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Correlated equilibria and communication equilibria in all-pay auctions*

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Abstract

We study cheap-talk pre-play communication in the static all-pay auctions. For the case of two bidders, all correlated and communication equilibria are payoff equivalent to the Nash equilibrium if there is no reserve price, or if it is commonly known that one bidder has a strictly higher value. Hence, in such environments the Nash equilibrium predictions are robust to pre-play communication between the bidders. If there are three or more symmetric bidders, or two symmetric bidders and a positive reserve price, then there may exist correlated and communication equilibria such that the bidders' payoffs are higher than in the Nash equilibrium. In these cases, pre-play cheap talk may affect the outcomes of the game, since the bidders have an incentive to coordinate on such equilibria.

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1 Introduction

An all-pay auction is a model of contest in which the participants expend resources trying to win a prize, and the prize goes to whoever spends the most. This model is important for studying various

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economic phenomena, especially lobbying and other rent-seeking activities (Hillman and Samet, 1987; Baye, Kovenock and de Vries, 1993). It is typically assumed that in the all-pay auction the bidders choose how much to bid without any prior contact with each other. Yet, in many situations it is difficult or impossible to prevent the bidders from engaging in cheap talk before the auction. Thus it is important to understand whether and how pre-play cheap-talk communication affects the outcomes of the all-pay auctions.

Competition in the all-pay auctions is typically intense. For example, if it is commonly known that the value of the good is the same for all bidders, then complete rent dissipation occurs in all Nash equilibria, i.e. the total expected payments of the bidders are equal to the value of the good, and each bidder gets a zero expected payoff. Thus, if pre-play communication is allowed, the bidders may want to try to coordinate their bidding in order to avoid cut-throat competition. However, because of the antagonistic nature of the all-pay auction it is unclear whether informative communication is possible. A bidder may not want to communicate his bidding intentions or privately known value truthfully to the opponents because this information could be used against him. Instead, each bidder, regardless of his value, might want to misguide the opponents into bidding less aggressively. We show that in some environments pre-play communication is indeed completely powerless, and the equilibrium outcomes of the game with communication are payoff equivalent to the equilibrium outcomes of the all-pay auction without communication. Perhaps more surprisingly, we also show that there are situations when pre-play communication helps the bidders to coordinate their behavior so that the intensity of bidding is reduced, and the bidders get higher payoffs than in the all-pay auction without communication.

To study the all-pay auction with pre-play communication in the environments with complete information we use the solution concept of correlated equilibrium (Aumann, 1974; 1987), and in the environments with incomplete information – communication equilibrium (Myerson, 1982). According to the revelation principle for games with communication, which is discussed in Section 2, the correlated and communication equilibria describe all possible outcomes that can be potentially achieved with the help of communication in a self-enforcing way.[1] In the all-pay auction models

[1] There are also other reasons to use correlated equilibrium as a solution concept. Correlated equilibrium has arguably more compelling epistemic foundations than Nash equilibrium (Aumann, 1987); it is easier for boundedly rational players to learn to play correlated equilibrium than Nash equilibrium (Hart and Mas-Colell, 2013). Communication equilibrium is one of the most popular ways of extending the concept of correlated equilibrium to the games with incomplete information (Forges, 1993; Bergemann and Morris, 2013).

that we study there is either a unique Nash equilibrium, or all Nash equilibria result in the same payoffs for the bidders. If it happens that in a given environment all correlated (communication) equilibria are payoff equivalent to the Nash equilibrium, then we can say that the Nash equilibrium prediction is robust to pre-play communication between the bidders. However, if there exist correlated (communication) equilibria that are not payoff equivalent to the Nash equilibrium, then pre-play communication may affect the outcomes of the game. In particular, if in such correlated (communication) equilibria the bidders get higher payoffs than in the Nash equilibrium, then they have an incentive to coordinate on the former. Different ways of organizing communication between the bidders to realize the outcomes of the correlated and communication equilibria are discussed in Section 5.

In Section 3 we study correlated equilibria in the all-pay auctions with complete information. We show that with two bidders the correlated equilibria are payoff equivalent to the Nash equilibrium when there is no reserve price, or if the bidders are asymmetric (Proposition 2). In such cases the all-pay auction is "strategically equivalent" to a particular zero-sum game, and for the two-player zero-sum games the correlated and Nash equilibria are known to be payoff equivalent (Moulin and Vial, 1978). It turns out that this strategic equivalence does not hold when there is a reserve price and the bidders are symmetric. For this case we construct correlated equilibria that are more profitable for the bidders than the Nash equilibrium (Example 1 and Proposition 3). When there are three or more symmetric bidders, such profitable correlated equilibria exist even when there is no reserve price (Example 2 and Proposition 4). The idea of the constructions is to introduce some imperfect negative correlation in the distribution of the bids. Say, when one of the bidders bids aggressively, then with a certain probability his opponents are "suggested" to bid zero, and thus save the cost of their bids.

In Section 4 we study communication equilibria in the all-pay auctions with independent private values. Similarly to the case of complete information, we show that with two bidders the communication equilibria are payoff equivalent to the Nash equilibrium when there is no reserve price (Proposition 6). That is, neither self-enforcing sharing of private information, nor correlation of play is possible in this case. However, in other cases there exist communication equilibria that are more profitable for the bidders than the Nash equilibrium. This is demonstrated for the case of two bidders and a positive reserve price (Example 3 and Proposition 7), and for the case of three or more bidders and no reserve price (Proposition 9). The constructions involve correlating the bidders' play in a way that is similar to the correlated equilibria in Section 3. The bidders also share some private information, but only to a limited extent because it is important to maintain enough uncertainty about the opponents' values and play for the construction to work.

Pre-play communication in auctions and contests is typically studied in context of collusion. For example, most of the studies of collusion in static auctions focus on a scenario when the bidders organize an explicit cartel that allows them to communicate, enforces coordinated behavior of the bidders in the auction, and facilitates exchange of side payments between the bidders.[2] The bidders' collusion that is self-enforcing is for the most part considered in the context of repeated auctions.[3] In such models the enforcement of the desired bidders' behavior is provided by the expectations of the future reaction of the opponents.

Only a few papers study collusion in static auctions when the behavior of the bidders in the auction cannot be directly controlled. Marshall and Marx (2007, 2009), and Lopomo, Marx and Sun (2011) study collusion in the first-price, second-price, and ascending-bid auctions under the following scenario. The bidders make reports to a "center"; based on these reports, the center privately recommends a bid to be made by each bidder, and requires payments from the bidders.[4] If we drop the possibility to exchange side payments before the auction, then such a model of collusion is equivalent to assuming that the bidders play some particular communication equilibrium. Lopomo, Marx and Sun (2011) show that in the first-price auction with discrete bids such a collusion is completely ineffective: all collusive equilibria are payoff equivalent to the unique Nash equilibrium.[5] However, Marshall and Marx (2007) show that in the second-price auction such a collusion works equally well as collusion in a model where the bidders behavior can be controlled by the cartel. In fact, in the second-price auctions viable collusion is possible even when the bidders cannot exchange side payments, i.e. there exist communication equilibria that are different from the Nash equilibria, and are more profitable for the bidders (Marshall and Marx, 2009).

In some cases it is reasonable to assume that the bidders can disclose private information about

[2] For example, Graham and Marshall (1987) study collusion in second-price auctions, and McAfee and McMillan (1992) study collusion in first-price auctions.

[3] For example, Aoyagi (2003) studies self-enforcing collusion with pre-play communication in repeated auctions.

[4] Lopomo, Marshall and Marx (2005) and Garratt, Tröger and Zheng (2009) study self-enforcing collusion without pre-auction side payments, but with a possibility of resale.

[5] See also Azacis and Vida (2010) for related results for the first-price auction with a continuum of bids.

their valuations in a verifiable way. Benoit and Dubra (2006), Hernando-Veciana and Tröge (2011), and Tan (2013) study the bidders' individual decisions to disclose information in winner-pay auctions. Kovenok, Morath and Münster (2010) and Szech (2011) study this problem in the all-pay auctions. The relation of such an approach to our approach is discussed in Section 4.1.

There are also many experimental studies of the effect pre-play communication in games. While we are unaware of any research that studies exactly our setting, there is some related work. For example, Harbring (2006) considers the effect of communication in a repeated all-pay auction with a cap on the maximal possible bids. Though there were only finitely many rounds, the bidders's behavior resembled collusive play in an infinitely repeated game, and the possibility of communication lead to lower bids and higher payoffs.[6] More generally, experimental research has revealed that pre-play communication often increases cooperation between the players beyond what is predicted by standard game-theoretic models, and this effect is attributed to a combination of norms, empathy, nonverbal cues, etc. (Camerer, 2003).

The rest of the paper is organized as follows. The model and the definitions of correlated and communication equilibria are in Section 2. The all-pay auctions with complete information and incomplete information are studied in Sections 3 and 4, respectively. Discussion is in Section 5. The proofs are relegated to the Appendix unless stated otherwise.

2 Model

There are $n \geq 2$ bidders. Bidder i chooses a bid b_i from a set of possible bids A_i. If there is no reserve price, then $A_i = [0, \infty)$. If there is a reserve price $r > 0$, then $A_i = \{0\} \cup [r, \infty)$, i.e., bidder i can either submit a "null" bid $b_i = 0$, or an "active" bid $b_i \geq r$.[7] If bidder i bids b_i, and the other bidders bid b_{-i}, then bidder i wins the good with probability $\rho_i(b_i, b_{-i})$. If there is no reserve price, then

$$\rho_i(b_i, b_{-i}) = \begin{cases} 0 & \text{if} \quad b_i < \max_{j \neq i} b_j \\ 1 & \text{if} \quad b_i > \max_{j \neq i} b_j \\ \frac{1}{\#\{k \colon b_k = b_i\}} & \text{if} \quad b_i = \max_{j \neq i} b_j \end{cases}$$

[6] For a survey of other experimental research on contests see Dechenaux, Kovenok and Sheremeta (2012).

[7] Alternatively one can keep the action set $A_i = [0, \infty)$, but this will result in an unnecessary multiplicity of equilibria because there will be multiple possible "inactive" bids.

If there is a reserve price $r > 0$, then

$$\rho_i(b_i, b_{-i}) = \begin{cases} 0 & \text{if} \quad b_i = 0 \text{ or } \{b_i \geq r \text{ and } b_i < \max_{j \neq i} b_j\} \\ 1 & \text{if} \quad b_i \geq r \text{ and } b_i > \max_{j \neq i} b_j \\ \frac{1}{\#\{k:\ b_k = b_i\}} & \text{if} \quad b_i \geq r \text{ and } b_i = \max_{j \neq i} b_j \end{cases}$$

We consider both complete and incomplete information environments.

Complete information. Bidder i has a valuation $v_i > 0$ for the good, and the bidders' values $(v_1, ..., v_n)$ are commonly known. If bidder i bids b_i, and the other bidders bid b_{-i}, then his payoff is $u_i(b_i, b_{-i}) = v_i \rho_i(b_i, b_{-i}) - b_i$.

In the complete information case we study correlated equilibria and Nash equilibria. To define a correlated equilibrium suppose there is a neutral trustworthy mediator who makes non-binding private recommendations (possibly stochastic) to each bidder of which bid to submit. The recommendations are made according to a *correlation rule* μ, which is a probability measure over the set of all possible bid profiles $A = \prod_{j=1}^{n} A_j$.[8] Each bidder then decides which bid to submit as a function of the mediator's recommendation. Thus a pure strategy of bidder i is $\widehat{b}_i : A_i \to A_i$.

Definition 1 *A correlation rule μ is a* correlated equilibrium *if each bidder finds it optimal to obey the mediator's recommendations:*

$$\int_A u_i(b)\, \mu(db) \geq \int_A u_i\left(\widehat{b}_i(b_i), b_{-i}\right) \mu(db) \qquad \textit{for every } i \textit{ and } \widehat{b}_i(\cdot).$$

The significance of the correlated equilibrium for studying all-pay auctions with communication is due to the revelation principle.[9] According to it, for any Nash equilibrium of a game that consists of some communication protocol followed by the all-pay auction, there exists an outcome equivalent correlated equilibrium of the all-pay auction. There is no loss of generality in requiring that for each player it is optimal to obey the mediator's recommendations.

[8] All considered sets and functions are Borel measurable; all considered probability measures are Borel, with topology of weak convergence.

[9] See Aumann (1974, 1987) and Myerson (1982). Cotter (1989) provides the revelation principle for the settings with large action and type spaces.

Let μ_i be the marginal probability measure of μ on A_i:

$$\mu_i(E_i) = \int_{E_i \times A_{-i}} \mu(db) \qquad \text{for every } E_i \subseteq A_i.$$

A *Nash equilibrium* is a correlated equilibrium μ^* such that each bidder's behavior is independent from the actions of the opponents, i.e., μ^* is a product of its marginals $\prod_{j=1}^n \mu_j^*$. Hence, both Nash and correlated equilibria are joint plans of actions that are individually self-enforcing, but correlated equilibrium allows for additional coordination by correlating recommendations to the bidders. When we encounter a Nash equilibrium, we write it as a profile of the individual mixed actions $(\mu_1^*, ..., \mu_n^*)$.

Incomplete information. Bidder i privately observes own value $v_i \in T_i = [\underline{v}_i, \overline{v}_i] \subset \mathbb{R}_+$. The value of bidder i is distributed according to a probability measure P_i on T_i, independently from the valuations of the other bidders. This information structure is assumed to be common knowledge. The payoff of bidder i with value v_i, who bids b_i, while the other bidders bid b_{-i}, is $u_i(b_i, b_{-i}; v_i) = v_i \rho_i(b_i, b_{-i}) - b_i$. Denote $T = \prod_{j=1}^n T_j$, and let P be a product measure $\prod_{j=1}^n P_j$, and $P_{-i} = \prod_{j \neq i} P_j$.

In the incomplete information case we study communication equilibria and Nash equilibria. To define a communication equilibrium suppose the bidders first privately report their values to a neutral trustworthy mediator, who then makes non-binding private recommendations (possibly stochastic) to each bidder of which bid to submit. The recommendations are made according to a *communication rule* μ, which is a family of probability measures $\{\mu(\cdot|v)\}_{v \in T}$. That is, for each profile of type reports v submitted to the mediator, $\mu(\cdot|v)$ is a probability measure over the set of all possible bid profiles A. Each bidder decides which type to report, and which bid to submit as a function of the mediator's recommendation. Thus a pure strategy of bidder i with value v_i specifies $\widehat{v}_i \in T_i$, the value to be reported, and $\widehat{b}_i : A_i \to A_i$, the rule for translating recommendations into bids.

Definition 2 *A communication rule μ is a* communication equilibrium *if P_i-a.e. type of each* *bidder finds it optimal to report the true type and obey the mediator's recommendations:*

$$\int_{T_{-i}} \left(\int_A u_i(b; v_i)\, \mu(db|v) \right) P_{-i}(dv_{-i}) \geq \int_{T_{-i}} \left(\int_A u_i\left(\widehat{b}_i(b_i), b_{-i}; v_i \right) \mu(db|\widehat{v}_i, v_{-i}) \right) P_{-i}(dv_{-i})$$

$$\textit{for every } i,\ P_i\textit{-a.e. } v_i,\ \textit{every } \widehat{v}_i,\ \textit{and } \widehat{b}_i(\cdot).$$

Similarly to the case of correlated equilibrium, the significance of communication equilibrium for studying all-pay auctions with communication in a setting with nonverfiable information is due to the revelation principle. For any Nash equilibrium of a game that consists of some communication protocol followed by the all-pay auction, there exists an outcome equivalent communication equilibrium of the all-pay auction. There is no loss of generality in requiring that for each player reporting the true type and obeying the mediator's recommendation is optimal.

Let $\mu_i(\cdot|v_i)$ be the marginal probability measure of μ on A_i conditional on v_i:

$$\mu_i(E_i|v_i) = \int_{T_{-i}} \left(\int_{E_i \times A_{-i}} \mu(db|v_i, v_{-i}) \right) P_{-i}(dv_{-i}) \qquad \text{for every } E_i \subseteq A_i.$$

A *Nash equilibrium* is a communication equilibrium μ^* such that each bidder's behavior is independent from the opponents' reports and actions, i.e., for every $v = (v_1, ..., v_n)$, $\mu^*(\cdot|v)$ is a product of marginals $\prod_{j=1}^n \mu_j^*(\cdot|v_j)$. Thus, relative to the Nash equilibrium, communication equilibrium allows for self-enforcing sharing of private information between the bidders, as well as for coordination via correlation of the recommended bids. When we encounter a Nash equilibrium, we write it as a profile $(\mu_1^*, ..., \mu_n^*)$, where μ_i^* is $\{\mu_i^*(\cdot|v_i)\}_{v_i \in T_i}$ for every i.

3 All-pay auctions with complete information

3.1 Two bidders

In this section we study and compare Nash equilibria and correlated equilibria of the all-pay auction under complete information. The Nash equilibria of this game are well understood, and we simply summarize the existing results. We are not aware, however, of any characterizations of the set of correlated equilibria of the all-pay auction.

In games with finite number of actions the set of correlated equilibria is defined by finitely many linear inequalities: if player i has $|A_i|$ possible actions, then there are $|A_i|(|A_i|-1)$ obedience constraints that ensure that he has no incentive to deviate from the recommended actions. It is thus straightforward to describe the extreme points of this set and to find the set of the players' payoffs achievable by the correlated equilibria. However, if each player has a continuum of possible actions, then there is a double continuum of obedience constraints, which is difficult to work with.[10] One possible approach is to discretize the action spaces and to use the linear programming tools. This path is pursued, for example, in Lopomo, Marx and Sun (2011) in their study of collusive schemes in the first price auction. In this paper we take a different route. For some cases we characterize correlated equilibria by exploiting a connection between the all-pay auction and a certain class of zero-sum games, and in other cases we construct correlated equilibria directly.

We begin with the case of two bidders. Denote the difference in the bidders' valuations by $\Delta v = v_1 - v_2$, and without loss of generality assume $\Delta v \geq 0$. To avoid uninteresting cases we assume that the valuations of both bidders are strictly above the reserve price $r \geq 0$.

Proposition 1 *In a complete information environment with two bidders:*

(i) If $r = 0$, there is a unique Nash equilibrium. Bidder 1 bids uniformly on $[0, v_2]$; bidder 2 bids 0 with probability $\frac{\Delta v}{v_1}$, and bids uniformly on $[0, v_2]$ otherwise. The bidders' payoffs are $U_1 = \Delta v$, $U_2 = 0$.

(ii) If $v_1 > v_2 > r > 0$, there is a unique Nash equilibrium. Bidder 1 bids r with probability $\frac{r}{v_2}$, and bids uniformly on $(r, v_2]$ otherwise; bidder 2 bids 0 with probability $\frac{\Delta v + r}{v_1}$, and bids uniformly on $(r, v_2]$ otherwise. The bidders' payoffs are $U_1 = \Delta v$, $U_2 = 0$.

(iii) If $v_1 = v_2 = v > r > 0$, there is a continuum of Nash equilibria. Bidder i bids 0 and r with probabilities $\alpha\frac{r}{v}$ and $(1-\alpha)\frac{r}{v}$ (where $\alpha \in [0,1]$), respectively, and bids uniformly on $(r, v]$ otherwise; bidder j bids 0 with probability $\frac{r}{v}$, and bids uniformly on $(r, v]$ otherwise. The bidders' payoffs are $U_1 = U_2 = 0$.

Proof. Part (i) follows from Proposition 2 in Hillman and Riley (1989), part (ii) from Proposition 1 in Bertoletti (2008), and part (iii) from Proposition 3 in Siegel (2012). ∎

[10] The principal-agent literature often uses a first-order approach for describing an agent's best response. This approach is not going to work here because a bidder's expected payoff is typically discontinuous in own bid.

The Nash equilibria of the complete information all-pay auctions exhibit "rent dissipation". The bidder with the lower valuation gets a zero payoff, while the bidder with the higher valuation gets a payoff equal to the difference in the valuations. In the case of symmetric bidders the rents are fully dissipated: the total payments of the bidders are equal to the value of the good, and each bidder gets a zero payoff.

In general the set of correlated equilibrium payoffs is at least as large as the convex hull of the payoffs of Nash equilibria: the players can use a public randomization device (or replicate it by a jointly controlled lottery) to coordinate on different Nash equilibria with different probabilities.[11] In the all-pay auction, however, this observation is not useful, because either the Nash equilibrium is unique, or all Nash equilibria yield the same payoffs for the bidders. In certain games (like "the chicken game") there exist correlated equilibrium payoffs outside of the convex hull of the Nash equilibrium payoffs, but the circumstances when this happens are not well understood.

It is known that the sets of correlated equilibrium payoffs and Nash equilibrium payoffs coincide in the two-player zero-sum games (Rosenthal, 1974). Regardless of whether we consider Nash or correlated equilibrium, each player has a strategy that guarantees him an expected payoff at least as large as his value of the game. Hence, by the minmax theorem, the players' expected payoffs must be equal to their respective values under either solution concept. While the all-pay auction game is not a zero-sum game, in some cases it turns out to be "strategically equivalent" to a particular zero-sum game (in a sense of Moulin and Vial, 1978). The next result takes advantage of this observation and shows that the bidders' correlated equilibrium payoffs are the same as under Nash equilibrium.[12]

Proposition 2 *In a complete information environment with two bidders, such that $r = 0$ or $v_1 > v_2$, every correlated equilibrium is payoff equivalent to the Nash equilibrium.*

Proof. Consider an auxiliary game with the same players and the same action spaces as in the all-pay auction, and with the payoffs derived from the all-pay auction payoffs for every $(b_i, b_j) \in A$

[11] For example, in the second-price auction without the reserve price there are many Nash equilibria: the truthful equilibrium, and infinitely many equilibria involving weakly dominated strategies (Blume and Heidhues, 2004). If the bidders correlate their play, then it is possible to sustain the following collusive scheme. Before the auction a designated winner is randomly chosen; during the auction the bidders coordinate on the equilibrium where the designated winner obtains the good for free by submitting a very high bid while the other bidders submit zero bids. See Section V.A in the working paper version of Marshall and Marx (2009).

[12] The result in Proposition 2 for the case $r = 0$ follows from a more general result, Proposition 6 in Section 4.1, that allows for incomplete information. However, the proof here is different and more intuitive.

as follows:

$$w_i(b_i, b_j) = \frac{1}{v_i} u_i(b_i, b_j) + \frac{1}{v_j} b_j - \frac{1}{2} = \rho_i(b_i, b_j) - \frac{1}{v_i} b_i + \frac{1}{v_j} b_j - \frac{1}{2} \tag{1}$$

Note that $\frac{1}{v_i}$ is strictly positive and $\frac{1}{v_j} b_j - \frac{1}{2}$ is independent of b_i. This implies that the best response of each bidder in the auxiliary game is the same as in the all-pay auction, and thus the two games have the same Nash equilibria and the same correlated equilibria.

Next we show that the auxiliary game is zero-sum when $r = 0$ or $v_1 > v_2$. If $r = 0$, then $\sum_{j=1}^{2} \rho_j(b) = 1$ for every $b \in A$, and thus $\sum_{j=1}^{2} w_j(b) = \sum_{j=1}^{2} \rho_j(b) - 1 = 0$ for every $b \in A$. If $r > 0$, then $\sum_{j=1}^{2} \rho_j(b) = 1$ for every b, except for $b = (0, 0)$. Note, however, that bid 0 is not rationalizable for bidder 1 when $v_1 > v_2$. This is because no rational bidder bids above his value, and thus bidder 1 strictly prefers to bid slightly above v_2 to bidding 0. Hence, although the auxiliary game is not zero-sum, it can be turned into a zero-sum game by removing bid 0 for bidder 1. This operation will not disturb the Nash equilibria or correlated equilibria because bidding 0 is not rationalizable for player 1, and is thus not played in either equilibrium.

We will use the following two properties of the zero-sum games: (i) the players' expected payoffs from any correlated equilibrium and from any Nash equilibrium of a zero-sum game are equal to their respective values of the game; (ii) if μ is a correlated equilibrium of a zero-sum game, then the pair of its marginals (μ_1, μ_2) is a Nash equilibrium. These properties have been established for finite games (Lemma 1 and Corollary 1 in Rosenthal, 1974), but it is straightforward to show that they also hold for zero-sum games with infinite strategy sets which have a Nash equilibrium.

Let (μ_1^*, μ_2^*) be the Nash equilibrium strategy profile, and (W_1^*, W_2^*) be the Nash equilibrium bidders' payoffs in the auxiliary zero-sum game. This Nash equilibrium is unique when $r = 0$ or $v_1 > v_2$ (Proposition 1). Then the expected payoff of player i from any correlated equilibrium μ in the all-pay auction is

$$\int_A u_i(b)\,\mu(db) = v_i \left(\int_A w_i(b)\,\mu(db) - \frac{1}{v_j} \int_A b_j \mu(db) + \frac{1}{2} \right) = v_i \left(W_i^* - \frac{1}{v_j} \int_{A_j} b_j \mu_j^*(db_j) + \frac{1}{2} \right)$$

where the first equality uses the definition of $w_i(\cdot)$ in (1); the second equality is true because $\int_A w_i(b)\,\mu(db) = W_i^*$ by property (i) mentioned above, (μ_1, μ_2) is a Nash equilibrium by property (ii), and $(\mu_1, \mu_2) = (\mu_1^*, \mu_2^*)$ by the uniqueness of the Nash equilibrium. Hence, every correlated equilibrium of the all-pay auction is payoff equivalent to the Nash equilibrium.[13] ■

One may conjecture that the payoff equivalence of Nash and correlated equilibria has something to do with the fact that the Nash equilibrium is unique when $r = 0$ or $v_1 > v_2$. While there may be some connection, the uniqueness of Nash equilibrium in general does not imply payoff equivalence of Nash and correlated equilibria.[14]

Lopomo, Marx and Sun (2011) provide a result of a similar kind for the first-price auction with two symmetric bidders and incomplete information. They show that collusion based on bid recommendations and pre-auction side payments is completely ineffective: every such collusive scheme is payoff equivalent to the unique Nash equilibrium of the auction. This implies that in the setting of Lopomo, Marx and Sun (2011) the correlated equilibria are also payoff equivalent to the unique Nash equilibrium. The first-price auction is not strategically equivalent to a zero-sum game, and the proof in Lopomo, Marx and Sun (2011) seems to rely on very different ideas.[15]

The case when $v_1 = v_2 = v$ and $r > 0$ is distinct. The proof of Proposition 2 cannot be extended to cover this case: though the all-pay auction can still be shown to be strategically equivalent to the auxiliary game, this game is no longer a zero-sum game because the bid profile $(b_1, b_2) = (0, 0)$ cannot be ruled out. (Indeed, in some Nash equilibria both bidders submit null bids with positive probability.) Next, we show that in this case there exist correlated equilibria that are not payoff equivalent to the Nash equilibrium. Paradoxically, the presence of the reserve price may help the bidders to avoid complete rent dissipation and thus be to the bidders' advantage.

Example 1 *Let $v_1 = v_2 = 1$, and $r \in (0, 1)$. The bidders are given recommendations according to*

[13] An alternative way to finish the proof is to use Theorem 3 from Moulin and Vial (1978), which shows that for any game that is strategically equilvalent to a zero-sum game there exist no "correlation scheme" that improves upon all Nash equilibrium payoffs for both players. The class of "correlation schemes" in Moulin and Vial (1978) includes correlated equilibria, as well as some other joint action plans that require certain commitment on the part of the players.

[14] See example on p.204 in Moulin and Vial (1978).

[15] Specifically, they formulate the collusive problem as a linear programming problem, and, by discretizing the bid spaces, manage to derive some properties of the dual problem which imply the result.

the following probability distribution, where "bid above r" means "bid uniformly on $(r, 1]$":

1's bid \ 2's bid	bid 0	bid above r
bid 0	0	$r(1-r)$
bid r	r^2	0
bid above r	$(1-r)r$	$(1-r)^2$

If bidder 1 is suggested to bid 0, then he knows that the opponent bids aggressively, and thus he is content to submit a null bid. If bidder 1 is suggested to bid r, then he knows that the opponent bids 0, and thus his best response is to bid r. If bidder 1 is suggested to bid above r, then his probability distribution over the opponent's bids is the same as in one of the Nash equilibria, and thus he is indifferent between all bids not higher than 1. Whether bidder 2 is suggested to bid 0 or to bid above r, he is indifferent between all bids not higher than 1.

Bidder 1 gets a payoff of $1-r$ when he is suggested to bid r, and a zero expected payoff otherwise. Hence, his ex ante payoff is $r^2(1-r)$. The expected payoff of bidder 2 is zero.

Let us compare the above correlated equilibrium with a Nash equilibrium for some $\alpha \in [0, 1]$ (described in part (iii) of Proposition 1). Under the Nash equilibrium bid profiles $(0, 0)$ and $(r, 0)$ are played with probabilities αr^2 and $(1-\alpha)r^2$, respectively, while under the correlated equilibrium $(0, 0)$ is never played, and $(r, 0)$ is played with probability r^2. Hence, under the correlated equilibrium the probability weight is shifted away from an unfortunate event (where both bidders bid zero and no one wins the good) to a nice event (where bidder 1 wins the good at a low price r). Next, under the Nash equilibrium the event when bidder 1 bids 0 and bidder 2 bids above r takes place with probability $\alpha r(1-r)$, and the event when bidder 1 bids r and bidder 2 bids above r takes place with probability $(1-\alpha)r(1-r)$. Under the correlated equilibrium the former event takes place with probability $r(1-r)$ and the latter event does not happen. Hence, under the correlated equilibrium the probability weight is shifted away from an unprofitable event (where bidder 1's bid r is wasted because bidder 2 bids above r) to a more profitable event (where bidder 1 bids 0 instead). Thus, the correlated equilibrium results in positive profits for bidder 1, while every Nash equilibrium features full rent dissipation.

The next result describes some other payoffs that can be achieved with correlated equilibria.[16] Note that for a given reserve price r the sum of the bidders' payoffs is constrained above by $v - r$, and thus the result implies that the bidders can approximate "ideal collusion" as reserve price r approaches v.

Proposition 3 *In a complete information environment with two bidders, such that $v_i = v$ for $i = 1, 2$ and $v > r > 0$, for every $(U_1, U_2) \in \mathbb{R}^2_+$ such that*

$$\begin{cases} U_1 + \frac{r^2}{v^2} U_2 \leq \frac{r^2(v-r)}{v^2} \\ \frac{r^2}{v^2} U_1 + U_2 \leq \frac{r^2(v-r)}{v^2} \end{cases}$$

there exists a correlated equilibrium that gives bidder i payoff U_i.

3.2 Three or more bidders

Here we consider the case of three or more bidders, and we restrict attention to the situations when the bidders are symmetric. Suppose each bidder has a valuation v that it is strictly above the reserve price $r \geq 0$. It is known that in this case there are many Nash equilibria, in every one of them complete rent dissipation takes place, and each bidder gets a zero payoff.[17]

Unlike in the case of two players, a connection between the all-pay auction and a certain class of zero-sum games is not going to allow us to obtain an analog of Proposition 2. In the zero-sum games with three or more players there is no minmax theorem to rely upon, and the sets of correlated equilibrium payoffs and Nash equilibrium payoffs no longer coincide. Hence, even though in the case of no reserve price it is possible to construct an auxiliary zero-sum game that is strategically equivalent to the all-pay auction, this does not imply that the correlated equilibria and Nash equilibria are payoff equivalent. Indeed, in the next example we describe a correlated equilibrium where the bidders get positive payoffs.

Example 2 *Let $n = 3$, $v = 1$, and $r = 0$. Consider the following symmetric correlation rule. First, a pair of bidders is randomly chosen, with each pair being equally likely to be chosen. Next, the*

[16] We conjecture that no other payoffs can be achieved by correlated equilibria, but we have not managed to prove this because of the technical difficulties outlined in the beginning of this section.

[17] This follows from Proposition 8 in Section 4.2. For a characterization of Nash equilibria in the asymmetric cases when there is no reserve price see Baye, Kovenock and de Vries (1996).

bidders receive private bid recommendations without being told whether they have been chosen. The bidder who is not chosen is recommended to bid 0*, and the chosen bidders are given recommendations according to the following probability distribution, where "bid low" means "bid uniformly on* $\left(0, \frac{1}{2}\right]$*", and "bid high" means "bid uniformly on* $\left(\frac{1}{2}, 1\right]$*":*

i's bid \ *j's bid*	*bid* 0	*bid low*	*bid high*
bid 0	0	$\frac{2}{26}$	0
bid low	$\frac{2}{26}$	$\frac{7}{26}$	$\frac{5}{26}$
bid high	0	$\frac{5}{26}$	$\frac{5}{26}$

If a bidder is suggested to bid high, then he knows that he competes against one chosen opponent who is equally likely to bid low or high. The probability of winning with bid $b \in [0, 1]$ *is equal to* b*, and thus the payoff from any such bid is* 0*.*

If a bidder is suggested to bid low, then he knows that he competes against one chosen opponent who either bids 0*, bids low, or bids high, with probabilities* $\frac{1}{7}$*,* $\frac{1}{2}$*, and* $\frac{5}{14}$*, respectively. The probability of winning with bid* $b > 0$ *is equal to* $\min\left\{b + \frac{1}{7}, \frac{5}{7}b + \frac{2}{7}\right\}$*, and thus the payoff from any* $b \in \left(0, \frac{1}{2}\right]$ *is* $\frac{1}{7}$*, and the payoff from any* $b \in \left(\frac{1}{2}, 1\right]$ *is below* $\frac{1}{7}$*.*

If a bidder is suggested to bid 0*, then he knows that either he was not chosen and thus faces two potentially active opponents, or that he was chosen but only his opponent was suggested to bid above* 0*. It is possible to show that the probability of winning with bid* $b > 0$ *is equal to* $\min\left\{\frac{14}{15}b^2 + \frac{8}{15}b, \frac{2}{3}b^2 + \frac{1}{3}\right\}$*, and thus the payoff from any* $b > 0$ *is nonpositive.*

In each case the bidder is willing to comply with the recommendation. Each chosen bidder gets an expected payoff of $\frac{1}{7}$ *when he is suggested to bid low (which happens with probability* $\frac{7}{13}$*), and a zero expected payoff otherwise. Each pair of bidders is equally likely to be chosen, and thus each bidder's ex ante payoff is* $\frac{2}{39}$*.*

This bid rotation correlation scheme holds together due to careful management of the amount of information revealed to each player. To see the basic idea, note first that there exists a Nash equilibrium such that two bidders bid uniformly on $(0, 1]$, and the third bidder bids 0. Second, suppose that in advance a mediator randomly chooses two bidders who are to take active roles in the above Nash equilibrium, and each bidder is privately informed of his role. Finally, suppose that with a small probability a mediator "cheats" one of the chosen bidders, and, instead of informing him that he is to take an active role, tells him to bid 0. If the probability of such "cheating" is sufficiently small, then the bidders will still be content to comply whenever they are recommended to bid 0. This "cheating" reduces the intensity of bidding, and thus raises the bidders' payoffs.[18]

The next result describes some other payoffs that can be obtained in symmetric correlated equilibria for any given reserve price r and any number of bidders $n \geq 3$.[19] In particular, the result implies that in the correlated equilibrium the bidders can avoid full rent dissipation. Even in the limit, as the number of bidders increases without bound, the sum of the bidders' expected payoffs does not have to go to zero (e.g., when $r = 0$, in the best constructed correlated equilibrium $nU \to \frac{2}{9}v$ as $n \to \infty$).

Proposition 4 *In a complete information environment with $n \geq 3$ symmetric bidders, such that $v_i = v$ for every i and $v > r \geq 0$, for every $U \in \left[0, \frac{2(v-r)}{n} \frac{(n-2)v^2+(n-2)vr+2nr^2}{(9n-14)v^2+(6n-8)vr+(n+6)r^2}\right]$ there exists a correlated equilibrium that gives each player payoff U.*

4 All-pay auctions with incomplete information

4.1 Two bidders

In this section we study and compare Nash equilibria and communication equilibria of the all-pay auction under incomplete information. The communication equilibrium solution concept is similar to the correlated equilibrium in that it allows for coordination between the players via correlation of the recommended actions. In addition, communication equilibrium gives the players possibilities to talk about their private information. Like in the case of complete information, we would like to know under what circumstances there exist communication equilibria that are not payoff equivalent to the Nash equilibrium, and, whenever such communication equilibria exist, we would like to understand how they work.

Characterizing communication equilibria in games with large action spaces is challenging, in

[18] The actual correlated equilibrium in Example 2 is slightly more involved: the active bidders are in addition recommended whether to bid high or low, and the probabilities of the mediator's profiles of recommendations are adjusted to ensure incentive compatibility.

[19] It is possible to construct correlated equilibria with asymmetric payoffs, but we do not present them here. We do not claim that the upper bound on the payoff in the presented symmetric correlated equilibria is the highest one could achieve.

much of the same way as characterizing correlated equilibria is, because one has to deal with many obedience incentive constraints. In addition, the players must be given incentives to report their types truthfully, and one has to worry about compound deviations when a player first misreports his type and then disobeys the recommended actions. For a class of environments we manage to demonstrate payoff equivalence between Nash equilibria and communication equilibria using an approach similar to that under complete information (Proposition 2 in Section 3). For another class of environments we build on the results on correlated equilibria from Section 3 and construct communication equilibria that are distinct from Nash equilibria.

First, we summarize some of the existing results on Nash equilibria with two bidders that we will refer to in this section.[20]

Proposition 5 *In an incomplete information environment with two bidders:*

(i) Let $r = 0$*, bidder i's value be continuously distributed on* $[0, 1]$ *with density that is continuously differentiable and positive on* $(0, 1)$*, independently of the opponent's value. There is a unique Nash equilibrium, this equilibrium is in pure strategies, and it is strictly monotonic.*

(ii) Let $r > 0$*, bidder i's value is* 0 *or* v *(such that* $v > r$*) with probabilities* p_i *and* $1 - p_i$*, independently of the opponent's value. Nash equilibrium exists. In every Nash equilibrium type* 0 *of each bidder gets a zero payoff, type* v *of each bidder gets a payoff of* $\max\{\overline{p}v - r, 0\}$*, where* $\overline{p} = \max\{p_1, p_2\}$*.*

Proof. Part (i) follows from Theorem 1 in Amann and Leininger (1996). See Appendix for the proof of part (ii). ■

Our first result on communication equilibria is about the case of no reserve price. Note that it involves rather mild restrictions on the distributions of the players valuations. Part (i) of Proposition 5 describes one set of sufficient conditions for existence of the unique Nash equilibrium, but there are also others.[21] Note that Nash equilibrium often fails to exist in the all-pay auction with no

[20] There exist other results on Nash equilibria of the all-pay auction with incomplete information, but many of them are about the case of interdependent valuations which is not covered in this paper. See, for example, Krishna and Morgan (1997), Lizzeri and Persico (2000), Siegel (2012).

[21] For example, the results of Siegel (2012) imply that the Nash equilibrium exists and is unique when there are finitely many strictly positive values for every bidder.

reserve price when the bidders' values are equal to zero with positive probability, and so ruling out such distributions does not seem very restrictive.

Proposition 6 *Consider an incomplete information environment with two bidders and no reserve price such that the values of the bidders are strictly positive with probability one, and there exists a unique Nash equilibrium. Then every communication equilibrium is interim payoff equivalent to the Nash equilibrium.*

The idea behind the result can be understood with the help of the connection between the all-pay auction and the auxiliary zero-sum game introduced in the proof of Proposition 2. Since the payoffs of the two games are related according to formula (1), it is easy to see that the best responses for each type of each bidder for the two games coincide, even when there is uncertainty about the opponent's value. Suppose, first, that the bidders are not allowed to communicate about their private information. Then we can consider the all-pay auction as a strategic form game, and, in a similar way as in Proposition 2, we can show that using correlated recommendations does not help to achieve payoffs different from the Nash equilibrium payoffs.

Next, suppose that the bidders are allowed to communicate about their private information. One would expect that in a zero-sum game the players are not too keen on truthfully revealing their private information because it may be used against them by the opponents. This is indeed confirmed by Proposition 6 that says that no payoff-consequential voluntary sharing of private information is possible, and this result can be viewed as a version of the "no trade" result (Milgrom and Stokey, 1982). Note that in any communication equilibrium a bidder can play the following strategy: (i) regardless of own type randomize over the type reports according to the prior probability distribution; (ii) regardless of the mediator's recommendations choose the same bids as in the Nash equilibrium. It turns out that in our auxiliary zero-sum game each player can guarantee himself at least his Nash equilibrium payoff by playing such a strategy. The no trade result then follows from the facts that the players have common prior, and that every allocation, including the Nash equilibrium outcome, is ex ante Pareto efficient (because the game is zero-sum).

Similarly to the case of complete information, a result analogous to Proposition 6 is likely to hold in some environments with strictly positive reserve price if we can rule out the case when both bidders choose null bids. This, for example, happens when there is no overlap in the supports of the bidders' valuations, say, $\overline{v}_2 < \underline{v}_1$, and the reserve price is low enough, $\underline{v}_1 > r$. Then bid 0 is not rationalizable for bidder 1 for any beliefs over the opponent's types, because he prefers to bid slightly above $\overline{v}_2$ to bidding 0. It remains an open question whether results analogous to Proposition 6 hold when the bidders have correlated or interdependent values.[22]

There exist some related results for the first-price auction under incomplete information. As mentioned in the previous section, Lopomo, Marx and Sun (2011) study a model of collusion with pre-auction communication, side payments and bid recommendations in the first-price auction with two bidders. They show that in a symmetric environment with two possible types (with or without reserve) and discrete bid spaces the collusive equilibria (and thus communication equilibria) are payoff equivalent to the unique Nash equilibrium.[23] Azacis and Vida (2010) study a similar environment in a model with continuum of bids. They show that several restricted versions of communication equilibrium are payoff equivalent to the Nash equilibrium, and they conjecture that the same is true for the canonical communication equilibrium.[24]

Kovenok, Morath and Münster (2010) consider the incentives of the bidders in the all-pay auction to share their private information which is assumed to be verifiable. First, each bidder decides whether to disclose his value to the opponent, after that the bidders play the all-pay auction according to Nash equilibrium given their updated beliefs. In the case when the bidders' disclosure decisions take place after they observe the realizations of their values there exist equilibria with full information disclosure as well as equilibria without any information sharing. In the model of Kovenok, Morath and Münster (2010) the bidders can hide their information but cannot lie about it, and this makes it is easier to achieve information revelation than in our setting. On the other hand, our model is more conducive to sustaining information revelation in the following respect. In Kovenok, Morath and Münster (2010) the bidders' payoffs following any disclosure decision are determined by the unique continuation Nash equilibria given the beliefs, but in our setting there

[22] If bidder i is uncertain about his valuation v_i, then a transformation of his payoff according to formula (1) is likely to change his best response, because in general $E[1/v_i] \neq 1/E[v_i]$. For the case of correlated values it is unclear how part (i) of the deviational strategy described in the previous paragraph has to be adjusted in order to guarantee a bidder his Nash equilibrium payoff.

[23] Lopomo, Marx and Sun (2011) check the robustness of the result by studying numerically other environments with two bidders.

[24] Azacis and Vida (2010) also present several results on the optimal collusive schemes in the first-price auction with omniscient mediator who is assumed to know the bidders values. In such a model the bidders can generally do better than in the Nash equilibrium without communication: the mediator selectively reveals information on the bidders' values to induce asymmetric beliefs which lead to less aggressive bidding. Bergemann, Brooks and Morris (2012) also study related constructions.

may be multiple continuation correlated equilibria, and thus the bidders' payoffs are not necessarily uniquely determined by the beliefs. We demonstrate in the next example how this feature allows to provide incentives for information revelation.

Example 3 *Let $r \in (0,1)$, bidder 1's value is 0 or 1 with probabilities p_1 and $1-p_1$, bidder 2's value is 1 with probability 1. By part (ii) of Proposition 5 the Nash equilibrium payoff of each bidder with value 1 is $\max\{p_1 - r, 0\}$.*

Consider the following scenario with pre-auction communication. Bidder 1 sends a cheap talk message to bidder 2, and then the bidders play the all-pay auction according to Nash equilibrium given the updated beliefs. It is easy to see that there is no cheap-talk equilibrium where bidder 1 truthfully reveals his type. If bidder 2 believes the announcement, then, after learning that $v_1 = 0$, bidder 2 bids r and expects to win with probability 1; after learning that $v_1 = 1$, the bidders play the Nash equilibrium that yields a zero payoff to each bidder. But then bidder 1 of type 1 can do better by reporting type 0, and then bidding slightly above r. This observation can be generalized to show that there are no cheap talk equilibria that result in payoffs that are different from the Nash equilibrium payoffs of the game without communication.[25]

This is no longer true if after a cheap talk announcement the bidders can correlate their play. Let $p_1 \in (0,r)$, so that the Nash equilibrium payoffs of the game without communication are zero for either type of bidder 1. Suppose type 0 of bidder 1 sends message m, and type 1 randomizes between messages m and m', so that the posterior beliefs that bidder 1's type is 0 following these two messages are $r + r^2(1-r)$ and 0, respectively. After message m the bidders play according to the Nash equilibrium, and after message m' according to the correlated equilibrium from Example 1. Type 0 of bidder 1 has no incentive to deviate because he is not interested in bidding anything other than 0. Type 1 of bidder 1 is willing to randomize between the messages because his expected payoff in either case is $r^2(1-r)$.

Next we show that in the situations with two sided uncertainty there also exist communication

[25] Here is a sketch of the argument. If following every message the posterior probability that bidder 1 is of type 0 is not higher than r, then after every message either type of bidder 1 gets zero payoff in the continuation Nash equilibrium. The prior belief p_1 must also be not higher than r, and hence the Nash equilibrium payoffs of the game without communication are the same. If the posterior beliefs following some messages are above r, then it is optimal for bidder 1 of type 1 to send messages that induce the highest possible belief that bidder 1's type is 0. However, since the equilibrium posterior beliefs must reflect the strategy of bidder 1, the highest posterior belief cannot be greater than the prior p_1. Hence, the posterior beliefs after every message must be equal to the prior, which implies payoff equivalence with the Nash equilibrium of the game without communication.

equilibria that result in the bidders' payoffs that are higher than in the Nash equilibrium. The behavior of the bidders of type v is coordinated in a way that is similar to the correlated equilibria in the complete information case, and when $p = 0$ the construction is identical to that in the proof of Proposition 3 for the case of symmetric payoffs.

Proposition 7 *Suppose there are two bidders and $r > 0$. Each bidder's value is 0 or v (such that $v > r$) with probabilities p and $1-p$, independently of the opponent's value. Then for every $p \in \left[0, \frac{r}{v}\right)$ there exists a communication equilibrium that gives each bidder of type v a positive payoff.*[26]

It is possible to show that in this environment all Nash equilibria are inefficient in a sense that the good sometimes remains unsold even though there is a bidder with value above the reserve price. This is because at least one bidder with value above the reserve price submits a null bid with positive probability.[27] However, the constructed communication equilibrium is efficient. If only one bidder has a value above the reserve price, then this bidder submits an active bid, and thus gets the good, with probability one; if both bidders have values above the reserve price, then with positive probability only one bidder submits an active bid.

Such a construction clearly involves some sharing of information about the values between the bidders. However, to provide the right incentives it is also important to maintain enough uncertainty about the opponents' values. For example, if it was known that the bidders' reports are revealed to their opponents with high probability, then it is possible to show that bidder of type v has a profitable deviation. The idea is similar to that in Example 3: reporting type 0 induces the opponent to bid at the reserve price, and thus it is profitable to report type 0 and then bid slightly above the reserve price. To make such a deviation unprofitable it is necessary that the bidders of type v bid aggressively enough when the opponent has reported type 0, and this is achieved through maintaining sufficient uncertainty about the opponent's type.

[26] It can be shown that an analogous result holds for the case when $p \in \left[\frac{r}{v}, 1\right]$ and r is sufficiently high. The proof is long, and thus not included in the paper.

[27] The inefficiency of Nash equilibrium is easy to observe when $p \approx 0$ and $r \approx v$. Efficiency requires that each bidder with value v submits an active bid, and thus the sum of the ex ante expected bids must be at least $2(1-p)r \approx 2v$. However, the bidders' gross ex ante payoff is only $v\left(1-p^2\right) \approx v$, which gives an impossibilty.

4.2 Three or more bidders

Here we continue to work with the symmetric independent case when each bidder's valuation can be either 0 or v. The Nash equilibrium payoffs when there are three or more bidders are described next.

Proposition 8 *Suppose there are $n \geq 3$ bidders and $r \geq 0$. Each bidder's value is 0 or v (such that $v > r$) with probabilities p and $1 - p$, independently of the opponents' values. Nash equilibrium exists. In every Nash equilibrium type 0 of each bidder gets a payoff of zero, type v of each bidder gets a payoff of* $\max\left\{p^{n-1}v - r, 0\right\}$.

Note that there can be no communication equilibrium such that some bidder gets a payoff below his Nash equilibrium payoff. Bidding 0 guarantees a payoff of (at least) zero; bidding the reserve price r leads to winning whenever all opponents have zero valuations, and thus guarantees a payoff of (at least) $p^{n-1}v - r$.

The next result demonstrates that there exist communication equilibria such that each bidder gets a payoff higher than in the Nash equilibrium. We focus on the case of no reserve price, but the construction can be extended to the case of positive reserve price as well.

Proposition 9 *In the environment described in Proposition 8, if $r = 0$, then for p sufficiently small there exists a communication equilibrium such that each bidder of type v gets a strictly higher payoff than in the Nash equilibrium.*

5 Discussion

We have shown that in the case of two bidders all correlated and communication equilibria are payoff equivalent to the Nash equilibrium if there is no reserve price, or if it is commonly known that one bidder has a strictly higher value. Hence, by the revelation principle for games with communication, the Nash equilibrium predictions in such cases are robust to pre-play communication between the bidders. Specifically, the bidders' expected payoffs and expected payments are unaffected by allowing them to communicate with each other prior to the auction using any mediated or unmediated communication protocol (the bidders' statements about their types cannot be verified).[28] It

[28] Another implication is that the Nash equilibrium prediction is robust to the bidders' having arbitrary correlated beliefs about payoff-irrelevant states of the world, as long as these beliefs are consistent with the common prior

will be interesting to see if these payoff equivalence results hold beyond the incomplete information environments with independent private values.

The results on payoff equivalence of Nash and correlated equilibria, and Nash and communication equilibria can be extended to related two-player games such as all-pay auctions with general cost functions (Siegel, 2009), as well as to models of contests where the determination of the winner stochastically depends on the amount of resources committed by the participants.[29] Using a payoff transformation similar to (1) in Proposition 2 it is possible to show that many versions of such models are strategically equivalent to particular zero-sum games; this strategic equivalence can then be used to establish payoff equivalence of predictions under the different solution concepts.[30]

We have demonstrated that in several particular settings with three or more symmetric bidders, and two symmetric bidders and a positive reserve price, there may exist correlated and communication equilibria that are not payoff equivalent to the Nash equilibrium payoffs. Specifically, the bidders' payoffs can be higher than in the Nash equilibrium, but never lower. This suggests that allowing the bidders to communicate before bidding may improve their payoffs, and, in case the bids represent socially unproductive expenses, communication may be unambiguously good for the society. On the other hand, if the bids represent transfers to the seller or some socially productive activities, then one should take into account that though communication may improve the bidders' payoffs and efficiency of the allocation, it may also result in less intense bidding. A characterization of all environments where there exist correlated and communication equilibria distinct from Nash equilibrium remains an open question. It may also be interesting to compute the set of all payoffs that can be achieved by the communication or correlated equilibria in such cases, and verify if it is true in general that the payoff of any bidder cannot fall below his Nash equilibrium payoff.

The correlated and communication equilibria describe all possible outcomes that can be potentially achieved with the help of communication in a self-enforcing way. It is important to know how exactly communication between the bidders can be organized if one wants to implement the outcome of some particular correlated or communication equilibrium. One natural approach is to find an extra player who is able to play the role of a mediator as described in Section 2. The

(Aumann, 1974).

[29] Such models include rent-seeking contests and models of conflict (Tullock, 1980; Hirshleifer, 1989), tournaments between workers (Lazear and Rosen, 1981), R&D contests (Baye and Hoppe, 2003), etc.

[30] The connection with zero-sum games may also be useful for obtaining more general results on existence, uniqueness, and payoff characterization of Nash equilibria in such games.

mediator should be able to communicate with each bidder privately, or, alternatively, he should be able to communicate with each bidder by encrypted messages according to a previously agreed upon code (Lehrer and Sorin, 1997). The mediator should also be able to commit to his communication strategy, but if this is impossible, then in some situations "strategic mediators" can be used as well.[31]

It may also be interesting to know if the outcomes of the correlated and communication equilibria can be implemented with a help of some unmediated communication procedure between the bidders.[32] Let us briefly discuss the case of two bidders. In our constructed correlated equilibria it is essential that the bidders remain uncertain about the strategies that are recommended to their opponents, and in the communication equilibria it is also important that the bidders are uncertain about the values of the opponents. Thus it is unlikely that the outcomes of these correlated and communication equilibria can be implemented by some simple unmediated communication procedure, whereby the bidders directly communicate with each other, because such communication cannot generate the desired correlated beliefs.[33] This implies that successful unmediated communication must use correlation devices and/or noisy communication channels.

To illustrate, here is one of many possible ways to implement the outcome of the correlated equilibrium in Example 1. Bidder 2 with probability r announces to bidder 1 that he will bid 0, and with probability $1 - r$ that he will bid above r, and then bids in the auction according to his announcement. The announcement is made in a foreign language such that bidder 1 is able to understand it with probability r, and the language ability of bidder 1 is known only to him. If bidder 1 understands the announcement of bidder 2, then he optimally responds to it, i.e. bids r if bidder 2 says he will bid 0, and bids 0 if bidder 2 says he will bid above r. If bidder 1 does not understand the announcement of bidder 2, he bids above r.[34]

[31] For example, Ivanov (2010) investigates how strategic mediators can be used in sender-receiver games.

[32] One can use the existing results on implementation of correlated and communication equilibria without a mediator for general games. See Forges (2009) for a survey. The constructions in these papers are for finite sets of actions, but they can be adapted to implement the correlated and communication equilibria constructed here.

[33] It also seems unlikely that there may exist other interesting correlated and communication equilibria that can be implemented by direct unnmediated communication before the all-pay auction. In the environment with complete information such pre-play communication only allows to achieve payoffs that are in the convex hull of the Nash equilibrium payoffs of the game without communication (Forges, 1990). We conjecture that it is impossible to improve upon the Nash equilibria using only such pre-play communication in the all-pay auction with incomplete information as well.

[34] Blume and Board (2013) introduced the idea that instead of communication via a noisy communication channel it is possible to use direct communication when there is uncertainty about the ability of the players to understand some messages.

6 Appendix

6.1 Proofs of Section 3

Proof of Proposition 3. Denote $u_i = \frac{1}{v-r} U_i$ for $i = 1, 2$, and fix $(u_1, u_2) \in \mathbb{R}_+^2$ such that $v^2 u_i + r^2 u_j \le r^2$. The bidders are given recommendations according to the following probability distribution, where "bid above r" means "bid uniformly on $(r, v]$".[35]

1's bid \ 2's bid	bid 0	bid r	bid above r
bid 0	0	u_2	$\frac{r}{v+r}(1-u_1-u_2)$
bid r	u_1	0	0
bid above r	$\frac{r}{v+r}(1-u_1-u_2)$	0	$\frac{v-r}{v+r}(1-u_1-u_2)$

Suppose bidder 1 is suggested to bid 0 and bids $b \in (r, v]$ instead.[36] Then his payoff is

$$\begin{aligned} & \left(\frac{u_2}{u_2 + \frac{r}{v+r}(1-u_1-u_2)} + \frac{\frac{r}{v+r}(1-u_1-u_2)}{u_2 + \frac{r}{v+r}(1-u_1-u_2)} \left(\frac{b-r}{v-r} \right) \right) v - b \\ = & \frac{r^2 u_1 + v^2 u_2 - r^2}{(v^2 - r^2)\left(u_2 + \frac{r}{v+r}(1-u_1-u_2) \right)} (v-b) \le 0 \end{aligned}$$

where the inequality follows from $r^2 u_1 + v^2 u_2 \le r^2$. If bidder 1 is suggested to bid r, then it is clearly optimal to comply, since the opponent bids 0 in such case. If bidder 1 is suggested to bid above r, then he is indifferent between all bids since the payoff from bidding $b \in [r, v]$ is

$$\left(\frac{r}{v} + \left(1 - \frac{r}{v}\right) \left(\frac{b-r}{v-r} \right) \right) v - b = 0$$

To summarize, bidder 1 gets a payoff of $v - r$ when he is suggested to bid r, and zero payoff otherwise. Hence, his ex ante payoff is $u_1 (v-r) = U_1$. Using a similar argument for bidder 2, we conclude that the considered correlation rule is a correlated equilibrium, and it achieves the desired

[35] It is straightforward to verify that the entries in table: (i) sum up to one; and (ii) are nonnegative. The latter is because

$$u_1 + u_2 = \frac{(v^2 u_1 + r^2 u_2) + (r^2 u_1 + v^2 u_2)}{v^2 + r^2} \le \frac{2r^2}{v^2 + r^2} < 1$$

where the first inequality follows from $v^2 u_i + r^2 u_j \le r^2$.

[36] Bidding exactly r is dominated by bidding slightly above r if there is a positive probability that the opponent bids r.

payoffs. ■

Proof of Proposition 4. Fix $U \in \left[0, \overline{U}\right]$, where $\overline{U} = \frac{2(v-r)}{n}\frac{(n-2)v^2+(n-2)vr+2nr^2}{(9n-14)v^2+(6n-8)vr+(n+6)r^2}$. Consider the following symmetric correlation rule. First, a pair of bidders is randomly chosen, with each pair being equally likely to be chosen. Next, the bidders receive private bid recommendations without being told whether they have been chosen. The bidders who are not chosen are recommended to bid 0, and the chosen bidders are given recommendations according to the following probability distribution, where "bid low" means "bid uniformly on $\left(r, \frac{1}{2}(v+r)\right]$", and "bid high" means "bid uniformly on $\left(\frac{1}{2}(v+r), v\right]$", and where $x = \frac{1}{v+r}\left(\frac{v-r}{4} - \frac{3v+r}{v}\frac{n}{8}U\right)$.[37]

i's bid \ j's bid	bid 0	bid low	bid high
bid 0	0	$\pi_l = \frac{2r}{v-r}x + \frac{v+r}{v-r}\frac{n}{2v}U$	$\pi_h = \frac{2r}{v-r}x$
bid low	$\pi_l = \frac{2r}{v-r}x + \frac{v+r}{v-r}\frac{n}{2v}U$	$\pi_{ll} = x + \frac{n}{2v}U$	$\pi_{hl} = x$
bid high	$\pi_h = \frac{2r}{v-r}x$	$\pi_{hl} = x$	$\pi_{hh} = x$

If a bidder is suggested to bid 0, then he knows that either he was not chosen (which happens with probability $\frac{n-2}{n}$), or that he was chosen but only his opponent was suggested to bid above 0 (which happens with probability $\frac{2}{n}(\pi_l + \pi_h)$).

If this bidder bids $b \in \left(r, \frac{1}{2}(v+r)\right]$ instead, then he has a chance to win only if none of his opponents bid high. In particular, bidder i could win if (i) he was not chosen, and one chosen bidder bids low (which happens with probability $\frac{n-2}{n}(2\pi_l)$); (ii) he was not chosen, and two chosen bidders bid low (which happens with probability $\frac{n-2}{n}\pi_{ll}$); (iii) he was chosen, and his opponent bids low (which happens with probability $\frac{2}{n}\pi_l$). The expected payoff this bidder is then

$$\left(\frac{\frac{n-2}{n}(2\pi_l) + \frac{2}{n}\pi_l}{\frac{n-2}{n} + \frac{2}{n}(\pi_l + \pi_h)}\left(\frac{b-r}{\frac{1}{2}(v+r) - r}\right) + \frac{\frac{n-2}{n}\pi_{ll}}{\frac{n-2}{n} + \frac{2}{n}(\pi_l + \pi_h)}\left(\frac{b-r}{\frac{1}{2}(v+r) - r}\right)^2\right) v - b \qquad (2)$$

[37] It is straightforward to verify that the entries in table: (i) sum up to one; and (ii) are nonnegative. The latter is because

$$\frac{v-r}{4} - \frac{3v+r}{v}\frac{n}{8}U \geq \frac{v-r}{4} - \frac{3v+r}{v}\frac{n}{8}\overline{U} = \frac{(v-r)^2(v+r)((3n-4)v+nr)}{2v((9n-14)v^2+(6n-8)vr+(n+6)r^2)} \geq 0$$

where the equality is by definition of $\overline{U}$.

Note that (2) is equal to $-r$ if $b = r$. If $b = \frac{1}{2}(v+r)$, then (2) becomes

$$\frac{\frac{n-2}{n}(2\pi_l + \pi_{ll}) + \frac{2}{n}\pi_l}{\frac{n-2}{n} + \frac{2}{n}(\pi_l + \pi_h)} v - \frac{1}{2}(v+r) = \frac{n}{8} \frac{(9n-14)v^2 + (6n-8)vr + (n+6)r^2}{(v-r)((n-2)v + nr + nU)} \left(U - \overline{U}\right) \leq 0 \quad (3)$$

where the inequality holds since $U \leq \overline{U}$. Since (2) is convex in b, this implies that it is nonpositive for every $b \in \left[r, \frac{1}{2}(v+r)\right]$.

If this bidder bids $b \in \left(\frac{1}{2}(v+r), v\right]$ instead, then he wins for sure if none of his opponents bid high, and has a chance to win otherwise. In particular, bidder i wins for sure if (i) he was not chosen, and none of the chosen bidders bid high (which happens with probability $\frac{n-2}{n}(2\pi_l + \pi_{ll})$); (ii) he was chosen, and his opponent does not bid high (which happens with probability $\frac{2}{n}\pi_l$). Also bidder i could win if (i) he was not chosen, and one chosen bidder bids high (which happens with probability $\frac{n-2}{n}(2\pi_h + 2\pi_{hl})$); (ii) he was not chosen, and two chosen bidders bid high (which happens with probability $\frac{n-2}{n}\pi_{hh}$); (iii) he was chosen, and his opponent bids high (which happens with probability $\frac{2}{n}\pi_h$). The expected payoff of this bidder is then

$$\begin{aligned}&\left(\frac{\frac{n-2}{n}(2\pi_l + \pi_{ll}) + \frac{2}{n}\pi_l}{\frac{n-2}{n} + \frac{2}{n}(\pi_l + \pi_h)} + \frac{\frac{n-2}{n}(2\pi_h + 2\pi_{hl}) + \frac{2}{n}\pi_h}{\frac{n-2}{n} + \frac{2}{n}(\pi_l + \pi_h)} \left(\frac{b - \frac{1}{2}(v+r)}{v - \frac{1}{2}(v+r)}\right) + \right. \\ &\left. + \frac{\frac{n-2}{n}\pi_{hh}}{\frac{n-2}{n} + \frac{2}{n}(\pi_l + \pi_h)} \left(\frac{b - \frac{1}{2}(v+r)}{v - \frac{1}{2}(v+r)}\right)^2\right) v - b\end{aligned} \quad (4)$$

Note that (4) is equal to (3) if $b = \frac{1}{2}(v+r)$, and (4) is equal to zero if $b = v$. Since (4) is convex in b, this implies that it is nonpositive for every $b \in \left(\frac{1}{2}(v+r), v\right]$.

If a bidder is suggested to bid low, then he knows that he is chosen, and faces exactly one chosen opponent. This opponent bids 0, low, or high with probabilities $\frac{\pi_l}{\pi_l + \pi_{ll} + \pi_{hl}}$, $\frac{\pi_{ll}}{\pi_l + \pi_{ll} + \pi_{hl}}$, and $\frac{\pi_{hl}}{\pi_l + \pi_{ll} + \pi_{hl}}$, respectively. The expected payoff of this bidder from bidding any $b \in \left(r, \frac{1}{2}(v+r)\right]$ is

$$\left(\frac{\pi_l}{\pi_l + \pi_{ll} + \pi_{hl}} + \frac{\pi_{ll}}{\pi_l + \pi_{ll} + \pi_{hl}} \left(\frac{b - r}{\frac{1}{2}(v+r) - r}\right)\right) v - b = \frac{U}{\frac{2}{n}(\pi_l + \pi_{ll} + \pi_{hl})} \geq 0$$

If he bids $b \in \left(\frac{1}{2}(v+r), v\right]$ instead, then his payoff is

$$\begin{aligned}
&\left(\frac{\pi_l + \pi_{ll}}{\pi_l + \pi_{ll} + \pi_{hl}} + \frac{\pi_{hl}}{\pi_l + \pi_{ll} + \pi_{hl}}\left(\frac{b - \frac{1}{2}(v+r)}{v - \frac{1}{2}(v+r)}\right)\right) v - b \\
= \; & \frac{U}{\frac{2}{n}(\pi_l + \pi_{ll} + \pi_{hl})} \frac{2(v-b)}{v-r} < \frac{U}{\frac{2}{n}(\pi_l + \pi_{ll} + \pi_{hl})}
\end{aligned}$$

If a bidder is suggested to bid high, then he knows that he is chosen, and faces exactly one chosen opponent. This opponent bids 0, low, or high with probabilities $\frac{\pi_h}{\pi_h + \pi_{hl} + \pi_{hh}}$, $\frac{\pi_{hl}}{\pi_h + \pi_{hl} + \pi_{hh}}$, and $\frac{\pi_{hh}}{\pi_h + \pi_{hl} + \pi_{hh}}$, respectively. The expected payoff of this bidder from bidding any $b \in \left(\frac{1}{2}(v+r), \frac{1}{2}v\right]$ is

$$\begin{aligned}
&\left(\frac{\pi_h + \pi_{hl}}{\pi_h + \pi_{hl} + \pi_{hh}} + \frac{\pi_{hh}}{\pi_h + \pi_{hl} + \pi_{hh}}\left(\frac{b - \frac{1}{2}(v+r)}{v - \frac{1}{2}(v+r)}\right)\right) v - b \\
= \; & \left(\frac{v+r}{2v} + \frac{v-r}{v}\left(\frac{b - \frac{1}{2}(v+r)}{v-r}\right)\right) v - b = 0.
\end{aligned}$$

If he bids $b \in \left(r, \frac{1}{2}(v+r)\right]$ instead, then his payoff is

$$\left(\frac{\pi_h}{\pi_h + \pi_{hl} + \pi_{hh}} + \frac{\pi_{hl}}{\pi_h + \pi_{hl} + \pi_{hh}}\left(\frac{b-r}{\frac{1}{2}(v+r) - r}\right)\right) v - b = \left(\frac{r}{v} + \frac{v-r}{v}\left(\frac{b-r}{v-r}\right)\right) v - b = 0$$

Each bidder gets a payoff of $\frac{U}{\frac{2}{n}(\pi_l + \pi_{ll} + \pi_{hl})}$ when he is suggested to bid low, and zero payoff otherwise. Hence, his ex ante payoff is U. Thus the considered correlation rule is a correlated equilibrium, and it achieves the desired payoffs. ■

6.2 Proofs of Section 4

Proof of Proposition 5. (ii) In every Nash equilibrium, type 0 of each bidder bids 0 and gets a zero payoff. Let us represent the equilibrium strategy of bidder i of type v as a distribution function $G_i : \{0\} \cup [r, \infty) \to [0, 1]$, let $\underline{b}_i$ and $\overline{b}_i$ be the infimum and the supremum of the support of his equilibrium bids, and let $U_i \geq 0$ be his equilibrium payoff.

Note that $U_i \leq v - \overline{b}_i$. Also note that bidder j of type v can secure a payoff arbitrarily close to $v - \overline{b}_i$ by bidding slightly above $\overline{b}_i$. Thus $U_j \geq v - \overline{b}_i$. Reversing the roles of i and j, and rearranging, we get $U_i = U_j = U$.

Next, note that bidder i of type v can secure a payoff arbitrarily close to $p_j v - r$ by bidding slightly above r, and thus winning when the opponent is of type 0. Hence, $U \geq \max\{p_1 v - r, p_2 v - r, 0\}$. If this inequality is strict, then neither bidder of type v bids 0 with positive probability, and thus $\underline{b}_i, \underline{b}_j \geq r$. Moreover, $U > \max\{p_1 v - r, p_2 v - r, 0\}$ implies that each bidder of type v must be winning with positive probability against the opponent of type v. Then $\underline{b}_i < \underline{b}_j$ is impossible, since bidder i who bids below $\underline{b}_j$ always loses against the opponent of type v. But $\underline{b}_i = \underline{b}_j = \underline{b}$ is impossible either: the requirement of winning with positive probability against the opponent of type v implies that both bidders bid $\underline{b}$ with positive probability, which cannot happen in equilibrium since each bidder could profitably deviate to a slightly higher bid. Thus $U = \max\{p_1 v - r, p_2 v - r, 0\}$.

Let $p_1 \leq p_2$. It is straightforward to check that the following is a Nash equilibrium. Types 0 of both bidders bid 0. Type v of bidder 1 bids 0 and r with probabilities $\frac{\min\{r - p_2 v, 0\}}{(1-p_1)v}$ and $\frac{p_2 - p_1}{1 - p_1}$, respectively, and bids uniformly on $(r, \min\{(1-p_2)v + r, v\}]$ otherwise; type v of bidder 2 bids 0 with probability $\frac{\min\{r - p_2 v, 0\}}{(1-p_2)v}$, and bids uniformly on $(r, \min\{(1-p_2)v + r, v\}]$ otherwise. ∎

Proof of Proposition 6. Denote by $U_i^*(v_i)$ the interim expected payoff of player i of type v_i in the Nash equilibrium (μ_1^*, μ_2^*), and by $U_i(v_i)$ the interim expected payoff of player i of type v_i in the communication equilibrium μ. Let $\mu_i(\cdot|v_i)$ be the marginal probability measure of μ on A_i conditional on v_i (defined in Section 2).

By the definition of Nash equilibrium, for every i and P_i-a.e. v_i, it is unprofitable to deviate to any strategy $\widetilde{\mu}_i(\cdot|v_i)$:

$$\begin{aligned} U_i^*(v_i) &= v_i \int_{T_j} \left(\int_{A_j} \int_{A_i} \rho_i(b)\, \mu_i^*(db_i|v_i) \mu_j^*(db_j|v_j) \right) P_j(dv_j) - \int_{A_i} b_i \mu_i^*(db_i|v_i) \\ &\geq v_i \int_{T_j} \left(\int_{A_j} \int_{A_i} \rho_i(b)\, \widetilde{\mu}_i(db_i|v_i) \mu_j^*(db_j|v_j) \right) P_j(dv_j) - \int_{A_i} b_i \widetilde{\mu}_i(db_i|v_i) \end{aligned} \tag{5}$$

By the definition of communication equilibrium, for every i and P_i-a.e. v_i, it is unprofitable to deviate to the following strategy: first, randomize over the type reports according to P_i, and then, regardless of the mediator's recommendation, choose bids according to some bidding strategy $\widetilde{\mu}_i(\cdot|v_i)$:

$$\begin{aligned} U_i(v_i) &= v_i\int_{T_j}\left(\int_A \rho_i(b)\,\mu(db|v_i,v_j)\right)P_j(dv_j) - \int_{A_i} b_i\mu_i(db_i|v_i) \qquad (6)\\ &\geq v_i\int_{T_j}\left(\int_A\left(\int_{A_i}\rho_i\left(\widehat{b}_i,b_j\right)\widetilde{\mu}_i(d\widehat{b}_i|v_i)\right)\int_{T_j}\mu(db|\widehat{v}_i,v_j)dP_i(d\widehat{v}_i)\right)P_j(dv_j) - \int_{A_i}\widehat{b}_i\widetilde{\mu}_i(d\widehat{b}_i|v_i)\\ &= v_i\int_{T_j}\left(\int_{A_j}\int_{A_i}\rho_i\left(\widehat{b}_i,b_j\right)\widetilde{\mu}_i(d\widehat{b}_i|v_i)\mu_j(db_j|v_j)\right)P_j(dv_j) - \int_{A_i}\widehat{b}_i\widetilde{\mu}_i(d\widehat{b}_i|v_i) \end{aligned}$$

For every i and $v_i \neq 0$, such that both (5) and (6) hold, perform the following operations. Take (5) with $\widetilde{\mu}_i = \mu_i$ and (6) with $\widetilde{\mu}_i = \mu_i^*$, add the two resulting inequalities, and divide by v_i. Then for every i and P_i-a.e. v_i we get

$$\begin{aligned} &\int_{T_j}\left(\int_{A_j}\int_{A_i}\rho_i(b)\,\mu_i^*(db_i|v_i)\mu_j^*(db_j|v_j) + \int_A \rho_i(b)\,\mu(db|v_i,v_j)\right)P_j(dv_j) \qquad (7)\\ \geq &\int_{T_j}\left(\int_{A_j}\int_{A_i}\rho_i(b)\,\mu_i(db_i|v_i)\mu_j^*(db_j|v_j) + \int_{A_j}\int_{A_i}\rho_i(b)\,\mu_i^*(db_i|v_i)\mu_j(db_j|v_j)\right)P_j(dv_j) \end{aligned}$$

Next, integrate (7) with respect to P_i over the set of types of bidder i for which inequality (7) holds. Note that the resulting inequality continues to hold even if we integrate over T_i because (7) is satisfied for P_i-a.e. v_i, and because we have assumed that $P_i(\{0\}) = 0$. Sum up the resulting inequalities over i:

$$\begin{aligned} &\int_T\left(\int_{A_j}\int_{A_i}\sum_{k=1,2}\rho_k(b)\,\mu_i^*(db_i|v_i)\mu_j^*(db_j|v_j) + \int_A\sum_{k=1,2}\rho_k(b)\,\mu(db|v_i,v_j)\right)P(dv) \qquad (8)\\ \geq &\int_T\left(\int_{A_j}\int_{A_i}\sum_{k=1,2}\rho_k(b)\,\mu_i(db_i|v_i)\mu_j^*(db_j|v_j) + \int_{A_j}\int_{A_i}\sum_{k=1,2}\rho_k(b)\,\mu_i^*(db_i|v_i)\mu_j(db_j|v_j)\right)P(dv) \end{aligned}$$

Since $\sum_{k=1,2}\rho_k(b) = 1$ for every b, inequality (8) holds as an equality. This implies that the following inequalities hold as equalities as well for P_i-a.e. v_i: inequality (5) when $\widetilde{\mu}_i = \mu_i$, and inequality (6) when $\widetilde{\mu}_i = \mu_i^*$.

Hence, for every i, P_i-a.e. v_i, the following is true for any $\widetilde{\mu}_i(\cdot|v_i)$:

$$\begin{aligned} U_i^*(v_i) &= v_i \int_{T_j} \left(\int_{A_j} \int_{A_i} \rho_i(b)\, \mu_i(db_i|v_i) \mu_j^*(db_j|v_j) \right) P_j(dv_j) - \int_{A_i} b_i \mu_i(db_i|v_i) \\ &\geq v_i \int_{T_j} \left(\int_{A_j} \int_{A_i} \rho_i(b)\, \widetilde{\mu}_i(db_i|v_i) \mu_j^*(db_j|v_j) \right) P_j(dv_j) - \int_{A_i} b_i \widetilde{\mu}_i(db_i|v_i) \end{aligned}$$

and

$$\begin{aligned} U_i(v_i) &= v_i \int_{T_j} \left(\int_{A_j} \int_{A_i} \rho_i(b_i, b_j)\, \mu_i^*(db_i|v_i) \mu_j(db_j|v_j) \right) P_j(dv_j) - \int_{A_i} b_i \mu_i^*(db_i|v_i) \\ &\geq v_i \int_{T_j} \left(\int_{A_j} \int_{A_i} \rho_i(b_i, b_j)\, \widetilde{\mu}_i(db_i|v_i) \mu_j(db_j|v_j) \right) P_j(dv_j) - \int_{A_i} b_i \widetilde{\mu}_i(db_i|v_i) \end{aligned}$$

This implies that (μ_i^*, μ_j) is a Nash equilibrium. In this equilibrium bidder i of type v_i gets payoff $U_i(v_i)$, and bidder j of type v_j gets payoff $U_j^*(v_j)$. Since by assumption the Nash equilibrium is unique, for every i we have $\mu_i = \mu_i^*$, and thus $U_i^*(v_i) = U_i(v_i)$ for every P_i-a.e. v_i.[38] ■

Proof of Proposition 7. We show that for $p \in \left[0, \frac{r}{v}\right)$ there exists a communication equilibrium such that each bidder of type v gets a payoff of $\frac{(r^2 - p^2 v^2)(v-r)}{v^2 + r^2 - 2pv^2}$. Consider the following symmetric communication rule. If a bidder reports type 0, then he is suggested to bid 0. If a bidder reports type v and his opponent reports type 0, then this bidder is suggested to bid r or "bid above r" (which means "bid uniformly on $(r, v]$"), with probabilities $\widehat{\pi} = \frac{(r-pv)(v+r)}{v^2+r^2-2pv^2}$ and $1 - \widehat{\pi}$, respectively. If both bidders report type v, then they are given recommendations according to the following probability distribution.[39]

1's bid \ 2's bid	bid 0	bid r	bid above r
bid 0	0	$\pi_r = \frac{(r-pv)r}{v^2+r^2-2pv^2}$	$\pi_h = \frac{(r-pv)(v-r)}{v^2+r^2-2pv^2}$
bid r	$\pi_r = \frac{(r-pv)r}{v^2+r^2-2pv^2}$	0	0
bid above r	$\pi_h = \frac{(r-pv)(v-r)}{v^2+r^2-2pv^2}$	0	$\pi_{hh} = \frac{(v-r)^2}{v^2+r^2-2pv^2}$

[38] If there are multiple Nash equilibria, then every communication equilibrium is interim payoff equivalent to some Nash equilibrium. We do not include this observation in Proposition 6 because we are not aware of any examples of multiple Nash equilibria in this setting.

[39] It is straightforward to verify that $\widehat{\pi} \in [0, 1]$ and that the entries in table: (i) sum up to one; and (ii) are nonnegative (since $r > pv$).

We need to check the incentives to tell the truth and to comply with the recommendations only for the bidders of type v, since the bidders of type 0 have no incentive to lie or to disobey.

If a bidder of type v has reported v and is suggested to bid 0, then he knows that his opponent must be of type v and bids r or above r, with probabilities $\frac{\pi_r}{\pi_r+\pi_h}$ and $\frac{\pi_h}{\pi_r+\pi_h}$, respectively. If this bidder bids $b \in (r, v]$ instead, then his payoff is[40]

$$\left(\frac{\pi_r}{\pi_r+\pi_h}+\frac{\pi_h}{\pi_r+\pi_h}\left(\frac{b-r}{v-r}\right)\right)v-b=\left(\frac{r}{v}+\frac{v-r}{v}\left(\frac{b-r}{v-r}\right)\right)v-b=0$$

If a bidder of type v has reported v and is suggested to bid r, then it is clearly optimal to comply, since the opponent bids 0, regardless of the type, in such case.

If a bidder of type v has reported v and is suggested to bid above r, then he knows that either his opponent is of type 0 and thus bids 0, or his opponent is of type v and bids 0 or high, with probabilities $\frac{p(1-\widehat{\pi})}{p(1-\widehat{\pi})+(1-p)(\pi_h+\pi_{hh})}$, $\frac{(1-p)\pi_h}{p(1-\widehat{\pi})+(1-p)(\pi_h+\pi_{hh})}$, and $\frac{(1-p)\pi_{hh}}{p(1-\widehat{\pi})+(1-p)(\pi_h+\pi_{hh})}$, respectively. The expected payoff of this bidder from bidding any $b \in [r, v]$ is

$$\begin{aligned}&\left(\frac{p\left(1-\widehat{\pi}\right)+\left(1-p\right)\pi_h}{p\left(1-\widehat{\pi}\right)+\left(1-p\right)\left(\pi_h+\pi_{hh}\right)}+\frac{\left(1-p\right)\pi_{hh}}{p\left(1-\widehat{\pi}\right)+\left(1-p\right)\left(\pi_h+\pi_{hh}\right)}\left(\frac{b-r}{v-r}\right)\right)v-b\\ =\ &\left(\frac{r}{v}+\frac{v-r}{v}\left(\frac{b-r}{v-r}\right)\right)v-b=0\end{aligned}$$

To summarize, if a bidder of type v truthfully reports his type and follows the recommendations, then he gets a payoff of $v-r$ when he is suggested to bid r, and zero payoff otherwise. Hence, his ex ante payoff is $\left(p\widehat{\pi}+\left(1-p\right)\pi_r\right)\left(v-r\right)=\frac{\left(r^2-p^2v^2\right)\left(v-r\right)}{v^2+r^2-2pv^2}$.

If a bidder of type v has reported 0, then he is suggested to bid 0. He knows that either his opponent is of type 0 and thus bids 0, or his opponent is of type v and bids r or above r, with probabilities p, $\left(1-p\right)\widehat{\pi}$, and $\left(1-p\right)\left(1-\widehat{\pi}\right)$, respectively. If this bidder bids $b \in (r, v]$, then his payoff is

$$\left(\left(p+\left(1-p\right)\widehat{\pi}\right)+\left(1-p\right)\left(1-\widehat{\pi}\right)\left(\frac{b-r}{v-r}\right)\right)v-b\leq\max\left\{\left(p+\left(1-p\right)\widehat{\pi}\right)v-r,0\right\}=\frac{\left(r^2-p^2v^2\right)\left(v-r\right)}{v^2+r^2-2pv^2}$$

where the inequality follows from the fact that payoff is a linear function of b and is thus maximized

[40] Bidding exactly r is dominated by bidding slightly above r if there is a positive probability that the opponent bids r.

either at $b = r$ or at $b = v$.

Thus the considered communication rule is a communication equilibrium, and it achieves the desired payoffs. ■

Proof of Proposition 8. The proof is similar to the proof of part (ii) of Proposition 5. In every Nash equilibrium, type 0 of each bidder bids 0 and gets a zero payoff. Let the equilibrium strategy of bidder i of type v be represented by distribution function $G_i : \{0\} \cup [r, \infty) \to [0,1]$, $\underline{b}_i$ and $\overline{b}_i$ be the infimum and the supremum of the support of his equilibrium bids, and $U_i \geq 0$ be his equilibrium payoff.

Note that $U_i \leq \prod_{k \neq i} \left(p + (1-p) G_k\left(\overline{b}_i\right)\right) v - \overline{b}_i$. Also note that bidder j of type v can secure a payoff arbitrarily close to $\prod_{k \neq i,j} \left(p + (1-p) G_k\left(\overline{b}_i\right)\right) v - \overline{b}_i$ by bidding slightly above $\overline{b}_i$. Thus $U_j \geq U_i$. Since this is true for every pair of i and j, we have $U_i = U$ for every i.

Next, note that bidder i of type v can secure a payoff arbitrarily close to $p^{n-1}v - r$ by bidding slightly above r, and thus winning when the opponent is of type 0. Hence, $U \geq \max\left\{p^{n-1}v - r, 0\right\}$. If this inequality is strict, then neither bidder bids 0 with positive probability, and thus $\underline{b}_i \geq r$ for every i. Moreover, $U > \max\left\{p^{n-1}v - r, 0\right\}$ implies that each bidder of type v must be winning with positive probability against some opponent of type v. Then it is impossible to have $\underline{b}_i < \underline{b} = \min_{j \neq i} \underline{b}_j$, since bidder i who bids below $\underline{b}$ always loses against the opponents of type v. But $\underline{b}_i = \underline{b}$ is impossible either: the requirement of winning with positive probability against some opponent of type v implies that the bidders who bid $\underline{b}$ must do so with positive probability, which cannot happen in equilibrium since each bidder could profitably deviate to a slightly higher bid. Thus $U = \max\left\{p^{n-1}v - r, 0\right\}$.

It is straightforward to check that the following is a Nash equilibrium. Type 0 of each bidder bids 0. Type v of each bidder bids 0 with probability $x = \frac{1}{1-p}\left(\max\left\{\left(\frac{r}{v}\right)^{\frac{1}{n-1}} - p, 0\right\}\right)$, and bids according to $G(b) = \frac{1}{1-p}\left(\left(\max\left\{p^{n-1} - \frac{r}{v}, 0\right\} + \frac{b}{v}\right)^{\frac{1}{n-1}} - p\right)$ on $\left(r, \min\left\{\left(1 - p^{n-1}\right)v + r, v\right\}\right]$. ■

Proof of Proposition 9. We show for sufficiently small $p > 0$ there exists a communication equilibrium such that each bidder of type v gets a payoff of $2p^{n-1}v$. Consider the following symmetric communication rule. If a bidder reports type 0, then he is suggested to bid 0. If exactly one bidder reports v, then this bidder is suggested to"bid low" (which means "bid uniformly on $\left(0, \frac{1}{2}v\right]$"). If $m > 1$ bidders report v, then a pair of bidders out of these m bidders is randomly chosen, with each pair being equally likely to be chosen. The bidders receive private bid recommendations without being told whether they have been chosen. The bidders who are not chosen are recommended to bid 0, and the chosen bidders are given recommendations according to the following probability distribution where "bid high" means "bid uniformly on $\left(\frac{1}{2}v, v\right]$", and where

$$g = \sum_{k=1}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} (1-p)^k\, p^{n-1-k} \left(\frac{2}{k+1}\right) = 2\left(\frac{1}{n}\frac{1-p^n}{1-p} - p^{n-1}\right)$$

is the probability that a bidder who submitted report v was chosen and that he is not the only one who submitted v.[41]

i's bid \ j's bid	bid 0	bid low	bid high
bid 0	0	$\pi_l = \frac{1}{g}p^{n-1}$	0
bid low	$\pi_l = \frac{1}{g}p^{n-1}$	$\pi_{ll} = \frac{1}{4} + \frac{1}{g}p^{n-1}$	$\pi_{hl} = \frac{1}{4} - \frac{1}{g}p^{n-1}$
bid high	0	$\pi_{hl} = \frac{1}{4} - \frac{1}{g}p^{n-1}$	$\pi_{hh} = \frac{1}{4} - \frac{1}{g}p^{n-1}$

We need to check the incentives to tell the truth and to comply with the recommendations only for the bidders of type v, since the bidders of type 0 have no incentive to lie or to disobey.

If a bidder of type v has reported v and is suggested to bid 0, then he knows that either he was not chosen (which happens with probability $1 - p^{n-1} - g$), or that he was chosen but only his opponent is suggested to bid above 0 (which happens with probability $g\pi_l$).

If this bidder bids $b \in \left(0, \frac{1}{2}v\right]$ instead, then he has a chance to win only if none of his opponents bid high. In particular, bidder i could win if (i) he was not chosen, and one chosen bidder bids low (which happens with probability $\left(1 - p^{n-1} - g\right) 2\pi_l$); (ii) he was not chosen, and two chosen bidders bid low (which happens with probability $\left(1 - p^{n-1} - g\right) \pi_{ll}$); (iii) he was chosen, and his opponent bids low (which happens with probability $g\pi_l$). The expected payoff of this bidder is then

$$\left(\frac{\left(1 - p^{n-1} - g\right) 2\pi_l + g\pi_l}{1-g}\left(\frac{b}{\frac{1}{2}v}\right) + \frac{\left(1 - p^{n-1} - g\right)\pi_{ll}}{1-g}\left(\frac{b}{\frac{1}{2}v}\right)^2\right) v - b \tag{9}$$

[41] It is straightforward to verify that the entries in table: (i) sum up to one; and (ii) are nonnegative for p sufficiently small (since $g = \frac{2}{n}$ when $p = 0$).

Note that (9) is equal to 0 if $b=0$. If $b=\frac{1}{2}v$, then (9) becomes

$$\left(\frac{\left(1-p^{n-1}-g\right)\left(2\pi_l+\pi_{ll}\right)+g\pi_l}{1-g}\right)v-\frac{1}{2}v=\left(\frac{\left(3\frac{1-p^{n-1}}{g}-\frac{9}{4}\right)p^{n-1}}{1-g}-\frac{1}{4}\right)v \tag{10}$$

Note that (10) is equal to $-\frac{1}{4}v$ if $p=0$, and that (10) is continuous in p. Thus for p small enough (9) is nonpositive at $b=0$ and at $b=\frac{1}{2}v$, and it is convex in b on $\left(0,\frac{1}{2}v\right]$, which implies that (9) is nonpositive for every $b\in\left(0,\frac{1}{2}v\right]$.

If this bidder bids $b\in\left(\frac{1}{2}v,v\right]$ instead, then he wins for sure if none of his opponents bids high, and has a chance to win otherwise. In particular, bidder i wins for sure if (i) he was not chosen, and none of the chosen bidders bid high (which happens with probability $\left(1-p^{n-1}-g\right)\left(2\pi_l+\pi_{ll}\right)$); (ii) he was chosen, and his opponent does not bid high (which happens with probability $g\pi_l$). Also bidder i could win if (i) he was not chosen, and one chosen bidder bids high (which happens with probability $\left(1-p^{n-1}-g\right)2\pi_{hl}$); (ii) he was not chosen, and two chosen bidders bid high (which happens with probability $\left(1-p^{n-1}-g\right)\pi_{hh}$). The expected payoff of this bidder is then

$$\begin{aligned}&\left(\frac{\left(1-p^{n-1}-g\right)\left(2\pi_l+\pi_{ll}\right)+g\pi_l}{1-g}+\frac{\left(1-p^{n-1}-g\right)2\pi_{hl}}{1-g}\left(\frac{b-\frac{1}{2}v}{\frac{1}{2}v}\right)+\right.\\&\left.+\frac{\left(1-p^{n-1}-g\right)\pi_{hh}}{1-g}\left(\frac{b-\frac{1}{2}v}{\frac{1}{2}v}\right)^2\right)v-b\end{aligned} \tag{11}$$

Note that (11) is equal to (10) if $b=\frac{1}{2}v$, and (11) is equal to zero if $b=v$. Thus for p small enough (11) is nonpositive at $b=\frac{1}{2}v$ and at $b=v$, and it is convex in b on $\left(\frac{1}{2}v,v\right]$, which implies that (11) is nonpositive for every $b\in\left(\frac{1}{2}v,v\right]$.

If a bidder of type v has reported v and is suggested to bid low, then he knows that either he faces no opponents (with probability p^{n-1}), or that he was chosen and faces one chosen opponent who bids 0, low, or high with probabilities $g\pi_l$, $g\pi_{ll}$, and $g\pi_{hl}$, respectively. The expected payoff of this bidder from bidding any $b\in\left(0,\frac{1}{2}v\right]$ is

$$\left(\frac{p^{n-1}+g\pi_l}{p^{n-1}+g\left(\pi_l+\pi_{ll}+\pi_{hl}\right)}+\frac{g\pi_{ll}}{p^{n-1}+g\left(\pi_l+\pi_{ll}+\pi_{hl}\right)}\frac{b}{\frac{1}{2}v}\right)v-b=\frac{2p^{n-1}v}{2p^{n-1}+\frac{1}{2}g}\geq 0$$

If he bids $b \in \left(\frac{1}{2}v, v\right]$ instead, then his payoff is

$$\begin{aligned}\left(\frac{p^{n-1}+g\left(\pi_l+\pi_{ll}\right)}{p^{n-1}+g\left(\pi_l+\pi_{ll}+\pi_{hl}\right)}+\frac{g\pi_{hl}}{p^{n-1}+g\left(\pi_l+\pi_{ll}+\pi_{hl}\right)}\left(\frac{b-\frac{1}{2}v}{\frac{1}{2}v}\right)\right)v-b &= \frac{2p^{n-1}v}{2p^{n-1}+\frac{1}{2}g}\left(\frac{v-b}{\frac{1}{2}v}\right) \\ &< \frac{2p^{n-1}v}{2p^{n-1}+\frac{1}{2}g}\end{aligned}$$

If a bidder of type v has reported v and is suggested to bid high, then he knows that he was chosen and faces one chosen opponent who bids low or high, with equal probabilities. Thus the expected payoff of this bidder from bidding any $b \in (0, v]$ is equal to zero.

To summarize, if a bidder of type v truthfully reports his type and follows the recommendations, then he gets a payoff of $\frac{2p^{n-1}v}{2p^{n-1}+\frac{1}{2}g}$ when he is suggested to bid low, and zero payoff otherwise. Hence, his ex ante payoff is $2p^{n-1}v$.

If a bidder of type v has reported 0, then he is suggested to bid 0. He knows that he faces no active opponents with probability p^{n-1}; one active opponent who bids low with probability $(n-1)(1-p)p^{n-2}+2d\pi_l$, where $d=\left(1-p^{n-1}-(n-1)(1-p)p^{n-2}\right)$; two active opponents who both bid low, both bid high, or one bids low and another high with probabilities $d\pi_{ll}$, $d\pi_{hh}$, and $2d\pi_{hl}$, respectively.

If this bidder bids $b \in \left(0, \frac{1}{2}v\right]$, then his payoff is

$$\left(p^{n-1}+\left((n-1)(1-p)p^{n-2}+d2\pi_l\right)\left(\frac{b}{\frac{1}{2}v}\right)+d\pi_{ll}\left(\frac{b}{\frac{1}{2}v}\right)^2\right)v-b \tag{12}$$

Note that if $b=0$, then (12) is equal to $p^{n-1}v$ which is smaller than the payoff from truthtelling $2p^{n-1}v$. If $b=\frac{1}{2}v$ then (12) becomes

$$\left(p^{n-1}+(n-1)(1-p)p^{n-2}+d\left(2\pi_l+\pi_{ll}\right)\right)v-\frac{1}{2}v \tag{13}$$

Note that (13) is equal to $-\frac{1}{4}v$ if $p=0$, and that (13) is continuous in p. Thus for p small enough (12) is smaller than $2p^{n-1}v$ at $b=0$ and at $b=\frac{1}{2}v$, and it is convex in b on $\left(0, \frac{1}{2}v\right]$, which implies that (13) is smaller than the payoff from truthtelling $2p^{n-1}v$ for every $b \in \left(0, \frac{1}{2}v\right]$.

If this bidder bids $b \in \left(\frac{1}{2}v, v\right]$, then his payoff is

$$\left(p^{n-1} + (n-1)(1-p)p^{n-2} + d(2\pi_l + \pi_{ll}) + 2d\pi_{hl}\left(\frac{b-\frac{1}{2}v}{\frac{1}{2}v}\right) + d\pi_{hh}\left(\frac{b-\frac{1}{2}v}{\frac{1}{2}v}\right)^2\right)v - b \quad (14)$$

Note that (14) is equal to (13) if $b = \frac{1}{2}v$, and (14) is equal to zero if $b = v$. Thus for p small enough (14) is smaller than $2p^{n-1}v$ at $b = \frac{1}{2}v$ and at $b = v$, and it is convex in b on $\left(\frac{1}{2}v, v\right]$, which implies that (14) is smaller than the payoff from truthtelling $2p^{n-1}v$ for every $b \in \left(\frac{1}{2}v, v\right]$.

Thus the considered communication rule is a communication equilibrium, and it achieves the desired payoffs. ■

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