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Fuzzy Set Ranking Methods and Multiple Expert Decision Making

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VERITAS ET UTILITAS

Fuzzy Set Ranking Methods and Multiple Expert Decision Making

The UNIVERSITY of WESTERN ONTARIO

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Abstract

The present report further investigates the multi-criteria decision making tool named Fuzzy Compromise Programming. Comparison of different fuzzy set ranking methods (required for processing fuzzy information) is performed. A complete sensitivity analysis concerning decision maker's risk preferences was carried out for three water resources systems, and compromise solutions identified. Then, a weights sensitivity analysis was performed on one of the three systems to see whether the rankings would change in response to changing weights. It was found that this particular system was robust to the changes in weights.

An inquiry was made into the possibility of modifying Fuzzy Compromise Programming to include participation of multiple decision makers or experts. This was accomplished by merging a technique known as Group Decision Making Under Fuzziness, with Fuzzy Compromise Programming. Modified technique provides support for the group decision making under multiple criteria in a fuzzy environment.

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1.0 Introduction

The field of statistics ended its monopoly as being the only field able to model imprecision and uncertainty. This change is credited to the introduction of fuzzy logic. The theory of fuzzy logic, unlike statistics (where uncertainty is modeled with randomness), represents imprecision by the fact that certain objects (or certain classes of objects) have poorly or ill-defined boundaries. It is assumed that "not all uncertainties easily fit the probabilistic classification", Bender and Simonovic (2000). As such, fuzzy logic is not a direct competitor to statistics, although many problems can be modeled both ways. However, it is argued that fuzzy logic is able to represent certain sorts of uncertainties better than statistics.

Some of the most popular examples of classes of objects where boundaries aren't clear (or are poorly defined) are "the class of all real numbers which are *much greater* than 1", "the class of *beautiful* women", or "the class of *tall* men." Note that the italicized words are ones which represent terms whose meaning isn't really clear (i.e., it's fuzzy). Multitudes of fuzzy terms exist in the natural language, such as *much better than, much smaller than, about or near, substantial*, *considerable*, *significant*, to just mention a few. Objects which are described by these fuzzy terms can, according to Bellman and Zadeh (1970), "convey information despite the imprecision of the meaning of the italicized words." Utilizing imprecise information of this type is the task of the field of fuzzy logic.

This type of information (i.e., fuzzy information) is represented by fuzzy sets, which assign grades of membership to objects within their universe of discourse. This means that a certain object (or a collection of objects) can belong to some larger class of objects with only partial membership. This is analogous or a “gray” or “shady” area, as we don’t really know if the objects in question belong to the class or not; all we know is that the objects partially belong to the class, and so we express that with membership functions. This way of dealing with imprecision is ideally suited for the area of modeling human decision making, as in many cases we process (although subconsciously) this fuzzy information on daily basis.

Human decision making usually deals with making a decision in presence of vague, incomplete or imprecise information. With the introduction of the theory of fuzzy sets, modeling decision making was made possible by yet another set of tools (in addition to ones based on statistics). One of these tools (Fuzzy Compromise Programming) is summarized in this report, and then supplemented by an additional feature - inclusion of multiple experts in the decision making process. In addition, a detailed study concerning fuzzy set ranking methods (which are necessary for processing results produced by Fuzzy Compromise Programming) is presented. It should be noted that inclusion of multiple experts in the decision making process, together with comparisons of different fuzzy set ranking methods are the two main objectives of this research

2.0 Background Information

This section briefly outlines the definitions of terms relevant to this report, as well as details of performing fuzzy arithmetic. Introduction and history of compromise programming is then given, together with a summary of Fuzzy Compromise Programming, as modified by Bender and Simonovic (1996, 2000).

2.1 Definitions

Definition 1. (Classical set)

Classical, or a crisp set, is one which assigns grades of membership of *either 0* or *1* to objects within their universe of discourse. To say it in another way, objects either belong to or do not belong to a certain class; or object either posses a certain property, or they do not; there is no middle ground. The type of a function that describes this is called a characteristic function.

Definition 2. (Fuzzy set)

A fuzzy set is one which assigns grades of membership *between 0* and *1* to objects within its universe of discourse. If X is a universal set whose elements are $\{x\}$, then, a fuzzy set A is defined by, its membership function,

$$\mu_A : X \rightarrow [0,1]\,, \qquad (1)$$

which assigns to every x a degree of membership μ_A in the interval $[0,1]$.

A fuzzy set can be represented by a continuous membership function $\mu_A(x)$, or by a set of discrete points. The latter is denoted by ordered pairs,

$$A = \{(x, \mu_A(x))\}, \quad x \in X\ . \qquad (2)$$

It is worth noting that a fuzzy set, whose degree of membership is only *0* and *1*, reduces to a crisp set.

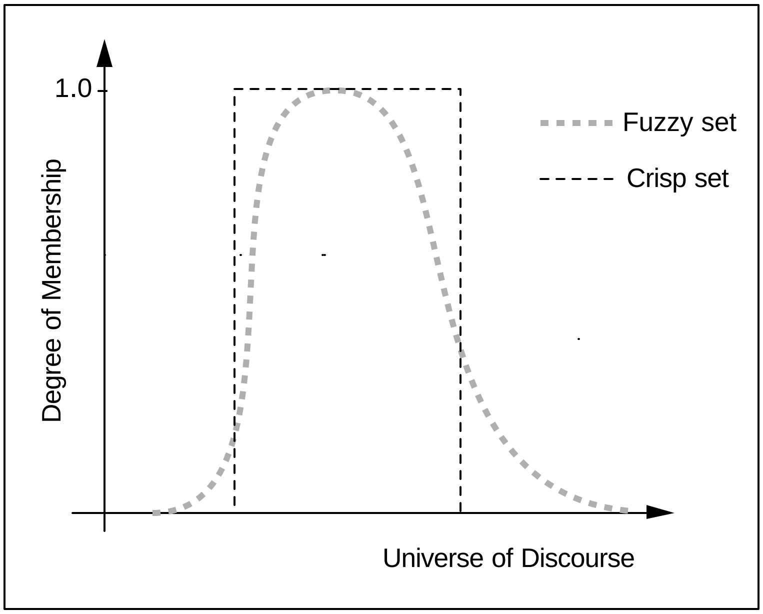

Figure 1. Illustration of a crisp and a fuzzy set

Definition 3. (Support of a fuzzy set)

Support of a fuzzy set *A* (written as *supp(A)*) is a (crisp) set of points in *X* for which μ_A is positive. An alternate way of saying this would be that the support of a fuzzy set *A* is the valid universe of discourse of *A* (i.e., all valid *x's*). Mathematically stated,

$$supp(A) = \{x \in X \mid \mu_A(x) > 0\}. \tag{3}$$

Synonyms of support are degree of fuzziness or a fuzzy spread.

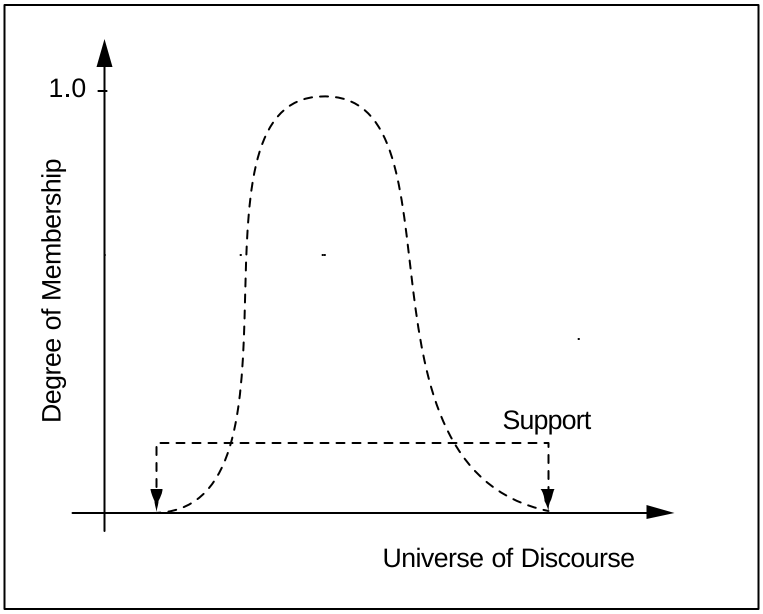

Figure 2. Support of a fuzzy set

Definition 4. (Normal fuzzy set)

A fuzzy set *A* is normal if its maximal degree of membership is unity (i.e., there must exist at least one *x* for which $\mu_A(x) = 1$). Of course, non-normal fuzzy sets have maximum degree of membership less than one.

Definition 5. (Convex fuzzy set)

A fuzzy set *A* is convex if and only if it satisfies the following property:

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq min(\mu_A(x_1), \mu_A(x_2)) \tag{4}$$

where λ is in the interval $[0,1]$, and $x_1 < x_2$. An example of a convex, as well as a non-convex fuzzy set is illustrated in Figure 3.

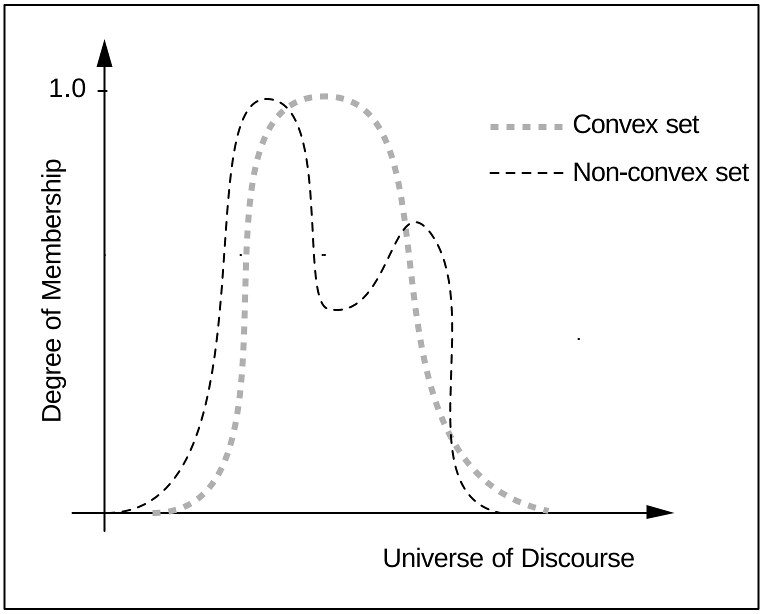

Figure 3. Convex and non-convex fuzzy sets

Remark: All fuzzy sets encountered in this report are both normal and convex.

Definition 6. (Intersection and union of fuzzy sets)

Intersection of fuzzy set A with fuzzy set B is:

$$\mu_{A \cap B}(x) = min(\mu_A(x), \mu_B(x)) \tag{5}$$

Union if two fuzzy sets is similarly defined:

$$\mu_{A \cup B}(x) = max(\mu_A(x), \mu_B(x)) \tag{6}$$

Note that intersection of two fuzzy sets is the largest fuzzy sets contained within A and B, and union is the smallest. See Figure 4 for clarification.

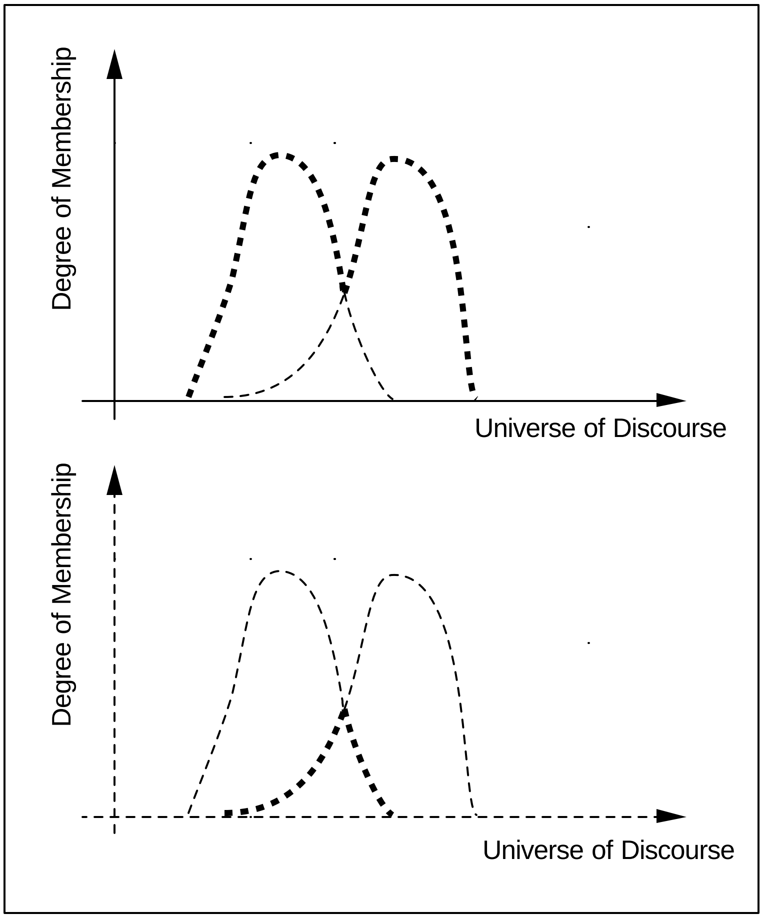

Figure 4. Union and intersection of two fuzzy sets

Definition 7. (Supremum and infimum of fuzzy sets)

Supremum, denoted by *sup*, is the largest possible value within given set, while infimum, denoted by *inf*, is the smallest value in a given set.

Definition 8. (λ-cut of a fuzzy set)

λ-cut of a fuzzy set is defined as crisp set A^{α} (or a crisp interval) for a particular degree of membership, α. Mathematically stated,

$$A^{\alpha} = [a^{\alpha}, b^{\alpha}] \tag{7}$$

where α, as before, can take on values between $[0,1]$.

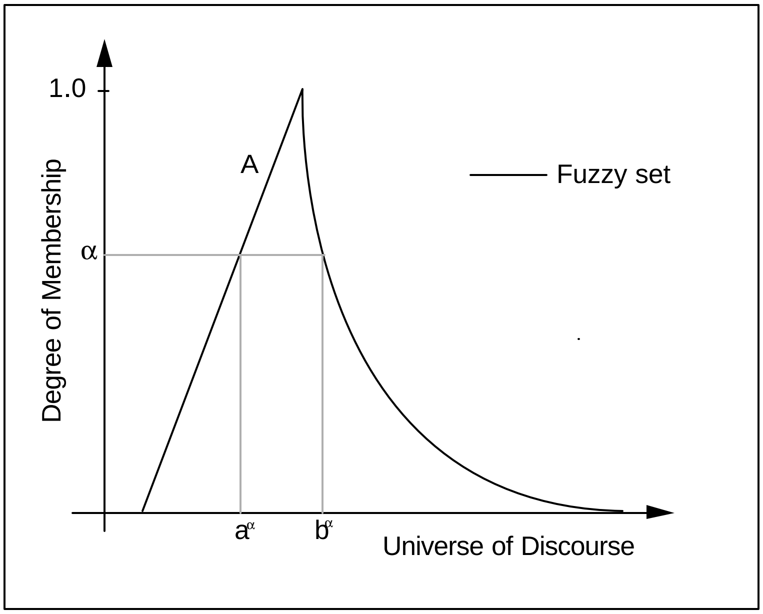

Figure 5. Illustration of an α-cut

Definition 9. (Fuzzy numbers)

A fuzzy number is a fuzzy set which is both normal and convex. In addition, the membership function of a fuzzy number must be piecewise continuous.

Most common types of fuzzy numbers are triangular and trapezoidal. Other types of fuzzy numbers are possible, such as bell-shaped or gaussian fuzzy numbers, as well as a variety of one sided fuzzy numbers. These will not be covered here. The interested reader is referred to a book by Klir and Yuan (1995) for more information on other types of fuzzy numbers. Triangular fuzzy numbers are defined by three parameters, while trapezoidal require four parameters.

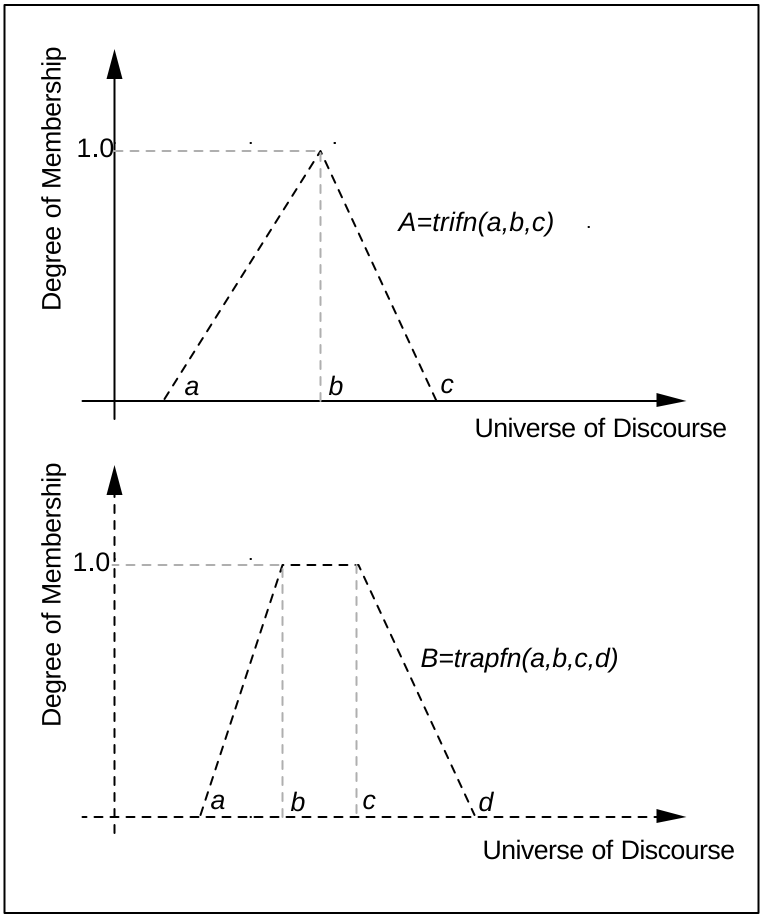

Figure 6. Triangular and trapezoidal fuzzy numbers

2.2 Fuzzy Arithmetic

A popular way to carry out fuzzy arithmetic operations is by way of interval arithmetic. This is possible because any α -cut of a fuzzy number is always an interval (see definition 8). Therefore, any fuzzy number may be represented as a series of intervals (one interval for every α-cut). In the Matlab code that was produced, 101 α -cuts (or intervals) were made, which means that α -cuts were made for $\alpha = 0, 0.01, 0.02, 0.03, \dots, 0.98, 0.99, 1.0$. Now, this means that there exist *101* intervals on which we are to perform interval arithmetic operations.

The basics of interval arithmetic are given next. For any two intervals, *[a, b]* and *[d, e]*, the arithmetic operations are performed in the following way:

Addition:	$[a, b] + [d, e] = [a+d, b+e];$	(8)
Subtraction:	$[a, b] - [d, e] = [a-e, b-d];$	(9)
Multiplication:	$[a, b] \cdot [d, e] = [\min(ad, ae, bd, be), \max(ad, ae, bd, be)];$	(10)
Power:	$[a, b]^{[d,e]} = [\min(a^d, a^e, b^d, b^e), \max(a^d, a^e, b^d, b^e)];$	(11)
Division:	$[a,b] / [d, e] = [\min(a/d, a/e, b/d, b/e), \max(a/d, a/e, b/d, b/e)],$ provided that $0 \notin [d, e]$.	(12)

Since any fuzzy number can be represented by a series of crisp intervals, we can then apply interval arithmetic operations (such as addition, subtraction, multiplication, division, power) and obtain an alternate way of performing fuzzy arithmetic. This is what most texts (and Matlab) consider as fuzzy arithmetic. In addition, this technique is *more* computationally efficient than brute force/dynamic search combination, but its downfall is that it cannot handle multi-modal fuzzy sets (i.e. multi-modal fuzzy sets cannot be expressed as intervals). An excellent text on fuzzy arithmetic is one by Kaufman and Gupta (1985); also, Klir and Yuan (1995) in their book cover the basics of fuzzy arithmetic rather well.

Note: Bender and Simonovic (1996) developed a different method of performing fuzzy arithmetic. Their method is based on brute force complimented with dynamic searches, which are used to lower computation time. An advantage of their method is that it's able to perform fuzzy arithmetic on all types of fuzzy sets, not just fuzzy numbers. However, even with dynamic searches, the method is *extremely* computationally intense.

Therefore, it can be concluded that if fuzzy arithmetic is required for non-convex (or multi-modal) fuzzy sets, brute force/dynamic search method should be used. If on the other hand, fuzzy arithmetic is required to be performed on fuzzy numbers, then application of interval arithmetic is sufficient.

2.3 History of Compromise Programming

Classical compromise programming was originally developed by Zeleny (1973), and since then, it has been applied (and modified) by many. Bardossy et al. (1985) modified compromise programming to form composite programming – a methodology that deals with problems of hierarchical nature (i.e., when certain criteria contain a number of sub-criteria). Composite programming works by applying the compromise programming equation to each sub-criterion, and then combines the compromise distance metrics of each sub-criterion to form a single composite distance metric (one composite distance metric for each objective of the problem). Its main strength is that it's able to assign a different distance metric exponent *p* for each sub-criterion, thus making the user "account for the analytical characteristics of statistical criteria versus the economic value of observation effort" (Bardossy et al. (1985), page 377). As such, composite programming is a natural extension of compromise programming.

Also, composite programming was further modified into fuzzy composite programming, which instead of crisp input variables, considers fuzzy variables. However, fuzzy composite programming (as of now) models only criteria values as fuzzy sets, while keeping the distance metric exponent p and the weights crisp.

Applications of fuzzy composite programming include that of Lee et al. (1991), Lee et al. (1992), Bardossy and Duckstein (1992), and Hagemeister et al. (1996).

Goicoechea et al. (1982) use compromise programming to evaluate a set of water resource systems subject to multiple (conflicting) criteria. Tkach and Simonovic (1997) use this same approach and apply compromise programming, together with Geographical Information Systems, to come up with spatial compromise programming - a methodology able to model spatial variability of criteria values. In addition, Bender and Simonovic (1996, 2000) fuzzified compromise programming (i.e., all parameters in the compromise programming equation were made into fuzzy sets) and came up with Fuzzy Compromise Programming.

2.4 Compromise Programming

Compromise programming is a mathematical programming technique that ranks a discrete set of solutions according to their distance from an ideal solution. This closeness is determined by some measure of distance. To see the meaning of this, consider the following example: Suppose two objectives are to be met for a maximization problem (one which seeks the maximization of all objectives when possible). Also, suppose that the objectives are 'protection of the environment' and 'development possibility'. Likewise, imagine that four available alternatives are present, from which one is to be chosen for implementation. Now, the ideal point (or the ideal alternative) would be one where both objectives are maximized. This point, in most practical cases is infeasible and as such, a compromise must be sought (i.e., if we are to have well a protected environment, the chance is that there will *not* be an excellent development possibility, and vise versa.) Compromise programming states that the best alternative is one which is closest to the ideal point. (Note that if *n* objectives are present, the distance metric is in n^{th} dimensional space.) Graphical representation of compromise programming is illustrated by Figure 7.

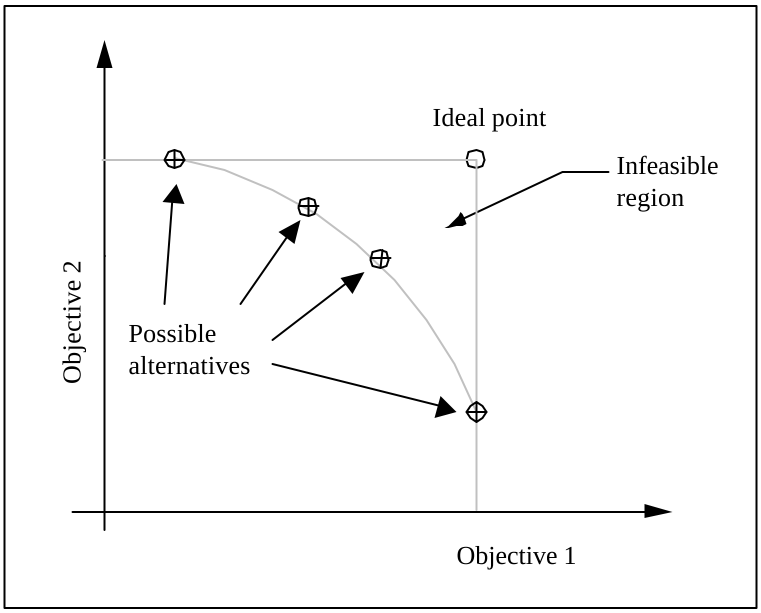

Figure 7. An illustration of compromise programming

The equation used to obtain a dimensionless distance metrics (one for every alternative) is:

$$L_j = \left[\sum_{i=1}^{n} \left\{ w_i^p \left(\frac{f_i^* - f_i}{f_i^* - f_i^-} \right)^p \right\} \right]^{\frac{1}{p}} \tag{13}$$

where:

$i = 1, 2, 3 ... n$ and represents n criteria or objectives;

$j = 1, 2, 3 ... m$ and represents m alternatives;

L_j is the distance metric of alternative j;

w_i corresponds to a weight of a particular criteria or objective;

p is a parameter ($p = 1, 2, \infty$);

f_i^* and f_i^- are the best and the worst value for criteria i, respectively;

f_i is the actual value of criterion i.

Of course, each criterion is to be given a level of importance, or a weight. The decision maker's preferences (concerning the criteria) are modeled with the variable w, noting that for compromise programming the weights are used simply to place emphasis on the important criteria. The parameter p is used to represent the importance of the maximal deviation from the ideal point. If $p = 1$, all deviations are weighted equally; if $p = 2$, the deviations are weighted in proportion to its magnitude. Typically, as p increases, so does the weighting of the deviations. For more information on the parameter p, consult Goicoechea et al. (1982), page 236-237.
Compromise programming equation is solved for $p = 1, 2$ and ∞, and then the alternatives are ranked. After the distance metrics are obtained, they are then sorted from smallest to largest, where the smallest represents the best compromise alternative.

2.5 Fuzzy Compromise Programming

Fuzzy Compromise Programming equation is obtained by fuzzifying the compromise programming equation (i.e., by fuzzifying equation (13)). Thus, instead of inputting crisp numbers into equation (13), fuzzy numbers are used instead; instead of using classical arithmetic, we have to resort to fuzzy arithmetic; instead of simply sorting distance metrics, fuzzy set ranking methods must be applied to sort the fuzzy distance metrics.

Differences between crisp and fuzzy numbers are outlined in section 2.1, and a treatment on fuzzy arithmetic is given in section 2.2. This section will present the motivation behind the transformation from compromise programming to Fuzzy Compromise Programming.

The fuzzification of criteria values is the main driving force for the mathematical transformation from crisp to fuzzy. Often the criteria values are subjective in nature, taking such form as “A is *roughly* as large as B”, “B is *much greater* than C”, and “C is *substantially* different from A”. There are techniques available (see Klir and Yuan (1995)) that allow generation of these fuzzy sets, thus preserving the information contained within the italicized words. If criteria values on the other hand are not subjective, it still may be worthwhile keeping them fuzzy. This is because of the inherit uncertainty associated with the criteria values themselves. For example, if the criterion such as cost comes out to be $25,000 it may be useful to model this with a fuzzy number such as “*about* $25,000”. Of course, the more we are certain about the true value of the cost, the lesser degree of fuzziness we assign to the fuzzy number.

By the same analogy, criteria weights should also be fuzzified because they, too, are subjective in nature. It is usually the stakeholders, the participants, or the decision makers that provide their individual weights concerning the criteria. Then, this information can be aggregated into appropriate collective weights, which are then used to obtain a compromise decision.

The same goes for the positive and negative ideals within equation (13), as they are very much subjective. Difficulties present themselves when assigning positive and negative ideals to such criteria as cost, for example. Different participants will most probably have a different idea of what the ideal cost should be, and so use of aggregation methods and fuzzy sets may be more accurate than simply using crisp averages. This way, more information is being preserved throughout the problem.

Last, but not least, is the fuzzified value of the distance metric exponent *p*. According to Bender and Simonovic (2000), "this is the most vague and imprecise element of the distance metric calculation." The exponent *p* is simply used for weighting deviations of criteria values from its ideal point. Of course, if it were known what the weighting of the deviations should be, the problem would be simple. However, such information in most practical situations is just not available, and so we resort to fuzzy sets in representing the parameter *p*.

Now that all terms of equation (13) are fuzzy sets, the resulting distance metrics, L_j, also become fuzzy sets. In order to determine the alternative that is closest to the ideal alternative, fuzzy distance metrics have to be ordered from smallest to largest. In other words, fuzzy distance metrics have to be sorted. It is noted that ranking fuzzy distance metrics (which are fuzzy numbers in our study) is not as straightforward as ranking crisp numbers. More detail on this is given in section 3.0.

3.0 Ranking Fuzzy Sets

Ordering of fuzzy quantities is based on extracting various features from fuzzy sets. These features may be a center of gravity, an area under the membership function, or various intersection points between fuzzy sets. A particular fuzzy set ranking method extracts a specific feature from fuzzy sets, and then ranks them [fuzzy sets] based on that feature. As a result, it is reasonable to expect that different ranking methods can produce different ranking order for the same sample of fuzzy sets. Intricacies like these make ranking fuzzy sets rather difficult – these are outlined in section 3.1. A brief survey of available ranking methods found in the literature is presented in section 3.2, together with selection criteria used to select methods for the application of our study. Lastly, details of these selected methods are given in section 3.3.

3.1 Problems with Ranking Fuzzy Quantities

All fuzzy set ranking methods can be categorized into two classes (after Yuan (1991)):

1) Methods which convert a fuzzy number to a crisp number by applying a mapping function F (i.e., if A is a fuzzy number, then $F(A) = a$, where a is a crisp number). Fuzzy numbers are then sorted by ranking crisp numbers (i.e., *a's*) produced by the mapping.

2) Methods which use fuzzy relations to compare pairs of fuzzy numbers, and then construct a relationship which produces a linguistic meaning of the comparison. The ordering results are something like 'fuzzy number A is *slightly better* than fuzzy number B'.

However, each methodology has its own advantages and disadvantages.

With 1), it has been argued that "by reducing the whole of our analysis to a single [crisp] number, we are loosing much of the information we have purposely been keeping throughout our calculations" (Freeling (1980), p.348). This methodology, on the other hand, produces a consistent ranking of all fuzzy sets considered (i.e., if A is ranked greater than B, and B is ranked greater than C, then A will always be much greater than C). Also, there will always exist a fuzzy set which is ranked as "best", "second best", "third best", and so on.

With 2), by keeping the comparisons linguistic, we are preserving the inherit fuzzy information of the problem. However, as Yuan (1991) points out, "it may not always be possible to construct total ordering among all alternatives based on pairwise fuzzy preference relations". This means that even if A is better than B, and B is better than C, A may not always be better than C.

Discouraging facts about fuzzy set ranking methods, unfortunately, do not end here. In their review, Bortolan and Degani (1985) find that for simple cases, most fuzzy set ranking methods produce consistent rankings. Difficult cases however, produce different rankings for different methods. This means that if membership functions overlap (or intersect) for some values of x, or if the supports of fuzzy numbers differ even slightly, different methods will most likely produce different rankings. This is discussed in detail in section 5.0.

3.2 Available ranking methods and selection criteria

Literature review reveals that multitudes of fuzzy set ranking methods exist. Papers by Bortolan and Degani (1985) as well as Wang and Kerre (2001a, 2001b) present a comprehensive survey of the available methods. From Bortolan-Degani and Wang-Kerre papers, the following seventeen methods were considered in our study:

Balwin and Guild (1979), Campos Ibanes and Munoz (1989), Chang and Lee (1994), Chen (1985), Chen and Klien (1997), Cheng (1998), Dubois and Prade (1983), Fortemps and Roubens (1996), Kim and Park (1990), Lee and Li (1988), Liou and Wang (1992), Matarazzo and Munda (2001), Modarres and Sadi-Nezhad (2001), Peneva and Popchev (1998), Yao and Wu (2000), Yager (1981), and Yoon (1996).

In order to select methods for application of ranking fuzzy distance metrics, it was decided that only methods which allow decision maker participation be selected. This participation is usually in the form of risk preferences, where the decision maker is allowed to specify the degree of risk with which he/she wishes to make the decision. Of the above methods, only nine included forms of risk preferences (Balwin and Guild (1979), Campos Ibanes and Munoz (1989), Chang and Lee (1994), Chen (1985), Chen and Klien (1997), Fortemps and Roubens (1996), Kim and Park (1990), Liou and Wang (1992), Peneva and Popchev (1998)).

To further narrow down these nine methods, the following selection criteria was used:

1) The ranking method must be able to rank fuzzy sets of various shapes (not just triangular and/or trapezoidal fuzzy sets).
2) Method should be able to rank fuzzy sets which are non-normal and non-convex.
3) The method must be able to rank several fuzzy sets. That is, the methods should not just compare two fuzzy alternatives, nor pick the best choice from the list.
4) There must exist a numeric preference relation that conveys which alternatives are most favoured.

5) There ought to exist a linguistic interpretation of the ranked alternatives (i.e. A4 is strongly better than A1, A2 and A3, and A3 is moderately better than A1 and A2, etc).
6) The preference relation from 5) must be rational. That is, if A is preferred to B, and B is preferred to C, then A should be preferred to C.

The table below shows how nine methods compare against properties 1) through 6).

Table 1. Fuzzy Set Ranking Methods vs. Properties

Methods/Properties	1)	2)	3)	4)	5)	6)
Chen and Klein (1997)	?	?	?	?		?
Chen (1985)	?	?	?	?		?
Fortemps and Roubens (1996)	?	?	?	?		?
Chang and Lee (1994)	?	?	?	?	?	?
Baldwin and Guild (1979)	?	?	?	?		?
Liou and Wang (1992)	?		?	?		?
Kim and Park (1990)	?		?	?		?
Peneva and Popchev (1998)			?	?		?
Campos and Gonzales (1989)	?	?	?	?		?

Based on the information above, only methods of Chen (1985) and Chang and Lee (1994) were selected for application of ranking fuzzy distance metrics. Other seven methods were rejected, for reasons that are outlined next.

Method of Chen and Klein (1997)

This method gives limited control to the decision maker in specifying his/her preferences. Further, it is pointed out that varying the decision maker participation "can change the magnitude of preference and indifference, but not the actual order" of the alternatives (Chen and Klein (1997), page 30). It is because of these two facts that the method is rejected.

Method of Peneva and Popchev (1998)

This method is rejected because it requires fuzzy quantities to be triangular. Fuzzy Compromise Programming produces fuzzy numbers which are not triangular, and so Peneva and Popchev's method can not be used for our study.

Method of Kim and Park (1990)

This method is extremely similar to Chen's (1985) method, (i.e., both are based on finding intersections of minimizing/maximizing sets with fuzzy numbers in question). The only difference between the two methods is in the specification of risk preferences – Chen's (1985) method does it by varying exponents of the maximizing and minimizing sets, while Kim and Park's (1990) method emphasizes intersections of minimizing/maximizing sets with fuzzy numbers differently. Chen's (1985) methods is presented in section 3.3.2, while Kim and Park's equations are given next:

Maximizing ($G_{max}(x)$) and minimizing ($G_{min}(x)$) sets are defined as:

$$G_{max}(x) = \begin{cases} \left[\dfrac{x - x_{min}}{x_{max} - x_{min}} \right] & x_{min} < x < x_{max} \\ zero & otherwise \end{cases} \tag{14}$$

$$G_{min}(x) = \begin{cases} \left[\dfrac{x_{max} - x}{x_{max} - x_{min}} \right] & x_{min} < x < x_{max} \\ zero & otherwise \end{cases} \tag{15}$$

The equation for ranking the fuzzy alternatives using Kim and Park's (1990) method is:

$$KP(i) = k\,sup(\mu_i(x) \cap G_{max}) + (1-k)\left[1 - sup(\mu_i(x) \cap G_{min})\right] \qquad (16)$$

where the constant k represents decision maker's preference, and has a valid range of $[0\ 1]$).
Kim and Park's (1990) method is rejected because of its close resemblance to Chen's (1985) method.

Method of Baldwin and Guild (1979)

Baldwin and Guild's (1979) method can give reasonable results only when fuzzy sets overlap. This feature makes the method not general enough for our purpose, and so it is rejected.

Method of Liou and Wang (1992)

The Total Integral Value (TIV), the ranking index developed by Liou and Wang (1992), has a similar form of Chang and Lee's (1994) index. The TIV is given by equation below:

$$TIV(A) = \beta \int \mu_{AL}^{-1}(\alpha)d\alpha + (1-\beta)\int \mu_{AR}^{-1}(\alpha)d\alpha \qquad (17)$$

where β can take on values between $[0,1]$ and is used as a parameter to express user preferences. In fact, the Total Integral Value is a special case of the Overall Existence Ranking Index of Chang and Lee (1994). It is for that reason that method of Liou and Wang (1992) is rejected.

Method of Campos Ibanes and Munoz (1987), and method of Fortemps and Roubens (1996)

Again, the methods of Campos Ibanes and Munoz (1987) and Fortemps and Roubens (1996) are special cases of Chang and Lee's (1994) Overall Existence Ranking Index. Therefore, it is reasonable to reject these two methods from our study.

3.3 Ranking Methods used in this report

This section presents details of Chang and Lee's (1994) and Chen's (1985) ranking indices. In addition, an alternate method based on the combination of Bender and Simonovic's (1996, 2000) Weighted Center of Gravity index and Cheng's (1998) distance method is developed for the purpose of checking results produced by Chang and Lee's (1994) and Chen's (1985) indices. Note: all risk preferences given here are for minimization problems, that is an optimist (or a risk taker) would prefer small distance metric values, while a pessimist (risk averse), to be safe, would expect higher values.

3.3.1 Method of Chang and Lee (1994)

Chang and Lee (1994) simplify their Overall Existence Ranking Index (OERI) for the use of convex fuzzy numbers (the type of fuzzy numbers encountered in our project). Equation (18) corresponds to their ranking index.

$$OERI(A) = \int_0^1 \omega(\alpha)\left[\chi_1(\alpha)\mu_{AL}^{-1}(\alpha) + \chi_2(\alpha)\mu_{AR}^{-1}(\alpha)\right]d\alpha \tag{18}$$

where $\chi_1(\alpha)$ and $\chi_2(\alpha)$ are the subjective type weighting indicating neutral, optimistic and pessimistic preferences of the decision maker, with the restriction that $\chi_1(\alpha) + \chi_2(\alpha) = 1$. Parameter $\omega(\alpha)$ is used to specify weights which are to be given to certain degrees of membership (if any). For example, sometimes degree of membership of around, say $\alpha = 0.5$ is valued the most, so then an appropriate equation could be specified to reflect that. (It is noted that in our comparison of alternatives, all degrees of membership were given equal weight, namely $\omega(\alpha) = 1$). Lastly, $\mu_{AL}^{-1}(\alpha)$ represents an inverse of the left part, and $\mu_{AR}^{-1}(\alpha)$ the inverse of the right part of the membership function. The limits of integration (and the limits of α) are $[0\ 1]$.

It is noted that linear and non-linear functions for the subjective type weighting are possible, thus giving the user more control in the ranking. For the present study however, only constants were used to represent risk preferences.

Then, for χ_1 values greater than *0.5*, the left side of the membership function is weighted more than the right side, which in turn makes the decision maker more optimistic. Of course, if the right side is weighted more, the decision maker is more of a pessimist (this is because he/she prefers larger distance metric values, which means the farther solution from the ideal solution). In summary, the risk preferences are: if $\chi_1 < 0.5$, the user is a pessimist (risk averse); if $\chi_1 = 0.5$, the user is neutral; and if $\chi_1 > 0.5$, the user is an optimist (risk taker).

3.3.2 Method of Chen (1985)

After obtaining *n* fuzzy sets, maximizing, $\mu_M(x)$, and minimizing, $\mu_m(x)$ sets are defined by the following equations:

$$\mu_M(x) = \left[\frac{w_i(x - x_{min})}{(x_{max} - x_{min})}\right]^r \quad x_{min} < x < x_{max} \qquad \text{(19)}$$
$$\textit{zero} \qquad \textit{otherwise}$$

$$\mu_m(x) = \left[\frac{w_i(x - x_{max})}{(x_{min} - x_{max})}\right]^r \quad x_{min} < x < x_{max} \qquad \text{(20)}$$
$$\textit{zero} \qquad \textit{otherwise}$$

where $w_i = sup(\mu_i(x))$, $w = inf(w_i)$, $x_{min} = inf(x)$, $x_{max} = sup(x)$ and the subscript i represents the i^{th} alternative. The participation of the decision maker is controlled by the constant *r*. If $r = 1$ the decision maker is conservative or neutral (see Figure 8); if $r = 0.5$ the decision maker is a risk taker, or an optimist, (see Figure 9), and if $r = 2$ the decision maker is risk averse, or a pessimist, (see Figure 10). Of course, values of *r* below *0.5* represent extreme optimism, while *r* values which are greater than *2* represent extreme pessimism.

To graphically represent equations (19) and (20), alternative 4 from Tisza River Basin example taken from Bender and Simonovic (1996) was chosen at random. Note that the maximizing set is shown in red (right most set), and the minimizing set in green (left most set).

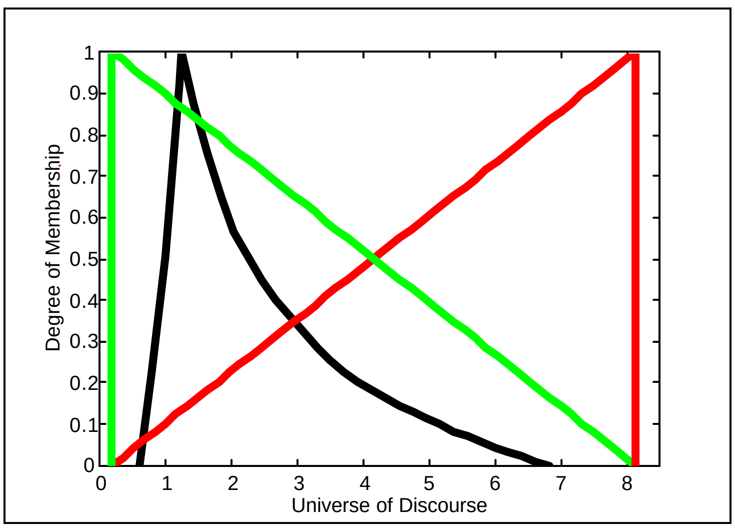

Figure 8. Chen's (1985) neutral preferences

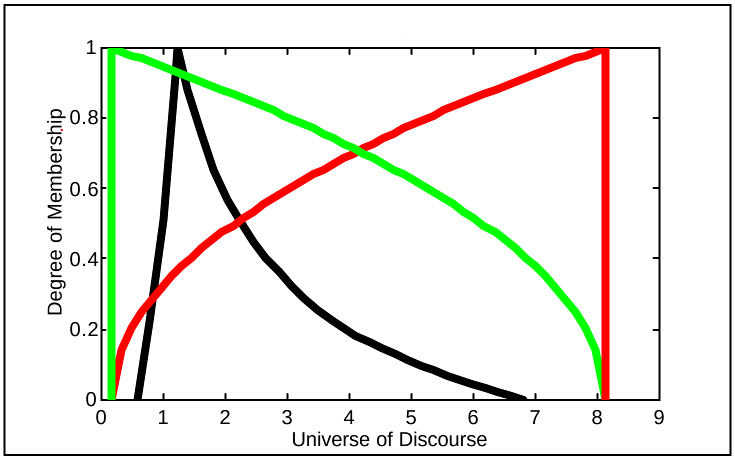

Figure 9. Chen's (1985) optimistic preferences

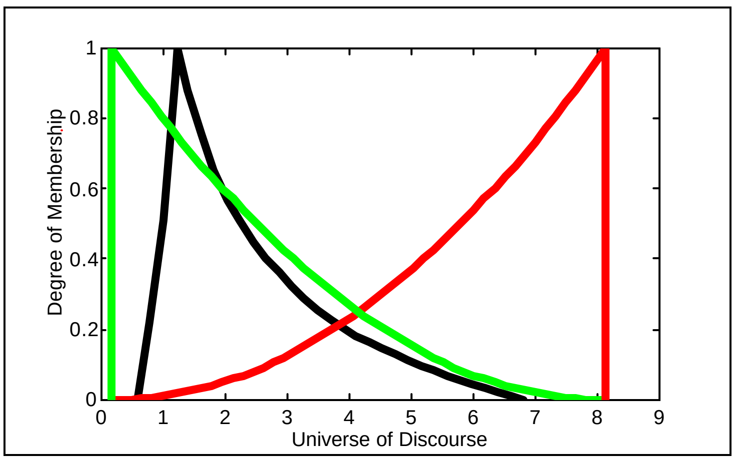

Figure 10. Chen's (1985) pessimistic preferences

To rank the alternatives, right ($U_M(A)$) and left ($U_m(A)$) utility values are calculated as follows:

$$U_M(A) = \sup(\mu_i(x) \cap \mu_M(x)) \quad (21)$$

$$U_m(A) = \sup(\mu_i(x) \cap \mu_m(x)) \quad (22)$$

$U_M(A)$ is the intersection of the maximizing set (red) with the right portion of the alternative in question (black), and $U_m(A)$ is the intersection of the minimizing set (green) with the left part of the membership function (black). The total utility value is then computed as:

$$U_T(A) = (U_M(A) + w - U_m(A))/2 \quad (23)$$

After this, the results are ordered from smallest to largest, smallest being the better alternative.

Comments:

Two concerns about this method must be noted. Firstly, the presence of an alternative with the membership function that is far to the left (or far to the right) from other alternatives, influences the way maximizing and minimizing sets are obtained. Therefore, by just one alternative being far away from the rest, increases (or decreases) the value of the parameter x_{max} (or x_{min}), which in turns shapes the maximizing and the minimizing sets. Liou and Wang (1992) also realized this, and showed in a four alternative system, that by moving one alternative to the left and then to the right changed the ranking value (and thus the ranking order) of all the alternatives within the system. Because of this, they deemed Chen's (1985) method illogical. However, if all alternatives are relatively close together, Chen's (1985) method can give reasonable results. Exactly how far apart the alternatives have to be before the method gives illogical results is not known at this time.

Secondly, since this method uses only two degrees of membership (the degrees of membership associated with the left and the right utility values), an objection can be raised that not enough fuzzy information is used in the ranking. This shortcoming is made explicit in section 5.0.

3.3.3 Modified Cheng's (1998) method (check method)

In addition to above methods, we are proposing a modification to an existing ranking method. The driving force for this modification is to end up with a method for ranking fuzzy sets which can check the results of the previous two methods. The argument for the modification is as follows:

Yager (1981) proposed a ranking index which is based on the area under the membership function. Yager's (1981) index is expressed as:

$$Y(A) = \frac{\int g(x)\mu_A(x)dx}{\int \mu_A(x)dx} \qquad (24)$$

where g(x) is a measure of the importance of the value of x.

Then, Bender and Simonovic (1996, 2000) modified Yager's (1981) index into the Weighted Center of Gravity (WCoG) index:

$$WCoG = \frac{\int g(x)\mu_A^q(x)dx}{\int \mu_A^q(x)dx} \qquad (25)$$

where exponent q is used to put more weight on higher membership values.

Cheng (1998) is developed a distance method similar to WCoG.

$$\bar{x}_0 = \frac{\int x\mu_A(x)dx}{\int \mu_A(x)dx} \qquad (26)$$

$$\bar{y}_0 = \frac{\int y\mu_A^{-1}(y)dy}{\int \mu_A^{-1}(y)dy} \qquad (27)$$

where the inverse of $\mu_A(x)$ is $\mu_A^{-1}(y)$.

The Ranking index of Cheng (1998) is computed as follows:

$$R(A) = \sqrt{\bar{x}_0^2 + \bar{y}_0^2} \qquad (28)$$

With all this in mind, the modification to Cheng's (1998) method is proposed. First, the indices $(\bar{x}_0, \bar{y}_0)$ have to be modified so that they take Yager's (1981) form.

$$\tilde{x}_0' = \frac{\int g(x)\mu_A(x)dx}{\int \mu_A(x)dx} \tag{29}$$

$$\tilde{y}_0' = \frac{\int g(y)\mu_A^{-1}(y)dy}{\int \mu_A(y)dy} \tag{30}$$

where functions *g(x)* and *g(y)* are the measure of importance of x and y respectively.

Then, the next step is the same as that of Bender and Simonovic (1996, 2000), in which Yager's (1981) index was modified to WCoG to include the exponent q.

$$\tilde{x}_0 = \frac{\int g(x)[\mu_A(x)]^q dx}{\int [\mu_A(x)]^q dx} \tag{31}$$

$$\tilde{y}_0 = \frac{\int g(y)\left[[\mu_{AL}^{-1}(y)]^q + [\mu_{AR}^{-1}(y)]^{(1+q_{\max}-q)}\right]dy}{\int \left[[\mu_{AL}^{-1}(y)]^q + [\mu_{AR}^{-1}(y)]^{(1+q_{\max}-q)}\right]dy} \tag{32}$$

Noting that the parameter q is used for the purpose of providing more weight to higher degrees of membership, we must assure that this is done in both x and y directions. The transition from (29) to (31) is identical to one performed by Bender and Simonovic (1996, 2000), and so it requires little explanation. Simply stated, equation (31) puts more emphasis on higher membership values (i.e., μ_A) by raising them to an exponent q. In other words, higher membership values give more weight in the ranking.

The transition from equation (30) to (32) however, requires explanation. In equation (32), x values (i.e., μ_A^{-1}) are raised to an exponent, not membership values. So, for convex fuzzy numbers we must consider left and right inverses separately, namely:

Case 1. For the left part of the fuzzy number, more weight should be provided to higher x values, because higher x values correspond to higher membership values. Thus, it is reasonable to raise μ_{AL}^{-1} to an exponent q.

Case 2. For the right part of the fuzzy number, more weight should be provided to lower x values, because lower x values correspond to higher membership values. To provide more emphasis on lower x values, we propose raising μ_{AR}^{-1} to an exponent ($q_{\max} + 1 - q$). By doing this, less and less emphasis is placed on higher x values (i.e., more and more emphasis is placed on lower x values). That way, higher membership values are weighed more in the ranking process.

Lastly, the parameter q_{max} represents the maximum value of the exponent q. (The author recommends using $q_{max} = 4$.)

Finally, the modified index is expressed by equation (33).

$$\tilde{R}(A) = \sqrt{\tilde{x}_0^2 + \tilde{y}_0^2} \tag{33}$$

Cheng (1998) shows that his method has benefits over other methods which use only x_0 as a point of reference for ranking. These benefits are preserved in the modified index, and an additional feature, parametric control is added.

Parametric control in a fuzzy set ranking method is needed because it is possible to conceive that two fuzzy sets can have the same centroid even if their supports were different. Thus, the presence of the exponent q takes into consideration degrees of fuzziness of the fuzzy sets to be compared. See Figure 11 for such a case. (Note: a ranking of A and B is given in section 5.0.)

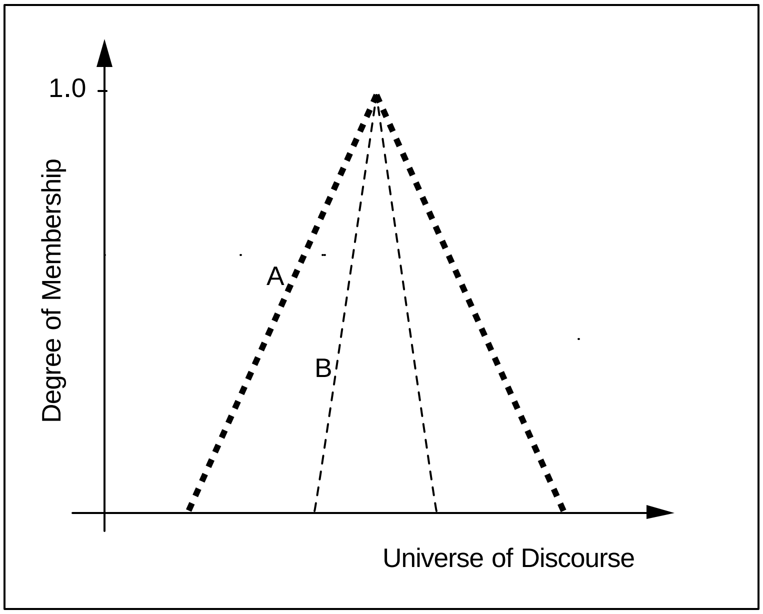

Figure 11. Fuzzy numbers with same centroids but different supports

In performing the sensitivity analysis, the levels of importance of x and y were represented by $g(x) = x^r$ and $g(y) = y^r$, respectively. The values of r that were used were $[1, 2, 3]$. Also, for each value of r, parameter q took on values of $[1, 2, 3, 4]$. For example, when $r = 1$, the fuzzy sets were ranked with $q = 1$, $q = 2$, $q = 3$ and so on. The same procedure was followed for other values of r.

This method, adapted by incorporating Yager (1981) and Cheng's (1998) indices, is to act as a check method for the above two methods. Modified Cheng's (1998) method is ideal for this task because it incorporates mapping functions in both x and y directions.

4.0 Fuzzy Compromise Programming for Multiple Decision Makers

There exists an array of ways to include multiple experts (or decision makers) into the decision making process via Fuzzy Compromise Programming. One method has been investigated for the application in this research – Group Decision Making Under Fuzziness. Other methodologies available are listed in the Recommendations for Future Research section.

4.1 Group Decision Making Under Fuzziness

Kacprzyk and Nurmi (1998) present a methodology which takes in opinions of *m* individuals concerning *n* crisp alternatives, and then outputs an alternative (or a set of alternatives) that are preferred by most individuals. Each individual is required to make a pairwise comparison between the alternatives; then a fuzzy preference relation matrix is constructed for each expert, results aggregated, and a group decision made. Please note that Kacprzyk and Nurmi's (1998) methodology can assign different experts different levels of importance (i.e., sometimes it makes sense that someone's opinion counts more than someone else's). In our study, everyone's opinion was counted the same. In addition, an overall degree of consensus of all participating individuals can also be calculated.

4.2 Group Decision Making Algorithm of Kacprzyk and Nurmi (1998)

Number of alternatives are denoted by subscripts $i, j = 1, 2, 3, \ldots n$ and number of individuals by subscript $k = 1, 2, 3, \ldots m$. In order to construct a fuzzy preference relation matrix for each individual, we must ask that person to compare every two alternatives in the system. For example, if there are three alternatives in the system (A1, A2 and A3), the individual must compare A1 to A2, A1 to A3, and A2 to A3, and tell us, for reach comparison, what alternative he/she prefers and to what degree. The options given to the individual are (from Kacprzyk and Nurmi (1998)):

$$\mu^k = \begin{cases} 1.0 & \text{if } Ai \text{ is definitely preferred to } Aj \\ c \in (0.5,1) & \text{if } Ai \text{ is slightly preferred to } Aj \\ 0.5 & \text{in the case of indifference} \\ d \in (0,0.5) & \text{if } Aj \text{ is slightly preferred to } Ai \\ 0.0 & \text{if } Aj \text{ is definitely preferred to } Ai \end{cases}$$

With the restrictions above, each individual is to construct a fuzzy preference relation matrix. For our three alternative example, a sample matrix for individual 1 may be:

$$r^{k=1} = \left[\begin{array}{c|ccc} & j=1 & 2 & 3 \\ \hline i=1 & 0 & 0.6 & 0.8 \\ 2 & 0.4 & 0 & 0.4 \\ 3 & 0.2 & 0.6 & 0 \end{array}\right]$$

Note: our individual 1 said that he/she preferred A1 to both A2 and A3, and A3 to A2, only slightly. Clearly, our individual thinks that A1 is the best option.

Once we obtain the fuzzy preference relation matrix from each individual, the aggregation of the results is performed in the following way. First, h_{ij} is calculated to see weather Ai defeats (in pairwise comparison) Aj ($h_{ij} = 1$) or not ($h_{ij} = 0$).

$$h_{ij}^k = \begin{cases} 1 & \text{if } r_{ij}^k < 0.5 \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

Then, we calculate

$$h_j^k = \frac{1}{n-1} \sum_{i=1, i \neq j}^{n} h_{ij}^k \tag{30}$$

which is the extent, from *0* to *1*, to which individual k is not against alternative Aj, where *0* standing for definitely not against to *1* standing for definitely against, through all intermediate values.

Next, we calculate

$$h_j = \frac{1}{m}\sum_{k=1}^{m} h_j^k \tag{31}$$

which expresses to what extent, from *0* to *1*, all individuals are not against alternative Aj.

Then, we compute

$$v_Q^j = \mu_Q(h_j) \tag{32}$$

which represents to what extent, from *0* to *1* as before, Q (most) individuals are not against alternative Aj. Q is a fuzzy linguistic quantifier, (in our case meaning "most") which is defined, after Zadeh (1983):

$$\mu_Q(x) = \begin{cases} 1 & if\ x \geq 0.8 \\ 2x - 0.6 & if\ 0.3 < x < 0.8 \\ 0 & if\ x \leq 0.3 \end{cases} \tag{33}$$

Lastly, the final result (fuzzy Q-core) is expressed as:

$$C_Q = \{(A1, v_Q^1), (A2, v_Q^2), (A3, v_Q^3), \ldots, (An, v_Q^n)\} \tag{34}$$

and is interpreted as a fuzzy set of alternatives that are not defeated by Q (most) individuals.

Similarly, fuzzy α /Q-core and fuzzy s/Q-core can be determined. The former is obtained by changing equation (29) into

$$h_{ij}^k(\alpha) = \begin{cases} 1 & if\ r_{ij}^k < \alpha \leq 0.5 \\ 0 & otherwise \end{cases} \tag{35}$$

and then performing all above steps as before. ($1 - \alpha$) represents a degree of defeat to which Ai defeats Aj; as such it is taken between *[0,0.5]*. The final result in this case is interpreted as a fuzzy set of alternatives that are not sufficiently (at least to a degree ($1 - \alpha$)) defeated by Q (most) individuals. The parameter α was arbitrarily chosen at *0.3*.

Fuzzy s/Q-core is determined by changing equation (29) to:

$$\hat{h}_{ij}^{k} = \begin{cases} 2(0.5 - r_{ij}^{k}) & if\ r_{ij}^{k} < 0.5 \\ 0 & otherwise \end{cases} \tag{36}$$

and, again, performing all above steps as before. With (36) above, strength is introduced into the defeat, and the final result interprets as a fuzzy set of alternatives that are not strongly defeated by Q (most) individuals.

4.3 Merging Group Decision Making with Fuzzy Compromise Programming

This section gives the algorithm used in including multiple experts in the decision making process that uses Fuzzy Compromise Programming.

1. Each decision maker is to specify his/her fuzzy weights concerning the importance of each criterion in the problem.
2. Then, for each expert, a set of fuzzy alternatives is generated via Fuzzy Compromise Programming.
3. After this, for each individual, a fuzzy preference relation matrix is generated (more on this later).
4. Finally, after everyone's fuzzy preference relation matrix is obtained, Q-core, α/Q-core and s/Q-core algorithms are performed, and a group decision is made.

4.4 Obtaining Individual Fuzzy Preference Relation Matrices

An individual fuzzy preference relation matrix is obtained via available ranking methods. Each individual's set of alternatives is ranked with a selected ranking method, and from the ranking values, the fuzzy preference relation matrix is obtained. (It is noted that neutral user preferences are used for the generation of this matrix.) The individual matrices were obtained in the following way:

First, a ranking method is called to rank the alternatives for each expert. Then, from all the ranking values for that expert, a difference is found for every two alternatives compared. To see what this means, consider the following. Suppose that a ranking method produces a vector of ranking values for each particular alternative, that is $ranV = \{r_{A1}, r_{A2}, r_{A3}, \ldots, r_{An}\}$. Then, a difference is found for every pair of r_{Ai} and r_{Ai+1}. From these differences in the ranking values, a fuzzy preference relation matrix is constructed. Then, if ($r_{Ai} - r_{Ai+1}$) is large and negative, that means that A1 is much more preferred than A2. Therefore, a fuzzy preference relation for this pair is given a value close to (or just less than) *1.0*. Similarly, if the difference is large and positive, meaning that A2 is much more preferred to A1, a value close to *0* is assigned for that particular pair. Of course, the if statements in the code cover all intermediate cases and thus assign values between *[0,1]* within the fuzzy preference relation matrix.

5.0 Results and Discussions

This section gives the results of all experiments performed in this study. The purpose of the first set of experiments (those of section 5.2) was to investigate the sensitivity of ranking fuzzy distance metrics with methods available in the literature. In other words, fuzzy distance metrics taken from case studies by Bender and Simonovic (1996) were ranked with three methods presented in section 3.3. Further, a complete sensitivity analysis concerning decision maker's risk preferences (from extreme pessimism to extreme optimism) was performed to investigate whether rankings would change in response to changing risk preferences. Case studies used were: Tisza River Basin, Yugoslavia Systems S1 and S2.

The purpose of the second set of experiments (those of section 5.3) was to investigate the sensitivity of criteria weights (the parameter w_i in equation (13)) to the problem of ranking fuzzy distance metrics. Four different sets of criteria weights were used in the Tisza River Basin example to observe how criteria weights influence rankings of the resulting fuzzy distance metrics. For each set of criteria weights, a sensitivity analysis concerning risk preferences was performed as well.

Third set (section 5.4) of experiments was set up to test the proposed methodology of including multiple experts into the decision making process via Fuzzy Compromise Programming. Each expert was allowed to specify his/her weights concerning the criteria of the problem, which were then used to form a group compromise decision.

5.1 Comments on features of fuzzy numbers that can influence ranking

Before the interpretation of the results takes place, few things must be noted. Firstly, few comments are made about different features of fuzzy numbers, such as the degree of fuzziness and proximity of a fuzzy number to the origin. After that, features of fuzzy number that make the rankings sensitive to risk preferences are given.

Simply stated, compromise programming favours smallest possible distance metrics. In the same way, Fuzzy Compromise Programming favours fuzzy distance metrics which are closer to the origin. That is to say, if every point on the left part of the membership function has a smaller distance metric value (for every α) than other fuzzy numbers, then that left part would be preferred. If the same is true for the right part, then definitely the fuzzy number in question will be ranked as smallest and therefore the best. (For example, see Figure 16. Expert 2 distance metrics. In this figure, A1 is always smaller than A5.)

Spreads on the other hand, have a more interesting effect on the ranking of fuzzy numbers. It is possible to conceive of system which has a fuzzy number with a quite large spread and relatively close to the origin, and also of a fuzzy number with a small spread and at the same time far from the origin. The alternative closer to the origin, despite its large spread, will be ranked as better than one further away. However, because of its large spread, its performance will be very unclear, but still better than the less vague (worse) performance of a fuzzy number further away. Of course, the opposite is also possible. We can imagine a fuzzy number with a small degree of fuzziness and relatively close to the origin, as well as a fuzzy number with a large degree of fuzziness and far away from the origin. In this case, the fuzzy number with a small spread and close to the origin will definitely be preferred to the other fuzzy number. Many cases of the latter kind are encountered in this study.

To see how the risk preferences affect the ranking of the results, please consider the example from Figure 11. Fuzzy numbers with same centroids but different supports. (For demonstration purposes, let the fuzzy set *A = trifn(1,5,9)*, and *B = trifn(3,5,7)*.) It is for cases like this that the ranking will depend on whether we chose an optimistic, pessimistic, or a neutral point of view. For Chang and Lee's (1994) method, if we are optimistic (i.e., weigh higher the left part of the membership function) we would get that A < B. If we are pessimistic and weigh higher the right part of the membership function, then the result A > B will be produced. Of course if we are neutral, a result A = B is produced. Chen's (1985) method, on the other hand produced a result of A = B, no matter what the risk preferences. However, since no symmetric fuzzy numbers were compared, this shortcoming did not play a role in this study; regardless, it is a flaw of the method.

Modified Cheng's method, also produced unreasonable results for this case. For example, for r = 1, the result of B > A, A > B, B > A and A > B was produced for q = 1, 2, 3 and 4 respectively. However, as will be seen in later sections, this discrepancy plays no role in ranking fuzzy distance metrics for our case studies, because our case studies are so robust, that they are almost insensitive to the ranking method.

Another feature that makes the rankings sensitive to risk preferences is the intersection of membership functions. More precisely, if the left part of one membership function intersects a left part of another membership function, then the ranking of these two alternatives will depend on the relative risk preference chosen. (This point is also covered by Chang and Lee (1994), page 5, and by Bender and Simonovic (1996), page 46, and so it won't be discussed further.) Therefore we must conclude that ranking order will be sensitive to risk preferences in cases where either one fuzzy number's support is contained within another fuzzy number's support, or if membership functions of fuzzy numbers intersect, or both.

5.2 Ranking Methods Applied to Case Studies from Bender and Simonovic (1996)

The results presented here include the applications of three selected ranking methods (with complete sensitivity analyses) to three case studies. The purpose of doing this investigation was to determine the variability (if any) of rankings, with application of different methods. The fuzzy distance metrics used in this part of the report were taken from Bender and Simonovic (1996), pages 94-105. The methods applied were that of Chang and Lee (1994), Chen (1985), together with modified Cheng's (1998) method.

5.2.1 Tisza River Basin Case Study

Information on this case study is given (in full detail) in Appendix B - Tisza River Basin. The distance metrics, as obtained by Bender and Simonovic (1996) are shown in Figure 12:

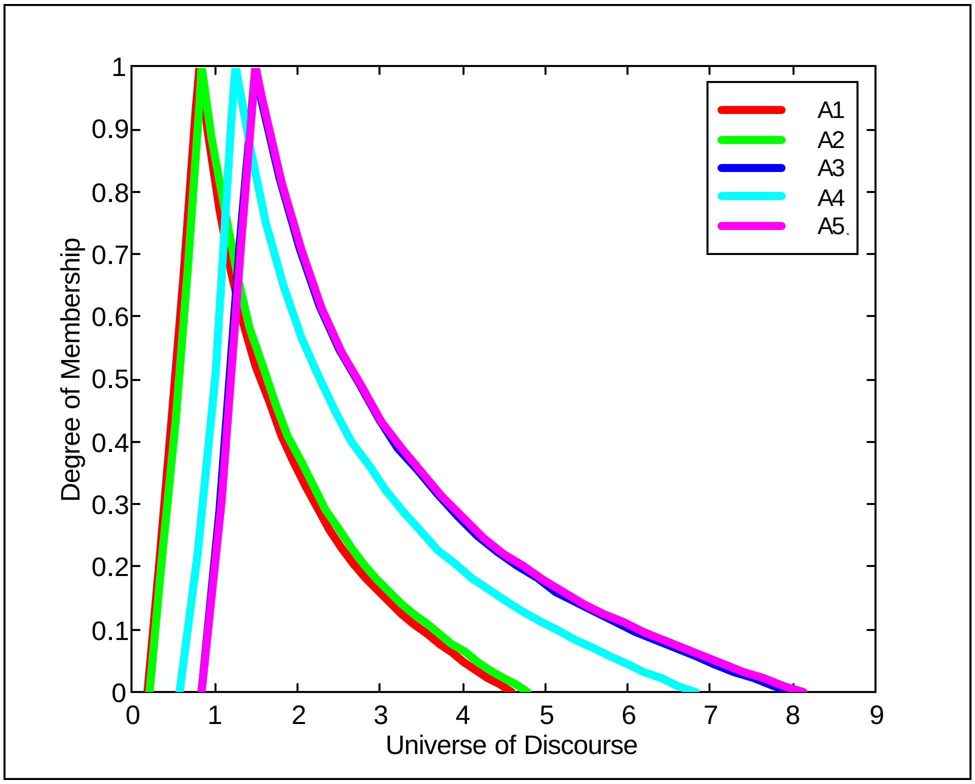

Figure 12. Tisza river basin distance metrics

All fuzzy set ranking methods produced identical rankings, namely [1 2 4 3 5]. In addition, the ranking order was not affected by the changes in decision maker's preferences. Sensitivity analysis was performed for all methods, and it was found that the ranking order still did not change. (Reason for this is given at the start of section 5.0.) By looking at Figure 12, it is observed that the first two alternatives, in addition to being very similar and having the smallest spreads, are closer to the origin than other alternatives. (Their closeness to the origin means they are favoured in Fuzzy Compromise Programming.) Alternatives 3 and 5 are also extremely similar in nature, but they have larger spreads and are slightly shifted to the right. It must be concluded that this is why they are consistently ranked last.

5.2.2 Yugoslavia System S1 Case Study

The background of this case study is not given in the appendix, as the purpose with this information is to investigate the consistency of the rankings by applying different fuzzy set ranking methods. Again, the distance metrics in fuzzy form, from Bender and Simonovic (1996) are:

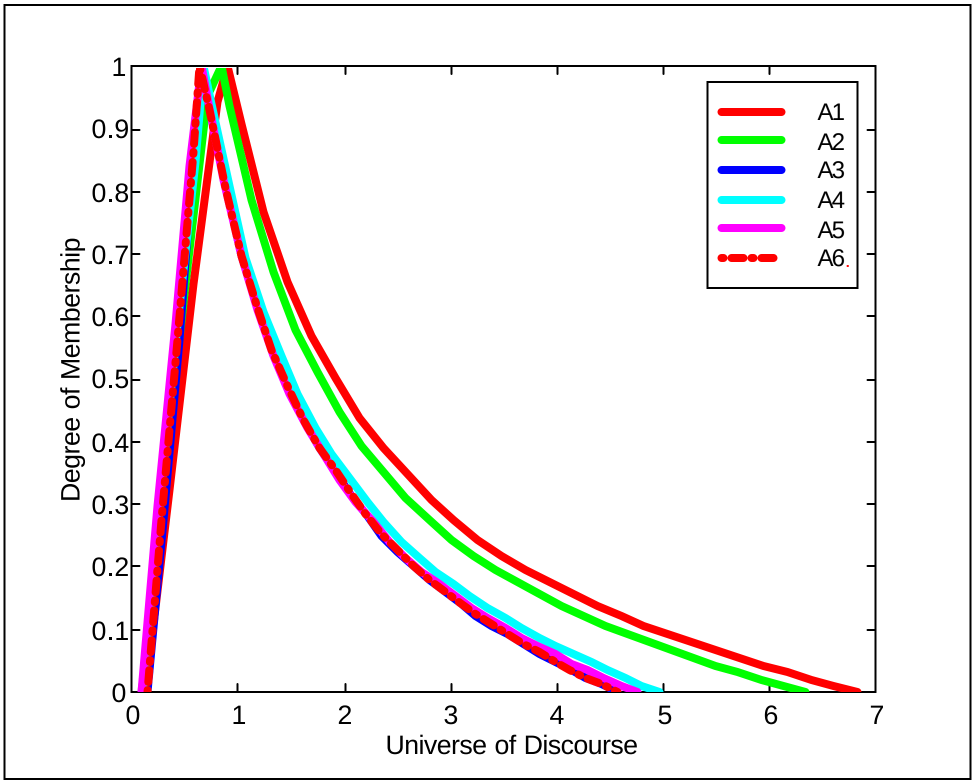

Figure 13. Yugoslavia System S1 distance metrics

In performing sensitivity analysis with Chen's (1985) method, the ranking order of [5 6 3 4 2 1] was produced for all values of *r*, except in the case of extreme optimism ($r < 0.1$), which produced [6 5 3 4 2 1]. It is noted that the degree to which alternative 6 was preferred to 5, was not significant, and thus does not pose a major problem.

Chang and Lee's (1994) method, however, produced more variations. For χ_1 values of 0.3, 0.5 and 0.7, the ranking was identical to that produced by Chen's (1985) method. Cases of extreme optimism and extreme pessimism produced different rankings, which was not observed in the application of Chen's method. It does make sense that if most weight is placed on only one part of the membership function, the ranking will be based mainly on than one part. If then, these parts are different from each other, it is reasonable to expect different rankings. Note that alternatives in Yugoslavia System S1 are more different from each other than are the alternatives in the Tisza River Basin example. It is expected that this is why the largest difference in rank are showing for this system.

With modified Cheng's (1998) method, most of the time the ranking of [5 6 3 4 2 1] is observed. This is again consistent with the results that were obtained previously. Alternatives 1, 2 and 4 were always ranked as worst, whereas alternatives 5, 6 and 3 were ranked in all possible combinations. This variation is, no doubt, due to weighting of parameters *r* and *q*. Regardless of the variation, this method provided an adequate check (that alternatives 5, 6 and 3 are among the best ones).

5.2.3 Yugoslavia System S2 Case Study

The only difference between System S1 and System S2 is that the former contains six, while the latter contains eight alternatives. The criteria for both systems were identical. As before, the distance metrics, from Bender and Simonovic (1996) are:

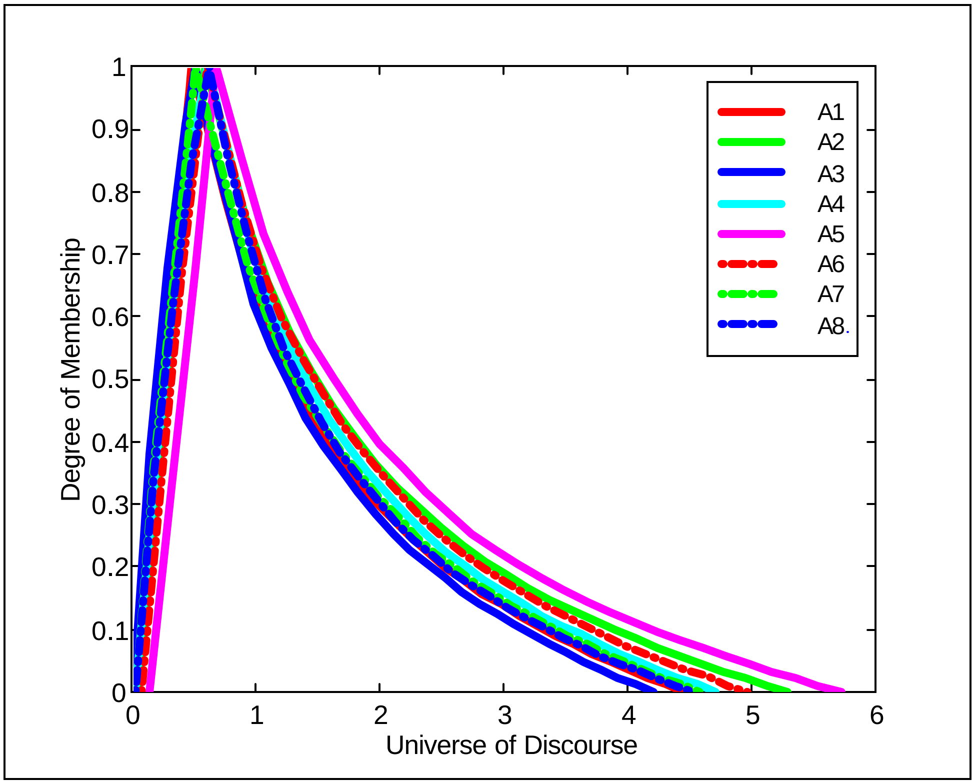

Figure 14. Yugoslavia System S2 distance metrics

As in the previous sub-section, Chen's (1985) method produced the identical ranking for all values of *r* which were greater than 0.1, namely [3 1 7 8 4 2 6 5]. For values of *r* smaller than *0.1*, the ranking of [1 3 7 8 4 2 6 5] was observed. Again, the degree of preference for alternative 1 over alternative 3 was so small, that it can be deemed insignificant.

The only variations with Chang and Lee's (1994) approach were the ranking of the three worst solutions. The best five solutions were always consistently ranked as [3 1 7 8 4].

With modified Cheng's (1998) method, five best solutions, namely [3 1 7 8 4] were ranked consistently for every case considered. Some deviation in the ranking existed for the three worst alternatives [2 6 5], but this was not significant enough to cause worrying. As before, this system had a much lower spread than S1, and that is why significant amount of deviation was not present

5.3 Weights Sensitivity Analysis for the Tisza River Basin Example

As previously mentioned, all relevant background information for this case study is given in Appendix B. The fuzzy input data, taken from Bender and Simonovic (1996) is given in Appendix C. The definition of fuzzy one was *trifn(0.99,1.0,1.01)*, and the fuzzy exponent p = *trifn(1,2,2)*. These two parameters affect the shape of the resulting fuzzy distance metrics.

By weights sensitivity analysis it is meant that fuzzy weights were varied for the Tisza River Basin example, and then a sensitivity analysis concerning decision maker's risk preferences was performed (i.e., from extreme pessimism, to neutral, to extreme optimism). Four sets of weights were considered, and as such, four sets of fuzzy distance metrics were generated. (Note that all other fuzzy input, such as p, f_i^*, f_i^- and f_i were held constant.) To investigate the variability of the rankings, weights from four experts were used. The experts were:

Expert 1, and Expert 4: held viewpoints somewhere in between the extremes of Expert 2 and 3;

Expert 2: had a mind set of someone who places emphasis on the protection of the environment, and very little on the development;

Expert 3: possessed strong opinions in favour of development, with very little concern to the environment;

Every expert was asked to rank the importance of the criteria using a scale 1 - 5, with 1 being least important and 5 most important, and all values in between. From this data, triangular fuzzy weights were constructed in the following manner:

1 = trifn(0.0,0.1,0.2); *2 = trifn(0.2,0.3,0.4)*; *3 = trifn(0.4,0.5,0.6)*; *4 = trifn(0.6,0.7,0.8)*; and *5 = trifn(0.8,0.9,1.0)*. If importance of the criteria was indicated by a number like *3.5*, the corresponding fuzzy weight was *trifn(0.5,0.6,0.7).*

The table below lists the importance ranking of each expert for each criteria.

Table 2. Expert weights

Criteria #	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
Expert 1	3	3	4	4	4	4	4	4	3.5	4	3.5	4
Expert 2	1	3	5	3	2	2	5	1	5	5	2	2.5
Expert 3	5	4	3	5	5	5	1	2	2	2	5	5
Expert 4	5	3	5	2	2	5	4	2	4	3	3	2

As a result of application of Fuzzy Compromise Programming, the Figures 15 - 18 represent four sets of fuzzy distance metrics.

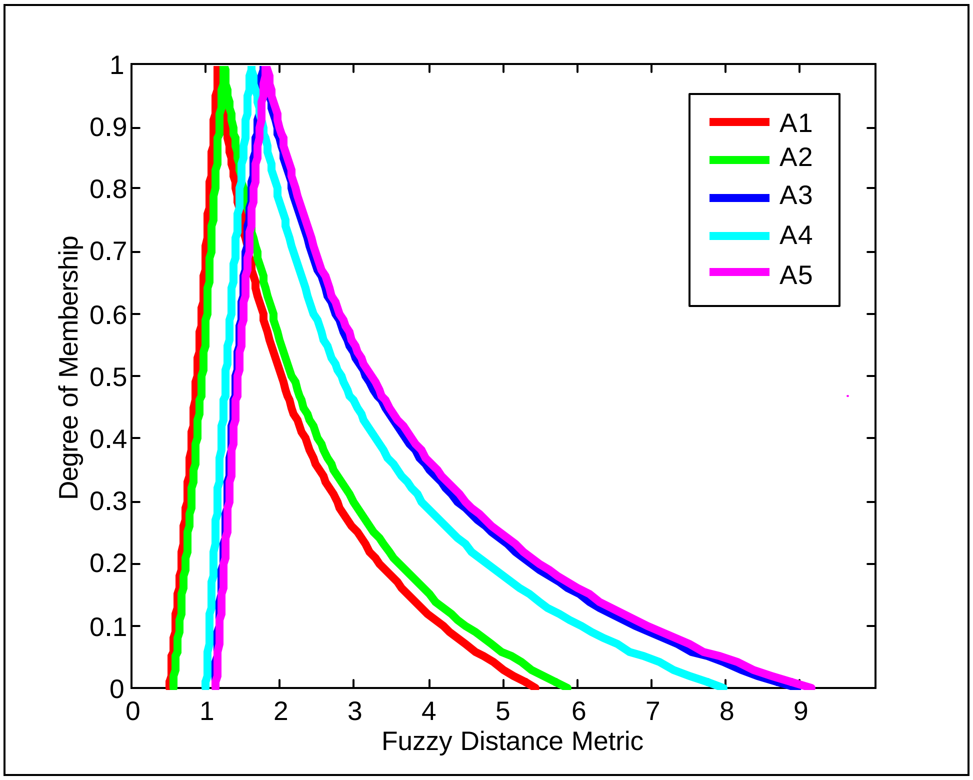

Figure 15. Expert 1 distance metrics

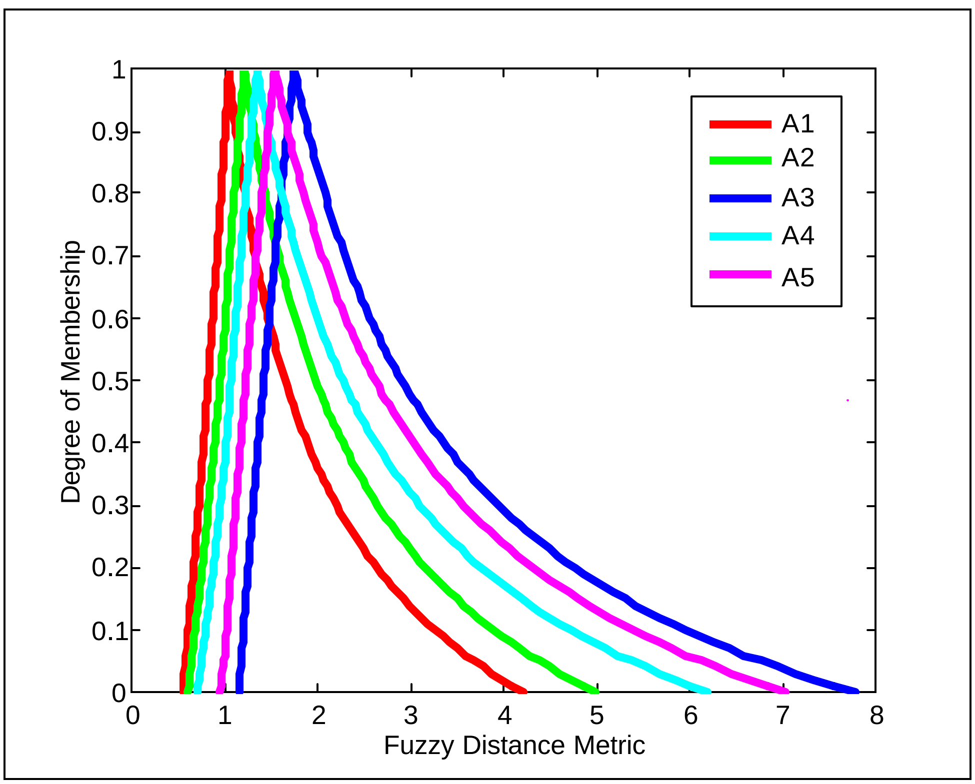

Figure 16. Expert 2 Distance Metrics

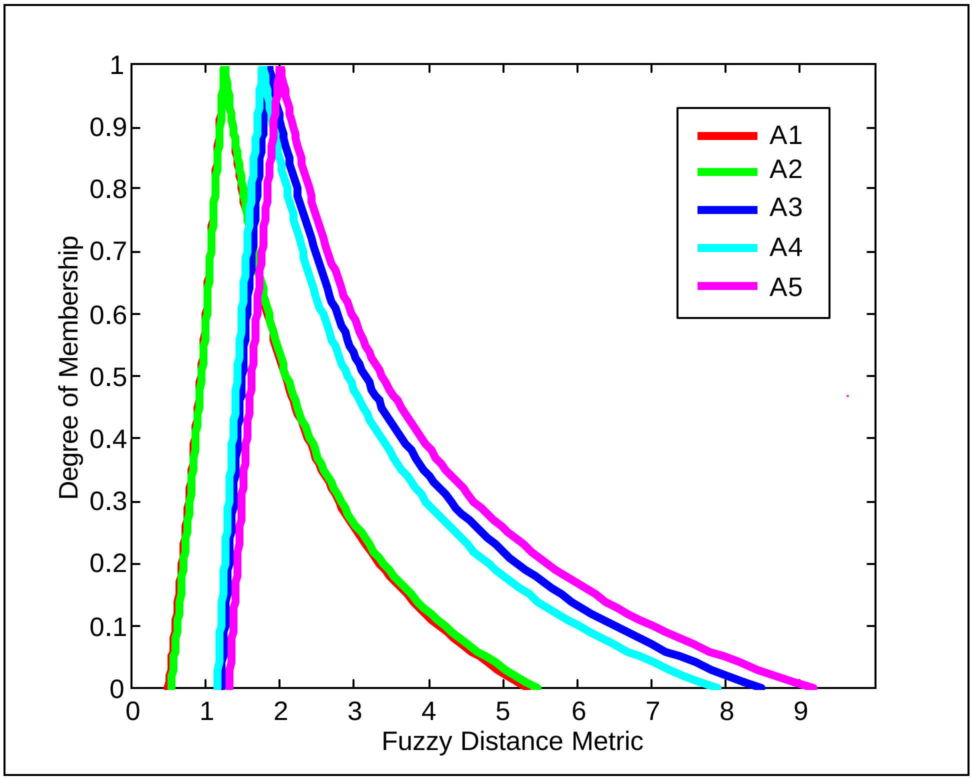

Figure 17. Expert 3 Distance Metrics

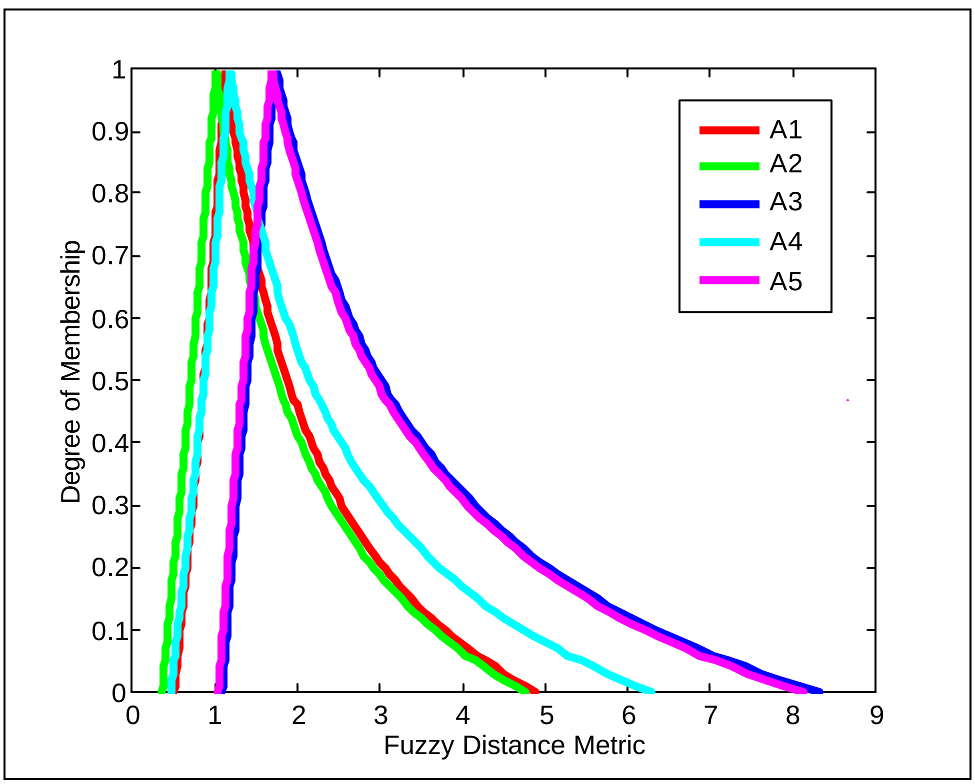

Figure 18. Expert 4 Distance Metrics

5.3.1 Results and Comments of the Weights Sensitivity Analysis

The results of the ranking for each expert are given next:

Expert 1: [1 2 4 3 5]; Expert 2: [1 2 4 5 3]; Expert 3: [1 2 4 3 5]; Expert 4: [2 1 4 5 3].

The actual ranking values for all cases considered are given in Appendix D [floppy disk]. It is noted that by changing the weights, some variation is, indeed, present. However, because of the robustness of the Tisza River Basin system alternatives, even these variations were not large. Note that in performing expert risk preference sensitivity analysis, it was observed that all rankings were *insensitive* to user preferences for every method. This fact can be explained by observing that, for every expert, features which are expected to change the rankings were not present in the four sets of fuzzy distance metrics. In addition, the circumstances in which Chen's (1985) (and modified Cheng's (1998)) method can not give adequate rankings were also not present. As such, insensitivity to risk preferences is deemed quite reasonable.

Also, it is noted that alternatives 1 and 2 are extremely similar. That is, they are consistently closer towards the origin than other alternatives - a feature that compromise programming favours. Because of this, they are consistently ranked as the two best alternatives. Their degree of fuzziness, again being the smallest from the set, indicates that their relative degree of performance is quite good. Alternatives 3, 4 and 5 on the other hand, have roughly the same shape, but greater degrees of fuzziness (greater than alternatives 1 and 2) - this indicates that their degree of performance is quite vague. Also, they are farther away from the origin that alternatives 1 and 2, and so are consistently ranked as the three worst alternatives.

5.4 Results of the Overall Group Decision

Four sets of data were taken from sub-section 5.2 and then applied to group decision making algorithms outlined in section 4.0. The results were not at least unexpected. They are:

Q-core: {1.0, 1.0, 0, 0.4, 0}; which represents the degrees that alternatives 1 and 2 were not at all defeated (in pairwise comparison).

α/Q-core: {0.7750, 0.6500, 0, 0, 0}; which gives the degrees that alternatives 1 and 2 were not sufficiently defeated (to a degree of 0.7).

s/Q-core: {0.7250, 0.5625, 0, 0, 0}; which expresses the degrees that alternatives 1 and 2 were not strongly defeated.

Results outputted by this methodology concern only the best alternatives, or ones that were not defeated in pairwise comparison. As such, no information is given about the three worst alternatives. Regardless, a final decision can now be made. Alternative 1 is the best overall water resources option for the Tisza River Basin.

6.0 Recommendations for Future Research

Even with the final decision of the previous paragraph, more analysis should be performed before anything is done to the basin. For example, in performing the weights sensitivity analysis, only weights were changed by each expert, while keeping other parameters (such as fuzzy one, fuzzy *p*, positive and negative ideals) constant. Perhaps, in the future each expert should be allowed to determine their own positive and negative ideals, together with their own definition of the fuzzy *p*. That, it is anticipated, will make Fuzzy Compromise Programming more realistic in modeling human decision making.

Also, only one methodology for including multiple decision makers was implemented in the group decision process. As was mentioned previously, other methodologies are also available, and so they should be used. One such methodology is suggested by Bender and Simonovic (1996), and it involves adjusting fuzzy weights and fuzzy criteria values to include views and opinions of multiple experts. Essentially, this adjustment produces a set of data that corresponds to an opinion of the entire group, which is then inputted into the Fuzzy Compromise code and the results sorted appropriately. Some work on aggregation operators – which could be used to aggregate individual opinions into a single, group opinion was done by Despic and Simonovic (2000). As such, it could probably be applied to our group decision making problem.

In addition, Cheng (1999) as well as Ghyym (1999) present additional methodologies for including multiple experts into the fuzzy decision environment. It is suggested that methodologies listed in this section be seriously considered for future work in this area.

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Appendix A – Tisza River Basin Example

The Tisza River Basin example was studied with the purpose of determining an optimal long range (60 years) water resources system that is best suited for the region. In developing the alternatives, David and Duckstein (1976) considered twelve criteria, many of which were subjective. The table below shows the alternatives, together with criteria values for each system in the study.

Table 2. Original Criteria Values used by David and Duckstein (1976)

	Criteria	I	II	III	IV	V
1	Total Annual Cost (10^9 Forints/year)	99.6	85.7	101.1	95.1	101.8
2	Probability of Water Shortage	4	19	50	50	50
3	Energy (Reuse Factor)	0.7	0.5	0.01	0.1	0.01
4	Land and Forrest Use (1000 ha)	90	80	80	60	70
5	Water Quality	Very Good	Good	Bad	Very Bad	Fair
6	Recreation	Very Good	Good	Fair	Bad	Bad
7	Flood Protection	Good	Excellent	Fair	Excellent	Bad
8	Manpower Impact	Very Good	Very Good	Good	Fair	Fair
9	Environmental Architecture	Very Good	Good	Bad	Good	Fair
10	Development Possibility	Very Good	Good	Fair	Bad	Fair
11	International Cooperation	Very Easy	Easy	Fairly Difficult	Difficult	Fairly Difficult
12	Sensitivity	Not Sensitive	Not Sensitive	Very Sensitive	Sensitive	Very Sensitive

A short description of the criteria is outlined next:

1) Total Annual Cost includes the cost of construction together with operation and maintenance of the system.

2) Probability of Water Shortage criterion is self explanatory, and thus will not be elaborated on.

3) Energy Reuse Factor is a ratio of generated energy (hydroelectric power) to the consumed energy (i.e., water pumping costs) of the system. Therefore, this ratio should be as high as possible.

4) Land and Forrest Use is amount of land and forests that are required by the

system. This will be needed for reservoirs (storage) and canals (transfer). Thus, the less of the land and forests is used in developing the region, the better.

Note criteria 5-10 are evaluated with linguistic terms *{excellent, very good, good, fair, bad}*; criterion 11 is evaluated with *{very easy, easy, fairly difficult, difficult}*, and criterion 12 with *{very sensitive, sensitive, fairly sensitive, not sensitive}*.

5) Water Quality includes the quality of drinking water, as well as the overall quality of water found in the rivers and streams within the basin.

6) Recreation is again a self-explanatory criterion. It is an important in that as the region develops and population increases, this criterion will directly have an impact on the quality of life of the public with in the river basin. As such, it should be as best as possible.

7) Adequate Flood Protection for the Tisza River (and its tributaries) should be provided. The social and economic consequences of floods can be quite severe, and so as much of protection as possible should be provided.

8) Manpower Impact is a criterion that has two parts. First, the number of persons needed for the construction and operation of the system should be as low as possible (due to monetary reasons). On the other hand, the persons employed should receive an adequate yearly income.

9) Environmental Architecture includes the preservation of the existing environment, including natural habitats for the various species of animals, fish and insects. Also, the esthetics of the region should be preserved as well.

10) Development Possibility is a social criterion that must be considered due to the fact that the population of the region will most likely increase within the next fifty

years. Again, as before, it should be as best as possible. (Note that criterion 9 is in direct conflict to criterion 10!)

11) International Cooperation is a factor that concerns the regions' neighbours, and thus is extremely important. It is measured with the degree of difficulty the implementation of the system is likely to raise international concern. Of course, the more international concern, the worse.

12) Sensitivity criterion is one which requires most explanation. The water resources system to be implemented should be flexible enough to accommodate a variety of requirements, which can not be known at the present time. The system should be able to link itself with another system, which might be built sometime in the future. Also, "it should be able to cope with several types of uncertainties, such as the natural uncertainty inherit in forecasting, the strategic uncertainty due to unknown future allocation policy, the economic uncertainty pertaining to the cost and loss functions... " (David and Duckstein (1976), p.738).

Figures A1 and A2 show the location of the alternatives, and the following paragraphs briefly describe each alternative water resources system. (Maps are courtesy www.maps.com.)

Figure A19. Locations of Alternatives I and III

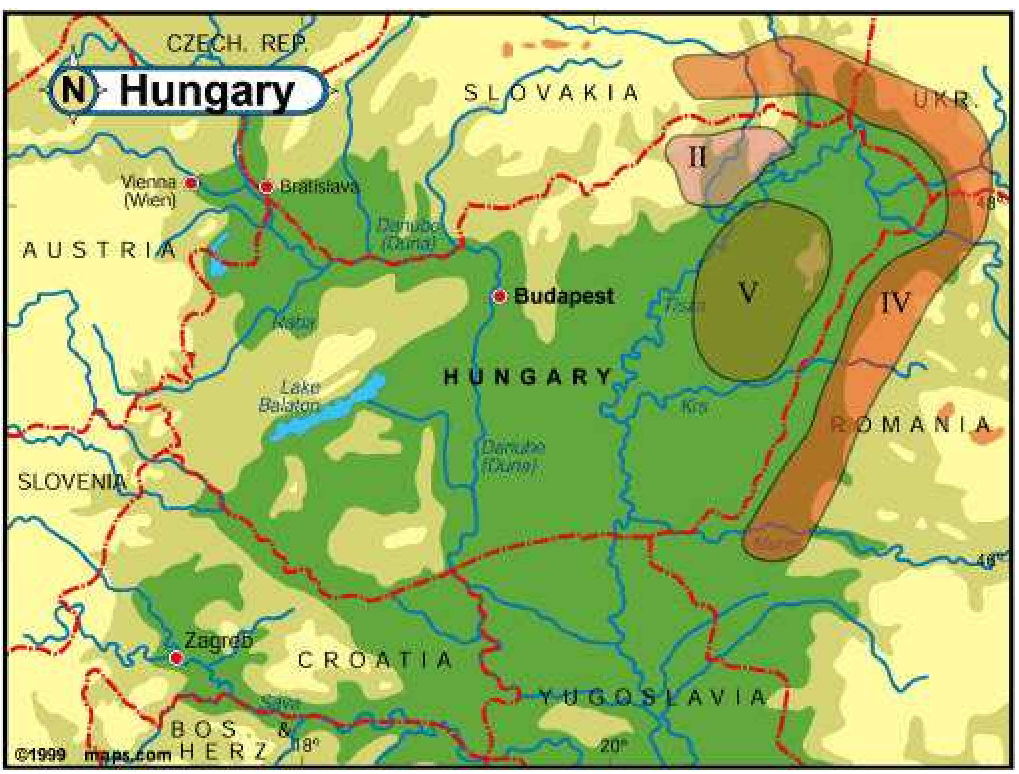

Figure A20. Location of Alternatives II, IV and V

(The following text is taken from reference David and Duckstein (1976))

System I - Danube-Tisza Interbasin Transfer Using a Multi-Purpose Canal-Reservoir System

The system used the water resources of both Tisza and Danube rivers. The water is transferred all year around from the Danube by a gravity canal in the flat area and by a pumped canal reservoir system in the mountains. There is enough allocated water in the Danube River for the present and the future; therefore the development and operation of the system does not depend to a great extent on international operation.

However, the system would consume large quantity of resources (e.g., land and forest resources for reservoir sites); it would not be of much help for flood control and drainage; and the quality of the Danube River is likely to decrease in the future, so that some treatment will be needed. The sensitivity of the system to these data is rather low.

System II - Pumped Reservoir System in the Northeastern Part of the Region

This pumped reservoir system supplied only from the Tisza River is developed [mainly] on the hilly region [of northeastern Hungary]. The system is also basically oriented toward water resources utilization, but the natural supply of water is available only four to five months per year. The system, which provides excellent flood protection, also consumes large quantities of resources. The water quality and the runoff condition are based on good international cooperation. Large peek pumping capacities are needed because the pumping time is generally limited to high water in the river. The system sensitivity to these data is not important.

System III - Flat Land Reservoir System

This system could be developed on the flat-land part of the region. The system using Tisza water would be composed of shallow flat land reservoirs 2 to 4 m deep, but only a limited area of 5.5 sq km could be used for reservoirs. A large quantity of land and forest resources is needed. The development and operation of the system is fairly difficult from both energy management and international cooperation viewpoints. The operation costs, especially for reuse, are quite high. The system is very sensitive to the basic data.

System IV - Mountain Reservoir System in Upper Tisza River Basin

This system would be located outside the country. It uses and regulates the water resources of the Tisza River by gravity. All storage capacity available in the framework of international cooperation is used, but not all the water resources. Excellent international cooperation must be initiated and maintained, which may be difficult. Because of international cooperation, difficulties would arise in evaluating costs. As a result of these uncertainties, the system is sensitive to data.

System V - Groundwater Storage System

The system would be developed mainly on the flat-land part of the region, especially on the eastern part. The system using the Tisza water and stored groundwater resources would be composed of underground storage spaces. But such spaces are limited; therefore, reuse [of water] would have to be high and salinity problems might arise in the future. Efficient use of the small storage space needs international cooperation so that water will be available to fill the reservoirs. Lastly, the system is very sensitive to uncertainties.

Appendix B – Tisza Fuzzy Input Data

Table 3, Tisza River Basin fuzzy weights, positive and negative ideals

w_i = *trifn(a, b, c)*			*f**			*f*		
0.5	0.66	0.75	25	30	30	0	0.5	0.5
0.5	0.66	0.75	99	100	100	0	1	1
0.5	0.66	0.75	0.9	1	1	0	0.1	0.1
0.5	0.66	0.75	0.99	1	1	0	0.1	0.1
0.5	0.66	0.75	0.9	1	1	0	0.1	0.1
0.5	0.66	0.75	0.9	1	1	0	0.1	0.1
0.25	0.33	0.5	99	100	100	0	0.1	0.1
0.25	0.33	0.5	0.9	1	1	0	0.1	0.1
0.5	0.66	0.75	0.9	1	1	0	0.1	0.1
0.5	0.66	0.75	0.9	1	1	0	0.1	0.1
0.25	0.33	0.5	0.9	1	1	0	0.1	0.1
0.25	0.33	0.5	0.9	1	1	0	0.1	0.1

Table 4. Tisza River Basin fuzzy criteria values

f_1			f_2			f_3			f_4			f_5		
10.3	10.4	10.5	24.2	24.3	24.4	8.8	8.9	9	14.8	14.9	15	8.1	8.2	8.3
94	96	98	79	81	83	48	50	52	48	50	52	48	50	52
0.6	0.7	0.8	0.4	0.5	0.6	0	0.1	0.2	0.6	0.7	0.8	0.2	0.3	0.4
0.69	0.7	0.71	0.49	0.5	0.51	0	0.01	0.02	0.09	0.1	0.11	0	0.01	0.02
0.6	0.7	0.8	0.4	0.5	0.6	0.2	0.3	0.4	0	0.1	0.2	0	0.1	0.2
0.4	0.5	0.6	0.8	0.9	1	0.2	0.3	0.4	0.8	0.9	1	0	0.1	0.2
8	10	12	18	20	22	18	20	22	38	40	42	28	30	32
0.6	0.7	0.8	0.6	0.7	0.8	0.4	0.5	0.6	0.2	0.3	0.4	0.2	0.3	0.4
0.6	0.7	0.8	0.4	0.5	0.6	0	0.1	0.2	0.4	0.5	0.6	0.2	0.3	0.4
0.7	0.8	0.9	0.5	0.6	0.7	0.3	0.4	0.5	0.1	0.2	0.3	0.3	0.4	0.5
0.6	0.7	0.8	0.4	0.5	0.6	0.2	0.3	0.4	0	0.1	0.2	0.2	0.3	0.4
0.1	0.2	0.3	0.1	0.2	0.3	0.7	0.8	0.9	0.5	0.6	0.7	0.7	0.8	0.9

Appendix C – Results

Table 5. Tisza River Basin results

Chang and Lee's (1994)			Chen's (1985) method		
χ_1 = 0.1	χ_1 = 0.5	χ_1 = 0.9	r = 2	r = 1	r = 0.5
1	1	1	1	1	1
2	2	2	2	2	2
4	4	4	4	4	4
3	3	3	3	3	3
5	5	5	5	5	5

Table 6. Yugoslavia System S1 results

Chang and Lee's (1994)			Chen's (1985) method		
χ_1 = 0.1	χ_1 = 0.5	χ_1 = 0.9	r = 2	r = 1	r = 0.5
6	5	5	5	5	5
3	6	4	6	6	6
5	3	6	3	3	3
4	4	3	4	4	4
2	2	2	2	2	2
1	1	1	1	1	1

Table 7. Yugoslavia System S2 results

Chang and Lee's (1994)			Chen's (1985) method		
χ_1 = 0.1	χ_1 = 0.5	χ_1 = 0.9	r = 2	r = 1	r = 0.5
3	3	3	3	3	3
1	1	1	1	1	1
7	7	7	7	7	7
8	8	8	8	8	8
4	4	4	4	4	4
6	6	2	2	6	6
2	2	6	6	2	2
5	5	5	5	5	5

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