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THE DISTRIBUTION OF THE STEIN-RULE ESTIMATOR IN A MODEL WITH NON-NORMAL DISTURBANCES

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The Distribution of the Stein-Rule Estimator in a Model with Non-Normal Disturbances

John L. Knight[1]

1. Introduction

In recent years the Stein-rule estimator has attracted a great deal of attention from econometricians. Ullah (1974) derived the exact moments of the estimator while Srivastava and Upadhyaya (1977) examined its various properties via the small σ asymptotic approach. This approach was also used by Ullah, Srivastava and Chandra (1983) to examine its properties under non-normal disturbances. The approximate distribution was derived by Ullah (1982) while most recently, Phillips (1984) has derived the exact distribution and given an alternative derivation of the moment formulae of Ullah (1974).

It is the purpose of this paper to extend the approach of Phillips (1984) to examine the distribution and moments of the estimator under the assumption the disturbances follow a non-normal distribution of the Edgeworth or Gram-Charlier type. We use the technique developed by Davis (1976) and used by the author in other contexts to examine the effects of non-normal disturbances (see, e.g. Knight (1983a, 1983b, 1984a, 1984b)).

2. The Model and Notation

Consider the linear regression model

$$y = X\beta + u \tag{1}$$

where y is a vector of T observations on a dependent variable, X is a $T \times m$

[1]This paper was written while the author was visiting the Department of Economics, University of Western Ontario. Thanks are due to the department for both financial and secretarial assistance.

observation matrix of full rank $m < T$ of non-random independent variables, and u is a vector of disturbances where each u_i $(i=1,\dots,T)$ are iid with some unknown non-normal distribution with mean 0 and variance σ^2. Following Phillips (1984) we assume that $T^{-1}X'X = I$ and thus the OLS estimator of β viz b is given by $b = T^{-1}X'y$ and the Stein-rule estimator given by

$$r = [1 - \frac{a}{T}(\frac{s}{b'b})]b$$

where $s = y'My$ with $M = I - X(X'X)^{-1}X'$ and a is a scalar constant. We further assume that $0 < a < 2(m-2)/(T-m+2)$ and $m \geq 3$.

Phillips using the above model with added assumption of normality for the u_i's derived, via the use of fractional calculus, the exact pdf of r. If we now allow the non-normal distribution of the u_i's to be well approximated by an Edgeworth or Gram Charlier distribution we can, by use of the technique of Davis (1976) in conjunction with the approach of Phillips (1984), readily derive the pdf and the moments of r.

3. The Density of the Stein-rule Estimator

In seeking to approximate the distribution of r under the assumption the u_i's are independently distributed with non-normal distributions we may apply the method of Davis (1976) as follows.

Step 1. Obtain the distribution $pdf(r|\eta)$ of r under the model

$$y = X\beta + \eta + u \tag{2}$$

where η is an arbitrary vector and the elements of u are normal with zero mean and variance σ^2.

Step 2. Compute the required distribution $pdf(r) = \underset{\eta}{E}(pdf(r|\eta))$ where the "expectation" is to be calculated as if the η_i's were independent random vectors with zero mean and variance and the same third and higher order cumulants as those of u_i's.

In carrying out Step 1 we can utilize the results and approach of Phillips (1984) quite extensively. We first note that under (2) and the assumption of normality we have that $b \sim N(\beta + \frac{1}{T} X'\eta, (\sigma^2/T)\cdot I)$, $s/\sigma^2 \sim \chi'^2(T-m),\lambda)$ where $\lambda = \frac{1}{2\sigma^2}\eta' M\eta$ and b and s are independent.

From Phillips (1984, equation (3)) we have the characteristic function (cf) of r given by

$$cf(t) = E(e^{itr})$$

$$= \int \exp(it'b - i(as/Tb'b)t'b)\,pdf(b)\,pdf(s)\,dbds$$

Now noting that

$$pdf(s) = pdf(\chi'^2\,(T-m,\lambda))\cdot\sigma^2$$

which may be written as a linear combination of central χ^2 we have

$$pdf(s) = \sum_{j=o}^{\infty} \frac{\lambda^j e^{-\lambda}}{j!}\cdot pdf(\chi^2(T-m+2j))\cdot\sigma^2 \tag{3}$$

Therefore we may readily apply the results of Phillips (1984, equations (4) through (11)) replacing (T-m) with (T-m+2j), β with $\beta^* = \beta + \frac{1}{T}X'\eta$ and (3) in place of pdf(s).

Thus equation (10) in Phillips (1984) now becomes:

$$cf(s) = \sum_{j=o}^{\infty} e^{-\lambda}\frac{\lambda^j}{j!}\exp[is(\beta' + \frac{1}{T}\eta' X)h - \sigma^2 s^2 h'h/2T]\{(1+2is\zeta_x)^{-(T-m+2j)/2} \cdot \exp[x'((\beta + \frac{1}{T}X'\eta) + is\sigma^2 h/T) + \sigma^2 x'x/2T]\}_{x=o} \tag{4}$$

where $\zeta_x = a\sigma^2 h'\partial x/T\Delta_x$ and we can readily find[2]

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[2]Note the pdf$(y|\eta)$ can also be considered the pdf of y under mis-specification of the form of excluding relevant explanatory variables. In this case if the true model is (1) and the estimated model $y = X_1\beta_1 + u_1$ where $u_1 = X_2\beta_2 + u$ then $\lambda = \frac{1}{2\sigma^2}\beta_2'\beta_2$.

$$\text{pdf}(y = h'r \mid \eta) = \left(\frac{T}{2\pi\sigma^2 h'h}\right)^{1/2} \sum_{j=o}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} \cdot$$

$$\cdot \sum_{k=o}^{\infty} \frac{((T-m+2j)/2)_k}{k!} \quad [(-2\zeta_x)^k \cdot$$

$$[(\partial z)^k \exp\{-T(y-\beta^{*\prime}h-\sigma^2 x'h/T-z)^2/2\sigma^2 h'h\}]_{z=o}$$

$$\cdot \exp\{x'\beta^* + \sigma^2 x'x/2T]]_{x=o}$$

where

$$\beta^* = \beta + \frac{1}{T} X'\eta = \beta + \Psi$$

Thus

$$\text{pdf}(y = h'r \mid \eta) = \left(\frac{T}{2\pi\sigma^2 h'h}\right)^{1/2} \sum_{j=o}^{\infty} \sum_{k=o}^{\infty} \frac{((T-m+2j)/2)_k}{j!k!} \quad [(-2\zeta_x)^k \cdot$$

$$\cdot \ [(\partial z)^k \exp\{-T(y-\beta'h-\sigma^2 x'h/T-z-\Psi'h)^2/2\sigma^2 h'h\}]_{z=o} \cdot \tag{5}$$

$$\cdot \ \lambda^j e^{-\lambda} \exp[x'\beta + \sigma^2 x'x/2T + x'\Psi]]_{x=o}$$

Now (5) may be rewritten as

$$\text{pdf}(y \mid \eta) = \left(\frac{T}{2\pi\sigma^2 h'h}\right)^{1/2} \sum_{k=o}^{\infty} \sum_{j=o}^{\infty} \frac{\left(\frac{T-m+2k}{2}\right)_j \left(\frac{T-m}{2}\right)_k}{j!k!\left(\frac{T-m}{2}\right)_j} \lambda^j e^{-\lambda} \cdot$$

$$[(-2\zeta_x)^k (\partial z)^k \exp[\frac{-T}{2\sigma^2 h'h} \{A^2 - 2\Psi'hA + (\Psi'h)^2\}] \cdot \tag{6}$$

$$\cdot \ \exp(x'\beta + \sigma^2 x'x/2T + x'\Psi)\,]_{\substack{z=o \\ x=o}}$$

where

$$A = (y-\beta'h-\sigma^2 x'h/T-z)\,] \ .$$

Noting that

$$\sum_{j=o}^{\infty} \frac{\left(\frac{T-m}{2}+k\right)_j}{\left(\frac{T-m}{2}\right)_j j!} \lambda^j = {}_1F_1\left(\frac{T-m}{2}+k, \ \frac{T-m}{2}, \ \lambda\right)$$

and also that

$$e^{-\lambda}{}_1F_1(\frac{T-m}{2}+k,\ \frac{T-m}{2},\ \lambda) = {}_1F_1(-k,\ \frac{T-m}{2},\ -\lambda)$$

we have

$$\text{pdf}(y|\eta) = (\frac{T}{2\pi\sigma^2 h'h})^{1/2} \sum_{k=o}^{\infty} \{\frac{((T-m)/2)_k}{k!} [(-2\zeta_x)^k [(\partial z)^k \cdot \exp[\frac{-T}{2\sigma^2 h'h}\{A^2 - 2\Psi' hA + (\Psi' h)^2\}]]_{z=o} \cdot \exp(x'\beta + \sigma^2 x'x/2T + x'\Psi)]_{x=o} \cdot {}_1F_1(-k,\ \frac{T-m}{2},\ -\lambda) \quad (7)$$

Equation (7) completes Step 1. In order to perform Step 2 it is necessary to take expectations with respect to η of (7). We note that η is involved in Ψ and also in λ. Although the associated exponentials can be expanded and term by term expectations derived the process seems very complicated. To overcome these complications and to facilitate the taking of expectations we can further apply the differential operator to isolate the terms in Ψ. This is easily achieved by noting

$$\exp\{\frac{-T}{2\sigma^2 h'h}\{-2A\Psi' h + (\Psi' h)^2\} + x'\Psi\}$$

$$= \{\exp[\frac{-T}{2\sigma^2 h'h}(-2Ah'\partial q + (h'\partial q)^2) + x'\partial q] \cdot e^{q'\Psi}\}_{q=o}$$

Therefore

$$\text{pdf}(y|\eta) = (\frac{T}{2\pi\sigma^2 h'h})^{1/2} \sum_{k=o}^{\infty} \frac{((T-m)/2)_k}{k!} [(-2\zeta_x)^k [(\partial z)^k \cdot [\exp\{\frac{-T}{2\sigma^2 h'h}[-2Ah'\partial q + (h'\partial q)^2] + x'\partial q\} e^{q'\Psi}]]]_{\substack{q=o\\x=o\\z=o}} \cdot {}_1F_1(-k,\ (T-m)/2,\ -\lambda) \quad (8)$$

Thus to find the unconditional density and hence Step 2 we now only require to consider the expectations with respect to η of the terms in the expansion of

$$e^{q'\Psi} \cdot {}_1F_1(-k, (T-m)/2, -\lambda)$$

i.e., terms of the form

$$\underset{\eta}{E}((q'\Psi)^j \lambda^{\ell}) = \underset{\eta}{E}((\frac{1}{T} q'X'\eta)^j \cdot (\eta' M \eta / 2\sigma^2)^{\ell})$$

For a Gram-Charlier expansion and correction terms for skewness and kurtosis we require $2\ell + j \le 4$.

From Appendix A we have the required expectations and substitution into (8) yields the unconditional pdf of y given by

$$\begin{aligned} \text{pdf}(y) = (\frac{T}{2\pi\sigma^2 h'h})^{1/2} \sum_{k=o}^{\infty} \{ \frac{((T-m)/2)_k}{k!} [(-2\zeta_x)^k [(\partial z)^k \cdot \\ \cdot \exp[-T(y-\beta'h-\sigma^2 x'h/T-z)^2/2\sigma^2 h'h] \cdot \\ \cdot [\exp\{[T(2(y-\beta'h-\sigma^2 x'h/T-z)h'\partial q - (h'\partial q)^2)/2\sigma^2 h'h] + x'\partial q\} \cdot \\ \cdot \{1 + K_3[\frac{1}{3!} \Sigma p_j^3 - ((-k)_1 / \frac{(T-m)}{2}) \Sigma p_j M_{jj}] + \\ + K_4[\frac{1}{4!} \Sigma p_j^4 + ((-k)_2 / 2(\frac{T-m}{2})_2) \Sigma M_{jj}^2 - ((-k)_1 / 2(\frac{T-m}{2})_1) \Sigma p_j^2 M_{jj}]\} \\ \cdot \exp(x'\beta + \sigma^2 x'x/2T)]]]_{\substack{x=o \\ z=o \\ q=o}} \end{aligned} \qquad (9)$$

where p_j is the j^{th} element in the vector $\frac{1}{T} Xq$ and M_{jj} is the j^{th} diagonal element in the matrix M. In order to simplify the above expression it is necessary to evaluate the derivatives with respect to q at the point $q = 0$. That is we require

$$\begin{aligned} \exp[B_1 h'\partial q - B_2(h'\partial q)^2 + x'\partial q]\{1 + C_1 \Sigma p_j^3 + C_2 \Sigma p_j M_{jj} + \\ + C_3 \Sigma p_j^4 + C_4 \Sigma M_{jj}^2 + C_5 \Sigma p_j^2 M_{jj}\}]_{q=o} \end{aligned} \qquad (10)$$

where

$$C_1 = K_3/3!; \; C_2 = 2kK_3/(T-m)$$

$$C_3 = K_4/4!; \; C_4 = K_4(-k)_2/2!(\frac{T-m}{2})_2$$

$$C_5 = K_4k/(T-m); \; B_1 = 2TA/2\sigma^2h'h; \; B_2 = T/2\sigma^2h'h$$

From Appendix B we have that (10) reduces to

$$1 + \frac{K_3}{3!}S_1 + \frac{K_4}{4!}S_2$$

where S_1 and S_2 are given in (B.6), (B.7) of Appendix B.

Therefore under non-normality the pdf of y is given by

$$\text{pdf}(y) = \left(\frac{T}{2\pi\sigma^2h'h}\right)^{1/2} \sum_{k=o}^{\infty} \{\frac{((T-m)/2)_k}{k!} [(-2\zeta_x)^k[(\partial z)^k \cdot \qquad (11)$$

$$\cdot \exp[-T(y-\beta'h-\sigma^2x'h/T-z)^2/2\sigma^2h'h] \cdot \{1 + $$

$$+ K_3S_1/3! + K_4S_2/4! \}\exp(x'\beta + \sigma^2x'x/2T)]]_{\substack{x=o\\z=o}}$$

Clearly, when $K_3 = K_4 = 0$, i.e., errors are normal, equation (11) reduces to that found by Phillips (1984) equation (12).

4. Moments Under Non-Normality

Exact moment formulae may be found in a number of ways. We first need to find $E(y^p|\eta)$. This may be done as in Phillips (1984), by directly integrating the $\text{pdf}(y|\eta)$ or alternatively differentiating the characteristic function (4). A third approach is to use the technique of Ullah (1974) and specialize it to our case of non-normality.

Using the cf (4) we have

$$E(y^p|\eta) = (-i)^p \left.\frac{\partial^p cf(s)}{\partial s^p}\right]_{s=o}$$

For the mean it is readily seen that the appropriate differentiation and evaluation at s=o and x=o gives

$$E(y|\mathcal{T}) = \beta^{*\prime}h - \sum_{j=o}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!}(T-m+2j)\,[\zeta_x \exp(x'\beta^* + \sigma^2 x'x/2T)\,]_{x=o}$$

Now noting that

$$e^{-\lambda}\sum \frac{\lambda^j}{j!}(T-m+2j) = (T-m) + 2\lambda$$

and using results in Phillips (1984, equations (15) to (21)) we have[3]

$$E(y|\mathcal{T}) = \beta^{*\prime}h - \frac{1}{2}(T-m+2\lambda)ah'\beta^* e^{-\theta^*}\frac{\Gamma(m/2)}{\Gamma(\frac{m}{2}+1)}\;{}_1F_1(\tfrac{m}{2}, \tfrac{m}{2}+1,\ \theta^*) \qquad (12)$$

where

$$\theta^* = T\beta^{*\prime}\beta^*/2\sigma^2$$

Considering the second moment, i.e., $E(y^2|\mathcal{T})$ we have from differentiating cf(s) in (4)[4,5]

$$\begin{aligned} E(y^2|\mathcal{T}) &= (\beta^{*\prime}h)^2 + \sigma^2 h'h/T \\ &\quad -2(T-m+2\lambda)\,[\zeta_x(\beta^{*\prime}h + \sigma^2 x'h/T)e^{x'\beta^* + \sigma^2 x'x/2T}]_{x=o} \qquad (13) \\ &\quad + [(T-m)(T-m+2) + 4\lambda(T-m+1+\lambda)\,]\,[\zeta_x^2\ e^{x'\beta^* + \sigma^2 x'x/2T}]_{x=o} \end{aligned}$$

---

[3]Note Phillips (1984) changes his notation from m to n. Thus in equations (17) through (22) n should be replaced by m for consistency with the rest of the paper. Also note there is a square missing in the exponent in equation (13) of Phillips (1984).

[4]Note that in equation (13) $(T-m+2\lambda) = E(\chi'^2(T-m,\lambda)$ and $(T-m)(T-m+2) + 4\lambda(T-m+1+\lambda) = E(\chi'^2(T-m,\lambda))^2$.

[5]As mentioned earlier the moments of the individual elements of r may be found alternatively using results of Ullah (1974). If we wished to use (13) it is of course necessary to find expressions for the terms in square brackets which is a complicating feature of this approach.

As a means of examining the effects on the moments of the non-normality assumption we will only examine the mean via (12). Thus we now require the expectation with respect to η of (12). This will complete the second step in the Davis (1976) procedure.

We first note that since $\beta^* = \beta + \frac{1}{T} X'\eta$ we have $\theta^* = \theta + \phi$ where $\theta = T\beta'\beta/2\sigma^2$ and $\phi = (\eta' X\beta/\sigma^2) + (\eta' XX'\eta/2T\sigma^2)$. Next we note that using results in Slater (1960, p. 23)

$$e^{-\theta^*} {}_1F_1(\tfrac{m}{2}, \tfrac{m}{2}+1, \theta^*) = e^{-\theta+\phi} {}_1F_1(\tfrac{m}{2}, \tfrac{m}{2}+1, \theta+\phi)$$

$$= e^{-\theta} \sum_{n=o}^{\infty} \frac{(1)_n(-\phi)^n}{(\frac{m}{2}+1)_n \, n!} {}_1F_1(\tfrac{m}{2}, \tfrac{m}{2}+1+n, \theta)$$

Therefore (12) may be written alternatively as:

$$E(y|\eta) = \beta' h + \frac{1}{T} \eta' Xh - \frac{1}{2} (T-m+2\lambda) ah'(\beta + \frac{1}{T} X'\eta) \cdot$$

$$\cdot \; e^{-\theta} \sum_{n=o}^{\infty} \frac{(1)_n(-\phi)^n \Gamma(m/2)}{n! \; \Gamma(\frac{m}{2}+1+n)} \cdot {}_1F_1(\tfrac{m}{2}, \tfrac{m}{2}+1+n, \theta)$$

and using notation introduced by Ullah (1974) by letting

$f_{o,1+n} = e^{-\theta} \dfrac{\Gamma(m/2)}{\Gamma(\frac{m}{2}+1+n)} \; {}_1F_1(\tfrac{m}{2}, \tfrac{m}{2}+1+n, \theta)$ we have

$$E(y|\eta) = \beta' h + \frac{1}{T} \eta' Xh - \frac{1}{2} (T-m+2\lambda) ah'(\beta + \frac{1}{T} X'\eta) \cdot \sum_{n=o}^{\infty} (-\phi)^n f_{0,n+1} \qquad (14)$$

As with the pdf it is now necessary to consider expectations with respect to η. This will involve

$E(-\phi)^n$ for $n=0,1,2,3,4$

$E(\frac{1}{T} h'X'\eta(-\phi)^n)$ for $n=0,1,2,3$

$E(\lambda\phi^n)$, $n=0,1,2$ and $E(\lambda \frac{1}{T} h'X'\eta\phi^n)$, $n=0,1$.

These expectations are given in Appendix C and substitution into (14) gives:

$$
\begin{aligned}
E(y) = \beta' h - \frac{1}{2}(T-m) a h' \beta f_{0,1} \\
&+ K_3 \Big\{ \frac{2f_{03}}{\sigma^4} \Sigma\, g_i M_{ii} - \frac{f_{0,4}}{\sigma^6} \Sigma\, g_i^3 \\
&\quad + \frac{a(T-m) f_{0,2}}{2\sigma^2} \Sigma\, \ell_i G_{ii} - \frac{a(T-m) f_{0,3}}{2\sigma^4} \Sigma\, \ell_i g_i^2 \\
&\quad + \frac{a h' \beta f_{0,2}}{2\sigma^4} \Sigma\, g_i M_{ii} - \frac{a f_{0,1}}{2\sigma^2} \Sigma\, \ell_i M_{ii} \Big\} \\
&+ K_4 \Big\{ \frac{f_{0,3}}{\sigma^4} \Sigma\, G_{ii}^2 - \frac{3 f_{0,4}}{\sigma^6} \Sigma\, g_i^2\, G_{ii} + \frac{f_{0,5}}{\sigma^8} \Sigma\, g_i^4 \\
&\quad - \frac{(T-m) a f_{0,3}}{\sigma^4} \Sigma\, g_i \ell_i G_{ii} - \frac{(T-m) a f_{0,3}}{2\sigma^6} \Sigma\, \ell_i g_i^3 \\
&\quad + \frac{a h' \beta f_{0,2}}{2\sigma^4} \Sigma\, G_{ii} M_{ii} - \frac{a h' \beta f_{0,3}}{2\sigma^6} \Sigma\, g_i^2\, M_{ii} \\
&\quad + \frac{a f_{0,2}}{2\sigma^4} \Sigma\, \ell_i g_i M_{ii} \Big\}
\end{aligned}
$$

where $\ell = \frac{1}{T} Xh$; $g = X\beta$, $G = \frac{1}{2T} XX'$ and $M = I - X(X'X)^{-1}X'$.

We see immediately that when $K_3 = K_4 = 0$, i.e., the errors are normally distributed the mean collapses to that found by Ullah (1974) and Phillips (1984).

5. Conclusion

The previous sections have shown the usefulness of the Davis (1976) technique to examine the behaviour of estimations, etc., under a non-normality assumption on the errors. By extending the results of Phillips (1984) we are able to give explicit representation of the pdf with corrections for both skewness and kurtosis. The extension of the technique to examine moments is straightforward however, as noted, the technique of Ullah (1974) may prove easier to apply than the direct approach of Phillips (1984).

Appendix A

Expectation required for Section 3.

If we let $p = \frac{1}{T} Xq$ then we have

$$\mathop{E}_{\eta} (p'\eta) = 0$$

$$\mathop{E}_{\eta} ((p'\eta)^2) = 0$$

$$\mathop{E}_{\eta} ((p'\eta)^3) = K_3 \sum_i p_i^3$$

$$\mathop{E}_{\eta} ((p'\eta)^4) = K_4 \sum_i p_i^4$$

$$\mathop{E}_{\eta} ((p'\eta)(\eta' M\eta)) = \mathop{E}_{\eta} (\sum_i p_i \eta_i^3 M_{ii}) = K_3 \sum_i p_i M_{ii}$$

$$\mathop{E}_{\eta} (\eta' M\eta) = 0$$

$$\mathop{E}_{\eta} ((p'\eta)^2(\eta' M\eta)) = \mathop{E}_{\eta} (\sum_i p_i^2 M_{ii} \eta_i^4) = K_4 \sum_i p_i^2 M_{ii}$$

$$\mathop{E}_{\eta} ((\eta' M\eta)^2) = K_4 \sum M_{jj}^2$$

Appendix B

Evaluation of the derivatives with respect to q required in Section 3.[a]

$$\exp[B_1 h'\partial q - B_2(h'\partial q)^2 + x'\partial q] \cdot C_1 \sum_i (q'X_i/T)^3$$

$$= \sum_{j=o}^{\infty} \frac{1}{j!} (B_1 h'\partial q - B_2(h'\partial q)^2 + x'\partial q)^j \; C_1 \sum_i (q'X_i'/T)^3$$

We note

$$\Delta_j = [B_1 h'\partial q - B_2(h'\partial q)^2 + x'\partial q]^j \; C_1 \sum_i (q'X_i'/T)^3, \qquad j=1,2,3,\ldots$$

For j=1, $\Delta_1 = C_1\{3B_1\Sigma(q'X_i'/T)^2(h'X_i'/T) - 6B_2\Sigma(q'X_i'/T)(h'X_i'/T)^2 + 3\Sigma(q'X_i'/T)^2(x'X_i'/T)\}$

For j=2; $\Delta_2 = C_1\{6B_1^2\Sigma(q'X_i'/T)(h'X_i'/T)^2 + 12B_1\Sigma(q'X_i'/T)(h'X_i'/T)(x'X_i'/T)$

$+ 6\Sigma(q'X_i'/T)(x'X_i'/T)^2 - 12B_1B_2\Sigma(h'X_i'/T)^3 - 12B_2\Sigma(h'X_i'/T)^2(x'X_i'/T)\}$

For j=3; $\Delta_3 = C_1\{6B_1^3\Sigma(h'X_i'/T)^3 + 18B_1^2\Sigma(h'X_i'/T)^2(x'X_i'/T) + 18B_1\Sigma(h'X_i'/T)(x'X_i'/T)^2$

$+ 6\Sigma(x'X_i'/T)^3\}$

For $j \geq 4$ $\quad \Delta_j = 0$

Thus

$$\exp[B_1 h'\partial q - B_2(h'\partial q)^2 + x'\partial q]C_1\Sigma(q'X_i'/T)^3]_{q=0}$$

$$= C_1\{(B_1^3 - 6B_1B_2)\Sigma(h'X_i'/T)^3 + (3B_1^2 - 6B_2)\Sigma(h'X_i'/T)(x'X_i'/T) \qquad \text{(B.1)}$$

$$+ 3B_1\Sigma(h'X_i'/T)(x'X_i'/T)^2 + \Sigma(x'X_i'/T)^3\}$$

Next consider

$$\exp[B_1 h'\partial q - B_2(h'\partial q)^2 + x'\partial q]C_2\Sigma(q'X_i'/T)M_{ii}$$

$$= \sum_{j=o}^{\infty} \frac{1}{j!} (B_1 h'\partial q - B_2(h'\partial q)^2 + x'\partial q)^j C_2\Sigma(q'X_i'/T)M_{ii}$$

Again letting

$$\Delta_j = [B_1 h'\partial q - B_2(h'\partial q)^2 + x'\partial q]^j C_2\Sigma(q'X_i'/T)M_{ii}$$

---

[a] Note that in this section $(q'X_i'/T) = p_i$ where X_i is the i^{th} row of X.

we have

$$j=1,\quad \Delta_1 = C_2\{B_1\Sigma(h'X_i'/T)M_{ii} + \Sigma(x'X_i'/T)M_{ii}\}$$

$$j\geq 2,\quad \Delta_j = 0$$

Thus

$$\exp[B_1h'\partial q - B_2(h'\partial q)^2 + x'\partial q]C_2\Sigma(q'X_i'/T)M_{ii}]_{q=0} \tag{B.2}$$

$$= C_2\{B_1\Sigma(h'X_i'/T)M_{ii} + \Sigma(x'X_i'/T)M_{ii}\}$$

Consider now

$$\exp[B_1h'\partial q - B_2(h'\partial q)^2 + x'\partial q]\cdot C_3\Sigma(q'X_i'/T)^4$$

$$= \sum_{j=o}^{\infty}\frac{1}{j!}[B_1h'\partial q - B_2(h'\partial q)^2 + x'\partial q]^jC_3\Sigma(q'X_i'/T)^4 = \sum_{j=o}^{\infty}\frac{1}{j!}\Delta_j$$

Thus

$$j=1,\quad \Delta_1 = C_3\{4B_1\Sigma(q'X_i'/T)^3(h'X_i'/T) - 12B_2\Sigma(q'X_i'/T)^2(h'X_i'/T)^2 + 4\Sigma(q'X_i'/T)^3(x'X_i'/T)\}$$

$$j=2;\quad \Delta_2 = C_3\{12B_1^2\Sigma(q'X_i'/T)^2(h'X_i'/T)^2 + 12\Sigma(q'X_i'/T)^2(x'X_i'/T)^2 + 24B_1\Sigma(q'X_i'/T)^2(h'X_i'/T)(x'X_i'/T) - 48B_1B_2\Sigma(q'X_i'/T)(h'X_i'/T)^3 - 48B_2\Sigma(q'X_i'/T)(h'X_i'/T)^2(x'X_i'/T) + 24B_2^2\Sigma(h'X_i'/T)^4\}$$

$$j=3,\quad \Delta_3 = C_3\{24B_1^3\Sigma(q'X_i'/T)(h'X_i'/T)^3 + 72B_1^2\Sigma(q'X_i'/T)(h'X_i'/T)^2(x'X_i'/T) + 72B_1\Sigma(q'X_i'/T)(h'X_i'/T)(x'X_i'/T)^2 + 24\Sigma(q'X_i'/T)(x'X_i'/T)^3 - 72B_1^2B_2\Sigma(h'X_i'/T)^4 - 144B_1B_2\Sigma(h'X_i'/T)^3(x'X_i'/T) - 72B_2\Sigma(h'X_i'/T)^2(x'X_i'/T)^2\}$$

$$j=4,\quad \Delta_4 = C_3\{24B_1^4\Sigma(h'X_i'/T)^4 + 96B_1^3\Sigma(h'X_i'/T)^3(x'X_i'/T) + 96B_1^2\Sigma(h'X_i'/T)^2(x'X_i'/T)^2 + 96B_1\Sigma(h'X_i'/T)(x'X_i'/T)^3 + 24\Sigma(x'X_i'/T)^4\}$$

$$j\geq 5,\quad \Delta_j = 0$$

Thus

$$\exp[B_1h'\partial q - B_2(h'\partial q)^2 + x'\partial q]C_3\Sigma(q'X_i'/T)^4]_{q=o}$$

$$= C_3\{(B_1^4 + 12B_1^2B_2 + 12B_2^2)\Sigma(h'X_i'/T)^4 + 4(B_1^3 - 6B_1B_2)\Sigma(h'X_i'/T)^3(x'X_i'/T)$$
$$+ 4(B_1^2 - 3B_2)\Sigma(h'X_i'/T)^2(x'X_i'/T)^2 + 4B_1\Sigma(h'X_i'/T)(x'X_i'/T)^3 +$$
$$+ \Sigma(x'X_i'/T)^4\} \tag{B.3}$$

Next

$$\exp[B_1h'\partial q - B_2(h'\partial q)^2 + x'\partial q]\cdot C_4\Sigma M_{ii}^2]_{q=o} = C_4\Sigma M_{ii}^2 \tag{B.4}$$

Further

$$\exp[B_1h'\partial q - B_2(h'\partial q)^2 + x'\partial q]\cdot C_5\Sigma(q'X_i'/T)^2M_{ii}$$

$$= \sum_{j=o}^{\infty}\frac{1}{j!}[B_1h'\partial q - B_2(h'\partial q)^2 + x'\partial q]^j\cdot C_5\Sigma(q'X_i'/T)^2M_{ii}$$

Using Δ_j as before we have

$$j=1;\ \Delta_1 = 2C_5\{B_1\Sigma(q'X_i'/T)(h'X_i'/T)M_{ii} - B_2\Sigma(h'X_i'/T)M_{ii} +$$
$$+ \Sigma(q'X_i'/T)(x'X_i'/T)M_{ii}\}$$

$$j=2,\ \Delta_2 = 2C_5\{B_1^2\Sigma(h'X_i'/T)^2M_{ii} + 2B_1\Sigma(h'X_i'/T)(x'X_i'/T)M_{ii} + \Sigma(x'X_i'/T)^2M_{ii}\}$$

$$j\geq 3,\ \Delta_j = 0$$

Thus

$$\exp[B_1h'\partial q - B_2(h'\partial q)^2 + x'\partial q]C_5\Sigma(q'X_i'/T)^2M_{ii}]_{q=o}$$
$$= C_5\{(B_1^2 - 2B_2)\Sigma(h'X_i'/T)^2M_{ii} + 2B_1\Sigma(h'X_i'/T)(x'X_i'/T)M_{ii} + \Sigma(x'X_i'/T)M_{ii}\} \tag{B.5}$$

Therefore

$$\exp[B_1h'\partial q - B_2(h'\partial q)^2 + x'\partial q]\{1 + C_1\Sigma(q'X_i'/T)^3 + C_2\Sigma(q'X_i'/T)M_{ii}$$
$$+ C_3\Sigma(q'X_i'/T)^4 + C_4\Sigma M_{ii}^2 + C_5\Sigma(q'X_i'/T)^2M_{ii}\} \tag{B.6}$$

can be found by adding (B.1) to (B.5) and using the facts that

$$C_1 = K_3/3!\,;\ C_2 = 2K_3k/(T-m)\,;\ C_3 = K_4/4!$$

$$C_4 = K_4(-k)_2/2\ \left(\tfrac{T-m}{2}\right)_2\,;\ C_5 = K_4k/(T-m)$$

Thus (B.6) can be shown to equal:

$$1 + \frac{K_3}{6} S_1 + \frac{K_4}{4!} S_2$$

where

$$\begin{aligned} S_1 = {} & (B_1^3 - 6B_1B_2)\Sigma(h'X_i'/T)^3 + 3(B_1^2 - 2B_2)\Sigma(h'X_i'/T)^2(x'X_i'/T) \\ & + 3B_1\Sigma(h'X_i'/T)(x'X_i'/T)^2 + \Sigma(x'X_i'/T)^3 + (12k/(T-m))B_1\Sigma(h'X_i'/T)M_{ii} \\ & + (12k/(T-m))\Sigma(x'X_i'/T)M_{ii} \end{aligned} \tag{B.7}$$

$$\begin{aligned} S_2 = {} & (B_1^4 + 12B_1^2B_2 + 12B_2^2)\Sigma(h'X_i'/T)^4 + 4(B_1^3 - 6B_1B_2)\Sigma(h'X_i'/T)^3(x'X_i'/T) \\ & + 4(B_1^2 - 3B_2)\Sigma(h'X_i'/T)^2(x'X_i'/T)^2 + 4B_1\Sigma(h'X_i'/T)(x'X_i'/T)^3 + \\ & + \Sigma(x'X_i'/T)^4 + (12(-k)_2/(\tfrac{T-m}{2})_2)\Sigma M_{ii}^2 \\ & + (48k/(T-m))[(B_1^2 - 2B_2)\Sigma(h'X_i'/T)M_{ii} + 2B_1\Sigma(h'X_i'/T)(x'X_i'/T)M_{ii} + \\ & \quad + \Sigma(x'X_i'/T)M_{ii}] \end{aligned} \tag{B.8}$$

Appendix C

Expectations required in Section 4.

Let $\phi = \frac{1}{\sigma^2}\eta' X\beta + \frac{1}{2T\sigma^2}\eta' XX'\eta = \frac{1}{\sigma^2}(g'\eta + \eta' G\eta)$

Then

$$\underset{\eta}{E}(\phi) = 0$$

$$\underset{\eta}{E}(\phi^2) = \frac{1}{\sigma^4}\{2K_3 \sum_i g_i G_{ii} + K_4 \sum_i G_{ii}^2\}$$

$$\underset{\eta}{E}(\phi^3) = \frac{1}{\sigma^6}\{K_3 \Sigma g_i^3 + 3K_4 \Sigma g_i^2 G_{ii}\}$$

$$\underset{\eta}{E}(\phi^4) = \frac{1}{\sigma^8}\{K_4 \Sigma g_i^4\}$$

$$E(\frac{1}{T} h' X' \eta \phi) = E(\ell' \eta \cdot \phi)$$

$$= \frac{1}{\sigma^2} K_3 \Sigma \ell_i G_{ii}$$

$$E(\ell' \eta \phi^2) = \frac{1}{\sigma^4} \{K_3 \Sigma \ell_i g_i^2 + 2K_4 \Sigma g_i \ell_i G_{ii}]$$

$$E(\ell' \eta \phi^3) = \frac{1}{\sigma^6} [K_4 \Sigma \ell_i g_i^3]$$

$$E(\lambda\phi) = \frac{1}{2\sigma^4} E(\eta' M \eta (g'\eta + \eta' G \eta))$$

$$= \frac{1}{2\sigma^4} [K_3 \Sigma g_i M_{ii} + K_4 \Sigma G_{ii} M_{ii}]$$

$$E(\lambda\phi^2) = \frac{1}{2\sigma^6} E(\eta' M \eta (g'\eta + \eta' G \eta)^2)$$

$$= \frac{1}{2\sigma^6} K_4 \Sigma g_i^2 M_{ii}$$

$$E(\lambda \ell' \eta \phi) = \frac{1}{2\sigma^4} E((\eta' M \eta) \ell' \eta (g'\eta + \eta' G \eta))$$

$$= \frac{1}{2\sigma^4} K_4 \Sigma \ell_i g_i M_{ii}$$

$$E(\lambda \ell' \eta) = \frac{1}{2\sigma^2} E(\eta' M \eta \ell' \eta)$$

$$= \frac{1}{2\sigma^2} K_3 \Sigma \ell_i M_{ii}$$

References

Davis, A. W. (1976), "Statistical Distributions in Univariate and Multivariate Edgeworth Populations," Biometrika 63, 661-670.

James, W. and C. Stein (1961), "Estimation with Quadratic Loss," in Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability (University of California Press, Berkeley, CA), 361-379.

Knight, J. L. (1983a), "Non-normal Disturbances and the Distribution of the Durbin-Watson Statistic," Paper presented at the European Meeting of the Econometric Society, Pisa.

Knight, J. L. (1983b), "Non-normal Errors and the Distribution of OLS and 2SLS Structural Estimators," Paper presented at the Winter Meeting of the Econometric Society, San Francisco.

Knight, J. L. (1984a), "The Moments of OLS and 2SLS When The Disturbances are Non-normal," Journal of Econometrics (forthcoming).

Knight, J. L. (1984b), "The Joint Characteristic Function of Linear and Quadratic Forms of Non-normal Variables," Sankhya A (forthcoming).

Phillips, P. C. B. (1984), "The Exact Distribution of the Stein-rule Estimator," Journal of Econometrics 25, 123-131.

Slater, L. J. (1960), Confluent Hypergeometric Functions (Cambridge University Press, Cambridge).

Srivastava, V. K. and S. Upadhyaya (1977), "Properties of Stein-Like Estimators in Regression Models when Disturbances are Small," Journal of Statistical Research 11, 5-21.

Ullah, A. (1974), "On the Sampling Distribution of Improved Estimators for Coefficients in Linear Regression," Journal of Econometrics 2, 143-150.

Ullah, A. (1982), "The Approximate Distribution of the Stein-rule Estimator," Economics Letters 10, 305-308.

Ullah, A., V. K. Srivastava and R. Chandra (1983), "Properties of Shrinkage Estimators in Linear Regression When Disturbances are Not Normal," Journal of Econometrics 21, 389-402.