepare for these two paragraphs, use Lemma B.5(b) to define $\dot{t}^\sharp$ such that $t^\sharp = \dot{C}_{\dot{t}^\sharp}$.

Since $t^\sharp \neq \{\}$, Lemma B.5(a) implies that $\dot{t}^\sharp \neq W$. Thus, $\dot{p}(\dot{t}^\sharp)$ is well-defined. Thus by Lemma B.2(a), we may let $\dot{c}$ be such that $\dot{p}(\dot{t}^\sharp){\cap}\dot{c} = \dot{t}^\sharp$. Let $\dot{t} = \dot{p}(\dot{t}^\sharp)$. By the last two sentences and (80) in the forward direction,

$$\dot{c}{\in}\dot{C}_{\dot{t}^\sharp} \text{ and } \dot{C}_{\dot{t}}{=}\dot{C}_{\dot{t}^\sharp}{\smallsetminus}\{\dot{c}\} \; .$$

By Lemma B.5(b), we may let $t = \dot{C}_{\dot{t}}$. Further, by (76), we may let $c = \dot{c}$. By the previous three sentences and the definition of $\dot{t}^\sharp$, we have

$$c{\in}t^\sharp \text{ and } t{=}t^\sharp{\smallsetminus}\{c\} \; .$$

Hence there exists a $c{\in}t^\sharp$ such that $t^\sharp{\smallsetminus}\{c\}{\in}T$.

Now consider any $c'{\in}t^\sharp$ such that $t^\sharp{\smallsetminus}\{c'\}{\in}T$. Let $t' = t^\sharp{\smallsetminus}\{c'\}$. By the last two sentences we have

$$c'{\in}t^\sharp \text{ and } t'{=}t^\sharp{\smallsetminus}\{c\} \; .$$

By Lemma B.5(b), we may let $\dot{t}'$ be such that $t' = \dot{C}_{\dot{t}'}$. Further, by (76), we may let $\dot{c}' = c'$. The last three sentences and the definition of $\dot{t}^\sharp$ two paragraphs ago yield

$$\dot{c}'{\in}\dot{C}_{\dot{t}^\sharp} \text{ and } \dot{C}_{\dot{t}'}{=}\dot{C}_{\dot{t}^\sharp}{\smallsetminus}\{\dot{c}'\} \; .$$

Thus by (80) in the reverse direction, we have that $\dot{t}'{=}\dot{p}(\dot{t}^\sharp)$ and $\dot{t}'{\cap}\dot{c}'{=}\dot{t}^\sharp$. By substituting out $\dot{t}'$, we arrive at $\dot{p}(\dot{t}^\sharp){\cap}\dot{c}'{=}\dot{t}^\sharp$. Thus by the uniqueness in Lemma B.2(a) and the definition of $\dot{c}$ at the start of the previous paragraph, we have $\dot{c}' = \dot{c}$. Thus, by the definition of $\dot{c}'$ in this paragraph and the definition of c in the previous one, we have $c' = c$.

(1e). Note that

$$\text{(81a)}\qquad (\forall \dot{t}^1, \dot{t}^2)\ \dot{F}(\dot{t}^1){=}\dot{F}(\dot{t}^2) \text{ or } \dot{F}(\dot{t}^1){\cap}\dot{F}(\dot{t}^2){=}\varnothing$$

$$\Leftrightarrow\ (\forall \dot{t}^1, \dot{t}^2)\ F(\dot{C}_{\dot{t}^1}){=}F(\dot{C}_{\dot{t}^2}) \text{ or } F(\dot{C}_{\dot{t}^1}){\cap}\dot{F}(\dot{C}_{\dot{t}^2}){=}\varnothing$$

$$\text{(81b)}\qquad \Leftrightarrow\ (\forall t^1, t^2)\ F(t^1){=}F(t^2) \text{ or } F(t^1){\cap}F(t^2){=}\varnothing\ ,$$

where the first equivalence follows from Lemma B.5(d), and the second follows from Lemma B.5(a,b). (81a) is the assumed (9e), and (81b) is the desired (1e).

No-trivial-moves. Take any t. By Lemma B.5(b), we may let $\dot{t}$ be such that $t = \dot{C}_{\dot{t}}$.

On the one hand, suppose there is a w such that $\dot{t} = \{w\}$. Then $\dot{p}^{-1}(\dot{t}) = \varnothing$. Thus by (9c), $\dot{F}(\dot{t}) = \varnothing$. Thus by Lemma B.5(d), $F(\dot{C}_{\dot{t}}) = \varnothing$. So by the definition of $\dot{t}$, $F(t) = \varnothing$.

On the other hand, suppose that there is not a w such that $\dot{t} = \{w\}$. Then by `S2` equation (4) and `S2` Lemma A.3(b), $\dot{p}^{-1}(\dot{t})$ has at least two elements. Thus by (9c), $\dot{F}(\dot{t})$ has at least two elements. Thus by Lemma B.5(d), $F(\dot{C}_{\dot{t}})$ has at least two elements. So by the definition of $\dot{t}$, $F(t)$ has at least two elements.

The last two paragraphs have shown that $F(t) = \varnothing$ or $|F(t)| \geq 2$. Hence $|F(t)| \neq 1$.

(b). I must show that $((C_i)_i, N)$ satisfies (5) and has no-trivial-moves. Both (5a) and no-trivial-moves were proved in part (a).

(5b). This follows from (12b) and the fact that each $C_i = \dot{C}_i$ by definition.

(5c). Take any i and any t. By Lemma B.5(b), there exists $\dot{t}$ such that $\dot{C}_{\dot{t}} = t$. By (12c),

$$\dot{F}(\dot{t}){\subseteq}\dot{C}_i \text{ or } \dot{F}(\dot{t}){\cap}\dot{C}_i{=}\varnothing\ .$$

Since $\dot{F}(\dot{t}) = F(\dot{C}_{\dot{t}})$ by Lemma B.5(d), this is equivalent to

$$F(\dot{C}_{\dot{t}}){\subseteq}\dot{C}_i \text{ or } F(\dot{C}_{\dot{t}}){\cap}\dot{C}_i{=}\varnothing\ .$$

By the definitions of $\dot{t}$ and C_i, this is equivalent to

$$F(t){\subseteq}C_i \text{ or } F(t){\cap}C_i{=}\varnothing \ .$$

□

Proof B.7 (for Theorem 2)**.** Part (a) follows from Lemma B.6(b). The remaining parts follow from Lemma B.5. □

APPENDIX C. FOR THEOREM 3 AND ITS COROLLARY

C.1. THERE AND BACK AGAIN, FROM BOTH SIDES

Proof C.1 (for Lemma 4.1)**.** For notational ease, define

$$(W, \dot{N}, (\dot{C}_i)_i) := \widehat{\mathtt{Z}}[((C_i)_i, N)] \ . \tag{82}$$

Then by the definition of $\widehat{\mathtt{Z}}$,

$$W = Z \ , \tag{83a}$$

$$\dot{N} = \{Z_n|n\} \ , \text{ and} \tag{83b}$$

$$(\forall i)\ \dot{C}_i = \{Z_c|c{\in}C_i\} \ , \tag{83c}$$

where Z, $(Z_n)_n$, and $(Z_c)_c$ are derived from $((C_i)_i, N)$ by (14).

Next define

$$((C'_i)_i, N') := \widehat{\mathtt{C}}[(W, \dot{N}, (\dot{C}_i)_i)] \ . \tag{84}$$

Then by the definition of $\widehat{\mathtt{C}}$,

$$(\forall i)\ C'_i = \dot{C}_i \text{ and} \tag{85a}$$

$$N' = \{\dot{C}_{\dot{n}}|\dot{n}\} \ , \tag{85b}$$

where $(\dot{C}_{\dot{n}})_{\dot{n}}$ is derived from $(W, \dot{N}, (\dot{C}_i)_i)$ by (16).

First I argue

$$(\forall i)\ C'_i = \dot{C}_i = \{\ Z_c \mid c{\in}C_i\ \} = \{\ \delta(c) \mid c{\in}C_i\ \} = \delta(C_i) \ . \tag{86}$$

The first equality is (85a), the second equality is (83c), and the third equality follows from the definition of δ. Second I argue

$$\begin{aligned} N' &= \{\ \dot{C}_{\dot{n}} \mid \dot{n}\ \} \\ &= \{\ \{\dot{c}|\dot{c}{\supseteq}\dot{n}\} \mid \dot{n}\ \} \\ &= \{\ \{Z_c|Z_c{\supseteq}Z_n\} \mid n\ \} \\ &= \{\ \{Z_c|c{\in}n\} \mid n\ \} \end{aligned} \tag{87}$$

$$= \{\ \{\delta(c)|c{\in}n\}\ |\ n\ \}$$
$$= \{\ \delta(n)\ |\ n\ \}\ .$$

The first equality is (85b), and the second equality follows from the definition of $(\dot{C}_{\dot{n}})_{\dot{n}}$. The third equality follows from (83b), (83c), and Theorem 1(b–c). The fourth follows from Lemma A.3(c). The fifth follows from the definition of δ.

Since $\delta = (Z_c)_c$ by definition, δ is a bijection by Theorem 1(c). This, (86), and (87) imply that $((C_i)_i, N)$ and $((C'_i)_i, N')$ are equivalent by the choice renaming δ.

Finally, recall $\Phi = ((C_i)_i, N)$ by the definition of Φ in the lemma statement. Thus $\widehat{\mathtt{C}}{\circ}\widehat{\mathtt{Z}}[\Phi] = ((C'_i)_i, N')$ by the previous sentence, (82), and (84). Thus the last three sentences imply that Φ and $\widehat{\mathtt{C}}{\circ}\widehat{\mathtt{Z}}[\Phi]$ are equivalent by the choice renaming δ. □

Proof C.2 (for Lemma 4.2)**.** For notational ease, define

(88) $$((C_i)_i, N) := \widehat{\mathtt{C}}[(W, \dot{N}, (\dot{C}_i)_i)]\ .$$

Thus, by the definition of $\widehat{\mathtt{C}}$,

(89a) $$(\forall i)\ C_i = \dot{C}_i \text{ and}$$

(89b) $$N = \{\dot{C}_{\dot{n}}|\dot{n}\}\ .$$

where $(\dot{C}_{\dot{n}})_{\dot{n}}$ is derived from $(W, \dot{N}, (\dot{C}_i)_i)$ by (16). By Theorem 2(a),

(90) $((C_i)_i, N)$ is a no-trivial-move (13) choice-set form (5) .

Next define

(91) $$(W', \dot{N}', (\dot{C}'_i)_i) := \widehat{\mathtt{Z}}[((C_i)_i, N)]\ .$$

Thus, by the definition of $\widehat{\mathtt{Z}}$,

(92a) $$W' = Z\ ,$$

(92b) $$\dot{N}' = \{Z_n|n\}\ ,$$

(92c) $$(\forall i)\ \dot{C}'_i = \{Z_c|c{\in}C_i\}\ .$$

where Z, $(Z_n)_n$, and $(Z_c)_c$ are derived from $((C_i)_i, N)$ by (14).

For future use on several occasions, I argue

(93) $$Z = N{\smallsetminus}F^{-1}(C)$$
$$= \{\ n\ |\ (\nexists n^\sharp)\ n{\subset}n^\sharp\ \}$$
$$= \{\ \dot{C}_{\dot{n}}\ |\ (\nexists \dot{n}^\sharp)\ \dot{C}_{\dot{n}}{\subset}\dot{C}_{\dot{n}^\sharp}\ \}$$

$$= \{\ \dot{C}_{\dot{n}} \mid (\nexists \dot{n}^\sharp)\ \dot{n} \supset \dot{n}^\sharp\ \}$$
$$= \{\ \dot{C}_{\dot{n}} \mid (\exists w)\ \dot{n} = \{w\}\ \}$$
$$= \{\ \dot{C}_{\{w\}} \mid w\ \}\ .$$

The first equality is the definition of Z. The second equality follows from (90) and `S1` Corollary 5.2(c). The third equality follows from (89b) and Theorem 2(b). The fourth equality follows from Lemma B.4(b). The fifth equality follows from (6a) and (6c).

First I argue

$$\begin{aligned}(\forall i)\ \dot{C}'_i &= \{\ Z_c \mid c \in C_i\ \}\\ &= \{\ \{z \mid c \in z\} \mid c \in C_i\ \}\\ &= \{\ \{z \mid \dot{c} \in z\} \mid \dot{c} \in \dot{C}_i\ \}\\ &= \{\ \{\dot{C}_{\{w\}} \mid \dot{c} \in \dot{C}_{\{w\}}\} \mid \dot{c} \in \dot{C}_i\ \}\\ &= \{\ \{\dot{C}_{\{w\}} \mid w \in \dot{c}\} \mid \dot{c} \in \dot{C}_i\ \}\\ &= \{\ \{\theta(w) \mid w \in \dot{c}\} \mid \dot{c} \in \dot{C}_i\ \}\\ &= \{\ \theta(\dot{c}) \mid \dot{c} \in \dot{C}_i\ \}\end{aligned}$$

The first equality holds by (92c), the second holds by the definition of Z_c, and the third holds by (89a). The fourth equality holds by (93) and Theorem 2(b). The fifth equality holds because $\dot{C}_{\{w\}} = \{\dot{c} \mid \dot{c} \supseteq \{w\}\} = \{\dot{c} \mid w \in \dot{c}\}$. The sixth equality holds by the definition of θ.

Second I argue

$$\begin{aligned}\dot{N}' &= \{\ Z_n \mid n\}\\ &= \{\ \{\ z \mid n \subseteq z\ \} \mid n\}\\ &= \{\ \{\ z \mid \dot{C}_{\dot{n}} \subseteq z\ \} \mid \dot{n}\}\\ &= \{\ \{\ \dot{C}_{\{w\}} \mid \dot{C}_{\dot{n}} \subseteq \dot{C}_{\{w\}}\ \} \mid \dot{n}\}\\ &= \{\ \{\ \dot{C}_{\{w\}} \mid \dot{n} \supseteq \{w\}\ \} \mid \dot{n}\}\\ &= \{\ \{\ \dot{C}_{\{w\}} \mid w \in \dot{n}\ \} \mid \dot{n}\}\\ &= \{\ \{\ \theta(w) \mid w \in \dot{n}\ \} \mid \dot{n}\}\\ &= \{\ \theta(\dot{n}) \mid \dot{n}\}\ .\end{aligned}$$

The first equality holds by (92b), and the second holds by the definition of $(Z_n)_n$. The third holds by (89b) and Theorem 2(b). The fourth holds by (93) and Theorem 2(b) again. The fifth holds by Lemma B.4(b). The seventh holds by the definition of θ.

Third I argue that $\theta = (\dot{C}_{\{w\}})_w$ is a bijection from W onto W'. I do so in four steps. [1] Trivially, $(\{w\})_w$ is a bijection from W onto $\{\{w\}|w\}$. [2] $(\dot{C}_{\{w\}})_{\{w\}}$ is a bijection from $\{\{w\}|w\}$ onto $\{\dot{C}_{\{w\}}|w\}$. This holds because [a] its domain $\{\{w\}|w\}$ is a subset of $\dot{N}$ by (12a), (9a), and (6c), [b] $\dot{N}$ is the domain of $(\dot{C}_{\dot{n}})_{\dot{n}}$, and [c] the latter function is a bijection by Theorem 2(b). [3] Steps [1] and [2] together imply that $\theta = (\dot{C}_{\{w\}})_w$ is a bijection from W onto $\{\dot{C}_{\{w\}}|w\}$. [4] $\{\dot{C}_{\{w\}}|w\} = Z = W'$ by (93) and (92a).

The conclusions of the last three paragraphs show that $(W, \dot{N}, (\dot{C}_i)_i)$ and $(W, \dot{N}', (\dot{C}'_i)_i)$ are equivalent by the outcome renaming θ. Recall $\dot{\Phi} = (W, \dot{N}, (\dot{C}_i)_i)$ by the definition of $\dot{\Phi}$ in the lemma statement. Thus $\widehat{\mathsf{Z}}{\circ}\widehat{\mathsf{C}}[\Phi] = (W, \dot{N}', (\dot{C}'_i)_i)$ by the last sentence, (88), and (91). Thus the last three sentences imply that $\dot{\Phi}$ and $\widehat{\mathsf{Z}}{\circ}\widehat{\mathsf{C}}[\dot{\Phi}]$ are equivalent by the outcome renaming θ. □

C.2. For Theorem 3

Lemma C.3. *Suppose that $\Phi := ((C_i)_i, N)$ and $\Phi' := ((C'_i)_i, N')$ are two choice-set forms (5) that are equivalent (20) by the choice renaming δ. Then the following hold.*

(a) Φ has no-trivial-moves (13) iff Φ' has no-trivial-moves. (I take this as self-evident.)

(b) Suppose Φ and Φ' have no-trivial-moves. Define the conversion operator $\widehat{\mathsf{Z}}$ by (18). Then $\widehat{\mathsf{Z}}[\Phi]$ is equivalent (22) to $\widehat{\mathsf{Z}}[\Phi']$ by the outcome-renaming function θ defined by $\theta(w) = \{\delta(c)|c{\in}w\}$ at each outcome w in the W of $\widehat{\mathsf{Z}}[\Phi]$. (Proof below.)

Proof. (b). This lengthy first paragraph derives four preliminary results about $\Phi = ((C_i)_i, N)$ and $\Phi' = ((C'_i)_i, N')$. From $((C_i)_i, N)$ derive $C = \cup_i C_i$, F (3), Z, $(Z_n)_n$, and $(Z_c)_c$ (14). Similarly from $((C'_i)_i, N')$ derive $C' = \cup_i C'_i$, F' (3), Z', $(Z'_n)_n$, and $(Z'_c)_c$ (14). (20b) and the bijectivity of δ imply that

(94) $\{\ (n, \delta(n))\ |\ n\ \}$ is a bijection from N onto N'

(where $\delta(n) = \{\delta(c)|c{\in}n\}$ by definition). Further, since $Z \subseteq N$, (94) implies that $\{\ (z, \delta(z))\ |\ z\ \}$ is a bijection from Z onto $\{\ \delta(z)\ |\ z\ \}$. Note that

$$\begin{aligned}\{\ \delta(z)\mid z\,\} &= \{\ \delta(n)\mid n{\in}N{\smallsetminus}F^{-1}(C)\ \}\\ &= \{\ \delta(n)\mid (\nexists n^\sharp)\, n{\subset}n^\sharp\ \}\\ &= \{\ \delta(n)\mid (\nexists n^\sharp)\, \delta(n){\subset}\delta(n^\sharp)\ \}\\ &= \{\ n'\mid (\nexists n'^\sharp)\, n'{\subset}n'^\sharp\ \}\\ &= \{\ n'\mid n'{\in}N'{\smallsetminus}(F')^{-1}(C)\ \}\\ &= Z'\ ,\end{aligned}$$

where the first equality holds by the definition of Z, the second holds by `S1` Corollary 5.2(c), the third holds by the assumed bijectivity of δ, the fourth holds by changing variables from $(n,n^\sharp)$ to $(n',n'^\sharp)$ via (94), the fifth holds by `S1` Corollary 5.2(c) again, and the last holds by the definition of Z'. The last two sentences imply that

(95) $\{\ (z,\delta(z))\mid z\,\}$ is a bijection from Z onto Z'

(where $\delta(z)=\{\delta(c)|c{\in}z\}$ by definition). Next, note that

(96) $$(\forall n)\ \{\ \delta(z)\mid z{\in}Z_n\ \}=\{\ \delta(z)\mid n{\subseteq}z\ \}$$
$$=\{\ \delta(z)\mid \delta(n){\subseteq}\delta(z)\ \}=\{\ z'\mid \delta(n){\subseteq}z'\ \}=Z'_{\delta(n)}\ ,$$

where the first equality follows from the definition of $(Z_n)_n$, the second follows from the bijectivity of δ, the third is a change of variables via (95), and the last follows from the definition of $(Z'_{n'})_{n'}$ since $\delta(n)\in N'$ by (94). Finally, note that

(97) $$(\forall c)\ \{\ \delta(z)\mid z{\in}Z_c\ \}=\{\ \delta(z)\mid c{\in}z\ \}$$
$$=\{\ \delta(z)\mid \delta(c){\in}\delta(z)\ \}=\{\ z'\mid \delta(c){\in}z'\ \}=Z'_{\delta(c)}\ ,$$

where the first equality follows from the definition of $(Z_c)_c$, the second follows from the bijectivity of δ, the third is a change of variables via (95), and the last follows from the definition of $(Z'_{c'})_{c'}$ since $\delta(c)\in C'$ by (20a).

Let

(98) $$(W,\dot N,(\dot C_i)_i) := \widehat{\mathtt{Z}}[\Phi] = (\ Z,\{Z_n|n\},(\{Z_c|c{\in}C_i\})_i\ )\ ,$$

where the second equality follows from the definition of $\widehat{\mathtt{Z}}$. Similarly, let

(99) $$(W',\dot N',(\dot C'_i)_i) := \widehat{\mathtt{Z}}[\Phi'] = (\ Z',\{Z'_{n'}|n'\},(\{Z'_{c'}|c'{\in}C'_i\})_i\ )\ ,$$

where the second equality follows from the definition of $\widehat{\mathtt{Z}}$.

I first show that θ is a bijection from W onto W'. By (95) and the first components of (98) and (99),

$$\{\ (w, \delta(w))\ |\ w\ \} \text{ is a bijection from } W \text{ onto } W'$$

(where $\delta(w) = \{\delta(c)|c{\in}w\}$ by definition). This bijection equals θ by the definition of θ.

To derive (22a), I argue

$$\begin{aligned}
&\{\ \theta(\dot{n})\ |\ \dot{n}\ \}\\
&= \{\ \{\ \theta(w)\ |\ w{\in}\dot{n}\ \}\ |\ \dot{n}\ \}\\
&= \{\ \{\ \theta(z)\ |\ z{\in}Z_n\ \}\ |\ n\ \}\\
&= \{\ \{\ \delta(z)\ |\ z{\in}Z_n\ \}\ |\ n\ \}\\
&= \{\ Z'_{\delta(n)}\ |\ n\ \}\\
&= \{\ Z'_{n'}\ |\ n'\ \}\\
&= \dot{N}'\ .
\end{aligned}$$

The second equality follows from the first and second components of (98) and from Theorem 1(b). The third follows from the definition of θ. The fourth follows from (96), the fifth from (94), and the sixth from the second component of (99).

To derive (22b), take any i. I argue

$$\begin{aligned}
&\{\ \theta(\dot{c})\ |\ \dot{c}{\in}\dot{C}_i\ \}\\
&= \{\ \{\ \theta(w)\ |\ w{\in}\dot{c}\ \}\ |\ \dot{c}{\in}\dot{C}_i\ \}\\
&= \{\ \{\ \theta(z)\ |\ z{\in}Z_c\ \}\ |\ c{\in}C_i\ \}\\
&= \{\ \{\ \delta(z)\ |\ z{\in}Z_c\ \}\ |\ c{\in}C_i\ \}\\
&= \{\ Z'_{\delta(c)}\ |\ c{\in}C_i\ \}\\
&= \{\ Z'_{c'}\ |\ c'{\in}C'_i\ \}\\
&= \dot{C}'_i\ .
\end{aligned}$$

The second equality follows from the first and third components of (98) and from Theorem 1(c). The third follows from the definition of θ. The fourth follows from (97). The fifth follows from (20a) and the bijectivity of δ. The sixth follows from the third component of (99). $\square$

Lemma C.4. *Suppose* $\dot\Phi := (W, \dot N, (\dot C_i)_i)$ *and* $\dot\Phi' := (W', \dot N', (\dot C'_i)_i)$ *are two concise* AR* *outcome-set forms (12) which are equivalent (22) by the outcome renaming* θ*. Define the conversion operator* $\widehat{\mathtt{C}}$ *by (19). Then* $\widehat{\mathtt{C}}[\dot\Phi]$ *and* $\widehat{\mathtt{C}}[\dot\Phi']$ *are equivalent (20) by the choice-renaming function* δ *defined by* $\delta(c) = \{\theta(w)|w{\in}c\}$ *at each choice* c *in the* $\cup_i C_i$ *of* $\widehat{\mathtt{C}}[\dot\Phi]$.

Proof. This first paragraph derives some preliminary results about $(W, \dot N, (\dot C_i)_i)$ and $(W', \dot N', (\dot C'_i)_i)$. From the first form, derive $\dot C{=}\cup_i \dot C_i$ and $(\dot C_{\dot n})_{\dot n}$ (16). From the second form, derive $\dot C'{=}\cup_i \dot C'_i$, and $(\dot C'_{\dot n'})_{\dot n'}$ (16). Because $\theta{:}W{\rightarrow}W'$ is assumed to be a bijection, (22a) implies

(100) $\{\ (\dot n, \theta(\dot n))\ |\ \dot n\ \}$ is a bijection from $\dot N$ onto $\dot N'$

(where $\theta(\dot n) = \{\theta(w)|w{\in}\dot n\}$ by definition). Similarly, (22b) and (12b) imply

(101) $\{\ (\dot c, \theta(\dot c))\ |\ \dot c\ \}$ is a bijection from $\dot C$ onto $\dot C'$

(where $\theta(\dot c) = \{\theta(w)|w{\in}\dot c\}$ by definition). Finally, note

(102)
$$(\forall \dot n)\ \{\theta(\dot c)|\dot c{\in}\dot C_{\dot n}\} = \{\theta(\dot c)|\dot c{\supseteq}\dot n\} = \{\theta(\dot c)|\theta(\dot c){\supseteq}\theta(\dot n)\}$$
$$= \{\dot c'|\dot c'{\supseteq}\theta(\dot n)\} = \dot C'_{\theta(\dot n)}\ ,$$

where the first equality holds by the definition of $\dot C_{\dot n}$, the second holds because θ is a bijection, the third is a change of variables from $\dot c$ to $\dot c'$ via (101), and the last follows from the definition of $(\dot C'_{\dot n'})_{\dot n'}$ since $\theta(\dot n) \in \dot N'$ by (100).

Let

(103)
$$((C_i)_i, N) := \widehat{\mathtt{C}}[\dot\Phi] = ((\dot C_i)_i, \{\dot C_{\dot n}|\dot n\})\ ,$$

where the second equality follows from the definition of $\widehat{\mathtt{C}}$. Similarly, let

(104)
$$((C'_i)_i, N') := \widehat{\mathtt{C}}[\dot\Phi'] = ((\dot C'_i)_i, \{\dot C'_{\dot n'}|\dot n'\})\ ,$$

where the second equality follows from the definition of $\widehat{\mathtt{C}}$.

I first show that δ is a bijection from C onto C', where $C{=}\cup_i C_i$ and $C'{=}\cup_i C'_i$ by definition. By the definition of C, the first component of (103), and the definition of $\dot C$, we have $C = \cup_i C_i = \cup_i \dot C_i = \dot C$. Similarly, by the definition of C', the first component of (104), and the definition of $\dot C'$, we have $C' = \cup_i C'_i = \cup_i \dot C'_i = \dot C'$. Thus by (101) and the last two sentences,

$$\{\ (c, \theta(c))\ |\ c\ \}\text{ is a bijection from } C \text{ onto } C'$$

(where $\theta(c) = \{\theta(w)|w{\in}c\}$ by definition). This bijection equals δ by the definition of δ.

To show (20a), take any i. I argue

$$\begin{aligned}\delta(C_i) &= \{\ \delta(c)\ |\ c{\in}C_i\ \} = \{\ \theta(c)\ |\ c{\in}C_i\ \}\\ &= \{\ \theta(\dot{c})\ |\ \dot{c}{\in}\dot{C}_i\ \} = \dot{C}'_i = C'_i\ .\end{aligned}$$

The second equality follows from the definition of δ. The third equality follows from the first component of (103). The fourth equality follows from (22b). The fifth equality follows from the first component of (104).

To show (20b), I argue

$$\begin{aligned}&\{\ \delta(n)\ |\ n\ \}\\ &= \{\ \{\ \delta(c)\ |\ c{\in}n\ \}\ |\ n\ \}\\ &= \{\ \{\ \theta(c)\ |\ c{\in}n\ \}\ |\ n\ \}\\ &= \{\ \{\ \theta(\dot{c})\ |\ \dot{c}{\in}\dot{C}_{\dot{n}}\ \}\ |\ \dot{n}\ \}\\ &= \{\ \dot{C}'_{\theta(\dot{n})}\ |\ \dot{n}\ \}\\ &= \{\ \dot{C}'_{\dot{n}'}\ |\ \dot{n}'\ \}\\ &= N'\ .\end{aligned}$$

The second equality follows from the definition of δ. The third equality follows from both components of (103) and Theorem 2(b). The fourth equality holds by (102). The fifth holds by (100). The last holds by the second component of (104). □

Proof C.5 (for Theorem 3)**.** *(a).* This paragraph shows that $\widehat{\mathtt{Z}}$ is a well-defined function from the class of choice-set forms with no-trivial-moves, into the class of concise $\mathtt{AR}^*$ outcome-set forms, given the theorem statement's definitions of equality. Accordingly, take any choice-set form Φ with no-trivial-moves. By Lemma C.3(a), Φ is well-defined given the concept of equality for choice-set forms. Further, by Theorem 1(a), the definition of $\widehat{\mathtt{Z}}$, and Lemma C.3(b), $\widehat{\mathtt{Z}}[\Phi]$ is a well-defined concise $\mathtt{AR}^*$ outcome-set form given the concepts of equality for [1] choice-set forms and [2] concise $\mathtt{AR}^*$ outcome-set forms.

This paragraph shows that $\widehat{\mathtt{C}}$ is a well-defined function from the class of concise $\mathtt{AR}^*$ outcome-set forms, into the class of choice-set forms with no-trivial-moves, given the theorem statement's definitions of equality. Accordingly, take any concise $\mathtt{AR}^*$ outcome-set form $\dot{\Phi}$. By Theorem 2(a), the definition of $\widehat{\mathtt{C}}$, and Lemma C.4, $\widehat{\mathtt{C}}[\dot{\Phi}]$ is a well-defined choice-set form, with no-trivial-moves, given the concepts of equality for [1] concise $\texttt{AR}^*$ outcome-set forms and [2] choice-set forms.

(b–c). Recall the definitions of $\widehat{\texttt{Z}}$ and $\widehat{\texttt{C}}$. Lemma 4.1 shows, for any choice-set form Φ with no-trivial-moves, that $\widehat{\texttt{C}}\circ\widehat{\texttt{Z}}[\Phi] = \Phi$. Similarly, Lemma 4.2 shows, for any concise $\texttt{AR}^*$ outcome-set form $\dot{\Phi}$, that $\widehat{\texttt{Z}}\circ\widehat{\texttt{C}}[\dot{\Phi}] = \dot{\Phi}$. The last two paragraphs and the last two sentences imply parts (b) and (c). □

C.3. For Corollary 5.1

Lemma C.6. *Suppose that $\bar{\Phi}$ and $\bar{\Phi}'$ are two $\texttt{OR}^*$ choice-sequence forms ($\texttt{S1}$ equation (6)) that are equivalent (26) by the choice renaming δ. Then the following hold.*

(a) $\bar{\Phi}$ has no-absent-mindedness ($\texttt{S1}$ equation (16)) iff $\bar{\Phi}'$ has no-absent-mindedness. (I take this as self-evident.)

(b) $\bar{\Phi}$ has no-trivial-moves (25) iff $\bar{\Phi}'$ has no-trivial-moves. (I take this as self-evident.)

Lemma C.7. *Suppose that $\bar{\Phi}$ and $\bar{\Phi}'$ are two $\texttt{OR}^*$ choice-sequence forms ($\texttt{S1}$ equation (6)) that have no-absent-mindedness ($\texttt{S1}$ equation (16)). Define the conversion operator $\widehat{\texttt{R}}$ by $\texttt{S1}$ equation (17). Then $\bar{\Phi}$ and $\bar{\Phi}'$ are equivalent (26) by the choice renaming δ iff $\widehat{\texttt{R}}[\bar{\Phi}]$ and $\widehat{\texttt{R}}[\bar{\Phi}']$ are equivalent (20) by the same δ. (I take this as self-evident.)*

Lemma C.8. *Suppose that $\bar{\Phi}$ is an $\texttt{OR}^*$ choice-sequence form ($\texttt{S1}$ equation (6)) with no-absent-mindedness ($\texttt{S1}$ equation (16)). Define the operator $\widehat{\texttt{R}}$ by $\texttt{S1}$ equation (17). Then $\bar{\Phi}$ has no-trivial-moves (25) iff $\widehat{\texttt{R}}[\bar{\Phi}]$ has no-trivial-moves (13).*

Proof. Define $((C_i)_i, \bar{N}) := \bar{\Phi}$ and derive its $\bar{T}$ ($\texttt{S1}$ equation (2)) and $\bar{F}$ ($\texttt{S1}$ equation (3)). Then define $((C_i)_i, N) := \widehat{\texttt{R}}[\bar{\Phi}]$ and derive its T (2) and F (3). I argue

$$\begin{aligned} & (\forall t)\ |F(t)| \neq 1 \\ \Leftrightarrow\ & (\forall \bar{t})\ |F(R(\bar{t}))| \neq 1 \\ \Leftrightarrow\ & (\forall \bar{t})\ |\bar{F}(\bar{t})| \neq 1\ . \end{aligned}$$

The first equivalence holds because $R|_{\bar{T}}$ is a bijection from $\bar{T}$ onto T by $\texttt{S1}$ Theorem 1(b,c). The second equivalence holds by $\texttt{S1}$ Theorem 1(e). □

Proof C.9 (for Corollary 5.1)**.** Let

$\bar{D}$:= { $\texttt{OR}^*$ choice-sequence forms with no-absent-mindedness and no-trivial-moves (25) } ,

D := { choice-set forms with no-trivial-moves (13) } , and

$\dot{D}$:= { concise $\texttt{AR}^*$ outcome-set forms } .

By Lemmata C.6 and C.3(a), these three classes are well-defined given the concepts of equality in the corollary statement.

(a-b). By S1 Theorem 2 and Lemma C.7, $\widehat{\texttt{R}}$ is a well-defined bijection from the class of $\texttt{OR}^*$ choice-sequence forms with no-absent-mindedness, onto the class of choice-set forms, given the concepts of equality in the corollary statement. Hence Lemma C.8 implies that $\widehat{\texttt{R}}|_{\bar{D}}$ is a bijection from $\bar{D}$ onto D. By definition, $\widehat{\texttt{R}}_* = \widehat{\texttt{R}}|_{\bar{D}}$.

(c-d). Part (b) shows that $\widehat{\texttt{R}}_*$ is a bijection from $\bar{D}$ onto D. Theorem 3(b) shows that $\widehat{\texttt{Z}}$ is a bijection from D onto $\dot{D}$. Parts (c) and (d) follow immediately. □

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