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1. Introduction

In a system of simultaneous structural equations, if the equation to be estimated has only one jointly dependent variable on the right hand side, the scalar coefficient of the right hand jointly dependent variable can be estimated by the two-stage least-squares (2SLS) estimation procedure proposed by Theil (1961). The exact sampling distribution of this estimator has been analyzed by Richardson (1968), Anderson and Sawa (1971) and Mariano (1969). The density function obtained is extremely complicated and is in terms of special mathematical functions.

In this paper we propose to analyze the form of the sampling distribution of the 2SLS estimator of the same scalar parameter by considering its exact moments and their approximations. The exact moments are complicated functions again, but useful approximations to them (for small sample analysis) can be obtained in a straightforward manner. The advantage of this approach is that it opens the way to analyzing exact moments in more general cases where there are two or more endogenous variables on the right hand side of the equation to be estimated. In fact, in a separate study Aman Ullah (1970) has already provided the exact means of the 2SLS estimators of coefficients of two endogenous variables on the right hand side of the equation. The approximations of moments (to order $1/\theta$ for mean and

$1/\theta^2$ for variance etc., where θ can be interpreted to be order T which is the number of observations) have been obtained by employing asymptotic expansions of confluent hypergeometric functions involved. These approximations turn out to be identical with those obtained by Nagar (1959).

The main results of this paper can be summarized as follows. Up to a certain order of approximation, the sampling distribution of the 2SLS estimator of the scalar coefficient estimated tends to be asymmetric and leptokurtic. The extent of departure from normality depends on the true value of the scalar coefficient--called γ. If γ is positive the distribution will tend to be positively skewed and negatively skewed if γ is negative. If $\gamma = 0$ the distribution will be symmetric up to the order of approximation considered. Similarly the extent of departure from mesokurticity toward leptokurticity will depend on the true magnitude of γ. It is interesting to note that the number of excluded exogenous variables (called K_2) from the equation being estimated does not affect the skewness and kurtosis coefficients up to the order of approximations considered. Finally, it should be noted that in the special case considered if $K_2 = 2$, i.e., if two exogenous variables have been excluded from the equation, the 2SLS estimator of the scalar coefficient being estimated is unbiased.

2. Model Specification and the Two-Stage Least-Squares (2SLS) Estimation Procedure

2.1. The Complete Structural System

Let us write a complete system of M linear structural equations, in matrix form, as

$$Y_w\Gamma + XB = U, \tag{2.1}$$

where Y_w and X are T x M and T x K matrices of observations on M jointly dependent and K predetermined variables, respectively; Γ and B are M x M and K x M matrices of structural coefficients, respectively; and U is a T x M matrix of structural disturbances.

The reduced form of this system is obtained by post multiplying both sides of (2.1) by Γ^{-1}, provided Γ is non-singular:

(2.2) $Y_w = X\Pi_w + \bar{V}_w$, where $\Pi_w = - B\Gamma^{-1}$ and $\bar{V}_w = U\Gamma^{-1}$,

so that Π_w and $\bar{V}_w$ are K x M and T x M matrices of reduced form coefficients and reduced form disturbances, respectively.

We shall assume that the elements of X are nonstochastic and fixed in repeated samples, thus there are no lagged endogenous variables present in the system. Further,

(2.3) rank of $X = K \leq T$,

i.e., the columns of X are linearly independent of each other, and $X'X$ is non-singular. We require $\lim_{T\to\infty} \frac{1}{T} X'X$ to be a positive definite matrix.

We also assume that the T rows of U are independent random drawings from an M-dimensional normal population with mean vector zero and covariance matrix Σ, thus

(2.4) $E\ u_i(t) = 0$ for all $i = 1,\ldots,M$ and $t = 1,\ldots,T$,

and

(2.5) $\frac{1}{T} E(U'U) = \Sigma$,

$u_i(t)$ being the element in the t^{th} row and i^{th} column of U .

We assume that Σ is positive definite.

It follows that

(2.6) $\frac{1}{T} E(\bar{V}_w' \bar{V}_w) \equiv \Omega = \Gamma'^{-1} \Sigma \Gamma^{-1}$

is also positive definite.

It can be shown that it is always possible to transform the variables and parameters of the given structural system (2.1) such that the covariance matrix of the transformed reduced form disturbances is an identity matrix. Since Σ is positive definite we can obtain a non-singular square matrix Ψ such that

(2.7) $\Sigma^{-1} = \Psi'\Psi$.

Let us then define

(2.8) $P = \Gamma \Psi'$

and observe that P is an M x M non-singular matrix if Γ is non-singular.

The transformed structural system may now be written as

(2.9) $Y_w^* \Gamma^* + XB = U$

where

(2.10) $Y_w^* = Y_w P$ and $\Gamma^* = P^{-1}\Gamma = \Psi'^{-1}$

are the matrices of transformed jointly dependent variables and their coefficients.

We note that the predetermined variables and their coefficients are unaffected by the above transformation.

The transformed reduced form equation is

(2.11) $Y_w^* = X \Pi_w^* + \bar{V}_w^*$

where

(2.12) $\Pi_w^* = - B \Gamma^{*-1} = \Pi_w P$ and $\bar{V}_w^* = U \Gamma^{*-1} = \bar{V}_w P$

are the K x M and T x M matrices of transformed reduced form coefficients and disturbances, respectively. We get

(2.13) $\frac{1}{T} E(\bar{V}_w^{*\prime} \bar{V}_w^*) = P' \Omega P = P' \Gamma'^{-1} \Sigma \Gamma^{-1} P = I$.

2.2. Single Equation of the System

Each structural equation of the complete system represents a certain economic hypothesis. It is then obvious that not all variables of the complete system will be represented in every equation. In fact the order condition of identifiability--which is in terms of exclusion of some variables from every equation--states the maximum number of variables that can be included in the equation:

> "the number of variables excluded from the equation must not be smaller than the number of equations in the system less one".

Let us suppose $m + 1 \leq M$ jointly dependent and $K_1 \leq K$ predetermined variables enter the equation under consideration. Then after omitting the excluded variables and normalising the coefficients, we may write the equation as

$$y = Y\gamma + X_1\beta + u \tag{2.14}$$

where y is a column vector of observations on the left hand jointly dependent variable, Y is a T x m matrix of observations on the right hand jointly dependent variables, X_1 is a T x K_1 matrix of observations on the right hand predetermined variables, γ and β are coefficient vectors and u is the disturbance vector.

The complete reduced form (2.2) of the system may now be written as

$$y = X\pi + \bar{v} = X_1\pi_1 + X_2\pi_2 + \bar{v} \tag{2.15}$$

$$Y = X\Pi + \bar{V} = X_1\Pi_1 + X_2\Pi_2 + \bar{V} \tag{2.16}$$

$$Y_2 = X\Pi^* + \bar{V}_2 = X_1\Pi_1^* + X_2\Pi_2^* + \bar{V}_2 \tag{2.17}$$

where

$$(2.18)\qquad \begin{cases} \pi = \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \quad \Pi = \begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} \text{ and } \Pi^* = \begin{pmatrix} \Pi_1^* \\ \Pi_2^* \end{pmatrix} \\ X = (X_1 \quad X_2)\ . \end{cases}$$

Thus (2.15) represents that part of the complete reduced form which corresponds to the left hand jointly dependent variable of (2.14), (2.16) gives the reduced form corresponding to the right hand jointly dependent variables of that equation. The last equation (2.17) is that part of the complete reduced form which corresponds to jointly dependent variables of the system excluded from (2.14).

The identifiability requirement of the equation (2.14) may be expressed as

$$(2.19)\qquad \pi_2 = \Pi_2\,\gamma\ ,$$

and the rank condition of identifiability of (2.14) as

$$(2.20)\qquad \text{rank } \Pi_2 = m\ .$$

It should be noted that the transformation of the structural equations, discussed in the preceding section, leaves the identifiability requirement (2.19) unaffected.

2.3. Two-Stage Least-Squares (2SLS) Estimation

The 2SLS estimator of the parameter vector $\binom{\gamma}{\beta}$ of (2.14) is given by

$$(2.21)\qquad \binom{c}{b} = \begin{bmatrix} Y'Y - V'V & Y'X_1 \\ X_1'Y & X_1'X_1 \end{bmatrix}^{-1} \begin{pmatrix} Y' - V' \\ X_1' \end{pmatrix} y$$

where

$$(2.22) \qquad V = Y - X(X'X)^{-1} X'Y$$

is the least squares estimator of $\overline{V}$ defined in (2.16). It follows that

$$(2.23) \qquad Y'Y - V'V = Y'M^*Y, \quad M^* = X(X'X)^{-1}X'$$

and

$$(2.24) \qquad Y - V = M^*Y .$$

If we premultiply both sides of (2.21) by

$$(2.25) \qquad \begin{bmatrix} Y'Y - V'V & Y'X_1 \\ X_1'Y & X_1'X_1 \end{bmatrix}$$

and solve for c and b we get

$$(2.26) \qquad c = (Y' N Y)^{-1} Y' Ny$$

$$(2.27) \qquad b = (X_1'X_1)^{-1}X_1'(y - Yc)$$

where

$$(2.28) \qquad N = M^*-M_1^*, \quad M^* = X(X'X)^{-1}X' \quad \text{and} \quad M_1^* = X_1(X_1'X_1)^{-1}X_1' .$$

Further, since N is a T x T idempotent symmetric matrix with

$$(2.29) \qquad \text{rank } N = \text{tr } N = K - K_1 = K_2 ,$$

there exists a $T \times K_2$ orthogonal matrix R such that

$$(2.30) \qquad N = RR' \quad \text{and} \quad R'R = I ,$$

where I is a $K_2 \times K_2$ unit matrix.

Then the 2SLS estimator c of γ can be expressed as

$$(2.31) \qquad c = (Z'Z)^{-1} Z'z$$

where

(2.32) $Z = R'Y$ and $z = R'y$,

are linear orthogonal transformations of Y and y, respectively.

The corresponding estimator b of β is obtained from (2.27).

3. Exact Moments of the Two-Stage Least-Squares (2SLS) Estimator - Special Case (m = 1)

Suppose there is only one jointly dependent variable included on the right hand side of the equation to be estimated (2.14); that is, Y is a column vector, say y_1, and then γ is simply a scalar coefficient. In this case let us write the equation (2.14) as

(3.1) $$y = \gamma y_1 + X_1\beta + u \ .$$

The 2SLS estimators of γ and β are

(3.2) $$c = \frac{y_1' N y}{y_1' N y_1} = \frac{z_1' z}{z_1' z_1}$$

and

(3.3) $$b = (X_1'X_1)^{-1} X_1'(y - cy_1)$$

where

(3.4) $z_1 = R'y_1$ and $z = R'y$

as in (2.32).

Let us write the reduced form corresponding to y and y_1 as

(3.5) $$y = X_1\pi^* + X_2\pi + \bar{v} = \bar{y} + \bar{v}, \quad \bar{y} = Ey = X_1\pi^* + X_2\pi$$

and

(3.6) $$y_1 = X_1\pi_1^* + X_2\,\pi_1 + \bar{v}_1 = \bar{y}_1 + \bar{v}_1, \quad \bar{y}_1 = Ey_1 = X_1\pi_1^* + X_2\,\pi_1$$

where

(3.7) $$E\bar{v} = E\bar{v}_1 = 0 \ ;$$

and assuming that the structural system has been already so transformed that the covariance matrix of the reduced form disturbances is unity, we have

$$(3.8) \quad E\,\bar{v}\bar{v}' = I, \quad E\,\bar{v}_1\bar{v}_1' = I \quad \text{and} \quad E\,\bar{v}_1\bar{v}' = 0 ,$$

I being a T x T unit matrix and 0 is a T x T zero matrix.

In this special case the identifiability requirement stated in (2.19) can be expressed as

$$(3.9) \qquad \pi = \gamma\,\pi_1$$

where π and π_1 are column vectors defined in (3.5) and (3.6) and γ is the scalar coefficient of (3.1).

According to the assumption of normality of structural disturbances (mentioned in Section 2.1) we observe that the elements of y and y_1 are independently normally distributed with means specified in (3.5) and (3.6), and

$$(3.10) \qquad E(y - \bar{y})(y - \bar{y})' = I = E(y_1 - \bar{y}_1)(y_1 - \bar{y}_1)'$$

and

$$(3.11) \qquad E(y - \bar{y})(y_1 - \bar{y}_1)' = 0 .$$

Also the elements of z and z_1, being linear functions of those of y and y_1, respectively, are independently normally distributed with means given by

$$(3.12) \qquad \begin{cases} E\,z = R'(E\,y) = R'\,\bar{y} = \bar{z} & \text{and} \\ E\,z_1 = R'(E\,y_1) = R'\,\bar{y}_1 = \bar{z}_1 & ; \end{cases}$$

and covariances

$$(3.13) \qquad \begin{cases} E(z - \bar{z})(z - \bar{z})' = E(z_1 - \bar{z}_1)(z_1 - \bar{z}_1)' = I & \text{and} \\ E(z - \bar{z})(z_1 - \bar{z}_1)' = 0 . \end{cases}$$

If we write z_i and z_{1i} for the i^{th} $(i = 1,\ldots,K_2)$ elements of z and z_1, respectively; and correspondingly, $\bar{z}_i$ and $\bar{z}_{1i}$ for the i^{th} elements of the mean vectors $\bar{z}$ and $\bar{z}_1$, we have

$$(3.14)\quad \begin{cases} E\, z_i = \bar{z}_i \\ E\, z_i^2 = 1 + \bar{z}_i^2 \\ E\, z_i^3 = 3\, \bar{z}_i + \bar{z}_i^3 \\ E\, z_i^4 = 3 + 6\, \bar{z}_i^2 + \bar{z}_i^4 \end{cases}$$

and

$$(3.15)\quad \begin{cases} E\, z_{1i} = \bar{z}_{1i} \\ E\, z_{1i}^2 = 1 + \bar{z}_{1i}^2 \\ E\, z_{1i}^3 = 3\, \bar{z}_{1i} + \bar{z}_{1i}^3 \\ E\, z_{1i}^4 = 3 + 6\, \bar{z}_{1i}^2 + \bar{z}_{1i}^4 \, . \end{cases}$$

The first four moments of the 2SLS estimator c of the scalar coefficient γ are given by

$$(3.16)\qquad E\, c = E\left(\frac{\sum_i z_{1i}\, z_i}{\sum_i z_{1i}^2}\right) = \sum_i (E\, z_i)\, E\left(\frac{z_{1i}}{\sum_i z_{1i}^2}\right)$$

$$= \sum_i \bar{z}_i\, E\left(\frac{z_{1i}}{\sum_i z_{1i}^2}\right)$$

(3.17) $$E\, c^2 = E\left(\frac{\sum_{i,j} z_{1i}\, z_{1j}\, z_i\, z_j}{(\sum_i z_{1i}^2)^2}\right)$$

$$= E\left(\frac{\sum_i z_{1i}^2\, z_i^2 + \sum_{i\neq j} z_{1i}\, z_{1j}\, z_i\, z_j}{(\sum_i z_{1i}^2)^2}\right)$$

$$= \sum_i (E\, z_i^2)\, E\left(\frac{z_{1i}^2}{(\sum_i z_{1i}^2)^2}\right) + \sum_{i\neq j} Ez_i Ez_j\; E\left(\frac{z_{1i}\, z_{1j}}{(\sum_i z_{1i}^2)^2}\right)$$

$$= \sum_i (1 + \bar{z}_i^2)\, E\left(\frac{z_{1i}^2}{(\sum_i z_{1i}^2)^2}\right) + \sum_{i\neq j} \bar{z}_i\, \bar{z}_j\, E\left(\frac{z_{1i}\, z_{1j}}{(\sum_i z_{1i}^2)^2}\right),$$

(3.18) $$E\, c^3 = E\left(\frac{\sum_{i,j,k} z_{1i}\, z_{1j}\, z_{1k}\, z_i\, z_j\, z_k}{(\sum_i z_{1i}^2)^3}\right)$$

$$= E\left(\frac{\sum_i z_{1i}^3\, z_i^3 + 3\sum_{i\neq j} z_{1i}^2 z_{ij} z_i^2 z_j + \sum_{i\neq j\neq k} z_{1i} z_{1j} z_{1k} z_i z_j z_k}{(\sum_i z_{1i}^2)^3}\right)$$

$$= \sum_i (E\, z_i^3)\, E\left(\frac{z_{1i}^3}{(\sum_i z_{1i}^2)^3}\right) +$$

$$+ 3\sum_{i\neq j} (E\, z_i^2)(E\, z_j)\, E\left(\frac{z_{1i}^2\, z_{1j}}{(\sum_i z_{1i}^2)^3}\right) +$$

$$+ \sum_{i \neq j \neq k} (E\, z_i)(E\, z_j)(E\, z_k)\; E\left(\frac{z_{1i}\, z_{1j}\, z_{1k}}{(\sum_i z_{1i}^2)^3}\right),$$

and

$$\text{(3.19)} \qquad E\, c^4 = E\left(\frac{\sum_{i,j,k,\ell} z_{1i}\, z_{1j}\, z_{1k}\, z_{1\ell}\, z_i\, z_j\, z_k\, z_\ell}{(\sum z_{1i}^2)^4}\right)$$

$$= \sum_i (E\, z_i^4)\; E\left(\frac{z_{1i}^4}{(\sum_i z_{1i}^2)^4}\right) +$$

$$+ 4 \sum_{i \neq j} (E\, z_i^3)(E\, z_j)\; E\left(\frac{z_{1i}^3\, z_{1j}}{(\sum_i z_{1i}^2)^4}\right) +$$

$$+ 3 \sum_{i \neq j} (E\, z_i^2)(E\, z_j^2)\; E\left(\frac{z_{1i}^2\, z_{1j}^2}{(\sum_i z_{1i}^2)^4}\right) +$$

$$+ 6 \sum_{i \neq j \neq k} (E\, z_i^2)(E\, z_j)(E\, z_k)\; E\left(\frac{z_{1i}^2\, z_{1j}\, z_{1k}}{(\sum_i z_{1i}^2)^4}\right) +$$

$$+ \sum_{i \neq j \neq k \neq \ell} (E\, z_i)(E\, z_j)(E\, z_k)(E\, z_\ell)\; E\left(\frac{z_{1i}\, z_{1j}\, z_{1k}\, z_{1\ell}}{(\sum_i z_{1i}^2)^4}\right)$$

where $i,j,k,\ell = 1,\ldots,K_2$.

The moments of z_i required in (3.16)-(3.19), are already given in (3.14). Further, let us observe that

$$(3.20)\qquad E\left(\frac{z_{1i}}{(\Sigma z_{1i}^2)^r}\right) = E\left(\frac{z_{1i} - \bar{z}_{1i} + \bar{z}_{1i}}{(\sum_i z_{1i}^2)^r}\right)$$

$$= E\left(\frac{z_{1i} - \bar{z}_{1i}}{(\sum_i z_{1i}^2)^r}\right) + \bar{z}_{1i}\, E\left(\frac{1}{(\sum_i z_{1i}^2)^r}\right),$$

and

$$(3.21)\qquad E\left(\frac{z_{1i}-\bar{z}_{1i}}{(\sum_i z_{1i}^2)^r}\right) = (2\pi)^{-K_2/2} \int_{z_{11}}\cdots\int_{z_{1K_2}} \frac{z_{1i}-\bar{z}_{1i}}{(\sum_i z_{1i}^2)^r}\, e^{-\frac{1}{2}\sum_i (z_{1i}-\bar{z}_{1i})^2}\, dz_{11} \cdots dz_{1K_2}$$

$$= (2\pi)^{-K_2/2} \underbrace{\int_{z_{11}}\cdots\cdots\int_{z_{1K_2}}}_{(\text{excluding } z_{1i})} \left\{\int_{z_{1i}} \frac{z_{1i}-\bar{z}_{1i}}{(\sum_i z_{1i}^2)^r}\, e^{-\frac{1}{2}(z_{1i}-\bar{z}_{1i})^2}\, dz_{1i}\right\} \times$$

$$\times\; e^{-\frac{1}{2}\sum_{j\neq i}(z_{1j}-\bar{z}_{1j})^2}\; \underbrace{dz_{11}\cdots dz_{1K_2}}_{(\text{excluding } dz_{1i})}$$

where $-\infty < z_{11},\ldots,z_{1K_2} < \infty$.

The integral within curl brackets in the second equality of (3.21) can be written as[1]

[1]It can be easily verified that the conditions for inverting the order of integration and differentiation are met in this case.

$$(3.22) \qquad \int_{z_{1i}} \frac{z_{1i} - \bar{z}_{1i}}{(\sum_i z_{1i}^2)^r} e^{-\frac{1}{2}(z_{1i} - \bar{z}_{1i})^2} dz_{1i} =$$

$$= \int_{z_{1i}} \frac{\partial}{\partial \bar{z}_{1i}} \left\{ \frac{1}{(\sum_i z_{1i}^2)^r} e^{-\frac{1}{2}(z_{1i} - \bar{z}_{1i})^2} \right\} dz_{1i}$$

$$= \frac{\partial}{\partial \bar{z}_{1i}} \int_{z_{1i}} \frac{1}{(\sum_i z_{1i}^2)^r} e^{-\frac{1}{2}(z_{1i} - \bar{z}_{1i})^2} dz_{1i} ,$$

and in turn

$$(3.23) \qquad E\left(\frac{z_{1i} - \bar{z}_{1i}}{(\sum_i z_{1i}^2)^r}\right) = \frac{\partial}{\partial \bar{z}_{1i}} \left\{ E\left(\frac{1}{(\sum_i z_{1i}^2)^r}\right)\right\} .$$

Thus, essentially, to evaluate (3.20), we require $E[1/(\sum_i z_{1i}^2)^r]$ and its partial derivative with respect to $\bar{z}_{1i}$.

It can be shown (the results are stated in the Appendix C) that all expectations required in (3.17)-(3.19) can be expressed in terms of $E[1/(\sum_i z_{1i}^2)^r]$ and its partial derivatives, with respect to $\bar{z}_{1i}$, of higher order.

Since $z_{11},\ldots,z_{1K_2}$ are independent normal variates with

$$(3.24) \qquad E\, z_{1i} = \bar{z}_{1i} \quad \text{and} \quad \text{Var}\, z_{1i} = 1, \qquad i = 1,\ldots,K_2 ,$$

$W = \sum_{i=1}^{K_2} z_{1i}^2$ is distributed according to 'non-central chi-square' distribution with K_2 degrees of freedom and

$$(3.25)\qquad \theta = \frac{1}{2} \sum_{i=1}^{K_2} \bar{z}_{1i}^{2} = \frac{1}{2} \bar{z}_1' \bar{z}_1$$

as the parameter of noncentrality. Using (3.12) we have

$$(3.26)\qquad \theta = \frac{1}{2} \bar{z}_1' \bar{z}_1 = \frac{1}{2} \bar{y}_1' R R' \bar{y}_1 = \frac{1}{2} \bar{y}_1' N \bar{y}_1$$

$$= \frac{1}{2}(X_1 \pi_1^* + X_2 \pi_1)' N (X_1 \pi_1^* + X_2 \pi_1)$$

$$= \frac{1}{2} \pi_1' \pi_1$$

provided $X'X = I$.[1]

It is well known (cf. Rao (1965), p. 146) that

$$(3.27)\qquad E\, W^{-r} = 2^{-r} \frac{\Gamma(\frac{K_2}{2} - r)}{\Gamma(\frac{K_2}{2})} e^{-\theta}\, {}_1F_1(\frac{K_2}{2} - r;\ \frac{K_2}{2};\ \theta)$$

provided $\frac{K_2}{2} > r$, where ${}_1F_1(\quad)$ is a confluent hyper-geometric function. Using a result of Slater (1960), p. 15:[2]

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[1] Thus, in general, θ is of order T .

[2]
$$\frac{d^s}{dx^s} [e^{-x}\, {}_1F_1(a;\ c;\ x)] =$$

$$= (-1)^s \frac{\Gamma(c - a + s)}{\Gamma(c - a)} \frac{\Gamma(c)}{\Gamma(c + s)} e^{-x}\, {}_1F_1(a;\ c;\ x),$$

for $\qquad s = 1,2,3,\ldots$.

(3.28) $$\frac{d^s}{d\theta^s} E\, W^{-r} = (-1)^s\, 2^{-r} \frac{\Gamma(r+s)}{\Gamma(r)} \frac{\Gamma(\frac{K_2}{2} - r)}{\Gamma(\frac{K_2}{2} + s)}\, e^{-\theta}\; {}_1F_1(\frac{K_2}{2} - r;\; \frac{K_2}{2} + s;\; \theta)$$

for $r = 1,2,3,\ldots$ and $s = 1,2,3,\ldots$.

From this we obtain the results stated in the Appendix B.

Finally the expectations required in (3.16)-(3.18) are stated in Appendix C.

Using these results given in respective appendices, we have:

(3.29) $$E\, c = \gamma\, \theta\, e^{-\theta}\, f_{01} ,$$

(3.30) $$E\, c^2 = e^{-\theta}\left[(\frac{1}{2}\,\theta + \gamma^2\theta^2) f_{02} + \frac{1}{2}(\frac{1}{2}\, K_2 + \gamma^2\theta) f_{-1,1}\right]$$

(3.31) $$E\, c^3 = \frac{1}{4}\,\theta\,\gamma\, e^{-\theta}\left[2\,\theta(3 + 2\,\theta\,\gamma^2) f_{03} + 3(K_2 + 2 + 2\,\theta\,\gamma^2) f_{-1,2}\right]$$

(3.32) $$E\, c^4 = \frac{1}{4}\,\theta^2\, e^{-\theta}\left[(2\,\theta\,\gamma^2 + 3)^2 - 6\right] f_{04} +$$

$$+ \frac{3}{4}\,\theta\, e^{-\theta}\left[4\,\theta^2\gamma^4 + 2(K_2 + 5)\theta\,\gamma^2 + (K_2 + 2)\right] f_{-1,3}$$

$$+ \frac{3}{16}\, e^{-\theta}\left[4\,\theta^2\gamma^4 + 4(K_2 + 2)\theta\,\gamma^2 + K_2(K_2 + 2)\right] f_{-2,2} ,$$

where

(3.33) $$f_{\mu\nu} = \frac{\Gamma(\frac{K_2}{2} + \mu)}{\Gamma(\frac{K_2}{2} + \nu)}\; {}_1F_1(\frac{K_2}{2} + \mu;\; \frac{K_2}{2} + \nu;\; \theta)$$

and writing $\mu = 0$, $\nu = 1$ we get f_{01}, and so on.

4. Approximations to the Exact Moments of the 2SLS Estimator of the Scalar Coefficient γ in (3.1)

If θ is positive and large, using the asymptotic expansion of the confluent hypergeometric function, we get

$$(4.1) \qquad E\,c = \gamma\left[1 + (1 - \frac{K_2}{2})\frac{1}{\theta} + (1 - \frac{K_2}{2})(2 - \frac{K_2}{2})\frac{1}{\theta^2} + \ldots\right]$$

and

$$(4.2) \qquad E\,c^2 = \gamma^2 + \left[\frac{1+\gamma^2}{2} + 2(1 - \frac{K_2}{2})\gamma^2\right]\frac{1}{\theta} + \left[(1 - \frac{K_2}{4}) + \right.$$

$$\left. + (2 - \frac{K_2}{2})\gamma^2 + 3(1 - \frac{K_2}{2})(2 - \frac{K_2}{2})\,\gamma^2\right]\frac{1}{\theta^2} + \ldots .$$

Therefore, the variance of c is

$$(4.3) \qquad V\,c = E\,c^2 - (E\,c)^2$$

$$= \frac{1+\gamma^2}{2}\;\frac{1}{\theta} + \left[(1 - \frac{K_2}{4}) + (3 - K_2)\gamma^2\right]\frac{1}{\theta^2} + \ldots$$

and the second moment of c about the true parameter value γ is

$$(4.4) \qquad E(c - \gamma)^2 = E\,c^2 - 2\,\gamma\,Ec + \gamma^2$$

$$= \frac{1+\gamma^2}{2}\,\frac{1}{\theta} + (1 - \frac{K_2}{4})\left[1 + 4(1 - \frac{K_2}{4})\gamma^2\right]\frac{1}{\theta^2} + \ldots .$$

Further, we have the following results on the third and fourth moments of c:

$$(4.5) \qquad E\,c^3 = \gamma^3 + 3\left[\gamma\,\frac{1+\gamma^2}{2} + (1 - \frac{K_2}{2})\gamma^3\right]\frac{1}{\theta} +$$

$$+ 3\left[(2 - \frac{K_2}{2})\gamma + \left\{3(1-\frac{K_2}{4}) + 2(1 - \frac{K_2}{2})(2 - \frac{K_2}{2})\right\}\gamma^3\right]\frac{1}{\theta^2} + \ldots;$$

and

$$(4.6)\qquad E(c - E\,c)^3 = 3\,\gamma\,\frac{1 + \gamma^2}{2}\,\frac{1}{\theta^2} + \dots\,,$$

$$(4.7)\qquad E(c - E\,c)^4 = \frac{3}{4}(1 + \gamma^2)^2\,\frac{1}{\theta^2} + $$

$$+ \frac{3}{4}(1 + \gamma^2)(1 + 4\,\gamma^2)(6 - K_2)\,\frac{1}{\theta^3} + \dots\,.$$

The Pearsonian coefficients of the skewness and kurtosis of the distribution of c are as follows:

$$(4.8)\quad \sqrt{\beta_1}\ (c) = 3\,\sqrt{2}\ \frac{\gamma}{\sqrt{1 + \gamma^2}}\ \frac{1}{\sqrt{\theta}}$$

and

$$(4.9)\quad \beta_2\ (c) = 3 + 6\,\frac{1 + 6\,\gamma^2}{1 + \gamma^2}\,\frac{1}{\theta}\,,$$

up to order $1/\theta$.

As should be expected

$$(4.10)\qquad \sqrt{\beta_1}\ (c) \to 0 \quad \text{and} \quad \beta_2\ (c) \to 3$$

as $\theta \to \infty$. It follows that (up to the required order of approximation) the departure from symmetry and mesokurticity of the distribution of c depends on γ and θ but not on K_2 (number of excluded exogenous variables in (3.1)). The distribution of c will tend to be positively skewed if true γ is positive and negatively skewed if γ is negative. Further the distribution of c will tend to be leptokurtic in small samples.

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APPENDIX

A. Confluent Hypergeometric Functions

A confluent hypergeometric function is defined as

$$(A.1) \qquad {}_1F_1(a;\ c;\ x) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{x^n}{n!}$$

and it is an "absolutely convergent" series for all values of a, c and x excluding c = 0, -1, -2,... .[1]

Since ${}_1F_1(a;\ c;\ x)$ is absolutely convergent we can differentiate the right hand side of (A.1) term by term. Thus

$$(A.2) \qquad \frac{d^s}{dx^s}\left[e^{-x}\ {}_1F_1(a;\ c;\ x)\right] =$$

$$= (-1)^s \frac{\Gamma(c-a+s)}{\Gamma(c-a)} \frac{\Gamma(c)}{\Gamma(c+s)} e^{-x}\ {}_1F_1(a;\ c+s;\ x)$$

for s = 1,2,3,... .[2]

The following recurrence relations should be noted:

$$(A.3) \qquad c\left[{}_1F_1(a;\ c;\ x) - {}_1F_1(a-1;\ c;\ x)\right] = x\ {}_1F_1(a;\ c+1;\ x),$$

$$(A.4) \qquad (1+a-c)\ {}_1F_1(a;\ c;\ x) + (c-1)\ {}_1F_1(a;\ c-1;\ x) = a\ {}_1F_1(a+1;\ c;\ x),$$

$$(A.5) \qquad (1+a-c)(a-c)\ {}_1F_1(a;\ c+1;\ x) + 2(1+a-c)c\ {}_1F_1(a;\ c;\ x) + c(c-1)\ {}_1F_1(a;\ c-1;\ x) = a(a+1)\ {}_1F_1(a+2;\ c+1;\ x),$$

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[1] Cf. Slater (1960), p. 2.

[2] Cf. Slater (1960), p. 15.

$$\text{(A.6)}\quad (1+a-c)(a-c)(a-c-1)\,{}_1F_1(a;\; c+2;\; x) +$$
$$+\,3(1+a-c)(a-c)(c+1)\,{}_1F_1(a;\; c+1;\; x) +$$
$$+\,3(1+a-c)(c+1)c\,{}_1F_1(a;\; c;\; x) +$$
$$+\,(c+1)\,c\,(c-1)\,{}_1F_1(a;\; c-1;\; x)$$
$$= a(a+1)(a+2)\,{}_1F_1(a+3;\; c+2;\; x),$$

$$\text{(A.7)}\quad (1+a-c)(a-c)(a-c-1)(a-c-2)\,{}_1F_1(a;\; c+3;\; x)$$
$$+\,4(1+a-c)(a-c)(a-c-1)(c+2)\,{}_1F_1(a;\; c+2;\; x)$$
$$+6(1+a-c)(a-c)(c+2)(c+1)\,{}_1F_1(a;\; c+1;\; x)$$
$$+4(1+a-c)(c+2)(c+1)c\,{}_1F_1(a;\; c;\; x)$$
$$+\,(c+2)(c+1)\,c\,(c-1)\,{}_1F_1(a;\; c-1;\; x)$$
$$= a(a+1)(a+2)(a+3)\,{}_1F_1(a+4;\; c+3;\; x)\,.$$

The relations (A.3) and (A.4) are given in Slater (1960), p.19--equations (2.2.4) and (2.2.3), respectively. The relations (A.5)-(A.7) can be verified by substituting relevant confluent hypergeometric series and equating coefficients of $x^n/n!$ on both sides of the respective relations.

B. <u>Derivatives of</u> $E\,W^{-r}$

The following results should be noted.

$$\text{(B.1)}\quad \frac{\partial}{\partial \bar{z}_{1i}}\, EW^{-r} = \left(\frac{d}{d\theta}\, EW^{-r}\right) \frac{\partial\theta}{\partial \bar{z}_{1i}}$$

$$\text{(B.2)}\quad \frac{\partial^2}{\partial \bar{z}_{1i}^2}\, EW^{-r} = \left(\frac{d^2}{d\theta^2}\, EW^{-r}\right)\left(\frac{\partial\theta}{\partial\bar{z}_{1i}}\right)^2 + \left(\frac{d}{d\theta}\, EW^{-r}\right) \frac{\partial^2\theta}{\partial\bar{z}_{1i}^2}$$

(B.3) $$\frac{\partial^2}{\partial \bar{z}_{1i}\,\partial \bar{z}_{1j}} EW^{-r} = \left(\frac{d^2}{d\theta^2} EW^{-r}\right) \frac{\partial \theta}{\partial \bar{z}_{1i}} \frac{\partial \theta}{\partial \bar{z}_{1j}}$$
$i \neq j$

(B.4) $$\frac{\partial^3}{\partial \bar{z}_{1i}^3} EW^{-r} = \left(\frac{d^3}{d\theta^3} EW^{-r}\right)\left(\frac{\partial \theta}{\partial \bar{z}_{1i}}\right)^3 + 3\left(\frac{d^2}{d\theta^2} EW^{-r}\right) \frac{\partial^2 \theta}{\partial \bar{z}_{1i}^2} \frac{\partial \theta}{\partial \bar{z}_{1i}}$$

(B.5) $$\frac{\partial^3}{\partial \bar{z}_{1i}^2\,\partial \bar{z}_{1j}} EW^{-r} = \left(\frac{d^3}{d\theta^3} EW^{-r}\right)\left(\frac{\partial \theta}{\partial \bar{z}_{1i}}\right)^2 \frac{\partial \theta}{\partial \bar{z}_{1j}} + \left(\frac{d^2}{d\theta^2} EW^{-r}\right) \frac{\partial^2 \theta}{\partial \bar{z}_{1i}^2} \frac{\partial \theta}{\partial \bar{z}_{1j}}$$
$i \neq j$

(B.6) $$\frac{\partial^3}{\partial \bar{z}_{1i}\,\partial \bar{z}_{1j}\,\partial \bar{z}_{1k}} EW^{-r} = \left(\frac{d^3}{d\theta^3} EW^{-r}\right) \frac{\partial \theta}{\partial \bar{z}_{1i}} \frac{\partial \theta}{\partial \bar{z}_{1j}} \frac{\partial \theta}{\partial \bar{z}_{1k}}$$
$i \neq j \neq k$

(B.7) $$\frac{\partial^4}{\partial \bar{z}_{1i}^4} EW^{-r} = \left(\frac{d^4}{d\theta^4} EW^{-r}\right)\left(\frac{\partial \theta}{\partial \bar{z}_{1i}}\right)^4 + 6\left(\frac{d^3}{d\theta^3} EW^{-r}\right)\left(\frac{\partial^2 \theta}{\partial \bar{z}_{1i}^2}\right)\left(\frac{\partial \theta}{\partial \bar{z}_{1i}}\right)^2$$
$$+ 3\left(\frac{d^2}{d\theta^2} EW^{-r}\right)\left(\frac{\partial^2 \theta}{\partial \bar{z}_{1i}^2}\right)^2 ,$$

(B.8) $$\frac{\partial^4}{\partial \bar{z}_{1i}^3\,\partial \bar{z}_{1j}} EW^{-r} = \left(\frac{d^4}{d\theta^4} EW^{-r}\right)\left(\frac{\partial \theta}{\partial \bar{z}_{1i}}\right)^3 \left(\frac{\partial \theta}{\partial \bar{z}_{1j}}\right) +$$
$i \neq j$
$$+ 3\left(\frac{d^3}{d\theta^3} EW^{-r}\right) \frac{\partial^2 \theta}{\partial \bar{z}_{1i}^2} \frac{\partial \theta}{\partial \bar{z}_{1i}} \frac{\partial \theta}{\partial \bar{z}_{1j}}$$

$$\text{(B.9)} \quad \underset{i \neq j}{\frac{\partial^4}{\partial \bar{z}_{1i}^2 \, \partial \bar{z}_{1j}^2}} EW^{-r} = \left(\frac{d^4}{d\theta^4} EW^{-r}\right)\left(\frac{\partial\theta}{\partial \bar{z}_{1i}}\right)^2 \left(\frac{\partial\theta}{\partial \bar{z}_{1j}}\right)^2 +$$

$$+ \left(\frac{d^3}{d\theta^3} EW^{-r}\right)\left(\frac{\partial\theta}{\partial \bar{z}_{1j}}\right)^2 \left(\frac{\partial^2\theta}{\partial \bar{z}_{1i}^2}\right) + \left(\frac{d^3}{d\theta^3} EW^{-r}\right)\left(\frac{\partial\theta}{\partial \bar{z}_{1i}}\right)^2 \left(\frac{\partial^2\theta}{\partial \bar{z}_{1j}^2}\right) +$$

$$+ \left(\frac{d^2}{d\theta^2} EW^{-r}\right) \frac{\partial^2\theta}{\partial \bar{z}_{1j}^2} \frac{\partial^2\theta}{\partial \bar{z}_{1i}^2}$$

$$\text{(B.10)} \quad \underset{i \neq j \neq k}{\frac{\partial^4}{\partial \bar{z}_{1i}^2 \, \partial \bar{z}_{1j} \, \partial \bar{z}_{1k}}} EW^{-r} \; \left(\frac{d^4}{d\theta^4} EW^{-r}\right)\left(\frac{\partial\theta}{\partial \bar{z}_{1i}}\right)^2 \left(\frac{\partial\theta}{\partial \bar{z}_{1j}}\right)\left(\frac{\partial\theta}{\partial \bar{z}_{1k}}\right) +$$

$$+ \left(\frac{d^3}{d\theta^3} EW^{-r}\right) \frac{\partial^2\theta}{\partial \bar{z}_{1i}^2} \frac{\partial\theta}{\partial \bar{z}_{1j}} \frac{\partial\theta}{\partial \bar{z}_{1k}}$$

$$\text{(B.11)} \quad \underset{i \neq j \neq k \neq \ell}{\frac{\partial^4}{\partial \bar{z}_{1i} \, \partial \bar{z}_{1j} \, \partial \bar{z}_{1k} \, \partial \bar{z}_{1\ell}}} EW^{-r} = \left(\frac{d^4}{d\theta^4} EW^{-r}\right) \frac{\partial\theta}{\partial \bar{z}_{1i}} \frac{\partial\theta}{\partial \bar{z}_{1j}} \frac{\partial\theta}{\partial \bar{z}_{1k}} \frac{\partial\theta}{\partial z_{1\ell}}$$

C. Evaluation of Mathematical Expectations Required in Section 3

In Section 3 we demonstrated the relationship

$$E\left(\frac{z_{1i}}{(\Sigma z_{1i}^2)^r}\right) = \frac{\partial}{\partial \bar{z}_{1i}} \left\{ E\left(\frac{1}{(\Sigma z_{1i}^2)^r}\right)\right\} + \bar{z}_{1i}\, E\left(\frac{1}{(\sum_i z_{1i}^2)^r}\right)$$

and we stated that all expectations required in (3.17)-(3.19) can be expressed in terms of

$$E\left(\frac{1}{(\sum_i z_{1i}^2)^r}\right)$$

and its partial derivatives with respect to $\bar{z}_{1i}$. Let us state the following results which can be obtained in a straightforward manner: Writing

(C.1) $W = \Sigma z_{1i}^2$

for the sake of brevity, we have

(C.2) $E(z_{1i} - \bar{z}_{1i})^2 W^{-r} = \left(\frac{\partial^2}{\partial \bar{z}_{1i}^2} + 1\right) EW^{-r}$,

(C.3) $E(z_{1i} - \bar{z}_{1i})(z_{1j} - \bar{z}_{1j})W^{-r} = \frac{\partial^2}{\partial \bar{z}_{1i}\, \partial \bar{z}_{1j}} EW^{-r}$
$i \neq j$

(C.4) $E(z_{1i} - \bar{z}_{1i})^3 W^{-r} = \left(\frac{\partial^3}{\partial \bar{z}_{1i}^3} + 3\frac{\partial}{\partial \bar{z}_{1i}}\right) EW^{-r}$

(C.5) $E(z_{1i} - \bar{z}_{1i})^2(z_{1j} - \bar{z}_{1j})W^{-r} = \left(\frac{\partial^3}{\partial \bar{z}_{1i}^2\, \partial \bar{z}_{1j}} + \frac{\partial}{\partial \bar{z}_{1j}}\right) EW^{-r}$
$i \neq j$

$$\text{(C.6)} \quad \underset{i\neq j\neq k}{E}(z_{1i} - \bar{z}_{1i})(z_{1j}-\bar{z}_{1j})(z_{1k}-\bar{z}_{1k})W^{-r} = \frac{\partial^3}{\partial\bar{z}_{1i}\,\partial\bar{z}_{1j}\,\partial\bar{z}_{1k}} EW^{-r}$$

$$\text{(C.7)} \quad E(z_{1i} - \bar{z}_{1i})^4 W^{-r} = \left(\frac{\partial^4}{\partial\bar{z}_{1i}^{\,4}} + 6\frac{\partial^2}{\partial\bar{z}_{1i}^{\,2}} + 3\right) EW^{-r}$$

$$\text{(C.8)} \quad \underset{i\neq j}{E}(z_{1i} - \bar{z}_{1i})^3(z_{ij} - \bar{z}_{1j})W^{-r} = \left(\frac{\partial^4}{\partial\bar{z}_{1i}^3\,\partial\bar{z}_{1j}} + 3\frac{\partial^2}{\partial\bar{z}_{1i}\,\partial\bar{z}_{1j}}\right) EW^{-r}$$

$$\text{(C.9)} \quad \underset{i\neq j}{E}(z_{1i} - \bar{z}_{1i})^2(z_{1j} - \bar{z}_{1j})^2 W^{-r} =$$

$$= \left(\frac{\partial^4}{\partial\bar{z}_{1i}^2\,\partial\bar{z}_{1j}^2} + \frac{\partial^2}{\partial\bar{z}_{1i}^2} + \frac{\partial^2}{\partial\bar{z}_{1j}^2} + 1\right) EW^{-r}$$

$$\text{(C.10)} \quad \underset{i\neq j\neq k}{E}(z_{1i} - \bar{z}_{1i})^2 (z_{1j} - \bar{z}_{1j})(z_{1k} - \bar{z}_{1k}) W^{-r} =$$

$$= \left(\frac{\partial^4}{\partial\bar{z}_{1i}^2\,\partial\bar{z}_{1j}\,\partial\bar{z}_{1k}} + \frac{\partial^2}{\partial\bar{z}_{1j}\,\partial\bar{z}_{1k}}\right) EW^{-r}$$

and

$$\text{(C.11)} \quad \underset{i\neq j\neq k\neq \ell}{E}(z_{1i} - \bar{z}_{1i})(z_{1j} - \bar{z}_{1j})(z_{1k} - \bar{z}_{1k})(z_{1\ell} - \bar{z}_{1\ell})W^{-r} =$$

$$= \frac{\partial^4}{\partial\bar{z}_{1i}\,\partial\bar{z}_{1j}\,\partial\bar{z}_{1k}\,\partial\bar{z}_{1\ell}} EW^{-r}$$

The expectations required in Section 3 may now be stated as follows.

(C.12) $$E(z_{1i}\, W^{-r}) = 2^{-r}\, \bar{z}_{1i}\, e^{-\theta}\, f_{-r+1,1}$$

where $f_{-r+1,1}$ has been obtained from (3.33) by writing $\mu = -\, r+1$ and $\nu = 1$.

Next

(C.13) $$E(z_{1i}^2\, W^{-r}) = E(z_{1i} - \bar{z}_{1i})^2\, W^{-r} + 2\, \bar{z}_{1i}\, E(z_{1i} - \bar{z}_{1i})W^{-r} + \bar{z}_{1i}^2\, EW^{-r}$$

$$= \left(\frac{\partial^2}{\partial \bar{z}_{1i}^2} + 1\right) EW^{-r} + 2\, \bar{z}_{1i}\, \frac{\partial}{\partial \bar{z}_{1i}}\, EW^{-r} + \bar{z}_{1i}^2\, EW^{-r}$$

$$= 2^{-r}\, e^{-\theta}\, [\bar{z}_{1i}^2\, f_{-r+2,2} + f_{-r+1,1}];$$

similarly, for $i \neq j$,

(C.14) $$E(z_{1i}\, z_{1j}\, W^{-r}) = 2^{-r}\, \bar{z}_{1i}\, \bar{z}_{1j}\, e^{-\theta}\, f_{-r+2;2}$$

where $f_{-r+2,2}$ has been obtained from (3.33) by writing $\mu = -\, r+2$ and $\nu = 2$.

Further we have

(C.15) $$E(z_{1i}^3\, W^{-r}) = 2^{-r}\, \bar{z}_{1i}^3\, e^{-\theta}\, f_{-r+3,3} + 3 \times 2^{-r}\, \bar{z}_{1i}\, e^{-\theta}\, f_{-r+2,2}\,,$$

(C.16) $$E(z_{1i}^2\, z_{1j}\, W^{-r}) = 2^{-r}\, \bar{z}_{1i}^2\, \bar{z}_{1j}\, e^{-\theta}\, f_{-r+3,3} + 2^{-r}\, \bar{z}_{1j}\, e^{-\theta}\, f_{-r+2,2}$$
$i \neq j$

(C.17) $$E(z_{1i}\, z_{1j}\, z_{1k}\, W^{-r}) = 2^{-r}\, \bar{z}_{1i}\, \bar{z}_{1j}\, \bar{z}_{1k}\, e^{-\theta}\, f_{-r+3,3}$$
$i \neq j \neq k$

(C.18) $$E(z_{1i}^4\, W^{-r}) = 2^{-r}\, \bar{z}_{1i}^4\, e^{-\theta}\, f_{-r+4,4} + 6 \times 2^{-r}\, \bar{z}_{1i}^2\, e^{-\theta}\, f_{-r+3,3} + $$

$$+ 3 \times 2^{-r}\, e^{-\theta}\, f_{-r+2,2}$$

(C.19) $$E(z_{1i}^3 z_{1j} W^{-r}) = 2^{-r} \bar{z}_{1i}^3 \bar{z}_{1j} e^{-\theta} f_{-r+4,4} + + 3 \times 2^{-r} \bar{z}_{1i} \bar{z}_{1j} e^{-\theta} f_{-r+3,3} \quad (i \neq j)$$

(C.20) $$E(z_{1i}^2 z_{1j}^2 W^{-r}) = 2^{-r} \bar{z}_{1i}^2 \bar{z}_{1j}^2 e^{-\theta} f_{-r+4,4} + + 2^{-r}(\bar{z}_{1i}^2 + \bar{z}_{1j}^2) e^{-\theta} f_{-r+3,3} + 2^{-r} e^{-\theta} f_{-r+2,2} \quad (i \neq j)$$

(C.21) $$E(z_{1i}^2 z_{1j} z_{1k} W^{-r}) = 2^{-r} \bar{z}_{1i}^2 \bar{z}_{1j} \bar{z}_{1k} e^{-\theta} f_{-r+4,4} + 2^{-r} \bar{z}_{1j} \bar{z}_{1k} e^{-\theta} f_{-r+3,3} \quad (i \neq j \neq k)$$

(C.22) $$E(z_{1i} z_{1j} z_{1k} z_{1\ell} W^{-r}) = 2^{-r} \bar{z}_{1i} \bar{z}_{1j} \bar{z}_{1k} \bar{z}_{1\ell} e^{-\theta} f_{-r+4,4} \quad (i \neq j \neq k \neq \ell)$$