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CONCISELY SPECIFYING CHOICES IN AN OUTCOME-SET FORM

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Abstract. Von Neumann and Morgenstern (1944) specify both nodes and choices as sets of outcomes. This outcome-set formulation is extended to the infinite horizon by the discrete extensive forms of Alós-Ferrer and Ritzberger (2013).

I propose to restrict such outcome-set forms with a new assumption called "conciseness". Conciseness requires that choices be defined in an economical fashion. I find broad classes of infinite-horizon forms that violate conciseness. Yet, I show that every outcome-set form can be equivalently re-defined so as to satisfy conciseness. Thus the assumption of conciseness can increase mathematical tractability at no cost to game theorists.

1. Introduction

1.1. Motivation

Von Neumann and Morgenstern (1944, Sections 9 and 10) specify both nodes and choices as sets of outcomes. In particular, each node is specified as the set of outcomes that remain conceivable at that point in the tree. Then each choice is specified as a set of outcomes that can remain conceivable after the choice is made.

Their formulation is limited to finite horizons. But recently, it has been insightfully extended to the infinite horizon by the discrete extensive forms of Alós-Ferrer and Ritzberger (2013 henceforth `AR`). [The

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present paper always assumes discreteness. Still more general formulations that do not satisfy discreteness are developed in Alós-Ferrer and Ritzberger (2005, 2008).]

In this paper, I propose to restrict the AR definition with an additional assumption known as conciseness. Conciseness requires that choices are specified in an economical fashion. In particular, it rules out choices that are not feasible from any node. Further, it prevents a feasible choice from containing outcomes that are already inconceivable at the node(s) from which the choice is feasible.

The AR definition implies conciseness in almost all finite-horizon forms. Thus conciseness is almost redundant in the context of von Neumann and Morgenstern (1944). In contrast, I show by example that there are broad classes of infinite-horizon forms in which the AR definition fails to imply conciseness. Thus conciseness is far from redundant in the larger context of AR.

Nonetheless, I argue that conciseness imposes no loss of generality that would be of concern to game theorists. I do this by showing that every AR form can be equivalently re-specified in a way that satisfies conciseness. This is the upshot of Theorems 1 and 2 below.

This paper contributes to a larger agenda. In essence, Streufert (2015b) finds a triple equivalence between (a) the outcome-set forms of AR that satisfy this paper's conciseness, (b) the choice-set forms of Streufert (2015a), and (c) the choice-sequence forms of Osborne and Rubinstein (1994). This paper supports that larger agenda: By justifying the assumption of conciseness, it shrinks the class of AR forms under consideration, and thereby enables the construction of a one-to-one correspondence between (a) and (b).

In addition, this paper reformulates the AR form to make it more directly comparable to (b) and (c). This additional contribution is of secondary importance.

1.2. Overview

Section 2. This section reformulates the definition of an AR form in order [1] to conserve notation and [2] to make the concept more directly comparable to (b) and (c) above. I call my reformulation an "AR* outcome-set form".

To be somewhat more precise, Theorem 1 nontrivially shows that the class of AR* forms is equal to the class of AR forms except for two minor considerations. First, an AR* form assumes that every outcome is contained in its own singleton (terminal) node. AR call this property "completeness" and argue that it imposes no loss of generality of concern to game theorists.

Second, an AR* form requires that simultaneous moves be specified indirectly (by means of multiple information sets). In contrast, an AR form also allows simultaneous moves to be specified directly (by having multiple agents move at the same information set). Since the latter does not extend the scope of strategic situations that can be modelled, I disallow it so that I can conveniently identify agents with information sets.

Sections 3 and 4. In order to introduce Sections 3 and 4, I need to explain the difference between an AR* form (discussed above) and an AR* "preform". A preform specifies both nodes and choices as sets of outcomes. In contrast, a form is a preform together with an allocation of choices to players. Thus a preform can be regarded as a one-player form. Section 3 concerns preforms, while Section 4 concerns forms.

Section 3.1 defines conciseness. In particular, a preform is said to be "concise" if every outcome in every choice is contained in at least one node from which the choice is feasible. Conciseness does two things. First, it rules out choices that are not feasible from any node. Second, it prohibits a somewhere-feasible choice from containing outcomes that are outside of (i.e. already inconceivable from) all the nodes at which the choice is feasible. I call such an outcome an "immaterial" member of the choice.

Section 3.2 considers preforms with a finite horizon. In this context, the definition of a preform implies conciseness except for one trivial consideration.

Section 3.3 considers preforms with an infinite horizon. Here I find (a) broad classes of preforms with many nowhere-feasible choices, and (b) broad classes of preforms with many somewhere-feasible choices that each have many immaterial outcomes. Thus conciseness is a substantial mathematical restriction.

Section 3.4 nonetheless argues that conciseness imposes no loss of generality that would be of concern to game theorists. I do so by showing that any preform can be equivalently re-defined as a concise preform. The conversion process naturally removes all nowhere-feasible choices, and also removes all immaterial outcomes from every somewhere-feasible choice.

Section 4 incorporates the above results about preforms into results about forms. This is relatively straightforward. Proposition 4.1 shows that conciseness is essentially redundant in the context of finite-horizon forms. Then I show that there are broad classes of non-concise infinite-horizon forms (this follows easily from Section 3.3 because a preform can be seen as a one-player form). Finally, Theorem 2 shows that any form can be equivalently re-defined as a concise form.

Concatenating Theorems 1 and 2. Theorems 1 and 2 can be concatenated to provide the central result stated casually in the fifth paragraph of Section 1.1. By Theorem 1, any complete `AR` form without directly specified simultaneous moves is an `AR`* form. By Theorem 2, any `AR`* form can be re-defined so as to satisfy conciseness. In this sense, the class of `AR` forms can be reformulated and reduced to the class of concise `AR`* forms. This tractability is the paper's main contribution.

2. `AR`* forms

Sections 2.1, 2.2, and 2.3 together define `AR`* forms. Then Theorem 1 of Section 2.4 shows that an `AR`* form is essentially equivalent to an `AR` form.

2.1. Trees specify nodes

Let W be an arbitrary set. Call a member w of the set W an *outcome*. An `AR`* *outcome-set tree* is a pair $(W, \dot{N})$ such that

(1a) $\dot{N}$ is a collection of subsets of W containing W but not $\varnothing$,

(1b) $$(\forall \dot{n}^1 \neq \dot{n}^2)\ \dot{n}^1 \supset \dot{n}^2 \text{ or } \dot{n}^2 \supset \dot{n}^1 \text{ or } \dot{n}^1 \cap \dot{n}^2 = \varnothing \ ,$$

(1c) $$\dot{N} \supseteq \{\{w\}|w\} \ ,$$

(1d) $$\dot{N} \supseteq \{ \cap \dot{N}^* \mid \dot{N}^* \text{ is a nonempty chain in } \dot{N} \} \ ,$$

(1e) $$\text{and } \dot{N} \subseteq \dot{T} \cup \{\{w\}|w\} \ ,$$

where $\dot{T}$ is defined by[1]

(2) $$\dot{T} := \{\ \dot{n} \mid \{\dot{n}^\flat | \dot{n}^\flat \supset \dot{n}\} \text{ is finite } \}\ ,$$

and $\{\{w\}|w\}$ is the collection of singletons of the form $\{w\}$. A member $\dot{n}$ of the collection $\dot{N}$ is called a *node*.

(1a) requires that nodes are nonempty and that the set W itself is a node. Thus (1a) implies that $W \neq \varnothing$. Accordingly, the smallest AR* trees have a singleton W and $\dot{N} = \{W\}$.

For (1b), say that $\dot{n}^1$ *precedes* $\dot{n}^2$, and that $\dot{n}^2$ *succeeds* $\dot{n}^1$, whenever $\dot{n}^1 \supset \dot{n}^2$. (1b) states that if two distinct nodes have a nonempty intersection, then either the first precedes the second or the second precedes the first.

(1c) is called *completeness* (AR page 92). It requires that every singleton is a node. Define a *terminal* node to be a node without a successor. Since $\varnothing \notin \dot{N}$ by (1a), (1c) immediately implies that the singleton nodes coincide with the terminal nodes. Although the collection $\{\{w\}|w\}$ of terminal nodes $\{w\}$ is in one-to-one correspondence with the set W of outcomes w, the collection $\{\{w\}|w\}$ and the set W are distinct.

For (1d), recall that a *chain* in $\dot{N}$ is a collection $\dot{N}^*$ of nodes $\dot{n}$ such that for all distinct $\dot{n}^1$ and $\dot{n}^2$ in $\dot{N}^*$ either $\dot{n}^1 \supset \dot{n}^2$ or $\dot{n}^2 \supset \dot{n}^1$. (1d) requires that the intersection of every nonempty chain is a node. Since a node cannot be empty by (1a), this implies that the intersection of every nonempty chain of nodes is nonempty.

(1e) requires that every node either has a finite number of predecessors or is a terminal node. The converse of (1e) is implied by the definition of $\dot{T}$ and (1c). Hence every AR* tree satisfies

(3) $$\dot{N} = \dot{T} \cup \{\{w\}|w\}\ .$$

For notational ease, let $\dot{X}$ denote the collection of nonterminal nodes. In other words, define

(4) $$\dot{X} := \dot{N} \setminus \{\{w\}|w\}\ .$$

By replacing $\dot{N}$ in (4) with the right-hand side of (3) one obtains

(5) $$\dot{X} = \dot{T} \setminus \{\{w\}|w\}\ .$$

Thus $\dot{X} \subseteq \dot{T}$. Accordingly, I will denote an arbitrary nonterminal node by $\dot{t} \in \dot{X}$.

[1] I use the superscript $^\flat$ to suggest a predecessor, and the superscript $^\sharp$ to suggest a successor.

The following lemma shows that the class of AR* trees equals the class of complete AR trees. Its proof is nontrivial because the definition (1) of an AR* tree is some distance from AR Definitions 1 and 5.

The completeness appearing in the lemma is an insubstantial qualification because of the discussion on AR page 92. That discussion, in turn, is based upon the underlying result of Alós-Ferrer and Ritzberger (2005, Proposition 10).

Lemma 2.1. $(W,\dot{N})$ *is an* AR* *outcome-set tree (1) iff it is a complete (1c) discrete game tree (*AR *Definitions 1 and 5 at* $N{=}\dot{N}$*). (Proof B.4.)*

In light of (3), it is useful to partition the class of AR* trees into the three subclasses of trees satisfying

$$\dot{T} \supseteq \{\{w\}|w\} \ , \tag{6a}$$

$$\dot{T}\cap\{\{w\}|w\} = \varnothing \ , \text{ and} \tag{6b}$$

$$\text{neither of the above .} \tag{6c}$$

[(6a) and (6b) together would imply $\{\{w\}|w\}{=}\varnothing$, which contradicts (1a)'s implication that $W{\neq}\varnothing$.] A *finite-horizon* tree is a tree satisfying (6a). Here every terminal node has a finite number of predecessors. An *infinite-horizon* tree is a tree satisfying (6b) or (6c). Here there is at least one terminal node that has an infinite number of predecessors. Given (6b), every terminal node has an infinite number of predecessors. Examples of such games appear in Section 3.3. Given (6c), some terminal nodes have a finite number of predecessors and others have an infinite number of predecessors.

Finally, let $(W,\dot{N})$ be an AR* tree (1) with its $\dot{T}$ (2). Then define its *immediate predecessor* function $\dot{p}:\dot{T}\smallsetminus\{W\}\rightarrow\dot{T}$ by

$$\dot{p}(\dot{t}) := \min\{\dot{t}^\flat|\dot{t}^\flat\supset\dot{t}\} \ . \tag{7}$$

Lemma A.1 shows that the function $\dot{p}$ is well-defined. (The analogous claim on AR page 80 is immediate.) Further, Lemma A.3 shows (a) that the function $\dot{p}$ is onto the set $\dot{X}$ of nonterminal nodes, and (b) that every nonterminal node $\dot{t}{\in}\dot{X}$ is partitioned by the set $\dot{p}^{-1}(\dot{t})$ of its immediate successors. (The first paragraph of Remark B.6 observes that the analogous proposition in AR contains a minor mistake.)

2.2. Preforms specify choices

An $\mathtt{AR}^*$ *outcome-set preform* is a triple $(W,\dot{N},\dot{C})$ such that

(8a) $(W,\dot{N})$ is an $\mathtt{AR}^*$ outcome-set tree (1) ,

(8b) $\dot{C}$ is a collection of nonempty subsets of W ,

(8c) $(\forall \dot{t}{\in}\dot{X})\ \dot{p}^{-1}(\dot{t}) = \{\ \dot{t}{\cap}\dot{c} \mid \dot{c}{\in}\dot{F}(\dot{t})\ \}$,

(8d) $(\forall \dot{t}{\in}\dot{X})$ the members of $\dot{F}(\dot{t})$ are disjoint , and

(8e) $(\forall \dot{t}^1,\dot{t}^2)\ \dot{F}(\dot{t}^1){=}\dot{F}(\dot{t}^2)$ or $\dot{F}(\dot{t}^1){\cap}\dot{F}(\dot{t}^2){=}\varnothing$,

where $\dot{T}$, $\dot{X}$, and $\dot{p}$ are derived from $(W,\dot{N})$ by (2), (4), and (7), and where $\dot{F}$ is defined by

(9) $\dot{F} := \{\ (\dot{t},\dot{c}) \mid \dot{c}{\not\supseteq}\dot{t}$ and $(\exists \dot{t}^\sharp{\in}\dot{p}^{-1}(\dot{t}))\ \dot{c}{\supseteq}\dot{t}^\sharp\ \}$.

A member $\dot{c}$ of the collection $\dot{C}$ is called a *choice*. $\dot{F}$ is called the *feasibility correspondence*. Accordingly $\dot{F}(\dot{t})$ is called the set of choices that are *feasible* from $\dot{t}$.

(8a) states that a preform incorporates a tree. (8b) says that choices (like nodes) are nonempty sets of outcomes. (8c) states that the immediate successors of a node are the intersections of the node with the choices that are feasible from that node. This implies that the collection $\dot{X}$ of nonterminal nodes equals the domain $\dot{F}^{-1}(\dot{C})$ of the feasibility correspondence $\dot{F}$ (Lemma A.4). (8d) states that the collection of feasible choices at any node consists of choices that are disjoint from one another.

(8e) enables an implicit specification of agents (i.e. information sets). This implicit specification is analogous to that of $\mathtt{AR}$ (page 82), and the idea itself can be traced back to Alós-Ferrer and Ritzberger (2005, page 791). In particular, let an *agent* be a member of

(10) $\dot{H} := \{\ \dot{F}^{-1}(\dot{c}) \mid \dot{c}\ \}$.

Thus each agent $\dot{F}^{-1}(\dot{c})$ is the collection of nodes from which a choice $\dot{c}$ is feasible. By Lemma A.5, (8e) is equivalent to

$$(\forall \dot{c}^1,\dot{c}^2)\ \dot{F}^{-1}(\dot{c}^1){=}\dot{F}^{-1}(\dot{c}^2) \text{ or } \dot{F}^{-1}(\dot{c}^1){\cap}\dot{F}^{-1}(\dot{c}^2){=}\varnothing\ .$$

Thus (8e) assures that agents are disjoint from one another. Further,

$$\cup\dot{H} = \cup\{\ \dot{F}^{-1}(\dot{c}) \mid \dot{c}\ \} = \dot{F}^{-1}(\dot{C}) = \dot{X}\ ,$$

where the last equality holds by Lemma A.4. Thus, by the last two sentences, every nonterminal node is assigned to exactly one agent.

2.3. Forms specify players

Let I be an arbitrary set, and call $i \in I$ a *player*. An $\mathtt{AR}^*$ *outcome-set form* is a triple $(W, \dot{N}, (\dot{C}_i)_i)$ such that

(11a) $(W, \dot{N}, \cup_i \dot{C}_i)$ is an $\mathtt{AR}^*$ outcome-set preform (8) ,

(11b) $(\forall i{\neq}j)\ \dot{C}_i{\cap}\dot{C}_j{\cap}\dot{F}(\dot{X}) = \varnothing$, and

(11c) $(\forall i)(\forall \dot{t}{\in}\dot{X})\ \dot{F}(\dot{t}){\subseteq}\dot{C}_i$ or $\dot{F}(\dot{t}){\cap}\dot{C}_i{=}\varnothing$.

where $\dot{T}$, $\dot{X}$, and $\dot{F}$ are derived from $(W, \dot{N}, \cup_i \dot{C}_i)$ by (2), (4), and (9).

A form uses the individual choice collections $\dot{C}_i$ to assign a preform's choices to individual players i. Accordingly, a preform can be understood as a single-player form. To be precise, $(W, \dot{N}, \dot{C})$ is a preform iff $(W, \dot{N}, (\dot{C}))$ is a form, provided that $(\dot{C}_i)_i = (\dot{C})$ is taken to mean that $I = \{1\}$ and $\dot{C}_1 = \dot{C}$.

For (11b), say that a choice $\dot{c}$ is *nowhere-feasible* if $\dot{F}^{-1}(\dot{c}) = \varnothing$ and that it is *somewhere-feasible* if $\dot{F}^{-1}(\dot{c}) \neq \varnothing$. By Lemma A.4, $\dot{F}(\dot{X})$ is the set of somewhere-feasible choices. Accordingly, (11b) states that a somewhere-feasible choice can be in no more than one $\dot{C}_i$. Since $\dot{F}(\dot{X}) \subseteq \cup_i \dot{C}_i$ by the definition of $\dot{F}$, this implies that every somewhere-feasible choice is in exactly one C_i.

For (11c), define

$$(\dot{H}_i)_i := (\{\, \dot{F}^{-1}(\dot{c}) \,|\, \dot{c}{\in}\dot{C}_i \,\})_i \ .$$

Each $\dot{H}_i$ is the set of agents (10) assigned to player i. Lemma A.6 uses (11c) to show that a nonempty agent can be assigned to no more than one $\dot{H}_i$. Further,

$$\cup_i \dot{H}_i = \cup_i \{\, \dot{F}^{-1}(\dot{c}) \,|\, \dot{c}{\in}\dot{C}_i \,\} = \{\, \dot{F}^{-1}(\dot{c}) \,|\, \dot{c}{\in}\dot{C} \,\} = \dot{H} \ ,$$

where the last equality holds by (10). Thus, by the last two sentences, every nonempty agent is in exactly one $\dot{H}_i$.[2]

2.4. Equality with AR forms

Theorem 1 will show that the class of $\mathtt{AR}^*$ forms is equal to the class of $\mathtt{AR}$ forms except for two minor considerations. First, $\mathtt{AR}^*$ forms are assumed to be complete. This restriction is insubstantial, as discussed in the paragraph before Lemma 2.1.

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[2]All agents are nonempty if every choice is somewhere-feasible. This is one consequence of conciseness (below).

Second, an AR* form requires that simultaneous moves be specified indirectly (that is, by means of multiple information sets). In contrast, an AR form also allows simultaneous moves to be specified directly (that is, by having multiple agents move at the same information set). Ritzberger (2002, pages 103-104) carefully discusses these two alternatives in a framework similar to this one. Further, AR Examples 8 and 9 convincingly illustrate some of the advantages of direct specification.

However, since direct specification does not expand the scope of strategic situations that can be modelled, I have chosen to disallow it in order to conveniently identify agents with information sets. Formally, an AR form is said to be *without directly specified simultaneous moves* if

$$\ddot{J} \text{ is singleton-valued}\ , \tag{12}$$

where the set-valued function $\ddot{J}$ is defined by AR Definition 6 (DEF.ii) (symbols with two dots are taken directly from AR, and most such symbols are confined to Appendix B).

Theorem 1. $(W, \dot{N}, (\dot{C}_i)_i)$ *is an* AR* *outcome-set form (11) iff it is a complete (1c) discrete extensive form (*AR *Definition 6 at* $N{=}\dot{N}$ *and* $(C_i)_i{=}(\dot{C}_i)_i$*) without directly specified simultaneous moves (12). (Proof B.10.)*

The proof of Theorem 1 is nontrivial, in part because an AR* form is defined in terms of the feasibility correspondence $\dot{F}$ (9) while an AR form is defined in terms of the extended predecessor correspondence "P" (AR pages 81–82). Lemma B.7 plays a key role in this conversion.

3. Concise AR* preforms

3.1. Definition

Let $(W, \dot{N}, \dot{C})$ be an AR* preform (8) and let $\dot{F}$ be its feasibility correspondence (9). Then $(W, \dot{N}, \dot{C})$ is said to be *concise* iff

$$(\forall \dot{c})\ \dot{c} \subseteq \cup \dot{F}^{-1}(\dot{c})\ . \tag{13}$$

Thus conciseness means that every choice is covered by the collection of nodes from which it is feasible.

Conciseness can be understood as a pair of restrictions. To see the first restriction, recall that every choice $\dot{c}$ is nonempty by (8b). Thus conciseness implies that every choice has a nonempty collection

$\dot{F}^{-1}(\dot{c})$ of nodes from which it is feasible. In other words, it prohibits nowhere-feasible choices. Examples of nowhere-feasible choices appear in Section 3.2 (if $W \in \dot{C}$) and in Section 3.3 (in Example 2's family of examples).

To see the second restriction, consider a somewhere-feasible choice. In other words, consider a choice $\dot{c}$ for which $\dot{F}^{-1}(\dot{c})$ is nonempty. Then consider any $\dot{t} \in \dot{F}^{-1}(\dot{c})$. By definition, $\dot{c} \in \dot{F}(\dot{t})$, and thus $\dot{c} \cap \dot{t}$ is an immediate successor of $\dot{t}$ by (8c). Notice that the outcomes in $\dot{c} \setminus \dot{t}$ are immaterial to this construction. Accordingly, the outcomes in $\dot{c} \setminus \cup \dot{F}^{-1}(\dot{c})$ are immaterial to such a construction at any node in $\dot{F}^{-1}(\dot{c})$. To formalize this, let $\dot{c} \cap \cup \dot{F}^{-1}(\dot{c})$ be the set of *material* outcomes in $\dot{c}$, so that $\dot{c} \setminus \cup \dot{F}^{-1}(\dot{c})$ becomes the set of *immaterial* outcomes in $\dot{c}$. Conciseness implies that somewhere-feasible choices cannot contain immaterial outcomes. Examples of immaterial outcomes appear in Section 3.3 (in Example 3's family of examples).

It will often be useful to have a formal statement of this two-part characterization of conciseness. First, Lemma A.4 implies that $\dot{F}(\dot{X})$ is the collection of somewhere-feasible choices. Thus the equality $\dot{F}(\dot{X}) = \dot{C}$ is equivalent to the absence of nowhere-feasible choices. Second, define the *material-part* function $\mathtt{M}$, from the domain $\dot{F}(\dot{X})$, by

$$\mathtt{M}(\dot{c}) = \dot{c} \cap \cup F^{-1}(\dot{c}) \ . \tag{14}$$

Accordingly, the statement $(\forall \dot{c} \in \dot{F}(\dot{X}))\ \mathtt{M}(\dot{c}) = \dot{c}$ is equivalent to the absence of immaterial outcomes in somewhere-feasible choices. The following lemma puts these two parts together. Its proof is easy.

Lemma 3.1. *Suppose* $(W, \dot{N}, \dot{C})$ *is an* AR* *outcome-set preform (8) with its* $\dot{X}$ *(4),* $\dot{F}$ *(9), and* $\mathtt{M}$ *(14). Then* $(W, \dot{N}, \dot{C})$ *is concise (13) iff* $\dot{C} = \dot{F}(\dot{X})$ *and* $(\forall \dot{c})\ \mathtt{M}(\dot{c}) = \dot{c}$. *(Proof C.1.)*

3.2. ALMOST ALL FINITE-HORIZON PREFORMS ARE CONCISE

Lemma 3.2. *Suppose that* $(W, \dot{N}, \dot{C})$ *is a finite-horizon (6a)* AR* *outcome-set preform (8). Then* $(W, \dot{N}, \dot{C})$ *is concise (13) iff* $W \notin \dot{C}$. *(Proof C.3.)*

Conciseness implies $W \notin \dot{C}$ regardless of the finite-horizon assumption. In other words, any preform $(W, \dot{N}, \dot{C})$ with $W \in \dot{C}$ is not concise.

To see this, simply note that in any such preform, W is a nowhere-feasible choice because

(15) $$\dot{F}^{-1}(W) = \{\ \dot{t} \mid W \not\supseteq \dot{t} \text{ and } (\exists \dot{t}^\sharp {\in} \dot{p}^{-1}(\dot{t}))\ W \supseteq \dot{t}^\sharp\ \} = \varnothing\ .$$

The first equality follows from the definition (9) of $\dot{F}$, and the second follows from the impossibility of $W \not\supseteq \dot{t}$.

The converse is more difficult and uses the finite-horizon assumption. Essentially, deriving conciseness requires showing that every w in every $\dot{c}$ belongs to some node from which $\dot{c}$ is feasible. Consider the chain consisting of $\{w\}$ and all its predecessors. It can be shown that the desired node is the immediate predecessor of the largest member of the chain that is yet a subset of $\dot{c}$. Accordingly, if $\{w\}$ is the only member of the chain that is a subset of $\dot{c}$, the desired node is the immediate predecessor of $\{w\}$. The existence of this immediate predecessor is implied by the finite-horizon assumption. However, this argument cannot be extended to the infinite horizon, since there, a terminal node $\{w\}$ might not have an immediate predecessor.

3.3. Examples of non-concise infinite-horizon preforms

This subsection exhibits numerous examples of non-concise infinite-horizon preforms. Some have many nowhere-feasible choices. Others have many somewhere-feasible choices that each have many immaterial outcomes. These examples are important because they demonstrate that violations of conciseness are liberally allowed by the conditions (8) that define an AR* preform.

Throughout this subsection and all its examples, let W^0 be the Cantor set, as defined for instance in Rudin (1976, page 41). The Cantor set is the set of real numbers in $[0,1]$ that can be expressed in base 3 without the use of the digit 1.

This paragraph merely defines a convenient way to specify subsets of W^0. First, let S be the set consisting of (a) the empty sequence $\{\}$ and (b) all finite nonempty sequences consisting of 0's and 2's. For notational ease, write a nonempty sequence in S without punctuation. For example, write $(2,0)$ as 20. Then define $\mathtt{D}{:}\,S{\rightarrow}\mathcal{P}(W^0)$ by (a) letting $\mathtt{D}(\{\}) = W^0$ and (b) for each $s{\in}S{\smallsetminus}\{\{\}\}$ letting $\mathtt{D}(s)$ be the set of numbers in W^0 that have a base-3 expansion beginning with (decimal)

s. For example,

$$\mathtt{D}(20) = [.20\bar{0}, .20\bar{2}]\cap W^0 = [.20, .21]\cap W^0$$

(where all numbers are expressed in base 3). Notice that $\mathtt{D}$ is injective. Hence, $\mathtt{D}$ is a bijection from its domain S onto its range $\{\mathtt{D}(s)|s\}$.

Then let

$$\dot{N}^0 := \{\mathtt{D}(s)|s\} \cup \{\{w\}|w\in W^0\}\ ,$$

and call $(W^0, \dot{N}^0)$ the *Cantor-set tree*. Lemma D.1 shows [1] that the Cantor-set tree is an $\mathtt{AR}^*$ tree, [2] that its $\dot{T}^0$ and $\dot{X}^0$ are both equal to $\{\mathtt{D}(s)|s\}$, [3] that its $\dot{p}^0$ has domain $\dot{T}^0\smallsetminus\{W^0\} = \{\mathtt{D}(s)|s\neq\{\}\}$, and [4] that this $\dot{p}^0$ satisfies

$$(\forall s\neq\{\})\ \dot{p}^0(\mathtt{D}(s)) = \mathtt{D}(s_-)$$

where s_- is the sequence derived from $s\neq\{\}$ by omitting its last component. For example, $\mathtt{D}(20)$ is an element of $\dot{T}^0\smallsetminus\{W^0\}$, and

$$\dot{p}^0(\mathtt{D}(20)) = \ \mathtt{D}(20_-) = \mathtt{D}(2)\ .$$

The rest of this subsection endows the Cantor-set tree $(W^0, \dot{N}^0)$ with various choice collections $\dot{C}$ in order to create various $\mathtt{AR}^*$ preforms of the form $(W^0, \dot{N}^0, \dot{C})$. The discussion centers on three relatively simple examples. Example 1 is concise. Example 2 is non-concise because of a nowhere-feasible choice. Example 3 is non-concise because of an immaterial outcome in a somewhere-feasible choice.

Example 1. Suppose the choice collection is $\{\mathtt{D}(s)|s\neq\{\}\}$. Lemma D.4 shows that the triple

$$(W^0, \dot{N}^0,\ \{\mathtt{D}(s)|s\neq\{\}\}\ )$$

is a concise $\mathtt{AR}^*$ preform. The lemma also shows that this preform's $\dot{F}^1$ (the superscript is for Example 1) is defined by

$$(\forall s)\ \dot{F}^1(\mathtt{D}(s)) = \{\ \mathtt{D}(s\oplus 0),\ \mathtt{D}(s\oplus 2)\ \}$$

where $\oplus$ is the concatenation operator. For instance,

$$\dot{F}^1(\mathtt{D}(2)) = \{\ \mathtt{D}(20),\ \mathtt{D}(22)\ \}\ .$$

Thus the feasible choices at the node $\mathtt{D}(2)$ are identical to the node's immediate successors. Every other nonterminal node $\mathtt{D}(s)$ is similar.

Example 2. Suppose the choice collection is $\{\mathtt{D}(s)|s\neq\{\}\}\cup\{\{1\}\}$. In other words, take the choice collection $\{\mathtt{D}(s)|s\neq\{\}\}$ of Example 1, and introduce the choice $\{1\}$. (Recall that $1\in W^0$ since $1=.\bar{2}$.) Lemma D.7 shows that the triple

$$(\,W^0, \dot{N}^0,\ \{\mathtt{D}(s)|s{\neq}\{\}\}{\cup}\{\{1\}\}\,)$$

is an $\mathtt{AR}^*$ preform. The proof is nontrivial. The lemma also shows that the preform is not concise because $\{1\}$ is a nowhere-feasible choice. To see this, note that the preform's $\dot{F}^2$ (the superscript is for Example 2) satisfies

$$(\forall \dot{c})\ \ (\dot{F}^2)^{-1}(\dot{c}) = \{\ \dot{t}\ |\ \dot{c}{\not\supseteq}\dot{t} \text{ and } (\exists \dot{t}^\sharp{\in}(\dot{p}^0)^{-1}(\dot{t}))\ \dot{c}{\supseteq}\dot{t}^\sharp\ \}$$

by (9). By the last inclusion, every somewhere-feasible choice $\dot{c}$ must contain at least one node $\dot{t}^\sharp$ with a finite number of predecessors. In this example, the choice $\{1\}$ does not subsume a node with a finite number of predecessors because every such node is an infinite set of the form $\mathtt{D}(s)$. Thus $\{1\}$ is nowhere-feasible. Thus the preform is not concise.

This second example is representative of a large class of non-concise preforms. In particular, let $\dot{C}^+$ be any nonempty collection of nonempty countable subsets of W^0. The previous paragraph considered the special case $\dot{C}^+ = \{\{1\}\}$. Alternatively, $\dot{C}^+$ could be the uncountable collection $\{\{w\}|w{\in}W^0\}$. Or, $\dot{C}^+$ could be the uncountable collection consisting of all two-element subsets of W^0. For any such collection $\dot{C}^+$, Lemma D.6 shows that

$$(W^0, \dot{N}^0, \{\mathtt{D}(s)|s{\neq}\{\}\}{\cup}\dot{C}^+)$$

is a non-concise $\mathtt{AR}^*$ preform in which all the members of $\dot{C}^+$ are nowhere-feasible. The lemma's proof establishes nowhere-feasibility in a manner similar to the previous paragraph: by assumption, each member of $\dot{C}^+$ is countable, and thus, it cannot subsume an uncountably infinite set of the form $\mathtt{D}(s)$.

Example 3. Suppose $\dot{C}$ is $\{\mathtt{D}(s)|s{\notin}\{\{\},22\}\}\cup\{\mathtt{D}(22){\cup}\{.02\}\}$. In other words, take the choice collection $\{\mathtt{D}(s)|s{\neq}\{\}\}$ of Example 1, and replace the choice $\mathtt{D}(22)$ with the choice $\mathtt{D}(22){\cup}\{.02\}$. Lemma D.3(b) shows that the triple

$$(W^0, \dot{N}^0,\ \{\mathtt{D}(s)|s{\notin}\{\{\},22\}\}\cup\{\mathtt{D}(22){\cup}\{.02\}\}\,)$$

is an $\mathtt{AR}^*$ preform. The proof is nontrivial.

Further, Lemma D.3(a) shows that this preform's $\dot{F}^3$ satisfies

(16a) $$\dot{F}^3(\mathtt{D}(2)) = \{\ \mathtt{D}(20),\ \mathtt{D}(22)\cup\{.02\}\ \}\ ,$$

(16b) $$\{\mathtt{D}(2)\} = (\dot{F}^3)^{-1}(\mathtt{D}(20))\ ,\ \text{and}$$

(16c) $$\{\mathtt{D}(2)\} = (\dot{F}^3)^{-1}(\mathtt{D}(22)\cup\{.02\})\ .$$

In accord with (16a) and (8c), the immediate successors of $\mathtt{D}(2)$ are

$$\mathtt{D}(2)\cap\mathtt{D}(20) = \mathtt{D}(20)\ \text{and}$$
$$\mathtt{D}(2)\cap(\mathtt{D}(22)\cup\{.02\}) = \mathtt{D}(22)\ .$$

In this fashion, the choice $\mathtt{D}(20)$ restricts the set of conceivable outcomes from $\mathtt{D}(2)$ to $\mathtt{D}(20)$, and similarly, the choice $\mathtt{D}(22)\cup\{.02\}$ restricts the set of conceivable outcomes from $\mathtt{D}(2)$ to $\mathtt{D}(22)$. However, .02 is an immaterial outcome in the choice $\mathtt{D}(22)\cup\{.02\}$ because (a) it does not belong to (i.e. is inconceivable from) $\mathtt{D}(2)$ and (b) $\mathtt{D}(2)$ is, by (16c), the only node from which $\mathtt{D}(22)\cup\{.02\}$ is feasible. Such immaterial outcomes are prohibited by conciseness. Hence the preform is not concise.

This example is representative of a large class of non-concise preforms. In particular, let $\mathtt{E}$ be any set-valued function from $\{s|s{\neq}\{\}\}$ such that [1] $\mathtt{E}(0) = \mathtt{E}(2) = \varnothing$, and [2] for every s with at least two components, $\mathtt{E}(s)$ is a countable subset of $\mathtt{D}(\ominus s)$, where $\ominus s$ is the sequence that is obtained from s by changing its first component. For example, one could set $\mathtt{E}(22) = \{.02\}$ and $\mathtt{E}(s) = \varnothing$ everywhere else. Note that $\mathtt{E}(22) \subseteq \mathtt{D}(02) = \mathtt{D}(\ominus 22)$, as the definition of $\mathtt{E}$ requires. (This specification of $\mathtt{E}$ leads to Example 3.) A second alternative would be to set

(17) $$\mathtt{E}(22) = \{.02, .022, .0222, .02222, ...\}$$

and $\mathtt{E}(s) = \varnothing$ everywhere else. Again, $\mathtt{E}(22) \subseteq \mathtt{D}(02) = \mathtt{D}(\ominus 22)$. A third alternative would be to set

(18) $$\mathtt{E}(s) = \{\ .\ominus s,\ .\ominus s{\oplus}2,\ .\ominus s{\oplus}22,\ \ominus s{\oplus}222, ...\}$$

at every s with at least two components, and to set $\mathtt{E}(0) = \mathtt{E}(2) = \varnothing$ (for instance, the $\mathtt{E}(22)$ specified by (18) equals the $\mathtt{E}(22)$ specified by (17)). Again, for every s with at least two components, $\mathtt{E}(s) \subseteq \mathtt{D}(\ominus s)$, as the definition of $\mathtt{E}$ requires.

Lemma D.2 shows that, for any such function $\mathtt{E}$, (a) the triple

$$(\ W^0, \dot{N}^0, \{\mathtt{D}(s)\cup\mathtt{E}(s)|s{\neq}\{\}\}\ )$$

is an $\mathtt{AR}^*$ preform and (b) at every s, $\mathtt{E}(s)$ is the set of immaterial outcomes in the choice $\mathtt{D}(s)\cup\mathtt{E}(s)$. Consequently the preform fails to be concise if at least one $\mathtt{E}(s)$ is nonempty. Proving (a) is a nontrivial task. In contrast, proving (b) is relatively easy. Consider any nonempty s. Essentially, $\mathtt{E}(s) \subseteq \mathtt{D}(\ominus s)$ implies that every element of $\mathtt{E}(s)$ is outside the node $\dot{p}^0(\mathtt{D}(s))$ from which $\mathtt{D}(s)\cup\mathtt{E}(s)$ is feasible. Because of this, every element of $\mathtt{E}(s)$ is an immaterial outcome in the choice $\mathtt{D}(s)\cup\mathtt{E}(s)$. Hence the non-emptiness of any $\mathtt{E}(s)$ implies non-conciseness.

3.4. Conciseness is costless to game theorists

The previous subsection showed by example that conciseness is a considerable mathematical restriction. In contrast, the following lemma shows that conciseness imposes no loss of generality that would concern game theorists. It does so by showing that every preform can be naturally and equivalently re-defined as a concise preform. This is explained in detail after the lemma.

Lemma 3.3. *Let $(W, \dot{N}, \dot{C})$ be an $\mathtt{AR}^*$ outcome-set preform (8) with its $\dot{T}$ (2), $\dot{X}$ (4), and $\dot{F}$ (9). To re-define this preform, let $\dot{C}^{\mathtt{M}} := \mathtt{M}(\dot{F}(\dot{X}))$, where $\mathtt{M}$ is defined by (14). Then the following hold.*

(a) $(W, \dot{N}, \dot{C}^{\mathtt{M}})$ is a concise (13) $\mathtt{AR}^*$ outcome-set preform.

(b) $\mathtt{M}$ is a bijection from $\dot{F}(\dot{X})$ onto $\dot{C}^{\mathtt{M}}$, and

(c) $\{\, (\dot{t}, \mathtt{M}(\dot{c})) \,|\, (\dot{t}, \dot{c}){\in}\dot{F} \,\}$ equals the $\dot{F}^{\mathtt{M}}$ (9) from $(W, \dot{N}, \dot{C}^{\mathtt{M}})$.

(Proof C.7.)

First consider the definition of the new choice collection $\dot{C}^{\mathtt{M}}$ in the lemma's second sentence. This definition could be restated as

$$\dot{C}^{\mathtt{M}} = \{\ \mathtt{M}(\dot{c}) \mid \dot{c}{\in}\dot{F}(\dot{X})\ \}\ .$$

This definition takes $\dot{C}$ to $\dot{C}^M$ into two steps. First, the original choices in $\dot{N}\smallsetminus\dot{F}(\dot{X})$ are excluded. In other words, all nowhere-feasible choices are removed. Second, every original choice $\dot{c}{\in}\dot{F}(\dot{X})$ is converted to $\mathtt{M}(\dot{c})$. In other words, every somewhere-feasible choice loses all its immaterial outcomes.

The examples of Section 3.3 illustrate this two-step definition. In Example 2's family, the first step removes all the nowhere-feasible choices in $\dot{C}^+$, and the second step is vacuous. In this fashion, each preform in Example 2's family is converted to Example 1. (Full details are in Lemma D.6 and Example 1's definition.) Meanwhile, in Example 3's family, the first step is vacuous, and the second step removes all the immaterial outcomes from every somewhere-feasible choice. In particular, each choice $\texttt{D}(s){\cup}\texttt{E}(s)$ becomes $\texttt{M}(\texttt{D}(s){\cup}\texttt{E}(s)) = \texttt{D}(s)$. In this fashion, each preform in Example 3's family is converted to Example 1. (Full details are in Lemma D.2 and Example 1's definition.)

Part (a) of Lemma 3.3 shows that this two-step procedure will convert any preform into a concise preform. In other words, it shows [1] that a preform remains a preform after its nowhere-feasible choices and immaterial outcomes have been removed, and [2] that the resulting preform is concise. For instance, all of the above examples are converted to Example 1 (by the previous paragraph), and Example 1 is a concise preform (by Lemma D.4).

Parts (b) and (c) of Lemma 3.3 describe the sense in which the original preform and the new preform are "equivalent". In particular, part (b) shows that there is a one-to-one correspondence between the collection $\dot{F}(\dot{X}) \subseteq \dot{C}$ of original somewhere-feasible choices and the collection $\dot{C}^{\texttt{M}}$ of new choices. Then part (c) relates the original feasibility correspondence $\dot{F}$ to the new feasibility correspondence $\dot{F}^{\texttt{M}}$. Although all the parts of the lemma are natural, they are not easily proved.

4. Concise AR* forms

Let a *concise* $\texttt{AR}^*$ *outcome-set form* be an $\texttt{AR}^*$ outcome-set form (11) whose preform (11a) is concise (13). In other words, a concise $\texttt{AR}^*$ outcome-set form is a triple $(W, \dot{N}, (\dot{C}_i)_i)$ such that

(19a) $(W, \dot{N}, \cup_i \dot{C}_i)$ is a concise (13) $\texttt{AR}^*$ outcome-set preform (8) ,

(19b) $$(\forall i{\neq}j)\ \dot{C}_i{\cap}\dot{C}_j{\cap}\dot{F}(\dot{X}) = \varnothing \text{ , and}$$

(19c) $$(\forall i)(\forall \dot{t}{\in}\dot{X})\ \dot{F}(\dot{t}){\subseteq}\dot{C}_i \text{ or } \dot{F}(\dot{t}){\cap}\dot{C}_i{=}\varnothing \text{ ,}$$

where $\dot{T}$, $\dot{X}$, and $\dot{F}$ are derived from (2), (4), and (9).

The following proposition shows that conciseness is essentially vacuous for finite-horizon forms. This proposition follows easily from Lemma 3.2, which concerned preforms rather than forms.

Proposition 4.1. *Suppose that* $(W, \dot{N}, (\dot{C}_i)_i)$ *is an* $\texttt{AR}^*$ *outcome-set form (8) with a finite horizon (6a). Then* $(W, \dot{N}, (\dot{C}_i)_i)$ *is a concise* $\texttt{AR}^*$ *outcome-set form (19) iff* $W \notin \cup_i \dot{C}_i$. *(Proof C.8.)*

Recall the many non-concise infinite-horizon preforms of Section 3.3. Since a preform is a one-player form, these preforms also show that there are many non-concise infinite-horizon forms. These examples are important because they demonstrate that violations of conciseness are liberally allowed by the conditions (11) that define an AR* form. Hence, by Theorem 1, such violations of conciseness are liberally allowed by the definition of an AR form.

Nevertheless, the following theorem shows that conciseness imposes no loss of generality that would concern game theorists. It does so by showing that every form can be naturally and equivalently re-defined as a concise form. The theorem itself is an extension of Lemma 3.3, which concerned preforms rather than forms. Most of the work was done there, and the discussion following that lemma can be readily adapted to interpret the theorem here.

Theorem 2. *Let* $(W, \dot{N}, (\dot{C}_i)_i)$ *be an* AR* *outcome-set form (11), with its* $\dot{T}$ *(2),* $\dot{X}$ *(4), and* $\dot{F}$ *(9). To re-define this form, let*

$$(\dot{C}_i^{\mathtt{M}})_i := (\mathtt{M}(\dot{C}_i \cap \dot{F}(\dot{X})))_i \ ,$$

where M *is defined by (14). Then the following hold.*

(a) $(W, \dot{N}, (\dot{C}_i^{\mathtt{M}})_i)$ *is a concise* AR* *outcome-set form (19).*

(b) M *is a bijection from* $\dot{F}(\dot{X})$ *onto* $\cup_i \dot{C}_i^{\mathtt{M}}$*.*

(c) $\{ (\dot{t}, \mathtt{M}(\dot{c})) \,|\, (\dot{t}, \dot{c}) \in \dot{F} \}$ *equals the* $\dot{F}^{\mathtt{M}}$ *(9) from* $(W, \dot{N}, (\dot{C}_i^{\mathtt{M}})_i)$*.*

(Proof C.9.)

Finally, I include a useful but very minor result. The following characterization of a concise form is slightly simpler than its definition (19). Specifically, (19b) can be simplified to (20b) in the presence of conciseness.

Proposition 4.2. $(W, N, (\dot{C}_i)_i)$ *is a concise* AR* *outcome-set form (19) iff*

(20a) $(W, \dot{N}, \cup_i \dot{C}_i)$ *is a concise (13)* AR* *outcome-set preform (8)* ,

(20b) $(\forall i \neq j)\ \dot{C}_i \cap \dot{C}_j = \varnothing$, *and*

(20c) $(\forall i)(\forall \dot{t} \in \dot{X})\ \dot{F}(\dot{t}) \subseteq \dot{C}_i$ *or* $\dot{F}(\dot{t}) \cap \dot{C}_i = \varnothing$,

where $\dot{T}$*,* $\dot{X}$*, and* $\dot{F}$ *are derived from (2), (4), and (9). (Proof C.10.)*

Appendix A. Basic Lemmata

This appendix provides some basic lemmata for the $\mathtt{AR}^*$ framework. Most of these lemmata are discussed briefly in Section 2, and the remainder are used in proving the ones that are discussed there. None of the lemmata rely on $\mathtt{AR}$, and none of them refer to conciseness.

A.1. For $\mathtt{AR}^*$ trees

Lemma A.1. *If $(W,\dot{N})$ satisfies (1a) and (1b) in the definition of an $\mathtt{AR}^*$ outcome-set tree, then $\dot{p}$ (7) is well-defined.*

Proof. Define $\dot{T}$ (2), and take any $\dot{t}\in\dot{T}\smallsetminus\{W\}$.

This paragraph makes three observations about $\{\dot{t}^\flat|\dot{t}^\flat\supset\dot{t}\}$. First, $\{\dot{t}^\flat|\dot{t}^\flat\supset\dot{t}\}$ is finite by $\dot{t}\in\dot{T}$ and the definition of $\dot{T}$. Second, $\{\dot{t}^\flat|\dot{t}^\flat\supset\dot{t}\}$ is nonempty because $\dot{t}\neq W$ by assumption and thus W is an element of the set by (1a). Third, $\{\dot{t}^\flat|\dot{t}^\flat\supset\dot{t}\}$ is a chain. To see this, (a) take any two distinct $\dot{t}^1$ and $\dot{t}^2$ in $\{\dot{t}^\flat|\dot{t}^\flat\supset\dot{t}\}$, (b) note that $\dot{t}^1\cap\dot{t}^2\neq\varnothing$ since both sets subsume $\dot{t}$, and (c) conclude that one of $\dot{t}^1$ and $\dot{t}^2$ precedes the other by (1b).

By the previous paragraph, $\{\dot{t}^\flat|\dot{t}^\flat\supset\dot{t}\}$ is a finite nonempty chain. Hence its minimum exists and is an element of $\dot{T}$. □

Lemma A.2. *Suppose $(W,\dot{N})$ is an $\mathtt{AR}^*$ outcome-set tree (1) with its $\dot{T}$ (2), $\dot{X}$ (4), and $\dot{p}$ (7). Then if $\dot{t}\in\dot{X}$ is such that $\dot{t}\ni w$, there exists $\dot{t}^\sharp\in\dot{p}^{-1}(\dot{t})$ such that $\dot{t}^\sharp\ni w$.*

Proof. Suppose $\dot{t}\in\dot{X}$ is such that $\dot{t}\ni w$. Since $\dot{t}\ni w$, we have $\dot{t}\supseteq\{w\}$. Thus, by $\dot{t}\in\dot{X}$ and the definition of $\dot{X}$, we have $\dot{t}\supset\{w\}$.

On the one hand, suppose there is not an $\dot{n}^+$ such that $\dot{t}\supset\dot{n}^+\supset\{w\}$. The last two sentences imply $\{w\}\in\dot{T}\smallsetminus\{W\}$, which is the domain of $\dot{p}$. The last three sentences imply $\dot{t}=\dot{p}(\{w\})$. Thus we may let $\dot{t}^\sharp$ be $\{w\}$.

On the other hand, suppose there is a $\dot{n}^+$ such that $\dot{t}\supset\dot{n}^+\supset\{w\}$. Let

$$\dot{N}^+ := \{\,\dot{n}\,|\,\dot{t}\supset\dot{n}\supseteq\dot{n}^+\,\}\ .$$

The remainder of this paragraph makes three observations about $\dot{N}^+$. First, $\dot{N}^+$ is nonempty because it contains $\dot{n}^+$. Second, $\dot{N}^+$ is a chain. To see this, [a] take two distinct elements $\dot{n}^1$ and $\dot{n}^2$ of $\dot{N}^+$, [b] note that their intersection contains w since $w\in\dot{n}^+$, and [c] conclude that either $\dot{n}^1\supset\dot{n}^2$ or $\dot{n}^2\supset\dot{n}^1$ holds by (1b). Third, $\dot{N}^+$ is finite. To see this, note [a] that $\dot{n}^+\in\dot{X}$ by virtue of its being a predecessor of $\{w\}$, [b] that this implies $\dot{n}^+ \in \dot{T}$ by (5), and [c] that this implies n^+ has a finite number of predecessors by the definition of $\dot{T}$.

By the previous paragraph, $\dot{N}^+$ is a nonempty finite chain. Hence it contains its own union $\cup\dot{N}^+$. Hence by the definitions of $\dot{N}^+$ and $\dot{n}^+$,

$$\dot{t} \supset \cup\dot{N}^+ \;, \tag{21a}$$

$$(\nexists\dot{n})\; \dot{t} \supset \dot{n} \supset \cup\dot{N}^+ \;, \text{ and} \tag{21b}$$

$$\cup N^+ \supseteq n^+ \supset \{w\} \;. \tag{21c}$$

(21a,b) imply $\cup\dot{N}^+ \in \dot{T}\smallsetminus\{W\}$, which is the domain of $\dot{p}$. (21a,b) and the last sentence imply $\dot{t} = \dot{p}(\cup\dot{N}^+)$. This and (21c) allow us to let $\dot{t}^\sharp$ be $\cup\dot{N}^+$. □

Lemma A.3. *Suppose* $(W, \dot{N})$ *is an* AR* *outcome-set tree (1) with its* $\dot{T}$ *(2),* $\dot{X}$ *(4), and* $\dot{p}$ *(7). Then the following hold.*

(a) $\dot{p}$ *is onto* $\dot{X}$*.*

(b) $(\forall\dot{t}\in\dot{X})$ $\dot{p}^{-1}(\dot{t})$ *is a partition of* $\dot{t}$ *that has at least two elements.*

Proof. (a). To see that $\dot{p}$ is into $\dot{X}$, take any $\dot{t}\in\dot{T}\smallsetminus\{W\}$. By Lemma A.1, $\dot{p}(\dot{t})$ exists. Further, $\dot{p}(\dot{t}) \supset \dot{t}$ by the definition of $\dot{p}$. Thus $\dot{p}(\dot{t}) \in \dot{X}$ by the definition of $\dot{X}$.

To see that $\dot{p}$ is onto $\dot{X}$, take any $\dot{t}\in\dot{X}$. Since $\dot{t}$ is a node, there exists $w\in\dot{t}$. Thus Lemma A.2 implies the existence of a $\dot{t}^\sharp\in\dot{p}^{-1}(\dot{t})$.

(b). Take any $\dot{t}\in\dot{X}$. On the one hand, the definition of $\dot{p}$ implies that every $\dot{t}^\sharp\in\dot{p}^{-1}(\dot{t})$ satisfies $\dot{t} \supset \dot{t}^\sharp$. Hence $\dot{t} \supseteq \cup\dot{p}^{-1}(\dot{t})$. On the other hand, Lemma A.2 implies that for every $w\in\dot{t}$ there exists a $\dot{t}^\sharp \in \dot{p}^{-1}(\dot{t})$ such that $w\in\dot{t}^\sharp$. Hence $\dot{t} \subseteq \cup\dot{p}^{-1}(\dot{t})$. By putting these two together we obtain

$$\dot{t} = \cup\dot{p}^{-1}(\dot{t}). \tag{22}$$

This paragraph shows that $\dot{t}$ is partitioned by $\dot{p}^{-1}(\dot{t})$. Given (22), it remains to show that the elements of $\dot{p}^{-1}(\dot{t})$ are nonempty and disjoint. Each element of $\dot{p}^{-1}(\dot{t})$ in nonempty simply because each is a node and nodes are nonempty by (1a). To show disjointness, suppose that $\dot{t}^1$ and $\dot{t}^2$ were two distinct elements of $\dot{p}^{-1}(\dot{t})$ that are not disjoint. (1b) would then imply that either $\dot{t}^1\supset\dot{t}^2$ or $\dot{t}^1\supset\dot{t}^2$. Without loss of generality assume $\dot{t}^1\supset\dot{t}^2$. Then $\dot{t}\supset\dot{t}^1\supset\dot{t}^2$, where the first inclusion follows from $\dot{t}^1\in\dot{p}^{-1}(\dot{t})$ and the second follows from the previous sentence. This contradicts $\dot{t}^2\in\dot{p}^{-1}(\dot{t})$.

Finally, this paragraph shows that $\dot{p}^{-1}(\dot{t})$ contains at least two elements. Since $\dot{t}$ is nonempty simply because it is a node, (22) implies the existence of some $\dot{t}^1 \in \dot{p}^{-1}(\dot{t})$. By the definition of $\dot{p}$, $\dot{t}{\supset}\dot{t}^1$, and thus, we may take $w \in \dot{t}{\smallsetminus}\dot{t}^1$. By (22) again, there exists $\dot{t}^2 \in \dot{p}^{-1}(\dot{t})$ such that $w{\in}\dot{t}^2$. Thus since $w{\notin}\dot{t}^1$ by the definition of $\dot{t}^1$, $\dot{t}^1$ and $\dot{t}^2$ are distinct. □

A.2. For AR* preforms

Lemma A.4. *Suppose $(W, \dot{N})$ is an* AR* *outcome-set tree (1) with its $\dot{X}$ (4). Let $\dot{C}$ be any set, and derive $\dot{F}$ by (9). Then the following hold.*

(a) $\dot{F}^{-1}(\dot{C}) \subseteq \dot{X}$.

(b) $\dot{F}^{-1}(\dot{C}) = \dot{X}$ if $(W, \dot{N}, \dot{C})$ satisfies (8c) in the definition of an AR* *outcome-set preform.*

Proof. Also derive $\dot{T}$ (2) and $\dot{p}$ (7).

(a). Take any $\dot{t}$ in the domain $\dot{F}^{-1}(\dot{C})$ of $\dot{F}$. Then there exists a $\dot{c}$ such that $(\dot{t}, \dot{c}) \in \dot{F}$. Hence the definition of $\dot{F}$ implies the existence of an $\dot{t}^\sharp \in \dot{p}^{-1}(\dot{t})$. Hence the definition of $\dot{p}$ implies $\dot{t}^\sharp \subset \dot{t}$. Hence the definition of $\dot{X}$ implies $\dot{t} \in \dot{X}$.

(b). I need only show the converse of part (a). Accordingly, take any $\dot{t} \in \dot{X}$. Then by Lemma A.3(a), $\dot{p}^{-1}(\dot{t})$ is nonempty. Hence by (8c), $\dot{F}(\dot{t})$ is nonempty. Hence $\dot{t}$ is in the domain $\dot{F}^{-1}(C)$ of $\dot{F}$. □

Lemma A.5. *If $\dot{F} \subseteq \dot{T}{\times}\dot{C}$, the following are equivalent.*

(a) $(\forall \dot{c}, \dot{c}')$ $\dot{F}^{-1}(\dot{c}){=}\dot{F}^{-1}(\dot{c}')$ or $\dot{F}^{-1}(\dot{c}){\cap}\dot{F}^{-1}(\dot{c}'){=}\varnothing$.

(b) $(\forall \dot{t}, \dot{t}')$ $\dot{F}(\dot{t}){=}\dot{F}(\dot{t}')$ or $\dot{F}(\dot{t}){\cap}\dot{F}(\dot{t}'){=}\varnothing$.

Proof. Every t, c, and F in this proof should have a dot over it. I have removed the dots to make reading easier. By inspection, the following seven statements are equivalent.

(23a) $$(\exists c, c')\ F^{-1}(c) \neq F^{-1}(c') \text{ and } F^{-1}(c){\cap}F^{-1}(c') \neq \varnothing.$$

$$(\exists c^1, c^2)\ F^{-1}(c^2){\smallsetminus}F^{-1}(c^1) \neq \varnothing \text{ and } F^{-1}(c^2){\cap}F^{-1}(c^1) \neq \varnothing.$$

$$(\exists c^1, c^2, t^1, t^2)\ t^1{\in}F^{-1}(c^2),\ t^1{\notin}F^{-1}(c^1),\ t^2{\in}F^{-1}(c^2), \text{ and } t^2{\in}F^{-1}(c^1).$$

$$(\exists c^1, c^2, t^1, t^2)\ (t^1, c^1){\notin}F \text{ and } \{(t^1, c^2), (t^2, c^1), (t^2, c^2)\}{\subseteq}F.$$

$$(\exists c^1, c^2, t^1, t^2)\ c^1{\in}F(t^2),\ c^1{\notin}F(t^1),\ c^2{\in}F(t^2), \text{ and } c^2{\in}F(t^1).$$

$$(\exists t^1, t^2)\ F(t^2)\smallsetminus F(t^1) \neq \varnothing \text{ and } F(t^2)\cap F(t^1) \neq \varnothing.$$

(23b) $$(\exists t, t')\ F(t) \neq F(t') \text{ and } F(t)\cap F(t') \neq \varnothing.$$

(23a) is the negation of (a), and (23b) is the negation of (b). □

A.3. For AR* forms

Lemma A.6. *Suppose that* $(W, \dot{N}, (\dot{C}_i)_i)$ *is an* AR* *outcome-set form (11) with its* $\dot{F}$ *(9). Then the members of* $(\{\, \dot{F}^{-1}(\dot{c}){\neq}\varnothing \,|\, \dot{c}{\in}\dot{C}_i \,\})_i$ *are disjoint.*

Proof. Suppose there exists $i^1 \neq i^2$, $\dot{c}^1 \in \dot{C}_{i^1}$, and $\dot{c}^2 \in \dot{C}_{i^2}$ such that $\dot{F}^{-1}(\dot{c}^1)\cap\dot{F}^{-1}(\dot{c}^2) \neq \varnothing$. Take $\dot{t} \in \dot{F}^{-1}(\dot{c}^1)\cap\dot{F}^{-1}(\dot{c}^2)$. I now gather four facts about $\dot{F}(\dot{t})$. [1] Since $\dot{c}^1 \in \dot{F}(\dot{t})$, $\dot{F}(\dot{t}) \neq \varnothing$. [2] Since $\dot{t} \in \dot{F}^{-1}(\dot{c}^1)$, $\dot{t} \in \dot{F}^{-1}(C)$. Hence $\dot{t} \in \dot{X}$ by Lemma A.4(a). Hence $\dot{F}(\dot{t}) \subseteq \dot{F}(\dot{X})$. [3] Since $\dot{c}^1 \in \dot{F}(\dot{t})\cap\dot{C}_{i^1}$, (11c) implies that $\dot{F}(\dot{t}) \subseteq \dot{C}_{i^1}$. [4] Similarly, since $\dot{c}^2 \in \dot{F}(\dot{t})\cap\dot{C}_{i^2}$, (11c) implies that $\dot{F}(\dot{t}) \subseteq \dot{C}_{i^2}$. These four facts imply that $\dot{F}(\dot{t})$ is a nonempty subset of $\dot{F}(\dot{X})\cap\dot{C}_{i^1}\cap\dot{C}_{i^2}$. This contradicts (11b). □

Appendix B. Connection with AR

This appendix proves Theorem 1 (Section 2.4), which shows that AR* forms are essentially identical to AR forms. En route, Lemma 2.1 relates AR* trees to AR trees. Symbols with double dots are taken directly from AR. Virtually all such symbols are confined to this appendix. The only exception is the $\ddot{J}$ appearing in Section 2.4.

B.1. For Lemma 2.1

Lemma B.1. *Suppose that* $(W, \dot{N})$ *satisfies (1a) and (1c), and derive its* $\dot{T}$ *(2) and* $\dot{X}$ *(4). Set the* AR N *to* $\dot{N}$*. Then*

(a) $\dot{X} = \ddot{X}$*, where* $\ddot{X}$ *is defined by* AR *page 80.*

Further, suppose $(W, \dot{N})$ *also satisfies (1e). Then*

(b) $\dot{T} = \ddot{F}(\dot{N})$*, where* $\ddot{F}(\dot{N})$ *is defined by* AR *page 80.*

Proof. (a). Suppose $\dot{n} \in \dot{X}$. By the definition of $\dot{X}$, $\dot{n}$ cannot be a singleton. Also note [1] $\dot{n} \neq \varnothing$ by (1a), so [2] there is some $\hat{w} \in \dot{n}$, and so [3] $\dot{n} \supseteq \{\hat{w}\} \in \dot{N}$ by (1c). By the last two sentences, $\dot{n} \supset \{\hat{w}\} \in \dot{N}$ and thus $\dot{n} \in \ddot{X}$ by the definition of $\ddot{X}$.

Conversely, take any $\dot{n} \in \ddot{X}$. Then by the definition of $\ddot{X}$, there exists $\dot{n}^\sharp$ such that $\dot{n} \supset \dot{n}^\sharp$. Since $\dot{n}^\sharp$ cannot be empty by (1a), $\dot{n}$ is not a singleton. Hence $\dot{n} \in \dot{X}$ by the definition of $\dot{X}$.

(b). Suppose $\dot{n} \in \dot{T}$. Then by the definition of $\dot{T}$, either $\dot{n}$ has no predecessors or $\dot{n}$ has a finite positive number of predecessors. In the first case, $\dot{n} = W$ by (1a). Thus $\dot{n} \in \ddot{F}(\dot{N})$ by the definition of $\ddot{F}(\dot{N})$. In the second case, $\{\dot{n}^\flat | \dot{n}^\flat \supset \dot{n}\}$ has a minimum, which implies that $\dot{n}$ is "finite" in the sense of `AR` page 80. This implies that $\dot{n} \in \ddot{F}(\dot{N})$ by the definition of $\ddot{F}(\dot{N})$.

Conversely, take any $\dot{n} \in \ddot{F}(\dot{N})$. Then by the definition of $\ddot{F}(\dot{N})$, either $\dot{n} = W$ or $\dot{n}$ is "finite" in the sense of `AR` page 80. In the first case, $\dot{n} \in \dot{T}$ by the definition of $\dot{T}$. In the second case, the definition of "finiteness" allows us to let $\dot{n}^*$ be the minimum of $\{\dot{n}^\flat | \dot{n}^\flat \supset \dot{n}\}$. Note [1] $\dot{n}^* \supset \dot{n}$, so [2] $\dot{n}^* \notin \{\{w\}|w\}$ since $\dot{n}$ is nonempty by (1a), and so [3] $\dot{n}^* \in \dot{T}$ by (1e). Therefore, the definitions of $\dot{n}^*$ and $\dot{T}$ imply that $\dot{n} \in \dot{T}$. □

Lemma B.2. *Suppose that* $(W, \dot{N})$ *is an* `AR`* *outcome-set tree (1) with its* $\dot{T}$ *(2) and* $\dot{X}$ *(4). Then the following hold (set the* `AR` N *to* $\dot{N}$*).*

(a) $\dot{X} = \ddot{X}$*, where* $\ddot{X}$ *is defined on* `AR` *page 80.*

(b) $\dot{T} = \ddot{F}(\dot{N})$*, where* $\ddot{F}(\dot{N})$ *is defined on* `AR` *page 80.*

(c) $(W, \dot{N})$ *is a complete (1c) discrete game tree (*`AR` *Definitions 1 and 5).*

Proof. (a,b). These two parts follow from Lemma B.1.

(c). First I show that $(W, \dot{N})$ is a game tree (`AR` Definition 1). I do this in four steps. [1] The first three lines of `AR` Definition 1 are implied by (1a). [2] `AR` Definition 1 (GT.i) in the forward direction is implied by (1d) together with the fact that $\varnothing \notin \dot{N}$ by (1a). [3] To derive `AR` Definition 1 (GT.i) in the reverse direction, consider any $\dot{N}^* \subseteq \dot{N}$ and for which there exists $w \in \cap \dot{N}^*$ (set their h to $\dot{N}^*$). Take any distinct $\dot{n}^1$ and $\dot{n}^2$ in $\dot{N}^*$. By the existence of w, $\dot{n}^1 \cap \dot{n}^2 \neq \varnothing$. Hence $\dot{n}^1 \supset \dot{n}^2$ or $\dot{n}^2 \supset \dot{n}^1$ by (1b). Thus $\dot{N}^*$ is a chain. [4] To derive `AR` Definition 1 (GT.ii), take any w and w' and let $x = \{w\}$ and $x' = \{w'\}$. These are members of $\dot{N}$ by (1c).

Next I show that $(W, \dot{N})$ is discrete (`AR` Definition 5). I do this in three steps. [1] No node is "strange" in the sense of `AR` Definition 2. To see this, take any $\dot{n} \in \dot{N} \smallsetminus \{W\}$ (set their x to $\dot{n}$ and their N to $\dot{N}$). Then their $(\uparrow x) \smallsetminus \{x\}$ becomes $\{\dot{n}^\flat | \dot{n}^\flat \supset \dot{n}\}$. I must show that this collection has an infimum in $\dot{N}$. By AR Definition 1 (GT.i) in the reverse direction (which was derived in the previous paragraph), $\{\dot{n}^\flat|\dot{n}^\flat\supset\dot{n}\}$ is a chain. Hence by (1d), $\cap\{\dot{n}^\flat|\dot{n}^\flat\supset\dot{n}\}$ is a node. This node is the infimum of $\{\dot{n}^\flat|\dot{n}^\flat\supset\dot{n}\}$. [2] $(W,\dot{N})$ is regular in the sense of AR Definition 4. This follows immediately from [1]. [3] $(W,\dot{N})$ is discrete. To show this, I apply AR Theorem 1. By the previous paragraph and [2], $(W,\dot{N})$ is a regular game tree. Further, condition (d) of AR Theorem 1 holds because of part (a) and because $\dot{X}\subseteq\dot{T}$ by (5). Hence AR Theorem 1(d⇒a) implies that $(W,\dot{N})$ is discrete.

Finally, completeness (1c) is directly assumed as part of the definition of an AR* outcome-set tree. □

Lemma B.3. *Suppose* $(W,\dot{N})$ *is a complete (1c) discrete game tree (*AR *Definitions 1 and 5 at* $N{=}\dot{N}$*). Derive its* $\ddot{X}$ *and* $\ddot{F}(\dot{N})$ *by* AR *page 80. Then the following hold.*

(a) $\ddot{X}=\dot{X}$*, where* $\dot{X}$ *is defined by (4).*

(b) $\ddot{F}(\dot{N})=\dot{T}$*, where* $\dot{T}$ *is defined by (2).*

(c) $(W,\dot{N})$ *is an* AR* *outcome-set tree (1).*

Proof. First, I show (i) that $(W,\dot{N})$ satisfies (1a) and (1c), and (ii) that part (a) holds. (1a) follows from the first three lines of AR Definition 1. (1c) has been assumed directly. Thus Lemma B.1(a) implies part (a).

Second, I show two intermediate results: (i) that $(W,\dot{N})$ is regular in the sense of AR Definition 4, and (ii) that $\ddot{X}\subseteq\dot{T}$. By AR Definition 5 and AR Proposition 3 in the forward direction, the assumed discreteness of $(W,\dot{N})$ implies that $(W,\dot{N})$ is regular. Because of regularity and discreteness, AR Theorem 1 (a⇒d) implies that $(\forall\dot{n}\in\ddot{X})$ $\{\dot{n}^\flat|\dot{n}^\flat\supset\dot{n}\}$ is finite. By the definition of $\dot{T}$, this is equivalent to $\ddot{X}\subseteq\dot{T}$.

Third, I show (i) that $(W,\dot{N})$ satisfies (1e), and (ii) that part (b) holds. By part (a) and by the second conclusion of the previous paragraph, $\dot{X}\subseteq\dot{T}$. Hence

$$\dot{N}=(\dot{N}\smallsetminus\{\{w\}|w\})\cup\{\{w\}|w\}=\dot{X}\cup\{\{w\}|w\}\subseteq\dot{T}\cup\{\{w\}|w\}\ ,$$

where the second equation holds by the definition of $\dot{X}$ and the set inclusion holds by the previous sentence. This result is (1e). Finally, (1e), the first conclusion of the first paragraph, and Lemma B.1(b) together imply part (b).

Finally, consider part (c). (1a), (1c), and (1e) have already been established by the first and third paragraphs. To derive (1b), suppose that $\dot{n}^1$ and $\dot{n}^2$ are distinct and have a nonempty intersection. Since $\dot{n}^1$ and $\dot{n}^2$ have a nonempty intersection, the reverse direction of AR Definition 1 (GT.i) implies that $\{\dot{n}^1,\dot{n}^2\}$ is a chain. Thus the distinctness of $\dot{n}^1$ and $\dot{n}^2$ implies that either $\dot{n}^1\supset\dot{n}^2$ or $\dot{n}^2\supset\dot{n}^1$.

It remains to derive (1d). Accordingly, take any nonempty chain $\dot{N}^* \subseteq \dot{N}$. Suppose $\dot{N}^*$ is a finite chain. Then $\cap\dot{N}^*$ is the smallest node in $\dot{N}^*$. This implies $\cap\dot{N}^* \in \dot{N}$ by the assumption that $\dot{N}^* \subseteq \dot{N}$. Hence we may suppose henceforth that

$$\dot{N}^* \text{ is an infinite chain .} \tag{24}$$

By AR Definition 1 (GT.i) in the forward direction, there exists some w^1 such that $\cap\dot{N}^* \ni w^1$. Suppose that $\cap\dot{N}^* = \{w^1\}$. Then $\cap\dot{N}^* \in \dot{N}$ by completeness. Hence we may suppose henceforth that

$$\cap\dot{N}^* \supset \{w^1\} . \tag{25}$$

The remainder of this proof shows that this leads to a contradiction.

By the regularity derived in the first conclusion of the second paragraph,

$$\dot{n}^1 := \inf\{\dot{n}|\dot{n}\supset\{w^1\}\}$$

is a well-defined node. Note that

$$\cap\dot{N}^* \supseteq \cap\{\dot{n}|\dot{n}\supseteq\{w^1\}\} \supseteq \inf\{\dot{n}|\dot{n}\supseteq\{w^1\}\} = \dot{n}^1 , \tag{26}$$

where the first set inclusion follows from (25) and the equality is the definition of $\dot{n}^1$. ($\cap\dot{N}^*$ and $\cap\{\dot{n}|\dot{n}\supseteq\{w^1\}\}$ may or may not be nodes.)

Since $\{w^1\}$ is a node by completeness, the definition of $\dot{n}^1$ implies that $\dot{n}^1 \supseteq \{w^1\}$. Suppose $\dot{n}^1 \supset \{w^1\}$. Then $\dot{n}^1 \in \ddot{X}$ by the definition of $\ddot{X}$. Yet $\dot{n}^1 \notin \dot{T}$ by (24), (26), and the definition of $\dot{T}$. The last two sentences contradict $\ddot{X} \subseteq \dot{T}$, which is the second conclusion of the second paragraph. Hence we may suppose henceforth that

$$\dot{n}^1 = \{w^1\} . \tag{27}$$

By (25), there exists w^2 such that

$$w^2 \neq w^1 \text{ and} \tag{28a}$$

$$\cap\dot{N}^* \ni w^2 . \tag{28b}$$

Further, (28a), (27), and the definition of $\dot{n}^1$ imply the existence of a $\dot{n}^+$ such that

$$\dot{n}^+ \supset \{w^1\} \text{ and} \tag{29a}$$

$$\dot{n}^+ \not\ni w^2 \ . \tag{29b}$$

Now consider any $\dot{n}^*$ in the chain $\dot{N}^*$. By (25) and (29a), we have $w^1 \in \dot{n}^* \cap \dot{n}^+$. By (28b) and (29b), we have $w^2 \in \dot{n}^* \smallsetminus \dot{n}^+$. The last two sentences and (1b) (which has already been derived) imply that $\dot{n}^* \supset \dot{n}^+$. Since this holds for all $\dot{n}^* \in \dot{N}^*$, we have $\cap \dot{N}^* \supseteq \dot{n}^+$.

By (24), the last sentence, and the definition of $\dot{T}$, we have that $\dot{n}^+ \notin \dot{T}$. Yet by (29a), we have that $\dot{n}^+ \in \ddot{X}$. The last two sentences contradict $\ddot{X} \subseteq \dot{T}$, which is the second conclusion of the second paragraph. □

Proof B.4 (for Lemma 2.1)**.** Lemma 2.1 follows immediately from Lemma B.2(c) and Lemma B.3(c). □

B.2. Further observations about trees

The remark and two lemmata in this section are concerned only with trees. They will be used to support Section B.3's proofs about forms.

Lemma B.5. *Suppose that $(W, \dot{N})$ is an* AR* *outcome-set tree (1). Derive $\dot{p}$ (7), $\ddot{F}(\dot{N})$ (*AR *page 80 at $N{=}\dot{N}$), and $\ddot{p}$ (*AR *page 80). Then*
(a) $\dot{p} = \ddot{p}|_{\ddot{F}(\dot{N}) \smallsetminus \{W\}}$ and
(b) $\dot{p} \cup \{(W, W)\} = \ddot{p}$.

Proof. Define $\dot{T}$ by (2). By Lemma B.2(b), $\dot{T} = \ddot{F}(\dot{N})$.

(a). Since $\dot{T} = \ddot{F}(\dot{N})$, the definition of $\dot{p}$ and the definition of $\ddot{p}|_{\ddot{F}(\dot{N}) \smallsetminus \{W\}}$ in AR page 80 equation (2) together imply that $\dot{p} = \ddot{p}|_{\dot{T} \smallsetminus \{W\}}$.

(b). Since $\dot{T} = \ddot{F}(\dot{N})$, AR page 80 defines the domain of $\ddot{p}$ to be all of $\ddot{F}(\dot{N})$ and sets $\ddot{p}(W) = W$. Thus $\dot{p} \cup \{(W, W)\} = \ddot{p}$. □

Remark B.6. AR Proposition 1(a) appears to have a minor mistake. To be precise, suppose $(W, \dot{N})$ is a complete (1c) discrete game tree (AR Definitions 1 and 5 at $N{=}\dot{N}$), and derive its $\ddot{F}(\dot{N})$ and $\ddot{p}$ by AR page 80. The proposition claims that every nonterminal node x is partitioned by $\ddot{p}^{-1}(x)$. However, the initial node W is not partitioned by $\ddot{p}^{-1}(W)$ (given W is nonterminal). Rather, (i) W is nontrivially partitioned by

$(\ddot{p}|_{\ddot{F}(\dot{N})\smallsetminus\{W\}})^{-1}(W)$ and (ii) $\ddot{p}^{-1}(W)$ is the union of this partition and $\{W\}$. The latter is not a partition because its elements are not disjoint.

This paragraph proves claims (i) and (ii). To prove (i), note that $(W,\dot{N})$ is an AR* outcome-set tree by Lemma 2.1, and derive its $\dot{p}$ by (7). Lemma A.3(b) implies that $\dot{p}^{-1}(W)$ is a partition of W with at least two elements. Hence Lemma B.5(a) implies that $(\ddot{p}|_{\ddot{F}(\dot{N})\smallsetminus\{W\}})^{-1}(W)$ is a partition of W with at least two elements. To prove (ii), note that $W \in \ddot{F}(\dot{N})$ and that $\ddot{F}(\dot{N})$ is the domain of $\ddot{p}$. Hence (ii) is equivalent to $\ddot{p}(W) = W$. This equality is part of the definition of $\ddot{p}$ on AR page 80.

In accord with its Proposition 1(a), AR interprets $\ddot{p}^{-1}(x)$ as the collection of immediate successors of a nonterminal node x (see for example the discussion of (DEF.ii) after Definition 6). In light of the above, I replace $\ddot{p}^{-1}(x)$ with $(\ddot{p}|_{\ddot{F}(\dot{N})\smallsetminus\{W\}})^{-1}(x)$ as the need arises. This expression appears on several occasions, including Lemma B.7, equation (43), and equation (49). In effect, I prohibit W from ever being regarded as a successor.

Lemma B.7. *Suppose that $(W,\dot{N})$ is a complete (1c) discrete game tree (AR Definitions 1 and 5 at $N{=}\dot{N}$). Next define its $\ddot{X}$, $\ddot{F}(\dot{N})$, $\ddot{p}$, and $\ddot{P}$ by AR pages 80 and 82. Then*

$$\ddot{P} = \{\ (a,x){\in}\mathcal{P}(W){\times}\ddot{X} \mid a{\not\supseteq}x \textit{ and } (\exists \dot{n}{\in}(\ddot{p}|_{\ddot{F}(\dot{N})\smallsetminus\{W\}})^{-1}(x))\ a{\supseteq}\dot{n}\ \}\ .$$

Proof. The lemma's conclusion is equivalent to

$$(\forall a{\in}\mathcal{P}(W)) \quad \ddot{P}(a) = \{\ x{\in}\ddot{X} \mid a{\not\supseteq}x \text{ and } (\exists \dot{n}{\in}(\ddot{p}|_{\ddot{F}(\dot{N})\smallsetminus\{W\}})^{-1}(x))\ a{\supseteq}\dot{n}\ \}\ . \tag{30}$$

This will be proven by considering two cases.

First consider $a = W$. Here I will argue that the empty set is on both sides of the equality in (30). The right-hand side is

$$\{\ x{\in}\ddot{X} \mid W{\not\supseteq}x \text{ and } (\exists \dot{n}{\in}(\ddot{p}|_{\ddot{F}(\dot{N})\smallsetminus\{W\}})^{-1}(x))\ W{\supseteq}\dot{n}\ \}\ .$$

No x can belong to this set because W is a superset of every node x. Meanwhile, the left-hand side $\ddot{P}(W)$ equals

$$\{\ x{\in}\dot{N} \mid (\exists y{\subseteq}W)\ \{\dot{n}|\dot{n}{\supseteq}x\}{=}\{\dot{n}|\dot{n}{\supseteq}y\}{\smallsetminus}\{\dot{n}|\dot{n}{\subseteq}W\}\ \}$$

by the definition of $\ddot{P}$ on AR page 82. No x can satisfy the equality defining this set because (a) $\{\dot{n}|\dot{n}{\supseteq}x\}$ is nonempty for every x (since

W belongs to it) and yet (b) $\{\dot{n}|\dot{n}{\supseteq}y\}{\smallsetminus}\{\dot{n}|\dot{n}{\subseteq}W\}$ is empty (since every node $\dot{n}$ belongs to $\{\dot{n}|\dot{n}{\subseteq}W\}$).

Second consider any $a \in \mathcal{P}(W){\smallsetminus}\{W\}$. Consider `AR` Proposition 2. The discreteness (`AR` Definition 5) of $(W,\dot{N})$ implies up-discreteness. Further, completeness (1c) implies that a is an element of the "$A(T)$" defined at the start of `AR` Proposition 2. Thus `AR` Proposition 2(b) implies that

$$(31) \qquad \ddot{P}(a) = \{\ \ddot{p}(\dot{n}) \mid \dot{n}{\in}\ddot{N}(a){\cap}\ddot{F}(\dot{N})\ \}\ ,$$

where `AR` equation (6) defines

$$(32) \qquad \ddot{N}(a)\ :=\ \{\ \dot{n} \mid a{\supseteq}\dot{n} \text{ and } (\nexists\dot{n}^\flat)\ a{\supseteq}\dot{n}^\flat{\supset}\dot{n}\ \}\ .$$

This paragraph argues that

$$\begin{aligned}(33) \qquad & \ddot{N}(a){\cap}\ddot{F}(\dot{N}) \\ &= \{\ \dot{n}{\in}\ddot{F}(\dot{N}) \mid a{\supseteq}\dot{n} \text{ and } (\nexists\dot{n}^\flat)\ a{\supseteq}\dot{n}^\flat{\supset}\dot{n}\ \} \\ &= \{\ \dot{n}{\in}\ddot{F}(\dot{N}){\smallsetminus}\{W\} \mid a{\supseteq}\dot{n} \text{ and } (\nexists\dot{n}^\flat)\ a{\supseteq}\dot{n}^\flat{\supset}\dot{n}\ \} \\ &= \{\ \dot{n}{\in}\ddot{F}(\dot{N}){\smallsetminus}\{W\} \mid a{\supseteq}\dot{n} \text{ and } a \not\supseteq \ddot{p}(\dot{n})\ \}\ .\end{aligned}$$

The first equality holds by (32). To prove the second equality, it must be argued that the left-hand set is included within the right-hand set. Accordingly, consider any $\dot{n}{\in}\ddot{F}(\dot{N})$ that satisfies $a{\supseteq}\dot{n}$. Because $a{\neq}W$ by assumption, $a{\supseteq}\dot{n}$ implies $\dot{n}{\neq}W$. Thus $\dot{n}{\in}\ddot{F}(\dot{N}){\smallsetminus}\{W\}$. Finally, the third equality follows from the definition of $\ddot{p}$ over $\ddot{F}(\dot{N}){\smallsetminus}\{W\}$, as stated in `AR` page 80 equation (2).

The remainder of the proof establishes the equality in (30) by arguing that

$$\begin{aligned} & \ddot{P}(a) \\ &= \{\ \ddot{p}(\dot{n}) \mid \dot{n}{\in}\ddot{N}(a){\cap}\ddot{F}(\dot{N})\ \} \\ &= \{\ \ddot{p}(\dot{n}) \mid \dot{n}{\in}\ddot{F}(\dot{N}){\smallsetminus}\{W\},\ a{\supseteq}\dot{n},\ a \not\supseteq \ddot{p}(\dot{n})\ \} \\ &= \{\ \ddot{p}(\dot{n}) \mid a \not\supseteq \ddot{p}(\dot{n}),\ a{\supseteq}\dot{n},\ \dot{n}{\in}\ddot{F}(\dot{N}){\smallsetminus}\{W\}\ \} \\ &= \{\ x{\in}\ddot{X} \mid a{\not\supseteq}x \text{ and } (\exists\dot{n}{\in}(\ddot{p}|_{\ddot{F}(\dot{N}){\smallsetminus}\{W\}})^{-1}(x))\ a{\supseteq}\dot{n}\ \}\ .\end{aligned}$$

The first equality holds by (31), the second equality holds by (33), the third equality is a rearrangement, and the final equality is proved by the next two paragraphs.

On the one hand, take any $\ddot{p}(\dot{n})$ in the set on the final equation's left-hand side. Let $x = \ddot{p}(\dot{n})$. This x is in the right-hand set because [1]

$x \in \ddot{X}$ by the definition of $\ddot{p}$, [2] $a \not\supseteq x$ by the left-hand fact that $a \not\supseteq \ddot{p}(\dot{n})$ and the definition of x, [3] $\dot{n} \in (\ddot{p}|_{\ddot{F}(\dot{N}) \smallsetminus \{W\}})^{-1}(x)$ by the definition of x and the left-hand fact that $\dot{n} \in \ddot{F}(\dot{N}) \smallsetminus \{W\}$, and [4] $a \supseteq \dot{n}$ because this is itself a left-hand fact.

On the other hand, take any x in the right-hand set. Then

$$a \not\supseteq x \ , \tag{34}$$

and there exists $\dot{n}$ in $(\ddot{p}|_{\ddot{F}(\dot{N}) \smallsetminus \{W\}})^{-1}(x)$ such that

$$a \supseteq \dot{n} \ . \tag{35}$$

Further, since $\dot{n}$ is in $(\ddot{p}|_{\ddot{F}(\dot{N}) \smallsetminus \{W\}})^{-1}(x)$, we have both

$$\dot{n} \in \ddot{F}(\dot{N}) \smallsetminus \{W\} \text{ and} \tag{36a}$$

$$x = \ddot{p}(\dot{n}) \ . \tag{36b}$$

Note that $\ddot{p}(\dot{n})$ is in the left-hand set because [1] $a \not\supseteq \ddot{p}(\dot{n})$ by (34) and (36b), [2] $a \supseteq \dot{n}$ by (35), and [3] $\dot{n} \in \ddot{F}(\dot{N}) \smallsetminus \{W\}$ by (36a). Thus x is in the left-hand set by (36b). □

B.3. For Theorem 1

Lemma B.8. *Suppose $(W, \dot{N}, (\dot{C}_i)_i)$ is an* `AR`* *outcome-set form (11). Then the following hold.*

(a) $(W, \dot{N})$ is a complete (1c) discrete game tree (`AR` *Definitions 1 and 5 at $N = \dot{N}$).*

(b) $(W, \dot{N}, (\dot{C}_i)_i)$ is a complete discrete extensive form (`AR` *Definition 6 at $N = \dot{N}$ and $(C_i)_i = (\dot{C}_i)_i$) without directly specified simultaneous moves (12).*

Proof. (a). Let $\dot{C} = \cup_i \dot{C}_i$. By (11a) in the definition of an `AR`* form, $(W, \dot{N}, \dot{C})$ is an `AR`* preform (8). Thus by (8a) in the definition of an `AR`* preform, $(W, \dot{N})$ is an `AR`* tree (1). Thus by Lemma 2.1, $(W, \dot{N})$ is a complete discrete game tree.

(b). Since completeness was shown in part (a), it remains to be shown that $(W, \dot{N}, (\dot{C}_i)_i)$ is a discrete extensive form (`AR` Definition 6) without directly specified simultaneous moves. The next eight paragraphs establish `AR` Definition 6. The last paragraph shows the absence of directly specified simultaneous moves.

This paragraph shows the opening four lines of AR Definition 6. $(W,\dot{N})$ is a discrete game tree by part (a). Further, every $\dot{C}_i$ consists of nonempty unions of nodes by (8b) and (1c).

This paragraph collects the four identities listed in (37). Define $\dot{T}$, $\dot{X}$, $\dot{p}$, and $\dot{F}$ by (2), (4), (7), and (9). Further, by part (a), we may define $\ddot{X}$, $\ddot{F}(\dot{N})$, $\ddot{p}$, and $\ddot{P}$ by AR pages 80 and 82.[3] By Lemmata B.2(a), B.2(b), and B.5(a),

$$\text{(37a)} \qquad \ddot{X} = \dot{X} \ ,$$

$$\text{(37b)} \qquad \ddot{F}(\dot{N}) = \dot{T} \ , \text{ and}$$

$$\text{(37c)} \qquad \ddot{p}|_{\ddot{F}(\dot{N})\smallsetminus\{W\}} = \dot{p} \ ,$$

Further, the remainder of this paragraph argues that

$$\begin{aligned} \text{(37d)} \quad & \ddot{P}|_{\dot{C}} \\ &= \{\ (\dot{c},x){\in}\dot{C}{\times}\ddot{X} \mid \dot{c}{\not\supseteq}x \text{ and } (\exists\dot{n}{\in}(\ddot{p}|_{\ddot{F}(\dot{N})\smallsetminus\{W\}})^{-1}(x))\ \dot{c}{\supseteq}\dot{n}\ \} \\ &= \{\ (\dot{c},x){\in}\dot{C}{\times}\dot{X} \mid \dot{c}{\not\supseteq}x \text{ and } (\exists\dot{n}{\in}\dot{p}^{-1}(x))\ \dot{c}{\supseteq}\dot{n}\ \} \\ &= \{\ (\dot{c},\dot{t}){\in}\dot{C}{\times}\dot{X} \mid \dot{c}{\not\supseteq}\dot{t} \text{ and } (\exists\dot{n}{\in}\dot{p}^{-1}(\dot{t}))\ \dot{c}{\supseteq}\dot{n}\ \} \\ &= \dot{F}^{-1} \ . \end{aligned}$$

The first equality holds by Lemma B.7, part (a), and the fact that $\dot{C} \subseteq \mathcal{P}(W)$ by (8b). The second equality holds by (37a) and (37c). The third equality holds because $\dot{X} \subseteq \dot{T}$ by (5) (the symbol $\dot{t}$ is reserved for members of $\dot{T}$). The final equality holds by the definition (9) of $\dot{F}$ and Lemma A.4(a).

This paragraph shows AR Definition 6 (DEF.i). Accordingly, take any $\dot{c}$ and $\dot{c}'$ such that $P(\dot{c}){\cap}P(\dot{c}') \neq \varnothing$ and $\dot{c} \neq \dot{c}'$ (the argument here does not require that $\dot{c}$ and $\dot{c}'$ belong to the same $\dot{C}_i$). By (37d), $P(\dot{c}){\cap}P(\dot{c}')$ $\neq \varnothing$ implies

$$\text{(38)} \qquad \dot{F}^{-1}(\dot{c}){\cap}\dot{F}^{-1}(\dot{c}') \neq \varnothing \ .$$

By (8e) and Lemma A.5(b⇒a), we have

$$\dot{F}^{-1}(\dot{c}){=}\dot{F}^{-1}(\dot{c}') \text{ or } \dot{F}^{-1}(\dot{c}){\cap}\dot{F}^{-1}(\dot{c}'){=}\varnothing \ .$$

[3] Symbols with double dots are taken directly from AR. Although this is usually natural, both $\dot{F}$ and $\ddot{F}$ appear in this proof (as well as the next proof). $\dot{F}$ and $\ddot{F}$ are completely unrelated. While $\dot{F}$ is the feasibility correspondence (9) in an AR* form, $\ddot{F}(\dot{N})$ is a collection of nodes that is defined on AR page 80.

Thus (38) implies $\dot{F}^{-1}(\dot{c}) = \dot{F}^{-1}(\dot{c}')$, which implies $P(\dot{c}) = P(\dot{c}')$ by (37d) again. Further, (38) also implies the existence of a $\dot{t}$ such that $\{\dot{c},\dot{c}'\} \subseteq F(\dot{t})$. Thus $\dot{c} \neq \dot{c}'$ implies $\dot{c}\cap\dot{c}' = \varnothing$ by (8d).

This and the next four paragraphs show AR Definition 6 (DEF.ii). Since $\ddot{X} = \dot{X}$ by (37a), it suffices to consider an arbitrary member of $\dot{X}$. Since $\dot{X} \subseteq \dot{T}$ by (5), I can denote this arbitrary member by $\dot{t}$ (the symbol $\dot{t}$ is reserved for members of $\dot{T}$). This $\dot{t}$ is fixed through the next three paragraphs.

This paragraph argues that $\{i|\dot{F}(\dot{t})\cap\dot{C}_i\neq\varnothing\}$ is a singleton. Since $\dot{t}\in\dot{X}$ by the previous paragraph, and since $\dot{X}$ is the domain of $\dot{F}$ by Lemma A.4, $\dot{F}(\dot{t})$ is nonempty. Further, by the definition of $\dot{F}$, $\dot{F}$ assumes values in $\dot{C}=\cup_i\dot{C}_i$. Thus the last two sentences imply that $\{i|\dot{F}(\dot{t})\cap\dot{C}_i\neq\varnothing\}$ has at least one element. Finally, (11b) implies that the set can have no more than one element.

This paragraph proves (41). Note that

$$(39)\qquad (\forall i)\;\; \ddot{A}_i(\dot{t}) = \{\dot{c}\in\dot{C}_i|\dot{t}\in P(\dot{c})\} = \{\dot{c}\in\dot{C}_i|\dot{t}\in\dot{F}^{-1}(\dot{c})\}$$
$$= \{\dot{c}\in\dot{C}_i|\dot{c}\in\dot{F}(\dot{t})\} = \dot{F}(\dot{t})\cap\dot{C}_i\ ,$$

where the first equality is the definition of $\ddot{A}_i$ in AR Definition 6 (DEF.ii) and the second equality follows from (37d). Thus

$$(40)\qquad \ddot{J}(\dot{t}) = \{i|\ddot{A}_i(\dot{t})\neq\varnothing\} = \{i|\dot{F}(\dot{t})\cap\dot{C}_i\neq\varnothing\}\ ,$$

where the first equality is the definition of $\ddot{J}$ in AR Definition 6 (DEF.ii), and the second equality holds by the previous sentence. Hence the previous paragraph implies that

$$(41)\qquad \ddot{J}(\dot{t}) \text{ is a singleton .}$$

Let i^* be the unique element of $\ddot{J}(\dot{t})$. By (40), $\dot{F}(\dot{t})\cap\dot{C}_{i^*}\neq\varnothing$. Thus by (11c),

$$(42)\qquad \dot{F}(\dot{t}) \subseteq \dot{C}_{i^*}\ .$$

Therefore

$$(43)\qquad (\ddot{p}|_{\ddot{F}(\dot{N})\setminus\{W\}})^{-1}(\dot{t})$$
$$= \dot{p}^{-1}(\dot{t})$$
$$= \{\ \dot{t}\cap\dot{c}\ |\ \dot{c}\in\dot{F}(\dot{t})\ \}$$
$$= \{\ \dot{t}\cap\dot{c}\ |\ \dot{c}\in\dot{F}(\dot{t})\cap\dot{C}_{i^*}\ \}$$
$$= \{\ \dot{t}\cap(\cap_{i\in\{i^*\}}\dot{c}_i)\ |\ (\dot{c}_i)_{i\in\{i^*\}} \in \{i^*\}\times(\dot{F}(\dot{t})\cap\dot{C}_{i^*})\ \}$$

$$= \{\ \dot{t}\cap(\cap_{i\in\{i^*\}}\dot{c}_i)\ |\ (\dot{c}_i)_{i\in\{i^*\}} \in \{i^*\}\times\ddot{A}_{i^*}(\dot{t})\ \}$$
$$= \{\ \dot{t}\cap(\cap_{i\in\{i^*\}}\dot{c}_i)\ |\ (\dot{c}_i)_{i\in\{i^*\}} \in \Pi_{i\in\{i^*\}}\ddot{A}_i(\dot{t})\ \}$$
$$= \{\ \dot{t}\cap(\cap_{i\in\ddot{J}(\dot{t})}\dot{c}_i)\ |\ (\dot{c}_i)_{i\in\ddot{J}(\dot{t})} \in \Pi_{i\in\ddot{J}(\dot{t})}\ddot{A}_i(\dot{t})\ \}$$
$$= \{\ \dot{t}\cap(\cap_{i\in\ddot{J}(\dot{t})}\dot{c}_i)\ |\ (\dot{c}_i)_{i\in\ddot{J}(\dot{t})} \in \ddot{A}(\dot{t})\ \}\ ,$$

where the first equality holds by (37c), the second equality holds by (8c), the third follows from (42), the fourth is a rearrangement, the fifth follows from (39) at $i = i^*$, the sixth is a rearrangement, the seventh follows from (41) and the definition of i^*, and the last follows from the definition of $\ddot{A}$ in `AR` Definition 6 (DEF.ii).

Equation (43) suffices to prove `AR` Definition 6 (DEF.ii) under the presumption that $\ddot{p}^{-1}(W)$ on the left-hand side of (DEF.ii) was not meant to contain W itself (see the third paragraph of Remark B.6).

Further, the above argument established (41) for any member $\dot{t}$ of $\ddot{X}$. This shows the absence of directly specified simultaneous moves (12). □

Lemma B.9. *Suppose* $(W, \dot{N}, (\dot{C}_i)_i)$ *is a complete (1c) discrete extensive form (*`AR` *Definition 6 at* $N{=}\dot{N}$ *and* $(C_i)_i{=}(\dot{C}_i)_i$*) without directly specified simultaneous moves (12). Then the following hold.*

(a) $(W, \dot{N})$ *is an* `AR`* *outcome-set tree (1).*

(b) $(W, \dot{N}, \cup_i\dot{C}_i)$ *is an* `AR`* *outcome-set preform (8).*

(c) $(W, \dot{N}, (\dot{C}_i)_i)$ *is an* `AR`* *outcome-set form (11).*

Proof. (a). By the second line of `AR` Definition 6, $(W, \dot{N})$ is a discrete game tree. This, the assumption of completeness, and Lemma 2.1 together imply that $(W, \dot{N})$ is an `AR`* outcome-set tree.

(b). This paragraph collects the four identities listed in (44). Let $\dot{C}{=}\cup_i\dot{C}_i$. Derive $\ddot{X}$, $\ddot{F}(\dot{N})$, $\ddot{p}$, and $\ddot{P}$ from $(W, \dot{N}, (\dot{C}_i)_i)$ by `AR` pages 80 and 82. Further, by part (a) and Lemma A.1, we may define $\dot{T}$, $\dot{X}$, $\dot{p}$, and $\dot{F}$ by (2), (4), (7), and (9). By Lemmas B.3(a), B.3(b), and B.5(a),

(44a) $$\ddot{X} = \dot{X}\ ,$$

(44b) $$\ddot{F}(\dot{N}) = \dot{T}\ ,\ \text{and}$$

(44c) $$\ddot{p}|_{\ddot{F}(\dot{N})\smallsetminus\{W\}} = \dot{p}\ .$$

Further, the remainder of this paragraph argues that

$$\text{(44d)}\quad \ddot{P}|_{\dot{C}}$$
$$= \{\ (\dot{c},x){\in}\dot{C}{\times}\ddot{X}\ |\ \dot{c}{\not\supseteq}x \text{ and } (\exists\dot{n}{\in}(\ddot{p}|_{\ddot{F}(\dot{N})\smallsetminus\{W\}})^{-1}(x))\ \dot{c}{\supseteq}\dot{n}\ \}$$
$$= \{\ (\dot{c},x){\in}\dot{C}{\times}\dot{X}\ |\ \dot{c}{\not\supseteq}x \text{ and } (\exists\dot{n}{\in}\dot{p}^{-1}(x))\ \dot{c}{\supseteq}\dot{n}\ \}$$
$$= \{\ (\dot{c},\dot{t}){\in}\dot{C}{\times}\dot{X}\ |\ \dot{c}{\not\supseteq}\dot{t} \text{ and } (\exists\dot{n}{\in}\dot{p}^{-1}(\dot{t}))\ \dot{c}{\supseteq}\dot{n}\ \}$$
$$= \dot{F}^{-1}\ .$$

The first equality follows from Lemma B.7 because [1] $(W,\dot{N})$ is an `AR` discrete game tree by the second line of `AR` Definition 6, [2] completeness has been assumed, and [3] $\dot{C} \subseteq \mathcal{P}(W)$ by the third and fourth lines of `AR` Definition 6. The second equality holds by (44a) and (44c). The third equality holds because $\dot{X} \subseteq \dot{T}$, which follows from (5), which in turn follows from part (a) (the symbol $\dot{t}$ is reserved for elements of $\dot{T}$). The final equality holds by the definition of $\dot{F}$ and Lemma A.4(a).

I now show the five components of the definition (8) of an `AR`* outcome-set preform. (8a) follows from part (a). (8b) follows from the third and fourth lines of `AR` Definition 6.

(8c). Take any $\dot{t} \in \dot{X}$. By (44a), $\dot{t} \in \ddot{X}$. Note that

$$\text{(45)}\qquad (\forall i)\ \ \ddot{A}_i(\dot{t}) = \{\dot{c}{\in}\dot{C}_i|\dot{t}{\in}\ddot{P}(\dot{c})\} = \{\dot{c}{\in}\dot{C}_i|\dot{t}{\in}\dot{F}^{-1}(\dot{c})\}$$
$$= \{\dot{c}{\in}\dot{C}_i|\dot{c}{\in}\dot{F}(\dot{t})\} = \dot{F}(\dot{t}){\cap}\dot{C}_i\ .$$

where the first equality is the definition of $\ddot{A}_i$ from `AR` Definition 6 (DEF.ii), the second equality holds by (44d), and the last two equalities are rearrangements. Further,

$$\text{(46)}\qquad \ddot{J}(\dot{t}) = \{i|\ddot{A}_i(\dot{t}){\neq}\varnothing\} = \{i|\dot{F}(\dot{t}){\cap}\dot{C}_i{\neq}\varnothing\}\ .$$

where the first equality is the definition of $\ddot{J}$ from `AR` Definition 6 (DEF.ii), and the second equality holds by (45).

$\ddot{J}(\dot{t})$ is a singleton by the assumed absence of directly specified simultaneous moves (12). Let i^* be its member. Then by (46),

$$\text{(47)}\qquad (\forall i{\neq}i^*)\ \ \dot{F}(\dot{t}){\cap}\dot{C}_i = \varnothing\ .$$

Further,

$$\text{(48)}\qquad \dot{F}(\dot{t}){\cap}C_{i^*} = (\dot{F}(\dot{t}){\cap}\dot{C}_{i^*}) \cup {\textstyle\bigcup_{i\neq i^*}}(F(\dot{t}){\cap}\dot{C}_i)$$
$$= \dot{F}(\dot{t}) \cap {\textstyle\bigcup_i}\dot{C}_i = \dot{F}(\dot{t})\ ,$$

where the first equality holds by the previous sentence, and the last holds because $\dot{F}$ is defined to assume values in $\dot{C}{=}\cup_i\dot{C}_i$.

Finally, (8c) holds by

$$(49)\qquad \begin{aligned}\dot{p}^{-1}(\dot{t}) &= (\ddot{p}|_{\ddot{F}(\dot{N})\smallsetminus\{W\}})^{-1}(\dot{t}) \\ &= \{\ \dot{t}\cap(\cap_{i\in\ddot{J}(\dot{t})}\dot{c}_i)\mid(\dot{c}_i)_{i\in\ddot{J}(\dot{t})}\in\ddot{A}(\dot{t})\ \} \\ &= \{\ \dot{t}\cap(\cap_{i\in\ddot{J}(\dot{t})}\dot{c}_i)\mid(\dot{c}_i)_{i\in J(\dot{t})}\in\Pi_{i\in\ddot{J}(\dot{t})}\ddot{A}_i(\dot{t})\ \} \\ &= \{\ \dot{t}\cap(\cap_{i\in\{i^*\}}\dot{c}_i)\mid(\dot{c}_i)_{i\in\{i^*\}}\in\Pi_{i\in\{i^*\}}\ddot{A}_i(\dot{t})\ \} \\ &= \{\ \dot{t}\cap\dot{c}\mid\dot{c}\in\ddot{A}_{i^*}(\dot{t})\ \} \\ &= \{\ \dot{t}\cap\dot{c}\mid\dot{c}\in\dot{F}(\dot{t})\cap\dot{C}_{i^*}\ \} \\ &= \{\ \dot{t}\cap\dot{c}\mid\dot{c}\in\dot{F}(\dot{t})\ \}\ ,\end{aligned}$$

whose eight equalities are justified as follows. The first equality holds by (44c). The second equality holds by AR Definition 6 (DEF.ii) under the presumption that $\ddot{p}^{-1}(W)$ on left-hand side of (DEF.ii) was not meant to contain W itself (see the third paragraph of Remark B.6). The third equality holds by the definition of $\ddot{A}$ in AR Definition 6 (DEF.ii), and the fourth by the definition of i^* above. The fifth is a rearrangement, the sixth holds by (45), and the seventh holds by (48).

(8d). Suppose that $\dot{t}$, $\dot{c}^1$, and $\dot{c}^2$ were such that $\{\dot{c}^1,\dot{c}^2\}\subseteq\dot{F}(\dot{t})$ and $\dot{c}^1\cap\dot{c}^2\neq\varnothing$. Then since $\dot{t}$ belongs to both $\dot{F}^{-1}(\dot{c}^1)$ and $\dot{F}^{-1}(\dot{c}^2)$, we have $\dot{F}^{-1}(\dot{c}^1)\cap\dot{F}^{-1}(\dot{c}^2)\neq\varnothing$. Thus by (44d), we have $\ddot{P}(\dot{c}^1)\cap\ddot{P}(\dot{c}^2)\neq\varnothing$. This and the assumption $\dot{c}^1\cap\dot{c}^2\neq\varnothing$ imply $\dot{c}^1=\dot{c}^2$ by AR Definition 6 (DEF.i).

(8e). By Lemma A.5, (8e) is equivalent to

$$(\forall\dot{c}^1,\dot{c}^2)\ \dot{F}^{-1}(\dot{c}^1){=}\dot{F}^{-1}(\dot{c}^2)\text{ or }\dot{F}^{-1}(\dot{c}^1)\cap\dot{F}^{-1}(\dot{c}^2){=}\varnothing\ .$$

To prove this, take any $\dot{c}^1$ and $\dot{c}^2$. On the one hand, if $\dot{c}^1{=}\dot{c}^2$, the first contingency holds. On the other hand, suppose $\dot{c}^1{\neq}\dot{c}^2$. Further suppose that the second contingency fails: $\dot{F}^{-1}(\dot{c}^1)\cap\dot{F}^{-1}(\dot{c}^2){\neq}\varnothing$. By (44d), $\ddot{P}(\dot{c}^1)\cap\ddot{P}(\dot{c}^2){\neq}\varnothing$. Thus by $\dot{c}^1{\neq}\dot{c}^2$ and AR Definition 6 (DEF.i), $\ddot{P}(\dot{c}^1){=}\ddot{P}(\dot{c}^2)$. Hence by (44d) again, $\dot{F}^{-1}(\dot{c}^1){=}\dot{F}^{-1}(\dot{c}^2)$. Thus the first contingency holds.

(c). I show the three components of the definition (11) of an AR* outcome-set form. (11a) holds by part (b).

(11b). Consider any $\dot{c} \in \dot{F}(\dot{X})$. Then there exists $\dot{t} \in \dot{X}$ such that $\dot{c} \in \dot{F}(\dot{t})$. Since $\dot{t} \in \dot{X}$, we may repeat the first two paragraphs in the above argument for (8c), in order to obtain an i^* for which (47) holds. (47) implies (11b).

(11c). Take any i and any $\dot{t} \in \dot{X}$. Since $\dot{t} \in \dot{X}$, we may repeat the first two paragraphs in the above argument for (8c), in order to obtain an i^* for which (47) and (48) hold. If $i = i^*$, then (48) implies $\dot{F}(\dot{t}) \subseteq \dot{C}_i$. If $i \neq i^*$, then (47) implies $\dot{F}(\dot{t}) \cap \dot{C}_i = \varnothing$. □

Proof B.10 (for Theorem 1)**.** The theorem follows immediately from Lemma B.8(b) and Lemma B.9(c). □

APPENDIX C. GENERAL RESULTS ABOUT CONCISENESS

This appendix proves Theorem 2 (Section 4) as well as all the lemmata and propositions in Sections 3 and 4. Like every appendix except Appendix B, it does not rely on `AR`.

C.1. FOR `AR`* PREFORMS

Proof C.1 (for Lemma 3.1)**.** I will argue that $(W, \dot{N}, \dot{C})$ is not concise iff $\dot{C} \neq \dot{F}(\dot{X})$ or $(\exists \dot{c} \in \dot{F}(\dot{X}))$ $\mathtt{M}(\dot{c}) \neq \dot{c}$. By the definition (13) of conciseness, non-conciseness is equivalent to

(50a) $$(\exists \dot{c} \notin \dot{F}(\dot{X}))\ \dot{c} \not\subseteq \cup \dot{F}^{-1}(\dot{c}) \text{ or}$$

(50b) $$(\exists \dot{c} \in \dot{F}(\dot{X}))\ \dot{c} \not\subseteq \cup \dot{F}^{-1}(\dot{c})\ .$$

(50a) is equivalent to the existence of a $\dot{c} \notin \dot{F}(\dot{X})$ because [1] $\dot{c} \neq \varnothing$ for every $\dot{c}$ and [2] $\dot{F}^{-1}(\dot{c}) = \varnothing$ for every $\dot{c} \notin \dot{F}(\dot{X})$ since $\dot{X}$ is the domain of $\dot{F}$ by Lemma A.4. The existence of a $\dot{c} \notin \dot{F}(\dot{X})$ is equivalent to $\dot{C} \neq \dot{F}(\dot{X})$ because $\dot{C}$ is always a superset of $\dot{F}(\dot{X})$ by the definition of $\dot{F}$. Meanwhile, (50b) is equivalent to $(\exists \dot{c} \in \dot{F}(\dot{X}))$ $\dot{c} \cap \cup \dot{F}^{-1}(\dot{c}) \subset \dot{c}$. By the definition of `M`, this is equivalent to $(\exists \dot{c} \in \dot{F}(\dot{X}))$ $\mathtt{M}(\dot{c}) \neq \dot{c}$. □

Lemma C.2. *(a) Suppose $(W, \dot{N})$ is an `AR`* outcome-set tree (1) with its $\dot{T}$ (2) and $\dot{p}$ (7). Further suppose $W \supset \dot{t}$. Then there exists a unique integer $K \geq 1$ such that the sequence $(\dot{p}^k(\dot{t}))_{k=1}^K$ is well-defined and $W = \dot{p}^K(\dot{t})$.*

(b) Suppose $(W, \dot{N}, \dot{C})$ is an `AR` outcome-set preform (8) with its $\dot{T}$ (2), $\dot{p}$ (7), and $\dot{F}$ (9). Further suppose $W \supset \dot{c} \supseteq \dot{t}$ and define K as*

in part (a). Then there exists an integer L such that $K \geq L \geq 1$ and $\dot{c} \in \dot{F}(\dot{p}^L(\dot{t}))$.

Proof. (a). Define $\dot{X}$ (4). Since $\dot{t} \in \dot{T} \smallsetminus \{W\}$, Lemma A.1, Lemma A.3, and equation (5) together imply that $\dot{p}(\dot{t})$ exists and belongs to $\dot{T}$.

This paragraph considers any $k \geq 1$ and shows that if $\dot{p}^k(\dot{t})$ exists and belongs to $\dot{T}$, then either [1] $\dot{p}^k(\dot{t}) = W$ or [2] $\dot{p}^{k+1}(\dot{t})$ exists and belongs to $\dot{T}$. Accordingly, assume $\dot{p}^k(\dot{t})$ exists and belongs to $\dot{T}$. Suppose contingency [1] fails. Then $\dot{p}^k(\dot{t}) \in \dot{T} \smallsetminus \{W\}$, so Lemma A.1, Lemma A.3, and equation (5) together imply that $\dot{p}^{k+1}(\dot{t})$ exists and belongs to $\dot{T}$. Thus contingency [2] holds.

By beginning with the last sentence of the first paragraph, and iteratively applying the first sentence of the second paragraph, one finds that either [1] there exists a $K \geq 1$ such that $(\dot{p}^k(\dot{t}))_{k=1}^K$ is well-defined and $W = \dot{p}^K(\dot{t})$, or [2] the infinite sequence $(\dot{p}^k(\dot{t}))_{k \geq 1}$ is well-defined. Since each $\dot{p}^{k+1}(\dot{t}) \supset \dot{p}^k(\dot{t})$ by the definition of $\dot{p}$, the second contingency would imply that $\{ \dot{p}^k(\dot{t}) \mid k \geq 1 \}$ is an infinite collection of predecessors of $\dot{t}$. This contradicts the definition of $\dot{T}$. Hence the first contingency must hold.

It remains to be shown that K is unique. Accordingly suppose there were $K' > K$ such that both $\dot{p}^{K'}(\dot{t})$ and $\dot{p}^K(\dot{t})$ equal W. Then $W = \dot{p}^{K'-K}(W)$. This contradicts $\dot{p}^{K'-K}(W) \supset W$ which follows from $K' > K$ and the definition of $\dot{p}$.

(b). By the definition of K and part (a), $(\dot{p}^k(\dot{t}))_{k=1}^K$ is well-defined and $\dot{p}^K(\dot{t}) = W$. Define $\dot{p}^0(\dot{t}) = \dot{t}$ so that the sequence $(\dot{p}^k(\dot{t}))_{k=0}^K$ becomes well-defined. Note that

$$\dot{c} \not\supseteq \dot{p}^K(\dot{t}) \ , \tag{51}$$

because $\dot{p}^K(\dot{t}) = W$ and $W \supset \dot{c}$ by assumption. Also note that

$$\dot{c} \supseteq \dot{p}^0(\dot{t}) \tag{52}$$

because $\dot{p}^0(\dot{t}) = \dot{t}$ and $\dot{c} \supseteq \dot{t}$ by assumption. Because of (52), we may let $k^* = \max\{ k \geq 0 \mid \dot{c} \supseteq \dot{p}^k(\dot{t}) \}$. Further $k^* < K$ because of (51). Hence

$$\dot{c} \not\supseteq p^{k^*+1}(\dot{t}) \text{ and } \dot{c} \supseteq p^{k^*}(\dot{t}) \ .$$

Thus $\dot{c} \in \dot{F}(p^{k^*+1}(\dot{t}))$ by the definition of $\dot{F}$. Set $L = k^*+1$. □

Proof C.3. (for Lemma 3.2)

$\Rightarrow$ *direction.* See the paragraph following the lemma statement.

$\Leftarrow$ *direction.* Derive $\dot{F}$ (9) from $(W, \dot{N}, \dot{C})$. Suppose $W \notin \dot{C}$. Then take any $\dot{c}$ and any $w \in \dot{c}$. Note that $W \supset \dot{c} \supseteq \{w\}$ and that $\{w\} \in \dot{T}$ because of the finite-horizon assumption. Thus by Lemma C.2(b), there exists an integer $L{\geq}1$ such that $\dot{c} \in \dot{F}(\dot{p}^L(\{w\}))$. Equivalently, $\dot{p}^L(\{w\}) \in \dot{F}^{-1}(\dot{c})$. Thus $w \in \dot{p}^L(\{w\}) \subseteq \cup\dot{F}^{-1}(\dot{c})$. Since this has been shown to hold for any $\dot{c}$ and any $w \in \dot{c}$, $(W, \dot{N}, \dot{C})$ satisfies the definition of conciseness. □

Lemma C.4. *Suppose* $(W, \dot{N}, \dot{C})$ *is an* AR* *outcome-set preform (8) with its* $\dot{X}$ *(4) and* $\dot{F}$ *(9). Then the following hold.*
(a) $\dot{F}$ *equals the* $\dot{F}'$ *(9) derived from* $(W, \dot{N}, \dot{F}(\dot{X}))$.
(b) $(W, \dot{N}, \dot{F}(\dot{X}))$ *is an* AR* *outcome-set preform.*

Proof. Also derive $\dot{T}$ (2) and $\dot{p}$ (7). Note that $\dot{T}$, $\dot{X}$, and $\dot{p}$ depend only on the tree $(W, \dot{N})$.

(a). I argue

$$\begin{aligned} \dot{F} &= \{\ (\dot{t},\dot{c}) \mid \dot{c}{\not\supseteq}\dot{t} \text{ and } (\exists \dot{t}^\sharp{\in}\dot{p}^{-1}(\dot{t}))\ \dot{c}{\supseteq}\dot{t}^\sharp\ \} \\ &= \{\ (\dot{t},\dot{c}) \mid \dot{c}{\not\supseteq}\dot{t} \text{ and } (\exists \dot{t}^\sharp{\in}\dot{p}^{-1}(\dot{t}))\ \dot{c}{\supseteq}\dot{t}^\sharp\ \}|_{\dot{T}\times\dot{F}(\dot{X})} \\ &= \dot{F}'\ . \end{aligned}$$

The first equality is the definition of $\dot{F}$, and the third equality is the definition of $\dot{F}'$. To see the second equality, note that [1] $\dot{X}$ is the domain of $\dot{F}$ by Lemma A.4(b), and thus [2] $\dot{F}(\dot{X})$ is the range of $\dot{F}$.

(b). By assumption, $(W, \dot{N}, \dot{C})$ satisfies the five components of (8). This paragraph derives the five components for $(W, \dot{N}, \dot{F}(\dot{X}))$. (8a) follows from (8a) for $(W, \dot{N}, \dot{C})$ because (8a) only concerns the tree $(W, \dot{N})$. (8b) follow from (8b) for $(W, \dot{N}, \dot{C})$ because $\dot{F}(\dot{X}) \subseteq \dot{C}$. (8c-e) follow from (8c-e) for $(W, \dot{N}, \dot{C})$ by part (a). □

Lemma C.5. *Let* $(W, \dot{N}, \dot{C})$ *be an* AR* *outcome-set preform (8) with its* $\dot{T}$ *(2),* $\dot{X}$ *(4), and* $\dot{F}$ *(9). Let* $\dot{C}^{\mathtt{M}} := \mathtt{M}(\dot{F}(\dot{X}))$*. Then*
(a) M *is a bijection from* $\dot{F}(\dot{X})$ *onto* $\dot{C}^{\mathtt{M}}$*, and*
(b) $\{\,(\dot{t}, \mathtt{M}(\dot{c}))\,|\,(\dot{t},\dot{c}){\in}\dot{F}\,\}$ *equals the* $\dot{F}^{\mathtt{M}}$ *(9) from* $(W, \dot{N}, \dot{C}^{\mathtt{M}})$.

Proof. (a). By construction, M is a function from $\dot{F}(\dot{X})$ onto $\mathtt{M}(\dot{F}(\dot{X}))$, which is $\dot{C}^{\mathtt{M}}$ by definition. Thus it remains to be shown that M is injective. Accordingly, suppose that $\dot{c}^1$ and $\dot{c}^2$ are two members of $\dot{F}(\dot{X})$ such that $\mathtt{M}(\dot{c}^1) = \mathtt{M}(\dot{c}^2)$. I will show that $\dot{c}^1 = \dot{c}^2$.

I start with two observations. Firstly, since $\mathtt{M}(\dot{c}^1) = \mathtt{M}(\dot{c}^2)$, the definition of $\mathtt{M}$ implies

$$\dot{c}^1 \cap \cup\dot{F}^{-1}(\dot{c}^1) = \dot{c}^2 \cap \cup\dot{F}^{-1}(\dot{c}^2) \ . \tag{53}$$

Secondly, and independently, since $\dot{c}^1$ and $\dot{c}^2$ are members of $\dot{F}(\dot{X})$, we may take $\dot{t}^1$ and $\dot{t}^2$ such that $\dot{c}^1 \in \dot{F}(\dot{t}^1)$ and $\dot{c}^2 \in \dot{F}(\dot{t}^2)$.

This paragraph shows that $\dot{c}^2 \not\supseteq \dot{t}^1$. Suppose hypothetically that $\dot{c}^2 \supseteq \dot{t}^1$. Note that $W \supset \dot{c}^2$ since $\dot{c}^2 \not\supseteq \dot{t}^2$ by $\dot{c}^2 \in \dot{F}(\dot{t}^2)$ and the definition of $\dot{F}$. By the last two sentences, Lemma C.2(b) implies that there exists an integer $L{\geq}1$ such that $\dot{c}^2 \in \dot{F}(\dot{p}^L(\dot{t}^1))$. Note that

$$\cup\dot{F}^{-1}(\dot{c}^2) \supseteq \dot{p}^L(\dot{t}^1) \supseteq \dot{t}^1 \ ,$$

where the first inclusion holds because $\dot{F}^{-1}(\dot{c}^2) \ni \dot{p}^L(\dot{t})$ by the previous sentence, and the second inclusion holds by the definition of $\dot{p}$. Thus

$$\dot{c}^1 \supseteq \dot{c}^1 \cap \cup\dot{F}^{-1}(\dot{c}^1) = \dot{c}^2 \cap \cup\dot{F}^{-1}(\dot{c}^2) \supseteq \dot{t}^1 \ ,$$

where the first inclusion is obvious, the equality is (53), and the second inclusion follows from [1] the hypothetical assumption that $\dot{c}^2 \supseteq \dot{t}^1$ and [2] the previous sentence. The previous sentence contradicts $\dot{c}^1 \not\supseteq \dot{t}^1$, which follows from $\dot{c}^1 \in \dot{F}(\dot{t}^1)$, which follows from the definition of $\dot{t}^1$.

Now define $\dot{t}^{1^\sharp} = \dot{t}^1 \cap \dot{c}^1$. Since $\dot{c}^1 \in \dot{F}(\dot{t}^1)$ by the definition of $\dot{t}^1$, (8c) implies

$$\dot{t}^{1^\sharp} \in \dot{p}^{-1}(\dot{t}^1) \ . \tag{54}$$

Further,

$$\dot{c}^2 \supseteq \dot{c}^2 \cap \cup\dot{F}^{-1}(\dot{c}^2) = \dot{c}^1 \cap \cup\dot{F}^{-1}(\dot{c}^1) \supseteq \dot{c}^1 \cap \dot{t}^1 = \dot{t}^{1^\sharp} \ . \tag{55}$$

where the first inclusion is obvious, the first equality holds by (53), the second inclusion holds because $\dot{F}^{-1}(\dot{c}^1) \ni \dot{t}^1$ by the definition of $\dot{t}^1$, and the final equality holds by the definition of $\dot{t}^{1^\sharp}$. By the last paragraph, (54), and (55), we have that $\dot{c}^2 \not\supseteq \dot{t}^1$, $\dot{t}^{1^\sharp} \in \dot{p}^{-1}(\dot{t}^1)$, and $\dot{c}^2 \supseteq \dot{t}^{1^\sharp}$. Thus the definition of $\dot{F}$ implies $\dot{c}^2 \in \dot{F}(\dot{t}^1)$.

At this point, we have [1] $\dot{c}^1 \in \dot{F}(\dot{t}^1)$ by the definition of $\dot{t}^1$, [2] $\dot{c}^2 \in \dot{F}(\dot{t}^1)$ by the previous sentence, [3] $\dot{c}^1 \supseteq \dot{t}^{1^\sharp}$ by the definition of $\dot{t}^{1^\sharp}$, and [4] $\dot{c}^2 \supseteq \dot{t}^{1^\sharp}$ by (55). Facts [3] and [4] imply that $\dot{c}^1 \cap \dot{c}^2$ is nonempty. Hence facts [1] and [2] imply that $\dot{c}^1 = \dot{c}^2$ by (8d) applied at $\dot{t}^1$.

(b). This paragraph shows $\dot{F}^{\mathtt{M}} \subseteq \{\, (\dot{t}, \mathtt{M}(\dot{c})) \,|\, (\dot{t},\dot{c}) \in \dot{F} \,\}$. Take any $(\dot{t}, \dot{c}^{\mathtt{M}}) \in \dot{F}^{\mathtt{M}}$. By the definition (9) of $\dot{F}^{\mathtt{M}}$, there is some $\dot{t}^{\sharp}$ such that

(56a) $$\dot{t} = \dot{p}(\dot{t}^{\sharp}) \ ,$$

(56b) $$\dot{c}^{\mathtt{M}} \not\supseteq \dot{t} \ , \text{ and}$$

(56c) $$\dot{c}^{\mathtt{M}} \supseteq \dot{t}^{\sharp} \ .$$

By the definition of $\dot{C}^{\mathtt{M}}$, there is a $\dot{c}$ such that (a) $\dot{c} \in \dot{F}(\dot{X})$ and (b) $\dot{c}^{\mathtt{M}} = \mathtt{M}(\dot{c})$. By (a) and (15),

(57) $$W \supset \dot{c} \ .$$

By (b) and the definition of $\mathtt{M}$, $\dot{c}^{\mathtt{M}} = \dot{c} \cap \cup \dot{F}^{-1}(\dot{c})$. By this equality, (56b) and (56c) imply

(58) $$\dot{c} \cap \cup \dot{F}^{-1}(\dot{c}) \not\supseteq \dot{t} \text{ and}$$

(59) $$\dot{c} \cap \cup \dot{F}^{-1}(\dot{c}) \supseteq \dot{t}^{\sharp} \ .$$

(59) trivially implies that

(60) $$\dot{c} \supseteq \dot{t}^{\sharp} \ .$$

Lemma C.2(b), (57), and (60) together imply the existence of an $L \geq 1$ such that $\dot{c} \in \dot{F}(\dot{p}^L(\dot{t}^{\sharp}))$. Hence $\dot{F}^{-1}(\dot{c}) \ni \dot{p}^L(\dot{t}^{\sharp})$. Thus

$$\cup \dot{F}^{-1}(\dot{c}) \supseteq \dot{p}^L(\dot{t}^{\sharp}) \supseteq \dot{p}(\dot{t}^{\sharp}) = \dot{t} \ ,$$

where the first inclusion follows from the last sentence, the second inclusion follows from the definition of $\dot{p}$, and the final equality is (56a). This and (58) imply that $\dot{c} \not\supseteq \dot{t}$. (56a), (60), and the previous sentence imply that $(\dot{t}, \dot{c}) \in \dot{F}$ by the definition of $\dot{F}$.

Conversely, this paragraph shows $\dot{F}^{\mathtt{M}} \supseteq \{\, (\dot{t}, \mathtt{M}(\dot{c})) \,|\, (\dot{t},\dot{c}) \in \dot{F} \,\}$. Take any $(\dot{t}, \dot{c}) \in \dot{F}$. By the definition of $\dot{F}$, there is some $\dot{t}^{\sharp}$ such that

(61a) $$\dot{t} = \dot{p}(\dot{t}^{\sharp}) \ ,$$

(61b) $$\dot{c} \not\supseteq \dot{t} \ , \text{ and}$$

(61c) $$\dot{c} \supseteq \dot{t}^{\sharp} \ .$$

(61b) implies that $\dot{c} \cap \cup \dot{F}^{-1}(\dot{c}) \not\supseteq \dot{t}$. By the definition of $\mathtt{M}$, this is equivalent to

(62) $$\mathtt{M}(\dot{c}) \not\supseteq \dot{t} \ .$$

Further note that

$$\cup \dot{F}^{-1}(\dot{c}) \supseteq \dot{t} = \dot{p}(\dot{t}^{\sharp}) \supseteq \dot{t}^{\sharp} \ ,$$

where the first inclusion holds since $\dot{F}^{-1}(\dot{c}) \ni \dot{t}$ by the definition of $\dot{t}$, the equality is (61a), and the last inclusion holds by the definition of $\dot{p}$. The previous sentence and (61c) imply that

$$\dot{c} \cap \cup\dot{F}^{-1}(\dot{c}) \supseteq \dot{t}^{\sharp} \ .$$

By the definition of $\mathtt{M}$, this implies $\mathtt{M}(\dot{c}) \supseteq \dot{t}^{\sharp}$. (61a), (62), and the previous sentence imply that $(\dot{t}, \mathtt{M}(\dot{c})) \in \dot{F}^{\mathtt{M}}$ by the definition of $\dot{F}^{\mathtt{M}}$. □

Lemma C.6. *Let $(W, \dot{N}, \dot{C})$ be an $\mathtt{AR}^*$ outcome-set preform (8) with its $\dot{X}$ (4) and $\dot{F}$ (9). Let $\dot{C}^{\mathtt{M}} := \mathtt{M}(\dot{F}(\dot{X}))$. Then $(W, \dot{N}, \dot{C}^{\mathtt{M}})$ is a concise (13) outcome-set preform.*

Proof. The next five paragraphs will show that $(W, \dot{N}, \dot{C}^{\mathtt{M}})$ is an $\mathtt{AR}^*$ outcome-set preform by deriving the five parts of (8). The final three paragraphs will show conciseness. Note that the five parts of (8) hold for $(W, \dot{N}, \dot{F}(\dot{X}))$ because $(W, \dot{N}, \dot{F}(\dot{X}))$ is an $\mathtt{AR}^*$ outcome-set preform by Lemma C.4(b).

*(*8a). This follows from (8a) for $(W, \dot{N}, \dot{F}(\dot{X}))$ simply because (8a) only concerns the tree $(W, \dot{N})$.

*(*8b). Take any $\dot{c}^{\mathtt{M}}$. By the definition of $\dot{C}^{\mathtt{M}}$ there exist $\dot{c}$ and $\dot{t}$ such that $c^{\mathtt{M}} = \mathtt{M}(\dot{c})$ and $\dot{c} \in \dot{F}(\dot{t})$. I argue $c^{\mathtt{M}}$ is a subset of W because

$$\dot{c}^{\mathtt{M}} = \mathtt{M}(\dot{c}) = \dot{c} \cap \cup\dot{F}^{-1}(\dot{c}) \subseteq \dot{c} \subseteq W \ .$$

The first equality holds by the definition of $\dot{c}$ and the second equality holds by the definition of $\mathtt{M}$. The final inclusion holds by (8b) for $(W, \dot{N}, \dot{F}(\dot{X}))$. Further, I argue $c^{\mathtt{M}}$ is nonempty because

$$\dot{c}^{\mathtt{M}} = \mathtt{M}(\dot{c}) = \dot{c} \cap \cup\dot{F}^{-1}(\dot{c}) \supseteq \dot{c} \cap \dot{t} \neq \varnothing \ .$$

The first equality holds by the definition of $\dot{c}$ and the second equality holds by the definition of $\mathtt{M}$. The inclusion holds because $\cup\dot{F}^{-1}(\dot{c}) \supseteq \dot{t}$ because $\dot{F}^{-1}(\dot{c}) \ni \dot{t}$ by the definitions of $\dot{c}$ and $\dot{t}$. The inequality holds because $\dot{c} \cap \dot{t}$ is an immediate successor of $\dot{t}$ by [1] $\dot{c} \in \dot{F}(\dot{t})$ from the definition of $\dot{c}$ and $\dot{t}$ and [2] (8c) for $(W, \dot{N}, \dot{F}(\dot{X}))$.

*(*8c). I argue that

$$\begin{aligned}(\forall \dot{t} \in \dot{X}) \ \ \dot{p}^{-1}(\dot{t}) &= \{\ \dot{t} \cap \dot{c} \mid \dot{c} \in \dot{F}(\dot{t})\ \} \\ &= \{\ \dot{t} \cap \dot{c} \cap \cup\dot{F}^{-1}(\dot{c}) \mid \dot{c} \in \dot{F}(\dot{t})\ \} \\ &= \{\ \dot{t} \cap \mathtt{M}(\dot{c}) \mid \dot{c} \in \dot{F}(\dot{t})\ \}\end{aligned}$$

$$= \{\ \dot{t}\cap\mathtt{M}(\dot{c})\mid \mathtt{M}(\dot{c})\in\dot{F}^{\mathtt{M}}(\dot{t}) \text{ and } \dot{c}\in\dot{F}(\dot{X})\ \}$$
$$= \{\ \dot{t}\cap\dot{c}^{\mathtt{M}}\mid \dot{c}^{\mathtt{M}}\in\dot{F}^{\mathtt{M}}(\dot{t})\ \}\ .$$

The first equality is (8c) for $(W, \dot{N}, \dot{F}(\dot{X}))$. The second holds because, for any $\dot{c} \in \dot{F}(\dot{t})$, we have $\dot{F}^{-1}(\dot{c}) \ni \dot{t}$ and thus $\cup\dot{F}^{-1}(\dot{c}) \supseteq \dot{t}$. The third holds by the definition of $\mathtt{M}$. To see the fourth equality, note [1] the left-hand predicate is equivalent to the first right-hand predicate by Lemma C.5(b) and [2] the left-hand predicate implies the second right-hand predicate since $\dot{t}\in\dot{X}$. The fifth equality holds by Lemma C.5(a).

*(*8d*)*. Consider any $\dot{t} \in \dot{X}$ and any distinct $\dot{c}^{\mathtt{M}}_A$ and $\dot{c}^{\mathtt{M}}_B$ in $\dot{F}^{\mathtt{M}}(\dot{t})$. By Lemma C.5(a),

$$\mathtt{M}^{-1}(\dot{c}^{M}_A) \neq \mathtt{M}^{-1}(\dot{c}^{M}_B)\ . \tag{63}$$

Further, by Lemma C.5(b),

$$\mathtt{M}^{-1}(\dot{c}^{M}_A) \in \dot{F}(\dot{t}) \text{ and } \mathtt{M}^{-1}(\dot{c}^{M}_B) \in \dot{F}(\dot{t})\ . \tag{64}$$

I argue

$$\begin{aligned} & \dot{c}^{\mathtt{M}}_A \cap \dot{c}^{\mathtt{M}}_B \\ =\ & \mathtt{M}(\mathtt{M}^{-1}(\dot{c}^{M}_A)) \cap \mathtt{M}(\mathtt{M}^{-1}(\dot{c}^{\mathtt{M}}_B)) \\ =\ & (\mathtt{M}^{-1}(\dot{c}^{\mathtt{M}}_A) \cap \cup F^{-1}(\mathtt{M}^{-1}(\dot{c}^{\mathtt{M}}_A)) \cap (\mathtt{M}^{-1}(\dot{c}^{\mathtt{M}}_B) \cap \cup F^{-1}(\mathtt{M}^{-1}(\dot{c}^{\mathtt{M}}_B)) \\ \subseteq\ & \mathtt{M}^{-1}(\dot{c}^{\mathtt{M}}_A) \cap \mathtt{M}^{-1}(\dot{c}^{\mathtt{M}}_B) \\ =\ & \varnothing\ . \end{aligned}$$

The first equality follows from Lemma C.5(a). The second follows from the definition of $\mathtt{M}$. The set inclusion follows from elementary manipulation. The last equality follows from (63), from (64), and from (8d) for $(W, \dot{N}, \dot{F}(\dot{X}))$.

*(*8e*)*. Take any two $\dot{t}^1$ and $\dot{t}^2$. By (8e) for $(W, \dot{N}, \dot{F}(\dot{X}))$,

$$\dot{F}(\dot{t}^1) = \dot{F}(\dot{t}^2) \ \text{ or } \ \dot{F}(\dot{t}^1)\cap\dot{F}(\dot{t}^2) = \varnothing\ .$$

By Lemma C.5(a), this implies (and is in fact equivalent to)

$$\mathtt{M}(\dot{F}(\dot{t}^1)) = \mathtt{M}(\dot{F}(\dot{t}^2)) \ \text{ or } \ \mathtt{M}(\dot{F}(\dot{t}^1))\cap\mathtt{M}(\dot{F}(\dot{t}^2)) = \varnothing\ .$$

By Lemma C.5(b), this implies (and is in fact equivalent to),

$$\dot{F}^{\mathtt{M}}(\dot{t}^1) = \dot{F}^{\mathtt{M}}(\dot{t}^2) \ \text{ or } \ \dot{F}^{\mathtt{M}}(\dot{t}^1)\cap\dot{F}^{\mathtt{M}}(\dot{t}^2) = \varnothing\ .$$

Conciseness. First, I argue

$$\dot{F}^{\mathtt{M}}(\dot{X}) = \mathtt{M}(\dot{F}(\dot{X})) = \dot{C}^{\mathtt{M}}\ . \tag{65}$$

The first equality holds by Lemma C.5(b). The second equality is the definition of $\dot{C}^{\mathtt{M}}$.

Second, let $\mathtt{M}^{\mathtt{M}}$ be the material-part function (14) derived from $(W, \dot{N}, \dot{C}^{\mathtt{M}})$. I argue

$$(66) \qquad (\forall \dot{c}^{\mathtt{M}}) \;\; \mathtt{M}^{\mathtt{M}}(\dot{c}^{\mathtt{M}}) = \dot{c}^{\mathtt{M}} \cap \cup(\dot{F}^{\mathtt{M}})^{-1}(\dot{c}^{\mathtt{M}})$$
$$= \dot{c}^{\mathtt{M}} \cap \cup\dot{F}^{-1}(\mathtt{M}^{-1}(\dot{c}^{\mathtt{M}}))$$
$$= \mathtt{M}(\mathtt{M}^{-1}(\dot{c}^{\mathtt{M}})) \cap \cup\dot{F}^{-1}(\mathtt{M}^{-1}(\dot{c}^{\mathtt{M}}))$$
$$= \mathtt{M}^{-1}(\dot{c}^{\mathtt{M}}) \cap \cup\dot{F}^{-1}(\mathtt{M}^{-1}(\dot{c}^{\mathtt{M}})) \cap \cup\dot{F}^{-1}(\mathtt{M}^{-1}(\dot{c}^{\mathtt{M}}))$$
$$= \mathtt{M}^{-1}(\dot{c}^{\mathtt{M}}) \cap \cup\dot{F}^{-1}(\mathtt{M}^{-1}(\dot{c}^{\mathtt{M}}))$$
$$= \mathtt{M}(\mathtt{M}^{-1}(\dot{c}^{\mathtt{M}}))$$
$$= \dot{c}^{\mathtt{M}} \,,$$

The first equality in (66) is the definition of $\mathtt{M}^{\mathtt{M}}$. The second equality holds by Lemma C.5(b), and the third holds by Lemma C.5(a). The fourth and sixth equalities hold by the definition of $\mathtt{M}$. The last equality holds by Lemma C.5(a).

By Lemma 3.1, (65) and (66) imply that $(W, \dot{N}, \dot{C}^{\mathtt{M}})$ is concise. □

Proof C.7 (for Lemma 3.3)**.** Part (a) follows from Lemma C.6. Parts (b) and (c) follow from Lemma C.5. □

C.2. For $\mathtt{AR}^*$ forms

Proof C.8 (for Proposition 4.1)**.** By definition (in the sentence before (19)), the form $(W, \dot{N}, (\dot{C}_i)_i)$ is concise iff its preform $(W, \dot{N}, \cup_i \dot{C}_i)$ is concise. Thus it must be shown that $(W, \dot{N}, \cup_i \dot{C}_i)$ is concise iff $W \notin \cup_i C_i$. This follows immediately from Lemma 3.2 because of the finite-horizon assumption. □

Proof C.9 (for Theorem 2)**.** As in the theorem's first sentence, let $(W, \dot{N}, (\dot{C}_i)_i)$ be an $\mathtt{AR}^*$ outcome-set form (11), with its $\dot{T}$ (2), $\dot{X}$ (4), and $\dot{F}$ (9). As in the theorem's second and last sentences, derive $\mathtt{M}$ by (14), let $(\dot{C}_i^{\mathtt{M}})_i = (M(\dot{C}_i \cap \dot{F}(\dot{X}))_i$, and derive $\dot{F}^{\mathtt{M}}$ (9) from $(W, \dot{N}, (\dot{C}_i^{\mathtt{M}})_i)$. Further, let $\dot{C} = \cup_i \dot{C}_i$ and $\dot{C}^{\mathtt{M}} = \cup_i \dot{C}_i^{\mathtt{M}}$. Note that

$$(67) \qquad \dot{C}^{\mathtt{M}} = \cup_i \dot{C}_i^{\mathtt{M}}$$
$$= \cup_i \mathtt{M}(\dot{C}_i \cap \dot{F}(\dot{X}))$$
$$= \mathtt{M}(\cup_i [\dot{C}_i \cap \dot{F}(\dot{X})])$$

$$
\begin{aligned}
&= \mathtt{M}((\cup_i \dot{C}_i) \cap \dot{F}(\dot{X})) \\
&= \mathtt{M}(\dot{C} \cap \dot{F}(\dot{X})) \\
&= \mathtt{M}(\dot{F}(\dot{X})) \ ,
\end{aligned}
$$

where first equality is the definition of $\dot{C}^{\mathtt{M}}$, the second equality follows from the definition of $(\dot{C}_i^{\mathtt{M}})_i$, the fifth equality follows from the definition of $\dot{C}$, and the last holds because $\dot{F}(\dot{X}) \subseteq \dot{C}$ by the definition of $\dot{F}$.

This paragraph applies Lemma 3.3. By (11a) and the definition of $\dot{C}$, $(W, \dot{N}, \dot{C})$ is an $\mathtt{AR}^*$ outcome-set preform (8), as assumed by the lemma. The lemma's definitions of $\dot{T}$, $\dot{X}$, and $\dot{F}$ coincide with the definitions in the first sentence of this proof. The lemma's definition of $\dot{C}^{\mathtt{M}}$ coincides with (67). The lemma's definition of $\dot{F}^{\mathtt{M}}$ coincides with the definition in the second sentence of this proof. Hence, the lemma allows us to conclude that

(68a) $\quad (W, \dot{N}, \dot{C}^{\mathtt{M}})$ is a concise (13) $\mathtt{AR}^*$ outcome-set preform ,

(68b) $\quad \mathtt{M}$ is a bijection from $\dot{F}(\dot{X})$ onto $\dot{C}^{\mathtt{M}}$, and

(68c) $$\dot{F}^{\mathtt{M}} = \{\, (\dot{t}, \mathtt{M}(\dot{c})) \,|\, (\dot{t}, \dot{c}) \in \dot{F} \,\} \ .$$

(a). I will show that $(W, \dot{N}, (\dot{C}_i^{\mathtt{M}})_i)$ satisfies (19). To begin, (19a) for $(W, \dot{N}, (\dot{C}_i^{\mathtt{M}})_i)$ follows from (68a) and the definition of $\dot{C}^{\mathtt{M}}$.

To show (19b) for $(W, \dot{N}, (\dot{C}_i^{\mathtt{M}})_i)$, I argue that something slightly stronger holds, namely, that

$$
\begin{aligned}
(\forall i, j)\ \dot{C}_i^{\mathtt{M}} \cap \dot{C}_j^{\mathtt{M}} &= \mathtt{M}(\dot{C}_i \cap \dot{F}(\dot{X})) \cap \mathtt{M}(\dot{C}_j \cap \dot{F}(\dot{X})) \\
&= \mathtt{M}(\ \dot{C}_i \cap \dot{F}(\dot{X}) \cap \dot{C}_j \cap \dot{F}(\dot{X})\ ) \\
&= \mathtt{M}(\ \dot{C}_i \cap \dot{C}_j \cap \dot{F}(\dot{X})\ ) \\
&= \varnothing \ .
\end{aligned}
$$

The first equality follows from the definition of $(\dot{C}_i^{\mathtt{M}})_i$, the second follows from (68b), and the last follows from (11b) for $(W, \dot{N}, (\dot{C}_i)_i)$.

Finally, this and the next three paragraphs will show (19c) for $(W, \dot{N}, (\dot{C}_i^{\mathtt{M}})_i)$. Take any i and any $\dot{t} \in \dot{X}$. By (11c) for $(W, \dot{N}, (\dot{C}_i)_i)$,

(69) $$\dot{F}(\dot{t}) \subseteq \dot{C}_i \ \text{ or } \ \dot{F}(\dot{t}) \cap \dot{C}_i = \varnothing \ .$$

First I argue

(70)
$$
\begin{aligned}
&\dot{F}(\dot{t}) \subseteq \dot{C}_i \\
\Rightarrow\ &\dot{F}(\dot{t}) \subseteq \dot{C}_i \cap \dot{F}(\dot{X})
\end{aligned}
$$

$$\Rightarrow \mathtt{M}(\dot{F}(\dot{t})) \subseteq \mathtt{M}(\dot{C}_i \cap \dot{F}(\dot{X}))$$
$$\Rightarrow \mathtt{M}(\dot{F}(\dot{t})) \subseteq \dot{C}_i^{\mathtt{M}}$$
$$\Rightarrow \dot{F}^{\mathtt{M}}(\dot{t}) \subseteq \dot{C}_i^{\mathtt{M}} .$$

The first implication holds because $\dot{t}\in\dot{X}$ implies $\dot{F}(\dot{t})\subseteq\dot{F}(\dot{X})$. The second is obvious, the third holds by the definition of $\dot{C}_i^{\mathtt{M}}$, and the fourth holds by (68c).

Second I argue

(71)
$$\dot{F}(\dot{t})\cap\dot{C}_i = \varnothing$$
$$\Rightarrow \dot{F}(\dot{t})\cap\dot{F}(\dot{X}) \cap \dot{C}_i\cap\dot{F}(\dot{X}) = \varnothing$$
$$\Rightarrow \mathtt{M}(\ \dot{F}(\dot{t})\cap\dot{F}(\dot{X}) \cap \dot{C}_i\cap\dot{F}(\dot{X})\ ) = \varnothing$$
$$\Rightarrow \mathtt{M}(\dot{F}(\dot{t})\cap\dot{F}(\dot{X})) \cap \mathtt{M}(\dot{C}_i\cap\dot{F}(\dot{X})) = \varnothing$$
$$\Rightarrow \mathtt{M}(\dot{F}(\dot{t})\cap\dot{F}(\dot{X})) \cap \dot{C}_i^{\mathtt{M}} = \varnothing$$
$$\Rightarrow \mathtt{M}(\dot{F}(\dot{t})) \cap \dot{C}_i^{\mathtt{M}} = \varnothing$$
$$\Rightarrow \dot{F}^{\mathtt{M}}(\dot{t}) \cap \dot{C}_i^{\mathtt{M}} = \varnothing .$$

The first two implications are obvious. The third follows from (68b). The fourth follows from the definition of $\dot{C}_i^{\mathtt{M}}$. The fifth holds because $\dot{t}\in\dot{X}$ implies $\dot{F}(\dot{t})\subseteq\dot{F}(\dot{X})$. The last holds by (68c).

To conclude, (69), (70), and (71) together imply

$$\dot{F}^{\mathtt{M}}(\dot{t}) \subseteq \dot{C}_i^{\mathtt{M}} \text{ or } \dot{F}^{\mathtt{M}}(\dot{t}) \cap \dot{C}_i^{\mathtt{M}} = \varnothing ,$$

which is (19c) for $(W, \dot{N}, (\dot{C}_i^{\mathtt{M}})_i)$.

(b–c). These are identical to (68b) and (68c). □

Proof C.10 (for Proposition 4.2)**.** To put it succinctly, the proposition claims that (19) and (20) are equivalent.

Take any $(W, \dot{N}, (\dot{C}_i)_i)$ satisfying (19). (19a) is identical to (20a). (19c) is identical to (20c). Finally, since $(W, \dot{N}, \cup_i\dot{C}_i)$ is concise by (19a), Lemma 3.1 implies $\dot{F}(\dot{X}) = \cup_i\dot{C}_i$. Hence (19b)'s statement that $\dot{C}_i\cap\dot{C}_j\cap\dot{F}(\dot{X}) = \varnothing$ implies (20b)'s statement that $\dot{C}_i$ and $\dot{C}_j$ are disjoint.

Take any $(W, \dot{N}, (\dot{C}_i)_i)$ satisfying (20). (20a) is identical to (19a). (20b) implies (19b). (20c) is identical to (19c). □

Appendix D. Non-concise infinite-horizon examples

This appendix concerns the examples of Section 3.3. All are $\mathtt{AR}^*$ preforms. The most direct means of analyzing these examples is convoluted superficially. To begin, Lemma D.1 in Appendix D.1 establishes basic results about the Cantor-set tree.

Next, Lemma D.2 in Appendix D.2 concerns Example 3's family. Lemma D.3 concerns Example 3 itself. Lemma D.4 concerns Example 1, which happens to be a degenerate case within Example 3's family.

Finally, Lemma D.6 in Appendix D.3 concerns Example 2's family. It uses Lemma D.4 about Example 1 in its proof. Lemma D.7 concerns Example 2 itself.

D.1. The Cantor-set tree

Lemma D.1. *Let* $(W^0,\dot{N}^0)$ *be the Cantor-set tree that is defined along with* S *and* $\mathtt{D}$ *in Section* 3.3*. Then the following hold.*

(a) $\{\mathtt{D}(s)|s\}$ *is the* $\dot{T}^0$ *(2) of* $(W^0,\dot{N}^0)$*.*

(b) $(W^0,\dot{N}^0)$ *is an* $\mathtt{AR}^*$ *outcome-set tree (1).*

(c) $\{\mathtt{D}(s)|s\}$ *is the* $\dot{X}^0$ *(4) of* $(W^0,\dot{N}^0)$*.*

(d) $\{\ (\mathtt{D}(s^\sharp),\mathtt{D}(s))\ |\ s^\sharp{\in}\{s{\oplus}0,s{\oplus}2\}\ \}$ *is the* $\dot{p}^0$ *(7) of* $(W^0,\dot{N}^0)$*.*

Proof. (a). Since $\dot{N}^0=\{\mathtt{D}(s)|s\}\cup\{\{w\}|w\}$ by definition, it suffices to show that [1] every $\mathtt{D}(s)$ has a finite number of predecessors and [2] every $\{w\}$ has an infinite number of predecessors. [1] Take any s. On the one hand, if $s=\{\}$, $\mathtt{D}(s)=W^0$, which has no predecessors. On the other hand, if $s=(s_i)_{i=1}^m$ has m elements, the collection of its predecessors is $\{\mathtt{D}(\{\})\}\cup\{\mathtt{D}((s_i)_{i=1}^j)|1{\leq}j{<}m\}$, which is finite. [2] Take any $\{w\}$. Since $w\in W^0$, it has a infinite base-3 expansion $(x_i)_{i\geq 1}$ listing 0's and 2's. Thus the collection of $\{w\}$'s predecessors is $\{\mathtt{D}((x_i)_{i=1}^j)|1{\leq}j\}$, which is infinite.

(b). Since $\dot{N}^0=\{\mathtt{D}(s)|s\}\cup\{\{w\}|w\}$ by definition, conditions (1a) and (1c) follow by inspection, the fact that $\mathtt{D}(\{\})=W^0$, and the fact that every $\mathtt{D}(s)$ is nonempty.

(1b). Take any $\dot{n}^1$ and $\dot{n}^2$ in $\dot{N}^0$. I consider three cases. [1] Suppose that either $\dot{n}^1$ or $\dot{n}^2$ equals W^0. Then distinctness implies that exactly one of the nodes is W^0 and that this node precedes the other. [2] Suppose that either n^1 or n^2 is a singleton. Then distinctness and

nonempty intersection imply that exactly one of the nodes is a singleton and that this node succeeds the other. [3] Suppose that neither node is W^0 and that neither node is a singleton. Then there exist nonempty s^1 and s^2 such that $\dot{n}^1 = \mathtt{D}(s^1)$ and $\dot{n}^2 = \mathtt{D}(s^2)$. Four subcases arise: [a] $s^1 = s^2$, [b] $s^1 \subset s^2$, [c] $s^1 \supset s^2$, or [d] there exists some i, no larger than the minimum of the lengths of s^1 and s^2, such that $s^1_i \neq s^2_i$. In case [a], $n^1{=}n^2$. In case [b], $\dot{n}^1 \supset \dot{n}^2$. In case [c], $\dot{n}^1 \subset \dot{n}^2$. In case [d], $\dot{n}^1{\cap}\dot{n}^2 = \varnothing$.

(1d). Let $\dot{N}^* \subseteq \dot{N}^0$ be a nonempty chain. On the one hand, suppose $\dot{N}^*$ contains a singleton $\{w'\}$. Because $\dot{N}^*$ is a chain of nonempty sets, $\cap\dot{N}^* = \{w'\}$. Therefore, since $\dot{N}^0$ contains all singletons, $\cap\dot{N}^* \in \dot{N}^0$. On the other hand, suppose $\dot{N}^*$ contains no singletons. Then by the definition of $\dot{N}^0$, we have the existence of a collection S^* such that $\dot{N}^* = \{\mathtt{D}(s)|s{\in}S^*\}$. Since $\dot{N}^*$ is a nonempty chain, and since $\mathtt{D}(s^1){\supseteq}D(s^2)$ iff $s^1{\subseteq}s^2$, S^* is also a nonempty chain. If S^* is finite, then $\cap\dot{N}^*$ is $\mathtt{D}(\cup S^*)$, where $\cup S^*$ is the longest element of S^*. If S^* is infinite, then $\cap\dot{N}^*$ is the singleton containing the w whose base-3 representation is (decimal) $\cup S^*$. Therefore, regardless of whether S^* is finite or infinite, $\cap\dot{N}^* \in \dot{N}^0$.

(1e). This follows from part (a) and the definition of $\dot{N}^0$.

(c). I argue

$$\dot{X}^0 = \dot{N}^0 \smallsetminus \{\{w\}|w\} = \{\mathtt{D}(s)|s\}\cup\{\{w|w\}\}\smallsetminus\{\{w|w\}\} = \{\mathtt{D}(s)|s\}\ .$$

The first equality is the definition of $\dot{X}^0$. The second holds by the definition of $\dot{N}^0$. The third holds because $\{\mathtt{D}(s)|s\}\cap\{\{w\}|w\} = \varnothing$, which holds because every $\mathtt{D}(s)$ has more than one element.

(d). I argue

$$\begin{aligned}\dot{p}^0 &= \{\ (\dot{t}^\sharp, \dot{t})\ |\ \dot{t}^\sharp{\neq}W^0,\ \dot{t}{=}\min\{\dot{t}'|\dot{t}'{\supset}\dot{t}^\sharp\}\ \}\\ &= \{\ (\mathtt{D}(s^\sharp), \mathtt{D}(s))\ |\ \mathtt{D}(s^\sharp){\neq}W^0,\ \mathtt{D}(s){=}\min\{\mathtt{D}(s')|\mathtt{D}(s'){\supseteq}\mathtt{D}(s^\sharp)\}\ \}\\ &= \{\ (\mathtt{D}(s^\sharp), \mathtt{D}(s))\ |\ s^\sharp{\neq}\{\},\ s{=}\max\{s'|s'{\subset}s^\sharp\}\ \}\\ &= \{\ (\mathtt{D}(s^\sharp), \mathtt{D}(s))\ |\ s^\sharp{\in}\{s{\oplus}0, s{\oplus}2\}\ \}\ .\end{aligned}$$

The first equality is the definition of $\dot{p}^0$. The second follows from part (a). The third and fourth are rearrangements. □

D.2. EXAMPLE 3'S FAMILY

Lemma D.2. *(Example 3's family) Let* $(W^0, \dot{N}^0)$ *be the Cantor-set tree that is defined along with* S *and* $\mathtt{D}$ *in Section* 3.3. *Then let* $\mathtt{E}$ *be a function from* $\{s|s\neq\{\}\}$ *such that [1]* $\mathtt{E}(0) = \mathtt{E}(2) = \varnothing$ *and [2] for every* s *with at least two elements,* $\mathtt{E}(s)$ *is a countable subset of* $\mathtt{D}(\ominus s)$*, where* $\ominus s$ *is the sequence that is obtained from* s *by changing its first element. Finally, let* $\Phi^{\mathtt{E}}$ *denote the triple* $(W^0, \dot{N}^0, \{\mathtt{D}(s)\cup\mathtt{E}(s)|s\neq\{\}\})$*. Then the following hold.*

(a) $\{\ (\mathtt{D}(s), \mathtt{D}(s^\sharp)\cup\mathtt{E}(s^\sharp))\ |\ s^\sharp\in\{s\oplus 0, s\oplus 2\}\ \}$ *is the* $\dot{F}^{\mathtt{E}}$ *(9) of* $\Phi^{\mathtt{E}}$*.*

(b) $\Phi^{\mathtt{E}}$ *is an* $\mathtt{AR}^*$ *outcome-set preform (8).*

(c) $\{\ (\mathtt{D}(s)\cup\mathtt{E}(s), \mathtt{D}(s))\ |\ s\neq\{\}\ \}$ *is the* $\mathtt{M}^{\mathtt{E}}$ *(14) of* $\Phi^{\mathtt{E}}$*.*

(d) $\Phi^{\mathtt{E}}$ *is concise (13) iff* $(\forall s\neq\{\})\ \mathtt{E}(s) = \varnothing$*.*

Proof. Derive $\dot{T}$ (2), $\dot{X}$ (4), and $\dot{p}^0$ (7) from $(W^0, \dot{N}^0)$.

(a). Take any $\dot{t}$ and any $\dot{c}\in\{\mathtt{D}(s)\cup\mathtt{E}(s)|s\neq\{\}\}$. I argue that

$$(\dot{t}, \dot{c}) \in \dot{F}^{\mathtt{E}}$$

$$\Leftrightarrow (\exists \dot{t}^\sharp)\ \dot{t}{=}\dot{p}^0(\dot{t}^\sharp),\ \dot{c}\not\supseteq\dot{t},\ \text{and}\ \dot{c}\supseteq\dot{t}^\sharp$$

$$\Leftrightarrow (\exists \dot{t}^\sharp)\ (\exists s, s^\sharp)\ \dot{t}{=}\mathtt{D}(s),\ \dot{t}^\sharp{=}\mathtt{D}(s^\sharp),\ s^\sharp\in\{s\oplus 0, s\oplus 2\},\ \dot{c}\not\supseteq\dot{t},\ \text{and}\ \dot{c}\supseteq\dot{t}^\sharp$$

$$\Leftrightarrow (\exists s, s^\sharp)\ \dot{t}{=}\mathtt{D}(s),\ s^\sharp\in\{s\oplus 0, s\oplus 2\},\ \dot{c}\not\supseteq\mathtt{D}(s),\ \text{and}\ \dot{c}\supseteq\mathtt{D}(s^\sharp)$$

$$\Leftrightarrow (\exists s)\ \dot{t}{=}\mathtt{D}(s)\ \text{and either}$$
$$\dot{c}\not\supseteq\mathtt{D}(s)\ \text{and}\ \dot{c}\supseteq\mathtt{D}(s\oplus 0)\ \text{or}$$
$$\dot{c}\not\supseteq\mathtt{D}(s)\ \text{and}\ \dot{c}\supseteq\mathtt{D}(s\oplus 2)$$

$$\Leftrightarrow (\exists s, s_c)\ \dot{t}{=}\mathtt{D}(s),\ \dot{c}{=}\mathtt{D}(s_c)\cup\mathtt{E}(s_c),\ \text{and either}$$
$$\mathtt{D}(s_c)\cup\mathtt{E}(s_c)\not\supseteq\mathtt{D}(s)\ \text{and}\ \mathtt{D}(s_c)\cup\mathtt{E}(s_c)\supseteq\mathtt{D}(s\oplus 0)\ \text{or}$$
$$\mathtt{D}(s_c)\cup\mathtt{E}(s_c)\not\supseteq\mathtt{D}(s)\ \text{and}\ \mathtt{D}(s_c)\cup\mathtt{E}(s_c)\supseteq\mathtt{D}(s\oplus 2)$$

$$\Leftrightarrow (\exists s, s_c)\ \dot{t}{=}\mathtt{D}(s),\ \dot{c}{=}\mathtt{D}(s_c)\cup\mathtt{E}(s_c),\ \text{and either}$$
$$\mathtt{D}(s_c)\not\supseteq\mathtt{D}(s)\ \text{and}\ \mathtt{D}(s_c)\supseteq\mathtt{D}(s\oplus 0)\ \text{or}$$
$$\mathtt{D}(s_c)\not\supseteq\mathtt{D}(s)\ \text{and}\ \mathtt{D}(s_c)\supseteq\mathtt{D}(s\oplus 2)$$

$$\Leftrightarrow (\exists s, s_c)\ \dot{t}{=}\mathtt{D}(s),\ \dot{c}{=}\mathtt{D}(s_c)\cup\mathtt{E}(s_c),\ \text{and either}$$
$$s_c\not\subseteq s\ \text{and}\ s_c\subseteq s\oplus 0\ \text{or}$$
$$s_c\not\subseteq s\ \text{and}\ s_c\subseteq s\oplus 2$$

$\Leftrightarrow$ $(\exists s, s_c)$ $\dot{t}{=}\mathtt{D}(s)$, $\dot{c}{=}\mathtt{D}(s_c){\cup}\mathtt{E}(s_c)$, and either

$s_c{=}s{\oplus}0$ or

$s_c{=}s{\oplus}2$.

The first equivalence is the definition (9) of $\dot{F}^E$, and the second follows from Lemma D.1(d). The next two are rearrangements. The fifth holds because $\dot{c}$ was taken to be an element of $\{\mathtt{D}(s){\cup}\mathtt{E}(s)\}$. The sixth will be justified in the following paragraph. The last two are rearrangements.

To justify the sixth equivalence, this paragraph shows that, for any s^1 and s^2, and for any countable set $E{\subseteq}W$, $\mathtt{D}(s^1){\cup}E{\supseteq}\mathtt{D}(s^2)$ iff $\mathtt{D}(s^1){\supseteq}\mathtt{D}(s^2)$. The reverse direction is obvious. To show the contrapositive of the forward direction, suppose that $\mathtt{D}(s^1){\not\supseteq}\mathtt{D}(s^2)$. Either [1] there exists an i (no larger than the minimum of the lengths of s^1 and s^2) such that $s^1_i{\neq}s^2_i$ or [2] there does not. In [1], $\mathtt{D}(s^1){\cap}\mathtt{D}(s^2){\neq}\varnothing$. Hence $\mathtt{D}(s^2){\smallsetminus}\mathtt{D}(s^1)$ is uncountable because it equals $\mathtt{D}(s^2)$. In [2], the assumption $\mathtt{D}(s^1){\not\supseteq}\mathtt{D}(s^2)$ implies that $s^1{\supset}s^2$, which implies that $\mathtt{D}(s^2){\smallsetminus}\mathtt{D}(s^1)$ is uncountable. In either case, the uncountability of $\mathtt{D}(s^2){\smallsetminus}\mathtt{D}(s^1)$ and the countability of $\mathtt{E}$ imply $E{\not\supseteq}\mathtt{D}(s^2){\smallsetminus}\mathtt{D}(s^1)$. Hence $\mathtt{D}(s^1){\cup}E{\not\supseteq}\mathtt{D}(s^2)$.

(b). (8a) holds by Lemma D.1(b). (8b) holds by inspection.

(8c). By Lemma D.1(c), we may let $\mathtt{D}(s)$ be an arbitrary element of $\dot{X}^0$. I argue that

$$
\begin{aligned}
& \{\ \mathtt{D}(s){\cap}\dot{c} \mid \dot{c}{\in}\dot{F}^{\mathtt{E}}(s)\ \} \\
=\ & \{\ \mathtt{D}(s){\cap}(\mathtt{D}(s^\sharp){\cup}\mathtt{E}(s^\sharp)) \mid s^\sharp{\in}\{s{\oplus}0, s{\oplus}2\}\ \} \\
=\ & \{\ (\mathtt{D}(s){\cap}\mathtt{D}(s^\sharp)) \cup (\mathtt{D}(s){\cap}\mathtt{E}(s^\sharp)) \mid s^\sharp{\in}\{s{\oplus}0, s{\oplus}2\}\ \} \\
=\ & \{\ \mathtt{D}(s^\sharp) \cup (\mathtt{D}(s){\cap}\mathtt{E}(s^\sharp)) \mid s^\sharp{\in}\{s{\oplus}0, s{\oplus}2\}\ \} \\
=\ & \{\ \mathtt{D}(s^\sharp) \cup \varnothing \mid s^\sharp{\in}\{s{\oplus}0, s{\oplus}2\}\ \} \\
=\ & (\dot{p}^0)^{-1}(\mathtt{D}(s))\ .
\end{aligned}
$$

The first equality follows from part (a). The second equality is a rearrangement. The third holds because $\mathtt{D}(s){\supseteq}\mathtt{D}(s^\sharp)$. The fourth will be proved in the following paragraph. The last holds by Lemma D.1(d).

The fourth equality requires two cases. On the one hand, if $s{=}\{\}$, then $s^\sharp$ is either 0 or 2 and the definition of $\mathtt{E}$ states that both $\mathtt{E}(0)$ and $\mathtt{E}(2)$ are empty. On the other hand, suppose $s{\neq}\{\}$. I argue

$$\mathtt{D}(s){\cap}\mathtt{E}(s^\sharp) \subseteq \mathtt{D}(s){\cap}\mathtt{D}(\ominus s^\sharp) \subseteq \mathtt{D}(s){\cap}\mathtt{D}(\ominus s) = \varnothing\ .$$

The first inclusion follows from the definition of $\mathtt{E}$. The second inclusion follows from $\mathtt{D}(\ominus s^\sharp)\subseteq\mathtt{D}(\ominus s)$, which follows from $s^\sharp\in\{s\oplus0, s\oplus2\}$. The equality holds because s and $\ominus s$ differ in their first element.

(8d). By Lemma D.1(c), let $\mathtt{D}(s)$ be an arbitrary element of $\dot{X}^0$. By part (a), it only needs to be shown that the two elements of

$$\dot{F}^{\mathtt{E}}(\mathtt{D}(s)) = \{\ \mathtt{D}(s\oplus0)\cup\mathtt{E}(s\oplus0),\ \mathtt{D}(s\oplus2)\cup\mathtt{E}(s\oplus2)\ \}$$

are disjoint. On the one hand, if s=$\{\}$,

$$\begin{aligned} & [\mathtt{D}(\{\}\oplus0)\cup\mathtt{E}(\{\}\oplus0)] \cap [\mathtt{D}(\{\}\oplus2)\cup\mathtt{E}(\{\}\oplus2)] \\ = \ & [\mathtt{D}(0)\cup\mathtt{E}(0)] \cap [\mathtt{D}(2)\cup\mathtt{E}(2)] \\ = \ & \mathtt{D}(0) \cap \mathtt{D}(2) = \varnothing\ , \end{aligned}$$

where the second equality follows from the definition of $\mathtt{E}$. On the other hand, if $s\neq\{\}$,

$$\begin{aligned} & [\mathtt{D}(s\oplus0)\cup\mathtt{E}(s\oplus0)] \cap [\mathtt{D}(s\oplus2)\cup\mathtt{E}(s\oplus2)] \\ \subseteq\ & [\mathtt{D}(s\oplus0)\cup\mathtt{D}(\ominus s\oplus0)] \cap [\mathtt{D}(s\oplus2)\cup\mathtt{D}(\ominus s\oplus2)] \\ =\ & \varnothing\ , \end{aligned}$$

where the set inclusion holds by the definition of $\mathtt{E}$, and where the equality holds because $s\oplus0$, $\ominus s\oplus0$, $s\oplus2$, and $\ominus s\oplus2$ are four distinct sequences of the same length.

(8e). Part (a) implies that $(\dot{F}^{\mathtt{E}})^{-1}$ is single-valued. Thus condition (a) of Lemma A.5 holds. Hence Lemma A.5(a⇒b) implies (8e).

(c). Take any $s\neq\{\}$. I argue

$$\begin{aligned} & \mathtt{M}^{\mathtt{E}}(\mathtt{D}(s)\cup\mathtt{E}(s)) \\ = \ & [\mathtt{D}(s)\cup\mathtt{E}(s)] \cap \cup(\dot{F}^{\mathtt{E}})^{-1}(\mathtt{D}(s)\cup\mathtt{E}(s)) \\ = \ & [\mathtt{D}(s)\cup\mathtt{E}(s)] \cap \mathtt{D}(s_-) \\ = \ & [\mathtt{D}(s)\cap\mathtt{D}(s_-)] \cup [\mathtt{E}(s)\cap\mathtt{D}(s_-)] \\ = \ & \mathtt{D}(s) \cup [\mathtt{E}(s)\cap\mathtt{D}(s_-)] \\ = \ & \mathtt{D}(s) \cup \varnothing\ , \end{aligned}$$

where s_- is the sequence derived from $s\neq\{\}$ by omitting its last component. The first equality is the definition of $\mathtt{M}^{\mathtt{E}}$. The second follows from part (a). The third is a rearrangement. The fourth holds because $\mathtt{D}(s)\subseteq\mathtt{D}(s_-)$. The fifth must be justified in two cases. First, if s has one element, then $\mathtt{E}(s) = \varnothing$ by the definition of $\mathtt{E}$. Second, if s has more than one element, I argue

$$\mathtt{E}(s) \cap \mathtt{D}(s_-) \subseteq \mathtt{D}(\ominus s) \cap \mathtt{D}(s_-) \subseteq \mathtt{D}(\ominus s_-) \cap \mathtt{D}(s_-) = \varnothing \ .$$

The first inclusion follows from the definition of $\mathtt{E}$. The second inclusion holds because [1] $\ominus s_-$ is well-defined because s has more than one element and [2] $\mathtt{D}(\ominus s) \subseteq \mathtt{D}(\ominus s_-)$. The equality holds because $\ominus s_-$ and s_- differ in their first element.

(d). I argue

(72)
$$\begin{aligned} & \Phi^{\mathtt{E}} \text{ is concise} \\ \Leftrightarrow\ & [\ \dot{F}^{\mathtt{E}}(\dot{X}^0) = \{\mathtt{D}(s) \cup \mathtt{E}(s) | s \neq \{\}\} \text{ and} \\ & (\forall s \neq \{\})\ \mathtt{M}^{\mathtt{E}}(\mathtt{D}(s) \cup \mathtt{E}(s)) = \mathtt{D}(s) \cup \mathtt{E}(s)\ ] \\ \Leftrightarrow\ & (\forall s \neq \{\})\ \mathtt{M}^{\mathtt{E}}(\mathtt{D}(s) \cup \mathtt{E}(s)) = \mathtt{D}(s) \cup \mathtt{E}(s) \\ \Leftrightarrow\ & (\forall s \neq \{\})\ \mathtt{D}(s) = \mathtt{D}(s) \cup \mathtt{E}(s) \\ \Leftrightarrow\ & (\forall s \neq \{\})\ \mathtt{E}(s) = \varnothing \ . \end{aligned}$$

The first equivalence follows from Lemma 3.1. The second equivalence holds because

$$\begin{aligned} \dot{F}^E(\dot{X}^0) &= \dot{F}^E(\{\mathtt{D}(s) | s\}) \\ &= \cup\{\ \dot{F}^{\mathtt{E}}(\mathtt{D}(s))\ |\ s\ \} \\ &= \cup\{\ \{\mathtt{D}(s^\sharp) \cup \mathtt{E}(s^\sharp) | s^\sharp \in \{s \oplus 0, s \oplus 2\}\}\ |\ s\ \} \\ &= \{\ \mathtt{D}(s^\sharp) \cup \mathtt{E}(s^\sharp)\ |\ s^\sharp \neq \{\}\ \}\ , \end{aligned}$$

where the first equality follows from Lemma D.1(c), the third equality follows from this lemma's part (a), and the other two equalities are rearrangements. Returning to (72), the third equivalence follows from this lemma's part (c). The reverse direction of the fourth equivalence is obvious. The forward direction holds because

$$\mathtt{D}(s) \cap \mathtt{E}(s) = \varnothing \ .$$

Proving this requires two cases. If s has one element, then $\mathtt{E}(s) = \varnothing$ by the definition of $\mathtt{E}$. If s has more than one element, then $\mathtt{D}(s) \cap \mathtt{E}(s) \subseteq \mathtt{D}(s) \cap \mathtt{D}(\ominus s) = \varnothing$ by the definition of $\mathtt{E}$ and the fact that s and $\ominus s$ differ in their first element. □

Corollary D.3. *(Example 3) Let* $(W^0, \dot{N}^0)$ *be the Cantor-set tree that is defined along with* S *and* $\mathtt{D}$ *in Section* 3.3. *Let* Φ^3 *denote the triple*

$$(W^0, \dot{N}^0, \{\mathtt{D}(s)|s\notin\{\{\}, 22\}\} \cup \{\mathtt{D}(22)\cup\{.02\}\}\ )\ .$$

(a) The $\dot{F}^3$ *(9) of* Φ^3 *is*

$$\begin{aligned}&\{\ (\mathtt{D}(s), \mathtt{D}(s^\sharp))\,|\,s{\neq}2,\ \ s^\sharp{\in}\{s{\oplus}0, s{\oplus}2\}\ \}\\ \cup\ &\{\ (\mathtt{D}(2), \mathtt{D}(20)),\ \ (\mathtt{D}(2), \mathtt{D}(22)\cup\{.02\})\ \}\ .\end{aligned}$$

(b) Φ^3 *is an* $\mathtt{AR}^*$ *outcome-set preform (8).*
(c) .02 *is an immaterial outcome in the choice* $\mathtt{D}(22)\cup\{.02\}$*.*
(d) Φ^3 *is not concise (13).*

Proof. Define $\mathtt{E}$ by setting $\mathtt{E}(22) = \{.02\}$ and by setting $\mathtt{E}(s) = \varnothing$ at every other nonempty s. Since $\mathtt{E}(22) \subseteq \mathtt{D}(02) = \mathtt{D}(\ominus 2)$, this $\mathtt{E}$ satisfies the assumption of Lemma D.2. Thus we may apply Lemma D.2 at $\Phi^E = \Phi^3$. In particular, part (a) follows from Lemma D.2(a). Similarly, part (b) follows from Lemma D.2(b). Further, part (c) holds because $.02 \notin \mathtt{D}(2)$ and $\{\mathtt{D}(2)\} = (\dot{F}^3)^{-1}(\mathtt{D}(22)\cup\{.02\})$ by part (a). Finally, part (d) follows from part (c) and Lemma 3.1. (Alternatively, parts (c,d) can be derived from Lemma D.2(c,d).) □

Corollary D.4. *(Example 1) Let* $(W^0, \dot{N}^0)$ *be the Cantor-set tree that is defined along with* S *and* $\mathtt{D}$ *in Section* 3.3. *Let* Φ^1 *denote the triple*

$$(W^0, \dot{N}^0, \{\mathtt{D}(s)|s\notin\{\}\})\ .$$

(a) The $\dot{F}^1$ *(9) of* Φ^1 *is* $\{\ (\mathtt{D}(s), \mathtt{D}(s^\sharp))\,|\,s^\sharp{\in}\{s{\oplus}0, s{\oplus}2\}\ \}$*.*
(b) Φ^1 *is a concise (13)* $\mathtt{AR}^*$ *outcome-set preform (8).*

Proof. Define $\mathtt{E}$ by setting $\mathtt{E}(s) = \varnothing$ at every nonempty s. Since this $\mathtt{E}$ satisfies the assumption of Lemma D.2, we may apply Lemma D.2 at $\Phi^E = \Phi^1$. In particular, part (a) follows from Lemma D.2(a). Further, part (b) follows from Lemma D.2(b,d). □

D.3. Example 2's family

Lemma D.5 does not assume the Cantor-set tree used elsewhere in this Appendix D. Its argument is easier to see at an abstract level.

Lemma D.5. *Let* $(W,\dot{N},\dot{C})$ *be an* AR* *outcome-set preform (8) with its* $\dot{T}$ *(2) and* $\dot{F}$ *(9). Further let*

$$A^+ = \{\ W{\supseteq}a{\supset}\varnothing \mid (\nexists\dot{t})\ a{\supseteq}\dot{t}\ \}$$

and let $\dot{C}^+$ *be a nonempty subcollection of* A^+*. Then the following hold.*

(a) $\dot{F}$ *is also derived (9) from* $(W,\dot{N},\dot{C}{\cup}\dot{C}^+)$.

(b) $(W,\dot{N},\dot{C}{\cup}\dot{C}^+)$ *is a non-concise (13)* AR* *outcome-set preform (8) in which all the members of* C^+ *are nowhere-feasible.*

Proof. Derive $\dot{p}$ (7) from $(W,\dot{N})$. Derive $\dot{F}^+$ (9) from $(W,\dot{N},\dot{C}{\cup}\dot{C}^+)$. Before proceeding with parts (a) and (b), I argue

(73)
$$(\forall\dot{c}^+{\in}\dot{C}^+)\ \ (\dot{F}^+)^{-1}(\dot{c}^+) = \\ \{\ \dot{t} \mid \dot{c}^+{\not\supseteq}\dot{t} \text{ and } (\exists\dot{t}^\sharp{\in}\dot{p}^{-1}(\dot{t}))\ \dot{c}^+{\supseteq}\dot{t}^\sharp\ \} = \varnothing\ .$$

Accordingly, take any $\dot{c}^+{\in}\dot{C}^+$. The first equality follows from the definition of $\dot{F}^+$. The second equality follows from $(\nexists\dot{t}^\sharp)\ \dot{c}^+{\supseteq}\dot{t}^\sharp$, which follows from $\dot{C}^+{\subseteq}A^+$ and the definition of A^+.

(a). Note that

$$(\forall\dot{c}{\in}\dot{C})\ \ (\dot{F}^+)^{-1}(\dot{c}) = \{\ \dot{t} \mid \dot{c}{\not\supseteq}\dot{t} \text{ and } (\exists\dot{t}^\sharp{\in}\dot{p}^{-1}(\dot{t}))\ \dot{c}{\supseteq}\dot{t}^\sharp\ \} = \dot{F}^{-1}(\dot{c})\ ,$$

where the first equality follows from the definition of $\dot{F}^+$ and the second equality follows from the definition of $\dot{F}$. Also note that

$$(\forall\dot{c}^+{\in}\dot{C}^+{\smallsetminus}\dot{C})\ \ (\dot{F}^+)^{-1}(\dot{c}^+) = \varnothing = \dot{F}^{-1}(\dot{c}^+)\ ,$$

where the first equality follows from (73) and the second equality follows from the fact that $\dot{F}$ is a subset of $\dot{T}{\times}\dot{C}$. These two observations together imply that $(\forall\dot{c}'{\in}\dot{C}{\cup}\dot{C}^+)\ (\dot{F}^+)^{-1}(\dot{c}') = \dot{F}^{-1}(\dot{c}')$. Thus $\dot{F}^+ = \dot{F}$.

(b). By assumption, $(W,\dot{N},\dot{C})$ satisfies the five components of the definition (8) of a preform. This paragraph derives the five components for $(W,\dot{N},\dot{C}{\cup}\dot{C}^+)$. (8a) follows from (8a) for $(W,\dot{N},\dot{C})$ because (8a) only concerns the tree $(W,\dot{N})$. (8b) holds because [1] $\dot{C}$ is a collection of nonempty subsets by (8b) for $(W,\dot{N},\dot{C})$ and [2] $\dot{C}^+$ is a collection of nonempty subsets of W by $\dot{C}^+ \subseteq A^+$ and the definition of A^+. (8c-e) follow from (8c-e) for $(W,\dot{N},\dot{C})$ because of part (a).

(73) implies that every element of $\dot{C}^+$ is nowhere-feasible. Thus, since $\dot{C}^+ \neq \varnothing$ by assumption, $(W,\dot{N},\dot{C}{\cup}\dot{C}^+)$ has at least one nowhere-feasible choice. Hence, the preform is not concise by Lemma 3.1. □

Lemma D.6. *(Example 2's family) Let* $(W^0, \dot{N}^0)$ *be the Cantor-set tree that is defined along with* S *and* $\mathtt{D}$ *in Section* 3.3. *Next let* $\dot{C}^+$ *be a nonempty collection of nonempty countable subsets of* W^0. *Then*

$$(W^0,\, \dot{N}^0,\, \{\mathtt{D}(s)|s{\neq}\{\}\}{\cup}\dot{C}^+)$$

is a non-concise (13) $\mathtt{AR}^*$ *outcome-set preform (8) in which all the elements of* $\dot{C}^+$ *are nowhere-feasible choices.*

Proof. Define A^+ as in Lemma D.5. This paragraph shows that $\dot{C}^+$ (as defined in this lemma's statement) is a nonempty subcollection of the A^+. $\dot{C}^+$ is nonempty by definition. Further, Lemma D.1(a) implies

$$A^+ = \{\ W^0{\supseteq}a{\supset}\varnothing \mid (\nexists s)\ a{\supseteq}\mathtt{D}(s)\ \}\ . \tag{74}$$

Now take any $\dot{c}^+{\in}\dot{C}^+$. By (74), it suffices to show that [1] $\dot{c}^+$ is a nonempty subset of W^0 and that [2] $(\nexists s)$ $\dot{c}^+{\supseteq}\mathtt{D}(s)$. The former holds by the definition of $\dot{C}^+$. The latter holds because $\dot{c}^+$ is countable by the definition of $\dot{C}^+$ and because every $\mathtt{D}(s)$ is uncountable.

I now apply Lemma D.5 to $(W^0, \dot{N}^0, \{\mathtt{D}(s)|s\})$.[4] $(W^0, \dot{N}^0, \{\mathtt{D}(s)|s\})$ is a preform by Lemma D.4(b). This and the previous paragraph establish Lemma D.5's assumptions. Thus Lemma D.5(b) implies this lemma's conclusion. □

Corollary D.7. *(Example 2) Let* $(W^0, \dot{N}^0)$ *be the Cantor-set tree that is defined along with* S *and* $\mathtt{D}$ *in Section* 3.3. *Then*

$$(W^0, \dot{N}^0, \{\mathtt{D}(s)|s{\neq}\{\}\}{\cup}\{\{1\}\})$$

is a non-concise (13) $\mathtt{AR}^*$ *outcome-set preform (8) in which* $\{1\}$ *is a nowhere-feasible choice.*

Proof. Apply Lemma D.6 at $\dot{C}^+ = \{\{1\}\}$. □

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[4]Lemma D.6 applies Lemma D.5 to Example 1 to generate Example 2's family. Likewise, Lemma D.5 could be applied to Example 3. The resulting family would contain preforms with both [1] an immaterial outcome in a somewhere-feasible choice (like Example 3) and [2] at least one nowhere-feasible choice (like Example 2's family). Many similar families of non-concise preforms can be generated by applying Lemma D.5 to the many other preforms in Example 3's family.

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