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Math Is Not A Spectator Sport: Student-driven Problem Solving in the Classroom

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Summary

One of the most important overlooked components in mathematics education is ensuring that students understand the logic and proof process. This workshop addresses this gap in pedagogy by introducing university math lecturers to two strategies for teaching the experimental nature of math results. The first strategy (problem-solving methods) involves describing the reasoning behind mathematical proofs (i.e., a series of logical statements establishing a conclusion from given premises), and the second strategy (student-driven problem solving) puts undergraduate students in charge of solution development using guided discussion and collaboration. While the first method teaches the tools of proof and allows the instructor to share insight, set standards, and provide examples, the second method actively engages the students in discovery, investigation and experimentation, ultimately shifting the analysis, understanding, and evaluation into their hands. In this workshop, participants will learn about creating and guiding class discussion regarding mathematical proof and student-driven solving in which students work collectively to generate solutions and proofs. They will learn how to best present reasoning and experimentation behind the proofs in their classes and tutorials.

Keywords

undergraduate mathematics, teaching mathematical proofs, math-based discussions, student-driven learning

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One of the most important overlooked components in mathematics education is ensuring that students understand the logic and proof process. This workshop addresses this gap in pedagogy by introducing university math lecturers to two strategies for teaching the experimental nature of math results. The first strategy (problem-solving methods) involves describing the reasoning behind mathematical proofs (i.e., a series of logical statements establishing a conclusion from given premises), and the second strategy (student-driven problem solving) puts undergraduate students in charge of solution development using guided discussion and collaboration. While the first method teaches the tools of proof and allows the instructor to share insight, set standards, and provide examples, the second method actively engages the students in discovery, investigation and experimentation, ultimately shifting the analysis, understanding, and evaluation into their hands. In this workshop, participants will learn about creating and guiding class discussion regarding mathematical proof and student-driven solving in which students work collectively to generate solutions and proofs. They will learn how to best present reasoning and experimentation behind the proofs in their classes and tutorials.

KEYWORDS: undergraduate mathematics, teaching mathematical proofs, math-based discussions, student-driven learning

LEARNING OUTCOMES
By the end of the workshop, participants will be able to:
- analyse a proof and determine the methods of proof writing, and the associated modes of thought employed;
- identify areas in their classroom and subjects that could benefit from more explicit proof instructions and from student-driven problem solving; and
- coordinate student discussions for solving a math problem in class.

REFERENCE SUMMARIES
Brookfield, S. D. (2006). Preparing students for discussion. In *The Skillful Teacher: On Technique, Trust and Responsiveness in the Classroom* (pp. 115-131). San Francisco, CA: Jossey-Bass.

This is a key reference regarding classroom discussions written from the perspective of a teacher reconciling his in-class discussions with idealized scenarios. Math classes often do not involve discussion and consequently, there is little published material about running such activities. Brookfield offers a number of ways to get a discussion started among

students, several of which have potential in mathematics courses. As pre-workshop reading, participants can think about how the ideas broadly apply to their discipline, which will then stimulate discussion during the workshop itself.

The chapter examines how instructors should define participation in discussions, when to use discussions in the classroom, and how to go about starting and guiding these discussions. Brookfield notes that when the parameters of a discussion remain undefined, there is no rubric for achieving an end goal. In the case of mathematics, the goal would be a consensus on the approach to a proof or problem. Brookfield also articulates the many advantages to classroom discussions including, 1) engaging student in exploring diverse perspectives, 2) increasing intellectual agility and openness, and 3) connecting students to a topic. These three advantages show the most promise for adoption to a problem-based math class.

The paper will illustrate the advantages of classroom discussion, and provide a basis from which workshop participants can formulate math-centric plans for their own in-class discussions. The other references focus on the notion of proof, but this article ties in the "student-driven" part of the workshop's goals.

Selden, A. & Selden, J. (2009). Teaching proving by coordinating aspects of proofs with students' abilities. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and Learning Proof Across the Grades: A K-16 Perspective (Studies in Mathematical Thinking and Learning Series)* (pp. 339-354)*.* New York, NY: Routledge.

This paper deals with the structure of mathematical proof, analyzing proofs, and the ability of undergraduate students to construct math proofs. The paper begins by describing the methods of creating a proof and the component parts of a math proof (hierarchical structure, construction path, proof framework, formal-rhetorical, and problem-centered). The authors discuss the role of logic and understanding proof generation at the undergraduate level. They contrast some of the methods of proof writing, examine teaching aspects of proof writing, and explore how to approach teaching proof construction to undergraduate students.

This paper factors into the workshop in two ways. First, it provides insight into undergraduate understanding and reasoning with regards to logic and mathematical proof. Given that student-driven, learner-engaged problem solving is the ultimate goal in a mathematics class, this insight will be emphasized with workshop participants. The second contribution is more material, as identifying the parts of the mathematical proof discussed in this paper are one of the activities in the workshop.

Alcock, L. (2010). Mathematicians' perspectives on the teaching and learning of proof. In F. Hitt, D. Holton, & P. W. Thompson (Eds.), *Research in Collegiate Mathematics Education VII* (pp. 63-92). Washington, DC: American Mathematical Society and Conference Board of Mathematical Sciences.

Alcock's paper is a collection and analysis of different perspectives on the teaching and learning of proof writing. The four modes of thinking in creating a mathematical proof described by the author are the chief ideas used in this workshop. Workshop participants will be asked to focus on Table 1 (p. 78) from the paper as a reference guide to different modes of thinking (i.e., instantiation, structural thinking, creative thinking, and critical thinking). Instantiation involves understanding the problem or statement and the mathematical objects and relations involved. Structural thinking involves the deductive process, incorporating previous results and formal statements. Creative thinking consists of identifying the objects and their manipulations that form the crux of the proof. Lastly, critical thinking involves the checking of the proof by confirming logic, ensuring assertions of theorems are met and by looking for counter examples. Alcock also highlights teaching strategies for each mode of thought using interviews with math teachers.

Fukawa-Connelly, T. P. (2012). A case study of one instructor's lecture-based teaching of proof in abstract algebra: making sense of her pedagogical moves. *Educational Studies in Mathematics*, 81(3), 325-345.

This paper discusses proof writing in class and begins by pointing out deficiencies in the traditional lecture. The authors note that the spectrum of teaching in mathematics ranges from 'pure telling' to 'pure investigation' and claims that the ideal approach is somewhere in the middle. The authors rely on Seldon and Seldon (2009) and Alcock (2010) to discuss the structure of proof writing and modes of thought that accompany writing proofs. Workshop participants will be asked to refer to Table 1 (p. 330) as a guide to aspects of proof writing and modes of thought that support proof writing.

The authors also provide a sample exchange between a teacher and which inspired the original linear algebra and calculus scenarios created for this workshop (see Appendices A and B). Future facilitators could create additional examples to suit other branches of mathematics, different levels of courses, or even different disciplines.

CONTENT AND ORGANIZATION
Prior to the workshop, ask participants to read Chapter 7 in Brookfield (2006) on preparing students for discussion, and the articles by Alcock (2010) and Fukawa-Connelly (2012).

Duration (min)	Subject	Activity	Purpose
10	Introduction	As part of the introduction, provide a brief background and motivation for the workshop and the selection of the pre-workshop readings. Articulate the focus on student-driven problem solving. After the facilitator/workshop introduction, ask each participant to introduce themselves, their disciplinary background, and say something about why they are taking the workshop. Alternatively, use one of a number of small icebreaker games (many can be found online) to facilitate this introductory activity.	Gauge participant interest in participating, and articulate the goals and purpose of the workshop to the attendees. Engage participants, and create a relaxed and comfortable atmosphere.
45	Activity 1: Pre-Workshop Reading and Scenario Discussion	Break participants into two groups. Each group receives one of two scenarios: (1) a first year linear algebra class (Appendix A) or (2) a first year calculus class (Appendix B). The scenarios are dialogs between a professor and students going through a proof in class. Have groups work together to identify which aspects of proof writing and associated modes of thought are evident in their example scenario. Refer them to Tables 1 from Fukawa-Connelly (2012, p.	Encourage participants to work cooperatively with goal-oriented case study. Identify the types of proof writing and associated modes of thought that occur in the subject-specific class scenarios. Comment on the provided scenarios and apply principles of good teaching from the

		330) and Alcock (2010, p. 78) as reference guides. Ask groups to consider what they learned from Brookfield's (2006) chapter on preparing students for class discussion and discuss how they might approach teaching the proof in this scenario differently. See Appendix C for examples of guiding questions. Ask each group in turn to summarize their scenario, describe what aspects of proof and associated modes of thought were evident, and share any ideas that resulted from the ensuing discussion.	Brookfield chapter to the math-specific scenarios. Provide different perspectives on the classroom scenarios, allowing participants to broaden their understanding of how the material may apply to their own teaching.
25	Activity 2: Comparative Proof Mini-lessons	This activity involves two mini-lesson presentations by the workshop instructor (~5 minutes each). The first mini-lesson should follow a non-interactive lecture style or 'pure-telling' approach. The second mini-lesson should incorporate explicit proof instructions and student-driven problem solving. See Presentation Strategies for further guidance. Following the two mini-lessons, allow participants to reflect independently on the learning experience and compare it to previous learning experiences and preferred teaching practices (~5 minutes). See Appendix C for example reflection questions.	Characterize through demonstration and role-playing the two different teaching styles. Demonstrate the advantage of explicit proof presentation. Participants should be able to clearly follow the proof of the result and understand its derivation. Compare the two different teaching styles to their own approaches in class and reflect on ways to adapt or modify personal teaching practices.

		With the remaining time, give participants an opportunity to share their preliminary thoughts and ideas about the perceived advantages and disadvantages of the two teaching approaches.	Openly explore the benefits and challenges of teaching a math proof using student-driven problem solving versus a non-interactive approach.
30	Activity 3: Facilitating Student Discussions	The final part of the workshop focuses on using guided discussions for problem solving in math classes. To start, ask the group and write down any concerns (possible misconceptions) that participants have about using discussions in math classrooms. Try to address these concerns through the ensuing discussion. Introduce each of the following questions one at a time. Ask participants to think independently about the question, and share their thoughts with a partner. Then facilitate a group conversation by asking participants to contribute their perspectives and ideas. (1) What other topics or scenarios in math classes could benefit from discussion? Note: The facilitator should have several additional examples prepared to share if the participants are unable to generate ideas. (2) Why would a classroom discussion of a new mathematical technique or theorem be a valuable	Model the benefits of discussion by engaging participants in a discussion activity. Share the advantages of discussion in a math class including the beneficial impacts of student engagement and interaction. Generate strategies for incorporating discussion into math classes.

		use of class time, specifically to familiarize students with the concept? Why would some instructors suggest that discussion activities are a drain on in-class time? (3) When and how could we best incorporate discussions into our classes in order to maximize student learning? (4) What are the benefits/challenges of having an instructor provide the end result of a problem and allow class time for students to find the solution. Would adding a group participation grade make a difference in this context? See additional questions in Appendix C. Take notes on key participant contributions. E-mail the results of the discussion to the group following the workshop.	
10	Questions and Closing	To close the workshop, summarize the various discussions and activities, and answer any final questions concerning the day's content.	Provide a recap for the workshop and its key takeaway messages.
Total Time: 120 minutes			

PRESENTATION STRATEGIES

A standard math class setup is required for the proof mini-lesson activity (i.e., a board and writing implements such as chalk or dry-erase markers). This minimalist setup allows the workshop to take place within or outside a math department.

For the proof mini-lesson activity, the facilitator should work through two comparable problems in order to illustrate the differences between a non-interactive lecture style ("pure-tell method) and the explicit proof/student-driven teaching style. Due the varied and specialized nature of the prospective participants, facilitators should consider choosing a generic problem-solving question. Such questions may be found in the Galois and Hypatia Math Competitions run by the Centre for Education in Mathematics and Computing (CEMC). See examples at the following link: http://cemc.uwaterloo.ca/contests/past_contests.html. Relying on generic questions will help to ensure that message of the workshop is not lost due to subject unfamiliarity or differences in discipline.

Suggestions for Extending the Workshop

Creating additional proof scenarios could introduce more variety in the types of proof writing and the associated modes of thought that occur in subject-specific contexts. A greater number of scenarios will allow for more discussion and help participants to apply principles of good teaching from the Brookfield chapter to the math-specific scenarios.

Classroom activities that foster mathematical understanding and proof writing could be incorporated as a follow-up or secondary session. For example, see http://www.uwo.ca/tsc/graduate_student_programs/pdf/Great_Ideas_2014-AO.pdf.

One could also introduce "micro problem-solving sessions". Participants would prepare and present short (7-10 minute) lessons facilitating the discussion of a problem or detailing their thoughts and experimentation as they solved a particular problem. They could then receive feedback from their peers and the workshop facilitator. In this case, a two-day workshop format would allow for second attempts at teaching, giving participants the chance to try out new or adapted approaches.

Ultimately, the core of this workshop could transfer to other natural science disciplines. The references and scenarios would need to reflect the targeted discipline, but the overall premise of teaching students how to approach problems and work to solve them would remain. For more experimental disciplines (e.g., chemistry) the students might learn the intuition behind the scientific method and how to set up labs and experiments to test hypotheses.

APPENDIX A: Handout #1 - Sample Exchange in a Linear Algebra Class

TOPIC: Subspaces

Teacher: So, the statement I have put up on the board, "A plane through the origin is a subspace of $\mathbb{R}^3$." What do we feel? True or false?

Students: (Murmur to themselves)

Teacher: I think it's true, but how would we show it? What object that we just learned about is the statement talking about?

Student: S-subspaces?

Teacher: Correct! We're talking about subspaces. Specifically, we're saying that one special kind of plane is a subspace. How would we go about proving this? Hmm? ... Well, do we know of any theorems or propositions dealing with subspaces yet?

Student: No, we don't.

Teacher: Right, so it's a good idea in this case to go back to our definitions. Now, there are three properties that make something a subspace, can anyone name one?

Student: Has to have a zero?

Teacher: That's good, the set must contain the zero vector, or origin. Another?

Student: There's the one about addition.

Teacher: Very good, that's the second of our three properties. The set must be closed under vector addition. And the last property? ... The set must also be closed under scalar multiplication.

Teacher: Now, there are the three properties we need to prove to show this result. We want to show planes through the origin are subspaces. So we have to check each one of these properties. If just one fails, we'll know it's false. Someone pick a property and we'll try to verify it.

Student: The first one?

Teacher: Sure, let's show that the plane contains the zero vector. Anyone have any thoughts?

Student: It's in the definition?

Teacher: Yes indeed. Planes containing the origin do in fact contain the origin, which is the zero vector. That wasn't too scary. Let's try another property?

Student: Closed under addition?

Teacher: Ok, so what does closed under addition mean?

Student: If you take two vectors in the plane, and add them, you get another vector in the plane?

Teacher: Right! So we want to take two vectors in the plane. We'd start by writing down, 'let u and v be arbitrary vectors in our plane'. Now, we need to add them, but we don't really know what these vectors look like. We know they have three components, u = $[u_1; u_2; u_3]$ and v = $[v_1; v_2; v_3]$ cause we're in $\mathbb{R}^3$, so that might be something to write out. What's the one other thing we know about u and v.

Student: They are in the plane?

Teacher: Exactly. Now planes in $\mathbb{R}^3$ we learned a couple ways to write them. This was a few classes ago. Can anyone remember a form or equation of a plane?

Student: The general form.

Teacher: Bingo! And that looks like?

Same Student: ax + by + cz = d.

Teacher: Right, so we might say, 'let ax + by + cz = d be the general form of our plane that passes through the origin'. Then u and v must satisfy this. So we'd write $au_1 + bu_2 + cu_3 = d$ and $av_1 + bv_2 + cv_3 = d$.

Teacher: Do we know anything else about our general form?

Student: The normal is [a; b; c].

Teacher: Yes, that's true. But I was thinking more along the lines of, do we know any other vectors that satisfy the equation? ... How about [0; 0; 0]? The plane passes through the origin, so the origin should satisfy the equation. So then $a_0 + b_0 + c_0 = d$. What does that tell us?

Student: d = 0?

Teacher: Yup! So now we can rewrite our two previous equations. We'll write them one on top of the other.

$$au_1 + bu_2 + cu_3 = 0$$
$$av_1 + bv_2 + cv_3 = 0$$

Student: Then we add them?

Teacher: Right again! And we get,

$$a(u_1 + v_1) + b(u_2 + v_2) + c(u_3 + v_3) = 0$$

Which we recognize as the components of u+v. This shows that...?

Student: u + v is also on the plane.

Teacher: So now we've done two properties, the last is to show the plane is closed under scalar multiplication. What does that mean we need to do?

Student: Take an arbitrary vector and multiply it by a scalar? Show that that's still in the plane?

Teacher: So we'll write down, 'let w be an arbitrary vector in the plane and let k be an arbitrary scalar'. Now, any suggestions for how we proceed?

Student: Same way as last time?

Teacher: Ok, so we know from our previous work that $aw_1 + bw_2 + cw_3 = 0$. What do we want to show?

Student: That $a(kw_1) + b(kw_2) + c(kw_3) = 0$

Teacher: And how do we get from here to there? We have to introduce k somehow. ... Let's try multiplying the first equation by k. Then we get

$$k(aw_1 + bw_2 + cw_3) = k(0)$$

Which simplifies to

$$k(aw_1 + bw_2 + cw_3) = 0$$

What do we do next?

Student: Expand the left hand side?

Teacher: Right which, when we rearrange, gives us,
$akw_1 + bkw_2 + ckw_3) = 0$ So are we done? What have we shown?

Student: That planes through the origin contain zero, are closed under addition and

are closed under multiplication by arbitrary scalars.

Teacher: And those are the three properties that show...?

Student: Planes through the origin are subspaces of $\mathbb{R}^3$

APPENDIX B: Handout #2 - Sample Exchange in a Calculus Class

TOPIC: Optimization

Teacher: So, on last year's midterm we were asked to do the following question: "You are given a 36 m length of fencing, and are tasked with building a rectangular enclosure against a stone wall. Determine the dimensions of the rectangle which encloses the largest area." Now, what kind of problem is this?

Student: Optimization?

Teacher: Right, and how did you know that?

Same Student: It's asking us to maximize or minimize something under constraints that it has given us.

Teacher: Couldn't have worded it better myself! So when we're doing an optimization question we have about six steps to follow. The first is to write down an expression for the thing we're trying to optimize. So let's draw a picture here.... We'll let our dimensions be x and y.
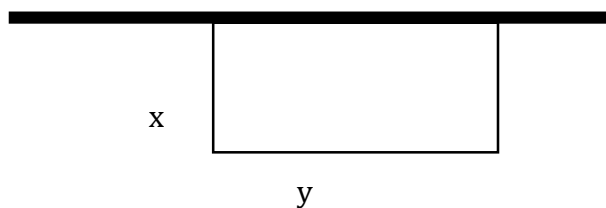
So a formula for the area is just?

Student: xy

Teacher: Yes, xy. Not to tricky yet. Now, the next thing to do is to identify our constraints. Anyone?

Student: The 36 m of fence.

Teacher: Right, and writing that mathematically, in terms of our variables, we have x+y+x = 2x+y = 36. With that done, we now use these constraints to simplify our formula. That is, we need to solve for one of x or y from constraint equation. So let's pick one, x or y?

Student: x?

Teacher: That sounds just fine, let's go with x. It's always good to go with your instincts and we're going to get to the same place either way. So if we rearrange for x we get?

Student: x = 18 - y/2 ?

Teacher: Yup.

$$2x = 36 - y \qquad x = (36 - y)/2 = 18 - y/2$$

Now that we have x, what should we do?

Student: Plug it into our other formula?

Teacher: Right! That's the next step, and we simplify.

$$xy = (18 - y/2)y = 18y - y^2/2$$

And this is the equation we wish to maximize.

Teacher: Now, how do we go about determining where this function is maximized?

Student: Set the derivative equal to 0.

Teacher: So let's take the derivative,

$$(18y - y^2/2)' = 18 - 2y/2$$

So we want to solve, $18 - y = 0$. And that is?

Student: $y = 18$.

Teacher: Right. This is a critical point. It doesn't mean it's a maximum, but it's a good candidate. If we had other critical points we'd need to find them too and work with them, but we only have the one.

Student: So isn't that it? 18 is the answer?

Teacher: Not yet, we'd still need to determine x, but before we get there we have other contenders, other points of interest. Does anyone know what I'm talking about? ... The endpoints! We have a constraint, $2x + y = 36$. Can y have negative length?

Students: No.

Teacher: Can y be 37m?

Students: No.

Teacher: Right, so there are bounds, $y = 0$ and $y = 36$.
We must check these endpoints, because they might be the maximum. I know $y = 18$ is critical, but for all we know it might be a minimum. So how do we check?

Student: Plug the points into our formula?

Teacher: Very good. So we figure out our function at $y = 0$, $y = 18$ and $y = 36$. We get, $0m^2$, $162m^2$, and $0m^2$ respectively. So it turns out $y = 18$ is the correct maximum, but we needed to show that. Maybe on this year's exam I'll make the endpoints the maximum. Hmmm?

Student: So now are we done?

Teacher: Almost, we are asked for the dimensions. We have y now we just need to find x. But we already have a formula for x in terms of y. So we plug in $y = 18$ and get,

$$x = 18 - y/2 = 18 - 9 = 9m$$

The last thing we should before we wrap up is to check and make sure of two things, that our x and y give us the same area we calculated with y alone, this checks our arithmetic a bit, and to make sure that x and y satisfy our constraint. So we check

$$xy = (9m)(18m) = 162m^2$$

and

$$36m = 2x + y = 2(9m) + 18m = 18m + 18m = 36m$$

Everything looks good, so we end with a statement saying that the maximum area is enclosed by dimensions $x = 9m$ and $y = 18$.

APPENDIX C: Example Reflection and Discussion Question Prompts for Workshop Activities

Activity 1: Pre-Reading and Scenario Discussion

- Which aspects of proof writing can you identify in these examples?
- Reflect on your own teaching and learning experiences. Do your lectures (or your instructor's lectures) articulate these aspects and emphasize their importance?
- Based on the readings, in what ways does teaching proofs enable/allow for active learning and student engagement?
- How would you improve the clarity of the examples and/or incorporate more of the methods of proofs?

Activity 2: Comparative Proof Mini-lessons

- What benefits and challenges are associated with each of the two mini-lessons?
- Putting yourself in the mindset of a student, which mini-lesson provides a better understanding of the material and why?
- Which mini-lesson style is more likely to incite student interest and engagement and why?
- How would you modify either mini-lesson technique to better support student learning?

Activity 3: Facilitating Student Discussions

- How can we reconcile the more exact nature of logic and math with the openness of a free discussion?
- How best can you apply best practices from Brookfield's (2006) article to add student-driven problem solving to your classroom?
- Should discussion-based, student-driven problem-solving always be the goal, and if not, when is it disadvantageous for students?