1971

An Aggregation Problem in the Comparison of Total Factor Productivities: The SAK Index, the Hybrid Index and Moorsteen's Theory

Yasushi Toda

20321

RESEARCH REPORT 7116

AN AGGREGATION PROBLEM IN THE COMPARISON OF TOTAL FACTOR PRODUCTIVITIES: THE SAK INDEX, THE HYBRID INDEX AND MOORSTEEN'S THEORY (Revised)

by

Yasushi Toda

June, 1971

AN AGGREGATION PROBLEM IN THE COMPARISON OF TOTAL FACTOR PRODUCTIVITIES: THE SAK INDEX, THE HYBRID INDEX AND MOORSTEEN'S THEORY

by Yasushi Toda*

I. Introduction

The purpose of this paper is to present two aggregate measurements for the change in total factor productivity and to analyze the possible discrepancy of them from a disaggregated measurement. We use the term, total factor productivity, as meaning the productivity of some combination of capital and labor in producing the net national products.

One way of comparing the total productivity of a country between two periods is to measure the change in net national products with the quantity index-number in constant commodity prices and to estimate separately the change in national value-added with the quantity index-number in constant factor prices. One then combines the two quantity index-numbers and calculates the ratio of these two. We may term this aggregate measurement as the SAK index.[1]

Let us measure the difference in total factor productivity from an early period chosen as the basis, say year 0, to a later period, say year τ. We assume that the national economy consists of two productive sectors, each

---

*The original draft of this paper was presented to Professor Herbert S. Levine's seminar on the soviet economy at Harvard. The author is grateful to Professors Hajime Oniki and Robert D. Warne for helpful discussions and to Professor Abram Bergson for permission to quote from an unpublished article.

of which produces a final commodity, say Y_1 and Y_2. Denoting the capital stock and the labor force available to this nation as K and L, the capital rental as R, the wage rate as W and the commodity prices as P_i's, the SAK index may be defined as

$$(1.1) \qquad \kappa^o = \frac{P_1^oY_1^T + P_2^oY_2^T}{P_1^oY_1^o + P_2^oY_2^o} \Bigg/ \frac{R^oK^T + W^oL^T}{R^oK^o + W^oL^o}$$

in terms of early prices and

$$(1.2) \qquad \kappa^T = \frac{P_1^TY_1^T + P_2^TY_2^T}{P_1^TY_1^o + P_2^TY_2^o} \Bigg/ \frac{R^TK^T + W^TL^T}{R^TK^o + W^TL^o}$$

in terms of late prices. Here an index is shown as the ratio of expenditures in constant prices. These indexs can also be shown as the ratio of the weighted arithmetic averages of quantity ratios in such a way as

$$(1.1)' \qquad \kappa^o = (\delta^o \frac{Y_1^T}{Y_1^o} + (1-\delta^o) \frac{Y_2^T}{Y_2^o}) \Bigg/ (\gamma^o \frac{K^T}{K^o} + (1-\gamma^o) \frac{L^T}{L^o})$$

and

$$(1.2)' \qquad \kappa^T = (\gamma^T \frac{K^o}{K^T} + (1-\gamma^T) \frac{L^o}{L^T}) \Bigg/ (\delta^T \frac{Y_1^o}{Y_1^T} + (1-\delta^T) \frac{Y_2^o}{Y_2^T})$$

where δ denotes the share of the final demand for the first commodity, $P_1Y_1/(P_1Y_1 + P_2Y_2)$, and γ the share of profit in the national income, $RK/(RK+WL)$.

Though not as frequently computed as the κ indexes, there is another aggregate measurement for the change in total factor productivity, which we may term the hybrid indexes.[2] Unlike the κ indexes which calculate the arithmetic averages both in the numerator and the denominator, the hybrid index takes a geometric average for primary factors while calculating an arithmetic average for final goods. It may be defined as

$$(1.3) \qquad \beta^{o} = \frac{P_1^o Y_1^\tau + P_2^o Y_2^\tau}{P_1^o Y_1^o + P_2^o Y_2^o} \Bigg/ \left(\frac{K^\tau}{K^o}\right)^{\gamma^o} \left(\frac{L^\tau}{L^o}\right)^{1-\gamma^o}$$

and [3]

$$(1.4) \qquad \beta^{\tau} = \frac{P_1^\tau Y_1^\tau + P_2^\tau Y_2^\tau}{P_1^\tau Y_1^o + P_2^\tau Y_2^o} \Bigg/ \left(\frac{K^\tau}{K^o}\right)^{\gamma^\tau} \left(\frac{L^\tau}{L^o}\right)^{1-\gamma^\tau}$$

Taking the idea from the aggregate Cobb-Douglas function, the formula in (1.3) and (1.4), however, makes a sectoral break-down of the national product and takes into account the possibility for the change in factor shares.

The rate of productivity change, however, may well vary among individual productive sectors. If one disregards external economies, the aggregate indexes are supposed to estimate a certain average of the sectoral productivity changes. To show this explicitly, we may start with the difference in residuals between two periods for each individual sector. We then average these sectoral residuals with some weights. It seems most natural to choose each sector's contribution to the net national product as weights. Denoting the residual accruing in the i-th sector in a late year τ relative to the base year 0 as A_i^τ/A_i^o,[4] the weighted-mean index may be defined as

$$(1.5) \qquad \alpha^{o} = \frac{P_1^o Y_1^o (A_1^\tau/A_1^o) + P_2^o Y_2^o (A_2^\tau/A_2^o)}{P_1^o Y_1^o + P_2^o Y_2^o} = \delta^o (A_1^\tau/A_1^o) + (1-\delta^o)(A_2^\tau/A_2^o)$$

with the expenditure shares in year 0 as weights. Similarly, using the expenditure shares in a late year τ as weights and turning a clock backward, another weighted-mean index may be defined as

$$(1.6) \qquad \alpha^{\tau} = \frac{P_1^\tau Y_1^\tau + P_2^\tau Y_2^\tau}{P_1^\tau Y_1^\tau (A_1^o/A_1^\tau) + P_2^\tau Y_2^\tau (A_2^o/A_2^\tau)} = (\delta^\tau (A_1^o/A_1^\tau) + (1-\delta^\tau)(A_2^o/A_2^\tau))^{-1}$$ [5]

Since the SAK index and the hybrid index are aggregate measurements in one form or another, the disaggregated estimation in the weighted-mean indexes may very well yield a different result, and the bias due to aggregation tends to occur. The major prupose of this study is to see whether and how the $\varkappa$'s and the β's are different from the α's.[6]

The following discussion is divided into five parts. Section II presents a static neoclassical model of production with two consumer goods and two primary factors and to construct the factor endowment frontiers within this theoretical framework. With this as a preparation, Section III shows diagrammatically the SAK index $\varkappa$, the hybrid index β, and the weighted index α in the commodity- and factor-spaces. Since the aggregation bias of the indexes for total factor productivity was first discussed in Moorsteen's 1961 article, we take up Moorsteen's theory in Section IV and compare it with our approach. Section V is devoted to the explanation of a duality between quantities and prices. There the index for total factor productivity is shown by the increase in factor rewards relative to commodity prices. The aggregation bias is discussed in the context of price indexes rather than quantity indexes. Section VI summarizes our findings. Throughout the paper we see the aggregation bias mainly in the simplest case in which residuals of the two sectors are assumed to be the same.

II. The Factor Endowment Frontier

As an analog to the isoquant of a production function, the factor endowment frontier traces minimum amounts of capital and labor needed to maintain a given productive capacity of a country. One may conceive of different frontiers, depending on what is meant by the productive capacity.

And we shall construct three of them. The first one is the variety of factor-mixes with which the community can produce the given levels of outputs that are actually produced. Alternatively one may fix the utility at the level actually attained by the community, while allowing the outputs of commodities to vary from the observed levels. Then one may trace the second locus of factor-mixes that are the necessary minimum amounts to generate the utility equal to the one at the observation.

The third locus lies somewhere between the first and second loci. Suppose the amounts of commodities are to change along an offer curve which starts from the observed output levels in the commodity space. Assuming that the amounts of commodities along the offer curve have to be produced domestically, one may ask what are the minimum amounts of factors for producing outputs on the offer curve. As will be seen presently, we call the first locus the S curve, the second one the B curve, and the last one the E curve.

We make the following restrictive assumptions: 1) The community indifference curves exist and are homothetic. 2) The production function without joint output is homogeneous of degree one with the capital and the labor as inputs. 3) The factor proportions of two sectors are irreversible. The first sector is capital-intensive, and the second sector labor-intensive regardless of the value of factor prices. 4) The possibility for the complete specialization of a community into producing one commodity is excluded. Both sectors operate at positive levels. 5) The capital and the labor are malleable and transferable from one sector to the other. 6) Their endowments are given and fully utilized. 7) The factor and commodity markets are in equilibrium at the observations in periods 0 and τ.

Figure 1 measures the amounts of capital and labor on two axes of the factor-quantity space. Suppose the factor endowments available to the community

Figure 1

Factor Endowment Frontiers

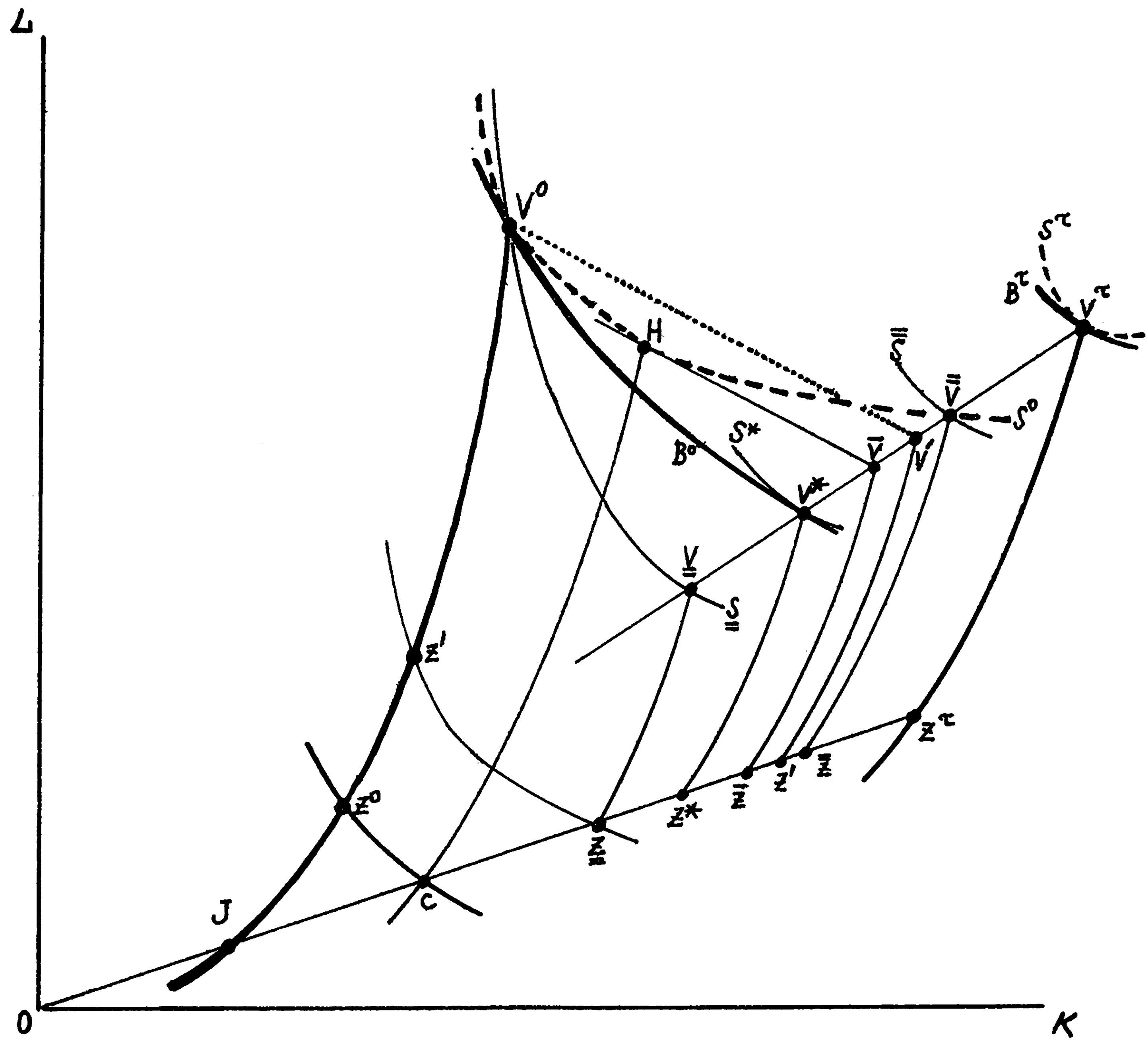
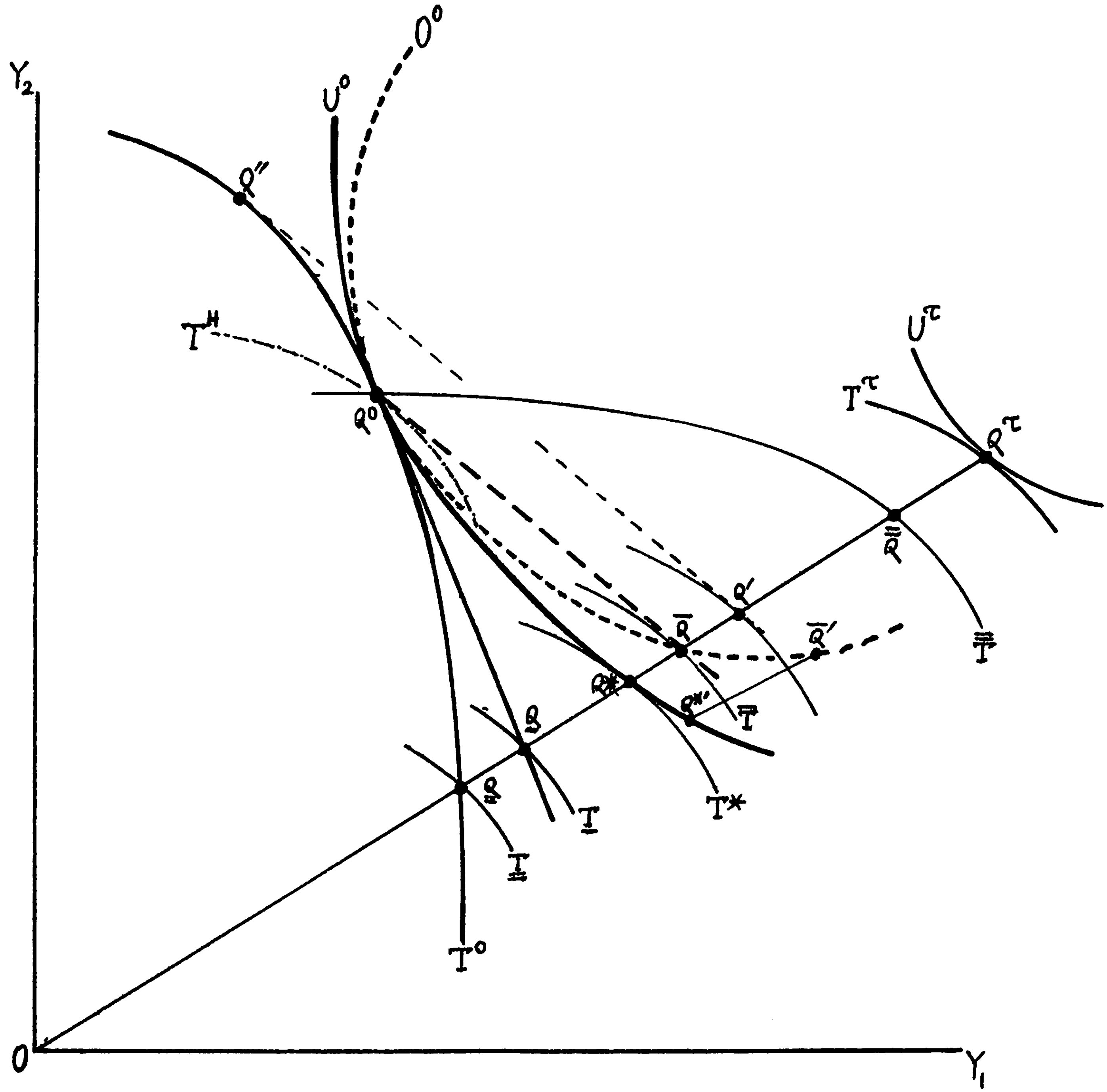


Figure 2

Production Possibilities in the Quantity Space



are given by the point V^o. We draw isoquants of the first sector with 0 as the origin, and isoquants of the second sector with V^o as the origin so that the efficiency locus OV^o may be traced from the assumptions 2 and 3. Suppose, further, the factors are allocated efficiently into the two sectors so that the allocation point may be found at Z^o on the locus.

First, if the total available labor decreases from the V^o endowment, how much capital should be added to the V^o in order for the outputs to remain at the observed levels? We draw the S^o curve to trace this factor substitution. Since the isoquants should remain the same, a point on the S^o curve, say H, corresponds with a point on the isoquants, say C. The two points lie on the same efficiency locus and the slope of the S^o curve at H is equal to the marginal rate of substitution of capital for labor at C.[7]

Figure 2 depicts the corresponding picture in the commodity space. The endowments at V^o generate the production possibility curve T^o. The system is in equilibrium at the observed output levels Q^o, where T^o is tangent to a community indifference curve U^o. With the factor endowments fixed at the observed V^o, suppose now that the output of Sector 1 rises and the output of Sector 2 falls . Along T^o curve the outputs change from the observed Q^o to, say $\underline{\underline{Q}}$. We select $\underline{\underline{Q}}$ so that it may be located on the same ray from the origin as point Q^τ, the observation in period τ.

To change outputs from Q^o to $\underline{\underline{Q}}$, the allocation point has to depart from Z^o and move along the efficiency locus OV^o toward, say Z'. As before, we ask the question: what are the minimum combinations of factor endowments in order to maintain the output levels at $\underline{\underline{Q}}$? The answer to this is the $\underline{\underline{S}}$ curve, whose slopes at V^o and $\underline{\underline{V}}$ are equal to the slopes of the isoquant at Z' and $\underline{\underline{Z}}$, respectively.[8] From our assumption of factor irreversal, the $\underline{\underline{S}}$ curve intersects with the S^o curve only once, at V^o.

Corresponding to our choice of point $\underline{\underline{Q}}$, we should also select the endowment point $\underline{\underline{V}}$ and the allocation point $\underline{\underline{Z}}$ in Figure 1 that are on the same rays from the origin as the observed V^{τ} and Z^{τ}, respectively. It is because in order for the commodity market to be in equilibrium at $\underline{\underline{Q}}$ under the late prices P_i^{τ}'s, the curve T^o should rotate around $\underline{\underline{Q}}$ to $\underline{\underline{T}}$ which is parallel to T^{τ} in period τ.[9] Accordingly, the allocation point leaves Z' and moves to $\underline{\underline{Z}}$ which has the factor intensities and the factor prices in common with the observation Z^{τ}. The endowment point moves along the $\underline{\underline{S}}$ curve to $\underline{\underline{V}}$ where the slope of the $\underline{\underline{S}}$ curve equals the observed wage-rental ratio in period τ, namely the slope of the S^{τ} curve at V^{τ}. This slope is flatter than the slope of S^o curve at V^o, the observed W^o/R^o ratio.[10]

Let us turn now to the second question. What should be the locus of factor-mixes which maintain the same utility level? Since $\underline{\underline{Q}}$ is less preferred to the observed Q^o by the community, the point such as Q* in Figure 2 which the community considers as indifferent to the observed point should be selected. We choose the point Q* which is in equilibrium under the same commodity prices as the point $\underline{\underline{Q}}$. The endowments on the $\underline{\underline{S}}$ curve which produce the outputs at $\underline{\underline{Q}}$ are obviously not sufficient to produce Q*. The $\underline{\underline{S}}$ curve has to shift outward in parallel-wise to gain the capacity for producing Q*. Along a straight line through the origin, the endowment point shifts from $\underline{\underline{V}}$ to, say V*, through which passes the S* curve to produce Q*.

Is there a limit to the outward shift of $\underline{\underline{S}}$ to attain the capacity to produce Q*? The endowment V* cannot go beyond $\bar{\bar{V}}$, which lies at the intersection of the S^o curve with the ray from the origin to another observation, V^{τ}. Since $\bar{\bar{V}}$ is on the S^o curve, this generates the production possibility curve $\bar{\bar{T}}$ passing through Q^o. The endowments V*, therefore, must be found somewhere between $\underline{\underline{V}}$ and $\bar{\bar{V}}$.[11] As the output levels change from Q^o to Q^* on the indifference curve U^o, the S^o curve shifts to S^*. One may draw the envelope B^o to this group of S's. At V^o and V^*, the B^o curve shares the equilibrium factor price ratios with S^o and S^*, but has a higher elasticity of factor substitution than the S's.[12]

So far we have defined three points in Figure 2: $\underline{\underline{Q}}$, $\bar{\bar{Q}}$ and Q^* which all have the equilibrium prices P_i^T's and lie along the ray OQ^T. For our discussion on index-numbers in the following section, it is important to introduce in addition the point of over-compensated variation in income, namely point $\bar{Q}$ which is at the intersection of this ray with the budget line passing through Q^o with the slope P_1^T/P_2^T.

What change should take place in the factor market corresponding to the change in equilibrium outputs from Q^o to $\bar{Q}$? In answer to this question, it is useful to trace the change in outputs in two steps: first, the production possibility curve T^o rotates around Q^o so that the new curve T^H may be tangent at Q^o to the new budget line $Q^o\bar{Q}$. Secondly, the T^H curve shifts to $\bar{T}$ with this budget line as an envelope in order that the equilibrium be reached at $\bar{Q}$.

Now in the first step the price ratio P_1/P_2 is falling, while outputs remain the same at Q^o. The wage-rental ratio will rise and the endowments will become more capital-intensive along the S^o curve. As the rotation ends at T^H, the movement along the S^o curve will stop at H, the point at which the slope of S^o equals W^T/R^T.[13] In the second step, unit costs of commodities remain the same and only the outputs change. Corresponding to this change, the sectoral factor intensities and the wage-rental ratio do not change, and only the endowments vary. The change in endowments takes place along the straight line from H to $\bar{V}$.[14]

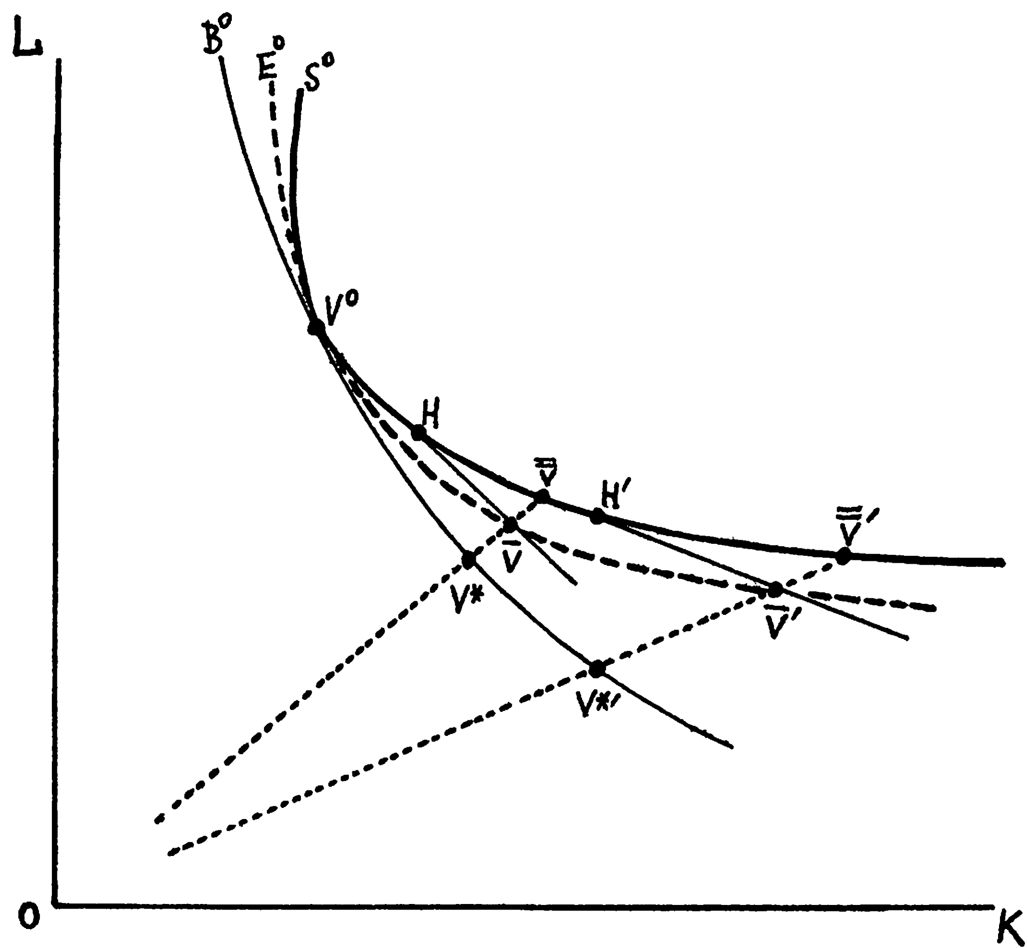
With those two steps combined, one finds that the change in the demand for commodities from Q^o to $\overline{Q}$ caused by the over-compensated income variation and the price variation from P_i^o's to P_i^T's requires an adjustment in the supply condition in such a way that the factor endowments and prices have to change from V^o to $\overline{V}$ via H. Needless to say, an S-curve and its envelope B-curve, say $\overline{S}$ and $\overline{B}$ (not shown), should be tangent at $\overline{V}$ to the straight line H $\overline{V}$ whose slope equals W^T/R^T. This is because these factor prices allocate the $\overline{V}$ endowments efficiently into two sectors at $\overline{Z}$, and produce the products at $\overline{Q}$ that bring the commodity market into equilibrium with the P_i^T prices (the slope of the straight line $Q^o\overline{Q}$).

We may follow the change in outputs from Q^o to $\overline{Q}$ in another way. In Figure 2 we draw an offer curve O^o along which lie all the overcompensation points stemming from Q^o. Suppose the demand has been shifting on this offer curve to $\overline{Q}$ and then further to $\overline{Q}'$. What are the required amounts of capital and labor, if this changing demand is to be satisfied by the shift in domestic production? We may find the answer in Figure 1a. In order to reach $\overline{Q}'$ from Q^o by two steps, the rotation of the production possibility curve T^o around Q^o has to go further than the previous rotation to T^H in Figure 2. Correspondingly, the wage-rental ratio becomes still higher and the endowment point passes over H and reaches H' in Figure 1a. In the second step, the outputs move along the budget line (not shown) connecting Q^o to $\overline{Q}'$, while unit costs of the products remain at the same levels. In Figure 1a, therefore, the endowments move to $\overline{V}'$ along a straight line which is tangent to the S^o curve at H'. In short, as the outputs change along the offer curve, the factor endowment should change along the E^o curve passing through V^o, $\overline{V}$ and $\overline{V}'$.[15]

If the output levels are fixed at Q^o and, therefore, the S^o curve

Figure 1a

Three Aggregate "Isoquants"



depicts the substitution between factor endowments, the factor endowment ratio shifts from the OV^{O} ray to the OH ray in response to the change in factor prices from W^{O}/R^{O} to W^{T}/R^{T}. If the output levels are not fixed, but are allowed to vary from Q^{O} to $\overline{Q}$ in accordance to the change in commodity prices, then the factor substitution becomes more elastic. The endowments shift from V^{O} to $\overline{V}$ in response to the same change in factor prices as above. As demonstrated algebraically by Amano (1964) and Jones (1965), the shift from V^{O} to H may be shown as Δ_S, the elasticity of substitution of the S^{O} curve:

(2.1) $\Delta_S = A_1\sigma_1 + A_2\sigma_2$

Likewise, the shift from V^{O} to $\overline{V}$ may be shown as Δ_B, the elasticity of substitution of the B^{O} curve:

(2.2) $\Delta_B = A_1\sigma_1 + A_2\sigma_2 + A_3\sigma_D$ [16]

Here, σ_1 and σ_2 are the elasticity of substitution of individual production functions, σ_D the price elasticity of demand, and the three A's are all positive and add up to unity. Hence, one finds $\Delta_S < \Delta_B$.

Let the products at $\overline{Q}$ be denoted as $\overline{Y}_1$ and $\overline{Y}_2$, and the endowments at $\overline{V}$ as $\overline{K}$ and $\overline{L}$. Since $\overline{Q}$ and Q^{O} are both on the budget line with the slope P_1^T/P_2^T, one has the budget equation: $P_1^T\overline{Y}_1 + P_2^T\overline{Y}_2 = P_1^TY_1^O + P_2^TY_2^O$. Since the outputs at $\overline{Q}$ are produced by the endowments at $\overline{V}$, the equivalence of the final demands to the total value-added tells that $P_1^T\overline{Y}_1 + P_2^T\overline{Y}_2 = R^T\overline{K} + W^T\overline{L}$. The right-hand side of this equation is shown by the straight line $H\overline{V}$ in Figure 1. If one draws a straight line with the slope W^T/R^T passing through the initial endowment at V^{O},[17] this straight line $V^{O}V'$ represents the national income $R^TK^O + W^TL^O$. As long as the S^{O} curve is convex to the origin, the straight line $V^{O}V'$ lies outward to the straight line $H\overline{V}$ with the same slope.

Therefore, one finds

(2.3) $$R^T K^o + W^T L^o > P_1^T Y_1^o + P_2^T Y_2^o$$

To derive (2.3), we assumed that the elasticity of substitution of a production function was positive and not zero. If both sectors have fixed input coefficients, however, the S^o curve itself is represented by two straight line segments crossing at point V^o at right angle. We may recall that the change in endowments from V^o to $\bar{V}$ involved a cost adjustment following the smooth S^o curve. But no longer is this adjustment of factor intensities due to the change in unit-cost necessary. In Figure 1 points $\bar{V}$ and $\bar{\bar{V}}$ coincide; in Figure 2 points $\bar{Q}$ and $\bar{\bar{Q}}$ coincide. One finds

(2.4) $$R^T K^o + W^T L^o = P_1^T Y_1^o + P_2^T Y_2^o$$

instead of (2.3).

III. Biases of the SAK and Hybrid Indexes

Based on our finding of the correspondence between the production possibility curve and the factor endowment frontier, we now discuss the difference of the aggregated $\varkappa$ and β indexes from the disaggregated α index in a simple setting where the residuals of two sectors are the same.

We first take up a SAK index $\varkappa^T$ as defined in (1.2). The product index in late year prices in (1.2) is graphically represented as the ratio of two parallel straight lines each passing through Q^T and Q^o in Figure 2 and thus equals the ratio of two line segments $OQ^T/O\bar{Q}$. Similarly, the factor index in late year prices in (1.2) is graphically represented as the ratio of two parallel straight lines each passing through V^T and V^o in Figure 1 and thus equals the ratio of two line segments OV^T/OV'. Therefore, the index is expressed as

(3.1) $\kappa^{\tau} = (OQ^{\tau}/O\bar{Q})/(OV^{\tau}/OV')$.

Endowed with factors at $\bar{V}$, however, the technology in period 0 produces efficiently the outputs at $\bar{Q}$. The endowments at V' which lies outside $\bar{V}$ are more than necessary to produce $\bar{Q}$. On the other hand, the observed endowment at V^{τ} is the necessary minimum for the technology in period τ to produce the observed Q^{τ}, since we assume the equilibrium at the observation. Thus the κ^{τ} index implies the inefficient utilization of the available endowments in year 0 and the full utilization in year τ. For this reason, κ^{τ} overestimates the increase in total factor productivity.

Unlike the ratio (3.1), a new ratio

(3.2) $(OQ^{\tau}/O\bar{Q})/(OV^{\tau}/O\bar{V})$

implies the full utilization in both periods. Since $O\bar{V} < OV'$, this ratio is smaller than κ^{τ} of (3.1). In what index-number form is this ratio represented? The output ratio of (3.2) is the same as that of (3.1) and thus compares the late observational point with the point of over compensated change in income from the early observation. In index-number form, therefore,

(3.3) $OQ^{\tau}/O\bar{Q} = (P_1^{\tau}Y_1^{\tau} + P_2^{\tau}Y_2^{\tau})/(P_1^{\tau}\bar{Y}_1 + P_2^{\tau}\bar{Y}_2)$.

The outputs in the numerator are produced by using the technology in period τ and allocating the endowments V^{τ} into two sectors at Z^{τ}. Therefore, $Y_1^{\tau} = F(K_1^{\tau}, L_1^{\tau}; \tau) = A_1^{\tau}\cdot\Phi(K_1^{\tau}, L_1^{\tau})$ and $Y_2^{\tau} = G(K_2^{\tau}, L_2^{\tau}; \tau) = A_2^{\tau}\cdot\Gamma(K_2^{\tau}, L_2^{\tau})$. The outputs in the denominator are produced by using the early technology and allocating the endowments $\bar{V}$ into two sectors at $\bar{Z}$. Denoting the factor inputs allocated at $\bar{Z}$ as $\bar{K}_i$ and $\bar{L}_i$, thus $\bar{Y}_1 = A_1^{o}\cdot\Phi(\bar{K}_1, \bar{L}_1)$ and $\bar{Y}_2 = A_2^{o}\cdot\Gamma(\bar{K}_2, \bar{L}_2)$. Therefore, the term in the denominator of (3.3) is

(3.4) $P_1^{\tau}\bar{Y}_1 + P_2^{\tau}\bar{Y}_2 = P_1^{\tau}\cdot A_1^{o}\cdot\Phi(\bar{K}_1, \bar{L}_1) + P_2^{\tau}\cdot A_2^{o}\cdot\Gamma(\bar{K}_2, \bar{L}_2)$.

Let the factor ratio of (3.2) be denoted as λ, namely, (3.5) $\lambda = OV^T/O\bar{V} = (R^T K^T + W^T L^T)/(R^T \bar{K} + W^T \bar{L})$.

As seen in Figure 1, the triangle $OV^T Z^T$ is similar to the triangle $O\bar{V}\bar{Z}$. The endowment ratio, the factor intensity of each sector and the factor prices are the same. The only difference is this scalor λ. Therefore, if the total endowments at $\bar{V}$ and the sectoral allocations at $\bar{Z}$ are multiplied with λ, the late observations at V^T and Z^T are obtained. That is, $\lambda\bar{K} = K^T$, $\lambda\bar{L} = L^T$, $\lambda\bar{K}_i = K_i^T$ and $\lambda\bar{L}_i = L_i^T$. Let the expenditure in (3.4) be multiplied with this scalor. Then one finds

$$(3.6) \qquad \lambda\cdot(P_1^T\bar{Y}_1 + P_2^T\bar{Y}_2) = P_1^T\cdot A_1^O\cdot\Phi(\lambda\bar{K}_1, \lambda\bar{L}_1) + P_2^T\cdot A_2^O\cdot\Gamma(\lambda\bar{K}_2, \lambda\bar{L}_2)$$
$$= P_1^T\cdot A_1^O\cdot\Phi(K_1^T, L_1^T) + P_2^T\cdot A_2^O\cdot\Gamma(K_2^T, L_2^T)$$
$$= P_1^T Y_1^T\cdot(A_1^O/A_1^T) + P_2^T Y_2^T\cdot(A_2^O/A_2^T) .$$

Therefore, the ratio which does not involve any under-utilization can be shown by the weighted-mean index defined in (1.6),

$$(3.7) \qquad (OQ^T/O\bar{Q})/(OV^T/O\bar{V}) = \frac{P_1^T Y_1^T + P_2^T Y_2^T}{P_1^T Y_1^T(A_1^O/A_1^T) + P_2^T Y_2^T(A_2^O/A_2^T}} = \alpha^T .$$

And, as already noted,

$$(3.8) \qquad \kappa^T > \alpha^T$$

by comparing (3.7) with (3.1).

One can make the same argument for the indexes in terms of the prices in the base year. The SAK index κ^O implies the full utilization of the available endowments in year 0 and the under-utilization in year τ. As compared with the weighted-mean index α^O, the index κ^O underestimates the increase in total factor productivity. Hence, one finds

$$(3.9) \qquad \kappa^O < \alpha^O.$$

Let us turn now to another aggregate estimation, the hybrid index β, and discuss its difference from the α index. Since the β index was designed by the analogy of the aggregate Cobb-Douglas production function, the factor endowment ratio in the β index assumes "aggregate isoquants" whose elasticity of substitution is unity. In the context of two-factor two-commodity model, what may be viewed as representing the "aggregate isoquant"? We have three candidates for it: the S, E and B curves in Figure 1a.

To take β^{τ} of (1.4) as an example, its factor index is defined as the ratio of the geometric average of the endowments at V^{τ}, $K(\tau)^{\gamma(\tau)} L(\tau)^{1-\gamma(\tau)}$, to the geometric average of the endowments at V^{o} with the same weights, $K(0)^{\gamma(\tau)} L(0)^{1-\gamma(\tau)}$. In general, the share of profit γ depends both on the endowment ratio K/L and on the state of technology, $A_i(t)$'s, which affects the factor price ratio. From our assumptions, however, γ is the function of K/L only. From the assumption of Hicks' neutrality, an efficiency locus, say OV^{o} in Figure 1, is fixed once the endowment is given at V^{o}, regardless of whether the technology in use is the one in period 0 or in period τ. Further, from the assumption of the equal rate of technological change between sectors, the allocation point remains unchanged at Z^{o}. The output-mix, the commodity price ratio and the factor price ratio all remain the same once the endowment-mix is given.

We may then construct a homogeneous function ϕ such as

$$(3.10) \qquad \phi(K,L) = L \cdot \phi(k,1) = \psi(U)$$

where ψ is an indicator for the community utility. If $k=k^{o}$, ϕ_L/ϕ_K equals the wage-rental ratio observed in period 0. If $k=k^{\tau}$, it equals the observation in period τ. In Figures 1 and 1a, the function(3.10) is represented by the B^{o}- curve. Its slope represents the equilibrium factor price ratio. It does not intersect with other B-curves. These properties are not shared by the S^{o} or E^{o} curve. Its slope does not represent the equilibrium factor prices except at the initial point V^o, as seen in Figure 1a. Two S-curves, S^o and $\bar{\bar{S}}$, intersect at point $\bar{\bar{V}}$ in Figure 1.[18]

Further, the aggregate isoquant in the β index assumes the unitary elasticity of substitution. Theoretically, any one of the S, E, and B curves may claim its elasticity to be unity. From an empirical point of view, however, it is unlikely that the S curve can claim this. As to the E or B curves, it is more conceivable that its elasticity is approximately unity. From (2.1), the elasticity of substitution of an S-curve is probably less than unity, since it is empirically more likely that the elasticity of substitution of individual production functions is lower than unity rather than higher than unity. Even if two production functions are of Cobb-Douglas, the elasticity of an S-curve is less than unity, From (2.2), however, the elasticity of substitution of a B-curve is unity if both production functions are of Cobb-Douglas and if the price elasticity of demand is unity. Even when σ_1 and σ_2 are less than unity, there still remains the possibility that Δ_B becomes unity provided that the demand for final goods is highly price-elastic to offset the low σ_1 and σ_2.

Let us assume the Cobb-Douglas production functions, the unitary demand elasticity and the intersectoral equality of the rate of technological change. Under these restrictive assumptions, the B-curves in period 0 and those in period τ coincide and their elasticity of substitution becomes always unity. Then the factor index in the β^τ index may be seen as comparing the B^τ and B^o curves in Figure 1. As before, the product index in the β^τ index may be seen as comparing two budget lines passing through Q^o and Q^τ with the common slope P_1^τ/P_2^τ in Figure 2. Then the β^τ index measures the ratio

(3.11) $\beta^\tau = (OQ^\tau/O\bar{Q})/(OV^\tau/OV^*)$.

This ratio, however, includes an infeasible production program in the initial period. Endowments at V* produce the outputs at Q* which are less than at $\overline{Q}$. The outputs at $\overline{Q}$ require the endowments to be at $\overline{V}$ which are more than at V*. Comparing (3.11) with (3.7), the inequality

(3.12) $\beta^{\tau} < \alpha^{\tau}$

is found. Similarly, the calculation of another hybrid index β^{o} involves an infeasible production in period τ. The bias of the hybrid index is thus shown by

(3.13) $\beta^{o} > \alpha^{o}$.[19]

As seen in (3.8), (3.9), (3.12) and (3.13), the hybrid indexes and the SAK index lie in the opposite side of inequalities with the weighted-mean index in the middle. If the prices and the factor shares in the base year 0 are used as is most often the case, our inequalities tell that the SAK index tends to underestimate, and the hybrid index tends to overestimate the residuals shown as the weighted-mean index.

Actually, the aggregation bias of the SAK index was realized by Moorsteen (1961), Moorsteen et al. (1966) and Bergson (1969) and, in its place, the hybrid index was designed to eliminate the bias. Our inequalities (3.12) and (3.13) indicate that their attempt might have overshot the target and created the aggregation bias in an opposite direction. Since we made extremely restrictive assumptions to reach (3.12) and (3.13), however, these inequality relationships may have to be reversed. Suppose the demand elasticity is higher than unity and the production functions are of Cobb-Douglas. Then the elasticity of a B-curve exceeds unity. The factor index of the β-index may be better represented by an E-curve than by a B-curve. If the elasticity of the E-curve happens to be unity, then the target is just hit and no aggregation bias arises if the total factor productivity is estimated by the hybrid index.

Whether (3.12) and (3.13) hold or not depends on the magnitudes of σ_1, σ_2 and σ_D, that is, upon the extent to which the demands for goods and factors are price-elastic.

IV. Moorsteen's Theory

An earlier article by Moorsteen (1961) discusses the bias of the SAK index toward what he calls a true index. His analysis follows a roundabout way. He first formulates a ratio of the product index to the factor index which the SAK index is supposed to approximate. Then, as the second step, the gap of a "true index" from this approximated SAK index is examined. How do his analyses fit in our diagrams of two factors and two goods?

First, his product index attempts to estimate the shift in production possibility curve from T^o to T^τ in Figure 2. Secondly, his factor endowment index is assumed to trace the shift in "isoquant" in the K-L space. It seems most likely that what he terms as "isoquant" is our S-curve.[20] Accordingly, Moorsteen's factor index traces the shift in "isoquant" from S^o to S^τ in Figure 1. With these two indexes combined, an index of total factor productivity which is supposed to approximate the SAK index estimates the shift in T-curve relative to the shift in S-curve along the rays from the origin to the observations. Since one can draw two rays (one to the observation in period 0 and the other to the observation in period τ), Moorsteen formulates two indexes. For our purpose, it sufficies to examine one of them. Let us take the one measured along the ray to the observation in period τ.

In Figure 1, the S^o curve intersects with the OV^τ ray at point $\bar{\bar{V}}$. In Figure 2, the T^o curve intersects with the OQ^τ ray at point $\underline{\underline{Q}}$. Denoting the endowments at $\bar{\bar{V}}$ as $\bar{\bar{K}}$ and $\bar{\bar{L}}$, and the outputs at $\underline{\underline{Q}}$ as $\underline{\underline{Y}}_i$'s, we may define one of Moorsteen's indexes

$$(4.1) \qquad \mu^T = \frac{P_1^T Y_1^T + P_2^T Y_2^T}{P_1^T \underline{\underline{Y}}_1 + P_2^T \underline{\underline{Y}}_2} \Bigg/ \frac{R^T K^T + W^T L^T}{R^T \bar{\bar{K}} + W^T \bar{\bar{L}}}$$

$$= (OQ^T/O\underline{\underline{Q}})/(OV^T/O\bar{\bar{V}}) \ .$$

As Moorsteen rightly observes and we also showed in Section II, the technology in period 0 applied to the endowments at $\bar{\bar{V}}$ generates the production possibility curve $\bar{\bar{T}}$ which passes through the observation Q^o.[21] The equilibrium point on the $\bar{\bar{T}}$ curve, however, is not Q^o but $\bar{\bar{Q}}$.[22]

The outputs are maintained at point $\underline{\underline{Q}}$ when the initial endowments at V^o are allocated into two sectors at Z'. As the relative commodity price P_1/P_2 rises along the T^o curve from Q^o to $\underline{\underline{Q}}$, the wage-rental ratio falls and thus makes the S^o curve pivot around V^o to the $\underline{\underline{S}}$ curve. In order for the market to be in equilibrium at $\underline{\underline{Q}}$, however, the production possibility curve T^o has to rotate to $\underline{\underline{T}}$. Correspondingly, the endowment point moves along the $\underline{\underline{S}}$ curve from V^o to $\underline{\underline{V}}$ to raise the wage-rental ratio and make the endowment ratio more capital intensive.

The true index formulated by Moorsteen in order to eliminate the aggregation bias should select the outputs at $\bar{\bar{Q}}$ rather than at $\underline{\underline{Q}}$ if $\bar{\bar{V}}$ is chosen as the endowment point. It is because the endowments $\bar{\bar{V}}$ divided into two sectors at $\bar{\bar{Z}}$ produces the outputs $\bar{\bar{Q}}$ in equilibrium. His "true index" $(OQ^T/O\bar{\bar{Q}})/(OV^T/O\bar{\bar{V}})$ is thus equivalent to our weighted-mean index α^T. By comparing this ratio with the definition (4.1), he correctly finds the upward bias of a μ index

$$(4.2) \qquad \mu^T > \alpha^T$$

since $O\bar{\bar{Q}} > O\underline{\underline{Q}}$ in Figure 2. Similarly, along the extensions of the rays OV^o

and OQ^o, one finds the downward bias of another μ index

(4.3) $\mu^o < \alpha^o$ [23]

In summary, when one uses the μ index as an approximation for the SAK index, the estimated result cannot avoid a bias. But this bias is in the same direction as the aggregation bias of the SAK index itself. This is all what one can say definitely.[24] While Moorsteen alludes to an aggregation bias of the SAK index in a roundabout way, we in the previous section found it directly.

V. Dual Indexes of Total Factor Productivity

In Section I the total factor productivity was defined as the increase in the quantities of final goods relative to the increase in the quantities of primary factors. The total factor productivity, however, may also be thought of as the increase in factor rewards relative to the increase in prices of the goods for final demands. This can easily be seen in terms of the Divisia index[25] and the SAK index. Let us take a SAK index (1.1), κ^o, as an example. Due to the equality of the final bill of goods with the total value-added, the denominator of the product index in κ^o, $P_1^oY_1^o + P_2^oY_2^o$, is equal to the denominator of the factor index, $R^oK^o + W^oL^o$. Thus this SAK index amounts to

(5.1) $$\kappa^o = (P_1^oY_1^T + P_2^oY_2^T)/(R^oK^T + W^oL^T).$$

In (5.1), divide the numerator by $P_1^TY_1^T + P_2^TY_2^T$, and the denominator by $R^TK^T + W^TL^T$. Due to the equivalence of the two concepts of the national income, this operation does not change κ^o. Then

$$(5.2) \qquad \varkappa^{o} = \frac{R^{T}K^{T} + W^{T}L^{T}}{R^{o}K^{T} + W^{o}L^{T}} \Bigg/ \frac{P_1^{T}Y_1^{T} + P_2^{T}Y_2^{T}}{P_1^{o}Y_1^{T} + P_2^{o}Y_2^{T}}$$

Thus $\varkappa^{o}$ is also expressed in the ratio of a factor price index to a product price index. The quantities of final products and primary factors used as "weights" are those in late year τ. These weights are contrary to the quantity expression of $\varkappa^{o}$ in (1.1), where the prices in initial year 0 are used as weights.[26] For the same reason, another SAK index $\varkappa^{T}$ is also expressed in the ratio of two price indexes:

$$(5.3) \qquad \varkappa^{T} = \frac{R^{T}K^{o} + W^{T}L^{o}}{R^{o}K^{o} + W^{o}L^{o}} \Bigg/ \frac{P_1^{T}Y_1^{o} + P_2^{T}Y_2^{o}}{P_1^{o}Y_1^{o} + P_2^{o}Y_2^{o}}$$

To see the total factor productivity from the price side, this section will examine the price relationship of final goods and primary factors. It is also hoped that by so doing more lights may be shed on the quantity relationship. Actually, our graphic demonstration of the SAK index is easier in this price comparison than in the previous quantity comparison. It is because we are attempting essentially to trace the amount of income change enough to compensate for the welfare effect of a given price change and to establish the point of compensated change in income both in the commodity space and the factor space. Naturally, one task is easier when the prices and the income rather than the quantities of goods and factors are treated as variables in the diagrams.

To depict the factor market, we measure the factor rewards, namely the wage rate and the rental rate, on the two axes in Figure 3. From the relationship between B- and S- curves in Figure 1, one may infer the existence of a curve tracing the various combinations of maximum rewards with a given utility level and the existence of another curve tracing the various combinations of

Figure 3

Factor Price Frontiers

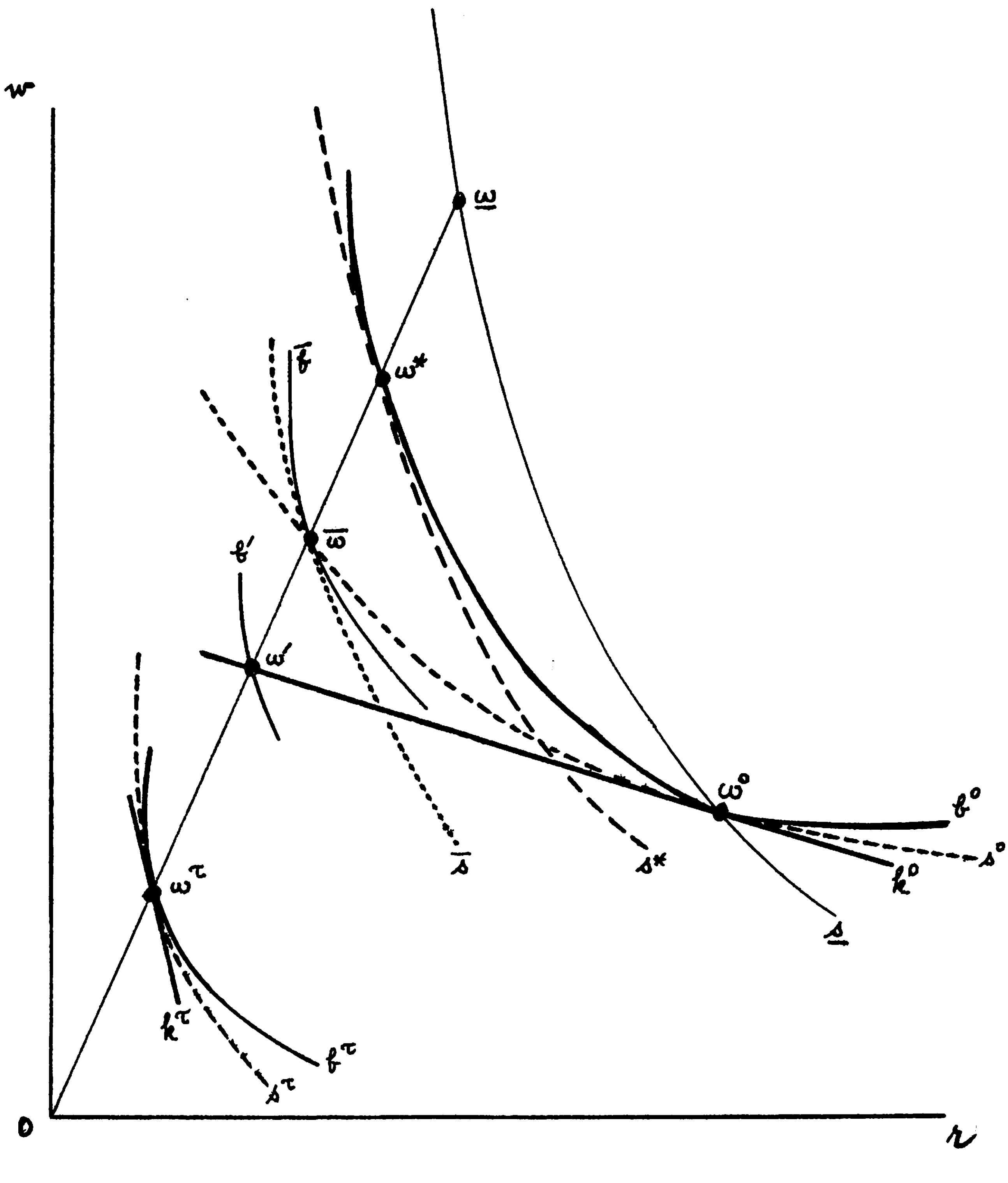
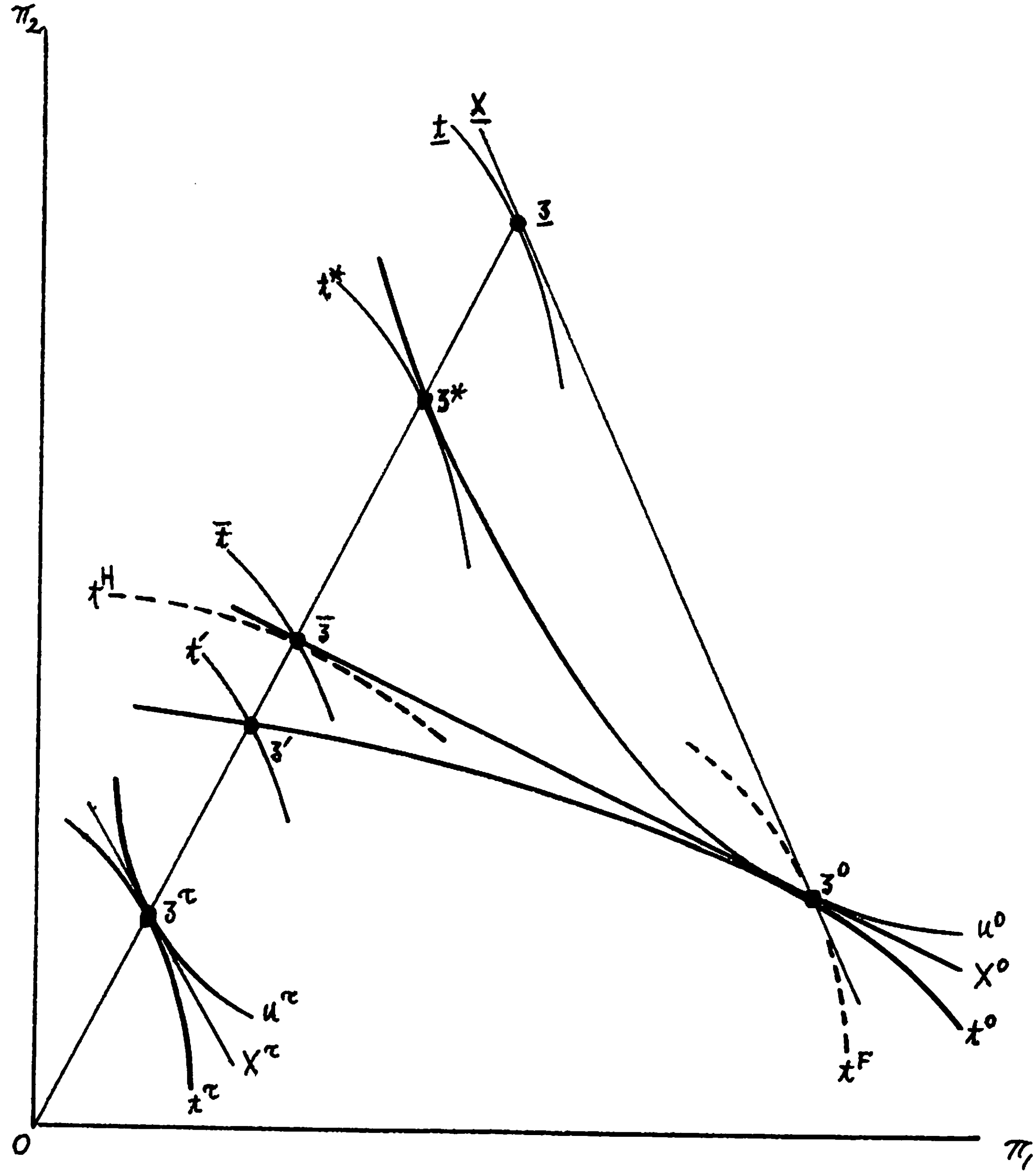


Figure 4

Production Possibilities in the Price-Income Space



maximum rewards with given output levels. As shown by b^o representing the former curve and by s^o representing the latter curve, the two curves share the slope, namely the endowment ratio $k^o = K^o/L^o$ and the former is an envelope to the latter at the equilibrium factor prices such as point ω^o. The only difference of the b-s relationship in Figure 3 from the B-S relationship in Figure 1 is that b^o is an envelope "from above" to s^o, whereas B^o was an envelope "from below" to S^o.[27] But we want to derive the b-s relationship directly, not in an indirect way from the B-S relationship. To do this, we need to investigate the price relationship of final goods.

For this purpose it is necessary to define community indifference curves and production possibility curves in terms of commodity prices and national income. Let us deflate the commodity prices, P_1 and P_2, with the value of national products, $M = P_1Y_1 + P_2Y_2$, and define the new variables $\pi_1 = P_1/M$ and $\pi_2 = P_2/M$.[28] With the π's on two axes, Figure 4 shows the indirect indifference curves such as u^o and u^T.[29]

The supply side is also described by π_1 and π_2, namely the unit costs of goods divided by the total value-added, $M = RK + WL$. When the factor endowments are fixed (say, at K^o and L^o or at K^T and L^T) and efficiently utilized, the relationship of π_1 and π_2 can be shown by a transformation curve, such as t^o or t^T in Figure 4, which is of the same shape as a production possibility curve in Figure 2 (namely, downward-sloping and concave to the origin).[30] An equilibrium point in the price-income space such as z^o in Figure 4 may be viewed as the point where the indirect utility is minimized under the given amounts of K and L. At this point, the slope of the tangency line X^o common to u^o and t^o represents the equilibrium output ratio Y_1^o/Y_2^o.

We also deflate the factor prices by M. The two axes of Figure 3 actually measure the deflated factor prices, $r = R/M$ and $w = W/M$. We shall now examine the correspondence between factor-prices and product-prices in three alternative settings which differ from one another by what is assumed to be fixed.

First, let us assume that the endowments are fixed at the initially observed K^o and L^o. Suppose the unit cost of production decreases in the first sector and increases in the second sector. In Figure 4, the initial equilibrium point z^o will move toward z' along the transformation curve t^o. Correspondingly, in Figure 3, the rental will fall and the wage rate will rise from the initial ω^o toward ω' along a straight line with the slope $k^o = K^o/L^o$.

Secondly, let us assume that the outputs remain at the initial levels, Y_1^o and Y_2^o. The iso-product line X^o is thus fixed. If the relative price of a capital-intensive good P_1/P_2 falls along this line from z^o to $\bar{z}$, the wage-rental ratio rises. Not only does this raise the capital intensity of individual sectors but, in view of the S^o curve in Figure 1, this also makes the factor endowment ratio more capital intensive. With the rising endowment ratio k, one can trace the rise in wage and the fall in rental by the s^o curve from ω^o to $\bar{\omega}$ in Figure 3.

Finally, assume that the utility remains at the initial level u^o. What will happen in the factor market if π's in the commodity market change from z^o to z^* along the indifference curve u^o? In the previous instance, the increase in endowment ratio was necessitated not by the change in output levels but solely by the fall in π_1/π_2. But now not only is the relative price of a capital-intensive good falling, but its relative output level is rising. With a given rise in W/R, the increase in k will thus be accelerated. Starting from ω^o, the factor prices will move toward ω^* along the b^o curve which has a steeper curvature than s^o.

Suppose now that the late observation in period τ is at z^τ in Figure 4. The market is in equilibrium at this point, as the transformation curve t^τ and the indifference curve u^τ are both tangent to the iso-product line X^τ (with the slope being equal to the observed Y_1^τ/Y_2^τ). Suppose also the equilibrium point observed in late period is at ω^τ in Figure 3 where the s^τ and b^τ curves share the slope $k^\tau = K^\tau/L^\tau$. Draw the rays from the origin, Oz^τ and $O\omega^\tau$. Along the extension of these rays we shall find the following three points.[31]

1) The point to be reached from the initial observation when the income is over-compensated:

The commodity prices and the income depart from the initial observation z^o and follow the iso-product line X^o. The transformation curve shifts from t^o to t^H. After the income and the prices have changed, the community can still buy the same bundle of goods as initially observed, since t^o and t^H share the X^o line. Corresponding to this change in the commodity market, the equilibrium point ω^o in the factor market moves toward $\bar{\omega}$ along the s^o curve. The wage-rental ratio rises and the endowment ratio becomes more capital-intensive.

To obtain an equilibrium in the commodity market at $\bar{z}$, the transformation curve t^H has to rotate around point $\bar{z}$ toward $\bar{t}$, and the iso-product line X^o to the line parallel to X^τ. This change in outputs with constant commodity-prices at $\bar{z}$ leaves the factor prices unchanged at $\bar{\omega}$ in Figure 3. But the factor endowment ratio becomes more capital intensive. The s^o curve pivots to $\bar{s}$ around $\bar{\omega}$.[32]

2) The point on the same indifference curve as the initial observation:

In the previous instance in which the equilibrium point moved along the iso-product line X^o, the income was overcompensated. In order for the utility to remain at the initial level, therefore, the income should be less than at $\bar{z}$. Let the output levels of two sectors be reduced at the same rate from $\bar{z}$, while maintaining the same commodity prices. Then the income falls and the equilibrium point changes along the ray from $\bar{z}$ to, say z^*, where the transformation curve t^* is tangent to the initial indifference curve u^o. Since only the scale of production declines with constant commodity prices, no change in the factor prices, the sectoral factor proportions k_i's and the endowment ratio k will take place. The curve $\bar{s}$ shifts parallelwise to, say s^*. The equilibrium point in the factor market moves from $\bar{\omega}$ to ω^* along the ray from the origin.

Now ω^o and ω^* represent two sets of factor prices and national income which corresponds with z^o and z^* in Figure 4. One may draw the curve b^o, which passes through ω^o and ω^* and forms an envelope to s^o and s^*. The b^o curve thus represents the variety of factor prices and income which are required to maintain the same utility.[33]

3) The point on the initial transformation curve:

As the point in the commodity market moves from z^o to z' on the transformation curve t^o, the point in the factor market moves from ω^o to ω' along a straight line with the slope equal to the initial endowment ratio k^o. To attain an equilibrium at z', the transformation curve has to rotate to t' to raise the output ratio to Y_1^T/Y_2^T. Accordingly, the endowment ratio at ω' increases to K^T/L^T.[34]

Having found three points[35] along the rays $O\omega^T$ and Oz^T, we now turn to the measurement of total factor productivity by comparing the change in deflated factor prices from ω^o to ω^T in Figure 3 with the change in deflated commodity prices from z^o to z^T in Figure 4. First, we compare ω^o and ω^T in index-number form with the endowment in period 0 as "weights".

(5.4) $(r^{\tau}K^{o} + w^{\tau}L^{o})/(r^{o}K^{o}+w^{o}L^{o})$

Secondly, we compare z^{o} and z^{τ} with the final outputs in period 0 as "weights",

(5.5) $(\pi_1^{\tau}Y_1^{o} + \pi_2^{\tau}Y_2^{o})/(\pi_1^{o}Y_1^{o} + \pi_2^{o}Y_2^{o})$.

Taking the ratio of (5.4) to (5.5) and substituting $r^{t} = R^{t}/M^{t}$, $w^{t} = W^{t}/M^{t}$ and $\pi_i^{t} = P_i^{t}/M^{t}$ (for $t=0, \tau$) into the ratio, one obtains

(5.6) $$\frac{r^{\tau}K^{o} + w^{\tau}L^{o}}{r^{o}K^{o} + w^{o}L^{o}} \Bigg/ \frac{\pi_1^{\tau}Y_1^{o} + \pi_2^{\tau}Y_2^{o}}{\pi_1^{o}Y_1^{o} + \pi_2^{o}Y_2^{o}} = \frac{R^{\tau}K^{o} + W^{\tau}L^{o}}{R^{o}K^{o} + W^{o}L^{o}} \Bigg/ \frac{P_1^{\tau}Y_1^{o} + P_2^{\tau}Y_2^{o}}{P_1^{o}Y_1^{o} + P_2^{o}Y_2^{o}}$$

The term in the right-hand side of (5.6) is the ratio of two price indexes which we showed in (5.3) as the dual expression of a SAK index $\varkappa^{\tau}$.

The left-hand side of the equation (5.6) may also be viewed as the $\varkappa^{\tau}$ index originally defined as the ratio of quantity indexes in (1.2). The reason is the following. Denote the deflated factor prices at point ω' as w' and r'. Since point ω' lies on the same straight line (with the slope k^{o}) as does point ω^{o}, one should have $r'K^{o} + w'L^{o} = r^{o}K^{o} + w^{o}L^{o}$. Therefore, the ratio (5.4) can be rewritten as the ratio

(5.7) $(r^{\tau}K^{o} + w^{\tau}L^{o})/(r'K^{o} + w'L^{o})$.

From our assumptions, ω' lies on the extension of the ray $O\omega^{\tau}$. Thus the factor prices at ω' are the same as those observed in period τ, W^{τ} and R^{τ}. The income level at ω' is such that under these factor prices the community could have employed K^{o} and L^{o} and used up the income. Therefore, the income equals $R^{\tau}K^{o} + W^{\tau}L^{o}$. Thus $r' = R^{\tau}/M'$ and $w' = W^{\tau}/M'$ where $M' = R^{\tau}K^{o} + W^{\tau}L^{o}$. In the numerator of (5.7), r^{τ} and w^{τ} are of course equivalent to R^{τ}/M^{τ} and W^{τ}/M^{τ} where $M^{\tau} = R^{\tau}K^{\tau} + W^{\tau}L^{\tau}$. Hence, the ratio (5.7) is equal to the ratio of two income levels,[36]

(5.8) $$\frac{M'}{M^{\tau}} = \frac{R^{\tau}K^{o} + W^{\tau}L^{o}}{R^{\tau}K^{\tau} + W^{\tau}L^{\tau}}$$

Turning now to the index of commodity prices (5.5), this index may be rewritten as

(5.9) $(\pi_1^T Y_1^o + \pi_2^T Y_2^o)/(\bar{\pi}_1 Y_1^o + \bar{\pi}_2 Y_2^o)$

where $\bar{\pi}_i$'s indicate point $\bar{z}$ in Figure 4. From our assumption, $\bar{z}$ should be on the ray $0z^T$. Then $\bar{\pi}_i$ equals $P_i^T/\bar{M}$ where $\bar{M}$ represents the overcompensated income $P_1^T Y_1^o + P_2^T Y_2^o$. We substitute $\bar{\pi}_i = P_i^T/\bar{M}$ and π_i^T/M^T into (5.9) and find that the ratio (5.9) equals the ratio of two income levels:[37]

$$(5.10) \qquad \frac{\bar{M}}{M^T} = \frac{P_1^T Y_1^o + P_2^T Y_2^o}{P_1^T Y_1^T + P_2^T Y_2^T}$$

Finally, take the ratio of (5.8) to (5.10) and obtain

$$\frac{R^T K^o + W^T L^o}{R^T K^T + W^T L^T} \Big/ \frac{P_1^T Y_1^o + P_2^T Y_2^o}{P_1^T Y_1^T + P_2^T Y_2^T} .$$

This is a SAK index κ^T as originally defined in (1.2). Therefore, Figures 3 and 4 demonstrate the equivalence of κ^T as the ratio of quantity indexes and κ^T as the ratio of price indexes.

Graphically the SAK index is shown as the ratio

(5.11) $\kappa^T = (0\omega^T/0\omega')/(0z^T/0\bar{z})$.

Here the factor prices at ω' are compared with the commodity prices at $\bar{z}$. But the commodity prices at $\bar{z}$ should correspond to the factor prices at $\bar{\omega}$ which are higher than those at ω'. One finds that at ω' the production costs are too low compared with the commodity prices and, therefore, the outputs are less than the optimal levels. Hence, the formula (5.11) contains the underutilization of factors in the initial period and thereby overestimates the improvement in productivity for the subsequent period. This conclusion is the same as the one we reached in Section III by quantity comparisons.

The hybrid index β may also be constructed as the ratio of price indexes. If one assumes that σ_1, σ_2 and σ_D are all unity, then the elasticity of substitution of a b-curve becomes unity. This property, together with the assumption of the intersectoral equality of residuals, assures that the profit share γ is the same in periods 0 and τ. The factor price index with geometric weights

$$\left(\frac{R(\tau)}{R(0)}\right)^{\gamma(0)} \cdot \left(\frac{W(\tau)}{W(0)}\right)^{1-\gamma(0)}$$

then compares the b^o and b^τ curves in Figure 3. By taking the ratio of this index to a product price index with arithmetic weights, we define a hybrid index:

$$(5.12) \qquad \beta^\tau = \left(\frac{R(\tau)}{R(0)}\right)^{\gamma(0)} \cdot \left(\frac{W(\tau)}{W(0)}\right)^{1-\gamma(0)} \Big/ \frac{P_1(\tau)\cdot Y_1(0) + P_2(\tau)\cdot Y_2(0)}{P_1(0)\cdot Y_1(0) + P_2(0)\cdot Y_2(0)}$$

This is equivalent to the hybrid index defined in (1.4) as the ratio of quantity indexes. Graphically the index (5.12) is shown as the ratio of line segments along the rays from the origin to the observations in period τ,

$$(5.13) \qquad \beta^\tau = (0\omega^\tau/0\omega^*)/(0z^\tau/0\bar{z}).$$[38]

The factor prices at ω^* are too high in relation to the commodity prices at $\bar{z}$, because in equilibrium point $\bar{\omega}$ (instead of ω^*) should correspond with point $\bar{z}$. With the factor prices and the commodity prices at ω^* and $\bar{z}$, respectively, the production cost exceeds the revenue. The scale of production has to be curtailed. The hybrid index shown in (5.13) thus contains the infeasible price-cost relation in the base period 0. Therefore, this index underestimates the subsequent improvement in technology.

The bias of the hybrid index is the same as found in Section III. As we noted there, however, this bias is highly susceptible of change, because the elasticity of the aggregate b-curve critically hinges on the elasticities: σ_1, σ_2 and σ_D.

VI Summary

We summarize major findings.

1) We assume no difference in the residual between sectors. Under this assumption, a consistent gap exists between a SAK index and a corresponding weighted-mean index unless the two production functions have fixed coefficients. The gaps are such that $\varkappa^O < \alpha^O$ and $\varkappa^T > \alpha^T$.

2) Moorsteen's index μ^T which measures the shift in production possibility curve relative to the shift in S-curve along the rays from the origin to the late observation has an upward bias to α^T. Thus the bias of Moorsteen's index to a weighted-mean index is found in the same direction as the bias of a SAK index to a weighted-mean index, though of course the two biases are generally of different magnitudes. We may thus uphold Moorsteen's theory that views the bias of a SAK index in analogy to the bias of a μ index.

3) Under the assumption in (1) plus the assumption of Cobb-Douglas production functions and the unitary elasticity of demand for final products, the B^O and B^T curves in Figure 1 are parallel to each other and their elasticity of substitution is unity. Under these assumptions, the gaps between hybrid indexes and weighted-mean indexes are such that $\beta^O > \alpha^O$ and $\beta^T < \alpha^T$ are found. This suggests that the hybrid index as a means for eliminating the bias of a SAK index leads to an aggregation bias in the opposite direction.

However, our argument on the aggregation bias of a hybrid index is certain only insofar as such restrictive assumptions as mentioned above are satisfied. If the production functions and the utility function are such that the demand schedules for final goods and primary factors are more price-elastic than the above assumptions imply, then the bias of the hybrid index to the weighted-mean index will be narrowed. The bias may even be reversed so that the hybrid index may stand in the same side as does the SAK index vis-à-vis the weighted-mean index.

4) Apart from the biases of two aggregate indexes to α, we may state more definitely about the difference between these two aggregate indexes. One should find $\beta^T > \kappa^T$ if the curve representing the weighted geometric average of K(0) and L(0) in the β index intersects with the OV^T ray inside point V'. This holds if this curve can be approximated by the B^O or E^O curve in Figures 1 and 1a. The inequality relation as above has to be reversed, however, if this curve resembles the S^O curve in Figure 1. But this is not likely the case, unless the elasticities of substitution of production functions are much higher than unity. Thus one will probably find $\beta^O > \kappa^O$ and $\beta^T < \kappa^T$. Although this study is not directly concerned with empirical estimations, it is worth noting that $\beta^O > \kappa^O$ is found in at least one empirical study.[39]

5) The improvement in total factor productivity may also be viewed as the increase in factor rewards relative to commodity prices. The index of total factor productivity is then defined as the ratio of a factor price index to a commodity price index. Based on the price relationship between factors and products in our static, neoclassical model, one can draw a factor price frontier and a transformation curve in the price-income space. In this space one then finds the same aggregation biases of a SAK index and a hybrid index as were shown in the quantity space.

6) A crucial assumption for these conclusions is the equality of sectoral residuals. Under this assumption, paired with the assumption of the Hicks neutral technological change, the static property is preserved so as to establish the link between the endowment ratio and the output-mixes, between the factor-price ratio and the commodity-price structure over periods.

Needless to say, the assumption of equal sectoral residuals is unrealistic. Once one removes this assumption (even with the Hicks neutrality still retained), however, many different possibilities arise and the systematic bias as was found above can no longer be held. Therefore, our results may still claim some relevance to the growth process only if (1) the residuals are not much different between sectors and (2) the endowment ratio significantly varies due to the high saving rate and the limited labor supply, and thus exerts a considerable influence to the change in output-mix and price structures.

Footnotes

[1]Schmookler - Abramovitz - Kendrick index, to follow the abbreviation by Domar (1961).

[2]Moorsteen et al. (1966) and Bergson (1969).

[3]To make an analogy to $\varkappa^{\tau}$ in (1.2)' more complete, β^{τ} may be shown as

$$\beta^{\tau} = \left(\frac{K^{o}}{K^{\tau}}\right)^{\gamma^{\tau}} \left(\frac{L^{o}}{L^{\tau}}\right)^{1-\gamma^{\tau}} \Bigg/ \left(\delta^{\tau} \frac{Y_1^{o}}{Y_1^{\tau}} + (1-\delta^{\tau}) \frac{Y_2^{o}}{Y_2^{\tau}}\right)$$

[4]Since the residuals are always defined as the ratios, we may conceive of A_1^{o}'s as unity.

[5]The national product index in prices of the base year 0 is equivalent to $\delta^{o}(Y_1^{\tau}/Y_1^{o}) + (1-\delta^{o})(Y_2^{\tau}/Y_2^{o})$. By replacing the ratios of net outputs with the sectoral residuals, one obtains α^{o}.

Similarly, the national product index in prices of a late year τ is equivalent to $(\delta^{\tau}(Y_1^{o}/Y_1^{\tau}) + (1-\delta^{\tau})(Y_2^{o}/Y_2^{\tau}))^{-1}$. Replacement of product ratios with residuals gives α^{τ}.

[6]Before proceeding further, it seems worth dwelling on the difference of our indexes in constant prices from the Divisia index (Jorgenson et al. (1967) and Sims (1969)). Since this index measures the difference in productivity of each period from the immediately preceding period, each index of the Divisia series takes commodity-and factor-prices of each period for the aggregation purpose. Applied to annual data, the indexes thus measure the annual rates of technological change with the weights varying from year to year. On the other hand, the indexes in constant prices in the form of either $\varkappa$, β or α compare two distinctive periods of time, years 0 and τ, that may be far apart. In intervening years there may have arisen emergencies where the goods were requisitioned and rationed, the prices fixed

at official rates and the primary factors mobilized without recourse to market forces. One should abandon the data in intervening years and pick up only the "normal" years. Moreover, if year 0 chosen as the base year for the κ, β and α indexes has a special significance for the economic history of a nation, these indexes can fairly claim the usefulness, even when the nation has been following a smooth developmental process in which prices have always reflected more or less accurately the supply and demand conditions and, therefore, one can meaningfully calculate the Divisia indexes. For example, the base year 0 may be the initial year of a "take-off period," the initial or terminal years of a five-year-plan, or the year when the recovery from the war damage had been completed and the development resumed a "normal" pattern. Then the economic indicators in all the remaining years, say τ, should be compared not only to year τ-1 but to this base year, in order to evaluate the performance in year τ. Furthermore, our indexes with year 0 as the base can be applied directly to international cross-section data with each country as an observation to measure the international difference in total factor productivity. It seems rather far-fetched to arrange the observations by order of per capita income of a nation or some other yardstick and to calculate the Divisia indexes.

If one estimates the change in total factor productivity in terms of the Divisia index, there does not arise any aggregation bias which is our major concern here. Denoting the rates at which the production functions F and G shift over time as $\hat{F}$ (= $(\partial F/\partial t)/F$) and $\hat{G}$, the Divisia index for the total productivity change in year τ may be shown as

$$\text{(i)} \quad \alpha(\tau) = \delta(\tau)\,\hat{F}(\tau) + (1-\delta(\tau))\,\hat{G}(\tau).$$

The rate of total productivity change may also be conceived of as the residual and thus may be expressed as the difference between the rates of increase in net outputs and the rates of increase in primary factors,

(ii) $\beta(\tau) = \delta(\tau)\ \hat{Y}_1(\tau) + (1-\delta(\tau))\ \hat{Y}_2(\tau) - \gamma(\tau)\ \hat{K}(\tau) - (1-\gamma(\tau))\ \hat{L}(\tau)$,

where a circumflex denotes the proportionate rate of change, e.g., $\hat{K} = (dk/dt)/K$. The two expressions, (i) and (ii), are equal as long as one holds the assumption of perfectly competitive markets. (The proof omitted here was given in Ohkawa (1967)).

Thus the estimation by the residual (ii) has no aggregation bias when one conceives of the technological change as an instantaneous change and approximates it by a Divisia index. In the context of the Divisia index, therefore, the discussion in this paper has no relevance. But, as we noted, our interest lies elsewhere.

[7] The S-curve, together with the B-curve to be introduced later, is a direct application of the Scitovsky and Bergson contours in welfare economics (Samuelson 1956 and Graaff 1957). An application of the Scitovsky contours to the production theory is also found in Vanek (1968).

The S^o curve is thus derived not by a parallel shift of the isoquant. The slope or the marginal rate of factor substitution is the same for the S^o curve and the iso-quant (between V^o and Z^o, between H and C). But the S^o curve and the isoquants generally have different elasticities of substitution.

[8] The slope of S-curves at V^o is steeper for $\underline{\underline{S}}$ than S^o, due to our assumption on the sectoral factor intensity. Since more factors are allocated to the capital-intensive Sector 1 at Z' than at Z^o, the relative factor price W/R is lower at Z' than at Z^o.

[9] As this rotation lowers the relative price of a capital-intensive good while keeping the output levels constant, the wage-rental ratio becomes higher and that makes each industry more capital-intensive.

[10]The slope of $\underline{\underline{S}}$ curve at $\underline{\underline{V}}$ is equal to the slope of the isoquant $Z'\underline{\underline{Z}}$ at $\underline{\underline{Z}}$ and, therefore, equal to the slope of the isoquant Z^oC at C. The slope of S^o curve at V^o equals the slope of the isoquant Z^oC at Z^o. Since W/R is higer at C than at Z^o, W/R is higher at $\underline{\underline{V}}$ on $\underline{\underline{S}}$ than at V^o on S^o.

[11]To see the mechanism more clearly, it may be useful to draw the $\bar{\bar{S}}$ curve which is parallel to $\underline{\underline{S}}$ and S* and intersects with S^o at $\bar{\bar{V}}$. The outward, parallel shift of the $\underline{\underline{S}}$ curve cannot go beyond this $\bar{\bar{S}}$ curve. The larger and similar triangle $O\bar{\bar{Z}}\bar{\bar{V}}$ to the triangle $O\underline{\underline{Z}}\underline{\underline{V}}$ indicates that the wage-rental ratio or the slope of the $\bar{\bar{S}}$ curve at $\bar{\bar{V}}$ is the same as the slope of the parallel S-curves at V* and $\underline{\underline{V}}$.

Since $\bar{\bar{V}}$ has the wage-rental ratio, the endowment ratio and the sectoral factor intensities in common with V* and $\underline{\underline{V}}$, the endowments at $\bar{\bar{V}}$ generates the production possibility curve whose equilibrium point is $\bar{\bar{Q}}$. This $\bar{\bar{Q}}$ shares the commodity price ratio and the output-mix with Q* and $\underline{\underline{Q}}$.

[12]From Figures 1 and 2, one can see the correspondences of a point to a curve. The endowments at V^o generate the production possibility curve T^o. As the endowments move along the B^o curve to V*, the production possibility curve shifts to T* with the U^o as an envelope. To see the picture in another way, the outputs at Q^o require the endowments shown by the S^o curve. As the outputs move along the U^o curve to Q*, the S^o curve shifts to S* with the B^o curve as an envelope.

Since a B-curve is derived from S-curves and an indifference curve, it also has the property of homotheticity. That is to say, B-curves are convex to the origin and do not intersect with each other. It is only the endowments on a B-curve and the factor prices represented by its slope that satisfy the equilibrium in both markets. Only once do a B-curve and an S-curve share the common slope at their tangency point. And this slope represents the equilibrium factor price ratio.

[13]Since the output levels are fixed at Q^o, the possible change in factor market may be seen by the change in allocation point along the Z^o isoquant and the change in endowments along the S^o curve. The decline in the price of a capital-intensive good and the rise in the price of a labor-intensive good as shown by the rotation of a production possibility curve will increase the wage-rental ratio. This change will make both sectors more capital-intensive, as seen by the change from Z^o to C. Then the endowments will become more capital intensive from V^o to H.

[14]The straight line $H\overline{V}$ is tangent to the S^o curve at H and has the slope equal to W^T/R^T.

The supply prices remain the same as the production possibility curve shifts from T^H to $\overline{T}$. Therefore, the wage-rental ratio and the sectoral factor intensities remain constant. This is illustrated in Figure 1, as the allocation points C and $\overline{Z}$ are on the same ray from the origin, and the straight lines CH and $\overline{Z}\,\overline{V}$ (not shown) are parallel to each other. While the sectoral factor intensities remain constant, more factors are transferred to the capital-intensive sector as the allocation point moves from C to $\overline{Z}$, reflecting the change in outputs. This makes the endowments more capital intensive, as seen by the movement from H to $\overline{V}$. Thus the second step may be considered as an application of Rybczynski (1955)'s Theory.

[15]As the slope of an offer curve does not represent the commodity prices, the slope of the E^o curve does not represent the factor prices. The equilibrium factor prices at $\overline{V}$, for example, is shown by the slope of the straight line $H\overline{V}$

which is equal to the slope of the B^o curve at V^*, but is not equal to the slope of the E^o curve at $\bar{V}$.

[16]This elasticity cannot be viewed as referring to the E^o curve. It is because the straight line $H\bar{V}$ whose slope equals W^T/R^T is not tangent to the E^o curve at $\bar{V}$.

[17]At V' the ray OV^T intersects with the straight line from V^o with the slope W^T/R^T. It is uncertain (and irrelevant to our problem) whether the point V' lies inside the point $\bar{\bar{V}}$ (as shown in Figure 1) or outside it. Incidentally, the endowments at V' will produce the equilibrium outputs at Q' in Figure 2. But this correspondence will be more easily shown in Section IV.

[18]Suppose the rate of technological improvement is unequal between the sectors. Then at the point where the ray OV^T intersects with the B^o curve constructed under the state of technology in period 0, the slope of this curve is not equal to the wage-rental ratio in period τ. Two B-maps constructed under different states of technology intersect with each other. But this property of the B curves is entirely different from the property of the S or E curves which do intersect with each other in the static framework.

[19]One may conceive of an index for total factor productivity where both the product index and the factor index are given in geometric averages. Instead of (1.2)' or (1.4), one then has

$$\left(\frac{Y_1(\tau)}{Y_1(0)}\right)^{\delta(\tau)} \cdot \left(\frac{Y_2(\tau)}{Y_2(0)}\right)^{1-\delta(\tau)} \cdot \left(\frac{K(\tau)}{K(0)}\right)^{-\gamma(\tau)} \cdot \left(\frac{L(\tau)}{L(0)}\right)^{\gamma(\tau)-1} .$$

Under the assumption of the unitary price-elasticity of demand for commodities and the fixed tastes, the consumers' expenditure shares remain unchanged, that is, $\delta(0) = \delta(\tau)$. The utility function has the Cobb-Douglas form. By assuming also Cobb-Douglas production functions, the index of total factor productivity formulated as above can be shown by the ratio (OQ^T/OQ^*) / (OV^T/OV^*). Then this index has no aggregation bias under the restrictive assumptions $\sigma_1 = \sigma_2 = \sigma_D = 1$. An empirical objection may have to be raised to this index, however, because there hardly exist any national product data calculated in geometric averages.

[20]An isoquant was considered to be "corresponding to one set of outputs on the production possibility schedule" (Moorsteen, 1961, p. 461). It traces the variety of factor combinations equivalent to the actual inputs--"equivalent in the sense that those inputs would just suffice for the production of the same output." (Moorsteen et al., 1966, p. 4).

[21]See Moorsteen (1961, p. 463) Figure A. His production possibility curve P"P" passing through points E and G' is equivalent to our $\bar{\bar{T}}$ curve passing through Q^O and $\bar{\bar{Q}}$. His production possibility curve PP passing through points E and G is equivalent to our T^O curve passing through points Q^O and $\underline{\underline{Q}}$.

[22]At $\bar{\bar{V}}$ in Figure 1, the S^O curve intersects with another S-curve, $\bar{\bar{S}}$, and the latter is tangent to the $\bar{\bar{B}}$ curve (not shown) with the common slope $\dot{W}^T/R^T$. And this slope gives the equilibrium factor prices at $\bar{\bar{V}}$.

[23]Unlike the inequalities (4.2) and (4.3), the inequality between a μ index and the corresponding SAK index is unclear. Comparing (4.1) with (3.1), one finds $\kappa^T/\mu^T = (OV'/O\bar{\bar{V}})$ / $(O\bar{Q}/O\underline{\underline{Q}})$. Since the amounts of net outputs at $\bar{Q}$ and $\underline{\underline{Q}}$ in terms of equilibrium prices P_i^T's are equal to the total value-added at $\bar{V}$ and $\underline{\underline{V}}$ in terms of W^T and R^T, respectively, one can rewrite the ratio as $\kappa^T/\mu^T = (OV'/O\bar{\bar{V}})$ / $(O\bar{V}/O\underline{\underline{V}})$. In Figure 1, it is clear that

$O\bar{V}/O\underline{\underline{V}} > 1$. On the other hand, it is unclear whether OV^* is greater or smaller than $O\bar{\bar{V}}$. Therefore, one cannot exclude the possibility that $OV^*/O\bar{\bar{V}}$ is even greater than $O\bar{V}/O\underline{\underline{V}}$. Hence, though $\mu^T > \kappa^T$ is more likely the case, yet the reverse possibility cannot be ruled out. Likewise, the inequality $\mu^o < \kappa^o$ might not hold.

[24]Moorsteen (1961) goes so far as to say that since the μ index has the bias, either upward or downward, to the α index, the output ratio should be measured along the OQ^o ray if the input ratio is measured along the OV^T ray. On the other hand, if the output ratios refer to the OQ^T ray, then the input ratio should be measured along the OV^o ray. In this way, he argues, one obtains true indexes.

We do not follow his argument in this regard. It was fortunate that the monumental empirical work by Moorsteen et al. (1966) did not use this method. As an experiment in which this method was used for an international comparison of productive efficiency, see Berliner (1964).

[25]The duality between the quantity index and the price index in terms of the Divisia index was discussed in Jorgenson et al. (1967).

[26]In the indexes with fixed prices such as κ^o, the weights for the quantity expression and the price expression should refer to different years, 0 and τ. In the Divisia index where the weights vary from year to year, however, the weights for the quantity expression and the price expression refer to the same year.

[27]The b^o curve may be considered as a static demonstration of the factor price frontier discussed in the literature such as Samuelson (1962), Bruno (1969) and Burmeister et al. (1970).

[28]Houthakker (1960), for example, gives a specific, indirect utility

function in the static framework with π's as variables.

[29]We assume the indirect community indifference curves to be homothetic. The closer to the origin, the more preferable is a u-curve.

[30]See Appendix.

[31]From the assumptions of a homothetic utility function and homogeneous production functions, the equilibrium points on the ray $O\omega^T$ share the endowment ratio k^T which is equal to the slope of all b curves.

[32]This change in the price-income space is translated into the change in the quantity space of Section II in the following manner. The shift of the transformation curve t^o to t^H is equivalent to the rotation of the production possibility curve T^o to T^H around Q^o in Figure 2. The rotation of the transformation curve t^H to $\bar{t}$ around $\bar{z}$ is equivalent to the shift of the production possibility curve T^H to $\bar{T}$ with the compensated budget line as an envelope.

The change in factor prices from ω^o to $\bar{\omega}$ along the s^o curve is equivalent to the change in endowments from V^o to H along the S^o curve in Figure 1. The rotation of the s^o curve to $\bar{s}$ around $\bar{\omega}$ may also be shown by the shift of S^o- curve to $\bar{S}$ (not shown) with $H\bar{V}$ line as an envelope.

[33]The outward movement of $\bar{z}$ and $\bar{\omega}$ to find the point which yields the same utility as the initial observation has a limit. In Figure 4 draw a straight line $\underline{X}$ which passes through z^o with the slope Y_1^T/Y_2^T. Denote as $\underline{z}$ its intersection with the ray Oz^T. It is clear that the transformation curve $\underline{t}$ which is tangent to the $\underline{X}$ line at $\underline{z}$ forms a limit to the outward shift of the $\bar{t}$ curve.

We may trace the change from z^o to $\underline{z}$ in two steps. The first is the rotation of the transformation curve t^o to t^F around z^o. The second is a shift of the transformation curve from t^F to $\underline{t}$ with the straight line $\underline{X}$ as an envelope.

In Figure 3, corresponding to the rotation in the first step, the s^o curve rotates to $\underline{s}$ around ω^o. The factor endowment ratio k becomes higher, but the factor prices and the national income remain at ω^o. As the second step, the wage-rental ratio and the factor endowment ratio both rise along the $\underline{s}$ curve. Corresponding to the equilibrium at $\underline{z}$ in Figure 4, the equilibrium is attained at $\underline{\omega}$ in Figure 3 where the slope of $\underline{s}$ is equal to the observed k^T. As the parallel shift of the $\bar{t}$ curve cannot go beyond $\underline{t}$, the $\underline{s}$ curve forms the limit to the parallel shift of the $\bar{s}$ curve. The curve s* must be found somewhere between $\bar{s}$ and $\underline{s}$.

Incidentally, the point $\underline{z}$ is the equilibrium point to be reached when the initial income at z^o is undercompensated to offset the welfare effect of the price changes from p_i^o's to P_i^T's. In Figure 2, this point is represented by $\underline{Q}$ which lies on the initial budget line and is in equilibrium under P_i^T's. (Samuelson 1953).

[34]These changes can also be traced in the quantity space of Figure 2. The change in π's from z^o to z' along the t^o curve in Figure 4 is equivalent in Figure 2 to the movement of Q^o along the T^o curve toward point Q'' where the marginal rate of transformation of the T^o curve equal the price ratio in late year, P_1^T/P_2^T. The rotation of the t^o curve to t' around z' is equivalent in Figure 2 to the movement from point Q'' to Q' along a budget line with the slope P_1^T/P_2^T. It is uncertain (and irrelevant to our problem) whether the income at this point is smaller (as shown in Figure 2) or greater than the income at point $\bar{\bar{Q}}$.

The change in factor rewards described in the text may be translated into the change in factor endowments in Figure 1 in the following way. As the factor rewards move from ω^o to ω' along the straight line in Figure 3, the allocation point in Figure 1 moves from Z^o toward the origin along the efficiency locus OV^o. This process will lead the allocation point to J where the factor price ratio rises to W^T/R^T. Correspondingly, the S^o curve pivots anti-clockwise around V^o (not shown), so as to make its slope at V^o equal to W^T/R^T. Now at ω' in Figure 3, the endowment ratio k^o has to rise to k^T (the slope of the $\bar{\bar{b}}$ curve) in order to restore an equilibrium. The equivalent movement in Figure 1 is shown by the change in endowment point from V^o to V' along an iso-cost line. The allocation point should accordingly move from J to Z' along the OZ^T ray. (The straight lines V^oJ and $V'Z'$ should be parallel to each other and to the line V^TZ^T.) It is uncertain whether the straight line V^oV' lies inward to point $\bar{\bar{V}}$ (as is actually drawn in Figure 1) or outward to $\bar{\bar{V}}$.

[35]If the production functions are those of fixed coefficients, s-curves are shown by straight lines. A b-curve (such as b^o and $\bar{b}$ in Figure 3) is then the envelope to a group of straight lines. In this case the s^o curve and the k^o line in Figure 3 coincide. There will be no distinction between points $\bar{\omega}$ and ω' in Figure 3. As explained in Section II, the equality (2.4) instead of the inequality (2.3) will then be found.

[36]This is graphically shown by the ratio $O\omega^T/O\omega'$ along the ray $O\omega^T$.

[37]This is graphically shown by the ratio $Oz^T/O\bar{z}$ along the ray Oz^T.

[38]As we noted, point ω^* is equivalent to point V^* in Figure 1; point $\bar{z}$ is equivalent to point $\bar{Q}$ in Figure 2. Therefore, the ratio $(O\omega^T/O\omega^*)/(Oz^T/O\bar{z})$ equals the ratio $(OQ^T/O\bar{Q})/(OV^T/OV^*)$ in Figures 1 and 2. Then the indexes (5.12) and (1.4) are equivalent.

[39]Moorsteen et al. (1966) gives the input series both in terms of geometric weights as shown in the denominator of a hybrid index and in terms of arithmetic weights as shown in the denominator of a SAK index. The factor shares or the factor prices in 1937 are assumed as constant. It turns out that the input series increases more for arithmetic aggregation than for geometric aggregation in post WW II period (Table 9-1, Rows 1 and 5). This implies $\beta^{o} > \varkappa^{o}$.

Appendix

The shape of the transformation curve in Figure 4 shows that $\frac{d\pi_1}{d\pi_2} < 0$ and $\frac{d^2\pi_2}{d\pi_1^2} < 0$. This may be proved as follows:

Denoting two homogeneous production functions in the intensity form, $Y_1 = L_1\ f(k_1)$ and $Y_2 = L_2\ g(k_2)$, the equality of the value of marginal product to the factor price is shown as

(1) $R = P_1\ f_k = P_2\ g_k$ and

(2) $W = P_1 \cdot (f - k_1 f_k) = P_2 \cdot (g - k_2 g_k)$.

We define π_i as P_i/M where $M = RK + WL$. By substituting (1) and (2) into the above definition of π_i's, one obtains

(3) $\pi_1 = 1/(f_k K + (f - k_1 f_k)L)$ and

(4) $\pi_2 = 1/(g_k K + (g - k_2 g_k)L)$.

If K and L are fixed, π_i is the function of k_i. Accordingly, π_i is the function of ω, the wage-rental ratio. It is because k_i is the function of ω due to the relation $\omega = \frac{f}{f_k} - k_1 = \frac{g}{g_k} - k_2$.

Differentiate (3) and (4) with respect to ω under constant K and L. This leads to

(5) $\frac{d\pi_1}{d\pi} = \frac{\pi_1}{\omega}\ (\frac{k}{\omega + k} - \frac{k_1}{\omega + k_1})$ and

(6) $\frac{d\pi_2}{d\omega} = \frac{\pi_2}{\omega}\ (\frac{k}{\omega + k} - \frac{k_2}{\omega + k_2})$, where $k = K/L$. Hence,

(7) $\frac{d\pi_1}{d\pi_2} = -\frac{\pi_1}{\pi_2}\ \frac{\omega + k_2}{\omega + k_1}\ \frac{k_1 - k}{k - k_2}$.

Here, the denominator and the numerator of the last term, $k_1 - k$ and $k - k_2$, are of the same sign. All the other terms are positive. Therefore, one obtains

(8) $\frac{d\pi_1}{d\pi_2} < 0.$

Actually, from the assumption of the full utilization of factor endowments, one finds $(k_1 - k)/(k - k_2) = L_2/L_1$. Further, $(\omega + k_2)/(\omega + k_1) = \frac{RK_2 + WL_2}{L_2} \Big/ \frac{RK_1 + WL_1}{L_1}$ and $\pi_1/\pi_2 = P_1/P_2$. Therefore, (7) may be rewritten as

$$\frac{d\pi_1}{d\pi_2} = -\frac{RK_2 + WL_2}{P_2} \Big/ \frac{RK_1 + WL_1}{P_1} .$$

Turning now to the second derivative, we differentiate (7) logarithmically with respect to ω under constant k. This leads to

(9) $$\frac{d^2\pi_1}{d\omega\, d\pi_2} \Big/ \frac{d\pi_1}{d\pi_2} = (\omega + k)\left(\frac{dk_1/d\omega}{(k_1 - k)(\omega + k_1)} + \frac{dk_2/d\omega}{(k - k_2)(\omega + k_2)} \right)$$

Combining (9) with (6), one obtains

$$\frac{d^2\pi_1}{d\pi_2^2} = \frac{d\pi_1}{d\pi_2} \frac{(\omega + k)^2 (\omega + k_2)}{\pi_2 (k - k_2)} \left(\frac{dk_1/d\omega}{(k_1 - k)(\omega + k_1)} + \frac{dk_2/d\omega}{(k - k_2)(\omega + k_2)} \right).$$

Here, $\frac{d\pi_1}{d\pi_2}$ is negative from (8). Both $\frac{dk_1}{d\omega}$ and $\frac{dk_2}{d\omega}$ are positive. Further, $k_1 - k$ and $k - k_2$ have the common sign. Hence,

$$\frac{d^2\pi_1}{d\pi_2^2} < 0.$$

References

Amano, Akihiro (1964), "Determinants of Comparative Costs: A Theoretical Approach," Oxford Economic Papers (New Series), Vol. 16, No. 3, pp. 389-400.

Bergson, Abram (1969), "Comparative Productivity and Efficiency in the USSR," (mimeo., January 15, 1969 with revisions made in July, 1969 and May, 1970).

Berliner, Joseph S. (1964), "The Static Efficiency of the Soviet Economy," American Economic Review, Papers and Proceedings, Vol. LIV, No. 3, pp. 480-89.

Bruno, M. (1969), "Fundamental Duality Relations in the Pure Theory of Capital and Growth," The Review of Economic Studies, Vol. XXXVI (1) No. 105, pp. 39-53.

Burmeister, Edwin and Kiyoshi Kuga (1970), "The Factor-price Frontier in a Neoclassical Multi-sector Model," International Economic Review, Vol. 11, No. 1, pp. 162-74.

Domar, Evsey D. (1961), "On the Measurement of Technological Change," Economic Journal, Vol. LXXI, No. 284, pp. 710-729.

Graaff, J. De V. (1957), Theoretical Welfare Economics, (Cambridge, U.K.).

Houthakker, H. S. (1960), "The Influences of Prices and Incomes on Household Expenditure," Bulletin of the International Statistical Institute, Vol. 37, No. 2, pp. 9-22.

Jones, Ronald W. (1965), "The Structure of Simple General Equilibrium Models," Journal of Political Economy, Vol. LXXIII, No. 6, pp. 557-72.

Jorgenson, D. W. and Z. Griliches (1967), "The Explanation of Productivity Change," Review of Economic Studies, Vol. 34, pp. 249-83.

Moorsteen, Richard H. (1961), "On Measuring Productive Potential and Relative Efficiency," Quarterly Journal of Economics, Vol. 75, pp. 451-467.

Moorsteen, Richard H. and Raymond P. Powell (1966), The Soviet Capital Stock, 1928-1962, (Irwin, Homewood, Illinois, 1966).

Ohkawa, Kazushi (1967), "Effects of the Change in Factor Allocations to the Growth of Aggregate Productivity," Keizaikenkyu, Vol. 18, No. 4, pp. 346-349.

Rybczynski, T. M. (1955), "Factor Endowment and Relative Commodity Prices," Economica, Vol. 22, No. 84, pp. 336-341. Reprinted in R. E. Caves and H. G. Johnson (eds.), Readings in International Economics, (Irwin, Homewood, Illinois, 1968), pp. 72-77.

Samuelson, P. A. (1953), "Consumption Theorems in Terms of Overcompensation rather than Indifference Comparisons," Economica, N.S. Vol. 20, pp. 1-9.

________(1956), "Social Indifference Curves," The Quarterly Journal of Economics, Vol. LXX, No. 1, pp. 1-22.

________(1962), "Parable and Realism in Capital Theory: The Surrogate Production Function," The Review of Economic Studies, Vol. XXIX, No. 3, pp. 193-206.

Sims, Christopher A. (1969), "Theoretical Basis for a Double Deflated Index of Real Value Added," The Review of Economics and Statistics, Vol. LI, No. 4, pp. 470-71.

Vanek, Jaroslav (1968), Maximal Economic Growth, (Ithaca, New York).